

(Joint with Beraldo, Lin)

Goal : Construct

$$\mathbb{L}_G : \text{DMod}(Bun_G) \longrightarrow \text{IndGhNilp}(LS_G)$$

Compatible with

- derived Satake
 - Es. series
- via gluing.

Recall in Sam's talk:

$$\mathbb{L}_G^{\text{temp}} : \text{DMod}(Bun_G)^{\text{temp}} \xrightarrow{\text{coeff}_G^{\text{enh}}} \text{QCoh}(LS_G^{\text{temp}})$$

Prop 1 $\exists !$ factor $\text{coeff}_G^{\text{enh}}$ s.t.

$$\begin{array}{ccc} \text{DMod}(Bun_G) & \xrightarrow{\text{coeff}_G^{\text{enh}}} & \text{QCoh}(LS_G^{\text{temp}}) \\ \downarrow \text{coeff}_G & & \uparrow \text{c} \\ \text{Whit}(Gr_G)_{\text{Ran}} & \xrightarrow[\text{c}]{G-S} & \text{Rep}(\check{G})_{\text{Ran}} \end{array}$$

Expect: $\text{coeff}_G^{\text{enh}} \stackrel{!}{=} \mathbb{F} \circ \mathbb{L}_G$

$$\mathbb{F} : \text{IndGhNilp} \longrightarrow \text{QCoh}$$

Idea:

$D\text{Mod}(\mathbb{R}u\mathfrak{g})$ / $\text{IndCohNilp}(LS_{\mathfrak{g}})$
can be "glued" from

$D\text{Mod}(\mathbb{R}u\mathfrak{m}_M)^{\text{temp}}$ / $\mathbb{Q}Gh(LS_{\check{M}})$

for standard Levi's M (or \check{M}) of
 G (or \check{G})

And $\mathbb{L}_M^{\text{temp}} : D\text{Mod}(\mathbb{R}u\mathfrak{m}_M)^{\text{temp}} \rightarrow \mathbb{Q}Gh(LS_{\check{M}})$

are compatible w/ the gluings.

s.s. Rank 1 case. ($G = \text{Sh}_2, \text{Gh}_2, \dots$)

Prop 2 (Arinkin - Gaitsgory, Bernstein),

"spectral gluing"

$$\begin{array}{ccc}
 \text{IndGh}_{\text{Nilp}}(LS_0^y) & \xrightarrow{\quad} & \text{IndGh}_0((LS_0^y)^{\wedge}_{LS_0^y}) \\
 \exists \uparrow \downarrow \Psi & \perp & \exists \uparrow \downarrow \Psi \\
 \text{QCh}(LS_0^y) & \xrightarrow[\ast\text{-pull}]{!\text{-push}} & \text{QCoh}((LS_0^y)^{\wedge}_{LS_0^y})
 \end{array}$$

is Cartesian. (The blue square is a pushout)

Remark: For $\mathcal{Y} \rightarrow \mathcal{Z}$ nice enough

$$\begin{array}{ccc}
 \text{IndCoh}(\mathcal{Z}^{\wedge}_{\mathcal{Y}}) & \xrightarrow[\text{oblv}_{\mathcal{D}}]{\text{ind}_{\mathcal{D}} \text{ generates}} & \text{QCoh}(\mathcal{Y}) \\
 \downarrow & \Downarrow & \downarrow \\
 \text{IndGh}(\mathcal{Z}^{\wedge}_{\mathcal{Y}}) & \xrightarrow[\text{!-pull}]{!\text{-push}} & \text{IndGh}(\mathcal{Y})
 \end{array}$$

