# Gl oh

Tuesday, June 16, 2020 12:32 PM

**Goal**: Formulate some math. conjectures (related to local Langlands for $GL(n)$) and "deduce" them from the formalism that Sam explained last time (Formalism 3d $N=4$ QFT)

**Notation**: $n \geq 1$

$\to W_n = \{ \mathcal{E}_1 \to \mathcal{E}_2 \to \dots \to \mathcal{E}_n \}$

$\mathcal{E}_i$ - local system of rank $i$ on $\mathcal{D}^* = \operatorname{Spec} K = \mathbb{C}((t))$

$\pi_n : W_n \to LS_n(\mathcal{D}^*)$

**Conjecture**

1. $D\text{-mod}(GR_{GL(n)})^{\text{Hecke}} \simeq \operatorname{IndCoh}(\pi_n^{-1}(\text{triv}))$

2. $\operatorname{Whit}(GL(n,K)) = \operatorname{QCoh}(W_n)$

3. $D\text{-mod}/GL(n,K)) = \operatorname{IndCoh}\left( W_n \underset{LS_n}{\times} W_n \right)$

Close relative of 1:

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$$\to S_n = \{ V_1 \to V_2 \to \dots \to V_{n-1} \to \mathbb{C}^n \}$$
$\dim V_i = i$

<u>Note:</u> $\pi_n^{-1}(\text{triv}) = \text{Maps}(D^*_{dR}, S_n)$

<u>Conj 1'</u>: $\text{D-mod}(Gr_{GL(n)})^{\text{Hecke}} \simeq$

$\text{D-mod}(S_n(K))$ $\quad T[GL(n)] = T(S_n)$

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Back to "physics"

Recall: given a 3d $N=4$ theory $T$
it has two top. twists (A and B)

$\mathcal{C}_C(T)$ and $\mathcal{C}_H(T)$ categories attached to $S^1$

Coulomb A $\quad$ Higgs B

$T^*$ – mirror dual interchanges A and B

<u>Often</u> we can start with some explicit $T$
and its $T^*$ is also explicit $\rightsquigarrow$ equiv. of cat.

Recall, that given a smooth "affine" stack Y

Recall, that given a smooth "affine" stack $Y$ ($Y = Z/G$ $Z$-smooth aff. $G$-reductive) one can construct $T(Y)$ (secretly it should depend only on $T^*Y$)

$\rightarrow \mathcal{C}_L(T) = D\text{-mod}(Y(K))$

$\rightarrow \mathcal{C}_H(T) = \mathrm{IndCoh}(\mathrm{Maps}(D^*_{dR}, Y))$

<u>One way</u> to obtain equivalences is to somehow claim that $T(Y)^* = T(Y^*)$ for some $Y^*$

Also, Sam defined $\forall$ reductive $G$ certain theory $T[G]$ (has $G$ flavour symmetry $\Leftrightarrow$ comes from a boundary cond. for $YM(G)$)

<u>Exercise on Sam's talk</u>:

1) $T[G]^* = T[G^\vee]$

2) Given $T$ - theory with $G$-symmetry one can form its S-dual $T^\vee$ has $G^L$-sym.

If $T$ has $G$-action $\rightsquigarrow$ $T/G$ "gauge $G$"

<u>ex.</u> if $G$ acts on $Y$

$T(Y)$ has $G$-symmetry

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$T(Y)/G = T(Y/G)$

$T^\vee = \left( T \times T[G] \,/\, \Delta G \right)^*$

$\shortparallel$

S-dual

3) $\underline{\mathcal{C}_C(T[G]) = \text{D-mod}(Gr_G)}^{Hecke}$

Given $T$ $\exists$ $\mathcal{C}_H, \mathcal{C}_C$ attached to $S^1$

$S^1 = \partial D$ gives objects $\mathcal{F}_H \in \mathcal{C}_H, \mathcal{F}_C \in \mathcal{C}_C$

$\mathcal{F}_C(T[G]) = R_{G^\vee}$ - D-module which corr. to reg. rep. of $G^\vee$ under geom. Satake

$\mathcal{C}_H(T[G]) = \text{D-mod}(Gr_{G^\vee})^{Hecke}$

For general reductive $G$

$T[G]$ is <u>not</u> of the form $T(Y)$ for any $Y$

If $G = GL(n)$ then it is.

