BFN2

Wednesday, May 12, 2021 12:29 PM

Last time:

G - reductive group (over $\mathbb{C}$)

V - rep. of G

$\rightsquigarrow$

$\mathcal{A}_{G,V}$ - ring object in $D_{G(\mathcal{O})}(Gr_G)$

$\Phi \sim G^\vee$- equiv. modules over $\mathrm{Sym}(\mathfrak{g}^\vee[-2])$

$\mathcal{A}_{G,V}$ corresponds to some algebra $\Phi(\mathcal{A}_{G,V})$ over $\mathrm{Sym}\,\mathfrak{g}^\vee[-2]$ with $G^\vee$-action

Also, we can consider $H^*_{G(\mathcal{O})}(\mathcal{A}_{G,V}) = A_{G,V}$

$\mathrm{Spec}\, A_{G,V} = \mathcal{M}_{C,V}$ - normal algebraic Poisson generically symplectic variety

$$\mathcal{M}_C = \mathrm{Whit}_{G^\vee}(\mathrm{Spec}\,\underset{\underset{G^\vee}{\circlearrowright}}{\Phi(\mathcal{A}_{G,V})})$$

Basic example 0: $V=0$. Who is $\mathcal{A}_{G,V}$ -?

$\mathcal{A}_{G,V} = \omega_{Gr_G}$ $\quad H^*_{G(\mathcal{O})}(\mathcal{A}_{G,V}) = H^{G(\mathcal{O})}_*(Gr_G)$

$$\mathcal{M}_C = \mathrm{Spec}\, H^{G(\mathcal{O})}_*(Gr_G) \simeq Z_{G^\vee} = \{(g,x) \mid g \in G^\vee,\ x \in \mathfrak{k}_{\mathfrak{g}^\vee},\ \mathrm{ad}_g(x) = x\}$$

$$\Phi(\mathcal{A}_{G,V}) = \Phi(\omega_{Gr_G}) = \mathbb{C}[W_{G^\vee}]$$

$W_{G^\vee}$ = twisted cotangent bundle of $G^\vee/U^\vee$

$\shortparallel$

$\mathrm{Whit}(T^*G^\vee)$

$$\parallel$$
$$\mathrm{whit}_{G^\vee}\left(T^* G^\vee \right)$$

Slight ~~generalization~~ generalization:

$$G_F \to \mathrm{Aut}_G(V)$$

$$\mathcal{B}_{G_F, V}$$
$$\parallel$$
$$\mathcal{A}_{G \times G_F, V} \rightsquigarrow \pi_!^{G(\mathcal{O})}\left(\mathcal{A}_{G \times G_F, V} \right)$$

$$Gr_G \times Gr_{G_F} \xrightarrow{\pi} Gr_{G_F}$$

$!$-stalk of $\mathcal{B}_{G_F, V}$ at $1 = A_{G, V}$

Example $G_F = GL(n)$

$$G = \prod_{i=1}^{n-1} GL(i)$$

$$V = \bigoplus_{i=1}^{n-1} \mathrm{Hom}\left(\mathbb{C}^i, \mathbb{C}^{i+1} \right)$$

$$\mathcal{M}_{C, G, V} = \mathcal{N}_{\mathfrak{gl}(n)} = \text{nilpotent cone in } \mathfrak{gl}(n)$$

$\mathcal{B}_{GL(n), V}$ – ring object in $D_{GL(n, \mathcal{O})}\left(Gr_{GL(n)} \right)$

Theorem $\mathcal{B}_{GL(n), V} \simeq \mathcal{R}_{GL(n)}$

<u>Theorem</u> $\mathcal{B}_{GL(n),V} \simeq \mathcal{R}_{GL(n)}$

$\mathcal{R}_{GL(n)}$ - perverse sheaf corresponding to $\mathbb{C}[GL(n)]$ under geometric Satake equivalence

---

Construction of Moore-Tachikawa varieties (a.k.a. Poisson snakes).

