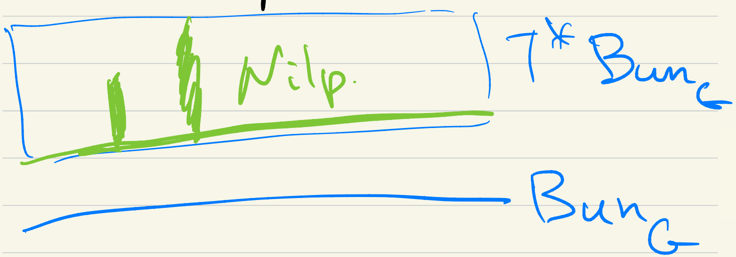
L3

## Summary:

Automorphic

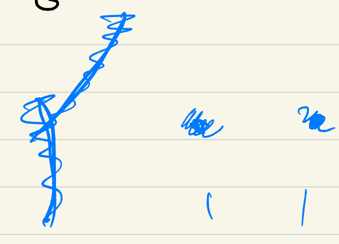
ℓ-adic
dR
Betti

$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$




Galois

$\mathrm{LocSys}^{\mathrm{res}}_{G^\vee}$




**Conjecture:** $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{res}}_{G^\vee})$.

**Theorem (Nadler–Yun):**

$\mathcal{F} \in Shv_{Nilp}(Bun_G) \Rightarrow$

$\mathcal{H}_V(\mathcal{F})$ is Hecke-lisse:

$\forall V$: $\mathrm{SingSupp}(\mathcal{H}_V(\mathcal{F})) \subset \underset{\substack{\cap \\ T^*Bun_G}}{Nilp} \times \underset{\substack{\cap \\ T^*X}}{\{0\}}$

**Th (reverse dir):**

$\mathcal{F} \in Shv(Bun_G)$,

$\forall V$: $\mathrm{SingSupp}(\mathcal{H}_V(\mathcal{F})) \subset T^*Bun_G \times \{0\}$

$\Downarrow$

$\mathcal{F} \in Shv_{Nilp}(Bun_G)$.

**Hecke functors:** $V \in Rep(\check{G})$:

$\mathcal{H}_V : Shv(Bun_G) \to Shv(Bun_G \times X)$.

## Corollaries:

- Hecke eigensheaves are in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$

- $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ $\circlearrowleft$ $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{res}}_{\check{G}})$ $\Longleftarrow$ Hecke action is lisse $\Downarrow$ Spectral decomposition Theorem

- $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is generated by images of Beilinson's spectral projectors $\Longleftarrow$ Construction of spectral projectors

Beilinson's spectral projectors $\Longrightarrow$ $\mathrm{D\text{-}mod}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ has reg. sing.

**Remark** (on Hecke-lisse property).

Two versions: $\mathcal{F} \in \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$

$\mathcal{H}_V(\mathcal{F}) \in \mathrm{Shv}_{Nilp \times \{0\}}(\mathrm{Bun}_G \times X)$ – geometric $\leftarrow \mathrm{Shv}_{\{0\}}(X)$

$\cap\ \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)$ – categorical

(Containment $\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}_{Nilp \times \{0\}}(\mathrm{Bun}_G \times X)$ is mostly formal, aside from completions and such).

In fact The two versions are equivalent. (important for spectral decomposition).

Key Lemma $X$ = projective, $\mathcal{Y}$ is arbitrary

$N \subset T^*\mathcal{Y}$ $\mathcal{F} \in Shv_{N \times \{0\}}(\mathcal{Y} \times X)$ – perverse (= constructible & in the heart)

Then $\mathcal{F}$ admits a filtration with $\mathcal{F}_i / \mathcal{F}_{i-1} \simeq \mathcal{F}^{(i)}_{\mathcal{Y}} \boxtimes \mathcal{F}^{(i)}_X$,

$\mathcal{F}^{(i)}_{\mathcal{Y}} \in Shv_N(\mathcal{Y})$, $\mathcal{F}^{(i)}_X \in Shv_{\{0\}}(X)$.

## Singular support

In case of $\ell$-adic sheaves in char p: Beilinson, Saito

An vanishing cycles.

**Key properties:**

$X$ = smooth.

