

Multiplication kernels

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Abstract

We introduce the notion of multiplication kernels of birational and D -module type and give various examples. We also introduce the notion of a semi-classical multiplication kernel associated with an integrable system and discuss its quantization. Finally, we discuss geometric and algebraic aspects of method of separation of variables, and describe hypothetically a cyclic D -module for the generalized multiplication kernels for Hitchin systems for groups GL_r .

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 4 |
| 1.1 | Informal explanation of the problem | 4 |
| 1.2 | Formal setups for functions and integrations | 6 |
| 1.3 | Multiplication formulas for special functions | 7 |
| 1.4 | Relations to Langlands correspondence and integrable systems | 8 |
| 1.5 | Multiplication kernels in other contexts | 10 |
| 1.6 | Other questions for functional analogue of tensor algebra | 11 |
| 1.7 | Content of the paper | 12 |
| 2 | Multiplication kernels of birational type | 13 |
| 2.1 | Formulation of the problem | 13 |
| 2.2 | One dimensional examples without auxiliary integration | 15 |
| 2.3 | One dimensional example with auxiliary integration | 18 |
| 2.4 | A hypothetical example of a generalized product | 20 |
| 2.5 | Some examples with 3 inputs and one output | 23 |
| 2.6 | On S_n -covariance of higher compositions of kernels | 24 |
| 2.7 | Direct images of cyclic D -modules at a generic point | 26 |
| 3 | Semi-classical kernels and their quantization | 27 |
| 3.1 | Semi-classical kernels associated with classical integrable systems | 27 |
| 3.2 | Quantization of semi-classical kernels and quantum integrable systems | 29 |
| 3.3 | Example corresponding to Hitchin systems for rank 2 bundles on the projective line with 4 regular singular points | 32 |
| 3.4 | Example corresponding to Hitchin systems for rank 2 bundles on the projective line with more than 4 regular singular points | 35 |
| 4 | Separation of variables | 38 |
| 4.1 | Families of functions in one variable, and generalized products | 38 |
| 4.2 | Semi-classical generalized products | 40 |
| 4.3 | Towards quantized generalized product | 42 |
| 4.4 | A generalization to other Poisson surfaces | 44 |

| | | |
|----------|--|-----------|
| 5 | Multiplication kernels associated with differential operators | 45 |
| 5.1 | General setup | 46 |
| 5.2 | A kernel associated with first order differential operators | 47 |
| 5.3 | The case of second order differential operators with 4 regular singular points . . | 49 |
| 5.4 | The case of second order differential operators with more than 4 regular singular points | 52 |
| 5.5 | The case of third order differential operators with 3 regular singular points . . . | 53 |
| | Acknowledgements | 55 |
| | References | 55 |

1 Introduction

1.1 Informal explanation of the problem

Suppose we have a collection of commuting linear operators¹ T_α acting on a finite dimensional vector space V . Assume that the joint spectrum of these operators is simple. Then we have a basis of V (of joint eigenvectors of T_α) defined up to permutation and rescaling. Assume furthermore that we choose a vector $v \in V$ which is cyclic, i.e. generates V as the module over the algebra generated by T_α . Then the basis of eigenvectors e_i is defined only up to permutations if we put the constraint $v = \sum_i e_i$ i.e. v has coordinates $(1, 1, \dots, 1)$. Alternatively, one may assume that a cyclic covector $u \in V^*$ is given and normalize basic elements by the condition $(u, e_i) = 1$.

The basis $\{e_i\}$ up to permutations can be encoded by the structure of a *commutative associative unital algebra* on V with the multiplication $e_i \cdot e_j = \delta_{i,j} e_i$ where $\delta_{i,j}$ is the Kronecker delta.

In this paper we deal with a functional analogue of this situation where vector space V is a space of functions (in a broad sense) in one or several variables, or more generally, V is a space of functions on a smooth or algebraic manifold. The typical example is $V = C^\infty(\mathbb{R}^n)$ where the commuting operators are derivations $\frac{\partial}{\partial x_k}$, $k = 1, \dots, n$. The continuous analogue of joint eigenvectors consists of Fourier modes $e_\lambda(x) = e^{ix \cdot \lambda}$, $\lambda \in \mathbb{R}^n$ where the normalization is given by $e_\lambda(0) = 1$. The multiplication is given by $e_\lambda(x) * e_\mu(x) = \delta(\lambda - \mu) e_\lambda(x)$ where $\delta(\lambda - \mu)$ is the Dirac delta². Notice that this multiplication is the additive convolution and can be written in terms of the standard multiplication of functions as

$$f * g(y) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta(x_1 + x_2 - y) f(x_1) g(x_2) dx_1 dx_2.$$

Let \mathcal{F} be a vector space of functions on a manifold X . We denote a typical element of \mathcal{F} by $f(x) \in \mathcal{F}$, where $x \in X$.

We want to study commutative associative multiplications $\mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}$ on the vector space \mathcal{F} . We write such multiplication in the form

$$f * g(y) = \int_{X \times X} K(x_1, x_2, y) f(x_1) g(x_2) dx_1 dx_2 \tag{1.1}$$

where $K(x_1, x_2, y)$ is a kernel of our multiplication and dx is a measure on X defined by a volume form. Commutativity of our multiplication means

$$K(x_1, x_2, y) = K(x_2, x_1, y) \tag{1.2}$$

¹A lot of interesting examples of such collections appears in the theory of integrable systems and in representation theory.

²We denote our multiplication in functional case by $*$ in order to distinguish it from usual pointwise product of functions.

and associativity of our multiplication means

$$\int_X K(x_1, x_2, y)K(y, x_3, z)dy = \int_X K(x_1, x_3, y)K(y, x_2, z)dy \quad (1.3)$$

In which way one can verify the associativity condition (1.3) if the kernel K is given in a closed form (for example, in terms of elementary functions)? In principle, one can suggest the following possibilities:

1. Compute explicitly the l.h.s. and the r.h.s. of the equation (1.3).
2. Find a change of variables $y \mapsto \tilde{y}$ which transforms the l.h.s. of the equation

$$K(x_1, x_2, y)K(y, x_3, z) dy = K(x_1, x_3, \tilde{y})K(\tilde{y}, x_2, z) d\tilde{y}$$

to its r.h.s. In this case the equation (1.3) also holds provided that X is a cycle.

3. Prove that the l.h.s and the r.h.s. of the equation (1.3) satisfies the same holonomic system of differential equations as a function in x_1, x_2, x_3, z . Strictly speaking, this does not mean that the equation (1.3) holds on the nose, but still can be considered as an associativity condition for a kernel K .

In this paper we do not deal with analysis as the possibility 1 suggests, and concentrate on algebraic side of the problem. For example, assuming that X is a small circle in \mathbb{C} around zero, when exploring possibility 1 we can reduce the integration to an algebraic operations if $K(x_1, x_2, y) \in \mathbb{C}[\frac{1}{y}][[x_1, x_2]]$, i.e. K is a power series in x_1, x_2 with coefficients polynomial in $\frac{1}{y}$.

Exploring the possibility 2 we assume that X is an algebraic variety and a change of variables $y \mapsto \tilde{y}$ defines a birational mapping $X \rightarrow X$.

Finally, in the possibility 3 we assume that X is a cycle and, therefore $\int_X \frac{\partial h}{\partial q} dq = 0$ for any h . This assumption reduces computations to algebraic manipulations in differential algebra.

More generally, the kernel K can be given by integration of an “elementary function” over some auxiliary variables. Namely, let Q be another manifold with a measure dq defined by a volume form. Assume that

$$K(x_1, x_2, y) = \int_Q K(x_1, x_2, y, q)dq. \quad (1.4)$$

In this case associativity condition takes a form

$$\int_{X \times Q \times Q} K(x_1, x_2, y, q_1)K(y, x_3, z, q_2)dydq_1dq_2 = \int_{X \times Q \times Q} K(x_1, x_3, y, q_3)K(y, x_2, z, q_4)dydq_3dq_4. \quad (1.5)$$

Finally, let X, Q both be algebraic varieties over \mathbb{C} . In this case we assume that

$$K = K_1^{s_1} \dots K_l^{s_l} \quad (1.6)$$

where $s_1, \dots, s_l \in \mathbb{C}$ are arbitrary parameters and K_1, \dots, K_l are either algebraic functions or exponential of algebraic functions³. These lead to the following

Definition 1.1.1. We say that $K(x_1, x_2, y, q)$ is a multiplication kernel of birational type if there exists a birational automorphism $(y, q_1, q_2) \rightarrow (\tilde{y}, \tilde{q}_3, \tilde{q}_4)$ of $X \times Q \times Q$ which transforms the l.h.s. of the equation

$$K(x_1, x_2, y, q_1)K(y, x_3, z, q_2)dydq_1dq_2 = K(x_1, x_3, \tilde{y}, \tilde{q}_3)K(\tilde{y}, x_2, z, \tilde{q}_4)d\tilde{y}d\tilde{q}_3d\tilde{q}_4$$

to its r.h.s. If $K = K_1^{s_1} \dots K_l^{s_l}$, then this birational automorphism should not depend on s_1, \dots, s_l .

