$$K_{x,r} = \bigoplus_{\langle x,\alpha\rangle + i \geq r} \mathfrak{g}_\alpha t^i$$

$$K_{x,r+} = \bigoplus_{\langle x,\alpha\rangle + i > r} \mathfrak{g}_\alpha t^i$$

$C_{\leq r}$ subcategory generated by $C^{K_{x,r+}}$

Last time:

Stated generation theorem

$D(G(\mathbb{C}(t))/K_{x,r+})^0$ form generators for $G(\mathbb{C}(t))$-rep$_{\leq r}$

$$D(G(\mathbb{C}(t))/K_{x,r+}) \circlearrowleft K_{x,r}/K_{x,r+}$$

$$D(G(\mathcal{O}(t))/K_{x,r+}) \times (K_{x,r}/K_{x,r+})^{*}$$

$$(K_{x,r}/K_{x,r+})^{*}$$

Semistable $:=$ L_x-closure does not contain the origin

$$P_x := K_{x,0}$$

$$L_x := P_x / K_{x,0+}$$

Stated computation of depth on $\hat{\mathfrak{g}}\text{-mod}^{crit}$

$$\hat{\mathfrak{g}}\text{-mod}^{crit}_{\leq r} \cong \hat{\mathfrak{g}}\text{-mod}^{crit}_{\widehat{\mathcal{O}_{B_r}}}$$

$$\hat{\mathfrak{g}}\text{-mod}^{crit}_{=r} \cong \hat{\mathfrak{g}}\text{-mod}^{crit}_{\widehat{\mathcal{O}}_{P_1} \setminus \widehat{\mathcal{O}}_{P_{<r}}}$$

Easier direction (Chen-Kamgarpour)

$$\hat{\mathfrak{g}}\text{-mod}^{crit}_{\leq r} \subseteq \hat{\mathfrak{g}}\text{-mod}^{crit}_{\widehat{\mathcal{O}}_{P_r}}$$

$$M \in \hat{\mathfrak{g}}\text{-mod}^{K_{\chi,r+},0}_{crit}$$

$K_{\chi,r+}$ acts nilpotently on M

If $K_{\chi,r+} \cdot v = 0$,

then $K_{\chi,r}/K_{\chi,r+}\vec{C}v$

has spectrum lying inside

$$(K_{\chi,r}/K_{\chi,r}^{+})^{*}_{ss}$$

Feigin-Frenkel center has associated graded $U(\mathfrak{g})^G((t))$

so x rational defines a $\mathbb{Q}/\mathbb{Z}$ grading on $\mathfrak{g}$.

grading on $\mathfrak{g}_\alpha$ is $\langle \alpha, x \rangle$

If x is rational this corresponds to some $\mathbb{Z}/n\mathbb{Z}$ grading on $\mathfrak{g}$.

$$\mathfrak{g} \cong \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{n-1}$$

$\mathfrak{g}_0$ is the Lie algebra of L_x

Vinberg:

GIT of $G_0 := L_x \circlearrowright \mathfrak{g}_1^*$

behaves like a coadjoint action.

More precisely:

i) $(\mathrm{Sym}^* \mathfrak{g}_1)^{L_x}$ is polynomial ring

ii) $p \in \mathfrak{g}_1^*$ is L_x-unstable iff all these polynomials vanish on p.

To link w/ Feigin-Frenkel center:

$$(\mathrm{Sym}\, \mathfrak{g})^G \to (\mathrm{Sym}\, \mathfrak{g}_1)^{G_0}$$

For inner gradings on $\mathfrak{g}$ (i.e. those from some x)

This map is finite

(i.e. RHS finite module over LHS)

$\Rightarrow \hat{\mathfrak{g}}\text{-mod}^{\kappa_{x,r+},0}_{crit}$ is supported on $\hat{\mathcal{O}}_{P_r} \setminus \hat{\mathcal{O}}_{P_{<r}}$.

So for instance if $r \in \mathbb{Z} + \frac{1}{n}$

$$\mathfrak{g}_1 \cong K_{x,r}/K_{x,r+}$$

$$\cong \bigoplus_{\langle \alpha, x\rangle + i = r} \mathfrak{g}_\alpha t^i$$

so a $\mathfrak{g}_\alpha$ appears (w/ some i)

iff $\langle \alpha, x\rangle = r$ inside $\mathbb{Q}/\mathbb{Z}$

gr. FF $\cong (\mathrm{Sym}\, \mathfrak{g})^G[t]$

action of $FF_{>r}$ on M this is necessarily trivial

Because $K_{x,rt}$ acts trivially on v

grading on FF

Oper $\nabla = \frac{d}{dt} + e + \sum f_i t^{m_i}$

regular term

slope of oper: $\max\left(\frac{-1-n_i}{m_i}\right)$

and m_i is the fundamental exponent corresponding α_0

Example: SL_2:

