Geometric Langlands conjecture with restricted variation.

Summary:

- New "flavor" of geometric Langlands — "universal"
- Structure results for automorphic sheaves — "spectral decompos."
- Categorical statement in ℓ-adic world, applies to "classical" statement. — "trace of Fr"

arxiv: 2010.01906

—, Gaitsgory, Kazhdan, Raskin, Rozenblyum, Varshavsky.

"Flavors" of Geometric Langlands Conjecture

X = curve, G = reductive group

Automorphic side | Galois side

sheaf on Bun_G ⇝ $G^\vee$ local system on X

"automorphic sheaves"

Hecke eigensheaves

$\mathrm{Bun}_G = \mathrm{Bun}_G(X)$ = stack of G-bundles on X

$\mathrm{LocSys}_{G^\vee}$ = stack of $G^\vee$ local systems.

"Fourier transform"

$$\{\text{Sheaves on } \mathrm{Bun}_G\} \underset{\text{equiv. of cat.}}{\simeq} \{\simeq \text{Coherent sheaves on } \mathrm{LocSys}_{G^\vee}\}.$$

ℓ-adic flavor:

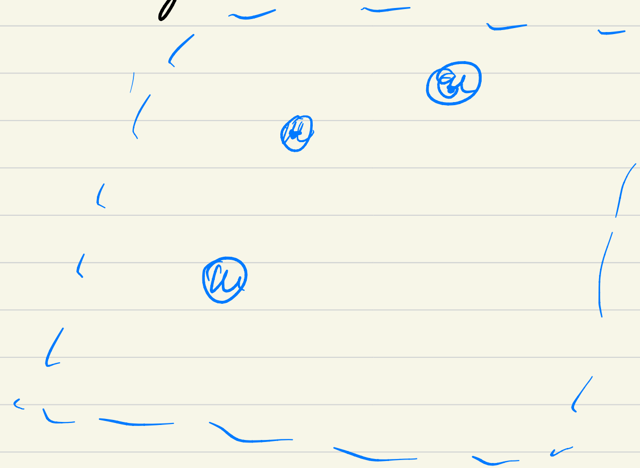
X/k ; Sheaves = ℓ-adic sheaves, $\ell \neq \mathrm{char}\, k$.

$k = \mathbb{F}_q$

$\mathrm{Shv}(\mathrm{Bun}_G)$.

$\mathrm{LocSys}_{\check{G}}^{\ell\text{-adic}}$ on X does not exist as a stack (only formally).

No categorical statement.



$\mathrm{LocSys}_{\check{G}}$ DNE

de Rham flavor X/k

char $k=0$, Sheaves = D-modules

GLC (de Rham) Conjecture

$$\text{D-mod}(\mathrm{Bun}_G) \simeq \cancel{\mathrm{QCoh}}(\mathrm{LocSys}^{dR}_{\check{G}})$$

IndCoh_{Nilp}.

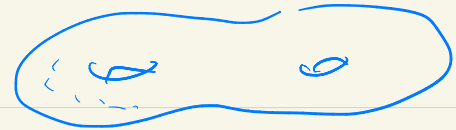
bundles + connection.

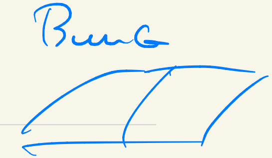
"

$\mathrm{LocSys}^{dR}_{\check{G}}$ is a (q.smooth) algebraic stack.

Rem: All categories are derived.

Betti flavor $X/\mathbb{C} = k$.





Sheaves = Sheaves (of ∞-dim spaces) in classical topology

$$\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X) = \mathrm{Hom}(\pi_1(X), \check{G}) / \check{G}$$

GLC (Betti) Conjecture (Ben Zvi-Nadler).

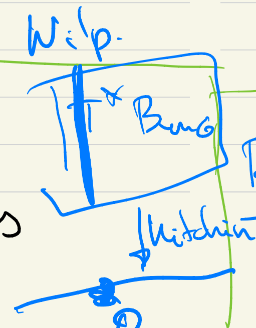
$$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}})$$

Lagrangian / symplectic

$$\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$$

Nilp = Nilpotent Higgs bundles.

$T^*\mathrm{Bun}_G$ = Higgs bundles



Rem $\mathrm{LocSys}_{\check{G}}(X)$ has dg structure if $X = \mathbb{P}^1$ (or G = reductive)

GLC (Betti) Conjecture (Ben Zvi - Nadler).

$$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}})$$

Remark RHS depends on topology of X only.
What about LHS?
(Ben Zvi - Nadler: "far from obvious from the definition").