The kernel $K(x_1, x_2, y)$ given by (1.4), (1.6) satisfies a holonomic system of differential equations in x_1, x_2, y i.e. it gives a holonomic D -module endowed with a cyclic vector. The associativity constraint (1.3) can be understood as an isomorphism between two holonomic D -modules in x_1, x_2, x_3, y with cyclic vectors. The operation of integration corresponds to the direct image of D -modules.

Definition 1.1.2. In the situation as above, we say that $K(x_1, x_2, y)$ is a multiplication kernel of D -module type.

Remark 1.1.1. The associativity conditions from Definition 1.1.2 can be formulated less abstractly as follows. Assume that $K(x_1, x_2, y)$ is a solution of a holonomic system of differential equations⁴ in x_1, x_2, y . Let I^{12} be the left ideal in the ring of differential operators in x_1, x_2, y, x_3, z generated by differential equations of $K(x_1, x_2, y)$. Let I^{23} be the left ideal in the same ring generated by differential equations of $K(y, x_3, z)$. It is clear that I^{23} is obtained from I^{12} by the change of variables $(x_1, x_2, y) \rightarrow (y, x_3, z)$. Define a left ideal I^{123} in the ring of differential operators in x_1, x_2, x_3, z by

$$I^{123} = (I^{12} + I^{23} + J) \cap R_{123}$$

where J is the *right* ideal generated by $\frac{\partial}{\partial y}$ and R_{123} is the ring of differential operators in x_1, x_2, x_3, z . By construction, I^{123} consists of differential equations for

$$\int_{X \times Q \times Q} K(x_1, x_2, y, q_1)K(y, x_3, z, q_2)dydq_1dq_2$$

where $X \times Q \times Q$ is a cycle.

Definition 1.1.2'. We say that $K(x_1, x_2, y)$ is a multiplication kernel of D -module type if I^{123} is invariant with respect to interchanging of x_2 and x_3 .

1.2 Formal setups for functions and integrations

In the previous informal definitions of multiplication kernel we use the notions of function, integration etc in non-rigorous way. There are various rigorous formalisms for the notions of

³In the case of general ground field we replace $K_i^{s_i}$ by $\chi_i(K_i)$ where χ_i are either multiplicative or additive characters of the ground field.

⁴We understand integral in (1.4) as direct image which means that our system of differential equations for $K(x_1, x_2, y)$ is a consequence of a holonomic system of differential equations for $K(x_1, x_2, y, q)$.

“explicit formulas” and “function” for an algebraic variety X defined over a field k , none of which is totally satisfactory.

1) If k is a local field (i.e. $k = \mathbb{R}, \mathbb{C}$, or a finite extension of \mathbb{Q}_p or $\mathbb{F}_p((t))$) and $\pi : Y \rightarrow X$ is a family of n -dimensional varieties over X , endowed with the volume element $vol \in \Gamma(Y, K_{Y/X})$ along fibers of π . Here $K_{Y/X} = \Lambda^n T_{Y/X}^*$ is the relative canonical line bundle. Then we obtain an \mathbb{R} -valued function on the set $X(k)$ given by $\pi_*(|vol|)$, the integration of the density $|vol|$ along fibers. Here we assume that the integral is convergent, at least for the generic point $x \in X(k)$.

One can twist the integral by additive and multiplicative characters applied to rational functions on Y (this is a formal replacement of exponentials and fractional powers). Moreover, it is enough to assume more generally that for some integer $N \geq 1$ we have an element of $\Gamma(Y, K_{Y/X}^{\otimes N})$, e.g. $vol^{\otimes N}$ if vol is defined up to multiplication by N -th root of 1.

This definition has several drawbacks. First, the same function on $X(k)$ can be presented as an integral in different ways, and it is not clear whether the equality always follows from a birational equivalence. Second, one expects that some interesting functions (for example in representation theory) do not have such an integral representation (see [6] for discussion). Third, there is a problem with S_n -covariance for n -fold compositions of multiplication kernels, see Section 2.6 for details.

2) If $char(k) = 0$, we can encode a “function” by a holonomic D -module endowed with a cyclic vector, see Section 2.7 for details. The drawback of this definition is that we can not distinguish functions f and cf where c is a non-zero constant.

3) We can forget the cyclic vector in the previous definition, and encode a function by an *equivalence class* of holonomic D -modules.

3') The previous definition can be transported to arbitrary characteristic, with holonomic D -modules replaced by motivic constructible sheaves.

4) Finally, if we are in the case of positive characteristic $p > 0$ and $k = \mathbb{F}_{p^n}$, then one can associate with motivic constructible sheaf a $\overline{\mathbb{Q}}^{CM}$ valued function⁵ on the finite set $X(\mathbb{F}_{p^n})$ given by the trace of Frobenius. Surprisingly, here we have again a well-defined function (as in 1)) although the information on the cyclic vector seems to be lost.

In all formalisms above one can speak about integrals (as direct images) and hence the associativity constraint (1.3) makes sense. Therefore, one can speak about multiplication kernels in different contexts.

1.3 Multiplication formulas for special functions

Let us discuss a dual viewpoint on multiplication kernels. The product $*$ defined by (1.3) on the space \mathcal{F} of functions gives by duality a coproduct Δ on the dual space \mathcal{F}^* of densities. The continuous basis $e_\lambda(x)$ of elementary projectors for $*$ gives the dual basis $e_\lambda^*(x)dx$ of \mathcal{F}^* . The

⁵Here $\overline{\mathbb{Q}}^{CM} \subset \overline{\mathbb{Q}}$ stands for the maximal totally real extension of \mathbb{Q} with added $\sqrt{-1}$. This notation comes from theory of complex multiplication for abelian varieties.

property $\Delta(e_\lambda^*(y)dy) = e_\lambda^*(x_1)dx_1 \otimes e_\lambda^*(x_2)dx_2$ is equivalent to⁶

$$e_\lambda^*(x_1)e_\lambda^*(x_2) = \int K(x_1, x_2, y)e_\lambda^*(y)dy$$

for all λ , where $K(x_1, x_2, y)$ is the same kernel as in (1.3) and, in particular, does not depend on λ .

Let us give several examples where M is one-dimensional and eigenfunctions $e_\lambda(x)$ are classical special functions. In these examples T_α consists of one differential operator T .

Example 1.3.1. $M = \mathbb{R}$, $T = \frac{d}{dx}$, $e_\lambda(x) = e^{\lambda x}$. We have $Te_\lambda(x) = \lambda e_\lambda(x)$, the normalization is defined by $e_\lambda(0) = 1$, and

$$e_\lambda(x_1)e_\lambda(x_2) = e_\lambda(x_1 + x_2) = \int_{\mathbb{R}} \delta(x_1 + x_2 - y)e_\lambda(y)dy.$$

Example 1.3.2. $M = \mathbb{R}_{>0}$, $T = x\frac{d}{dx}$, $e_\lambda(x) = x^\lambda$. We have $Te_\lambda(x) = \lambda e_\lambda(x)$, the normalization is defined by $e_\lambda(1) = 1$, and

$$e_\lambda(x_1)e_\lambda(x_2) = e_\lambda(x_1x_2) = \int_{\mathbb{R}} \delta(x_1x_2 - y)e_\lambda(y)dy.$$

Example 1.3.3. $M = \mathbb{R}_{>0}$, $T = \frac{d^2}{dx^2} + x^{-1}\frac{d}{dx}$, $e_\lambda(x) = J_0(\lambda x)$. We have $Te_\lambda(x) = -\lambda^2 e_\lambda(x)$, the normalization is defined by $\lim_{x \rightarrow 0} e_\lambda(x) = 1$, and

$$e_\lambda(x_1)e_\lambda(x_2) = \int_{|x_2-x_1|}^{x_2+x_1} \frac{e_\lambda(y)}{A(x_1, x_2, y)} \frac{ydy}{2\pi}.$$

Here

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!^2} \left(\frac{x}{2}\right)^{2m} = \frac{1}{2\pi i} \oint e^{x\frac{u-u^{-1}}{2}} \frac{du}{u}$$

is Bessel function and

$$A(x_1, x_2, y) = \frac{1}{4}(2x_1^2x_2^2 + 2x_1^2y^2 + 2x_2^2y^2 - x_1^4 - x_2^4 - y^4)^{\frac{1}{2}}$$

is the area of triangle with sides of the length x_1, x_2, y . This multiplication formula is called Sonine–Gegenbauer formula [4, 5].