$\mathcal{F} \in \mathrm{Shv}(X)$ (actual constructible, not ind-completed)

$\rightsquigarrow$

$\mathrm{SingSupp}(\mathcal{F}) \subset T^*X$.

conical, closed, $\dim = \frac{1}{2}\dim T^*X = \dim X$

No Lagrangian property!

For perverse objects (in the heart):

$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \Rightarrow$

$\mathrm{SingSupp}(\mathcal{F}_2) = \mathrm{SingSupp}(\mathcal{F}_1) \cup \mathrm{SingSupp}(\mathcal{F}_3)$.

# Functoriality:

$$f : X \to Y$$

$$\begin{tikzcd} & T^*Y \times X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ T^*X & & T^*Y \end{tikzcd}$$

$=: X \times_Y T^*Y$.

$\cup$ Sing Supp(F)

$\mathcal{F} \in Shv(X)$

1. Smooth pullback $f$ is smooth
   ($\varphi$ is closed embedding)

   $$\text{Sing Supp } f^*(\mathcal{F}) = \varphi \circ \psi^{-1}(\text{Sing Supp}(\mathcal{F}))$$

2. Proper pushforward $f$ is proper
   ($\psi$ is proper)

   $$\text{Sing Supp } f_*(\mathcal{F}) \subset \psi \circ \varphi^{-1}(\text{Sing Supp}(\mathcal{F})).$$

3. Compatibility with $\boxtimes$, $\mathbb{D}$

   $$\left(\text{Sing Supp } \mathbb{D}\mathcal{F} = \text{Sing Supp } \mathcal{F},\right.$$
   $$\left.\text{Sing Supp}(\mathcal{F} \boxtimes \mathcal{G}) = \text{Sing Supp } \mathcal{F} \times \text{Sing Supp } \mathcal{G}\right)$$

# Example of Nadler–Yun Th

$G = GL_2$ ; $V =$ standard rep.

$$F \in Shv_{Nilp}(Bun_G) \overset{?}{\Longrightarrow} H_V(F) \in Shv_{Nilp \times \{0\}}(Bun_G \times X).$$

length$(E/E') = 1$.

$(E \supset E' \mid E/E' \simeq \mathcal{O}_x)$ elementary modifications.

$\cap$

$$\begin{tikzcd} & \text{Hecke} \arrow[dl] \arrow[dr] & \\ F \quad Bun_G & & Bun_G \times X \quad H_V(F) \end{tikzcd}$$

$Bun_G \ni E$ rk 2 bundle on $X$

$Bun_G \times X \ni (E', x)$, $x = supp(E/E')$.

(all maps are smooth and proper)

$\Downarrow$

Sing Supp behaves well

In our case:

$$\left((E,A)\hookrightarrow(E',A'),\ E/_{E'}\simeq\mathcal{O}_x\right),$$

$$\begin{tikzcd}
 & \mathrm{HiggsHecke} \arrow[dl] \arrow[dr, squiggly] & \\
T^*\mathrm{Bun}_G & & T^*\mathrm{Bun}_G\times T^*X
\end{tikzcd}$$

$\xi = A|_{E/_{E'}}$

$T^*\mathrm{Bun}_G \ni (E,A)$ $\qquad$ $T^*\mathrm{Bun}_G\times T^*X \ni (E',A',x,\xi)$

$A: E\to E\otimes\Omega$ $\qquad$ $\xi\in T^*_xX.$

Nilp $\qquad$ $\{0\}.$

$$\begin{tikzcd}
 & & T^*_{Z_2}Z_1\times_{T^*Z_2}T^*_{Z_2}Z_3 \arrow[dl] \arrow[dr] \arrow[ddll, dashed, bend right] \arrow[ddrr, dashed, bend left] & & \\
 & T^*_{Z_2}Z_1 \arrow[dl] \arrow[dr] & & T^*_{Z_2}Z_3 \arrow[dl] \arrow[dr] & \\
T^*Z_1 & & T^*Z_2 & & T^*Z_3
\end{tikzcd}$$

$$\begin{tikzcd}
Z_1 & Z_2 \arrow[l, "\text{smooth proper}"'] \arrow[r, "\text{smooth proper}"'] & Z_3
\end{tikzcd}$$

Thm reduces to claim:

if $E \supset E'$      modification

compatible with $A: E \to E \otimes \Omega$

$A': E' \to E' \otimes \Omega$      For Iwahori direction: Sawin

and $A$ is nilpotent, then

1) $A'$ is nilpotent

2) $A|_{E/E'}: E/E' \to E/E' \otimes \Omega$ is zero.

