$$
\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, "\sim"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
D(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G"] \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G}}"'] \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[u, "\mathrm{Loc}_G"] \arrow[r, phantom, "\simeq"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}(D^\times))_{\mathrm{Ran}} \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G}}"']
\end{tikzcd}
$$

$$
D(\mathrm{Bun}_G)^{\mathrm{irred}} := D(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})
$$

$$
\begin{array}{ccc}
D(\mathrm{Bun}_G)^{\mathrm{irred}} & \longrightarrow & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}}) \\
& & \wr\!\parallel \\
& & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}})
\end{array}
$$

<u>Lemma</u> $D(\mathrm{Bun}_G)^{\mathrm{irred}} \subseteq D(\mathrm{Bun}_G)_{\mathrm{cusp}}$

$$D(\mathrm{Bun}_G)_{\mathrm{cusp}} := \left(D(\mathrm{Bun}_G)_{\mathrm{Eis}}\right)^{\perp}$$

Proof of Lemma

Need to show:

$$D(\mathrm{Bun}_G)_{\mathrm{Eis}} \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) = 0$$

$$\begin{tikzcd}
D(\mathrm{Bun}_M) \arrow[rr, "\mathrm{Eis}_!"] \arrow[dr] & & D(\mathrm{Bun}_G) \\
& D(\mathrm{Bun}_M) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{M}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check{P}}) \arrow[ur, "\mathrm{Eis}_!^{\mathrm{interm}}"'] &
\end{tikzcd}$$

$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$

Remark

$$D(\mathrm{Bun}_G)^{irred} \subseteq D(\mathrm{Bun}_G)_{cusp.}$$

In fact, this is an equality.

Cor $D(\mathrm{Bun}_G)^{irred} \subseteq D(\mathrm{Bun}_G)_{temp.}$

$$\begin{array}{ccc} D(\mathrm{Bun}_G)^{irred} & \xrightarrow[\sim]{\mathbb{L}_G^{irred}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{irred}) \\ D(\mathrm{Bun}_G) & \xleftarrow{\Phi_G} & \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}}) \end{array}$$

<u>Thm</u> $\mathbb{L}_G^{irred}$ is both a left and a right adjoint of Φ_G^{irred}.

$$V - 0/\mathbb{G}_m \overset{j}{\hookrightarrow} V/\mathbb{G}_m$$

$$D(Bun_G)^\vee =: D(Bun_G)_{co}$$

$$U \overset{j}{\hookrightarrow} Bun_G$$

Recall that $D(Bun_G)$ is compactly generated $j_!(\mathcal{F}_U)$, $\mathcal{F}_U \in D(U)^c$

$$D(Bun_G)_{co} := \mathrm{Ind}\begin{pmatrix} \text{subcat generated} \\ \text{by} \\ j_*(\mathcal{F}_U) \end{pmatrix}$$

$$\mathrm{PsId}_* : D(\mathrm{Bun}_G)_{co} \longrightarrow D(\mathrm{Bun}_G)$$
$$\uparrow$$
$$\mathrm{PsId}_! : D(\mathrm{Bun}_G)_{co} \xrightarrow{\sim} D(\mathrm{Bun}_G)$$

$$\mathrm{PsId}_* \longleftrightarrow \Delta_*(\omega_{\mathrm{Bun}_G}) \in D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \simeq D(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$
$$\mathrm{PsId}_! \longleftrightarrow \Delta_!(\omega_{\mathrm{Bun}_G})$$

$$\mathrm{PsId}_* \circ (\mathrm{PsId}_!)^{-1} : D(\mathrm{Bun}_G) \longrightarrow D(\mathrm{Bun}_G)$$

$$\begin{array}{ccc} \mathrm{IndCoh}_{\mathrm{Nilp}}(LS_{\check{G}}) & \longrightarrow & \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{g}}}(LS_{\check{G}}) \\ \downarrow & & \\ \mathrm{QCoh}(LS_{\check{G}}) & \xrightarrow{\otimes\, st} & \mathrm{QCoh}(LS_{\check{G}}) \end{array}$$

Thm (Lin Chen)

Exe $\Downarrow$ $\mathrm{Cone}(\mathrm{Ps}\,\mathrm{Id}_! \longrightarrow \mathrm{Ps}\,\mathrm{Id}_*)$ maps to $D(\mathrm{Bun}_G)_{\mathrm{Eis}}$

Cor $\mathrm{Ps}\,\mathrm{Id}_!\big|_{D(\mathrm{Bun}_G)_{\mathrm{cusp}}} \xrightarrow{\sim} \mathrm{Ps}\,\mathrm{Id}_*\big|_{D(\mathrm{Bun}_G)_{\mathrm{cusp}}}$.

