(Joint with Beraldo, Lin)

Goal: Construct

$$\mathbb{L}_G : \mathrm{DMod}(\mathrm{Bun}_G) \longrightarrow \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}})$$

compatible with

- derived Satake
- Eis. series

via gluing.

Recall in Sam's talk:

$$\mathbb{L}_G^{temp} : \mathrm{DMod}(\mathrm{Bun}_G)^{temp} \xrightarrow{\mathrm{coeff}_G^{enh}|_{temp}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

<u>Prop 1</u> $\exists !$ functor coeff_G^{enh} s.t.

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Bun}_G) \arrow[rr] \arrow[rr, bend right=5, "\mathrm{coeff}_G^{enh}"'] \arrow[d, "\mathrm{coeff}_G"] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[d, "\subseteq"] \\
\mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[rr, "\mathrm{C-S}", "\simeq"'] & & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[u, bend left]
\end{tikzcd}$$

expect: $\mathrm{coeff}_G^{enh} :\simeq \Psi \circ \mathbb{L}_G$

$$\Psi : \mathrm{IndCoh}_{Nilp} \twoheadrightarrow \mathrm{QCoh}$$

Idea:

$\mathrm{DMod}(\mathrm{Bun}_G)$ / $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$

can be "glued" from

$\mathrm{DMod}(\mathrm{Bun}_M)^{\mathrm{temp}}$ / $\mathrm{QCoh}(\mathrm{LS}_{\check{M}})$.

for standard Levi's M (or $\check{M}$) of

G (or $\check{G}$)

And $\mathbb{L}_M^{\mathrm{temp}} : \mathrm{DMod}(\mathrm{Bun}_M)^{\mathrm{temp}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{M}})$

are compatible w/ the gluings.

s.s. Rank 1 case. ($G = SL_2, GL_2, \dots$)

Prop 2 (Arinkin - Gaitsgory, Beraldo).

"spectral gluing"

$$\begin{tikzcd}
\mathrm{IndCoh}_{Nilp}(LS_{\check{G}}^{u}) \arrow[r] \arrow[d, "\Psi"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{IndCoh}_0\left((LS_{\check{G}}^{u})^{\wedge}_{LS_{\check{B}}}\right) \arrow[l, bend right, blue] \arrow[d, "\Psi"] \\
\mathrm{QCoh}(LS_{\check{G}}^{u}) \arrow[u, bend left, blue, "\Xi"] \arrow[r, "*\text{-pull}"'] & \mathrm{QCoh}\left((LS_{\check{G}}^{u})^{\wedge}_{LS_{\check{B}}}\right) \arrow[l, bend right, blue, "!\text{-push}"'] \arrow[u, bend left, blue, "\Xi"]
\end{tikzcd}$$

is Cartesian. (The blue square is a pushout)

Rmk: For $Y \to Z$ nice enough

$$\begin{tikzcd}
\mathrm{IndCoh}_0(Z^{\wedge}_{Y}) \arrow[r, "\mathrm{oblv}_D"'] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \mathrm{QCoh}(Y) \arrow[l, bend right, blue, "\mathrm{ind}_D"'] \arrow[d, hook] \\
\mathrm{IndCoh}(Z^{\wedge}_{Y}) \arrow[r, "!\text{-pull}"'] & \mathrm{IndCoh}(Y) \arrow[l, bend right, blue, "!\text{-push}"']
\end{tikzcd}$$

generates

Fact: This monad on $\mathrm{IndCoh}(Y)$ preserves $\mathrm{QCoh}(Y)$

$\mathrm{IndCoh}(Z^{\wedge}_{Y})$

"the category of qcoh. on Y equipped connections along $Y \to Z$"

$$\begin{tikzcd} \mathrm{QCoh}(\mathrm{LS}_{\check{B}}) \arrow[d, "\text{generate}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}) \arrow[l, "*\text{-pull}"'] \\ {} & \end{tikzcd}$$

<u>Cor 2</u>

$$\begin{tikzcd} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) & \mathrm{IndCoh}((\mathrm{LS}_{\check{G}})^{\wedge}_{\mathrm{LS}_{\check{B}}}) \arrow[l, "!\text{-push}"'] \\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, "\subseteq"] \arrow[ur, phantom, "\lrcorner"] & \mathrm{QCoh}((\mathrm{LS}_{\check{G}})^{\wedge}_{\mathrm{LS}_{\check{B}}}) \arrow[l, "!\text{-push}"'] \arrow[u, "\subseteq"'] \end{tikzcd}$$

is a pushout diagram

Prop 3. (Beraldo - C.) $\qquad$ s.s. rank $(G) = 1$

"automorphic gluing"

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Bun}_G) \arrow[r, "CT_B^{enh}"] \arrow[d, "\text{temperization}"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{I}(G,B) \arrow[d] \\
\mathrm{DMod}(\mathrm{Bun}_G)^{temp} \arrow[r, "CT"] & \mathcal{I}(G,B)^{temp}
\end{tikzcd}$$

is Cartesian.

