

An analytic version of the Langlands correspondence for complex curves

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Introduction

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The *unramified Langlands correspondence* for a curve X/\mathbb{F}_q :

on one side, **joint spectrum** of the commuting **Hecke operators** acting on the space of L^2 functions on the set of \mathbb{F}_q -points of the stack Bun_G of G -bundles on X ;

on the other side, **Galois data** associated to X and the Langlands dual group ${}^L G$.

If X is a curve over \mathbb{C} , the Langlands correspondence has been traditionally formulated in terms of **sheaves** rather than functions. It is usually referred to as *geometric* or *categorical*.

It turns out that there is a *function-theoretic (or analytic) version* for complex curves as well. The two versions complement each other.

Analogy: correlation functions in 2D conformal field theory are single-valued **bilinear combinations** of (multi-valued) **conformal** and **anti-conformal** blocks.

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Namely, it is possible to associate to Bun_G of X/\mathbb{C} (and more generally X/F , where F is a local field) a natural Hilbert space \mathcal{H}_G and define analogues of the Hecke operators acting on a dense subspace of \mathcal{H}_G . We conjecture that they give rise to mutually commuting normal compact operators on \mathcal{H}_G .

In the case $F = \mathbb{C}$, these Hecke operators commute with the global holomorphic differential operators on Bun_G introduced by Beilinson and Drinfeld, as well as their complex conjugates.

We conjecture that the joint spectrum of this commutative algebra (properly understood) can be identified with the set of ${}^L G$ -opers on X whose monodromy is in the split real form of ${}^L G$, up to conjugation (these play the role of the Galois data).

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Basic definitions

X – smooth projective irreducible curve over \mathbb{C}

$S \subset X$ – reduced divisor

K_X – canonical line bundle on X

G – connected simple algebraic group over \mathbb{C}

${}^L G$ – the Langlands dual group

$\text{Bun}_G = \text{Bun}_G(X, S)$ – algebraic stack of pairs (\mathcal{F}, r_S) , where \mathcal{F} is a G -bundle on X and r_S is a B -reduction of $\mathcal{F}|_S$

$\text{Bun}_G^s = \text{Bun}_G^s(X, S) \subset \text{Bun}_G(X, S)$ – substack of those stable pairs (\mathcal{F}, r_S) whose group of automorphisms is equal to the center $Z(G)$ of G

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Assumption:

$\text{Bun}_G^s(X, S)$ is *open and dense* in $\text{Bun}_G(X, S)$, i.e. one of the following cases:

- 1 the genus of X is greater than 1, and S is arbitrary;
- 2 X is an elliptic curve and $|S| \geq 1$;
- 3 $X = \mathbb{P}^1$ and $|S| \geq 3$.

The stack $\text{Bun}_G^s(X, S)$ is a $Z(G)$ -gerbe over a smooth algebraic variety $\text{Bun}_G^s(X, S)$.

K_{Bun} – the canonical line bundle on Bun_G

For simply-connected G , Beilinson and Drinfeld have constructed a square root $K_{\text{Bun}}^{1/2}$ of K_{Bun} . For a general G , their construction sometimes requires a choice of a square root of the canonical line bundle K_X on X . If so, we will make such a choice (however, the bundle $\Omega_{\text{Bun}}^{1/2}$ below does not depend on this choice).

We'll use the same notation for the restriction of this $K_{\text{Bun}}^{1/2}$ to Bun_G^s .

Given a holomorphic line bundle \mathcal{L} on a variety Y , let

$$|\mathcal{L}| := \mathcal{L} \otimes \overline{\mathcal{L}}$$

Set $\Omega_{\text{Bun}}^{1/2} := |K_{\text{Bun}}^{1/2}|$ – the line bundle of half-densities on Bun_G^s .

