1. Construction of the functor $\mathbb{L}_G$.
2. Localization from Kac-Moody modules
3. Spectral decomposition theorem
4. Localization and CT functors

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \curvearrowright D(\mathrm{Bun}_G)$$

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, shift right, "\mathrm{Loc}_{\check{G}}^{\infty}"'] \\
\mathrm{QCoh}(LS_{\check{G}}) \arrow[u, shift right, "\Gamma_{\check{G},\infty}"']
\end{tikzcd}$$

$$\mathrm{Loc}_\infty(V_x) = \mathcal{E}v_x(V)$$

$x \in X$

$V \in \mathrm{Rep}(\check{G})$

$$\mathcal{E}v_x : LS_{\check{G}} \longrightarrow pt/\check{G}$$

Prop $\Gamma_{\check{G},\infty}$ is fully faithful.

Thm The Hecke action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ on $D(\mathrm{Bun}_G)$ factors through $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$.

$$\begin{tikzcd}
D(\mathrm{Bun}_G) & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[ll, "W_!^{\mathrm{enh}}"'] \\
& & \mathrm{Vect} \arrow[u] \arrow[ull, "W_!"]
\end{tikzcd}$$

$$\begin{tikzcd}
D(\mathrm{Bun}_G) \arrow[rr, "\mathrm{coeff}_{\mathrm{vac}}^{\mathrm{enh.}}"] \arrow[drr, "\mathrm{coeff}_{\mathrm{vac}}"'] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[d, "\Gamma"] \\
& & \mathrm{Vect}
\end{tikzcd}$$

Previous talks:

$$\begin{tikzcd} & \mathrm{QCoh}(LS_{\check{G}}) \arrow[d, hook, shift right] \\ D(\mathrm{Bun}_G) \arrow[ur, "\mathrm{coeff}^{\mathrm{ql}}_{\mathrm{Vac}}"] \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(LS_{\check{G}}) \arrow[u, shift right] \end{tikzcd}$$

- $$\begin{tikzcd} \mathrm{Whit}(\mathrm{Gr})_{\mathrm{Ran}} \arrow[r, "\mathrm{FLE}^{\infty}", "\sim"'] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ D(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}"] \arrow[r, "\mathbb{L}_G"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(LS_{\check{G}}) \arrow[u, "\Gamma_{\check{G},\infty}"'] \end{tikzcd}$$

- $\mathbb{L}_G$ is compatible with derived Satake.

- $$\begin{tikzcd} D(\mathrm{Bun}_M) \arrow[r, "\mathbb{L}_M"] \arrow[d, "\mathrm{Eis}_!"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(LS_{\check{M}}) \arrow[d, "\mathrm{Eis}^{\mathrm{spec}}"] \\ D(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(LS_{\check{G}}) \end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{Whit}(\mathrm{Gr}_M)_{\mathrm{Ran}} \arrow[dl, "\omega J"'] \arrow[rr, "\mathrm{FLE}_M^{\infty}"] & & \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \arrow[dl, "\omega\mathrm{Ind}"'] \\
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[rr, "\mathrm{FLE}_G^{\infty}"] & & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} & \\
& D(\mathrm{Bun}_M) \arrow[uu] \arrow[dl, "\mathrm{Eis}_!"'] \arrow[rr, "L_M"] & & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \arrow[uu, "\Gamma_\varphi"'] \arrow[dl, "\mathrm{Eis}^{\mathrm{spec}}"] \\
D(\mathrm{Bun}_G) \arrow[uu, "\mathrm{coeff}_G"] \arrow[rr, "L_G"'] & & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[uu, "\Gamma_\varphi"] &
\end{tikzcd}$$

$$\begin{tikzcd}
D(\mathrm{Bun}_M) \arrow[r, "\mathbb{L}_M"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \\
D(\mathrm{Bun}_G) \arrow[u, "\mathrm{CT}_*"] \arrow[r, "\mathbb{L}_G"'] \arrow[ur, phantom, "\Rightarrow"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[u, "\mathrm{CT}^{\mathrm{spec}}"']
\end{tikzcd}$$