G - reductive

$\forall n>0$ want X_n with action of $(G^\vee)^n$

Poisson, generically symplectic, affine, action is Hamiltonian ($\exists\, \mathbb{G}_m$ action which scales the symp. form)

① $(X_n \times X_m)/\!\!/G^\vee \cong X_{n+m-2}$

one copy of $G^\vee$ which acts on both

② $\mathrm{Whit}_{G^\vee}(X_n) \simeq X_{n-1}$

<u>Generalization of 1</u>: (Higgs branches of Sicilian theories)

Let Σ be a oriented. surface of genus g

Let $x_1, \dots x_n \in \Sigma \rightsquigarrow X_{\Sigma, x_1, \dots x_n}$

Poisson, gen. symplectic, affine, action of $(G^\vee)^n$

$\Sigma, x_1 \dots x_n$

Poisson, gen. symplectic, affine, action of $(G^\vee)^n$

$$\Sigma_{1, x_1 \dots x_n} \underset{i,j}{*} \Sigma_{2, y_1 \dots y_m}$$ – get a surface with $n+m-2$ punctures

$i \in \{1, \dots n\}$

$j \in \{1 \dots m\}$

$$\tilde{\Sigma}, z_1 \dots z_{n+m-2}$$

$$X_{\tilde{\Sigma}_{z_1 \dots z_{n+m-2}}} = (X_1 \times X_2) /\!\!/ G^\vee$$

If all surfaces have genus 0 we get exactly one variety X_n $\forall n > 0$

Gluing condition is ~~pe~~ precisely ① before

Category: objects are finite collection of circles $\mathcal{C}$

Morphisms = cobordisms

$\mathcal{S}$: objects are reductive groups

$Mor(G_1, G_2)$ = Poisson ~~v~~ affine varieties with Ham. action of $G_1 \times G_2$

$$\underset{G_1 \times G_2}{X} \circ \underset{G_2 \times G_3}{Y} = X \times Y /\!\!/ G_2$$

Moore-Tachikawa: Given $G^\vee$ $\exists$ natural functor $\mathcal{C} \to \mathcal{S}$

$(G^\vee)^n$

$\mathcal{C} \to \mathcal{S}$

$$\underbrace{o \quad o \quad o \quad o}_{n} \longmapsto (G^\vee)^n$$

<u>Construction of</u> <u>X_n</u>

$$\underset{(G^\vee)^n}{\mathbb{C}[X_n]} = \Phi\Big(\underbrace{R_G \overset{!}{\otimes} R_G \otimes \ldots \overset{!}{\otimes} R_G}_{n-1}\Big)$$

$$\Phi(\mathcal{F}) = H^*_{G(\mathcal{O})}\big(R_G \overset{!}{\otimes} \mathcal{F}\big)$$

X_2 – variety with action of $G^\vee \times G^\vee$

$X_n * X_2 \simeq X_n$

X_2 – identity morphism from $G^\vee$ to $G^\vee$ in $\mathcal{S}$

$X_2 \simeq T^* G^\vee$

X – any variety with Hamiltonian action of $G^\vee$

$$X * X_2 = (X \times X_2)/\!/G^\vee \simeq X$$

$$\Phi(R_G) \simeq \mathbb{C}[T^* G^\vee]$$

$$X_1 = W_{G^\vee} = \mathrm{Whit}_{G^\vee}(T^* G^\vee)$$

Λ_1 $W_{G^\vee}$ $G^\vee$

$$\mathbb{C}[W_{G^\vee}] = \Phi(\omega_{Gr_G})$$

(<u>Remark</u> $W_{G^\vee} \simeq G^\vee \times K_{\mathfrak{g}^\vee}$)

<u>Theorem</u> (Easy) X_n defined as above

(as Spec $H^*_{G(\mathcal{O})}(Gr_G, \underbrace{\mathcal{A}_G \overset{!}{\otimes} \cdots \overset{!}{\otimes} \mathcal{A}_G}_{n})$)

satisfies the requirements 1 & 2 above

(BFN or Ginzburg – Kazhdan)

<u>Ex</u> $G^\vee = SL(2)$ $\quad n=3 \quad X_3 \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$G^\vee = SL(3)$ $\quad n=3 \quad X_3 =$ closure of the minimal orbit in E_6.

If $G^\vee = SL(3)$ $\quad n>3$ $\quad$ no explicit description

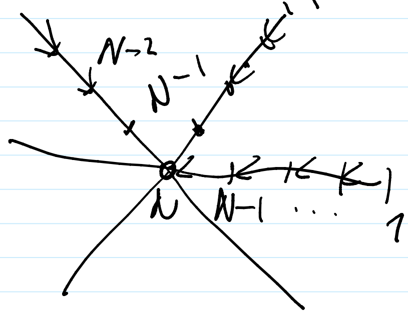
$G^\vee = SL(2)$ $\quad n>4$

<u>Theorem</u> Let $G = G^\vee = GL(N)$

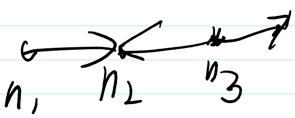
Then the algebra $H^*_{G(\mathcal{O})}(\mathcal{A}_G \overset{!}{\otimes} \mathcal{A}_G \overset{!}{\otimes} \cdots \overset{!}{\otimes} \mathcal{A}_G)$ is

finitely generated, integrally closed etc.