Oper $\frac{d}{dt} + e + ft^{-2}$

has slope $\frac{1}{2}$

Understand action of

$FE_{\underline{=r}}$ on M

What is $FE_{\underline{=r}}$? It is a set

of generators, one

for each m_i with

$m_i \cdot r$ integral.

$$U(\mathfrak{g})^{G} \longrightarrow U(\mathfrak{g}_1)^{G_0}$$

automatically fills generators with major not integral

$$\mathrm{Spec}(\mathrm{Sym}\, E^{\vee}) \longleftarrow \mathrm{Spec}(\mathrm{Sym}\, \mathfrak{g}_1)^{G_0}$$

this map is finite, so it sends non-origin points to non-origin points

The semistability condition $\Rightarrow$ nonorigin on RHS

Statement we want to prove

is nonorigin on LHS

$\mathrm{Sym}\ \mathfrak{g}_1 \circlearrowright V$

Claim: action of $FF_{\underline{=}r}$ factors as

$$\mathrm{Sym}\ FF_{\underline{=}r} \to \mathrm{Sym}\ \mathfrak{g}_1^{G_0} \to \mathrm{Sym}_{\mathfrak{g}_1} \circlearrowright V$$

$FF_{\underline{=}r}$ is in

$$\mathrm{Sym}\,(k_{x,r}/k_{x,r+})$$

$$+ (\ \text{—}\ ) \cdot k_{x,r+}$$

Generation theorem:

$D(G(\mathbb{C}(t))/K_{x,r+})^b$ generate $G(\mathbb{C}(t))$-rep$_{\geq r}$.

$\tau^{\geq r} D(G(\mathbb{C}(t))/K_{x,r+})$

$\cong D(G(\mathbb{C}(t))/K_{x,r+})^b$

$\tau^{\geq r} C := C/C_{<r}$.

Formally: $D(G(\mathbb{C}(t))/K_{x,r+})$ generate $G(\mathbb{C}(t))$-rep$_{<r}$

$\Rightarrow \tau^{\geq r}$ generate $G(\mathbb{C}(t))$-rep$_{\geq r}$

Combinatorial Statement:

$$Z = \bigcup_{Y} (K_{x,r} \cap K_{y,r})^{\perp}$$

$$Z' = \bigcup_{K_{y,r} \subseteq K_{x,r}} K_{y,r}^{\perp}$$

Call considered inside $(K_{x,r}/K_{x,r+})^{*}$)

Claim:

$$L_x \cdot Z = L_x \cdot Z' = (K_{x,r}/K_{x,r+})^{*}_{us}$$

First: $C = D(G(\mathbb{C}((t)))/K_{X,r+})$

$$D(G(\mathbb{C}((t)))/K_{X,r+})^{us} \subseteq \tau_{<r} C$$

We have, for any Y

$$D(G(\mathbb{C}((t)))/K_{Y,r}) \subseteq \tau_{<r} C$$

Reason: $K_{Y,r} = K_{Y,st}$ for some $s<r$

As long as $K_{X,r+} \subseteq K_{Y,r} \subseteq K_{X,r}$, then

$$D(G(\mathbb{C}((t)))/K_{Y,r})$$

$$\cong D(G(\mathbb{C}((t)))/K_{X,r+}) \times_{(K_{X,r}/K_{X,r+})^*} K_{Y,r}^{\perp}$$

Taking the union over all y, we get

$$\bigcup K_{y,r}^{\perp} = Z'$$

We can get a L_X action from convolution

$$D(G((t))/K_{y,r}) \times D(\mathrm{Gr}_{K_{y,r}}^{P_X}(K_{x,rt})) \longrightarrow D(G((t))/K_{x,rt})$$

& so the statement that

$$L_X \cdot Z' = \text{unstable locus}$$

implies inclusion

$$\tau_{\subset r} C \subseteq D\,C\,G((t))/(K_{x,rt})^{\omega}$$

suffices to show, for any y

$$D(K_{y,r}\backslash G((t))/K_{x,rt}) \subseteq C^{\omega}$$

Split $G((t))$ into cells

$$P_y w P_x, \quad w \in W_{aff}$$

Argument for $P_y P_x$:

$$D(K_{y,r}\backslash P_y P_x/(K_{x,rt}) \circlearrowleft$$

can commute $K_{y,r}$ past

P_y (by normality)

need to understand:

$$D(K_{y,r} \backslash P_x(K_{x,r+}))$$

$$w^{-1} K_{y,r} w = K_{wy,r}$$