Rmk:

$$\begin{tikzcd}
\mathcal{I}(G,B) \arrow[r, "i^!"'] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \mathrm{DMod}(\mathrm{Bun}_T) \arrow[l, bend right, "i_! \text{ generates}"'] \arrow[d, "\text{pull}"] \\
\mathrm{DMod}(\mathrm{Bun}_G^{B\text{-gen}}) \arrow[r, "!\text{-pull}"'] & \mathrm{DMod}(\mathrm{Bun}_B)
\end{tikzcd}$$

$\mathrm{Bun}_G^{B\text{-gen}}$ classifies G-torsors equipped with a generic B-reduction.

Cor 3

$$\begin{tikzcd}
 & & \mathrm{DMod}(\mathrm{Bun}_T) \arrow[d] \\
\mathrm{DMod}(\mathrm{Bun}_G) & & \mathcal{I}(G,B) \arrow[ll, "\mathrm{Eis}^{\mathrm{enh}}_{B}"'] \\
\mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[u, "\subseteq"] \arrow[urr, phantom, "\lrcorner"] & & \mathcal{I}(G,B)^{\mathrm{temp}} \arrow[ll] \arrow[u, "\subseteq"']
\end{tikzcd}$$

is a pushout diagram.

Goal: match each term and connecting functors in the colimit diagram.

$$\mathrm{DMod}(\mathrm{Bun}_G)^{temp} \xrightarrow{\mathbb{L}_G^{temp}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

$$\begin{array}{ccc} \mathrm{DMod}(\mathrm{Bun}_T) & \xrightarrow{\mathbb{L}_T^{temp}} & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}) \\ \downarrow \text{generates} & & \downarrow \text{generates} \\ \text{want: } \mathcal{I}(G,B) & \longrightarrow & \mathrm{IndCoh}_0((\mathrm{LS}_{\check{G}})^{\wedge}_{\mathrm{LS}_{\check{B}}}) \end{array}$$

sends tempered to qcoh.

Method: local - global

Def:

$$\mathrm{Sph}_G := \mathrm{DMod}(\mathfrak{L}^+G \backslash \mathfrak{L}G / \mathfrak{L}^+G)$$

$$\mathrm{Sph}_M := \mathrm{DMod}(\mathfrak{L}^+M \backslash \mathfrak{L}M / \mathfrak{L}^+M)$$

generates

$$[\mathrm{Sph}_{G,P} :=]\ \mathrm{SI}_P := \mathrm{DMod}(\mathfrak{L}^+G \backslash \mathfrak{L}G)^{\mathfrak{L}U \cdot \mathfrak{L}^+M}$$

"semi-infinite"

$$1 \to U \to P \to M \to 1$$

P : standard parabolic of G $\quad M = \mathrm{Levi}(P)$.

$$
\begin{tikzcd}
 & \mathrm{Grp} \arrow[dl] \arrow[dr] & \\
\mathrm{Gr}_G & & \mathrm{Gr}_M
\end{tikzcd}
$$

global $\qquad\qquad$ local.

$$
\begin{tikzcd}
\mathrm{DMod}(\mathrm{Bun}_M) \arrow[d, shift right] & \mathrm{Sph}_M \arrow[d, shift right] \\
\mathcal{I}(G,p) \arrow[u, shift right] & \mathrm{Sph}_L \circlearrowright \mathrm{SI}_p \circlearrowleft \mathrm{Sph}_M \arrow[u, shift right]
\end{tikzcd}
$$

<u>Rmk</u>: Sph_M, SI_p are factorization categories (over Ran)

We will write

$$\mathrm{Sph}_M := \mathrm{Sph}_{M,\mathrm{Ran}}$$

Lem 4 $\exists$ an equivalence

$$\underbrace{SI^{ren}_{P,Ran}}_{\mathrm{Sph}^{ren}_{M,Ran}} \otimes \mathrm{DMod}(\mathrm{Bun}_M) \xrightarrow{\sim} \mathcal{I}(G,P).$$

