Object of Study G simple

Category $C \circlearrowleft G((t))$

C $\quad B(G((t)))$-module

Filtration on C indexed by a discrete set $\subset \mathbb{Q}$

For instance, if $G = SL_2$

$$C_{\leq 0} \subseteq C_{\leq \frac{1}{2}} \subseteq C_{\leq 1} \cdots \subseteq C$$

$$\operatorname{colim} C_{\leq r} \cong C.$$

$\subseteq$: fully faithful subcat.

Note: $\subseteq^R$ continuous $G((t))$-equivariant

Motivations:

1. Defined by Moy–Prasad in arithmetic setting

irr. $V \circlearrowleft G(\mathbb{Q}_p)$

MP: associates a rational #, called depth

$\xleftrightarrow{\text{Langlands}}$ smallest r

Gal^r

upper ramification filtration

Geometrically:

$C_{\leq r}$ should be

$$L(LCC) \times_{LocSys(D^\times)} LocSys(D^\times)_{\leq r}$$

slope filtration on local systems

$$LocSys(D^\times)_{\leq 0} = LocSys(D^\times)_{RS}$$

More generally, quantum:

$$G((t))\text{-mod}_K \cong \check{G}((t))\text{-mod}_{\check{K}}$$

match up depth filtrations

Motivation #1: Functional Analysis

$\mathrm{Whit}: G(K)\text{-mod} \to \mathrm{DGCat}$

Raskin: Defines a sequence of functors $\mathrm{Whit}^{\leq n}$ such that

$$\operatorname{colim} \mathrm{Whit}^{\leq n} \cong \mathrm{Whit}$$

use this for instance to show

$\mathrm{Whit}\ \mathrm{inv} \simeq \mathrm{Whit}\ \mathrm{coinv}$

$\Rightarrow$ Whit commutes w/ all limits & colimits

$$\mathrm{Whit}(\mathcal{C}) \simeq \mathcal{C} \underset{G(K)}{\otimes} \mathrm{Whit}(G(K))$$

$$\mathrm{Whit}^{\leq n}(\mathcal{C}) \simeq \mathcal{C} \underset{G(K)}{\otimes} \mathrm{Whit}_{\leq n}(G(K))$$

This filtration on whittaker is depth filtration.

<u>Motivation 2</u>

Situation:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{fully faithful}} & \mathcal{D} \\ \circlearrowleft & & \circlearrowleft \\ G(\mathbb{C}(t)) & & G(\mathbb{C}(t)) \end{array}$$

Call a sequence of $G(\mathbb{C}((t)))$-categories

$C_1, C_2, \ldots,$ generating if

$\forall\, C \to D$

$$\mathrm{Hom}_{G(\mathbb{C}((t)))}(C_i, C) \to \mathrm{Hom}_{G(\mathbb{C}((t)))}(C_i, D)$$

equivalence for all C_i

$\Rightarrow C \to D$ equivalence.

Moy-Prasad theory produces such a generating set.

Definitions: Chen-Kamgarpour (not quite the same)

T max. torus $\subseteq G$

$X_*(T)$ cocharacter lattice

For a point $x \in X_*(T) \otimes \mathbb{R}$
and a rational $r \in \mathbb{Q}$

$K_{x,r}, K_{x,r+} \subseteq G((t))$

Lie algebras:

