Hecke operators for $X=\mathbb{P}^1$ with 4 points.

1. F local field, $\|\cdot\| : F^\times \to \mathbb{R}_+$ the norm. Residue field $\mathbb{F}_q$ [char $\mathbb{F}_q \neq 2$] (in non-archimedean case), $\|\pi\| = q^{-1}$. For $\mathbb{R}$ $\|x\| = |x|$
$\underset{\text{uniformizer}}{\pi}$ $\quad \mathbb{C}$ $\|x\| = |x|^2$.

2. $X = \mathbb{P}^1$, $G = PGL_2$, parabolic structures at $0, 1, t, \infty$, $t \in F$.
$Bun_0^s \cong \mathbb{P}^1 \setminus \{0,1,t,\infty\}$.
Namely, given $y \in \mathbb{P}^1 \setminus \{0,1,t,\infty\}$, can define bundle
$E_y = \mathcal{O} \oplus \mathcal{O}$ with parabolic str-res $0, 1, y, \infty$.
This is stable iff $y \neq 0, 1, t, \infty$

stable bundle: $s(E) = \frac{1}{2}\deg E + 1$
$s(L) = \deg L + \frac{1}{2}N$
$L \subset E \Rightarrow s(L) < s(E)$

3. Also $\forall X$ with at least one parabolic pt P, we have a canonical isomorphism

$\mathrm{Bun}_0^s \cong \mathrm{Bun}_1^s$ by doing the Hecke modif. at $\mathbb{P}$ using the parabolic line. We'll identify them using this isomorphism.

4. Also $\mathrm{Bun}_n^s \cong \mathrm{Bun}_{n+2}^s$ $\boxed{\text{at } \infty}$ by tensoring with $\mathcal{O}(1)$.

5. Let us write down the Hecke modification $\mathcal{H}_{x,s}$ $(x \in \mathbb{P}^1 \setminus \{0,1,t,\infty\}, s \in \mathbb{P}^1)$ in these coordinates.

Proposition. We have

$$\mathcal{H}_{x,s}(E_y) = E_z$$

where

$$z = \frac{(s-1)(st-xy)}{(s-x)(s-y)}.$$

6. Let H_x be the modified Hecke operator at x (usual one times $\| x(x-1)(x-t)\|^{-\frac{1}{2}}$.

We have a half-density $\|dx\|^{1/2}$ on $\mathbb{P}^1$. Using this half-density, we can identify the space $\mathcal{H}$ of square integrable half-densities on $\mathbb{P}^1(F)$ with $L^2(F, \|dx\|)$.

Corollary. The Hecke operator H_x has the form

$$\boxed{(H_x\psi)(y) = \int_F \psi\left(\frac{(s-1)(ts-xy)}{(s-x)(s-y)}\right)\left\|\frac{ds}{(s-x)(s-y)}\right\|}$$

7. Using this formula, we can compute the Schwartz kernel of the operator H_x. Namely, we can make a change of variable

$$z = \frac{(s-1)(ts-xy)}{(s-x)(s-y)}.$$

This is a double cover, so we'll get a factor of 2. Also integration will be not over the whole $\mathbb{P}^1(F)$ but only over the image of this map. The result is as follows.

Proposition. Let $f_t(x,y,z)$ be the following polynomial:

$$f_t(x,y,z) = (xy+xz+yz-t)^2 + 4(1+t-x-y-z)xyz.$$

Then

$$\boxed{(H_x\psi)(y) = 2\int_F \frac{\theta(f_t(x,y,z))\psi(z)}{\sqrt{\|f_t(x,y,z)\|}}\|dz\|}$$

where $\theta(a)=1$ if a is a square and $\theta(a)=0$ if not.

integral converges, log sing. at $0,1,t,\infty$ $\Leftarrow$ $\displaystyle\int_{|x|\leq 1} \frac{\|dx\|}{\sqrt{\|x\|\cdot\|x+\varepsilon\|}} \sim \log\left(\frac{1}{\varepsilon}\right)$, $\varepsilon\to 0$.

