GLC with restricted variation:

Conjecture: X = curve $/k$, $\check{G}$ = reductive $/E$

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{rs}}(\check{G})).$$

One of 3 contexts:

Spaces $/k$, sheaves $/E$:

1) ℓ-adic: $E = \overline{\mathbb{Q}}_\ell$, $\ell \neq \mathrm{char}\, k$
2) de Rham: $E = k$, $\mathrm{char}\, k = 0$
3) Betti: $k = \mathbb{C}$

Automorphic

$Shv_{Nilp}(Bun_G)$:

ind-completed

derived category of sheaves on Bun_G w/ s.supp $\subset$ Nilp

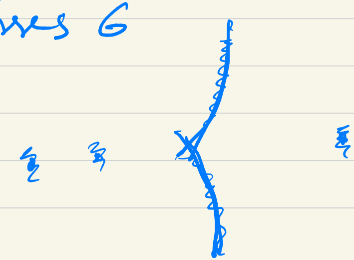
Automatically holonomic

Galois

$\underbrace{IndCoh_{Nilp}}_{\text{QCoh renormalized.}}(LocSys^{res}_{\check{G}})$

$LocSys^{res}_{\check{G}}$ = moduli stack of local systems with restricted variation:

$$\cong \bigsqcup_{\substack{\text{s-equivalence} \\ \text{classes } \sigma}} LocSys^{res,\sigma}_{\check{G}}$$



① Automorphic: "all interesting sheaves have nilpotent support"

Suppose $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$.

For any $V \in \mathrm{Rep}(G^\vee)$, we have $\mathcal{H}_V(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$

Hecke action.

By Satake

("universal" Hecke functor).

Example $\mathcal{F}$ = Hecke eigensheaf for eigenvalue $\mathcal{E} \in \mathrm{LocSys}$

$$\mathcal{H}_V(\mathcal{F}) \simeq \mathcal{F} \boxtimes V_{\mathcal{E}}.$$

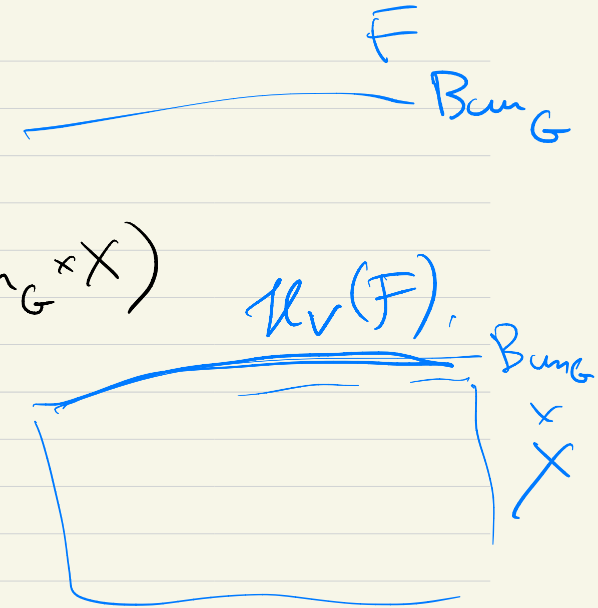
Theorem (Nadler–Yun):

If $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, then

$\forall V$: $\mathrm{SingSupp}(\mathcal{H}_V(\mathcal{F})) \subset T^*(\mathrm{Bun}_G \times X)$

$$\begin{array}{c} \mathrm{SingSupp}(\mathcal{H}_V(\mathcal{F})) \\ \cap \\ \mathrm{Nilp} \times \{0\} \subset T^*X. \end{array}$$

(i.e., $\mathcal{H}_V(\mathcal{F})$ is "lisse" along X)



Remark It is in fact true that $\mathcal{H}_V(\mathcal{F}) \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)$

"completed" ℓ-systems on X.

Theorem (converse claim)

Suppose $\mathcal{F} \in Shv(Bun_G)$ satisfies $\forall V \in Rep(G^\vee)$

$$\mathrm{SingSupp}\, \mathcal{H}_V(\mathcal{F}) \subset T^*Bun_G \times \{0\}$$ (Hecke-lisse)

Then $\mathcal{F} \in Shv_{Nilp}(Bun_G)$.

Corollary All Hecke eigensheaves are in $Shv_{Nilp}(Bun_G)$.

Proof of Thm: manipulations

with SingSupp $\quad T^*\mathrm{Bun}_G$ - Higgs bundles.

