$$
\begin{tikzcd}
KL(G)_{crit} \arrow[rr, "\widetilde{FLE}_{crit}"] \arrow[dr, "DS"'] & & QCoh(Op^{mon\text{-}free}_{\check{G}}(D^\times)) \arrow[dl, "{\Gamma(Op^{mon\text{-}fr}_{\check{G}}(D^\times), -)}"] \\
& Vect &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& Rep(\check{G}) \arrow[dl] \arrow[dr] & \\
Sph_G & \simeq & IndCoh(LS_{\check{G}}(D) \times_{LS_{\check{G}}(D^\times)} LS_{\check{G}}(D))
\end{tikzcd}
$$

$$
\begin{array}{ccc}
\mathbb{V}^\lambda_{crit} & \longmapsto & \mathcal{O}_{Op^\lambda_{\check{G}}} \\
\shortparallel & & \\
ind^{\hat{g}}_{G(\mathcal{O})}(V^\lambda) & &
\end{array}
$$

Thm $\underline{x} = x_1, \dots, x_n$

$$
\begin{tikzcd}
KL(G)_{crit,\underline{x}} \arrow[rr, "Loc"] \arrow[d, "(a)"'] & & D(Bun_G) \\
KL(G)_{crit,\underline{x}} \otimes_{QCoh(Op^{mon\text{-}free}_{\check{G}}(D^\times_{\underline{x}}))} & & QCoh(Op^{mon\text{-}free}_{\check{G}}(X-\underline{x})) \arrow[u, dotted]
\end{tikzcd}
$$

(5) $\cdots\to$ is compatible w/ Hecke action

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$$

$$\mathrm{Loc}(\mathbb{V}^0) = \mathrm{Diff}_{\mathrm{crit}}$$

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}_x), \mathcal{O}) \longrightarrow \Gamma(\mathrm{Op}_{\check{G}}(X), \mathcal{O})$$

<u>Thm</u> $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ action on $D(\mathrm{Bun}_G)$ factors through

$$\mathrm{Loc}_{\infty,\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)).$$

$K \in \ker(\mathrm{Loc}_{\infty,\check{G}})$

Want to show that the Hecke action of K on $D(\mathrm{Bun}_G)$ is zero.

It's enough to exhibit generators of

$D(\mathrm{Bun}_G)$ that are killed by K.

- $\mathrm{Eis}_!(D(\mathrm{Bun}_M))$
- $\mathrm{Loc}_{G,\mathrm{crit}}$

<u>Prop</u>

$\mathrm{Loc}_{\mathrm{crit}} \otimes \mathrm{Loc}_{\mathrm{crit}}$:

$$(KL(G)_{\mathrm{crit}} \otimes KL(G)_{\mathrm{crit}})_{\mathrm{Ran}} \to D(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

$$(\mathrm{Loc}_{\mathrm{crit}} \otimes \mathrm{Loc}_{\mathrm{crit}})(CDO_{\mathrm{crit}}) = \Delta_*(\omega_{\mathrm{Bun}_G})$$

$$\Delta : \mathrm{Bun}_G \longrightarrow \mathrm{Bun}_G \times \mathrm{Bun}_G \quad .$$

<u>Cor</u> $U \overset{j}{\subseteq} \mathrm{Bun}_G$

quasi-compact stack.

$$(KL(G)_{\mathrm{crit}})_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}_{\mathrm{crit}}} D(\mathrm{Bun}_G) \xrightarrow{j^*} D(U)$$

Verdier quotient.

$$P_{sp} \xRightarrow{\mathrm{Exe}} \mathrm{Cor}$$

Hint: $C_1 \xrightarrow{F} C_2$

F preserves compactness:

$$F^{op}: C_1^\vee \longrightarrow C_2^\vee$$

$$\shortparallel$$

$$(F^\vee)^L$$

$u_C \in C \otimes C^\vee$.

$$\begin{array}{c} (F \otimes F^{op})(u_{C_1}) \\ \uparrow \\ u_{C_2} \end{array} \in C_2 \otimes C_2^\vee$$

Lemma F is Verdier quotient iff this map is an iso.

Cor $\Longrightarrow$ generation.

<u>Cor</u> $D(\mathrm{Bun}_G)_{co} \xrightarrow{\mathrm{Ps\text{-}Id}^{naive}} D(\mathrm{Bun}_G)$

The essential image of this functor lies in the essential image of $\mathrm{Loc}_{\mathrm{coarse}}$.