<u>Claim</u> $T[GL(n)] = T(S_n)$

$S_n = \{ (V_1 \to V_2 \to \dots \to V_{n-1} \to \mathbb{C}^n) \}$

$\dim V_i = i$

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$Y$ - smooth stack $\rightsquigarrow$ $T(Y)$ - theory attached to $Y$

If $G$ acts on $T$ can "gauge the symmetry" $\rightsquigarrow T/G$

If $G$ acts on $Y$ then $G$ acts on $T(Y)$

$$T(Y)/G = T(Y/G)$$

$T[G]$ is typically not of the form $T(Y)$

$$\mathcal{C}_C(T[G]) = D\text{-mod}(GR_G)^{Hecke}$$

$G(K)$

**Claim** If $G = GL(n)$ then $T[G] = T(Y)$ for $Y = S_n = \{0 \to V_1 \to V_2 \to \cdots \to V_{n-1} \to \mathbb{C}^n\}$

$S_n$ has natural action of $GL(n)$

$$T[G]^* = T[G^\vee]$$

$$T[GL(n)]^* = T[GL(n)] \Rightarrow \mathcal{C}_C \cong \mathcal{C}_H$$

$$\mathcal{C}_H = \mathrm{IndCoh}(\pi_n^{-1}(\mathrm{triv}))$$

$$\pi_n : W_n \to LS_n$$

$$W_n = \{\mathcal{E}_1 \to \mathcal{E}_2 \to \cdots \to \mathcal{E}_n\}$$

$\mathcal{M}_n = \{ z_1, z_2, \dots, z_n \}$

$\mathcal{C}_C \simeq D\text{-mod}(S_n(K))$ (renormalized derived category)

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How can we guess that two theories are the same?

$T$ 3d $N=4$ theory

$\mathcal{C}_C \qquad \mathcal{C}_H$

$\mathcal{F}_C \in \mathcal{C}_C \qquad \mathcal{F}_H \in \mathcal{C}_H$

$\underbrace{\mathrm{Ext}^*(\mathcal{F}_C, \mathcal{F}_C)}_{\mathcal{A}_C} \qquad \underbrace{\mathrm{Ext}^*(\mathcal{F}_H, \mathcal{F}_H)}_{\mathcal{A}_H}$ commutative

$\mathrm{Spec}\,\mathcal{A}_C = \mathcal{M}_C$ Coulomb branch

$\mathrm{Spec}\,\mathcal{A}_H = \mathcal{M}_H$ Higgs branch

<u>Slogan</u> Given two theories $T_1$ and $T_2$ s.t. $\mathcal{M}_C(T_1) = \mathcal{M}_C(T_2)$, $\mathcal{M}_H(T_1) = \mathcal{M}_H(T_2)$ it is expected that $T_1 \sim T_2$.

What are $\mathcal{M}_H$ and $\mathcal{M}_C$ for $T[G]$?

$\mathcal{M}_H = \mathcal{N}_G$ nilpotent cone

$\mathcal{M}_C = \mathcal{N}_{G^\vee}$

$\mathcal{C}_C = D\text{-mod}(Gr_G)^{Hecke}$

$\mathcal{F}_C \in \mathcal{C}_C$ corresponds to $\mathbb{C}[G^\vee]$ under Satake

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Exercise $\mathcal{M}_C = \mathcal{N}_{G^\vee}$

looking for $Y$ s.t. $T[G] = T(Y)$

for this $Y$,

$$\mathcal{A}_H(Y) = \mathbb{C}[T^* Y]$$

$$Y = Z/G \qquad \mathbb{C}[T^*Y] = \mathbb{C}[\mu^{-1}(0)]^G \qquad T^* Z \xrightarrow{\mu} \mathfrak{g}^*$$