This operator was considered by Kontsevich in 2007. The following theorem, stated in his paper, is not too hard to prove.

Thm. The operator H_x is compact and self-adjoint. Moreover $[H_x, H_y]=0$ for all x,y.

Example. If $F=\mathbb{C}$ then

$$(H_x\psi)(y)=\frac{2}{\pi}\int_{\mathbb{C}} \frac{\psi(z)\,dz\,d\bar{z}}{|f_t(x,y,z)|}$$

Proof. It is clear that H_x is symmetric, since its Schwartz kernel is symmetric. Also H_x^2 is a Hilbert-Schmidt operator, i.e. its kernel $K_2(y,z)$ satisfies

$$\int |K_2(y,z)|^2 dy dz < \infty.$$

Γ^2 (this requires a computation).

Thus H_x^2 is compact, hence so is H_x. The identity $[H_x, H_y] = 0$ follows since Hecke modifications at x and y commute.

Remark. The operator H_x itself is not Hilbert-Schmidt, since

$$\int_{F^2} \frac{\|dy\,dz\|}{\|f_t(x,y,z)\|} = \int_{F^2} |K_1(y,z)|^2 \|dy\,dz\| \text{ is log-divergent.}$$

So H_x^2 is not trace class.

But $(H_x^2)^{1+\varepsilon}$ is trace class $\forall \varepsilon > 0$.

Note that we have an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on Bun_0^s by $S_0(y) = \frac{t}{y}$, $S_1(y) = \frac{y-1}{y-t}$.

$S_0 S_1 = S_1 S_0$

Proposition. $H_x \sim \|x\|^{-1} \log \|x\|$
$x \to \infty$

and similar asymptotics at $0, 1, t$:

$$H_x \sim \log \left\|\frac{1}{x}\right\| S_0, \quad x \to 0$$

$$H_x \sim \log \left\|\frac{1}{x-1}\right\| S_1, \quad x \to 1$$

$$H_x \sim \log \left\|\frac{1}{x-t}\right\| S_0 S_1, \quad x \to t$$

Spectral decomposition.

By the spectral theory of compact self-adjoint operators, the commuting operators H_x have a common eigenbasis ψ_n, $n \geq 0$ of $\mathcal{H}$, $\|\psi_n\| = 1$

$$H_x \psi_n = \beta_n(x) \psi_n.$$

distinct.

Can choose ψ_n real and positive near ∞, this fixes ψ_n uniquely.

The functions ψ_n are smooth (= locally constant in the non-archimedean case) outside $0, 1, t, \infty$.

Proposition. We have $\beta_n = c_n \psi_n$

$c_n > 0$

Thus $$\frac{2\theta(f_t(x,y,z))}{\sqrt{\|f_t(x,y,z)\|}} = \sum_{n=0}^{\infty} c_n \psi_n(x)\psi_n(y)\psi_n(z)$$

Proof. This follows since $f_t(x,y,z)$ is S_3-symmetric.

Cor. $$H_x H_y = \int_F \frac{2\theta(f_t(x,y,z))}{\sqrt{\|f_t(x,y,z)\|}} H_z \|dz\|.$$

(same action on eigenvectors)

Cor. $c_n = \psi_{n,\infty}^{-1}$ where

$$\psi_{n,\infty} = \lim_{x\to\infty} \frac{\psi_n(x)}{\|x\|^{-1}\log\|x\|}.$$

Thus

$$\boxed{\frac{2\theta(f_t(x,y,z))}{\sqrt{\|f_t(x,y,z)\|}} = \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)\psi_n(z)}{\psi_{n,\infty}}.}$$

Remark. The numbers c_n^2 are the eigenvalues of the positive operator $Q = \int_F H_x^2 \|dx\|$.

We have $\sum c_n^4 < \infty$, so Q is HS. In fact, I expect that Q is almost trace class, i.e. $\sum c_n^2$ is log divergent but $\sum c_n^{2+\varepsilon}$ is convergent $\forall \varepsilon > 0$.