It follows from the Theorem from the beginning that X_n is actually of beginning that Λ_n is actually of the form $\mathcal{M}_{C,G,V}$ for certain G and V.

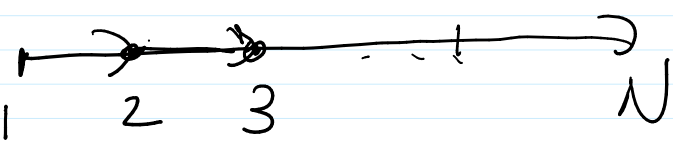


n legs



$G = \prod_i GL(N_i)$ over all vertices of the quiver

$V = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{N_i}, \mathbb{C}^{N_j})$



Consider n such legs

Glue them along the "N-th" vertex.

Elementary construction (Drinfeld, Ginzburg-Kazhdan, V. Lafforgue)

$X_1 = W_{G^\vee}$

$\underbrace{W_{G^\vee} \times \dots \times W_{G^\vee}}_{n}$ symplectic, has $(G^\vee)^n$-action

<u>Want</u>: Reduce dim by $(n-1) \cdot 2 \operatorname{rank} G^\vee$

$$\begin{array}{ccc} W_{G^\vee} & \circlearrowleft & Z_{G^\vee} \curvearrowleft G \\ \downarrow & & \downarrow \\ \mathfrak{c}_{\mathfrak{g}^\vee} = \mathfrak{t}/W & & \end{array}$$

group scheme

$Z_{G^\vee}$ action on $W_{G^\vee}$ commutes with $G^\vee$-action

$$Y_n = (\underbrace{W_{G^\vee} \times \dots \times W_{G^\vee}}_{n}) \,/\!/\, Z_{G^\vee}^{n-1}$$

$n=2$

$$\begin{array}{c} W_{G^\vee} \times W_{G^\vee} \\ \downarrow \pi \\ \mathfrak{c}_{\mathfrak{g}^\vee} \times \mathfrak{c}_{\mathfrak{g}^\vee} \end{array}$$

$$\pi^{-1}(\Delta_{\mathfrak{c}_{\mathfrak{g}^\vee}}) / Z_{G^\vee} = (W_{G^\vee} \times W_{G^\vee}) /\!/ Z_{G^\vee}$$

<u>Lemma</u> Y_n is quasi-affine

<u>Theorem</u> $\mathbb{C}[Y_n] = \mathbb{C}[X_n]$

Proof uses a theorem of Bapat

$$\mathbb{C}[X_n] = H^*_{G(\mathcal{O})}(\underbrace{\mathcal{R}_G \otimes \dots \otimes \mathcal{R}_G}_{n})$$

$$\cong$$

$$H^*_{G(\mathcal{O})}(\mathcal{R}_G) \otimes \dots \otimes H^*(\mathcal{R}_G)$$

$$H^*_{G(\mathcal{O})}(Gr) \dots$$

$\ldots, \quad H^*_{G(\mathcal{O})}(Gr) \ldots$

Non-derived tensor product!

Generalization You can try to play this game $\forall$ map $L \to M$ of reductive

(Previous case is about $L = G$, $M = G^{n-1}$ map = diagonal)

Want: construct $X_{L,M}$ – Poisson, generically symplectic affine variety with action of $L \times M$

Property $\text{Whit}_M(X_{L,M}) = W_L$

$X_{L,M}$ – "quasi-classical version" of the local Langlands lifting corresponding to $L^\vee \to M^\vee$

Definition of $X_{L,M}$ via GR works:

$$\mathbb{C}[X_{L,M}] = \Phi_{L^\vee}(i^! \mathcal{R}_{M^\vee})$$

$$i: Gr_{L^\vee} \to Gr_{M^\vee}$$

"Elementary" construction works as well. $Y_{L,M}$

Isomorphism $X_{L,M} \sim Y_{L,M}$ is not clear in general because we don't know an analog of Bapat's thm (works in a lot of examples) of Bapat's thm (works in a lot of examples)