Example: $G = GL(1) = G^\vee$ $\quad$ $\mathrm{Bun}_G = \mathrm{Pic}\,X$ - abelian variety. (Class FT) (CFT)

Correspondence

$$\left\{\begin{array}{c}\text{rk 1 local systems} \\ \text{on } \mathrm{Pic}\,X\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{rk 1 local systems} \\ \text{on } X\end{array}\right\}.$$

ℓ-adic $\quad \mathrm{Shv}^{\mathrm{constr}}(\mathrm{Pic}\,X)$ $\qquad$ $\mathrm{LocSys}_{G^\vee}$ is formal

de Rham $\quad \mathrm{D\text{-}mod}(\mathrm{Pic}\,X) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{dR}_{GL(1)}X)$ ext. of Jacobian by v. space.

Riemann-Hilbert (not alg.).

Betti $\quad \mathrm{Shv}^{\mathrm{Betti}}_0(\mathrm{Pic}\,X) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{GL(1)}X)$ $\simeq \mathbb{G}_m^{2g}$

∞-dim local systems.

Automorphic side $(k = \mathbb{C})$.

de Rham

$$
\begin{tikzcd}
\mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\
\mathrm{D\text{-}mod}^{\mathrm{hol}}(\mathrm{Bun}_G) \arrow[u, phantom, "\cup"] \arrow[d, "\text{R.-H.}", "\wr"'] & \mathrm{D\text{-}mod}^{\mathrm{hol}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[l, phantom, "\supset"] \arrow[d, "\text{R.H.}", "\wr"'] \\
\mathrm{Shv}^{\mathrm{constr}}(\mathrm{Bun}_G) & \mathrm{Shv}^{\mathrm{const}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[l, phantom, "\supset"] \\
& \mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[u, phantom, "\cap"]
\end{tikzcd}
$$

ℓ-adic

Betti

Automorphic side of the conjecture.

Remarks: 1. Betti = "∞-constr"

2. Ignoring irregular D-mods/Stokes data

Restricted version of GLC

Automorphic side:

$$\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) = \text{'}$$

Sheaves (ℓ-adic / classical / D-modules) on Bun_G
Betti Holonomic.

with sing supp $\subset$ Nilp.

Remarks: 1. Not really a "flavor" - exists in different flavors.
(but different flavors agree where they all make sense)
More like gcd of flavors.

2. Automorphic side is constructible / holonomic

$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Remarks 3. Obviously borrowed from Betti flavor (Ben Zvi - Nadler).

4. Singular support makes sense in ℓ-adic setting: thanks to Beilinson, Saito.

But: What is the Galois side?

Local systems with restricted variation

Definition (Betti flavor): R = ring. A family of reps $\rho : \pi_1(X,x) \to GL(n,R)$ has <u>restricted</u> <u>variation</u> if for any $\gamma \in \pi_1(X,x)$,

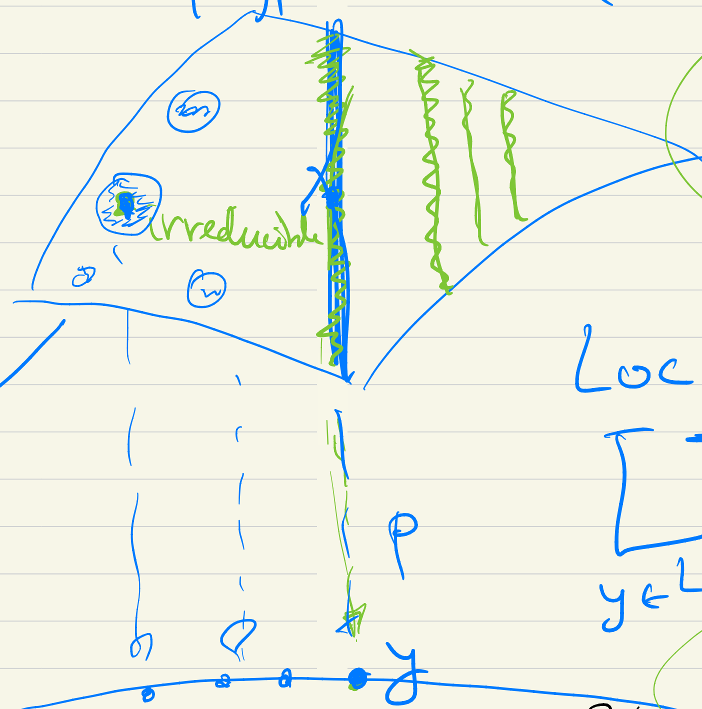
$\mathrm{tr}(\rho(\gamma)) \in R$ is locally constant on $(\mathrm{Spec}\, R)^{red}$

$\mathrm{LocSys}^{res}(X)$ = moduli of loc. systems with restricted var.

