$$
\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, phantom, "\simeq"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
D(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G"] \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G}}"'] \\
\mathrm{KL}(G)_{\mathrm{crit}} \arrow[u, "\mathrm{Loc}_G"'] \arrow[r, phantom, "\simeq"] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}(D^{\times}))_{\mathrm{Ran}} \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G}}"']
\end{tikzcd}
$$

Arinkin
Beraldo
Chen
Lin ← Faergeman
Raskin
G

$$D(\mathrm{Bun}_G)^{\mathrm{irred}} := D(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$$

Lemma: $D(\mathrm{Bun}_G)^{\mathrm{irred}} \subseteq D(\mathrm{Bun}_G)_{\mathrm{cusp}}$

$$D(\mathrm{Bun}_G)^{\mathrm{irred}} \xrightarrow{\mathbb{L}_G^{\mathrm{irred}}} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$$

<u>Thm</u> $\exists$ $D(\mathrm{Bun}_G) \xleftarrow{\Phi_G} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ such that

$\mathbb{L}_G^{\mathrm{irred}}$ and Φ_G^{irred} are adjoint both ways.

$$\mathbb{L}_G^{irred} \circ \Phi_G^{irred} = -\bigotimes_{\mathcal{O}_{LS^{irred}_{\check{G}}}} \mathcal{A}_G$$

Cor 1 $\mathcal{A}_G$ is a perfect complex.

$$(\mathbb{L} \circ \Phi)^R = \Phi^R \circ \mathbb{L}^R = \mathbb{L} \circ \Phi$$

Cor 2 $\mathcal{A}_G$ is self-dual

Theorem 2 $$(\mathcal{A}_G)_\sigma = C_\cdot(\{\text{generic oper structures on } \sigma\})$$ $\sigma \in LS^{irred}_{\check{G}}$

Cor 3 $\mathcal{A}_G$ lives in degree 0; $(\mathcal{A}_G)_\sigma =$ vector space spanned by $\pi_0(\{\text{generic oper structures}\})$

Cor 3'

$$\begin{tikzcd} \mathrm{Op}^{\mathrm{gen}}_{\check{G}}(X) \times_{\mathrm{LS}_{\check{G}}} \mathrm{LS}^{\mathrm{irred}}_{\check{G}} \arrow[d, "\text{proper map w/ contractible fibers}"] \\ \mathcal{Y} \arrow[d, "\text{finite étale}"] \\ \mathrm{LS}^{\mathrm{irred}}_{\check{G}} \end{tikzcd}$$

$$\begin{tikzcd} D(\mathrm{Bun}_G) & & D(\mathrm{Bun}_G)_{co} \arrow[ll, shift right=2, "\mathrm{Ps Id}_!"', "\sim"] \arrow[ll, shift left=2, "\mathrm{Ps Id}_*"] \arrow[ll, phantom, "\Downarrow"] \end{tikzcd}$$

- $\mathrm{Ps Id}_! \longrightarrow \mathrm{Ps Id}_*$ is an iso on the cuspidal subcategory

- $\mathrm{Ps Id}_! \circ \underset{\substack{\| \\ \mathrm{coeff}^\vee}}{\mathrm{Poinc}_*} = \mathrm{Poinc}_!$

$$\begin{tikzcd} D(\mathrm{Bun}_G) & D(\mathrm{Bun}_G)_{co} \arrow[l, "\mathrm{Ps Id}_!"'] & & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[ll, "\mathbb{L}_G^\vee"'] \arrow[lll, bend left, "\Phi_G"] \end{tikzcd}$$

Let's prove that Φ_G is the left adjoint of $\mathbb{L}_G$

$$\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, phantom, "\simeq"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
D(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G"] \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G}}"']
\end{tikzcd}
\implies (\mathbb{L}_G^{\mathrm{irred}})^L = \Phi_G^{\mathrm{irred}}$$

$$\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{Poinc}_!"'] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "\mathrm{loc}^{\mathrm{spec}}_{\check{G}}"] \\
D(\mathrm{Bun}_G) \arrow[d] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[l, "(\mathbb{L}_G)^L"'] \arrow[d] \\
D(\mathrm{Bun}_G)^{\mathrm{irred}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \arrow[l, "(\mathbb{L}_G^{\mathrm{irred}})^L"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G) \arrow[r, phantom, "\overset{\mathrm{id}}{=}"] \arrow[d, "\mathrm{Poinc}_!"] & \mathrm{Whit}(\mathrm{Gr}_G) \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{Poinc}_*"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "\mathrm{loc}^{\mathrm{spec}}_{\check{G}}"] \\
D(\mathrm{Bun}_G) \arrow[d] & D(\mathrm{Bun}_G)_{\mathrm{co}} \arrow[l, "\mathrm{Ps}\text{-}\mathrm{Id}_!"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[l, "(\mathbb{L}_G)^\vee"'] \arrow[d] \\
D(\mathrm{Bun}_G)^{\mathrm{irred}} & & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \arrow[ll, "\Phi_G^{\mathrm{irred}}"']
\end{tikzcd}$$