Thm This natural transformation is an iso.

$$\mathrm{Rep}(G) \xrightarrow{\mathrm{Loc}_{G,x}} \mathrm{QCoh}(\mathrm{LS}_G)$$

Sat_x

$$
\begin{tikzcd}
KL(G)_{crit} \arrow[r, "Loc_{G,crit}"'] & Dmod(Bun_G)_{crit} \arrow[d, equal] \\
 & D(Bun_G)
\end{tikzcd}
$$

$$KL(G)_{crit} = (\hat{\mathfrak{g}}_{crit}, G(O))$$

$$
\begin{tikzcd}
Rep(G(O)) \arrow[r] \arrow[d, "ind"'] & QCoh(Bun_G) \arrow[d, "ind"] \\
KL(G)_{crit} \arrow[r, "Loc_{G,crit}"] & D(Bun_G)
\end{tikzcd}
$$

$$Bun_G \xrightarrow{Ev_x} pt/G(O)$$

- Compatible w/ Hecke action.

<u>Thm</u> (critical FLE)

(a) $KL(G)_{crit} \xrightarrow{\sim} QCoh(Op_{\check{G}}^{mon\text{-}free})$

$$D_S \to \mathrm{Vect}$$

$$Op_{\check{\sigma}}(D) \subseteq Op_{\check{\sigma}}^{\text{non-free}} \subseteq Op_{\check{\sigma}}(D^\times)$$

$$Op_{\check{\sigma}}^{\text{non-free}} = \bigsqcup_{\lambda \in \Lambda^+} Op_{\check{\sigma}}^{\text{non-free},\lambda}$$

$$Op_{\check{\sigma}}^{\text{non-free}} = Op_{\check{\sigma}}(D^\times) \underset{LS_{\check{\sigma}}(D^\times)}{\times} LS_{\check{\sigma}}(D)$$

(b) Compatible w/ Hecke action

$$\mathbb{V}^\lambda \longrightarrow \mathcal{O}_{Op^\lambda_{\check{G}}}$$

$$\mathbb{V}^\lambda = \mathrm{ind}(V^\lambda)$$

<u>Cor</u> $KL(G)_{\mathrm{crit}}$ lives over $Op_{\check{G}}^{\text{non-free}}$

Thm

(a)
$$\begin{tikzcd}
\mathrm{KL}(G)_{crit, x_1 \ldots x_n} \arrow[r, "\mathrm{Loc}"] \arrow[d] & D(\mathrm{Bun}_G) \\
\mathrm{KL}(G)_{crit, x_1 \ldots x_n} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}, x_1 \ldots x_n})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X - x_1 \ldots x_n)) \arrow[ur, "\widetilde{\mathrm{Loc}}"] &
\end{tikzcd}$$

(b) $\widetilde{\mathrm{Loc}}$ is compatible with the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$.

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}_{\check{G},\infty}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \to$$
$$\to \mathrm{QCoh}(\mathrm{Op}^{\text{non-free}}_{\check{G}}(X - x_1 \ldots x_n))$$

$$\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, "\mathrm{FLE}^{\infty}", "\sim"'] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
D(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G"] \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma_{\check{G},\infty}"']
\end{tikzcd}$$

$$
\begin{tikzcd}
D(\mathrm{Bun}_G) \arrow[r, dash] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \\
\mathrm{KL}(G)_{\mathrm{crit}} \arrow[u, "\mathrm{Loc}_{G,\mathrm{crit}}"] \arrow[r, "\mathrm{FLE}^{\mathrm{crit}}"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}"']
\end{tikzcd}
$$

<u>Thm</u>

$$
\begin{tikzcd}
 & \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X - x_1 \ldots x_n) \arrow[dl] \arrow[dr, hook] & \\
\mathrm{Op}^{\text{mon-free}}_{\check{G}, x_1 \ldots x_n} & & \mathrm{LS}_{\check{G}}
\end{tikzcd}
$$