Prop. The spectrum of $\{H_x\}$ is simple and they have no common kernel.

This recovers the package of properties from Kontsevich's paper.

The complex and real case.

Let $\quad L = \partial_x\, x(x-1)(x-t)\,\partial_x + x.$

This is the Lamé operator (after so-called Halphen transform), i.e, $L \sim \partial_u^2 + \frac{1}{4}\wp(u,\tau).$

Proposition. $\left(L_y - L_z\right)\dfrac{1}{\sqrt{f_t(x,y,z)}} = 0$

$\Rightarrow [L, H_x] = 0$

Corollary. $L\psi_n(x) = \Lambda_n \psi_n(x)$

for $F = \mathbb{R}, \mathbb{C}$ and for $\mathbb{C}$ also

$$\overline{L}\,\psi_n(x) = \overline{\Lambda}_n\, \psi_n(x).$$

Consider now the case $F = \mathbb{C}$.

So $\quad \psi_n(x) = f_{1n}(x)\,\overline{f_{2n}(x)} + f_{2n}(x)\,\overline{f_{1n}(x)}$

where f_{1n}, f_{2n} are basic solutions of $Lf = \Lambda_n f.$

Since ψ_n is single-valued, there is an invariant nondegenerate Hermitian form on the monodromy repr. of the equation $Lf = \Lambda_n f$. The monodromy matrices at the 4 parabolic points are unipotent, hence this form must be indefinite. This implies that the monodromy repr. takes values in $SL_2(\mathbb{R}) = SU(1,1)$.

And conversely, if Λ is such that the monodromy takes values in $SL_2(\mathbb{R})$ then we get an eigenfunction (i.e. it belongs to L^2). Thus we get

Proposition. Eigenfunctions of $\{H_x\}$ correspond to $\Lambda \in \mathbb{C}$ such that the monodromy representation of $Lf = \Lambda f$ lands in $SL_2(\mathbb{R})$ (up to conjugation).

Remark. The positive eigenfunction (largest eigenvalue of H_x) corresponds to analytic uniformization of $\mathbb{CP}^1 \setminus \{0, 1, \infty, t\}$.

The other eigenvalues corr. to other real projective structures.

These other projective structures can be obtained from the uniformization one by a geometric procedure called grafting which was described by W. Goldman. These structures were also discussed in a paper by Faltings in 1983.

Details on analytic uniformization:

L. Takhtajan, arxiv:1407.1815

$J : \mathbb{C}_+ \longrightarrow \mathbb{CP} \setminus \{0, 1, t, \infty\}$

analytic uniformization map.

$J^{-1} : \mathbb{CP}^1 \setminus \{0, 1, t, \infty\} \to \mathbb{C}_+$

multivalued holomorphic function. Then $\psi_1(z) = \frac{1}{\sqrt{(J^{-1})'(z)}}$ and $\psi_2(z) = \frac{i \cdot J^{-1}(z)}{\sqrt{(J^{-1})'(z)}}$

are a basis of solutions of the Lamé equation $L\psi = \Lambda_0 \psi$

So the function

$$\beta(z,\bar z) = \psi_1(z)\overline{\psi_2(z)} + \psi_2(z)\overline{\psi_1(z)}$$

$$= \frac{2\,\mathrm{Im}\, J^{-1}(z)}{|J^{-1\prime}(z)|}.$$

satisfies $L\beta = \Lambda_0 \beta$

$\bar L \beta = \overline{\Lambda_0}\beta$

and $\beta > 0$

Then the Poincaré conformal metric looks like

$$\boxed{\beta^{-2}(z,\bar z)|dz|^2}$$

(curvature -1)

$\rightsquigarrow$ Λ_0 is the "Perron-Frobenius" eigenv.

Remark. In fact it is important that the equation

$$(L_y - L_z)\frac{1}{\sqrt{f_t(x,y,z)}} = 0$$

holds not just formally but "in the sense of distributions".