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Hilbert space

Let V_G – space of smooth compactly supported sections of $\Omega_{\text{Bun}}^{1/2}$ over Bun_G^s , and let

$\langle \cdot, \cdot \rangle$ – positive-definite Hermitian form on V_G given by

$$\langle v, w \rangle := \int_{\text{Bun}_G^s} v \cdot \bar{w}, \quad v, w \in V_G$$

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What kind of operators could act on the Hilbert space \mathcal{H}_G ?

- 1 holomorphic differential operators;
- 2 anti-holomorphic differential operators;
- 3 Hecke (integral) operators.

Challenges: Differential operators are **unbounded**. It is a highly non-trivial task to define their self-adjoint (or normal) extensions, which is necessary to be able to make sense of the notion of their joint spectra on \mathcal{H}_G (and there could be different choices).

Hecke operators are also initially defined on a dense subspace of \mathcal{H}_G . But we conjecture that they extend by continuity to **normal compact operators** on the entire \mathcal{H}_G . If one proves this, one gets a good **spectral problem** for both Hecke & differential operators since one can show that they commute (in the sense we'll discuss later).

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Holomorphic differential operators

Consider the case of simply-connected G and $|S| = \emptyset$.

Let \mathcal{D}_G be the sheaf of algebraic (hence holomorphic) differential operators acting on the line bundle $K_{\text{Bun}}^{1/2}$ on Bun_G .

$$D_G := \Gamma(\text{Bun}_G, \mathcal{D}_G)$$

Theorem 1 (Beilinson & Drinfeld)

$D_G \simeq \text{Fun Op}_{L_G}(X)$, where $\text{Op}_{L_G}(X)$ – space of L_G -opers on X .

Definition. An L_G -oper on a curve X is a holomorphic L_G -bundle with a holomorphic connection ∇ and a reduction to a Borel subgroup L_B which is in a special relative position with ∇ .

Example (to be discussed later). A PGL_2 -oper on X is a projective connection, i.e. a second-order holomorphic differential operator of the form $\partial_z^2 - v(z): K_X^{-1/2} \rightarrow K_X^{3/2}$.

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$D_G \simeq \text{Fun Op}_{L^1G}(X)$, where $\text{Op}_{L^1G}(X)$ – space of L^1G -opers on X .

Definition. An L^1G -oper on a curve X is a holomorphic L^1G -bundle with a holomorphic connection ∇ and a reduction to a Borel subgroup L^1B which is in a special relative position with ∇ .

Example (to be discussed later). A PGL_2 -oper on X is a projective connection, i.e. a second-order holomorphic differential operator of the form $\partial_z^2 - v(z): K_X^{-1/2} \rightarrow K_X^{3/2}$.

Holomorphic differential operators

Consider the case of simply-connected G and $|S| = \emptyset$.

Let \mathcal{D}_G be the sheaf of algebraic (hence holomorphic) differential operators acting on the line bundle $K_{\text{Bun}}^{1/2}$ on Bun_G .

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Anti-holomorphic differential operators

Complex conjugates of elements of D_G are global **anti-holomorphic** differential operators acting on $\overline{K}_{\text{Bun}}^{1/2}$.

They generate a commutative algebra \overline{D}_G .

$$\overline{D}_G \simeq \text{Fun } \overline{\text{Op}}_{L_G}(X)$$

$\mathcal{A}_G := D_G \otimes \overline{D}_G$ is a **commutative algebra** acting on C^∞ sections of the line bundle $\Omega_{\text{Bun}}^{1/2} = K_{\text{Bun}}^{1/2} \otimes \overline{K}_{\text{Bun}}^{1/2}$ on Bun_G^s .

Let \tilde{V}_G be the space of smooth sections of $\Omega_{\text{Bun}}^{1/2}$ on $\text{Bun}_G^{\text{vs}} \subset \text{Bun}_G^s$, the moduli space of **very stable G -bundles** (those \mathcal{F} which do not admit non-zero $\phi \in \Gamma(X, \mathfrak{g}_{\mathcal{F}} \otimes K_X)$ taking nilpotent values everywhere).