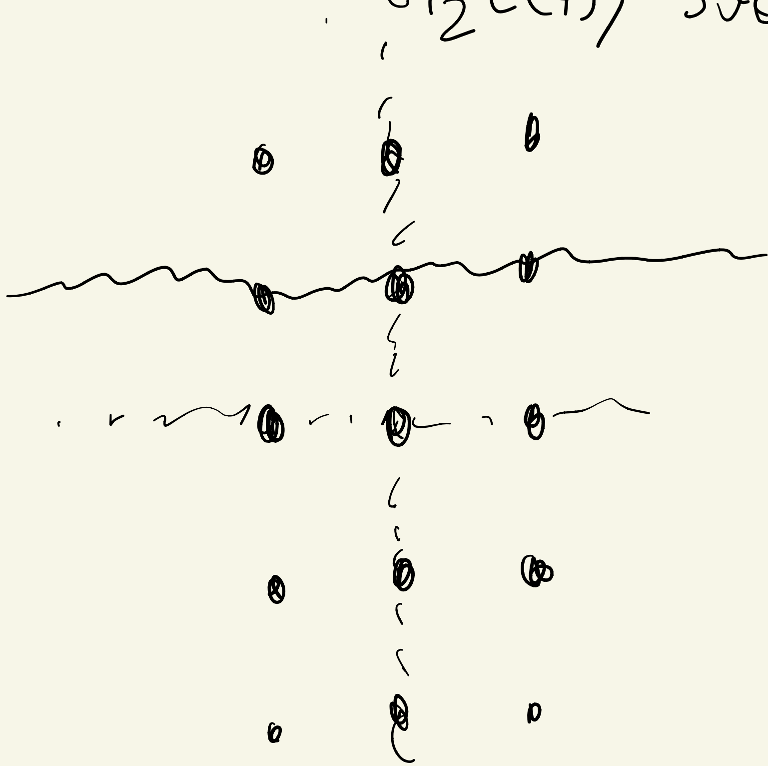
$$k_{x,r} = \bigoplus_{\langle \alpha, x \rangle + i \geq r} \mathfrak{g}_\alpha t^i$$

$$k_{x,r+} = \bigoplus_{\langle \alpha, x \rangle + i > r} \mathfrak{g}_\alpha t^i$$

Case of $x=0$, $r \in \mathbb{Z}_{\geq 0}$

$K_{0,r}$ = rth congruence subgroup

$K_{0,r+}$ = (r+1)th congr $sl_2\mathbb{C}[[t]]$ subgroup



$K_{0,1}$, $K_{0,1+}$

Example of SL_2 $\qquad r \in \mathbb{Z}$

$K_{0,r}$ as above

$$K_{\frac{1}{2}, r+\frac{1}{2}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$a \equiv 1 \pmod{t^{r+1}}$

$b \equiv 0 \pmod{t^{r}}$

$c \equiv 0 \pmod{t^{r+1}}$

$d \equiv$ [illegible] $\pmod{t^{r+1}}$

$$K_{\frac{1}{2}, \frac{1}{2}} = I_0$$

$$K_{\frac{1}{2}, \frac{1}{2}+} = [I_0, I_0]$$

$\mathcal{C}_{\leq r}$ " $:=$ " the $G((t))$-subcategory generated by $\mathcal{C}^{K_{x,r+}}$ over all x

Lemma (Ben-Zvi-Gunningham-Orem)

$$D(G((t))/K_{x,r+}) \otimes_{D(K_{x,r+}\backslash G((t))/K_{x,r+})} \mathcal{C}^{K_{x,r+}} \xrightarrow{a_{x,r}} \mathcal{C}$$

fully faithful

$a_{x,r}^R$ continuous & $G((t))$ equivariant

$K_{x,r}$ & K_{x,r^+} are normal inside $K_{x,0}$

General statement: As long as K is normal inside parahoric

$$D(G(\mathbb{F}_q)/K) \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{C}$$

$$D(K \backslash G(\mathbb{F}_q)/K)$$

Proof: K parahoric

Can write a bar complex

$$\iddots$$

$$\downarrow\downarrow\downarrow$$

$$D(G(C(t))/K) \otimes D(K^{G(C(t))}/K) \otimes C^K$$

$$\downarrow\downarrow$$

$$D(G(C(t))/K) \otimes C^K \to C$$

or:

$$\underbrace{D(G(C(t))/K) \otimes D(K^{G(C(t))})}_{\text{contract}} \otimes_{G(C(t))} D(^{G(C(t))}/K) \ldots$$