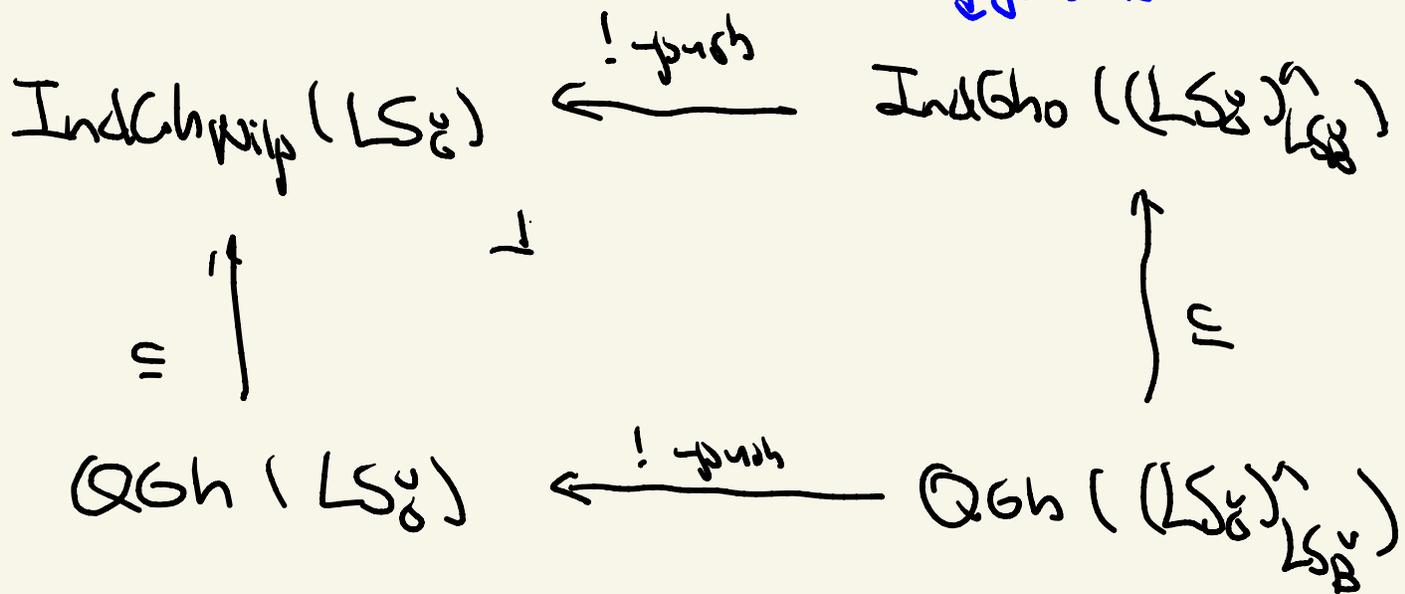
Fact: This monad on $\text{IndGh}(\mathcal{Y})$ preserves $\text{QCoh}(\mathcal{Y})$

$\text{IndGho}(\mathcal{Z}_Y)$

"the category of mod. on Y equipped
connections along $Y \rightarrow Z$ "

Cor 2

$\text{QGr}(LS_X) \xleftarrow{\text{pull}} \text{QGr}(LS_Y)$
 $\downarrow \text{generate}$

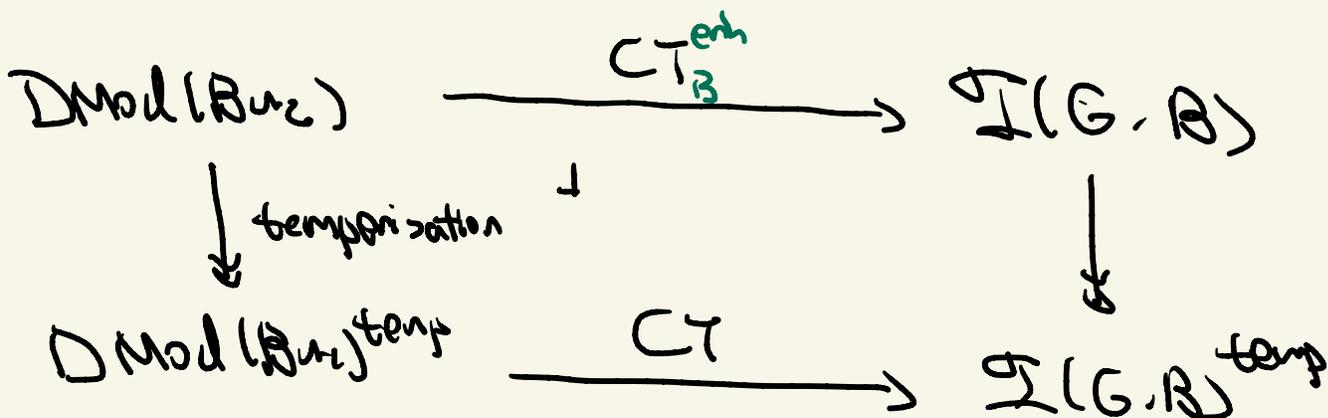


is a pushout diagram

Prop 3. (Borel - C.)