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“Doubling” of the quantum Hitchin system

Given a homomorphism $\Lambda : \mathcal{A}_G \rightarrow \mathbb{C}$, denote by $\tilde{V}_{G,\Lambda}$ the corresponding **eigenspace** of \mathcal{A}_G in \tilde{V}_G .

$\Lambda = (\chi, \mu)$, where $\chi \in \text{Op}_{LG}(X)$, $\mu \in \overline{\text{Op}}_{LG}(X)$.

If f is a non-zero element of $\tilde{V}_{G,(\chi,\mu)}$, then it satisfies two systems of **differential equations**:

$$(1) \quad P \cdot f = \chi(P)f, \quad P \in D_G$$

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System (1) is known as the **quantum Hitchin system**.

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The corresponding left \mathcal{D}_G -module

$$\Delta_\chi := \mathcal{D}_G \otimes_{D_G} \mathbb{C}_\chi$$

was introduced and studied by Beilinson and Drinfeld, who have proved that Δ_χ is a **Hecke eigensheaf** corresponding to the ${}^L G$ -oper χ under the geometric/categorical Langlands correspondence.

Moreover, they have shown that the restriction of Δ_χ to Bun_G^{vs} is a vector bundle with a projectively flat connection (of a rank that grows exponentially with the genus of X).

Local sections of Δ_χ over Bun_G^{vs} are local **holomorphic solutions** of system (1). They are **multi-valued** and the monodromy is rather complicated, which is why it's impossible to attach to a given χ a specific holomorphic half-form. (Even if there were single-valued solutions, it wouldn't be clear which one to choose.) Instead, we attach a whole **\mathcal{D}_G -module on Bun_G** to χ .

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Likewise, to $\mu \in \overline{\text{Op}}_{LG}(X)$ we attach an anti-holomorphic D -module $\overline{\Delta}_\mu$ whose local sections on Bun_G^{vs} are local **anti-holomorphic solutions** of system (2), also multi-valued.

However, if we look for *smooth solutions* of systems (1) and (2) simultaneously, it is possible that for some χ and μ there will be a single-valued solution, which can be written locally in bilinear form

$$f = \sum_{i,j} a_{ij} \phi_i(\mathbf{z}) \overline{\psi}_j(\overline{\mathbf{z}})$$

$\{\phi_i\}$ – local sections of Δ_χ

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This actually implies that $\dim \tilde{V}_{G,(\chi,\mu)} < \infty$.

Moreover, if Δ_χ is **irreducible and has regular singularities** (question posed by Beilinson and Drinfeld; follows from the results of D. Gaitsgory for $G = PGL_n$) and $\tilde{V}_{G,(\chi,\mu)} \neq 0$, then $\dim \tilde{V}_{G,(\chi,\mu)} = 1$.

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Conjecture 1

- 1 All $\tilde{V}_{G,(\chi,\mu)} \subset \mathcal{H}_G$
- 2 There is an orthogonal decomposition
$$\mathcal{H}_G = \widehat{\bigoplus}_{(\chi,\mu)} \tilde{V}_{G,(\chi,\mu)}$$
- 3 If $\tilde{V}_{G,(\chi,\mu)} \neq 0$, then $\mu = \tau(\bar{\chi})$, where τ is the Chevalley involution on ${}^L G$ and $\chi \in \text{Op}_{{}^L G}(X)_{\mathbb{R}}$.

Definition. $\text{Op}_{{}^L G}(X)_{\mathbb{R}}$ is the set of ${}^L G$ -opers on X such that the *monodromy representation* $\rho_{\chi} : \pi_1(X, p_0) \rightarrow {}^L G(\mathbb{C})$ is isomorphic to its *complex conjugate*, i.e. $\rho_{\chi} \simeq \bar{\rho}_{\chi}$.