1.4 Relations to Langlands correspondence and integrable systems

Theory of automorphic forms provides examples of commuting operators (Hecke operators). In the case of a curve C over finite field \mathbb{F}_q these operators act on the space of functions on the

⁶In the examples below we make an identification between \mathcal{F} and \mathcal{F}^* and write this formula in terms of e_λ .

countable set of isomorphism classes of G -bundles on C where G is a reductive group. In the case $G = GL_2$ there are multiplicity one theorems which guarantee that the joint spectrum is simple. In the old paper [7] of one of us, the multiplication kernel was written explicitly in a special case of rank 2 bundles on \mathbb{P}^1 with the parabolic structure in 4 points. Also, in the same paper, a kernel for the case of local field k was given by the formula

$$f * g(y) = \int_{y \in k, F_t(x, y, z) \in (k^*)^2} \frac{f(x_1)g(x_2)}{|F_t(x_1, x_2, y)|^{\frac{1}{2}}} |dy|$$

where

$$F_t(x_1, x_2, y) = (x_1x_2 + x_1y + x_2y - t)^2 + 4x_1x_2y(1 + t - x_1 - x_2 - y).$$

Recently, in the paper [1], Hecke operators in the case of curves over \mathbb{C} were defined (but not yet the multiplication kernels).

The joint spectrum of commuting integral Hecke operators for the case of curves over local fields is rather mysterious, and its relation to the usual Langlands program is quite unclear. In the case of a non-archimedean field k with the residue field \mathbb{F}_p , a finite “low frequency” part of the spectrum is presumably the same as the spectrum for the case of curves over finite fields and hence is related to Galois representations. For $k = \mathbb{C}$ the joint spectrum is expected to coincide [1] with the set of opers (roughly speaking, differential equations of rank r) with real monodromy. Similarly, for $k = \mathbb{R}$ the joint spectrum is expected to be the spectrum of the algebra of commuting differential operators on the set of \mathbb{R} -points of algebraic variety Bun_G , coming from the quantization of Hitchin integrable system.

In general, if H_1, \dots, H_n are commuting differential operators on n -dimensional manifold, then for any scalar parameters $\lambda_1, \dots, \lambda_n$ we have a holonomic system

$$(H_i - \lambda_i)\psi(x) = 0, \quad i = 1, \dots, n.$$

Let us denote $\psi_\lambda(x)$ a solution of this system where $\lambda = (\lambda_1, \dots, \lambda_n)$. In order to have unique solution one has to impose some normalization conditions. See Section 4 for details.

In the case $G = GL_r$ and arbitrary constraints at singularities, there exists a remarkable birational symplectomorphism between the phase space of Hitchin integrable systems and the cotangent bundle to $Sym^g C$ where g is the genus of the generic spectral curve or, equivalently, the dimension of the base of the integrable system. This construction is called the method of separation of variables [11]. It is expected that in the case $G = GL_r$ there exists an integral operator given by an explicit kernel, identifying functions on Bun_G and on $Sym^g C$. Moreover, eigenfunctions of commuting differential operators on Bun_G (or of Hecke operators) map to symmetric functions on $Sym^g C$ of the form $\phi_\lambda(x_1) \dots \phi_\lambda(x_g)$ which are external powers of functions in one variable.

In this presentation the multiplication kernel can be expressed by subsequent integration in terms of a more elementary kernel which we denote by $K_{g+1, g}$, which is a function of $2g + 1$ variables whereas the original multiplication kernel is a function of $3g$ variables. We will discuss in details this approach in Section 4.

1.5 Multiplication kernels in other contexts

The first group of questions we want to discuss here is related to commuting families of Hecke operators in theory of modular forms. The multiplicity one theorems in the theory of automorphic forms for group GL_r are valid not only for the case of curves over finite field, but also in the number field case. This leads e.g. to the following question concerning classical modular forms for group $SL(2, \mathbb{Z})$. For any $n \geq 1$ there are d_n Hecke eigenforms

$$f_i^{(n)}(q) = q + \sum_{j>1} a_{i,j}^{(n)} q^j$$

of weight n and level 1, where $a_{i,j}^{(n)}$, $i = 1, \dots, d_n$ are eigenvalues of Hecke operator $H_j^{(n)}$ and d_n is the dimension of the space of cusp forms of weight n . Coefficients of these forms are algebraic integers, not necessarily rational. Let us consider the generating series in 4 variables

$$K = \sum_{n \geq 1} t^n \sum_{i=1}^{d_n} f_i^{(n)}(q_1) f_i^{(n)}(q_2) f_i^{(n)}(q_3) \in \overline{\mathbb{Q}}[[q_1, q_2, q_3, t]].$$

One can show that $K \in \mathbb{Z}[[q_1, q_2, q_3, t]]$. We can also write K in terms of traces of products of Hecke operators as

$$K = \sum_{n, j_1, j_2, j_3 \geq 1} \text{tr}(H_{j_1}^{(n)} H_{j_2}^{(n)} H_{j_3}^{(n)}) q_1^{j_1} q_2^{j_2} q_3^{j_3} t^n.$$

It will be interesting to find a closed formula for K . A similar question can be asked about higher level modular forms and about Maass forms.

In order to explain the second group of questions, we start with an example. Let $X = [0, 1]$ be the unit interval in \mathbb{R} , and $K : X^3 \rightarrow \{0, 1\}$ be the characteristic function of the closed tetrahedron with vertices $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$.

$$K(x, y, z) = \begin{cases} 1, & \text{if } x \leq y + z, y \leq x + z, z \leq x + y, x + y + z \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the following is true: for any integer $n \geq 1$ introduce vector space

$$A_n = \mathbb{Q}^{X \cap \frac{1}{n}\mathbb{Z}} = \mathbb{Q}^{\{0, \frac{1}{n}, \dots, 1\}} = \mathbb{Q}^{n+1}$$

with the basis $e_x, x \in X \cap \frac{1}{n}\mathbb{Z}$. Define a product in A_n by

$$e_{x_1} \cdot e_{x_2} = \sum_{x_3 \in X \cap \frac{1}{n}\mathbb{Z}} K(x_1, x_2, x_3) e_{x_3}.$$

Then this product is commutative and associative. In fact, it is Verlinde algebra for sl_2 at level n .

The proof of associativity can be made independent of n . Namely, it follows from the existence of a piecewise-linear identification with integer coefficients (\mathbb{Z} PL in short) of two 5-dimensional polytopes fibered over X^4 . Let

$$P_1 = \{(x_1, x_2, x_3, x_4, y); (x_1, x_2, y) \in K, (y, x_3, x_4) \in K\},$$

$$P_2 = \{(x_1, x_2, x_3, x_4, y); (x_1, x_3, y) \in K, (y, x_2, x_4) \in K\}.$$

Define two maps $\pi_i : P_i \rightarrow X^4$, $i = 1, 2$ by $\pi_i(x_1, x_2, x_3, x_4, y) = (x_1, x_2, x_3, x_4)$. One can check that for all x_1, x_2, x_3, x_4 the fibers $\pi_1^{-1}(x_1, x_2, x_3, x_4)$ and $\pi_2^{-1}(x_1, x_2, x_3, x_4)$ are closed intervals of the same length, and can be identified by a shift. The resulting map $P_1 \rightarrow P_2$ is a \mathbb{Z} PL homeomorphism. This argument is similar to the cut-and-paste proof of associativity in the case of multiplication kernels for varieties over finite field studied in [7].

This example leads to several questions:

1. Generalize it to the case of other reductive groups,
2. Find other \mathbb{Z} PL examples of multiplication kernels,
3. Find the relation with multiplication kernels given by integral operators over non-archimedean fields,
4. Find similar formulas for multiplication kernels where numbers of integer points in polytopes is replaced by volumes of polytopes.

1.6 Other questions for functional analogue of tensor algebra

In this paper we study explicit associative commutative kernels using purely algebraic framework (see discussion after associativity condition (1.3)). This algebraic approach can be applied to other problems in functional analogue of tensor algebra.

1. Let $\mathcal{F}_1, \mathcal{F}_2$ be two spaces of functions, possibly on different manifolds. One can study explicit kernels for mappings

$$R : \mathcal{F}_1 \otimes \mathcal{F}_2 \rightarrow \mathcal{F}_2$$

subject to constraint $R(f_1, R(f_2, g)) = R(f_2, R(f_1, g))$ where $f_1, f_2 \in \mathcal{F}_1$, $g \in \mathcal{F}_2$. This condition means that all linear operators on \mathcal{F}_2 of the form $g \mapsto R(f, g)$ commute. Note that associative commutative kernels provide examples of this structure in the case $\mathcal{F}_1 = \mathcal{F}_2$ because operators of multiplication by a given element commute in commutative associative algebras.