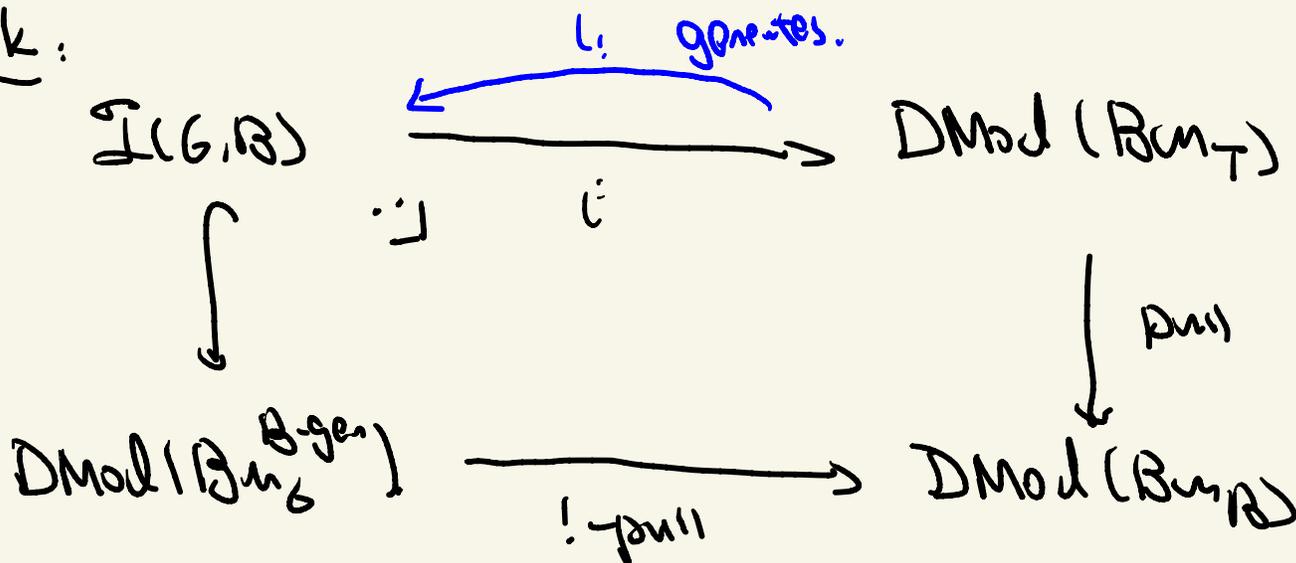
s.s. rank $(G) = 1$

"automorphic gluing"



is Cartesian.

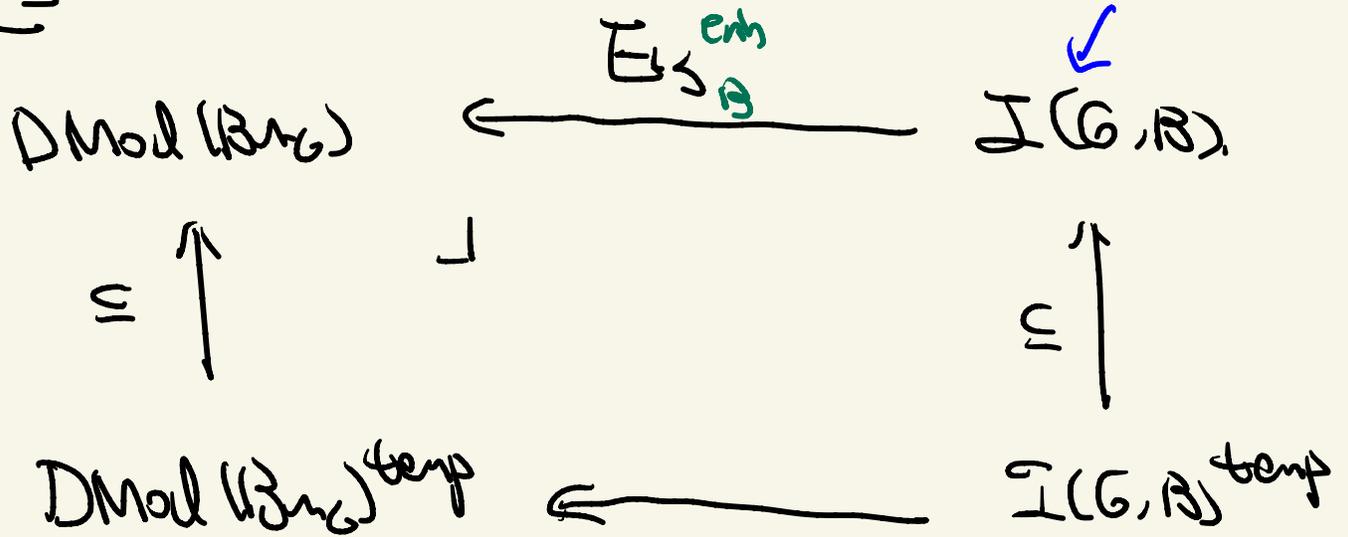
Rank:



