Hello

X - smooth, Z - target scheme

$\mathrm{Maps}^{gen}(X, Z)$

$$\mathrm{Hom}(S, \mathrm{Maps}^{gen}(X,Z))$$
$$\shortparallel$$
$$\mathrm{Hom}(U, Z)$$

$U \subseteq S \times X$
s.t. $\forall\, s \in S$ $\quad U \cap (s \times X) \neq 0$

$U_1 \xrightarrow{\varphi_1} Z$
are identified if
$U_2 \xrightarrow{\varphi_2} Z$

$$\varphi_1 |_{U_1 \cap U_2} = \varphi_2 |_{U_1 \cap U_2}$$

Example $Z = \mathbb{A}^1$

$$\mathrm{Hom}(pt, \mathrm{Maps}^{sm}(X, \mathbb{A}^1)) \simeq k(X)$$

1) $x \in X$

$$\mathrm{Maps}(X - x, \mathbb{A}^1)$$

$$\mathrm{Hom}(S, \mathrm{Maps}(X-x, \mathbb{A}^1))$$

$$\shortparallel$$

$$\mathrm{Hom}(S \times (X-x), \mathbb{A}^1)$$

$$\mathrel{\rotatebox{90}{$\simeq$}}$$

$$\mathcal{O}_S \otimes \Gamma(X-x, \mathcal{O})$$

V - (potentially inf. dim.)

$$\mathrm{Tot}(V)$$

$$\mathrm{Hom}(S, \mathrm{Tot}(V)) = \mathcal{O}_S \otimes V$$

$$\mathrm{Maps}(X - x, \mathbb{A}^1) = \mathrm{Tot}(\Gamma(X - x, \mathcal{O}))$$

$$\mathrm{Maps}^{sch}(X, \mathbb{A}^1) \neq \mathrm{Tot}(k(X))$$

$\uparrow$

<u>Exe</u>

<u>Observation</u> $U \xhookrightarrow[\text{open}]{} Z$

$$\mathrm{Maps}^{sch}(X, U) \xhookrightarrow[\text{open subfunctor}]{} \mathrm{Maps}^{sch}(X, Z)$$

$$V \xhookrightarrow[\text{closed}]{} Z$$

$$\mathrm{Maps}^{sch}(X, V) \xhookrightarrow[\text{closed}]{} \mathrm{Maps}^{sch}(X, Z)$$

$$\mathcal{Y} \longrightarrow C_\cdot(\mathcal{Y})$$

$$\mathrm{Sch}^{aff} \xrightarrow{C_\cdot(-)} \mathrm{Vect}$$

$$\begin{array}{c} \downarrow \\ \mathrm{PreStk} \end{array} \cdots \uparrow \text{left Kan extension}$$

I.e.,

$$C_\cdot(\mathcal{Y}) = \operatorname*{colim}_{S \to \mathcal{Y}} C_\cdot(S)$$

Tautologically $\mathcal{Y} \simeq \operatorname*{colim}_i \mathcal{Y}_i$

$$\operatorname*{colim}_i C_\cdot(\mathcal{Y}_i) \xrightarrow{\sim} C_\cdot(\mathcal{Y})$$

$$\mathrm{Maps}^{\mathrm{gen}}_s(\mathbb{P}^1, \mathbb{P}^1)_{\leq 1} =$$

$$\left\{ z \longrightarrow \frac{az+b}{cz+d} \right\} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$$

$$\begin{array}{c} \mathrm{Mat}(2\times 2)^{-0}/\mathbb{G}_m \\ \cong \\ \mathbb{P}^3 \end{array}$$

$$\begin{array}{ccc} \mathbb{P}^3 & \longrightarrow & \mathrm{Maps}^{\mathrm{gen}}_s(\mathbb{P}^1, \mathbb{P}^1)_{\leq p} \\ \cup & & \cup \end{array}$$

$$\mathbb{P}^3_{deg} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

$$\text{"} \qquad\qquad\qquad\qquad \text{"}$$

$$\{ad = bc\} \qquad\qquad \mathrm{Maps}_1^{sm}(\mathbb{P}^1, \mathbb{P}^1)_0$$

$$\mathrm{Maps}_3^{sm}(\mathbb{P}^1, \mathbb{P}^1)_{\deg 1} = \mathbb{P}^3 \underset{\mathbb{P}^1 \times \mathbb{P}^1}{\sqcup} \mathbb{P}^1$$

pushout

<u>Thm</u> $\quad Z \underset{\text{open}}{\subseteq} \mathbb{A}^n$

$$C_\cdot(\mathrm{Maps}^{sm}_{gs}(X, Z)) \xrightarrow{\sim} C_\cdot(pt)$$

Ran

$\mathrm{Hom}(S, \mathrm{Ran})$ = set of finite non-empty subsets of $\mathrm{Hom}(S, X)$

$$\mathrm{Ran} \simeq \operatorname*{colim}_{I \in (\mathrm{fSet}_{\mathrm{surj}})^{op}} X^I$$