2. Let $e_\lambda(x)$ be the set of joint eigenfunctions of a family of commuting operators on a space of functions \mathcal{F} . Here we do not need to choose a normalization. Define a mapping

$$R : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$$

by $R(e_\lambda(x) \otimes e_\mu(x)) = \delta(\lambda - \mu)e_\lambda(x) \otimes e_\lambda(x)$. One can study explicit kernels for this mapping for some interesting classes of commuting operators. Notice that R also satisfies to an analogue of associativity and commutativity constraints: R is invariant with respect to the action of

$S_2 \times S_2$ and the composition $R^{13} \circ R^{23}$ is invariant with respect to the action of $S_3 \times S_3$. A hypothetical example of such structure is given in Remark 3.3.2.

3. Given $g \geq 1$ one can study kernels for mapping

$$R: \mathcal{F}^{\otimes(g+1)} \rightarrow \mathcal{F}^{\otimes g}$$

invariant with respect to the action of $S_{g+1} \times S_g$ such that $R \circ (R \otimes Id_{\mathcal{F}})$ is invariant with respect to the action of $S_{g+2} \times S_g$. We call this structure a *generalized product*. Such a product induces a structure of an associative commutative algebra on the space $Sym^g \mathcal{F}$ (see Proposition 4.1.1).

4. One can study kernels for associative but not necessarily commutative multiplications.

5. Let $\mathcal{F}_1, \mathcal{F}_2$ be two spaces of functions. One can study kernels for two inverse linear mappings $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ and $\mathcal{F}_2 \rightarrow \mathcal{F}_1$. This question can be thought of as a fundamental question of integral geometry in the sense of Gelfand-Gindikin-Graev [2].

1.7 Content of the paper

In Section 2 we define the notion of multiplication kernel of birational type in more rigorous way and give examples. The most part of examples are related to the Hitchin systems for group GL_2 on the curve \mathbb{P}^1 with 4 or more regular singular points. Another way of constructing examples is to solve certain functional equations which is explained in Remark 2.2.2. Notice that Sections 2.2 - 2.5 can be read independently of other parts of the paper provided that the reader is fine with informal explanation of Definition 1.1.1 from Section 1.1.

In Section 3 we explain that classical integrable systems give commutative monoids in the Weinstein category of symplectic varieties and Lagrangian correspondences. We also discuss quantization which is a construction of a multiplication kernel of D -module type starting from a quantum integrable system. Our examples are again related with Hitchin systems. It seems to be an interesting and important problem to find explicitly multiplication kernels for various quantum integrable systems.

In Section 4 we describe semi-classical geometry of Sklyanin's method of separation of variables. The algebraic counterpart of this method is the notion of a generalized product, a map $Sym^{g+1}V \rightarrow Sym^gV$ satisfying an analog of associativity constraint. We also introduce a hypothetical construction of quantum generalized products via formal solutions of differential equations expanded in a chosen base point. All considerations in Section 4 can be generalized to trigonometric and elliptic difference equations which is beyond the standard geometric Langlands perspective.

In Section 5 we explain, and illustrate by examples, how to construct multiplication kernels satisfying the property (1.3) with M is a small circle $|z| = \varepsilon$, $0 < \varepsilon \ll 1$ starting with a differential operator. These kernels can also be considered as lifts of more abstract kernels from Section 4. Notice that Section 5 can be read independently of other parts of the paper.

2 Multiplication kernels of birational type

2.1 Formulation of the problem

Let us fix a ground field k of characteristic zero, and a commutative algebraic group⁷ A over k .

Consider the following category $\mathcal{C} = \mathcal{C}_{k,A}$. Its objects are smooth equidimensional varieties over k . The set of morphisms $\text{Hom}_{\mathcal{C}}(X_1, X_2)$ is defined as the set of equivalence classes of tuples $(Z, \pi_1 : Z \rightarrow X_1, \pi_2 : Z \rightarrow X_2, \text{vol}, \rho : Z \rightarrow A)$ where Z is an equidimensional smooth variety over k , π_1, π_2 are smooth morphisms (submersions), $\text{vol} \in \Gamma(Z, K_{Z/X_2})/(\pm 1)$ is a volume element along fibers of π_2 up to a sign⁸, and $\rho : Z \rightarrow A$ is an arbitrary map of varieties. The equivalence relation is generated by identifications of tuples

$$(Z, \pi_1, \pi_2, \text{vol}, \rho) \sim (U, \pi_1|_U, \pi_2|_U, \text{vol}|_U, \rho|_U) \quad (2.7)$$

for fixed X_1, X_2 where $U \subset Z$ is a Zariski open dense subvariety of Z .

Remark 2.1.1. If the ground field k is a local field, then the morphisms in \mathcal{C} can be thought of as formal integral operators depending on a generic character of locally compact abelian group $A(k)$. Namely, for a variety X/k denote by \mathcal{F}_X the space of \mathbb{C} -valued continuous functions on $X(k)$. A tuple $(Z, \pi_1 : Z \rightarrow X_1, \pi_2 : Z \rightarrow X_2, \text{vol}, \rho : Z \rightarrow A)$ as above gives a formal integral operator $K : \mathcal{F}_{\mathbb{S}_{\infty}} \rightarrow \mathcal{F}_{X_{\epsilon}}$ given by

$$K(f)(x_2) = \int_{z \in \pi_2^{-1}(x_2)(k)} f(\pi_2(z)) \cdot \chi(\rho(z)) \cdot |\text{vol}|_{\pi_2^{-1}(x_2)}$$

where $\chi : A(k) \rightarrow \mathbb{C}^*$ is a character. We ignore the convergence issues here. This heuristics explains the following definition of the composition in \mathcal{C} .

For two tuples

$$\begin{aligned} &(Z, \pi_1 : Z \rightarrow X_1, \pi_2 : Z \rightarrow X_2, \text{vol}, \rho : Z \rightarrow A), \\ &(Z', \pi'_1 : Z' \rightarrow X_2, \pi'_2 : Z' \rightarrow X_3, \text{vol}', \rho' : Z' \rightarrow A) \end{aligned}$$

their composition is given by the fibered product

$$Z'' = Z \times_{X_2} Z'$$

endowed with maps $\pi''_1 = \pi_1 \circ \pi_2^*(\pi'_1) : Z'' \rightarrow X_1$, $\pi''_3 = \pi'_2 \circ (\pi'_1)^*(\pi_2) : Z'' \rightarrow X_3$.

⁷In our examples A is the product of the additive group scheme \mathbb{G}_a and of several copies of the multiplicative group scheme \mathbb{G}_m .

⁸More generally, one can modify the definition by replacing $\text{vol} \in \Gamma(X, K_{Z/X_2})$ by its power $\text{vol}^{\otimes N} \in \Gamma(X, K_{Z/X_2}^{\otimes N})$ for $N \geq 1$.

This can be represented by the following commuting diagram:

$$\begin{array}{ccccc}
 & & Z'' & & \\
 & \swarrow^{\pi_1''} & & \searrow_{\pi_3''} & \\
 & Z & & Z' & \\
 \swarrow^{\pi_1} & & \searrow_{\pi_2} & & \swarrow^{\pi_2'} \\
 X_1 & & X_2 & & X_3 \\
 & \nwarrow_{\pi_1} & & \nwarrow_{\pi_2'} & \nwarrow_{\pi_3'} \\
 & & & &
 \end{array}$$

The volume element vol'' along fibers of π_3'' is obtained by the multiplication of volume elements along fibers of maps $Z'' \rightarrow Z'$ and $Z' \rightarrow X_3$. The map $\chi'' : Z'' \rightarrow A$ is defined as the product in group scheme A of maps $\rho \circ (\pi_1 : Z'' \rightarrow Z)$ and $\rho' \circ (\pi_2' : Z'' \rightarrow Z')$.

One can check that the composition is well-defined on equivalence classes of representatives of morphisms.

The identity morphism $id_{\mathcal{C}}(X)$ is given by $Z = X$, $pr_1 = pr_2 = id : X \rightarrow X$, $vol = 1$, and $\rho(x) = 0 \in A$.

One can see that two varieties X_1, X_2 are isomorphic as objects of \mathcal{C} iff they are birationally equivalent.

Remark 2.1.2. In the definition of \mathcal{C} one can omit the condition that π_1 is submersion and the equivalence relation generated by (2.7). In the modified category one loses the birational invariance. On the other hand, if we want to keep the equivalence relation (and birational invariance), then the composition is defined if we assume that π_1 is dominant on each component of Z . Passing to a Zariski open dense set $U \subset Z$, we can replace this condition by smoothness of π_1 .