$$\exists\ U \xhookrightarrow{\ \tilde{j}\ } \mathrm{Bun}_G$$

- U is quasi-compact

$$\begin{tikzcd}
D(\mathrm{Bun}_G) & D(U) \arrow[l, hook', "\tilde{j}_*"'] \arrow[r, bend left, "e_U^L"] & D(\mathrm{Bun}_G)_{\mathrm{cusp}} \arrow[l, hook', "e_U"'] \arrow[ll, bend left=40, "e"] \arrow[d, equal] \\
D(\mathrm{Bun}_G)_{\mathrm{co}} & D(U) \arrow[l, hook', "\tilde{j}_{*,\mathrm{co}} = (\tilde{j}^*)^\vee"'] \arrow[r, bend left, "e_U^L"] & D(\mathrm{Bun}_G)_{\mathrm{co,cusp}} \arrow[l, hook', "e_U"'] \arrow[ll, bend left=40, "e_{\mathrm{co}}"]
\end{tikzcd}$$

$$
\begin{tikzcd}
D(\mathrm{Bun}_G)^{\mathrm{irred}} \arrow[rr, "\mathbb{L}_G^{\mathrm{irred}}"] & & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \\
D(\mathrm{Bun}_G) \arrow[u, "e_u^L \circ \tilde{j}^*"] \arrow[rr, "\mathbb{L}_G"] & & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[u, "j^*"'] \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[u, "\mathrm{Loc}."'] \arrow[rr, phantom, "\simeq"] & & \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(\mathcal{D}^{\times})) \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G}}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
D(\mathrm{Bun}_G)^{\mathrm{irred}} \arrow[d, "(\tilde{j}^* \circ \mathrm{Loc}_G)^R \circ e_u"] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}}) \arrow[ll, "(\mathbb{L}\mathbb{L}_G^{\mathrm{irred}})^R"'] \arrow[d, "(j^* \circ \mathrm{Poinc}^{\mathrm{spec}})^R"] \\
\mathrm{KL}(G) \arrow[rr, phantom, "\simeq"] & & \mathrm{IndCoh}(\mathrm{Op})
\end{tikzcd}
$$

Goal:

$$
\begin{tikzcd}
D(\mathrm{Bun}_G)^{\mathrm{irred}} \arrow[d, "(\tilde{j}^* \circ \mathrm{Loc}_G)^R \circ e_u"] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}}) \arrow[ll, "\Phi_G^{\mathrm{irred}}"'] \arrow[d, "(j^* \circ \mathrm{Poinc}^{\mathrm{spec}})^R"] \\
\mathrm{KL}(G) \arrow[rr, phantom, "\simeq"] & & \mathrm{IndCoh}(\mathrm{Op})
\end{tikzcd}
$$

$$
\begin{tikzcd}
D(\mathrm{Bun}_G)_{\mathrm{cusp}} \arrow[d, "\mathrm{en}"] & D(\mathrm{Bun}_G)_{\mathrm{co,cusp}} \arrow[l, "\mathrm{PsId}^{\mathrm{irred}}_*"'] \arrow[d, "\mathrm{en}"] & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \arrow[l, "(\mathbb{L}^{\mathrm{irred}}_G)^\vee"'] \arrow[dd, "j_*"] \\
D(U) \arrow[d, "\tilde{j}_*"] \arrow[r, equal, "\mathrm{id}"] & D(U) \arrow[d, "\tilde{j}_{\mathrm{co},*}"] & \\
D(\mathrm{Bun}_G) & D(\mathrm{Bun}_G)_{\mathrm{co}} \arrow[l, "\mathrm{PsId}_*"'] \arrow[d, "\Gamma_G"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[l, "\mathbb{L}_{\check{G}}"'] \arrow[d, "(\mathrm{Poinc}^{\mathrm{spe}})^\vee"] \\
& \mathrm{KL}(G) \arrow[r, phantom, "\simeq"] & \mathrm{IndCoh}(\mathrm{Op})
\end{tikzcd}
$$

$$
\begin{tikzcd}
D(\mathrm{Bun}_G)^{\mathrm{irred}} \arrow[d, "\mathrm{en}"] & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \arrow[l, "\Phi_G^{\mathrm{irred.}}"'] \arrow[dd, "(\mathrm{Poin}^{\mathrm{spe}})^\vee \circ j_*"] \\
D(U) \arrow[d, "\Gamma_G \circ \tilde{j}_{\mathrm{co},*}"] & \\
\mathrm{KL}(G) \arrow[r, phantom, "\simeq"] & \mathrm{IndCoh}(\mathrm{Op})
\end{tikzcd}
$$

