Multiplication kernels

Talk by Maxim Kontsevich
Geometric Langlands
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Maxim's joint work with A. Odesski
(in preparation)
based on discussions with
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Let C be a cyclic group, $|C|=N$.
Consider the space
V of functions $C \to \mathbb{C}$.
On this space we have
operators $(T_a f)(x) = f(x+a)$,
$a \in C$.
These operators have eigenfunctions
$\psi_\lambda(x) = e^{\frac{2\pi i(\lambda, x)}{N}}$ such that
$\psi_\lambda(0) = 1$ and $(\lambda \in C^\vee = \mathrm{Hom}(C, \mathbb{Z}/N))$

$$T_a \psi_\lambda = e^{\frac{2\pi i(\lambda, a)}{N}} \psi_n$$ (to avoid confusion we should not identify C with $C^\vee$!)

We can form the addition kernel:

$$K(x,y,z) = \frac{1}{N} \sum_{\lambda \in \mathbb{Z}/N\mathbb{Z}} \psi_\lambda(x)\psi_\lambda(y)\psi_\lambda(z)$$

$$= \delta(x+y+z),$$

symmetric in x, y, z.

It is called this way because the function $K(x,y,-z)$ is the delta-function of the graph of addition, $z = x+y$. We have

$$\psi_\lambda(x)\psi_\lambda(y) = \frac{1}{N} \sum_z K(x,y,-z)\, \psi_\lambda(z).$$

This can be thought of as a commutative algebra structure defined by structure constants $K(x,y,-z)$, and this algebra A acts on the space W of functions $C^\vee \to \mathbb{C}$

by $(\psi(x) f)(\lambda) = \psi_\lambda(x) f(\lambda)$, so that

$$\psi(x)\psi(y) = \frac{1}{N}\sum_z K(x,y,-z)\,\psi(z).$$

Moreover, we have the trace $\mathrm{Tr}: A \to \mathbb{C}$ (the trace in the representation W divided by N)

$$\mathrm{Tr}(\psi(x)) = \sum_\lambda \psi_\lambda(x) = \delta_{x,0},$$ and

$$\mathrm{Tr}(\psi(x)\psi(y)) = \delta_{x,y},$$

$$\mathrm{Tr}(\psi(x)\psi(y)\psi(z)) = K(x,y,z).$$

Thus A is a Frobenius algebra. We can also form m-point functions

$$K_m(x_1, \dots, x_m) = \mathrm{Tr}(\psi(x_1)\cdots\psi(x_m)).$$

We have a natural isomorphism $V \xrightarrow{\sim} A$, $f \mapsto \sum_x f(x)\psi(x)$, and the module W has cyclic vector $1 \in W$, so that the map $\alpha \mapsto \alpha \cdot 1$ gives an isomorphism $A \xrightarrow{\sim} W$. The composite $V \to A \to W$ is the Fourier transform.

This story extends straightforwardly to the infinite setting, when the group C is replaced with the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. In this case, $V = L^2(S^1)$, $T_a = \exp(iaL)$ where $L = -i\frac{d}{dx}$, $C^\vee = \mathbb{Z}$, eigenfunctions $\psi_\lambda(x) = e^{2\pi i\lambda x}$, $L\psi_\lambda = \lambda\psi_\lambda$; $W = \ell_2(\mathbb{Z})$,

$$\psi_\lambda(x)\psi_\lambda(y) = \frac{1}{2\pi}\int K(x,y,-z)\,\psi_\lambda(z)\,dz$$

where $K(x,y,z) = \delta(x+y+z)$, so except for analytic details the story is the same. It also extends to the case

$C = \mathbb{R}$, $C^\vee = \mathbb{R}$. In fact we can take for C any locally compact abelian group, and this is just the story of Pontryagin duality.

Of course this example is pretty trivial, but there are more interesting ones, where addition is "non-deterministic", so the kernel $K(x,y,z)$ actually has full support. One of the simplest examples is the one of Sonine-Gegenbauer formula.

Namely, consider the operator

$$L = -\partial^2 - \frac{1}{x}\partial.$$

The solution of the equation $L\psi = \lambda^2\psi$ such that $\psi_\lambda(0)=1$ has the form

$$\psi_\lambda(x) = J_0(\lambda x),$$

where

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!^2}\left(\frac{x}{2}\right)^{2m} = \frac{1}{2\pi i}\oint e^{x\left(u-\frac{1}{u}\right)}\frac{du}{u}.$$

Theorem (The Sonine-Gegenbauer formula)

$$\int_0^\infty J_0(\lambda x) J_0(\lambda y) J_0(\lambda z)\, \lambda d\lambda = K(x,y,z)$$

$$x, y, z \geq 0$$

where

$$K(x,y,z)=\frac{1}{2\pi\Delta(x,y,z)}$$

where

$$\Delta(x,y,z)=\sqrt{(x+y+z)(x+y-z)(x+z-y)(y+z-x)}$$

is the area of a triangle with sides x,y,z.