$B_{u_6}^{\text{B-gen}}$ classifies G -torsors equipped with a generic B -reduction.

$D\text{Mod}(B_{\text{int}})$

Cor 3



\cong is a pushout diagram.

Goal: match each term and connecting functors in the cobordism diagram.

$$\text{DMod}(Bun_G)^{\text{temp}} \xrightarrow{\mathbb{L}_G^{\text{temp}}} \text{QCoh}(LS_G)$$

$$\text{DMod}(Bun_T) \xrightarrow{\mathbb{L}_T^{\text{temp}}} \text{QCoh}(LS_T)$$

↓ generates

↓ generates

want: $\Sigma(G, B) \xrightarrow{\quad} \text{IndCoh}((LS_G)^\wedge_{LS_T})$

sends temped to qcoh.

Method: local - global

Def:

$$\text{Sph}_G := \text{DMod}(\mathbb{Z}^+G \mid \mathbb{Z}G / \mathbb{Z}^+G)$$

$$\text{Sph}_M := \text{DMod}(\mathbb{Z}^+M \mid \mathbb{Z}M / \mathbb{Z}^+M)$$

generates (↙ ↗)

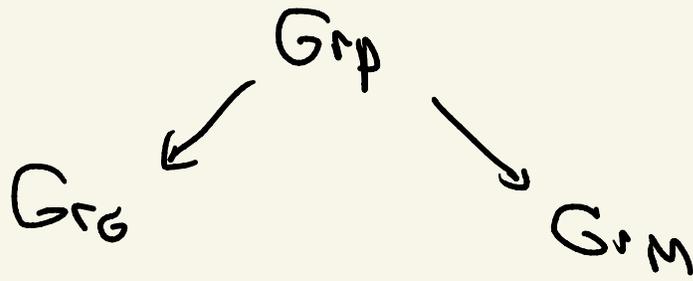
$$(\text{Sph}_{G^p} :=) \text{Sf}_p := \text{DMod}(\mathbb{Z}^+G \mid \mathbb{Z}G)^{\mathbb{Z}U \cdot \mathbb{Z}^+M}$$

"semi-infinite"

P : standard parabolic of G

$$1 \rightarrow U \rightarrow P \rightarrow M \rightarrow 1$$

$$M = \text{Levi}(P)$$



global

local

$\text{DMol}(\text{Bunps})$

SphM

$\downarrow \uparrow$

$\downarrow \uparrow$

$\mathcal{I}(G, \rho)$

$\text{Sph}_2 \curvearrowright \text{Sfp} \curvearrowleft \text{SphM}$

Remark: SphM, Sfp are factorization categories

(over $\mathcal{A}on$)

We will write

$$\text{SphM} := \text{SphM}, \mathcal{A}on$$

Lemma 4 \exists an equivalence

$$\underbrace{S\mathcal{I}_{P_{\text{ren}}}}_{\text{SphM, ren}} \otimes \text{DMod}(\text{Bun}_M) \xrightarrow{\cong} \mathcal{I}(G, \rho).$$

Construction of the functor:

$$G_{G, \text{ren}} \times \text{Bun}_M \xrightarrow{\pi} \text{Bun}_G^{\text{Augen}}$$

$$\left. \begin{array}{l} x: \text{closed points of } X \\ \mathcal{F}_M: M\text{-torsor on } X \\ \mathcal{F}_G: G\text{-torsor on } X \\ \mathcal{F}_G|_{X-x} \simeq G^M \times \mathcal{F}_M|_{X-x} \end{array} \right\} \mapsto \left. \begin{array}{l} \mathcal{F}_G \\ \text{P-structure} \\ \text{given by} \\ P_x^M \mathcal{F}_M \end{array} \right\}$$

$$\mathcal{F} \in S\mathcal{I}_{P, \text{ren}} \quad \mathcal{K} \in \text{DMod}(\text{Bun}_M)$$