Let's show that $K \in \ker(\mathrm{Loc}_{\infty,\check{G}})$ kills $\mathrm{Eis}_!$

$$\begin{tikzcd}
D(\mathrm{Bun}_M) \arrow[rr, "\mathrm{Eis}_!"] \arrow[dr] & & D(\mathrm{Bun}_G) \\
& D(\mathrm{Bun}_M) \underset{\mathrm{QCoh}(LS_{\check{M}})}{\otimes} \mathrm{QCoh}(LS_{\check{P}}) \arrow[ur, dashed, "\mathrm{Eis}_!^{int}"'] &
\end{tikzcd}$$

and $\mathrm{Eis}_!^{int}$ respects the action of

$$\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, "\mathrm{FLE}_\infty"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
{} \arrow[u, "\mathrm{coeff}"] & {} \arrow[u, "\Gamma_\infty"']
\end{tikzcd}$$

$$
\begin{tikzcd}
D(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \\
(\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{Ran}} \arrow[u, "\mathrm{Loc}_{\mathrm{crit}}"] \arrow[r, "\mathrm{FLE}_{\mathrm{crit}}"] & \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(\overset{\circ}{D}))_{\mathrm{Ran}} \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}"']
\end{tikzcd}
$$

$$
\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(\overset{\circ}{D}_{\underline{x}}) \leftarrow \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(X - \underline{x}) \rightarrow \mathrm{LS}_{\check{G}}(X)
$$

<u>Thm</u> The outer diagram commutes.

$$
\begin{tikzcd}
(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Rep}(\check{G}))_{\mathrm{Ran}} \arrow[r, "\mathrm{FLE}_{\mathrm{crit}} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mathrm{FLE}_{\infty}"'] & (\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(\overset{\circ}{D})) \otimes \mathrm{Rep}(\check{G}))_{\mathrm{Ran}} \arrow[d, "(A)"] \\
(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}(\mathrm{Gr}_G))_{\mathrm{Ran}} \arrow[r, "(B)"'] & \mathrm{Vect}
\end{tikzcd}
$$

Functor A:

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\overset{\circ}{D}{}^{\times}) \longrightarrow LS_{\check{G}}(D)$$

tensor up and take global sections on $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\overset{\circ}{D}{}^{\times})$

Functor B:

$$\mathrm{Whit}(\mathrm{Gr}_G) = \left(\mathrm{Dmod}(\mathrm{Gr}_G)\right)_{N((t)),\chi}$$

$$\begin{tikzcd}
\mathrm{Dmod}(\mathrm{Gr}_G) \otimes KL(G) \arrow[r] & \hat{\mathfrak{g}}\text{-mod} \arrow[r, "DS"] & \mathrm{Vect} \\
\left(\mathrm{Dmod}(\mathrm{Gr}_G)\right)_{N((t)),\chi} \otimes KL(G) \arrow[r] \arrow[urr, bend right=60, "(B)"'] & \left(\hat{\mathfrak{g}}\text{-mod}\right)_{N((t)),\chi} \arrow[ur] &
\end{tikzcd}$$

Why is (A) = (B) ?

(A): $\Gamma(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(D^\times), -)$

(B): $DS : KL(G) \longrightarrow \mathrm{Vect}$

Why does (B) correspond to the composite vertical functor on the geometric side.

$$
\begin{tikzcd}
\mathrm{Vect} & & \\
& D(\mathrm{Bun}_N^{\omega}) \arrow[r, "\Gamma_{dR}(\mathrm{Bun}_N^{\omega},\, -\otimes \mathrm{Ach})"] & \mathrm{Vect} \\
D(\mathrm{Bun}_G) \arrow[uu, "\mathrm{coeff}_{\mathrm{Vac}}"] \arrow[ur, "!\text{-pullback}"] & & \\
KL(G)_{\mathrm{Ran}} \arrow[u, "\mathrm{Loc}"] \arrow[uurr, "H^{ch} \circ DS"'] & &
\end{tikzcd}
$$

Exe.

(A)

$$\begin{tikzcd}
\mathrm{Vect} & \\
\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, "{\Gamma(\mathrm{LS}_{\check{G}}, -)}"] & \\
& \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X - \underline{x})) \arrow[ul] \\
\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\overset{\times}{D}_{\underline{x}})) \arrow[uu, "\mathrm{Poinc}^{\mathrm{spec}}"] \arrow[ur] &
\end{tikzcd}$$

.