Category \mathcal{C} carries the natural structure of a symmetric monoidal category. The tensor product on objects is given by the usual product of varieties. In the definition of the tensor product of morphisms we use the product in group scheme A .

Definition 2.1.1. A multiplication kernel of birational type is a commutative semigroup object⁹ in (\mathcal{C}, \otimes) .

Remark 2.1.3. This definition seems to be too general, as we get some pathological examples. The issue is related to the fact that in our heuristics with local field, we ignore the question of convergence. As a first approximation to a better definition (which takes the convergence into account) one can suggest the following.

Definition 2.1.2. A morphism $(Z, \pi_1 : Z \rightarrow X_1, \pi_2 : Z \rightarrow X_2, vol, \rho : Z \rightarrow A)$ in \mathcal{C} is called *geometrically convergent* if the following property holds. Consider generic point $x_2 \in X_2$. Then $\pi_2^{-1}(x_2)$ is a smooth variety which is mapped to $X_1 \times A$ by (π_1, ρ) . Let Y_{x_2} denote the image of a connected component of $\pi_2^{-1}(x_2)$. This is not necessarily a smooth variety, but it is nevertheless smooth at its generic point $y \in Y_{x_2}$. On the smooth variety

⁹In the framework of category (\mathcal{C}, \otimes) it is not reasonable to require an object to be a commutative *monoid*, which is a commutative semigroup object with a unit.

$V_{x_2, y} = (\pi_1, \rho)^{-1}(y) \subset Y_{x_2}$ we have a volume element $vol_{x_2, y}$ defined up to multiplication by a non-zero constant. Namely, $vol_{x_2, y}$ is defined as the ratio of $vol_{\pi_2^{-1}(x_2)}$ and a non-zero element in $\Lambda^d T_y^* Y_{x_2}$ where $d = \dim Y_{x_2}$. We demand that $vol_{x_2, y}$ extends to volume form without poles on some (or equivalently, on all) smooth compactification of $V_{x_2, y}$.

This definition of convergence is not completely satisfactory. For example, the composition of geometrically convergent morphisms is not necessarily geometrically convergent. This reflects the fact that in the case of local field and $A = 0$ the integral operator associated with a geometrically convergent morphism maps the space of bounded measurable functions to a larger space of unbounded measurable functions. One can not compose such operators in general.

On the other hand, there exist situations when the integral operator corresponding to a not geometrically convergent morphism gives a well-defined compact operator. This is related to the fact that integrals $\int_{V(k)} |vol|$ for a meromorphic volume element vol on algebraic variety V/K can be convergent even if vol has poles on $V(\bar{k})$. For example, the sphere $S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n; \sum_{i=1}^n x_i^2 = 1\}$ has finite volume.

For a multiplication kernel of birational type $\mu \in Hom_C(X \times X, X)$ one can ask that $\mu_2 = \mu$, and $\mu_3 = \mu \circ (\mu \otimes id_X) \in Hom_C(X^3, X), \dots, \mu_n \in Hom_C(X^n, X), \dots$ are geometrically convergent where μ_n corresponds to a product of length n .

2.2 One dimensional examples without auxiliary integration

Let $X = \mathbb{A}^1$, $Z = X^3$ or a ramified covering of X^3 . In this case we do not have auxiliary integration in the formulas for kernels.

Example 2.2.1. Let

$$K(x_1, x_2, y) = e^{c(x_1 x_2 y + y)} \frac{1}{y}$$

where c is a constant. Then we have

$$K(x_1, x_2, y) K(y, x_3, z) dy = K(x_1, x_3, \tilde{y}) K(\tilde{y}, x_2, z) d\tilde{y}$$

if $\tilde{y} = \frac{x_1 x_2 + x_3 z + 1}{x_1 x_3 + x_2 z + 1} y$.

The example 2.2.1 looks degenerate because μ_n are not geometrically convergent for $n \geq 4$. It looks plausible, however, that further examples in this Section, as well as in Sections 2.3, 2.4 are not degenerate in the sense that all μ_n , $n \geq 2$ are geometrically convergent.

Example 2.2.2. Here we assume $A = \mathbb{G}_a$ in notations of Section 2.1. Let

$$K(x_1, x_2, y) = e^{c\left(x_1 x_2 y + \frac{x_1}{x_2 y} + \frac{x_2}{x_1 y} + \frac{y}{x_1 x_2}\right)} \frac{1}{y}$$

where c is a constant. Then we have

$$K(x_1, x_2, y) K(y, x_3, z) dy = K(x_1, x_3, \tilde{y}) K(\tilde{y}, x_2, z) d\tilde{y}$$

if $\tilde{y} = \frac{x_1x_2+x_3z}{x_1x_3+x_2z}y$.

A closely related version of this kernel is

$$K(x_1, x_2, y) = e^{\frac{x_1x_2y+x_1+x_2+y}{\sqrt{x_1x_2y}}} \frac{1}{y}.$$

Example 2.2.3. Here $A = \mathbb{G}_m^2$. Introduce the notation (see also the formula for F_t in Section 1.4)

$$f_t(x, y, z) = (xy + yz + zx - t)^2 + 4xyz(1 + t - (x + y + z))$$

where $t \neq 0, 1$ is a parameter. Let

$$K(x_1, x_2, y) =$$

$$\left(\frac{x_1x_2(2y-1) - (x_1+x_2)y + t + w_{12}}{(x_1-1)(x_2-1)y} \right)^{c_1} \left(\frac{x_1x_2(2y-t) - t(x_1+x_2)y + t^2 + tw_{12}}{(x_1-t)(x_2-t)y} \right)^{c_2} \frac{1}{w_{12}}$$

where $w_{12} := f_t(x_1, x_2, y)^{1/2}$ and c_1, c_2 are arbitrary constants. Then

$$K(x_1, x_2, y)K(y, x_3, x_4) dy = K(x_1, x_3, \tilde{y})K(\tilde{y}, x_2, x_4) d\tilde{y} \quad (2.8)$$

where \tilde{y} is a function in x_1, x_2, x_3, x_4, y independent of c_1, c_2 .

More precisely, let $E_1 \subset \mathbb{A}^3$ be an affine elliptic curve given by¹⁰

$$w_{12}^2 = f_t(x_1, x_2, y), \quad w_{34}^2 = f_t(y, x_3, x_4)$$

and $E_2 \subset \mathbb{A}^3$ be an affine elliptic curve given by¹¹

$$w_{13}^2 = f_t(x_1, x_3, \tilde{y}), \quad w_{24}^2 = f_t(\tilde{y}, x_2, x_4).$$

Then there exists a unique birational mapping $\rho : E_1 \rightarrow E_2$ which transforms the l.h.s. of (2.8) to its r.h.s. In particular, j -invariants of the elliptic curves E_1, E_2 are equal. This birational mapping has the form

$$\rho : (y, w_{12}, w_{34}) \mapsto (\tilde{y}, w_{13}, w_{24})$$

where $(y, w_{12}, w_{34}) \in E_1$, $(\tilde{y}, w_{13}, w_{24}) \in E_2$ and $\tilde{y}, w_{13}, w_{24}$ are given by

$$\tilde{y} = \frac{(x_1 - x_2)(x_3 - x_4)y^2 - w_{12}w_{34} + (x_1x_2 - t)(x_3x_4 - t) + \frac{y}{(x_1-x_4)(x_2-x_3)}Q_1}{2(x_1 - x_3)(x_2 - x_4)y + \frac{2(x_1x_2 - x_3x_4)}{(x_1-x_4)(x_2-x_3)}Q}$$

$$w_{13} = \frac{(x_3 - x_4)w_{12}y - (x_1 - x_2)w_{34}y + \frac{w_{12}}{(x_1-x_3)(x_2-x_3)}Q_2 + \frac{w_{34}}{(x_1-x_3)(x_1-x_4)}Q_3}{2(x_2 - x_4)y + \frac{2(x_1x_2 - x_3x_4)}{(x_1-x_3)(x_1-x_4)(x_2-x_3)}Q}$$

¹⁰Here w_{12}, w_{34}, y are affine coordinates on \mathbb{A}^3 and t, x_1, x_2, x_3, x_4 are parameters.