Namely, for $F = \mathbb{C}$ we have

$$(L_y - L_z)\frac{1}{|f_t(x,y,z)|} = 0$$

$$(\bar{L}_y - \bar{L}_z)\frac{1}{|f_t(x,y,z)|} = 0$$

and for $F = \mathbb{R}$

$$(L_y - L_z)\frac{\theta(f_t(x,y,z))}{\sqrt{|f_t(x,y,z)|}} = 0$$

Subleading term

Recall that

$$H_x \sim \|x\|^{-1} \log\|x\|, \quad x \to \infty$$

Prop. $\exists$ (weak)

$$M = \lim_{x\to\infty}\left(\|x\|\, H_x - \log\|x\|\right),$$

an unbounded self-adjoint operator.

It has the Schwartz kernel

$$K_M(y,z) = \frac{2}{\|y-z\|} - \delta(y-z)\log\|y(y-1)(y-t)\|.$$

The operator M is closely related to the integral operator used by S. Ruijsenaars arXiv:0904.3250 to fix a self-adjoint extension of L for $F=\mathbb{R}$ (but his self-adjoint extension is slightly different from ours).

<u>The Schwartz space</u>

<u>Def.</u> The Schwartz space $S \subset \mathcal{H}$ is the space of elements having rapidly decaying Fourier coefficients with

respect to the basis ψ_n.

(2nd description of $\mathcal{S}$)
Prop. $\mathcal{S}$ is the space of functions smooth outside $0, 1, t, \infty$ and near these points behaving as $a(x) + b(x)\log\|x\|$ for $0, 1, t$ and $\frac{a(x)+b(x)\log\|x\|}{\|x\|}$ near ∞ where a, b are smooth. (for $F = \mathbb{C}$).

(3rd description of $\mathcal{S}$)
Prop. For $F = \mathbb{C}$, $\mathcal{S}$ is the space of $f \in \mathcal{H}$ such that $L^m \overline{L}^n f \in \mathcal{H}$. $\forall m,n$

Prop. L is an essentially normal operator on $\mathcal{S}$. This means that $L + \overline{L}$ and $\frac{1}{i}(L - \overline{L})$ are essentially self-adjoint and strongly commute.

The non-archimedean case.

Theorem (Kontsevich)

The eigenvalues of H_x are algebraic numbers.

Proof. Let $x_0 \in F$, $x_0 \neq 0, 1, t, \infty$.
Then $\forall n$,

$$\int_{\|x-x_0\| \leq q^n} H_x \, \|dx\|$$

is a finite rank operator.
Moreover, the space generated by $\underline{\mathbb{1}_{B(x_0, q^n)}}$ under $\{H_x\}$ is finite dimensional.
Thus eigenvalues of H_x are eigenvalues of finite rational matrices.

Example. Consider the space V generated under H_x by $\mathbb{1}_{B(x_0, q^{-1})}$ where $\|x_0\| = \|x_0 - 1\| = \|x_0 - t\| = 1$.

Proposition. $\dim V = q+5$. $\langle \mathbb{1}_{B(z,q^{-1})} \rangle$

Namely, $V = \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_q), \mathbb{C}) \oplus V'$, where V' is the 4-dim space spanned by "logarithmic tails" $\psi_0, \psi_1, \psi_t, \psi_\infty$

e.g.
$$\psi_0(y) = \mathbb{1}_{B(0,q^{-1})}(y) \log_q \|y^{-1}\|.$$

Eigenvalues: Take x with $\|x\| = \|x-1\| = \|x-t\| = 1$.

Then the eigenvalues of H_x on V are as follows:

$$\lambda_\pm = (1+q^{-1})\,\frac{1 \pm \sqrt{1 + \frac{32q}{(q+1)^2}}}{2}$$ Roots of $\lambda^2 - (1+q^{-1})\lambda + 8q^{-1} = 0$

(λ_+ is the largest, $\lambda_+ \sim 1 + O(q^{-1})$)

The other eigenvalues are much smaller.