Technical issue: need estimates

on $\quad \mathrm{SingSupp}\, \mathcal{H}_V(\mathcal{F}) \quad$ (in terms of $\mathrm{SingSupp}(\mathcal{F})$)

from below

(while Nadler-Yun direction

needs estimates from above).

QED.

Side remark

The "usual" Geometric Langlands conjecture (de Rham)

$$\text{D-mod}(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check{G}})$$

must respect Hecke action (better: put together as action of $\underline{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})}$)

So (not surprisingly),

"usual" conjecture $\Longrightarrow$ restricted conjecture
de Rham

$\Longleftarrow$
essentially equivalent.

② Spectral decomposition (Galois): "Hecke-lisse things live over LocSys^{res}"

Reminder: C = category

Betti / de Rham: Action of $\mathrm{Rep}(\check{G})$ on C "by Hecke functors" $\rightsquigarrow$ Module structure $\mathrm{QCoh}(\mathrm{LocSys}) \curvearrowright C$

"integration"

(compatible with fusion)

(Hecke functors $\leftrightsquigarrow$ tautological bundles in $\mathrm{QCoh}(\mathrm{LocSys})$.

Rem Pack this action as

$\forall n$: act by Hecke functors at n points

+ compatibility with restriction to diagonals. $C^m \to C^n$

"Spectral decomposition" Th

Let C be a category together with a restricted action of $\mathrm{Rep}(G^\vee)/X$

- Nice enough

(lisse action?).

- $\forall I$, $|I|<\infty$, given

$$\mathrm{Rep}(G^\vee)^{\otimes I} \otimes C \to C \otimes \mathrm{QLisse}(X)^{\otimes I}$$

"C-valued l. systems on X^I".

- Compatible with $\otimes$

I.e., monoidal functor

$$\mathrm{Rep}(G^\vee)^{\otimes I} \to \mathrm{End}(C) \otimes \mathrm{QLisse}(X)^{\otimes I}$$

- Compatible with $I \to J$

Then C acquires an (encodes all Hecke functors

$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{res}}_{\check{G}}) \circlearrowright C$ via tautological bundles:

$\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{res}}(X)) \otimes \underline{\underline{}}$

$\mathrm{QLisse}(X)^{\otimes I}$)

Proof Technical, but main idea is simple: (and familiar)

If $\mathrm{LocSys}^{\mathrm{res}}_{\check{G}}$ were affine,

we would need to construct $k[\mathrm{LocSys}^{\mathrm{res}}_{\check{G}}]$

from input data (restricted action).

Technicalities:

$\mathrm{LocSys}_{G^\vee}^{\mathrm{res}}$ is not affine.

Instead, it is

$\bigsqcup$ formal affine schemes $/G^\vee$.

Why "Spectral decomposition"?

Re: $\mathrm{LocSys}^{\mathrm{res}}_{\check G} = \bigsqcup_{\substack{\text{semisimple} \\ \sigma}} \mathrm{LocSys}^{\mathrm{res},\sigma}_{\check G}$

$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{res}}_{\check G}) \simeq \bigoplus_{\sigma} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{res},\sigma}_{\check G})$.

Hence: Any C decomposes with restricted action of $\mathrm{Rep}(\check G)$: $\overset{\mathrm{Th}}{\Longleftrightarrow}$ action of $\mathrm{LocSys}^{\mathrm{res}}_{\check G}$

$$C = \bigoplus_{\sigma} C^{\sigma},$$

C^{σ} lives over $\mathrm{LocSys}^{\mathrm{res},\sigma}_{\check G}$

(E.g.: σ is irreducible $\Rightarrow C^{\sigma}$ is almost "eigenspace")

In particular applies to

$C = \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$!

Automorphic.

So $\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$

$$\simeq \bigoplus_{\sigma} \left(\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)\right)^{\sigma}$$

[Frankly: This is not immediate for technical reasons,] but still true

E.g. σ is irreducible:

Galois side:

$\mathrm{LocSys}^{\mathrm{res},\sigma}_{\check{G}}$ is formal;

$$\left(\mathrm{LocSys}^{\mathrm{res},\sigma}_{\check{G}}\right)^{\mathrm{red}} = \mathrm{pt}/\mathrm{Aut}.$$

$\mathrm{QCoh}\left(\mathrm{LocSys}^{\mathrm{res},\sigma}_{\check{G}}\right)$ is generated by $\mathrm{QCoh}\left(\left(\mathrm{LocSys}^{\mathrm{res},\sigma}_{\check{G}}\right)^{\mathrm{red}}\right)$

Automorphic side:

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\sigma} \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

generated by σ-eigen objects.