Prop 1 (a) $\Gamma_G \cdot \tilde{j}_{\infty,*}$

$$D(U) \xrightarrow{\tilde{j}_{\infty,*}} D(\mathrm{Bun}_G)_\infty \xrightarrow{\Gamma_G} KL_G$$

is the right adjoint to

$$D(U) \xleftarrow{\tilde{j}^*} D(\mathrm{Bun}_G) \xleftarrow{\mathrm{Loc}_G} KL_G$$

(b) $\Gamma_G \cdot \tilde{j}_{\infty,*}$ is fully faithful $\Longleftrightarrow$ $\tilde{j}^* \cdot \mathrm{Loc}_G$ is Verdier quotient.

Delegated to next time

Prop 2

$$(\mathrm{Poinc}^{\mathrm{spec}})^{\vee} \circ j_* : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irr}}) \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{(\mathrm{Poinc}^{\mathrm{spec}})^{\vee}} \mathrm{IndCoh}(\mathrm{Op})$$

is the right adjoint of

$$j^* \circ \mathrm{Poinc}^{\mathrm{spec}} : \mathrm{IndCoh}(\mathrm{Op}) \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{j^*} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irr}})$$

Proof of proposition 2

$$\begin{tikzcd} & \mathrm{LS}_{\check{G}} \\ \mathrm{Op}^{\text{mon-free}}(D^{\times}) & \mathrm{Op}^{\text{mon-free}}_{p}(X-x) \arrow[l, "i"'] \arrow[u, "p"'] \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \\ \mathrm{IndCoh}^{*}(\mathrm{Op}^{\text{mon-free}}(D^{\times})) \arrow[r, "i^{*}"] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}(X-x)) \arrow[u, "p_{*}"'] \end{tikzcd}$$

$$
\begin{tikzcd}
 & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \arrow[d, "p^!"] \\
\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(D^\times)) & \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X-x)) \arrow[l, "(i^*)^\vee"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \arrow[d, "p^!"] \\
\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(D^\times)) & \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X-x)) \arrow[l, "i_*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(D^\times)) \arrow[r, "i^!"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(X-x)) \arrow[d, equal, "\mathrm{id}"] \\
\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(D^\times)) \arrow[r, "i^*"] & \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(X-x))
\end{tikzcd}
$$

$(\mathrm{DS}\circ\mathrm{DS})(\mathrm{CDO}(\check{G}))$

$$\begin{tikzcd}
\mathrm{IndCoh}^{*}\left(\mathrm{Op}_{\check{G}}^{\text{non-lue}}(D^{\times})\right) \arrow[r, "i^{*}"] \arrow[d, bend right] \arrow[d, bend left, "(\mathrm{DS}\circ\mathrm{DS})(\mathrm{CDO}_{G,\mathrm{crit}})"] & \mathrm{IndCoh}\left(\mathrm{Op}^{\text{non-lue}}(X-x)\right) \arrow[d, equal] \\
\mathrm{IndCoh}^{!}\left(\mathrm{Op}_{\check{G}}^{\text{non-lue}}(D^{\times})\right) \arrow[r, "i^{!}"] & \mathrm{IndCoh}\left(\mathrm{Op}^{\text{non-lue}}(X-x)\right)
\end{tikzcd}$$

[illegible] $\longrightarrow$

$$\mathrm{Op}^{u}(D) \hookrightarrow \mathrm{Op}^{u}(D^{\times})$$

$$\mathrm{Op}(D^{\times}) \longrightarrow \mathrm{Op}^{u}(D^{\times}) / \mathrm{Op}^{u}(D)$$

$$\bigoplus_{e} \Gamma(D, \Omega^{\otimes e})$$

$$
\begin{tikzcd}
\mathrm{Op}_{\check{G}} \arrow[r, hook] & \mathrm{LS}_{\check{G}} \times_{\mathrm{Bun}_{\check{G}}} \mathrm{Bun}_{\check{G}}^{\check{B}\text{-gen}} \arrow[r] \arrow[d] & \mathrm{Bun}_{\check{G}}^{\check{B}\text{-gen}} \arrow[d] \\
& \mathrm{LS}_{\check{G}} \arrow[r] & \mathrm{Bun}_{\check{G}}
\end{tikzcd}
$$