$\Leftrightarrow$ finiteness condition on

$$\rho(\gamma) : R^n \to R^n$$

S-equivalence class
(same semi-simplification)



$\mathrm{LocSys}^{\mathrm{Betti}}(X) = \mathcal{M}$

$\forall \gamma \in \pi_1(X,x)$

$\mathrm{tr}(\rho(\gamma))$

k

$\mathrm{LocSys}^{\mathrm{Betti,res}}(X) = \widehat{\mathcal{M}}_{p^{-1}(y)}$

$[\ ]$

$y \in \mathrm{LocSys}^{\mathrm{Betti,coarse}}$

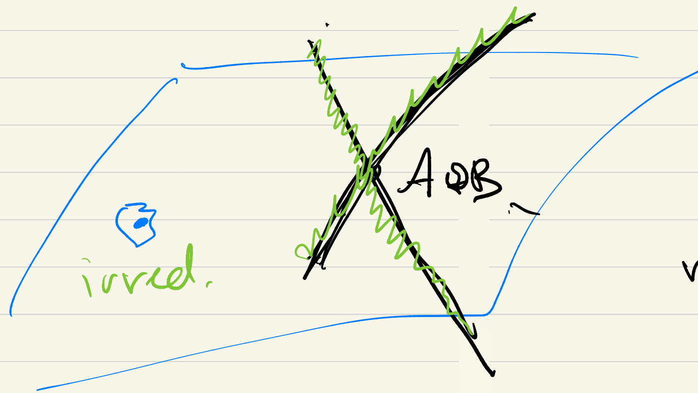
$\mathrm{LocSys}^{\mathrm{Betti,coarse}}(X)$

"$\mathrm{Spec}\,\Gamma(\mathrm{LocSys}^{\mathrm{Betti}}(X), \mathcal{O}_X)$"

generated by

$G = GL(2)$:

$$0 \to A \to E \to B \to 0$$



LocSys

$\mathrm{rk}\, A = \mathrm{rk}\, B = 1$

$$0 \to B \to E \to A \to 0$$

In general Consider any of our flavors:

X/k, coeffs $= E$, $\mathrm{char}\, E = 0$

$\mathrm{Shv}(X) =$ Ind-completion of $\mathrm{Shv}^{\mathrm{constr}}(X)$

Betti $k = \mathbb{C}$, any E $\mathrm{Shv}^{\mathrm{constr}}$ = constructible sheaves

de Rham $k = E$, $\mathrm{char}\, k = 0$ Shv^{dR} = holonomic D-mods

ℓ-adic $E/\mathbb{Q}_\ell$, $\ell \neq \mathrm{char}\, k$ $\mathrm{Shv}^{\ell\text{-adic}}$ = ℓ-adic constructible sheaves.

(actually, work with ind-completion).

Qlisse = (ind-completed) local systems

(cheating slightly).

Tannakian def:

R-family of $\check{G}$-l.sys. with $\quad \operatorname{Spec} R \to \operatorname{LocSys}^{res}$

res. variation:

a (right exact) monoidal

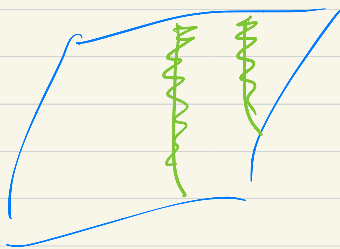
functor $\operatorname{Rep}(\check{G}) \to R\text{-mod}(\operatorname{QLisse}(X))$

Properties

$$\mathrm{LocSys}^{res} = \coprod_{\sigma \text{ is semi-simple}} \mathrm{LocSys}^{res,\sigma}$$

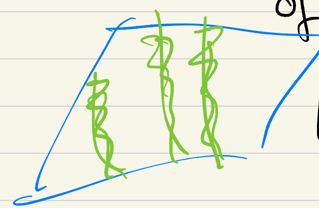
(semisimple = irreducible M-loc. system for Levi M).

Each $\mathrm{LocSys}^{res,\sigma}$ is a formal stack. ($\simeq$ formal completion of alg stack).



Betti

$$\mathrm{LocSys} \supset \mathrm{LocSys}^{res,}$$



de Rham

$$\begin{array}{c}\mathrm{LocSys} \\ \downarrow \\ \mathrm{LocSys}^{coarse}.\end{array}$$

R-H correspondence

③ GLC with restricted variation:

Conjecture:

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{rs}}(G)).$$