We have 0 with multiplicity 3 and the action of H_x on the orthogonal complement V_0 of this 5-dim space is the Drinfeld Hecke operator over a finite field (up to factor $(-q)$), so by Drinfeld's Bound these eigenvalues satisfy

$|\lambda| \leqslant 2q^{-1/2}.$

Larger number of points $(m+2=N \geqslant 5)$

Points $t_0=0, t_1, \dots, t_m, t_{m+1}=\infty$ on $\mathbb{P}^1$.

$Bun^s = Bun^s_0 \cup Bun^s_1$ $\dim = m-1$.

$S_i : Bun^s_0 \rightleftarrows Bun^s_1$, $S_i^2 = 1$, $S_i S_j = S_j S_i$

$i = 0, \dots, m+1$

Hecke modif. at t_i along the parabolic line.

We identify $Bun^s_0 \simeq Bun^s_1$ using S_{m+1}, then $S_0, \dots, S_m$ define an action of $(\mathbb{Z}/2)^{m+2}_0$ (sum of coordinates $=0$) on Bun^s_0

Parametrization:

$E \in Bun^s_0$, $E \cong \mathcal{O} \oplus \mathcal{O}$

Parabolic structures $(1,0), (1,y_1), \dots, (1,y_m), (0,1)$

$y_1, \dots, y_m \in F$.

Symmetry $(y_1, \dots, y_m) \to (\gamma y_1, \dots, \gamma y_m)$

So we obtain a birational parametrization of Bun_0^S by $\mathbb{P}^{m-1}$.

Hecke modification

Proposition $\mathcal{H}_{x,s}(E_y) = E_z$

$$\boxed{z_i = \frac{x y_i - s t_i}{y_i - s}}$$

$y = (y_1, \dots, y_m)$

$z = (z_1, \dots, z_m)$

Hilbert space

$\mathcal{H}$ = half-densities on $F\mathbb{P}^{m-1}$ = functions on F^m homogeneous of degree $-\frac{m}{2}$:

$$\psi(\lambda y) = \|\lambda\|^{-\frac{m}{2}} \psi(y)$$

(satisfying the L^2 condition).

$=$ functions on F^{m+1} (with coord. $y_0, .., y_m$) which are translation invariant and homogeneous of degree $-\frac{m}{2}$. In the last realization, it is convenient to write down the Hecke operator.

Parabolic pts $t_0, t_1, .., t_m, \infty$

Prop. The modified Hecke operator (by $\prod \|x-t_i\|^{1/2}$) has the form

$$\boxed{(H_x\psi)(y) = \int_F \psi\left(\frac{t_0-x}{s-y_0}, .., \frac{t_m-x}{s-y_m}\right) \left\| \frac{ds}{\prod_{i=0}^{m}(s-y_i)} \right\|}$$

Boundedness.

Thm. H_x is bounded on $\mathcal{H}$.

Proof. $\phi(w_1, .., w_{m-1}) = \psi(0, w_1, .., w_{m-1}, 1)$

$$\|\psi\|^2 = \|\phi\|^2_{L^2} = \int_{F^{m-1}} |\phi|^2 dv$$ ← Lebesgue measure.

$$H_x = \int_F U_{x,s} \left\| \frac{x(x-1)}{s(s-1)(s-x)} \right\|^{\frac{1}{2}} \|ds\|,$$

where

$$(U_{x,s}\phi)(u) = \left\| \prod_{i=1}^{m-1} \frac{s(s-1)(t_i - x)}{(s-x)(s-u_i)^2} \right\|^{\frac{1}{2}} \cdot$$

$$\phi\left(\frac{(t_1 s - u_1 x)(s-1)}{(s-u_1)(s-x)}, \dots, \frac{(t_{m-1}s - u_{m-1}x)(s-1)}{(s-u_{m-1})(s-x)} \right)$$

Key fact: $U_{x,s}$ is a unitary operator. (action of an element of PGL_2^{m-1} on a unitary repr.).