Z-affine

$$\mathrm{Maps}^{\mathrm{Ran}}(X, Z)$$

$$\|$$

$$\left\{ \underset{\mathrm{Ran}}{\underset{\cap}{\underline{x}}} ;\ \varphi : X - \underline{x} \longrightarrow Z \right\}$$

$$\begin{tikzcd} \mathrm{Maps}^{\mathrm{Ran}}(X, Z) \arrow[d] \\ \mathrm{Maps}^{\mathrm{gen}}(X, Z) \end{tikzcd}$$

<u>Exe</u> (1) This map induces an iso on $C_\cdot(-)$.

(2) This map is a universal

homological equivalence.

$$\begin{tikzcd} \mathrm{Maps}^{\mathrm{Ran}}(X, Z) \arrow[d] \\ \mathrm{Ran} \end{tikzcd}$$

← relative ind-scheme

$Z = \mathbb{P}^n$

$$\begin{tikzcd} \mathrm{QMaps}(X, \mathbb{P}^n) \arrow[r] \arrow[d] & \mathrm{Maps}^{\mathrm{Ran}}(X, \mathbb{P}^n) \\ \mathrm{QMaps}^{\mathrm{Ran}}(X, \mathbb{P}^n) \arrow[ur] & \end{tikzcd}$$

$$\mathrm{QMaps}^{\mathrm{Ran}}(X, \mathbb{P}^n) = \left\{ \underset{\mathrm{Ran}}{\frac{\underline{x}}{\cap}} \; ; \; \underset{\text{line bundle}}{\mathcal{L}} \; : \; \mathcal{L}|_{X - \underline{x}} \longrightarrow \mathcal{O}^{n+1}|_{X - \underline{x}} \right\}$$

Launch the proof

<u>Reduction 1.</u> $Z = \bigcup_i Z_i$ s.t. the contractibility statement is true for $Z_{i_1} \cap \dots \cap Z_{i_n}$. Then the contractibility statement is true for Z.

$$\mathrm{Maps}^{ka}(X, Z) \overset{\text{Zar. des.}}{=} \operatorname{colim} \mathrm{Maps}^{ka}(X, Z_{i_1} \cap \dots \cap Z_{i_n})$$

This allows us to reduce the statement to the case when Z is

$$\begin{array}{c} Z \\ \shortparallel \\ \mathbb{A}^n - V(f) \end{array}$$

$f \in k[x_1 \dots x_n]$

<u>Reduction 2</u>

$$\begin{tikzcd} & Z_1 \arrow[dl] \\ X_1 \arrow[d, "J"] & \\ X_2 & \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[d] \\ X \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Res}_{\pi}(Z_1) \arrow[d] \\ X_2 \end{tikzcd}$$

$$\mathrm{Sect}^{gen}(X_1, Z_1)$$
$$\wr\!\wr$$
$$\mathrm{Sect}^{gen}(X_2, \mathrm{Res}_{\pi}(Z_2))$$

$$Z_1 = \mathbb{A}^n \times X_1$$

$$\begin{tikzcd} \mathrm{Res}_{\pi} \arrow[d] \\ X_2 \end{tikzcd} \quad \text{generically} \simeq \mathbb{A}^{n \cdot \deg(\pi)}$$

$$\mathrm{Maps}^{gen}(X_1, U_1) \simeq \mathrm{Maps}^{gen}(X_2, U_2)$$

$$U_1 \subseteq \mathbb{A}^n \qquad\qquad U_2 \subseteq \mathbb{A}^{n \cdot \deg \pi}$$

This reduces to the case of
$X = \mathbb{P}^1$

$$Z = \mathbb{A}^n - V(f)$$

$$\mathrm{Maps}^{\mathrm{gen}}(\mathbb{A}^1, \mathbb{A}^n) \ni \underline{u} = (u_1 \dots u_n)$$
$$\cup\!\!\!\mid$$
$$\mathrm{Maps}^{\mathrm{gen}}(\mathbb{A}^1, \mathbb{A}^n)_{\leq d}$$