Construction of the functor :

$$\mathrm{Gr}_{G,Ran} \underset{x}{\times} \mathrm{Bun}_M \xrightarrow{\pi} \mathrm{Bun}_G^{P\text{-gen}}$$

$$\left\{\begin{array}{l} \underline{x} : \text{closed points of } X \\ \mathcal{F}_M : M\text{-torsor on } X \\ \mathcal{F}_G : G\text{-torsor on } X \\ \mathcal{F}_G|_{X-\underline{x}} \simeq G \overset{M}{\times} \mathcal{F}_M|_{X-\underline{x}} \end{array}\right\} \longmapsto \left\{\begin{array}{l} \mathcal{F}_G \\ P\text{-structure} \\ \text{given by} \\ P \overset{M}{\times} \mathcal{F}_M \end{array}\right\}$$

$\mathcal{F} \in SI_{P,Ran}$ $\quad \mathcal{K} \in \mathrm{DMod}(\mathrm{Bun}_M)$

$\pi_!(\mathcal{F} \mathbin{\widetilde{\boxtimes}} \mathcal{K})$ is well-defined and

is contained in $\mathcal{I}(G,P)$

Thm 5

(Factorization derived Satake)

(i) $\mathrm{Sph}_G^{ren} \overset{\otimes}{\simeq} \mathrm{IndCoh}^{!,ren}\left(\mathrm{LS}_{\check G}(D) \times_{\mathrm{LS}_{\check G}(\mathring{D})^\wedge} \mathrm{LS}_{\check G}(D)\right)$

as monoidal factorisation category

$G \leftarrow G \to G$

$\check G \leftarrow \check G \to \check G$

$\mathrm{LS}_{\check G \times \check G}(D) \times_{\mathrm{LS}_{\check G \times \check G}(\mathring{D})^\wedge} \mathrm{LS}_{\check G}(\mathring{D})^\wedge$

Rmk: • $\mathrm{Sph}_G^{ren} := \mathrm{Ind}(\mathrm{Sph}_G^{loc.c})$

• $\mathrm{LS}_{\check G}(D_x) \times_{\mathrm{LS}_{\check G}(\mathring{D}_x)} \mathrm{LS}_{\check G}(D_x) \simeq (pt \times_{\check{\mathfrak g}} pt)/\check G$

But the factorisation version is of infinite type

$\mathrm{IndCoh}^{!,ren}$ is developed by Raskin

(and Fn. C.)

• $\mathrm{LS}_{\check G}(\mathring{D})^\wedge := \left(\mathrm{LS}_{\check G}(\mathring{D})\right)^\wedge_{\mathrm{LS}_{\check G}(D)}$

(ii)

$$G \leftarrow P \rightarrow M \qquad \exists \text{ an equivalence}$$
$$\check{G} \leftarrow \check{P} \rightarrow \check{M}$$

$$\mathrm{Sph}_G^{ren} \curvearrowright \mathrm{SI}_P^{ren} \curvearrowleft \mathrm{Sph}_M^{ren} \qquad \mathrm{SI}_P^{ren} \simeq \underline{\mathrm{IndCoh}}^{!,ren}\left(\mathrm{LS}_{\check{G}\times\check{M}}(D) \underset{\mathrm{LS}_{\check{G}\times\check{M}}(\mathring{D})^{\wedge}}{\times} \mathrm{LS}_{\check{P}}(\mathring{D})^{\wedge}\right)$$

<u>compatible with actions in (i)</u>

$$\mathrm{LS}_{\check{G}}(D) \simeq \mathrm{Conn}_{\check{G}}(D) / \mathrm{gauge}_{\mathfrak{L}^+\check{G}}$$

$\mathrm{Conn}_{\check{G}}(D)$ $\quad \mathfrak{L}^+\check{G}$ $\quad$ pro-finite type.

$$\mathrm{IndCoh}^!(\mathrm{Conn}_{\check{G}}(D)) := \mathrm{Ind}(\mathrm{Coh}(\mathrm{Conn}_{\check{G}}(D)))$$

$$\mathrm{IndCoh}^!(\mathrm{Conn}_{\check{G}}(D) / \mathfrak{L}^+G)$$
$$\simeq \mathrm{Tot}\left(\mathrm{IndCoh}^!(\mathrm{Conn}_{\check{G}}(D)) \rightrightarrows \mathrm{IndCoh}^!(\mathfrak{L}^+\check{G} \times \mathrm{Conn}_{\check{G}}(D)) \cdots\right)$$

$$\mathrm{Coh}(\mathrm{Conn}_{\check G}(D)/\mathcal{L}^+G) \subseteq \mathrm{IndCoh}^!$$

subcat s.t. pullback to $\mathrm{Conn}_{\check G}(D)$ is coherent.

$$\mathrm{IndCoh}^{!,\mathrm{ren}} := \mathrm{Ind}(\mathrm{Coh}(\quad))$$

Write the theorem as

$$\mathrm{Sph}_G^{\mathrm{ren}} \simeq \check{\mathrm{Sph}}^{\mathrm{ren}}_{\check G}$$

$$\mathrm{SI}_P^{\mathrm{ren}} \simeq \check{\mathrm{SI}}^{\mathrm{ren}}_{\check P}$$