We expect that $\text{Op}_{{}^L G}(X)_{\mathbb{R}}$ is a *discrete subset* of $\text{Op}_{{}^L G}(X)$. This is known for ${}^L G = PGL_2$ (G. Faltings).

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associated to χ is contained, up to conjugation, in the *split real form* $L^G(\mathbb{R})$ of $L^G(\mathbb{C})$.

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Proving Conjecture 1 directly is a daunting task. This is where the third set of operators on \mathcal{H}_G – **integral Hecke operators** – comes in handy.

Though they are also initially defined on a dense subspace of \mathcal{H}_G (like diff. operators), we conjecture that, unlike the differential operators, they extend to (mutually commuting) continuous operators on the entire \mathcal{H}_G , which are moreover *normal* and *compact* with trivial common kernel.

If so, then by a general result of functional analysis, \mathcal{H}_G decomposes into a (completed) direct sum of mutually orthogonal *finite-dimensional eigenspaces* of the Hecke operators. Moreover, we can show that they commute with the differential operators, and so the Compactness Conjecture can be used to prove Conjecture 1.

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Hecke operators

In fact, Hecke operators can be defined for curves over any local field.

For non-archimedean local fields, these operators were essentially defined by [A. Braverman](#) and [D. Kazhdan](#) in

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For $G = PGL_2$, $X = \mathbb{P}^1$, Hecke operators were studied by [M. Kontsevich](#) in his paper *Notes on motives in finite characteristic* (2007). In his letters to us (2019) he conjectured compactness of averages of the Hecke operators over sufficiently many points.

The idea that Hecke operators over \mathbb{C} could be used to construct an analogue of the Langlands correspondence was suggested in 2018 by [R.P. Langlands](#), who attempted to construct them in the case when $G = GL_2$, X is an elliptic curve, and $S = \emptyset$ (however, for an elliptic curve X we can only define Hecke operators if $|S| \neq \emptyset$).

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For a dominant coweight λ of G , denote by

$$q : Z(\lambda) \rightarrow \text{Bun}_G \times \text{Bun}_G \times X$$

the *Hecke correspondence* attached to λ . Let

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be the projections, and set $q_i := p_i \circ q$.

The following is due to Beilinson–Drinfeld and Braverman–Kazhdan.

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There exists an isomorphism

$$a : q_1^*(K_{\text{Bun}}^{1/2}) \simeq q_2^*(K_{\text{Bun}}^{1/2}) \otimes \omega_2 \otimes q_3^*(K_X^{-\langle \lambda, \rho \rangle})$$

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$$U_G(\lambda) := \{\mathcal{F} \in \text{Bun}_G^s \mid (q_2(q_1^{-1}(\mathcal{F}))) \subset \text{Bun}_G^s\}$$

This is an open subset of Bun_G^s , which is **dense** if

$$\dim \text{Bun}_G = \dim G \cdot (g - 1) + \dim G/B \cdot |S| \quad (g > 1)$$

is sufficiently large. (For example, for $G = PGL_2$, $\lambda = \omega_1$, this is so if $\dim \text{Bun}_G > 1$.)

Assume that $U_G(\lambda) \subset \text{Bun}_G^s$ is **dense** and let $V_G(\lambda) \subset V_G$ be the subspace of half-densities f such that $\text{supp}(f) \subset U_G(\lambda)$.

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$$Z_{\mathcal{G},x} := (q_2 \times q_3)^{-1}(\mathcal{G} \times x), \quad \mathcal{G} \in \text{Bun}_G(\mathbb{C}), \quad x \in X(\mathbb{C})$$

It is compact and isomorphic to the closure $\overline{\text{Gr}}_\lambda$ of the $G[[z]]$ -orbit Gr_λ in the affine Grassmannian of G .