¹¹Here $w_{13}, w_{24}, \tilde{y}$ are affine coordinates on \mathbb{A}^3 and t, x_1, x_2, x_3, x_4 are parameters.

$$w_{24} = \frac{(x_1 - x_2)w_{34}y - (x_3 - x_4)w_{12}y + \frac{w_{34}}{(x_2-x_3)(x_2-x_4)}Q_4 + \frac{w_{12}}{(x_2-x_4)(x_1-x_4)}Q_5}{2(x_1 - x_3)y + \frac{2(x_1x_2-x_3x_4)}{(x_2-x_3)(x_2-x_4)(x_1-x_4)}Q}.$$

Here $Q, Q_1, Q_2, Q_3, Q_4, Q_5$ are irreducible polynomials in x_1, x_2, x_3, x_4, t defined by the following properties:

$$(x_2 - x_4)w_{13} - (x_1 - x_3)w_{24} = (x_3 - x_4)w_{12} - (x_1 - x_2)w_{34},$$

$$\rho : (0, x_1x_2 - t, x_3x_4 - t) \mapsto (0, x_1x_3 - t, x_2x_4 - t),$$

$$\rho : (1, x_1x_2 - x_1 - x_2 + t, x_3x_4 - x_3 - x_4 + t) \mapsto (1, x_1x_3 - x_1 - x_3 + t, x_2x_4 - x_2 - x_4 + t),$$

$$\rho : (t, x_1x_2 - tx_1 - tx_2 + t, x_3x_4 - tx_3 - tx_4 + t) \mapsto (t, x_1x_3 - tx_1 - tx_3 + t, x_2x_4 - tx_2 - tx_4 + t)$$

where

$$(0, x_1x_2 - t, x_3x_4 - t), (1, x_1x_2 - x_1 - x_2 + t, x_3x_4 - x_3 - x_4 + t),$$

$$(t, x_1x_2 - tx_1 - tx_2 + t, x_3x_4 - tx_3 - tx_4 + t) \in E_1,$$

$$(0, x_1x_3 - t, x_2x_4 - t), (1, x_1x_3 - x_1 - x_3 + t, x_2x_4 - x_2 - x_4 + t),$$

$$(t, x_1x_3 - tx_1 - tx_3 + t, x_2x_4 - tx_2 - tx_4 + t) \in E_2$$

are rational points of the elliptic curves.

Remark 2.2.1. The kernel in Example 2.2.3 depends on four points of \mathbb{P}^1 which are set to $0, 1, t, \infty$. After an arbitrary fractional linear transformation of the variables $x_1, x_2, x_3, x_4, y, \tilde{y}$ we obtain four pairwise distinct arbitrary points in \mathbb{P}^1 . Colliding some of these points we obtain various degenerations of the kernels in this family, including the kernel in Example 2.2.2.

Remark 2.2.2. Let

$$K(x_1, x_2, y) = \phi(x_1, x_2, y)^c \psi(x_1, x_2, y)$$

where ϕ, ψ are symmetric with respect to x_1, x_2 . Assume that there exists a function $\tilde{y}(x_1, x_2, x_3, y, z)$ independent of c such that

$$K(x_1, x_2, y)K(y, x_3, z) dy = K(x_1, x_3, \tilde{y})K(\tilde{y}, x_2, z) d\tilde{y}.$$

This condition gives a system of functional equations for the functions ϕ, ψ, \tilde{y} . For computational purposes it is convenient to set

$$\tilde{y} = y + q_1(x_1, x_2, y, z) \cdot (x_2 - x_3) + q_2(x_1, x_2, y, z) \cdot (x_2 - x_3)^2 + \dots$$

and assume that the equations

$$\phi(x_1, x_2, y)\phi(y, x_3, z) = \phi(x_1, x_3, \tilde{y})\phi(\tilde{y}, x_2, z),$$

$$\psi(x_1, x_2, y)\psi(y, x_3, z) dy = \psi(x_1, x_3, \tilde{y})\psi(\tilde{y}, x_2, z) d\tilde{y}$$

hold simultaneously.

Examples 2.2.1, 2.2.3 were obtained by solving this system of functional equations, and Examples 2.5.1, 2.5.2 below were obtained by solving the similar functional equations for kernels of the form

$$K(x_1, x_2, x_3, y) = \phi(x_1, x_2, x_3, y)^c \psi(x_1, x_2, x_3, y).$$

It looks that any solution of these functional equations is either listed in Examples 2.2.1, 2.2.3, 2.5.1, 2.5.2 or can be obtained as a limit of the family described in Example 2.2.3. It would be interesting to study similar functional equations for more general kernels.

2.3 One dimensional example with auxiliary integration

Let $X = \mathbb{A}^1$, $Z = \mathbb{A}^5$, $A = \mathbb{G}_m^4$. In this case we have auxiliary integration over 2-dimensional domain in the formulas for kernels. Fix $t \neq 0, 1$ as in Example 4. Let s_1, s_2, s_3, r be arbitrary constants symbolising a generic character of A .

Example 2.3.1. Let

$$K(x_1, x_2, y, q_1, q_2) = (x_1 x_2)^{1-s_1} ((x_1 - 1)(x_2 - 1))^{1-s_2} ((x_1 - t)(x_2 - t))^{1-s_3} F(u, v)$$

where

$$u = \frac{(x_1 - 1)(x_2 - 1)(y - 1)}{(t - 1)^2}, \quad v = \frac{(x_1 - t)(x_2 - t)(y - t)}{t(t - 1)^2}$$

and

$$F(u, v) = (1 - q_1 - q_2)^{s_1 - r - 1} q_1^{s_2 - r} q_2^{s_3 - r} (q_1 q_2 + v q_1 + u q_2)^{r - 2}.$$

Theorem 2.3.1. There exists a birational mapping

$$\mu : (y, q_1, q_2, q_3, q_4) \rightarrow (\tilde{y}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4)$$

depending on x_1, x_2, x_3, z , which transforms the l.h.s. of the equation

$$\begin{aligned} & K(x_1, x_2, y, q_1, q_2) K(y, x_3, z, q_3, q_4) dy \wedge dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4 = \\ & \pm K(x_1, x_3, \tilde{y}, \tilde{q}_1, \tilde{q}_2) K(\tilde{y}, x_2, z, \tilde{q}_3, \tilde{q}_4) d\tilde{y} \wedge d\tilde{q}_1 \wedge d\tilde{q}_2 \wedge d\tilde{q}_3 \wedge d\tilde{q}_4 \end{aligned} \quad (2.9)$$

to its r.h.s.

Proof. The l.h.s. of the equation (2.9) can be written as

$$f_1^{s_1} f_2^{s_2} f_3^{s_3} f_4^r \cdot h dy \wedge dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4$$

where

$$\begin{aligned} f_1 &= \frac{1 - q_1 - q_2}{x_1 x_2} \cdot \frac{1 - q_3 - q_4}{y x_3} \\ f_2 &= \frac{q_1}{(x_1 - 1)(x_2 - 1)} \cdot \frac{q_3}{(y - 1)(x_3 - 1)} \\ f_3 &= \frac{q_2}{(x_1 - t)(x_2 - t)} \cdot \frac{q_4}{(y - t)(x_3 - t)} \end{aligned}$$

$$f_4 = \frac{\left(q_1 q_2 + \frac{(x_1-t)(x_2-t)(y-t)}{t(t-1)^2} q_1 + \frac{(x_1-1)(x_2-1)(y-1)}{(t-1)^2} q_2 \right) \left(q_3 q_4 + \frac{(y-t)(x_3-t)(z-t)}{t(t-1)^2} q_3 + \frac{(y-1)(x_3-1)(z-1)}{(t-1)^2} q_4 \right)}{(1 - q_1 - q_2) q_1 q_2 \cdot (1 - q_3 - q_4) q_3 q_4}$$

$$h = \frac{x_1 x_2 (x_1 - 1) (x_2 - 1) (x_1 - t) (x_2 - t)}{\left(1 - q_1 - q_2 \right) \left(q_1 q_2 + \frac{(x_1-t)(x_2-t)(y-t)}{t(t-1)^2} q_1 + \frac{(x_1-1)(x_2-1)(y-1)}{(t-1)^2} q_2 \right)^2} \cdot$$

$$\frac{y x_3 (y - 1) (x_3 - 1) (y - t) (x_3 - t)}{\left(1 - q_3 - q_4 \right) \left(q_3 q_4 + \frac{(y-t)(x_3-t)(z-t)}{t(t-1)^2} q_3 + \frac{(y-1)(x_3-1)(z-1)}{(t-1)^2} q_4 \right)^2}$$

The r.h.s. of the equation (2.9) can be written as

$$\tilde{f}_1^{s_1} \tilde{f}_2^{s_2} \tilde{f}_3^{s_3} \tilde{f}_4^r \cdot \tilde{h} \, d\tilde{y} \wedge d\tilde{q}_1 \wedge d\tilde{q}_2 \wedge d\tilde{q}_3 \wedge d\tilde{q}_4$$

where $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4, \tilde{h}$ are obtained from f_1, f_2, f_3, f_4, h by swapping x_2 and x_3 and replacing y, q_1, q_2, q_3, q_4 by $\tilde{y}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4$.

