Recall $\{G\text{-schemes}\}$

$$\mathrm{Mor}(X,Y) = \mathrm{Shv}\left(\frac{X\times Y}{G}\right)$$

$$\mathrm{Mor}(X,Z) \otimes_{\mathrm{End}(X)} \mathrm{Mor}(Y,X) \to \mathrm{Mor}(Y,Z).$$

fully faithful

Apply this to get derived Satake

$$\mathcal{H} = G_F / G_O.$$

$$D_{G(O)}(\mathcal{H}) \xleftarrow{\sim} D({}_{G_O}\backslash^{G_F}/_{(I_0,\psi)})$$

$$D({}_{(I_0,\psi)}\backslash^{G_F}/_{(I_0,\psi)})_{\text{unip.}} \otimes D({}_{I_0,\psi}\backslash^{G_F}/_{G_O})$$

$I_0 \subset I$ - radical of Iwahori

$\psi : I_0 \to G_a$

I_0,ψ - equivariant - baby Whittaker

Here $D({}_{I_0,\psi}\backslash G_F/I_0,\psi)_{unip} \subset$

$D({}_{I_0,\psi}\backslash\!\backslash G_F/I_0,\psi)$ — full subcat.

generated by the image of the

averaging functor from

$D({}_{I^-}\backslash G_F/I^\circ,\psi)$. This behaves

as sheaves on an ind projective

quotient.

Derived Satake

$$D_{G(0)}(Gr) \simeq D\,Coh^{G^\vee}(\{1\} \underset{G^\vee}{\times} \{1\}).$$

1) Thm $D({}_{G_0}\backslash G_F/I_0,\psi) \simeq D^b \mathrm{Rep}(G^\vee)$

"geom.-c Casselmann-Shalika".

Frenkel
Gaitsgory
Vilonen 2000 with $\mathbb{C}$ coeffs.

B G. Mirkovic Riche, Rider
with modular c-nts.

$\mathrm{Perv}({}_{G_0}\backslash G_F/I_0,\psi) \simeq \mathrm{Rep}(G^\vee)$

---

2) (w. Riche, Rider in progress)

$D({}_{I_0,\psi}\backslash G_F/I_0,\psi)_{\mathrm{unip.}} \simeq$

$\simeq D^b \, \mathrm{Coh}^{G^\vee}_{\mathcal{U}}(\tilde{G}^\vee)$

Recall Gaitsgory's central functor $Z: \mathrm{Rep}(G^\vee) \to D_I(\mathcal{Fl})$

$$= D(I \backslash G_F / I)$$

Defined by nearby cycles

carries a tensor automorphism (monodromy) allowing to extend it to a functor from $\mathrm{Coh}^{G^\vee}_{free}(G^\vee)$

$$\mathcal{F} = V \otimes \mathcal{O}$$

Version: $\hat{Z}: \mathrm{Rep}(G^\vee) \to D(\dot{I} \backslash G_F / \dot{I})$

monodromic with unip. monodromy

Hope: version of this works for $\left({}_{J_0,\psi}\backslash G_F / J_0,\psi\right)_{\mathrm{unip}}$.

Not worked out.

Another approach

Consider

$$D\left({}_{I^-}\backslash G_F / I^-\right) = \mathcal{H}$$

$$I, \underset{\substack{| \\ \text{opposite}}}{I^-} \subset G(O)$$

Have obvious functors.

$$\begin{tikzcd}[column sep=large]
 & D\left({}_{I^-}\backslash G_F / I_0,\psi\right) \\
\mathcal{H} \arrow[ur] \arrow[r, "AV_{wh}"'] & D\left({}_{I_0,\psi}\backslash G_F / (I_0,\psi)\right)_{\mathrm{unip}}
\end{tikzcd}$$

$$\underset{\substack{|| \\ wh}}{{}_{I_0,\psi}}$$

Observations.

It's easy to describe the kernel of Av_{mh}

$$\mathrm{Im}(p_{ew})|_{\mathcal{H}} = \{ L_w \mid w \in W_{aff} \}$$

$$\hat{\mathcal{H}}$$

$$\ker(Av_{mh}) = \langle L_w \mid w \notin {}^f W^f \rangle$$

${}^f W^f$ - minimal in its 2-sided coset.

$$\ker(Av_{mh})^{\perp} =: \widehat{R}.$$

R can be described using $\hat{Z}$ & its properties.

Now $\mathcal{R}$ is "almost equivalent" to Wh.

More precisely.

Wh is naturally a category over $T^\vee/W$. (set. th. supp at 1).

Prop.

$$\mathcal{R} \simeq Wh \underset{Coh(T^\vee/W)}{\otimes} Coh\left(T^\vee \underset{T^\vee/W}{\times} T^\vee\right)$$

$W \times W$ acts on both sides.

Thm. (w. Riche, Rider)

Then $\mathcal{R} \simeq Coh^{G^\vee}_{\mathcal{U}}\left(G^\vee \underset{T^\vee/W}{\times} T^\vee \underset{T^\vee/W}{\times} T^\vee\right)$

W^2 eq-ntly $\Longrightarrow Wh \simeq Coh^{G^\vee}_{\mathcal{U}}(G^\vee$.

Prop $\Leftarrow$ formalism ~~of~~ of last lecture + work in progress with T. Deshpande.

modular version of a result of Ginzburg (for D-modules)

We show

$$D\left({}_{U,\psi}\backslash G/{}_{U,\psi}\right) \subseteq D\left(\check{T}/W\right),$$

explicit subcat.

G - reductive group

GIT quot

$$D\left({}_{U,\psi}\backslash G/{}_{U,\psi}\right)_{unip} \simeq D\,Coh_1\left(\check{T}/\!\!/W\right)$$

$$\cap$$

$$D\,Coh_T(W)$$

$$D(X/U_{?,\psi})_{unip} \otimes_{D({}_{U,\psi}\backslash G/U,\psi)_{unip}} D({}_{U,\psi}\backslash G/U^-)$$

$$\downarrow$$

$$D(X/U^-)$$

$$\otimes_{Coh(\tilde{T}/W)} Coh(\tilde{T})$$

Lands in the right orthogonal to kernel of the averaging functor

1 and is an eq-nce with that kernel at least

in our case))

Proof of the Thm.

Recall $D(\mathcal{B}\backslash G/B^-) \longrightarrow D(\mathcal{B}\backslash G/(U,\psi))$

The kernel is generated

by L_w, $w \neq 1$.

Its orthogonal is

gen-d by Ξ - big

projective - pro-

cover of L_1

(in the monodromic

setting - pro-object Ξ).

Our R is generated by

1) $R = \langle Z(V) * \widehat{\Xi} \rangle$.

2) in modular setting can restrict to $V = T_\lambda$ - tilting in $Rep(\check{G})$

lemma $\left(\Xi * Z(T_\lambda) \right)$ - tilting.

$\widehat{\Xi} * \widehat{Z}(T_\lambda)$ - free monodromic tilting.

$Ext^{>0}(\ , \ ) = 0$.

$Hom(\ \ \ ) \simeq Hom(T_\lambda \otimes O, T_\mu \otimes O)$

- proved by analysing the quotient

$D\left({}^{\mathcal{G}F}_{I}, {}_{I}\right) / \langle L_w \mid \ldots \rangle$

$w \neq 1$.

$\longleftrightarrow \quad Coh\left(\check{G}_{reg}\right)$

(or sheaves on Kostant slice)