Joint with Peter Scholze

Starting point: Naive question:
Is there some "relative" variant of perverse sheaves?

Setup: Fix a prime ℓ. Always work with $\mathbb{Z}[\frac{1}{\ell}]$-schemes. (All schemes qcqs.)

Three scenarios we consider:

A) Λ an ℓ-torsion ring, $\underline{D_{ét}(X,\Lambda)}$ = left completion of $D(X_{ét},\Lambda)$ $= \lim_{n\to\infty} D^{\geq -n}$

B) Λ as in A), but look at $\underline{D_{cons}(X,\Lambda)} \subset D_{ét}(X,\Lambda)$.

C) take $L/\mathbb{Q}_\ell$ an alg. extension, look at $\underline{D_{cons}(X, L \text{ or } \mathcal{O}_L)} \subset D_{proét}(X,\Lambda)$ = objects which become dualizable on a constructible stratification of X.

Just write $D(X)$ for the relevant category in each scenario.

Thm (H.-Scholze). Let $X \to S$ be a finitely presented map. Let $D(X)$ be as in scenarios A)-C). (In scenario C, assume that all constr. subsets of S have fin. many irred. compts.)

Then there is a t-structure ${}^{p/S}D^{\leq 0}(X)$, ${}^{p/S}D^{\geq 0}(X)$ on $D(X)$ characterized by the condition that $A \in D(X)$ lies in ${}^{p/S}D^{\leq 0}$ resp. ${}^{p/S}D^{\geq 0}$

iff $\forall\ \bar{s} \to S$, with fiber $X_{\bar{s}} \xrightarrow{h} X$, h^*A lies in ${}^{p}D^{\leq 0}(X_{\bar{s}})$ resp. ${}^{p}D^{\geq 0}(X_{\bar{s}})$.

(h^* for both!!)

This t-structure interpolates between two extremes:

i) If $X \to S = X$ is the identity, just get the standard t-structure on $D(X)$.
or naive

ii) If $S = \mathrm{Spec}\, k$ is a pt, get the usual perverse t-structure on $D(X)$.

In general, this t-structure has no good "finite length" properties.
However, it does have good properties along these lines after imposing another condition, namely the condition of being ULA.

Fix $f: X \to S$, $A \in D(X)$.

Intuition: A is universally locally acyclic (ULA) w.r.t. f if the coh. of $A|$ slices of a small ball in X is constant as the slices vary.

Definition: A is ULA w.r.t. f if $\forall\ \bar{x} \to X$, $\bar{t} \rightsquigarrow f(\bar{x})$ specialization, the nat. map

$$R\Gamma(X_{\bar{x}}, A) \longrightarrow R\Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{t}, A) \text{ is an isom.,}$$

and likewise after any base change on S.

Key things: 1) $X \to S$ smooth, Λ is ULA.

2) In reasonable situations, "any" sheaf is ULA over a dense open in the target. (Deligne)

Thm. Fix $f: X \to S$ as before.

1) If $A \in D(X)$ is f-ULA, then all relative perverse truncations of A are f-ULA.

$\Longrightarrow$ $\mathrm{Perv}(X/S)$ = heart of rel. t-structure on $D(X)$

comes with full subcat. $\mathrm{Perv}^{ULA}(X/S)$.

2) $\mathrm{Perv}^{ULA}(X/S)$ is stable under relative Verdier duality (i.e. $R\mathcal{H}om(-, f^! \Lambda)$).

$\rightsquigarrow$ E.g. in case C with $\Lambda = L$, $\mathrm{Perv}^{ULA}(X/S)$ is noetherian and Artinian.

3) $\mathrm{Perv}^{ULA}(X/S)$ stable under subquotients.

Key special case of main thm:

$S = \mathrm{Spec}\, V$, V a rank one valuation ring with rank one alg. closed fraction field ("AIC" valuation ring)

absolutely integrally closed

$|S| = \underset{s}{\cdot} \rightarrow \underset{\eta}{\cdot}$ $X \to S$ as before. $j: X_\eta \to X$, $i: X_s \to X$ as usual

By a result of Lurie, always can define a t-str. whose connective part consists of sheaves which lie in ${}^pD^{\leq 0}$ (every fiber). (HA 1.4.4.11)

In the present case, get ${}^{p/S}D^{\geq 0}(X) = A \in D(X)$ s.t. $j^*A \in {}^pD^{\leq 0}(X_\eta)$ and $i^*A \in {}^pD^{\leq 0}(X_s)$.

$\Rightarrow$ By general nonsense, $A \in {}^{P/S}D^{\geq 0}$ iff $j^*A \in {}^pD^{\geq 0}(X_\eta)$ and $i^!A \in {}^pD^{\geq 0}(X_s)$.

Claim: The latter pair of conditions is equivalent to: $j^*A, i^*A \in {}^pD^{\geq 0}$.

Key idea: Look at the exact triangle

$$i^! A \to i^* A \to \underbrace{i^* j_* j^* A}_{} \to$$

Assume $j^* A \in {}^pD^{\geq 0}$. ${}^pD^{\geq 0}$ Then $i^* j_* : D(X_\eta) \to D(X_s)$ is the nearby cycles functor, which in particular is perverse t-exact!

Thm of Gabber (Illusie '94)

The idea in general case is to reduce to this (very!) special case by descent arguments.

For this, we need very fine topologies:

Recall: v-topology $\subsetneq$ top. of universal submersions $\subsetneq$ arc-topology

$X \to Y$ cover if $\forall$ $\mathrm{Spec}\, V \to Y$, V val. ring, can lift after replacing V by some V'/V faithfully flat

$$\exists \quad \begin{tikzcd} X & \mathrm{Spec}\, V' \arrow[l, dashed] \arrow[d] \\ Y \arrow[from=1-1, d] & \mathrm{Spec}\, V \arrow[l] \end{tikzcd}$$

$X \to Y$ univ. submersion if $|X| \to |Y|$ is a q.t. map ~~after~~ any base change on Y.

Same as v-top., but only test via rank $(\leq)1$ val. rings.

<u>Thm</u> (Bhatt-Mathew, Gabber). In each of scenarios A)–C), $X \mapsto D(X)$ is a hyper v-sheaf of stable ∞-cats.

In scenarios B)–C), it is an arc-sheaf (Bhatt-Mathew)

In scenario A), it is a universally submersive sheaf (Gabber).

Idea of Pf: 1) $S = \operatorname{Spec} V$ rk 1 a.i.c valuation ring. OK ✓

2) $S = \operatorname{Spec} V$ a.i.c val. ring. Reduce to the previous case by approx. & descent.

3) S has each conn'd comp. $\simeq \operatorname{Spec} V$, V a.i.c val. ring.

Reduce to previous case by pure top. + Lemma: "the perverse coh. amplitude is a constructible fn on the base".

4) General S. Key pt: Can pick a v-hypercover $S_\bullet \to S$ with all S_n as in 3). Then already have t-str. you want on all $D(X \times_S S_n)$, and the pullbacks $D(X \times_S S_n) \to D(X \times_S S_m)$ are rel. perv. t-exact.

$\Rightarrow$ Formal to get the desired t-str. on $D(X) = \lim\limits_{n \in \Delta} D(X \times_S S_n)$.

S conn'd, $\bar{s} \to S$ S $\mathbb{Q}$-scheme, or $\bar{s}$ dominates gen. pt

$$\mathrm{Perv}^{ULA}(X/S) \longrightarrow \mathrm{Perv}(X_{\bar{s}}) \quad \text{fully faithful}$$