Let E be a curve in affine space \mathbb{A}^5 with coordinates y, q_1, q_2, q_3, q_4 given by

$$f_1 = C_1, \quad f_2 = C_2, \quad f_3 = C_3, \quad f_4 = C_4.$$

Let \tilde{E} be a curve in affine space \mathbb{A}^5 with coordinates $\tilde{y}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4$ given by

$$\tilde{f}_1 = C_1, \quad \tilde{f}_2 = C_2, \quad \tilde{f}_3 = C_3, \quad \tilde{f}_4 = C_4.$$

Here (C_1, C_2, C_3, C_4) is a generic point of $A = \mathbb{G}_m^4$.

Note that equations for the curve E can be written as

$$\begin{aligned} (1 - q_1 - q_2)(1 - q_3 - q_4) &= y \cdot \left(C_1 x_1 x_2 x_3 \right), \\ q_1 q_3 &= (y - 1) \cdot \left(C_2 (x_1 - 1) (x_2 - 1) (x_3 - 1) \right), \\ q_2 q_4 &= (y - t) \cdot \left(C_3 (x_1 - t) (x_2 - t) (x_3 - t) \right), \\ \left(q_1 q_2 + \frac{(x_1 - t)(x_2 - t)(y - t)}{t(t - 1)^2} q_1 + \frac{(x_1 - 1)(x_2 - 1)(y - 1)}{(t - 1)^2} q_2 \right) & \cdot \\ \left(q_3 q_4 + \frac{(y - t)(x_3 - t)(z - t)}{t(t - 1)^2} q_3 + \frac{(y - 1)(x_3 - 1)(z - 1)}{(t - 1)^2} q_4 \right) &= \\ y(y - 1)(y - t) \cdot \left(C_1 C_2 C_3 C_4 x_1 x_2 x_3 (x_1 - 1) (x_2 - 1) (x_3 - 1) (x_1 - t) (x_2 - t) (x_3 - t) \right) & \end{aligned} \quad (2.10)$$

and equations for the curve \tilde{E} are obtained from the equations (2.10) by interchanging x_2 and x_3 , and replacing y, q_1, q_2, q_3, q_4 by $\tilde{y}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4$. One can show by direct computation that E and \tilde{E} are elliptic curves with the same j -invariant. To show this we solve the first three equations in the system (2.10) with respect to q_1, q_2, y and substitute the result into the fourth equation. We obtain a plane cubic curve with coordinates q_3, q_4 , its genus and j -invariant can

be computed in a usual way. After that we observe that j -invariant is symmetric with respect to x_1, x_2, x_3 .

Observe that there exist rational points

$$(y, q_1, q_2, q_3, q_4) = \left(0, -\frac{C_2(t-1)}{z-1}, \frac{C_3t(t-1)}{z-t}, \frac{z-1}{t-1}, -\frac{z-t}{t-1}\right) \in E,$$

$$(\tilde{y}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4) = \left(0, -\frac{C_2(t-1)}{z-1}, \frac{C_3t(t-1)}{z-t}, \frac{z-1}{t-1}, -\frac{z-t}{t-1}\right) \in \tilde{E}.$$

This gives a birational mapping $\mu : E \rightarrow \tilde{E}$ such that

$$\mu\left(0, -\frac{C_2(t-1)}{z-1}, \frac{C_3t(t-1)}{z-t}, \frac{z-1}{t-1}, -\frac{z-t}{t-1}\right) = \left(0, -\frac{C_2(t-1)}{z-1}, \frac{C_3t(t-1)}{z-t}, \frac{z-1}{t-1}, -\frac{z-t}{t-1}\right).$$

One can also check that μ transforms $h \, dy \wedge dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4$ to $\tilde{h} \, d\tilde{y} \wedge d\tilde{q}_1 \wedge d\tilde{q}_2 \wedge d\tilde{q}_3 \wedge d\tilde{q}_4$.
□

2.4 A hypothetical example of a generalized product

Here we suggest a hypothetical and more complicated example related to the generalized products described in Section 4.1. We still have $X = \mathbb{A}^1$ with auxiliary integration over $(n+1)$ -dimensional cycle, but our commutative associative operation has $n+1$ inputs and n outputs.

Example 2.4.1. Let

$$K(x_1, \dots, x_{2n+1}, q_1, \dots, q_{n+1}) = (x_1 \dots x_{2n+1})^s u_1^{-k_1} \dots u_{n+1}^{-k_{n+1}} q_1^{2k_1-1} \dots q_{n+1}^{2k_{n+1}-1} \times \quad (2.11)$$

$$\left(1 + q_1 + \dots + q_{n+1}\right)^{s-k_1-\dots-k_{n+2}} \left(1 + \frac{u_1}{q_1} + \dots + \frac{u_{n+1}}{q_{n+1}}\right)^{s+k_1+\dots+k_{n+2}}$$

where

$$u_1 = \frac{(x_1-1)\dots(x_{2n+1}-1)t_1^2 \dots t_n^2}{x_1 \dots x_{2n+1} (t_1-1)^2 \dots (t_n-1)^2},$$

$$u_{i+1} = \frac{(x_1-t_i)\dots(x_{2n+1}-t_i) \prod_{j \neq i} t_j^2}{x_1 \dots x_{2n+1} (t_i-1)^2 \prod_{j \neq i} (t_j-t_i)^2}, \quad i = 1, \dots, n.$$

Here $t_1, \dots, t_n \neq 0, 1$ are pairwise distinct parameters and s, k_1, \dots, k_{n+2} are arbitrary constants.

Conjecture 2.4.1. There exists a birational mapping

$$(y_1, \dots, y_n, q_1, \dots, q_{2n+2}) \rightarrow (\tilde{y}_1, \dots, \tilde{y}_n, \tilde{q}_1, \dots, \tilde{q}_{2n+2})$$

which transforms the l.h.s. of the equation

$$K(x_1, \dots, x_{n+1}, y_1, \dots, y_n, q_1, \dots, q_{n+1}) K(x_{n+2}, y_1, \dots, y_n, z_1, \dots, z_n, q_{n+2}, \dots, q_{2n+2}) \times$$

$$dy_1 \wedge \dots \wedge dy_n \wedge dq_1 \wedge \dots \wedge dq_{2n+2} =$$

$$\pm K(x_1, \dots, x_{n+2}, \tilde{y}_1, \dots, \tilde{y}_n, \tilde{q}_1, \dots, \tilde{q}_{n+1}) K(x_{n+1}, \tilde{y}_1, \dots, \tilde{y}_n, z_1, \dots, z_n, \tilde{q}_{n+2}, \dots, \tilde{q}_{2n+2}) \times \\ d\tilde{y}_1 \wedge \dots \wedge d\tilde{y}_n \wedge d\tilde{q}_1 \wedge \dots \wedge d\tilde{q}_{2n+2} =$$

to its r.h.s. Here the r.h.s. is obtained from the l.h.s. by interchanging x_{n+1} and x_{n+2} , and replacing the variables $y_1, \dots, y_n, q_1, \dots, q_{2n+2}$ by $\tilde{y}_1, \dots, \tilde{y}_n, \tilde{q}_1, \dots, \tilde{q}_{2n+2}$.

Notice that in the Conjecture above $n > 1$ because if $n = 1$, then this family of kernels is essentially the same as in Example 5, so for $n = 1$ this is proved.

The kernel in this example can be written in more symmetric form

$$\int K(x_1, \dots, x_{2n+1}, q_1, \dots, q_{n+1}) dq_1 \dots dq_{n+1} = \\ \int \prod_{i=1}^{n+2} (q_{i,+}^{+k_i} q_{i,-}^{-k_i}) \cdot \left(\sum_{i=1}^{n+2} q_{i,+} \right)^{s-k_1-\dots-k_{n+2}} \left(\sum_{i=1}^{n+2} q_{i,-} \right)^{s+k_1+\dots+k_{n+2}} \cdot \frac{\prod_{i=1}^{n+2} \frac{dq_{i,+}}{q_{i,+}}}{\frac{d\lambda}{\lambda}}$$

where we integrate over $(n+1)$ -dimensional torus $\mathbb{G}_m^{n+2}/(\mathbb{G}_m)_{diag}$ with coordinates $q_{1,+}, \dots, q_{n+2,+}$ factorized by the diagonal action $q_{i,+} \mapsto \lambda q_{i,+}$ of \mathbb{G}_m . Variables $q_{1,-}, \dots, q_{n+2,-}$ are defined by

$$q_{i,+} q_{i,-} = \frac{\prod_{\alpha=1}^{2n+1} (x_\alpha - t_i)}{P'(t_i)^2}, \quad i = 1, \dots, n+2$$

where $P(u) = \prod_{i=1}^{n+2} (u - t_i)$. Here we use arbitrary pairwise distinct parameters t_1, \dots, t_{n+2} , the previous formula for the kernel $K(x_1, \dots, x_{2n+1}, q_1, \dots, q_{n+1})$ corresponds to the choice $t_{n+1} = 0$, $t_{n+2} = 1$.