<u>Lem 6</u>

$$\check{\mathrm{SI}}^{\mathrm{ren}}_{\check P} \underset{\check{\mathrm{Sph}}^{\mathrm{ren}}_{\check M}}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check M}) \xrightarrow{\sim} \mathrm{IndCoh}_0\left((\mathrm{LS}_{\check G})^{\wedge}_{\mathrm{LS}_{\check P}}\right)$$

$$\begin{tikzcd}
\mathrm{SI}_p^{\mathrm{ren}} \underset{\mathrm{Sph}_M^{\mathrm{ren}}}{\otimes} \mathrm{DMod}(\mathrm{Bun}_M) \arrow[r, "\sim"] \arrow[d] & \mathcal{I}(G,P) \arrow[d] \\
\check{\mathrm{SI}}_p^{\mathrm{ren}} \underset{\check{\mathrm{Sph}}_M}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check M}) \arrow[r, phantom, "\simeq"] & \mathrm{IndCoh}_0((\mathrm{LS}_{\check G})^{\wedge}_{\mathrm{LS}_{\check P}})
\end{tikzcd}$$

(Need $\mathrm{coeff}_M^{\mathrm{enh}} : \mathrm{DMod}(\mathrm{Bun}_M) \to \mathrm{QCoh}(\mathrm{LS}_{\check M})$
is compatible with derived Satake)

Lem 7:

$$\mathcal{I}(G,P) \longrightarrow \mathrm{IndCoh}_0((\mathrm{LS}_{\check G})^{\wedge}_{\mathrm{LS}_{\check P}})$$

sends tempered objects to QCoh .

$\mathrm{s.s.\,rank}(G) = \mathrm{s.s\,rank}(\check{G}) = 1$

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[d] & \mathrm{I}(G,B)^{\mathrm{temp}} \arrow[l, "\mathrm{Eis}"'] \arrow[d] \arrow[r, hook, "\subseteq"] \arrow[dl, phantom, "(*)"] & \mathrm{I}(G,B) \arrow[d] \\
\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) & \mathrm{QCoh}\big((\mathrm{LS}_{\check{G}})^{\wedge}_{\mathrm{LS}_{\check{B}}}\big) \arrow[l, "!\text{-push}"] \arrow[r, hook, "\subseteq"] & \mathrm{IndCoh}\big((\mathrm{LS}_{\check{G}})^{\wedge}_{\mathrm{LS}_{\check{B}}}\big)
\end{tikzcd}$$

<u>Prop 8</u>: $(*)$ commutes.

[Idea:

$$\begin{tikzcd}
\mathrm{D}(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[d, "\mathrm{coeff}^{\mathrm{enh}}_G"] \arrow[r, squiggly, shift right=2] & \mathrm{DMod}(\mathrm{Bun}_M)^{\mathrm{temp}} \arrow[l, "\mathrm{Eis}_!"'] \arrow[d, "\mathrm{coeff}^{\mathrm{enh}}_M"] \\
\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[r, squiggly, shift right=2] & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}) \arrow[l, "\mathrm{Eis}^{\mathrm{spec}}\ \mathrm{loc}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_G) \arrow[d, "\mathrm{c.s}"'] & \mathrm{Whit}(\mathrm{Gr}_M) \arrow[l, "\mathrm{coJac}"'] \arrow[d, "\mathrm{c.s.}"] \\
\mathrm{Rep}(\check{G}) & \mathrm{Rep}(\check{M}) \arrow[l, "\mathrm{coJac}"']
\end{tikzcd}$$

Conj:

$$\begin{tikzcd} D(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[r, "CT"] \arrow[d, "\mathrm{coeff}^{\mathrm{ext}}_0"] & D\mathrm{Mod}(\mathrm{Bun}_M)^{\mathrm{temp}} \arrow[d] \\ QCoh(LS_{\check{G}}) \arrow[r, "CT"'] & QCoh(LS_{\check{M}}) \end{tikzcd}$$

This is harder because it is not true on $D\mathrm{Mod}(\mathrm{Bun}_G)$

$I(G,P)^{P'\text{-temp}}$ $\quad P \subset P'$

$W(G, P \subset P')$

$W(G, P = P) \xrightarrow[\text{(Conj)}]{\simeq} \mathrm{Whit}_{G,P}$

$D(\mathrm{Bun}_G) = \lim W(G, P \subset P')$

contra in P

covar in P'

$\xrightarrow{\simeq} \mathrm{Lax}\lim W(G, P = P)$

$\xrightarrow{\leq}$ Caolin White G.P