The claim is obvious.

**Remark** In general need an extra step to show that computation of Sing Supp can be done orbit-by-orbit in Gr.

(for non-miniscule reps $V$)

Satake:

$$V \rightsquigarrow \mathrm{Sat}(V) \in \mathrm{Shv}(\mathrm{Gr})$$

## How about reverse direction?

For Higgs bundles, clear:

if $(E, A: E \to E \otimes \Omega)$

and $A$ is not nilpotent,

$\exists$ $x \in X$ and modification

$E' \hookrightarrow E$ s.t. $A(E') \subset E' \otimes \Omega$

and $A|_{E/E'} : E/E' \to E/E' \otimes \Omega$

is not zero.

$$\begin{tikzcd}
? \arrow[r, hook] \arrow[d] & T^*\mathrm{Hecke} \arrow[r] & ? \arrow[d] \\
T^*\mathrm{Bun}_G & & T^*\mathrm{Bun}_G \times T^*X
\end{tikzcd}$$

$(E, A) \dashrightarrow (E', A'), (x, \xi)$

$\mathrm{SingSupp}(\mathcal{F})$

But how is this enough?

char. cycle (in $\ell$-adic setting)

**Th** (Adapted from Saito's results):

Suppose $f : X \to Y$ (schematic).

$\mathcal{F} \in \mathrm{Shv}(X)$

$\mathrm{SingSupp}\, \mathcal{F} = N \subset T^*X$.

Consider

$(x, \xi) \in \boxed{\varphi^{-1}(N)} \subset T^*_Y X$.

$x \in X$

$\xi \in T^*_{f(x)} Y$

$$\begin{tikzcd} & \varphi^{-1}(N) \subset \begin{matrix} T^*Y \\ X \end{matrix} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ N \subset T^*X & & T^*Y \\ \mathrm{SingSupp}(\mathcal{F}) \arrow[u, phantom, "\cup"] & & \mathrm{SingSupp}(f_*\mathcal{F}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\mathcal{F}$ - - - - - - $f_*\mathcal{F}$

Suppose

1) $(x, \xi)$ is isolated on the fiber of $\psi$.

2) $\dim_{(x,\xi)} \varphi^{-1}(N) = \dim Y$.

(always? In char 0, 1) $\Rightarrow$ 2) for symplectic reasons)

Then $\psi(x, \xi) \in \mathrm{SingSupp}\,(f_*\mathcal{F})$.

Here's what we need:

$(E, A : E \to E \otimes \Omega)$ and $A$ is not nilpotent,

$\exists$ $x \in X$ and modification

$E' \hookrightarrow E$ s.t. $A(E') \subset E' \otimes \Omega$

and $\boxed{A|_{E/E'} : E/E' \to E/E' \otimes \Omega}$

is not zero.

if $\dim E/E' \geq 1$, should look $\mathrm{tr}(A|_{E/E'}) \neq 0$.

AND:

modification $(E', A') \hookrightarrow (E, A)$ is isolated ~~among~~

1) Upper modifications of $(E', A')$

2) Lower modifications of $(E, A)$.

at $x$

Lower modifications of $(E,A)$ at $x$ $\rightsquigarrow$ invariant subspaces of $A(x)$ on $E_x$.

Want to take an eigenspace, but don't know multiplicity in advance.

Take generalized eigenspace of $A(x)$ for non-zero eigenvalue

$A$ is not nilpotent.

$\exists x$ : $A(x)$ is not nilpotent

$\exists$ non-zero eigenvalue.

mult 1: elementary modification

mult 2: $E' = E(-x)$.

For $GL(n)$:

Linear algebra Lm

$A \in \mathfrak{gl}(n) \Rightarrow$ [generalized eigenspaces of $A$ are isolated in $Gr^A$.] conclusion.

In general:

Use isolated points on the affine Springer fiber.

Linear algebra model:

Suppose $V$, $\dim V = n$, $\mathrm{char}\, k > n$.

$A: V \to V$ is not nilpotent.

Then $\exists$ $W \subset V$ :

1) $A(W) \subset W$

2) $\mathrm{tr}(A|_W) \neq 0$

3) $W$ is isolated in $Gr(V)^A$

conclusion.