The results of Braverman–Kazhdan imply that for any $f \in V_G(\lambda)$ and $x \in X(\mathbb{C})$, the restriction of the pull-back $q_1^*(f)$ to $Z_{\mathcal{G},x}$ is a **well-defined measure** with values in the line $|\Omega_{\text{Bun}}|_{\mathcal{G}}^{1/2} \otimes |K_X|_x^{-\langle \lambda, \rho \rangle}$.

Hence for any $f \in V_G(\lambda)$, the integral

$$(\widehat{H}_\lambda(x) \cdot f)(\mathcal{G}) := \int_{Z_{\mathcal{G}}^x(F)} q_1^*(f),$$

is **absolutely convergent** for all $\mathcal{G} \in \text{Bun}_G^s(\mathbb{C})$ and belongs to the space \widehat{V}_G of smooth functions on $\text{Bun}_G^s(\mathbb{C})$.

Therefore this integral defines a **Hecke operator**

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Conjecture 2 (Compactness Conjecture)

- 1 For any $f \in V_G(\lambda)$ and $x \in X(\mathbb{C})$, the section $\widehat{H}_\lambda(x) \cdot f$ is square-integrable (i.e. belongs to \mathcal{H}_G) and hence we obtain an operator

$$H_\lambda(x) : V_G(\lambda) \rightarrow \mathcal{H}_G \otimes |K_X|_x^{-\langle \lambda, \rho \rangle}.$$

- 2 For any identification $(K_X^{1/2})_x \cong \mathbb{C}$, the corresponding operators $V_G(\lambda) \rightarrow \mathcal{H}_G$ extend to a family of commuting **compact normal** operators on \mathcal{H}_G , which we also denote by $H_\lambda(x)$.
- 3 $H_\lambda(x)^\dagger = H_{-w_0(\lambda)}(x)$.
- 4 $\bigcap_{\lambda, x} \text{Ker} H_\lambda(x) = \{0\}$.

Remark. We expect that integrals defining Hecke operators $H_\lambda(x)$ are absolutely convergent for all $f \in V_G$.

From now on we **assume** that Compactness Conjecture holds.

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Let \mathbb{H}_G be the **commutative algebra** generated by operators $H_\lambda(x)$, $\lambda \in \check{P}^+$, $x \in X$. Denote by $\text{Spec}(\mathbb{H}_G)$ its **spectrum**.

Corollary 3

There is an orthogonal decomposition

$$\mathcal{H}_G = \widehat{\bigoplus}_{s \in \text{Spec}(\mathbb{H}_G)} \mathcal{H}_G(s)$$

where $\mathcal{H}_G(s)$, $s \in \text{Spec}(\mathbb{H}_G)$, are the *finite-dimensional joint eigenspaces* of \mathbb{H}_G in \mathcal{H}_G .

Conjecture 3

Every $\mathcal{H}_G(s)$ is an **eigenspace** of \mathcal{A}_G .

Corollary 4

If $(\chi, \mu) \in \text{Spec } \mathcal{A}_G$, then $\mu = \tau(\bar{\chi})$ and $\chi \in \text{Op}_{L_G}^\gamma(X)_{\mathbb{R}}$.

$\text{Op}_{L_G}^\gamma(X)_{\mathbb{R}}$ – subset of real L_G -opers in a *component* of $\text{Op}_{L_G}(X)$.

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Remark. Recall that first we defined a Hecke operator $\widehat{H}_\lambda(x) : V_G(\lambda) \rightarrow \widehat{V}_G$.

The algebra \mathcal{A}_G naturally acts on both $V_G(\lambda)$ and \widehat{V}_G . Hence the commutators $[P, \widehat{H}_\lambda(x)]$, $P \in \mathcal{A}_G$, make sense.

We have $[P, \widehat{H}_\lambda(x)] = 0$, $\forall P \in \mathcal{A}_G$.

To see this, realize Bun_G as $G(X \setminus x) \backslash G(F_x) / G(\mathcal{O}_x)$.