$\pi_!(\mathcal{F} \hat{\otimes} \mathcal{K})$ is well-defined and

is contained in $\mathcal{I}(G, \rho)$

Thm 5

(Factorization derived Setake)

$$(i) \text{ Sph}_G^{\text{ren}} \cong \text{IndGh}^{\text{!,ren}} \left(\underset{\text{LS}_G^{\circ}(\mathbb{D})}{\text{LS}_G^{\circ}(\mathbb{D})} \times \text{LS}_G^{\circ}(\mathbb{D}) \right) \underset{\text{LS}_G^{\circ}(\mathbb{D})^{\wedge}}{\wedge}$$

as monoidal factorization category $\left. \right)$

$$G \leftarrow G \rightarrow G$$

$$\mathring{G} \leftarrow \mathring{G} \rightarrow \mathring{G}$$

$$\underset{\text{LS}_{G \times G}^{\circ}(\mathbb{D})}{\text{LS}_{G \times G}^{\circ}(\mathbb{D})} \times \underset{\text{LS}_{\mathring{G} \times \mathring{G}}^{\circ}(\mathbb{D})^{\wedge}}{\text{LS}_{\mathring{G} \times \mathring{G}}^{\circ}(\mathbb{D})^{\wedge}}$$

Ans: $\text{Sph}_G^{\text{ren}} := \text{Ind}(\text{Sph}_G^{\text{loc},c})$

$$\bullet \text{LS}_G^{\circ}(\mathbb{D}_x) \underset{\text{LS}_G^{\circ}(\mathbb{D}_x)}{\times} \text{LS}_G^{\circ}(\mathbb{D}_y) \cong (\text{pt} \times_{\mathring{G}} \text{pt}) / \mathring{G}$$

But the factorization version is of infinite type

$\text{IndGh}^{\text{!,ren}}$ is developed by Barvin

(and Fu. C.)

$$\bullet \text{LS}_G^{\circ}(\mathbb{D})^{\wedge} := (\text{LS}_G^{\circ}(\mathbb{D})) \underset{\text{LS}_G^{\circ}(\mathbb{D})}{\wedge}$$

(ii)

$$G \leftarrow P \rightarrow M$$

\exists an equivalence

$$\check{G} \leftarrow \check{P} \rightarrow \check{M}$$

$$\begin{array}{ccc}
 \text{Sph}_G^{\text{ren}} & \xrightarrow{\cong} & \text{IndGh}^{\text{!ren}} \left(\text{LS}_{\check{G} \times \check{M}}^{\check{G} \times \check{M}}(\mathbb{D}) \times \text{LSp}(\mathbb{D}^{\text{!}}) \right) \\
 \uparrow & & \uparrow \\
 \text{Sph}_G^{\text{ren}} & & \text{LS}_{\check{G} \times \check{M}}^{\check{G} \times \check{M}}(\mathbb{D})^{\text{!}}
 \end{array}$$

Compatible with actions in (i)

$$\text{LS}_{\check{G}}(\mathbb{D}) \cong \text{Conn}_{\check{G}}(\mathbb{D}) / \text{gauge}_{\mathbb{R}^+ \check{G}}$$

$\text{Conn}_{\check{G}}(\mathbb{D})$ $\mathbb{R}^+ \check{G}$ pro-finite type.

$$\text{IndGh}^{\text{!}}(\text{Conn}_{\check{G}}(\mathbb{D})) := \text{Ind}(\text{Coh}(\text{Conn}_{\check{G}}(\mathbb{D})))$$

$$\text{IndGh}^{\text{!}}(\text{Conn}_{\check{G}}(\mathbb{D}) / \mathbb{R}^+ \check{G})$$

$$\cong \text{Tot} \left(\text{IndGh}^{\text{!}}(\text{Conn}_{\check{G}}(\mathbb{D})) \rightrightarrows \text{IndGh}^{\text{!}}(\mathbb{R}^+ \check{G} \times \text{Conn}_{\check{G}}(\mathbb{D})) \right. \\ \left. \dots \right)$$

$\text{Coh}(\text{Conn}_G^{\text{red}}(D) / \mathbb{R}^+G)$

$$\subseteq \text{Ind}G^{\text{red}}$$

subset set pullback to $\text{Conn}_G^{\text{red}}(D)$ is
coherent.