Exercise

$$\mathcal{C}_H(Y) = \mathrm{IndCoh}(\mathrm{Maps}(D^*_{dR}, Y))$$

$\mathcal{F}_H$ = direct image of $\mathcal{O}$ from $\mathrm{Maps}(D_{dR}, Y)$

looking for $Y$ s.t

$$\mathbb{C}[T^*Y] = \mathbb{C}[\mathcal{N}_G]$$

$$Y = S_n = \{ V_1 \to V_2 \to \dots \to V_{n-1} \to \mathbb{C}^n \}.$$

$$= \underbrace{\prod_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1})}\Big/ \underbrace{\prod_{i=1}^{n-1} GL(i)}$$

Proposition $\mathcal{M}_C(S_n) \simeq \mathcal{N}_{GL(n)}$

Assume that $T$ has $G$ symmetry

On $\mathcal{M}_H$, $G$ will act

$\mathcal{M}_H(T/G) = \mathcal{M}_H /\!/ G$

What about $\mathcal{M}_C$? $\mathcal{M}_C = \operatorname{Spec} \mathcal{A}_C$

action of $G$ on $T$ produces a ring object $R$ in $D_{G(\mathcal{O})}(Gr_G)$ s.t. $\mathcal{A}_C$ is its !-stalk at $1$.

$\left[\begin{array}{l} T = T[G] \qquad R = \text{"Regular rep. of } G^\vee \text{"} \\ \text{!-stalk at } 1 = \mathbb{C}[\mathcal{N}_{G^\vee}] \end{array}\right.$

action of $G$ on $T$ $\rightsquigarrow$ action of $G(K)$ on $\mathcal{C}_C$ s.t. $\mathcal{F}_C$ is $G(\mathcal{O})$-equivariant.

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$\mathrm{Whit}(GL(n,K))$

let's pretend that local geom. Langlands just says that cat. with $G(K)$ action $\Leftrightarrow$ cat. which live over $LS_{G^\vee}(\mathcal{D}^*)$

cat. which we over $LS_{G^\vee}(D^\circ)$

Given a theory $T$ with $G$ action
$\rightsquigarrow T^\vee$ (S-dual.) with $G^\vee$-action

Gaiotto – Witten

$$T^\vee = (T \times T[G] / \Delta G)^*$$

Given $T$ with $G$ action $\rightsquigarrow \mathcal{C}_C$ has an action of $G(K)$

$$L_G : G(K)\text{-cat} \to \text{Cat over } LS_{G^\vee}$$

$$L_G(\mathcal{C}_C(T)) = \mathcal{C}_H(T^\vee / G^\vee)$$

General phenomenon: if $G$ acts on $T$ then $\mathcal{C}_H(T/G)$ lives over $LS_G$.

Make the case: $\mathcal{C}_C((T[G]/G)^*)$ = universal Langlands category for $G^\vee$.

$$\mathcal{C}_{G^\vee} \qquad L_{G^\vee}(\mathcal{C}) = \mathcal{C} \otimes_{G^\vee(K)} \mathcal{C}_{G^\vee}$$

$(T[G]/G)^*$ = S-dual of trivial theory.

$$\mathcal{C}_C((T[G]/G)^*) = \mathcal{C}_H(T[G]/G)$$

$e_C(\mathbb{T}[G]/G)) = e_H(\mathbb{T}[G]/G)$

$T \supset G \qquad e_C(T/G) = e_C(T)^{G(K)}$

$e_H(T/G) - ??$

If $T = T(Y)$ with $Y$ - smooth (affine) stack with $G$-action

$T(Y)/G = T(Y/G)$

Apply this to $T[GL(n)] \simeq T(S_n)$

$T[GL(n)]/GL(n) \simeq T(S_n/GL(n))$

$S_n/GL(n) = \{ V_1 \to V_2 \to V_3 \to \dots \to V_n \}$

$\dim V_i = i$

$e_H = \operatorname{Ind} \operatorname{Coh}( \underset{\substack{\parallel \\ W_n}}{\operatorname{Maps}( D^*_{dR}, S_n/GL(n))} )$