$$\downarrow\downarrow$$

$$D(G(C(t))/K) \otimes D(K^{G(C(t))}) \otimes_{G(C(t))} C$$

$C_{\leq r}$ full subcategory generated under cones by

$$D(\mathcal{C}(G(\mathcal{C}(t))/K_{x,rt}) \otimes_{\text{Hecke}} C^{K_{x,rt}}$$

as x ranges over all possible x

Lemma: Only finitely many x are needed

Example: $G = SL_2$

Filtration jumps at $\frac{1}{2}\mathbb{Z}$

For $r \in \mathbb{Z}$

$$C_{\leq r} \simeq C^{K_{0,r^+}} \underset{\text{Hecke}}{\otimes} D(K_{0,r^+})^{G(\mathcal{O}_s)}$$

For $r \in \mathbb{Z} + \frac{1}{2}$

$$C_{\leq r} \simeq C^{K_{\frac{1}{2},r^+}} \underset{\text{Hecke}}{\otimes} D(K_{\frac{1}{2},r^+})^{G(\mathcal{O}_s)}$$

Note: The set of r for which filtration jumps are those r whose denom divides a fund. exponent of G

$a_r : C_{\leq r} \hookrightarrow C$

a_r is fully faithful

a_r^R is continuous and

$G(\mathcal{O})$-equivariant

Applications: <u>Generation</u>

Conj (FG):

$$\hat{\mathfrak{g}}\text{-mod}^{reg}_{crit} \simeq D(Gr) \underset{pt/\check{G}}{\times} Op^{reg}$$

$U(\hat{\mathfrak{g}})_{crit}$ has a large center

$$Z(U_{crit}(\hat{\mathfrak{g}})) \cong \mathcal{O}(Op_{\check{G}}(D^\times))$$

$$\downarrow$$

$$\mathcal{O}(Op_{\check{G}}(D))$$

<u>Warning</u>: All not quite true

$\hat{\mathfrak{g}}\text{-mod}^{reg}_{crit}$ is modules where the center factors through $\mathcal{O}(Op_{\check{G}}(D))$

Sam Raskin proved FG for $G = SL_2$.

have a fully faithful embedding

$$\begin{tikzcd} D(Gr) \times_{pt/G} Op^{reg} \arrow[r] & \hat{\mathfrak{g}}\text{-mod}^{reg}_{crit} \end{tikzcd}$$

$G(\mathcal{O})$ acts on the left, $G(\mathcal{O})$ acts on the right (be careful)

Claim: For $G = SL_2$,

$\mathrm{Whit}(G(K))$, $D(G(K)/I)$

is a generating set.

So it suffices to show

$$\mathrm{Whit}(\mathrm{Gr}) \times_{pt/\check{G}} \mathrm{Op}^{reg} \to \mathrm{Whit}(\hat{\mathfrak{g}}\text{-mod}_{crit}^{reg})$$

$$D(I\backslash \mathrm{Gr}) \times_{pt/\check{G}} \mathrm{Op}^{reg} \to \hat{\mathfrak{g}}\text{-mod}_{crit}^{reg, I}$$

are equivalences.

We can describe every category here.

$\mathrm{Whit}(\mathrm{Gr})$ Frenkel–Gaitsgory–Vilonen

$D(I\backslash \mathrm{Gr})$ Arkhipov–Bezrukavnikov–Ginzburg

$\mathrm{Whit}(\hat{\mathfrak{g}}\text{-mod})$ Raskin

$\hat{\mathfrak{g}}\text{-mod}^{I}_{reg}$ Frenkel–Gaitsgory

Goal: Prove an analogous generation statement for all G.

$G((t))$-rep

Subcategories:

$G((t))\text{-rep}_{\leq r}$: $C_{\leq r} \xrightarrow{\sim} C$

$G((t))\text{-rep}_{\geq r}$: $C_{<r} \cong 0$

$G((t))\text{-rep}_{=r}$: intersection

I will write down generators for

$$G((t))\text{-rep}_{\geq r}$$

over all r.