Then $\widehat{H}_\lambda(x)$ acts from the *right*, whereas \mathcal{A}_G can be obtained from the action of the center of $\widetilde{U}(\widehat{\mathfrak{g}})_{\text{crit}}$ from the *left*.

However, to prove Conjecture 3 we need a stronger form of commutativity, and a crucial element in proving it is the system of **differential equations** satisfied by $\widehat{H}_\lambda(x)$ which we discuss below.

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With a choice of $K_X^{1/2}$, we can identify $\text{Op}_{L_G^{\text{ad}}}(X)$ with a specific component $\text{Op}_{L_G}^\gamma(X)$ in $\text{Op}_{L_G}(X)$.

An oper $\chi \in \text{Op}_{L_G}^\gamma(X)$ is a triple $(\mathcal{F}_{L_G}^\gamma, \mathcal{F}_{L_B}^\gamma, \nabla, \cdot)$, where $\mathcal{F}_{L_G}^\gamma$ is a specific L_G -bundle on X equipped with a reduction $\mathcal{F}_{L_B}^\gamma$ to a Borel subgroup $L_B \subset L_G$, and ∇_χ is a holomorphic connection on $\mathcal{F}_{L_G}^\gamma$, satisfying a *transversality condition* with respect to $\mathcal{F}_{L_B}^\gamma$.

Consider the case $G = SL_2$ (following Beilinson and Drinfeld).

$$\text{Op}_{SL_2}(X) = \bigsqcup_{\gamma \in \theta(X)} \text{Op}_{SL_2}^\gamma(X)$$

where $\theta(X)$ is the set of isomorphism classes of square roots of K_X .

Consider a component $\text{Op}_{SL_2}^\gamma(X)$ of $\text{Op}_{SL_2}(X)$.

$K_X^{1/2}$ – a square root of K_X in the isomorphism class γ .

\mathcal{V}_{ω_1} – the rank 2 vector bundle associated to $\mathcal{F}_{SL_2}^\gamma$. Then

$$0 \rightarrow K_X^{1/2} \rightarrow \mathcal{V}_{\omega_1} \rightarrow K_X^{-1/2} \rightarrow 0$$

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Each component $\mathrm{Op}_{SL_2}^\gamma(X)$ is isomorphic to $\mathrm{Op}_{PGL_2}(X)$. Here's an alternative description of this component.

A *projective connection* associated to $K_X^{1/2}$ is a second-order differential operator $P : K_X^{-1/2} \rightarrow K_X^{3/2}$ such that

- ① $\mathrm{symb}(P) = 1 \in \mathcal{O}_X$, and
- ② P is algebraically self-adjoint.

They form an affine space $\mathcal{P}roj_\gamma(X)$. Locally, $P = \partial_z^2 - v(z)$.

Lemma 5

There is a bijection $\mathrm{Op}_{SL_2}^\gamma(X) \simeq \mathcal{P}roj_\gamma(X)$

$$\chi \in \mathrm{Op}_{SL_2}^\gamma(X) \quad \mapsto \quad P_\chi \in \mathcal{P}roj_\gamma(X)$$

such that the section $s_{\omega_1} \in \Gamma(X, K_X^{-1/2} \otimes \mathcal{V}_{\omega_1})$ corresponding to the embedding $K_X^{1/2} \hookrightarrow \mathcal{V}_{\omega_1}$ satisfies $P_\chi \cdot s_{\omega_1} = 0$

(here we use the \mathcal{D}_X -module structure on \mathcal{V}_{ω_1} corresponding to ∇_X).

We will say that $\chi \in \text{Op}_{L^G}^\gamma(X)_{\mathbb{R}}$ if the monodromy representation $\rho_\chi : \pi_1(X, p_0) \rightarrow {}^L G(\mathbb{C})$ is isomorphic to its complex conjugate, i.e. $\rho_\chi \simeq \bar{\rho}_\chi$.