Here if there is no such triangle, the expression under the square root is <0, and we should put

$K(x,y,z)=0$.

The role of Fourier transform here is played by the Fourier-Bessel transform.

In fact, this story can be deformed: we can consider the operator

$$L_\alpha = -\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\alpha^2}{x^2}\right)$$

Then the solution of $L_\alpha \psi = \lambda^2 \psi$ with $\psi(0)=1$ is

$$\psi_\lambda(x) = J_\alpha(\lambda x).$$

Note that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x),$$

so for $\alpha = 1/2$ this specializes to the previous Fourier story (restricted to even functions).

Note: the minus in front of z in $K(x,y,-z)$ disappeared because we now work with even functions and modded out the symmetry $z \to -z$

This could be further deformed as follows.

Consider the operator

$$L = \partial\, x(x-1)(x-t)\, \partial + x, \quad 1<t<\infty$$

and consider solutions $\psi_\lambda(x)$ of $L\psi = \lambda\psi$ with $\psi_\lambda(0)=1$ which solve a suitable Sturm-Liouville problem.

There is a discrete spectrum, and

we have

$$\sum_{\lambda} \psi_\lambda(x)\psi_\lambda(y)\psi_\lambda(z) = K(x,y,z)$$

where

$$K(x,y,z) = \frac{1}{\sqrt{f_t(x,y,z)}}$$

$$f_t(x,y,z) = (xy+xz+yz-t)^2 + 4xyz(1+t-x-y-z).$$

As before, if this is <u>negative</u>, we are supposed to put

$K(x,y,z)=0$.

Thus we have commuting integral operators H_x given by

$$(H_x f)(y)=\int K(x,y,z)f(z)\,dz,$$

and $H_x\psi_\lambda=\psi_\lambda(x)\,\psi_\lambda$.

and $[H_x,H_y]=0$, $H_xH_y=\int K(x,y,z)H_z\,dz$.

The important thing is that since the kernel $K(x,y,z)$ is given by an algebraic formula, the operators H_x can be defined over

any local field F,
as well as functions
φ_λ (although the differential operator L
will not be defined
when the field is
non-archimedian.
One can also do this
over a finite field.

Uneducated comment from pasha:
The subtlety in the finite
field case is that we
cannot restrict ourselves
to the "set of full measure",
ignoring singularities,
at which we'll have correction terms, but

one expects that the story over $\mathbb{F}_p$ should be "level 1" of the story over $\mathbb{Q}_p$. One may also be able to define a geometric version of $K(x,y,z)$ in terms of ℓ-adic sheaves which will take care of this problem. In any case, the "level 1" idea works for the above example of H_x.

Note: the example of H_x corresponds to Langlands for $G = PGL_2$, $X = \mathbb{P}^1 \setminus (0,1,t,\infty)$.

This can in fact be discussed in the general setting of symmetric monoidal categories.

For motivation, consider a finite dimensional vector space V.

Suppose we have a commutative family of linear operators on V parametrized by a space $\mathcal{U}$. Thus we have a linear map $\phi: \mathcal{U} \otimes V \to V$ such that the map

$$\phi(1\otimes\phi): \mathcal{U}\otimes\mathcal{U}\otimes V \to V$$

is Σ_2-invariant, i.e. gives rise to a map

$\mathrm{Sym}\,\mathfrak{U} \otimes V \to V.$

Assume in addition that these operators act on V with simple spectrum.

So $V = \bigoplus_\lambda \mathcal{L}_\lambda$ eigenspaces, $\dim \mathcal{L}_\lambda = 1$.

Pick a vector $\psi_\lambda \neq 0$ in $\mathcal{L}_\lambda$.

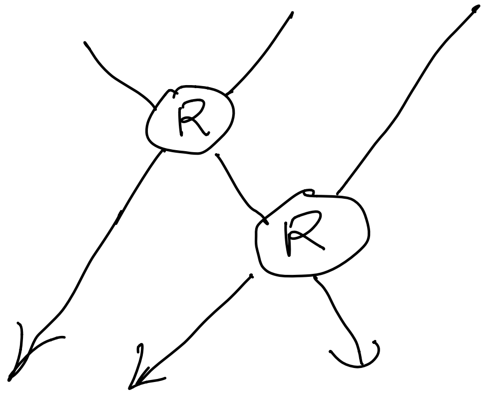
This gives rise to a linear map

$$R : V \otimes V \to V \otimes V$$

$$\psi_\lambda \otimes \psi_\mu \longrightarrow \begin{cases} 0, & \lambda \neq \mu \\ \psi_\lambda \otimes \psi_\lambda, & \lambda = \mu \end{cases}$$

which is $\Sigma_2 \times \Sigma_2$ invariant.

moreover, the map defined by the picture



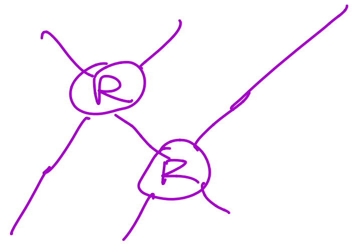
$V^{\otimes 3} \to V^{\otimes 3}$

is $\Sigma_3 \times \Sigma_3$-invariant.