Note: Formally

$$D(G((t))/K_{x,r+}) \text{ over all } x$$

generate $G((t))\text{-rep}_{\leq r}$

$$\tau^{\geq r} D(G((t))/K_{x,r+})$$

generate $G((t))\text{-rep}_{\geq r}$

$$D(G(\mathbb{C}(t))/K_{x,r^+}) \supseteq D(K_{x,r}/K_{x,r^+})$$

If $r > 0$,

$K_{x,r}/K_{x,r^+}$ is an affine space

w/ action of

$$L_x := K_{x,0}/K_{x,0^+}$$

By the Fourier transform,

$$D(K_{x,r}/K_{x,r^+}) \simeq D(\mathbb{C}^{\sim *})$$

$$(\mathfrak{k}_{x,r}/\mathfrak{k}_{x,r+})^{*,0} \subset (\mathfrak{k}_{x,r}/\mathfrak{k}_{x,r+})^*$$

L_x-semistable locus

points whose L_x orbit does not contain 0 in its closure.

Main Theorem

$$\tau^{\geq r} D(G((t))/K_{x,r+})$$

$$\simeq D(G((t))/K_{x,r+}) \times_{(\mathfrak{k}_{x,r}/\mathfrak{k}_{x,r+})^*_{dR}} {}^{0}$$

$$L_x\left(\bigcup_y (\mathfrak{k}_{x,r} \cap \mathfrak{k}_{y,r})^{\perp}\right) = \text{unstable locus}$$

Consequence

$$D(G((t))/(K_{x,rt})^{\circ}$$

is a set of generators for $G((t))$-rep$_{=r}$

Application 1 (computation of depth filtration on KM^{crit})

Thm $\hat{\mathfrak{g}}\text{-mod}^{crit}_{\leq r} \cong \hat{\mathfrak{g}}\text{-mod}^{crit}_{\widehat{Op}_{\leq r}}$

(Note: Chen-Kamgarpour conjecture Thm 3* (Chen))

$Op_{\leq r} \subset Op(D^\times)$

opers w/ slope at most r

pro-affine space

Chen-Kamgarpour proc (following BD in quantization 6.1.3)

$\mathrm{Ind}^{\mathfrak{g}((t))}_{K_{x,r^+}} \mathbb{C}$ central support lies in

$\underset{=}{\subset}$

$Op_{\leq r}$ locus,

Thm 2 (joint w. Raskin)

Conj FG is true for all G.

So we have a candidate depth filtration on ~~[illegible]~~ $\hat{\mathfrak{g}}\text{-mod}_{crit}$

$$\hat{\mathfrak{g}}\text{-mod}_{crit}^{\leq 0'} \subseteq \hat{\mathfrak{g}}\text{-mod}_{crit}^{\leq r'} \cdots$$

to show this is actually the depth filtration, it suffices to show

$$gr'_r \; \hat{\mathfrak{g}}\text{-mod}_{crit} \text{ is purely of depth } r$$

$\Longleftarrow$ for $s \not\geq r$

$$gr'_r \; \hat{\mathfrak{g}}\text{-mod}_{crit} \otimes_{G(O)} D(G(K)/K_s) \cong 0$$

$$\mathrm{gr}'_r\ \hat{\mathfrak{g}}\text{-mod}_{crit}^{K_{x,st},\,0} \cong \mathcal{O}$$

Why does $K_{x,st}^{ss}$ detect the corresponding part of the center

$G = SL_2$

~~$K_{x,st,r,0}$~~ $M \in \hat{\mathfrak{g}}\text{-mod}_{crit}^{K_{x,st}^{0}}$

if s is an integer.

This is a module w/ some element 1

w/ $\mathfrak{g}[[t]]\, t^{s+1} \cdot 1 = 0.$

And the action of

$\mathfrak{g}[[t]]\, t^{s} \cdot 1$

has a non-nilpotent character in $\mathfrak{g}^*$

The corresponding element of FF

Center look like,

$(et^s)\cdot(ft^s) + (ht^s)\cdot(ht^s) + (ft^s)\cdot(et^s)$
$+ \ldots \cdot (t^{s+1})$