According to Corollary 4, we expect that there is a map

$$\Phi : \text{Op}_{L^G}^\gamma(X)_{\mathbb{R}} \rightarrow \text{Spec}(\mathbb{H}_G)$$

(possibly multivalued) and we wish to describe it explicitly. This would give a description of the **eigenvalues of the Hecke operators**.

As x varies along X , the Hecke operators $H_\lambda(x)$ combine into a **section of the C^∞ line bundle $|K_X|^{-\langle \lambda, \rho \rangle}$** on X with values in operators $\mathcal{H}_G \rightarrow \mathcal{H}_G$. We denote it by H_λ .

Thus, each **eigenvalue of H_λ** defines a section of the C^∞ line bundle $|K_X|^{-\langle \lambda, \rho \rangle}$ on X .

We will now write an explicit formula for the eigenvalue $\Phi_\lambda(\chi)$ corresponding to $\chi \in \text{Op}_{L^G}^\gamma(X)_{\mathbb{R}}$.

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Recall $0 \rightarrow K_X^{1/2} \rightarrow \mathcal{V}_{\omega_1} \rightarrow K_X^{1/2} \rightarrow 0$

and $s_{\omega_1} \in \Gamma(X, K_X^{-1/2} \otimes \mathcal{V}_{\omega_1})$ corresponding to $K_X^{-1/2} \hookrightarrow \mathcal{V}_{\omega_1}$.

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as C^∞ flat bundles. Since $\mathcal{V}_{\omega_1} \simeq \mathcal{V}_{\omega_1}^*$, we get an Hermitian form

$$h_{\chi, \omega_1}(\cdot, \cdot) : (\mathcal{V}_{\omega_1}, \nabla_{\chi, \omega_1}) \otimes (\overline{\mathcal{V}}_{\omega_1}, \overline{\nabla}_{\chi, \omega_1}) \rightarrow (C_X^\infty, d)$$

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$$\Phi_{\omega_1}(\chi) = \pm h_{\chi, \omega_1}(s_{\omega_1}, \overline{s_{\omega_1}})$$

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Lemma 6

$h_{\chi, \omega_1}(s_{\omega_1}, \overline{s_{\omega_1}})$ is the unique, up to a scalar, *section* Φ of $|K_X|^{-1/2}$ which is a solution of the system

$$P_\chi \cdot \Phi = 0, \quad \overline{P}_\chi \cdot \Phi = 0$$

Hence we can prove Conjecture 4 by showing that the Hecke operator \widehat{H}_{ω_1} satisfies the same system of *second-order differential equations*.

We can do this by using a theorem of Beilinson–Drinfeld describing the action of the *Hecke functors* on the sheaf \mathcal{D}_G .

To see this, recall that $\chi \mapsto P_\chi : K_X^{-1/2} \rightarrow K_X^{3/2}$ and

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We can do this by using a theorem of Beilinson–Drinfeld describing the action of the *Hecke functors* on the sheaf \mathcal{D}_G .

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Explicit formula for $\Phi_\lambda(\chi)$ in general

In general, λ is a dominant coweight of G , which we can interpret as a dominant weight of ${}^L G$. Let V_λ be the irreducible finite-dimensional representation of ${}^L G$ with **highest weight** λ .

Let \mathcal{V}_λ be the **associated vector bundle** $\mathcal{F}_{L^G}^\gamma \times_{L^G} V_\lambda$ equipped with the **connection** $\nabla_{X,\lambda}$ induced by ∇_λ . Then $\mathcal{F}_{L^B}^\gamma$ defines a line subbundle of \mathcal{V}_λ , which is known to be isomorphic to $K_X^{(\lambda,\rho)}$.