Thus

$$\|H_x\| \le \int_F \left\| \frac{x(x-1)}{s(s-1)(s-x)} \right\|^{\frac{1}{2}} \|ds\| < \infty.$$

Also H_x are self-adjoint and pairwise commuting.

Compactness.

To prove that H_x is compact,

it suffices to check that we have

Theorem: for $n \geq 0$, H_x^n is a compact operator.

Proof: $d\nu_x(s) = \left\| \frac{x(x-1)}{s(s-1)(s-x)} \right\|^{1/2} \|ds\|$

$$H_x^n = \int_{F^n} \overbrace{U_{s_1,x} \cdots U_{s_n,x}}^{\in PGL_2^{m-1}} \, d\nu_x(s_1) \cdots d\nu_x(s_n).$$

$$\xi_n : \mathbb{A}^n \longrightarrow PGL_2^{m-1}$$

$$\xi_n(s_1, \dots, s_n) = \overbrace{U_{s_1,x} \cdots U_{s_n,x}}^{\in PGL_2^{m-1}}$$

$d\lambda = \xi_{n*}(\nu_x^{\otimes n})$ — measure on $PGL_2^{m-1}(F)$

$d\lambda = f_x(g)\, dg$ ← Haar measure on $PGL_2^{m-1}(F)$

↑ L^1-function

($n = 3m-3$ then ξ_n dominant)

W principal series repr

of $PGL_2(\mathbb{F})$ on half-densities on $\mathbb{P}^1$: $\Rightarrow W^{\otimes m-1}$

unitary rep ρ of $PGL_2^{m-1}(F)$.

$$H_X^n = \int_{PGL_2^{m-1}(F)} f_x(g)\,\rho(g)\,dg$$

Harish-Chandra thm: If ϕ on $PGL_2^{m-1}(F)$ is smooth with compact support then the operator

$$\int_{PGL_2^{m-1}(F)} \phi(g)\,\rho(g)\,dg$$

is trace class $\Rightarrow$ compact.

But f_x can be approximated in L^1 metric by smooth functions with compact support, so H_x^n can be approx. by compact operators, hence it is itself compact. ▨

Asymptotics:

$$H_x \sim \|x\|^{-1} \log\|x\|, \quad x \to \infty$$

$$H_x \sim \log\|x - t_i\|^{-1} S_i, \quad x \to t_i$$

Spectral decomposition:

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

$$H_x \psi = \beta_k(x) \psi$$

χ_k - character of $(\mathbb{Z}/2)_0^{m+1}$ on $\mathcal{H}_k$

$$\beta_k(x) \sim \log\|x - t_i\|^{-1} \cdot \chi_k(S_i), \quad x \to t_i.$$

$$\beta_{\mathbb{R}}(x) \sim \|x\|^{-1} \log \|x\|, \quad x \to \infty.$$

<u>Thm.</u> For large enough a the operator $|H_x|^a$ is trace class

$\Rightarrow |H_x|^{a/2}$ is Hilbert-schmidt.

It'd be interesting to determine for which a this is so. Clearly

$$a \geq 2\dim(\mathrm{Bun}) = 2(m-1).$$

since otherwise the kernel of $|H_x|^a$ does not have full support.

Archimedean case.

In the archimedian case

We have also the quantum Hitchin system on Bun_0^S.
This is the Gaudin system:

$$\sum_{\substack{0\le j\le m\\ j\ne i}} \frac{1}{t_i-t_j}\left(-(y_i-y_j)^2\partial_i\partial_j+(y_i-y_j)(\partial_i-\partial_j)+\frac{1}{2}\right) = \tilde{G}_i$$

$$[\tilde{G}_i,\tilde{G}_j]=0, \quad \sum_i \tilde{G}_i=0$$

$$\tilde{G}_i^*=\tilde{G}_i, \quad \sum t_i\tilde{G}_i=\frac{m}{4}$$

Let

$$G_i=\tilde{G}_i-\sum_{j\ne i}\frac{1}{2(t_i-t_j)}$$

Thm. (Universal oper equation)

$$\left(\partial_x^2+\sum_{j\ge 0}\frac{1}{x-t_i}\partial_x\right)H_x - H_x\sum_{i\ge 0}\frac{G_i}{x-t_i}=0.$$

Cor. The operator G_i acts by a scalar $\mu_{i,k}$ on $\mathcal{H}_k$ and $[G_i, H_x] = 0$.