Note that this data does not change under rescaling permuting φ_λ. and makes sense in any symmetric monoidal category.

<u>Definition.</u> A commutative pseudoalgebra (Pasha's bad terminology) is an object V of a symmetric monoidal category $\mathcal{C}$ with a

$\Sigma_2\times\Sigma_2$-invariant morphism $R: V\otimes V\to V\otimes V$ such that the morphism



is $\Sigma_3\times\Sigma_3$ invariant.

Note that V has no multiplication $V\otimes V\to V$ so it is not (yet) an algebra.

However, in the vector space example, suppose we pick a cyclic vector $v=\sum \varphi_\lambda \in V$ (it is equivalent to choosing scaling for each φ_λ). Then we get a map $\xi : \mathcal{U} \to V$

$\xi(u) = u \cdot v$, and $\operatorname{Ker} \xi \lhd \mathcal{U}$ is a ideal.

so we get $V \cong \mathcal{U}/\operatorname{Ker}\xi$ is a commutative associative algebra with 1.

To endow a general commutative pseudoalgebra in a symmetric monoidal category with a structure of a commutative associative algebra, we need to give a trace.

$T : V \to \mathbb{1}$.

Then we can define the product $\mu: V\otimes V \longrightarrow V$

$\mu = (1\otimes T)\circ R$,

so that $\mu\circ(1\otimes\mu): V^{\otimes 3}\to V$ is Σ_3-invariant, i.e., commutative and associative. We also have the inner product $B: V\otimes V \to \mathbb{I}$ where

$B_T = T\circ\mu$.

Let us say that T is <u>nondegenerate</u> if the form B_T is nondegenerate, i.e., V is rigid and B_T defines an isomorphism $V \xrightarrow[\sim]{B_T} V^*$.

In this case we can consider $\iota := B_T^{-1} \circ T^* : \mathbb{I} \to V^* \overset{B_T^{-1}}{\cong} V$, which is a unit in V, and V becomes a commutative associative unital Frobenius algebra with a nondegenerate trace.

So to make a commutative pseudoalgebra into a commutative associative Frobenius algebra with a nondegenerate trace, we need to give a nondegenerate trace $T : V \to \mathbb{I}$, which in our linear algebra model is just giving a cyclic vector in V^*, which is equivalent to fixing the scaling of the eigenvectors.

We can also do the same in a different way. Suppose V is a commutative pseudoalgebra and $\iota : \mathbb{I} \to V$. We say that ι is nondegenerate if V is rigid and

$$\tau_\iota = R(\iota \otimes \iota): \mathbb{I} \to V \otimes V$$

defines an isomorphism $V^* \xrightarrow{\tau_\iota} V$. Then we can define the trace T by $T = \tau_\iota^{-1} \circ \iota^*: V \overset{\tau_\iota^{-1}}{\cong} V^* \to \mathbb{I}$ which is a nondegenerate trace, and then ι becomes the unit of the corresponding algebra.

Also in this situation we get the n-point functions

$B_n : V^{\otimes n} \to \mathbb{1}$

$B_n = T \circ \mu \circ (1 \otimes \mu)(1^{\otimes 2} \otimes \mu) \cdots (1^{\otimes n-2} \otimes \mu)$

which are S_n-invariant.

So we get a 2-dim TQFT with values in $\mathcal{C}$.

Thus we should think of the notion of a commutative pseudoalgebra as a nondegenerate Frobenius algebra "without fixing the normalization of eigenvectors".

The story we are considering is this kind of story in the case when

V is ∞-dimensional (space of functions on some variety X), with basis $\psi_\lambda(x)$ (which could be discrete or continuous), e.g. Fourier $\psi_\lambda(x)=e^{i(\lambda,x)}$ and we have a multiplication law

$$(f*g)(z)=\int K(x,y,z)f(x)g(y)\,dx\,dy.$$

We also have the dual law (coming from $V\cong V^*$)

$$f_\lambda^*(x)\,f_\lambda^*(y)=\int K(x,y,z)\,f_\lambda^*(z)\,dz,$$

where f_λ^* is the dual basis to f_λ: distributions such that

$$(f_\lambda^*, f_\mu) = \delta_{\lambda\mu}.$$

All of this needs to be said with appropriate analytic details, but this is the general idea.

Problem. In the case of (geometric) Langlands describe

$$K(x,y,z) = \sum_n \psi_n(x)\psi_n(y)\psi_n(z).$$

$x \in Bun_G$

(say for GL_n when the spectrum of Hecke operators is simple).

Note: We need to find a good normalization of eigenfunctions to have a good answer.

Pasha's remark: In the case of 4 parabolic points, $X=\mathbb{P}^1$, $G=PGL_2$ can use asymptotics at one of them to normalize ψ_n. (over a local field).