Thus, we have embeddings

$$\kappa_\lambda : K_X^{(\lambda,\rho)} \hookrightarrow \mathcal{V}_\lambda$$

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$$(\mathcal{V}_\lambda, \nabla_{\chi, \lambda}) \simeq (\overline{\mathcal{V}}_\lambda, \overline{\nabla}_{\chi, \lambda})$$

of C^∞ flat bundles on X , and hence an Hermitian form

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Conjecture 5

For $\chi \in \text{Op}_{LG}^\gamma(X)_\mathbb{R}$, the section $\Phi_\lambda(\chi) \in \Gamma(X, |K_X^{-\langle \lambda, \rho \rangle}|)$ is equal to

$$\Phi_\lambda(\chi) = h_{\chi, \lambda}(s_\lambda, \overline{s_{-w_0(\lambda)}})$$

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We have proved Conjecture 5 (modulo Compactness Conjecture) for $G = SL_n$ and $\lambda = \omega_1$ (in this case $\overline{\text{Gr}}_\lambda = \text{Gr}_\lambda$).

The proof is based on a system of differential equations satisfied by the Hecke operator \widehat{H}_{ω_1} , which we derive from the theorem of Beilinson and Drinfeld describing the action of the corresponding Hecke functor on \mathcal{D}_G .

These differential equations imply a “strong commutativity” between the differential operators and the Hecke operators, which can be used to prove Conjecture 3.

We expect that all of this can be extended to the case of a general simple Lie group G .

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The the Hecke correspondence $Z(\lambda)$ gives rise to an “integral transform” functor \mathbf{H}_λ on the category of left D -modules. These are the *Hecke functors*. Consider for simplicity the case when $\overline{\text{Gr}}_\lambda$ is smooth (e.g. $G = PGL_n, \lambda = \omega_1$). Then

$$\mathbf{H}_\lambda(\mathcal{F}) := (q_2 \times q_3)_*^D q_1^*(\mathcal{F})$$

Recall that \mathcal{D}_G is the sheaf of twisted differential operators acting on the line bundle $\mathcal{L} = K_{\text{Bun}}^{1/2}$ on Bun_G :

$$\mathcal{D}_G = \mathcal{L} \otimes \mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}$$

Beilinson and Drinfeld have computed $\mathbf{H}_\lambda(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1})$.

To state their result, let $\mathcal{V}_\lambda^{\text{univ}}$ be the *universal vector bundle over* $\text{Op}_{L_G}^\gamma(X) \times X$ with a partial connection ∇^{univ} along X , such that

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Moreover, we have a map

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corresponding to the oper Borel reduction.

Let $\mathcal{V}_{\lambda, X}^{\text{univ}} := \pi_*(\mathcal{V}_\lambda^{\text{univ}})$, where $\pi : \text{Op}_{L_G}^\gamma(X) \times X \rightarrow X$. The connection ∇^{univ} makes $\mathcal{V}_{\lambda, X}^{\text{univ}}$ into a left \mathcal{D}_X -module.

The algebra $D_G \simeq \text{Fun Op}_{L_G}^\gamma(X)$ acts on $\mathcal{V}_{\lambda, X}^{\text{univ}}$ and commutes with \mathcal{D}_X .

The map $\kappa_\lambda^{\text{univ}}$ yields a map

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$$\mathbf{H}_\lambda(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \simeq (\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \boxtimes_{D_G} \mathcal{V}_{\lambda, X}^{\text{univ}}.$$

Moreover, the above isomorphism a gives rise to a map

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which corresponds to $\iota \boxtimes \kappa_{\lambda, X}^{\text{univ}}$, where $\iota : \mathcal{L}^{-1} \hookrightarrow \mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}$.

As shown by Beilinson and Drinfeld, this statement implies that Δ_X is a Hecke eigensheaf, which is a key result in the [geometric Langlands correspondence](#).

We claim that we can derive from this statement that the Hecke operator $\widehat{H}_\lambda(x)$ satisfies a differential equation, which is a key result in the [analytic Langlands correspondence](#).

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