Find $\tilde{u}$ - polynomial function.

$$\tilde{u}\cdot\underline{u} = (\tilde{u}\cdot u_1, \dots, \tilde{u}\cdot u_n)$$

$\forall i \quad \deg \tilde{u}\cdot u_i \leq d + \deg \tilde{u}$

$$\operatorname*{colim}_{d} \mathrm{Maps}^{\mathrm{gen}}(\mathbb{A}^1, \mathbb{A}^n)_{\leq d} \xrightarrow{\sim} \mathrm{Maps}^{\mathrm{gen.}}(\mathbb{A}^1, \mathbb{A}^n)$$

$$\begin{array}{c} \operatorname*{colim}_{d} \mathrm{Maps}^{\mathrm{gen}}(\mathbb{A}^1, \mathbb{A}^n)_{\leq d} \cap \mathrm{Maps}^{\mathrm{gen}}(\mathbb{A}^1, Z) \\ \downarrow \wr \\ \mathrm{Maps}^{\mathrm{gen}}(\mathbb{A}^1, Z) \end{array}$$

Theorem $\forall A$

$$\begin{array}{c} \mathrm{Maps}^{\mathrm{gr}}(\mathbb{A}^1, \mathbb{A}^n)_{\le d} \cap \mathrm{Maps}^{\mathrm{gr}}(\mathbb{A}^1, Z) \\ \downarrow \\ \mathrm{Maps}^{\mathrm{gr}}(\mathbb{A}^1, Z) \end{array}$$

is $\mathbb{A}^1$ - homotopic to the constant map

Proof of contractibility

$$\begin{array}{c} \operatorname{colim}_d H_k(\mathrm{Maps}^{\mathrm{gr}}(\mathbb{A}^1, \mathbb{A}^n)_{\le d} \cap \mathrm{Maps}^{\mathrm{gr}}(\mathbb{A}^1, Z)) \\ \downarrow \wr \\ H_k(\mathrm{Maps}^{\mathrm{gr}}(\mathbb{A}^1, Z)) \end{array}$$

However, Theorem $\Rightarrow \forall k > 0 \ \forall d$ the map

$$H_k(\mathrm{Maps}^{\mathrm{gr}}(\mathbb{A}^1, \mathbb{A}^n)_{\le d} \cap \mathrm{Maps}^{\mathrm{gr}}(\mathbb{A}^1, Z))$$
$$\big|$$

$H_{\ast}(\mathbb{A}^1, \mathbb{Z})^{\perp}$ is 0.

QED.

$$f = x_1^m + \sum_{m'<m} x_1^{m'} \cdot \tilde{f}_{m'}(x_2, \dots x_n)$$

$$d' \geq \left(\max_{m'<m} \deg(\tilde{f}_{m'})\right) \cdot d$$

Observation: $(u_1, \dots, u_n) \in \mathrm{Maps}^{\mathrm{alg}}(\mathbb{A}^1, V(f))$

Then if $\deg u_i \leq d$ $\Rightarrow$ $\deg u_1 \leq d'$
$2 \leq i$

Choose w a polynomial of degree $\geq \max(d, d')$

$$(u_1 \ldots u_n) \longrightarrow (1-t)(u_1 \ldots u_n) + t(w, 0, \ldots, 0).$$

$t \in \mathbb{A}^1$

Need to show that if

$(u_1 \ldots u_n) \in \mathbb{A}^n - V(f) \Longrightarrow$

$\forall t$

$$\underbrace{(1-t)(u_1 \ldots u_n) + t(w, 0 \ldots 0)}_{(\tilde{u}_1, \ldots, \tilde{u}_n)} \in \mathbb{A}^n - V(f).$$

<u>Case 1</u> $t = 0$ okay

<u>Case 2</u> $t \neq 0$

By observation if

$f(\tilde{u}_1, \ldots \tilde{u}_n) = 0 \Longrightarrow$

$\deg(\tilde{u}_1) \leq d'$

$\tilde{u}_1 = (1-t)u_1 + w$

$\deg \tilde{u}_1 = \deg w > d'$ ■

V – inf. dim. space over $\mathbb{R}$

$$V - 0 \longrightarrow V$$

$V' \subseteq V$
fin. dim.

$$\begin{tikzcd}
V' - 0 \arrow[r] \arrow[d, "\text{null-homotopic}"'] & V - 0 \\
V'' - 0 \arrow[ur, no head] &
\end{tikzcd}$$