Corollary The eigenvalues $\beta_k(x)$ satisfy the usual oper equation

$$\left(\partial_x^2 - \sum_{j\geq 0} \frac{1}{x-t_i}\partial_x\right)\beta_k(x) - \sum \frac{\mu_{i,k}}{x-t_i}\beta_k(x) = 0.$$

Thm. (i) The Hecke operators have a joint simple spectrum Σ on $\mathcal{H}$.

(ii) Let $\mathcal{R}$ be the set of real opers. Then $\forall \mu \in \mathcal{R}$ $\exists!$ up to scaling real analytic half-density defined outside the non-very-stable divisor on Bun_G° such that

$$G_i \psi_\mu = \mu_i \psi_\mu, \quad \overline{G_i} \psi_\mu = \overline{\mu_i} \psi_\mu.$$

(iii) The spectrum of Hecke operators Σ embeds into $\mathbb{R}$.

(iv) If $\psi \in \mathcal{H}$ is a joint eigenfunction of $G_i, \overline{G_i}$ as a distribution then $\psi \in \mathcal{H}_k$ for some k.

Schwartz space:

As before it is the space $\mathcal{S}$ of functions with rapidly decaying coefficients with respect to the basis ψ_k.

Thm. (1) $\mathcal{S} = \{ f \in \mathcal{H}, \ L\bar{M}f \in \mathcal{H}$ $\forall$ quantum Hitchin Hamiltonians $L, M \}$.

(2) Quantum Hitchin hamiltonians are essentially normal on $\mathcal{S}$

Example of 5 points $0, 1, s, t, \infty$

Bun_0^s = blowup of $\mathbb{P}^2$ at 5 points $(0,0,1), (0,1,0), (1,0,0)$
$(1,1,1), (s,t,1)$. $\boxed{(y,z,1) \in \mathbb{P}^2}$

The divisor D of singularities of the Hitchin system:

16 components:

1. Exceptional fibers (5)
2. 10 lines:

$y=0,1,s,\quad z=0,1,s$

$y=z,\quad ty=sz$

$(t-1)(y-1)=(s-1)(z-1)$

line at ∞

3. Quadric

$st(y-z)+(t-s)yz+sz-ty=0$

They have normal crossings and are permuted transitively by $S_5\ltimes(\mathbb{Z}/2)^5_0=W(D_5)$,

Stabilizer $=W(A_4)=S_5$

Crossings form Clebsch graph, 5-hypercube/central symmetry.

Theorem. 1. The basic solutions of the Hitchin system (of rank 4) behave near generic point of D as $1,1,1,\sqrt{z}$ and

near intersections as
$1, 1, \sqrt{z}, \sqrt{w}$

2. Single-valued eigenfunctions of G_i and $\overline{G_i}$ look there like $\boxed{\psi_0 + |z|\psi_1}$, ψ_i smooth (generically on D) and

$$\boxed{\psi_0 + |z|\psi_1 + |w|\psi_2}$$ (at intersections)

So they are continuous $\Rightarrow L^2$ and are eigenfunctions of Hecke operators.

3. Monodromy around components of D looks like $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$

near intersections also $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$

Expect a similar picture for $m+2>5$ points.