3 Projection from $\mathrm{Shv}(\mathrm{Bun}_G)$.

Construction of spectral projector (Beilinson).

Fix $F \in \mathrm{LocSys}_{G^\vee}$.

Define

$R_F : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G)$

s.t. 1) $R_F(-)$ is eigensheaf for F — Hecke

2) R_F is universal:

(left adjoint to $\left(\mathrm{Shv}(\mathrm{Bun}_G)\right)_{F\text{-eigen}} \to \mathrm{Shv}(\mathrm{Bun}_G)$) ("inclusion" of eigenspace)

not really.

Re: R_F defined explicitly via integrals of Hecke functors corresponding to regular rep.

Since $R_F(-)$ has eigenproperty, $\quad \bigoplus \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)^{\sigma}$

$R_F(-) \in \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$ $\;//\; \sigma$ (in fact: specific $\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)^{\sigma}$)

Moreover: $R_F(-)$ for varying F almost generate $\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$

Imagine There's no S-equivalence (no Eisenstein series and no reducible local systems, too).

Galois side:

$$\mathrm{LocSys}^{res} = \bigsqcup_{\sigma} \mathrm{LocSys}^{res,\sigma}$$

$$\mathrm{QCoh}(\mathrm{LocSys}^{res}) = \bigoplus \mathrm{QCoh}(\mathrm{LocSys}^{res,\sigma})$$

generated by

$$\bigoplus_{\sigma} \mathrm{QCoh}((\mathrm{LocSys}^{res,\sigma})^{red})$$

. . . .

(spectral decomp.)

Automorphic side

$$\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$$

$$\rotatebox{90}{=}$$

$$\bigoplus \underbrace{\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)^{\sigma}}_{\text{Generalized eigenspaces}}$$

generated by

$$\bigoplus \underbrace{\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)_{\sigma\text{-eigen}}}_{\text{eigenspaces.}}$$

$$\begin{array}{ccc}
\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{res}}) & \text{Spectral decomp.} \circlearrowright & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \\
\uparrow \text{generate} & & \uparrow \text{generate} \\
\bigoplus_\sigma \mathrm{QCoh}\left((\mathrm{LocSys}^{\mathrm{res},\sigma})^{\mathrm{red}}\right) & & \bigoplus_\sigma \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\sigma\text{-eigen}} \\
 & & \uparrow \text{Beilinson } R_\sigma \\
 & & \mathrm{Shv}(\mathrm{Bun}_G)
\end{array}$$

Correction. R_S

Because they are left-adjoint → generate

In fact: This works, even with S-equivalence:

Correction: $(\mathrm{LocSys}^{\mathrm{res},\sigma})^{\mathrm{red}}$ is not discrete.

Replace $F \in \mathrm{LocSys}^{\mathrm{res}}$ with

$S \to \mathrm{LocSys}^{\mathrm{res}}$

scheme

red.

$(\mathrm{LocSys}^{\mathrm{res},\sigma})$

Corollaries:

(1) $\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$ is compactly generated by objects

$$\left(\mathcal{R}_S(\delta_P),\ \ S \to \mathrm{LocSys}^{res},\ P \in \mathrm{Bun}_G \right)$$

"nice"

(2) In de Rham setting,

$\text{D-mod}_{Nilp}(\mathrm{Bun}_G)$ has regular singularities

(Because generators do).

Remark: Restricted correspondence $(/\mathbb{C})$.

$$
\begin{tikzcd}
\text{Betti} & \mathrm{Shv}^{\mathrm{Betti}}_{Nilp}(\mathrm{Bun}_G) \arrow[r, "\simeq"] \arrow[d, "\wr"', "\text{R-H}"] & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}^{\mathrm{res}}_{\mathrm{Betti}}) \arrow[dd, "\wr"', "\text{R-H}"] \\
 & \text{D-mod}^{\mathrm{Reg}}_{Nilp}(\mathrm{Bun}_G) \arrow[d, equal] & \\
\text{de Rham} & \text{D-mod}_{Nilp}(\mathrm{Bun}_G) \arrow[r, "\simeq"] & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}^{\mathrm{res}}_{dR})
\end{tikzcd}
$$

Rem (Last time, Dennis):

In Betti context, it should be easier to show that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is a top. invariant of X

(then for ∞-dim constructible objects, as in Ben Zvi–Nadler)

Outline (conjectural):

Use $R_S(\delta)$ as generators in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, Hom's between them are locally constant

X varies

Requires: using restricted theory for variable X (and f.-dim!).