$$\text{Ind}G^{\text{red}} := \text{Ind}(\text{Coh}(\quad))$$

Write the theorem as

$$\text{Sph}_G^{\text{red}} \cong \text{Sph}_{\text{ac}}^{\text{red}}$$

$$\text{SI}_P^{\text{red}} \cong \text{SI}_P^{\text{red}}$$

Lemma 6

$$\text{SI}_P^{\text{red}} \otimes_{\text{Sph}_M^{\text{red}}} \mathcal{O}\text{Coh}(LS_M) \cong \text{IndEho}(LS_G^{\text{red}})_{LS_P^{\text{red}}}$$

$$SI_p^{\text{ren}} \otimes_{SphM^{\text{ren}}} DMod(R_{\text{univ}}) \xrightarrow{\sim} \mathcal{I}(G, p)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ SC_p^{\text{ren}} \otimes_{SphM^{\text{ren}}} QGh(L_{\text{univ}}) & \simeq & IndGho((LS_{\mathbb{G}})_{LS_{\mathbb{G}}}) \end{array}$$

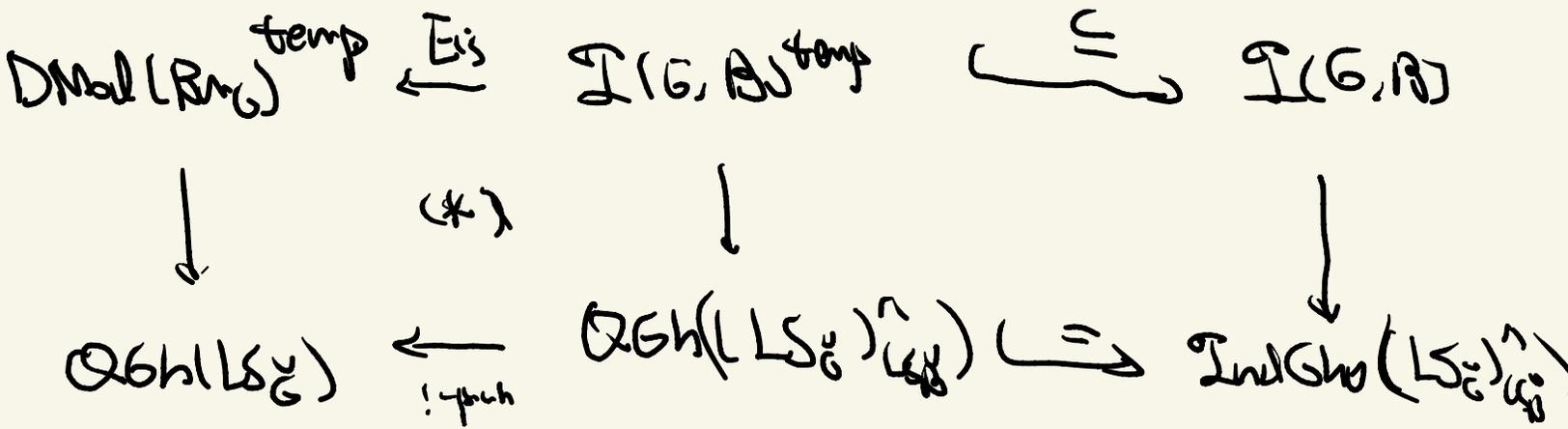
(Need coeff_M^{enh} : DMod(R_{univ}) → QGh(L_{univ})
 is compatible with derived Satake)

Lem 7 :

$$\mathcal{I}(G, p) \longrightarrow IndGho((LS_{\mathbb{G}})_{LS_{\mathbb{G}}})$$