Case 1: m is odd:

near generic point of the divisor D eigenfunctions are

$$\psi = \psi_0 + |z|^{m-2}\psi_1,$$

ψ_0, ψ_1 smooth

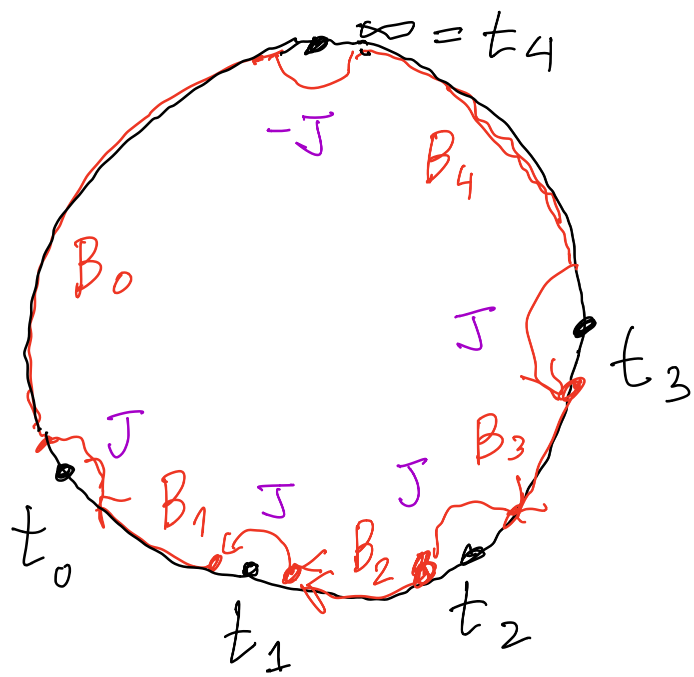
Case 2. m is even

$$\psi = \psi_0 + \psi_1 |z|^{m-2}\log|z|$$

ψ_0, ψ_1 smooth.

The real case.

Take $F=\mathbb{R}$, $t_0<t_1<\cdots<t_m$



Equation for eigenvalue

$$L(\mu)\beta = 0.$$

$$J = \begin{pmatrix} i & 0 \\ 1 & i \end{pmatrix}$$

$$B_j = \begin{pmatrix} 1 & b_j \\ 0 & -a_j \end{pmatrix} \qquad j \neq 0$$

$$B_0 = \begin{pmatrix} \lambda & b_0 \\ 0 & -a_0 \end{pmatrix}$$

$$\left.\begin{aligned} &\prod_{j=0}^{m+1} B_j J = -1 \\ &\prod_{j=0}^{m-1} B_j J^{-1} = -1 \end{aligned}\right\} \Rightarrow \lambda = \prod_j a_j^{-1}.$$

$2m+4$ unknowns $\Rightarrow$ $2(m-1)$ dim space of solutions.
6 equations

This is a parametrization of flat connections ∇.

So what are "real" opers?

<u>Def.</u> ∇ is a balanced connection if $a_j = 1$ $\forall j$, so $\lambda = 1$. I.e.

$$B_j = \begin{pmatrix} 1 & b_j \\ 0 & -1 \end{pmatrix}.$$

Then the two matrix equations above are equivalent. So we just have

$$\prod_{j=0}^{m+1} B_j J = -1.$$

This defines an $m-1$-dim. variety X_{m-1}.

<u>Thm.</u> Eigenvalues of Hecke operators $\longleftrightarrow$ balanced connections.

<u>Thm.</u> Balanced connections $\longleftrightarrow$ nonvanishing solutions of the T-system of level m:

$$T_i(k-1)\,T_i(k+1) = T_{i-1}(k)\,T_{i+1}(k)+1.$$

$0 \le i \le m$

$$T_0(k) = T_m(k) = 1$$

for all k. $T_1(k) = b_k$

Remark. Every nonvanishing solution of the T-system of level m is $m+2$-antiperiodic, i.e.

$$T_i(k+m+2) = T_{m-i}(k).$$

This is the Al. Zamolodchikov conjecture (Frenkel-Szenes)

More general versions

were one of the starting points of the theory of cluster algebras.

(Fomin - Zelevinsky)