Thm $\mathrm{Poinc}_! \simeq \mathrm{Ps}\,\mathrm{Id}_! \circ \underset{\underset{(\mathrm{coeff})^\vee}{\cong}}{\mathrm{Poinc}_*}$

$$D(\mathrm{Bun}_G) \xleftarrow{\mathrm{Ps\text{-}Id}_!} D(\mathrm{Bun}_G)_{co} \xleftarrow{\mathbb{L}_G^\vee} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$$

This is Φ_G.

<u>Thm 1</u> Φ_G^{irred} is the left adjoint of $\mathbb{L}_G^{irred}$

<u>Thm 2</u> Φ_G^{irred} is the right adjoint of $\mathbb{L}_G^{irred}$.

$$
\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, equal, "\mathrm{id}"] \arrow[d, "\mathrm{Poinc}_!"'] & \mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, "\sim"', no head] \arrow[d, "\mathrm{Poinc}_*"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}"] & \\
D(\mathrm{Bun}_G) & D(\mathrm{Bun}_G)_{\mathrm{co}} \arrow[l, "\mathrm{PsId}_!"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[l, "(\mathbb{L}_G)^\vee"'] \arrow[r, Rightarrow] & \Phi^{\mathrm{irred}}_G = (\mathbb{L}^{\mathrm{irred}}_G)^L
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, "\sim", no head] \arrow[d, "\mathrm{Poinc}_!"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}"] \\
D(\mathrm{Bun}_G) \arrow[d] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[l, "(\mathbb{L}_G)^L"'] \arrow[d] \\
{} & {}
\end{tikzcd}
$$

$$\begin{tikzcd}
 & \mathrm{Rep}(\check{G}) \arrow[dl] \arrow[d, two heads, "\mathrm{Loc}"] \\
D(\mathrm{Bun}_G) & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[l, shift right, "(\mathbb{L}_G)^L"'] \arrow[l, shift left, "\Phi_G"]
\end{tikzcd}$$

Thm 2 $\mathbb{L}_G^{\mathrm{irred}}$ is the left adjoint of Φ_G^{irred}.

$$\begin{array}{c} D(\mathrm{Bun}_G) \\ \Big\uparrow {\scriptstyle \mathrm{Loc}_G} \\ \mathrm{KL}(G)_{\mathrm{crit}} \end{array} \qquad\qquad \begin{array}{c} D(\mathrm{Bun}_G)_{\mathrm{co}} \\ \Big\downarrow {\scriptstyle \Gamma_G} \\ \mathrm{KL}(G)_{\mathrm{crit}} \end{array}$$

Let $U \overset{j}{\hookrightarrow} \mathrm{Bun}_G$ be q.c.

Lemma A

$(j^* \circ \mathrm{Loc}_G, \quad \Gamma_G \circ j_{*,\mathrm{co}})$ are adjoint.

$$
\begin{tikzcd}
\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \\
\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[u, "j^*"] \\
\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(\mathcal{D}^\times)). \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \arrow[d, "j_*"] \\
\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[d, "\mathrm{coeff}^{\mathrm{spec}}_{\check{G}}"] \\
\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(\mathcal{D}^\times)).
\end{tikzcd}
$$

<u>Lemma B</u> $\left(j^* \circ \mathrm{Poinc}^{\mathrm{spec}}_{\check{G}} \; , \; \mathrm{coeff}^{\mathrm{spec}}_{\check{G}} \circ j_* \right)$

Lemma A + Lemma B $\Rightarrow$ Thm 2

$$\begin{tikzcd} & LS_{\check{G}}(x) \\ Op^{\text{mon-free}}_{\check{G}}(D^\times) & Op^{\text{mon-free}}_{\check{G}}(X - x) \arrow[l, hook', "i"'] \arrow[u, "p"'] \end{tikzcd}$$

$$\mathrm{Poinc}^{spec} = p_* \cdot i^!$$

$$\mathrm{coeff}^{spec} = i_* \cdot p^!$$

$$(\mathrm{coeff}^{spec})^L = (p^!)^L \circ i^*$$

Want: • p_* is $(p^!)^L$

• $i^! \simeq i^* [\quad]$

game with determinant gerbes.

Possibility for next time:

- Lemma A + B $\Longrightarrow$ Thm 2.
- The shape of $\mathbb{L}_G^{irred}$, Φ_G^{irred} (is an étale algebra).

$$
\begin{tikzcd}
D(U) \arrow[r] & D(\mathrm{Bun}_G)^{irred} \arrow[r, shift left, "\mathbb{L}_G^{irred}"] \arrow[r, shift right, "(\Phi_G^{irred})^L"'] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{irred}) \\
& D(\mathrm{Bun}_G) \arrow[u] & \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}}) \arrow[u] \\
& KL(G)_{crit, Ran} \arrow[uul] \arrow[u, "\mathrm{Loc}_G"'] \arrow[r, "\sim"'] & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{mon\text{-}free}(D^\times))_{Ran} \arrow[u, "\mathrm{Poinc}^{spec}"']
\end{tikzcd}
$$