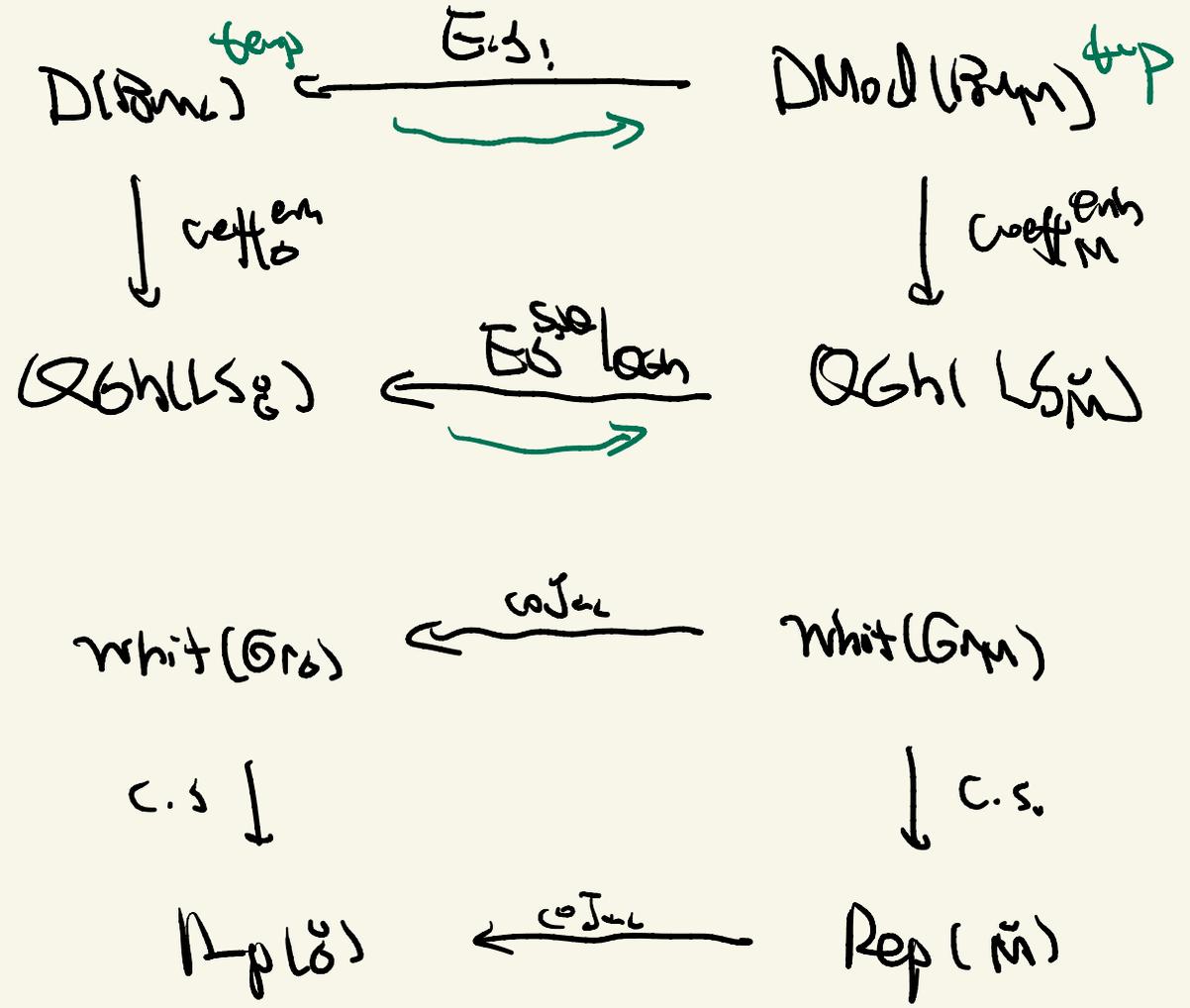
sends tempered objects to QGh .

$$\text{s.s. rank}(G) = \text{s.s. rank}(\check{G}) = 1$$



Prop 8: (*) commutes.

(Idea:



Conj:

$$D(\mathbb{R}_2)_{\text{top}} \xrightarrow{CT} D\text{Mod}(\mathbb{R}_2)_{\text{top}}$$

\downarrow coeff₀

$$\mathcal{R}Gh(\mathbb{K}_2) \xrightarrow{CT} \mathcal{R}Gh(\mathbb{L}_2)$$

\downarrow

This is harder because it is not true
on $D\text{Mod}(\mathbb{R}_2)$

$$\mathcal{I}(G, p)^{p'-\text{top}}$$

$$p \subset p'$$

$$w(G, p \subset p')$$

$$w(G, p = p) \xrightarrow[\text{Conj}]{\cong} \text{Whit}_{G,p}$$

$$D(\mathbb{R}_2) = \lim_{\substack{\text{center in } p \\ \text{cover in } p'}} w(G, p \subset p')$$

$$\xrightarrow{\cong} \text{Conj} \lim w(G, p = p)$$

↳ Caroline Whitgip