$\mathcal{E}_1 \to \mathcal{E}_2 \to \dots \to \mathcal{E}_n \qquad \mathcal{E}_i \in LS_i(\mathcal{D}^*)$

<u>Conj</u> $\operatorname{Whit}(GL(n,K)) \simeq \operatorname{QCoh}(W_n)$

<u>Ex.</u> $n=2 \qquad W_2 = \{ \mathcal{E}_1 \to \mathcal{E}_2 \}$

Fix $\mathcal{E}_2$ - irreducible

Fiber of $\operatorname{Whit}(GL(2,K))$ over an irreducible

Fiber of $\mathrm{Whit}(GL(2,K))$ over an irreducible $\mathcal{E}_2$ is supposed to be "irreducible and cuspidal category with $GL(2)$-action attached to $\mathcal{E}_2$"

$\mathcal{C}$ acted on by $GL(2)$ cuspidal

$\mathrm{Whit}(\mathcal{C}) = \mathrm{Vect}$

$$\begin{array}{c} W_2 \\ \downarrow \pi_2 \\ LS_2 \end{array} \qquad \pi_2^{-1}(\mathcal{E}_2) = LS_1$$

$\mathcal{C}$ is irreducible and cuspidal

$\mathcal{C} \simeq QCoh(LS_1) \simeq D\text{-mod}(K^*)$

$K^* \hookrightarrow GL(2,K) \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ equiv. of $K^*$-cat.

$\mathcal{C} \simeq D\text{-mod}(K^*) \qquad D\text{-mod}(K \setminus \{0\})$

<u>Conj</u> $\mathcal{C}$ - irreducible and cuspidal $/GL(2)$

$\mathcal{C} \simeq D\text{-mod}(K^*)$

<u>Remark</u> $\mathcal{C} = D\text{-mod}(GL(2,K)/\underset{\uparrow \text{center}}{K^*}(I^0, \chi))$

$I^0$ - unip. radical of Iwahori

$\chi$ - generic character

For this one I "checked" the equivalence.

For any $n$

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$\mathcal{C}$- irreducible and cuspidal with $GL(n,k)$ action

$$\mathcal{C}|_{GL(n-1,k)} \simeq \text{Whit}(GL(n-1,k))$$

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$$\text{D-mod}(GL(n,K)) = \text{IndCoh}(W_n \times_{LS_n} W_n)$$

equivalence of monoidal categories

That gives us an interpretation of $GL(n,K)$-cat on the "coherent" side.

On the other hand, Arinkin produced a probably correct formulation of local geometric Langlands.

Are the two conjectures compatible?

$$\begin{array}{c} W_n \\ \downarrow \\ LS_n \end{array}$$ is not proper. (otherwise Arinkin conj. would follow)

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$$\text{D-mod}(GR_{GL(n)}) \cong \text{IndCoh}(\pi_n^{-1}(pt/GL(n)))$$

$\mathcal{E}_1 \to \mathcal{E}_2 \to \dots \to \mathcal{E}_n$ - trivial

Bezrukavnikov give a coherent

Bezrukavnikov
(Arkhipov, Ginzburg)

give a coherent interpretation of

$$\text{D-mod}\left({}_{I_n}\backslash \mathrm{GR}_{GL(n)}\right) \simeq$$

Embarassing confession: I don't know how the two are compatible.

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$$\text{D-mod}\left({}_{I_n}\backslash \mathrm{GR}_{GL(n)}\right) = \mathrm{IndCoh}(?)$$

$$? \overset{j}{\underset{\text{open}}{\hookrightarrow}} \pi_n^{-1}(pt/GL(n))$$

The embedding functor is NOT $j_*$
(I don't know what it is)