The expression which we integrate is invariant under the diagonal action of \mathbb{G}_m . It is also invariant with respect to the group S_{n+2} acting on variables $q_{1,+}, \dots, q_{n+2,+}$ by permutations. One can achieve invariance with respect to a larger group $S_{n+2} \times (\mathbb{Z}/2)^{n+2}$ (where the i th generator of $(\mathbb{Z}/2)^{n+2}$ acts by interchanging of $q_{i,+}$ and $q_{i,-}$) by adding one extra variable of integration (and loosing geometric convergence).

$$\int K(x_1, \dots, x_{2n+1}, q_1, \dots, q_{n+2}) dq_1 \dots dq_{n+2} = C \int \prod_{i=1}^{n+2} (q_{i,+}^{+k_i} q_{i,-}^{-k_i}) \cdot \left(\sum_{i=1}^{n+2} (q_{i,+} + q_{i,-}) \right)^{2s} \cdot \prod_{i=1}^{n+2} \frac{dq_{i,+}}{q_{i,+}}$$

where we integrate over the $(n+2)$ -dimensional torus with coordinates $q_{1,+}, \dots, q_{n+2,+}$. Here C is independent of x_1, \dots, x_{2n+1} .

Indeed, our two expressions for the kernel can be written as

$$F_1 = \int \left(\sum_{i=1}^{n+2} a_i q_i + \sum_{i=1}^{n+2} \frac{b_i}{q_i} \right)^{\alpha+\beta} \cdot \prod_{i=1}^{n+2} q_i^{\lambda_i} \cdot \prod_{i=1}^{n+2} \frac{dq_i}{q_i}, \\ F_2 = \int \left(\sum_{i=1}^{n+2} a_i q_i \right)^{\alpha} \cdot \left(\sum_{i=1}^{n+2} \frac{b_i}{q_i} \right)^{\beta} \cdot \prod_{i=1}^{n+2} q_i^{\lambda_i} \cdot \prod_{i=1}^{n+2} \frac{dq_i}{q_i}$$

for some $a_i, b_i, \lambda_i, \alpha, \beta$ such that

$$\sum_{i=1}^{n+2} \lambda_i = \beta - \alpha.$$

Notice that the expressions F_1, F_2 both satisfy the same holonomic system of differential equations¹²:

$$\begin{aligned} \left(a_i \frac{\partial}{\partial a_i} - b_i \frac{\partial}{\partial b_i} + \lambda_i \right) F &= 0, \quad i = 1, \dots, n+2, \\ \left(\sum_{i=1}^{n+2} \left(a_i \frac{\partial}{\partial a_i} + b_i \frac{\partial}{\partial b_i} \right) - \alpha - \beta \right) F &= 0, \\ \frac{\partial^2 F}{\partial a_1 \partial b_1} &= \frac{\partial^2 F}{\partial a_2 \partial b_2} = \dots = \frac{\partial^2 F}{\partial a_{n+2} \partial b_{n+2}} \end{aligned} \tag{2.12}$$

which means that they should be in a sense equal.

The equality $F_2 = C F_1$ can be also shown as follows. Introduce an auxiliary function

$$\phi(A, B) = \int \delta\left(\sum_{i=1}^{n+2} a_i q_i - A\right) \cdot \delta\left(\sum_{i=1}^{n+2} \frac{b_i}{q_i} - B\right) \cdot \prod_{i=1}^{n+2} q_i^{\lambda_i} \cdot \prod_{i=1}^{n+2} \frac{dq_i}{q_i}.$$

The function $\phi(A, B)$ is homogeneous

$$\phi(qA, q^{-1}B) = q^{\beta-\alpha} \phi(A, B) \quad \text{for all } q \neq 0,$$

so we can write

$$\phi(A, B) = \phi_0(AB) \cdot A^{\beta-\alpha}$$

where $\phi_0(u) = \phi(1, u)$. We have

$$\begin{aligned} F_1 &= \int (A+B)^{\alpha+\beta} \phi(A, B) dA dB = \int (A+B)^{\alpha+\beta} \phi_0(AB) \cdot A^{\beta-\alpha} dA dB = \\ &= \int \left(A + \frac{u}{A} \right)^{\alpha+\beta} \phi_0(u) A^{\beta-\alpha} du \frac{dA}{A} = \int u^\beta \phi_0(u) du \cdot \int \frac{(1+v^2)^{\alpha+\beta}}{2v^{2\alpha}} \frac{dv}{v} \end{aligned}$$

where we made substitutions $B = \frac{u}{A}$ and $A = v\sqrt{u}$.

Similar computation for F_2 gives

$$F_2 = \int A^\alpha B^\beta \phi(A, B) dA dB = \int A^\alpha \left(\frac{u}{A} \right)^\beta \phi_0(u) A^{\beta-\alpha} du \frac{dA}{A} = \int u^\beta \phi_0(u) du \cdot \int \frac{dA}{A}.$$

Therefore, F_1 and F_2 are both equal to the integral $\int u^\beta \phi_0(u) du$ multiplied by a constant independent of $a_i, b_i, i = 1, \dots, n+2$. Moreover, removing the integral $\int \frac{dA}{A}$ from our final expression for F_2 we cure its geometrical divergence.

¹²Such expressions and the corresponding D -modules belong to a class of A-hypergeometric functions [3].

2.5 Some examples with 3 inputs and one output

Here we give examples of products μ_3 with 3 inputs and one output with look a bit pathological but still satisfy an analog of associativity and commutativity conditions: μ_3 is S_3 -invariant and $\mu_3 \circ (\mu_3 \otimes id)$ is S_5 -invariant.

Example 2.5.1. Let

$$K(x_1, x_2, x_3, y) = e^{c(x_1x_2x_3y+y)} \frac{1}{y}$$

where c is a constant. Then we have

$$K(x_1, x_2, x_3, y) K(y, x_4, x_5, z) dy = K(x_1, x_2, x_4, \tilde{y}) K(\tilde{y}, x_3, x_5, z) d\tilde{y}$$

if $\tilde{y} = \frac{x_1x_2x_3+x_4x_5z+1}{x_1x_2x_4+x_3x_5z+1}y$.

This example is similar to Example 1 and it is degenerate in the same sense.

Example 2.5.2. Let

$$K(x_1, x_2, x_3, y) = \left(\frac{1}{y} + \frac{1}{y} \sqrt{1 + x_1x_2x_3y} \right)^c \frac{1}{\sqrt{1 + x_1x_2x_3y}}$$

where c is an arbitrary constant. Then

$$K(x_1, x_2, x_3, y) K(y, x_4, x_5, z) dy = K(x_1, x_2, x_4, \tilde{y}) K(\tilde{y}, x_3, x_5, z) d\tilde{y} \quad (2.13)$$

where \tilde{y} is a function in $x_1, x_2, x_3, x_4, x_5, y, z$ independent of c .

More precisely, let $C_1 \subset \mathbb{A}^3$ be a rational affine curve given by¹³

$$w_{123}^2 = 1 + x_1x_2x_3y, \quad w_{45}^2 = 1 + yx_4x_5z$$

and $C_2 \subset \mathbb{A}^3$ be a rational affine curve given by¹⁴

$$w_{124}^2 = 1 + x_1x_2x_4\tilde{y}, \quad w_{35}^2 = 1 + \tilde{y}x_3x_5z.$$

Then the formulas

$$w_{124} = \frac{(x_1x_2 - x_5z)x_4w_{123} + (x_3 - x_4)x_1x_2w_{45}}{x_1x_2x_3 - x_4x_5z},$$

$$w_{35} = \frac{(x_3 - x_4)x_5z w_{123} + (x_1x_2 - x_5z)x_3w_{45}}{x_1x_2x_3 - x_4x_5z},$$

$$\tilde{y} = \frac{2(x_3 - x_4)(x_1x_2 - x_5z)(w_{123}w_{45} - 1) + x_3x_4(x_1x_2 - x_5z)^2y + x_1x_2x_5z(x_3 - x_4)^2y}{(x_1x_2x_3 - x_4x_5z)^2}$$

define a birational mapping $C_1 \rightarrow C_2$ which transforms the l.h.s. of (2.13) to its r.h.s.

¹³Here w_{123}, w_{45}, y are affine coordinates on \mathbb{A}^3 and $x_1, x_2, x_3, x_4, x_5, z$ are parameters.

¹⁴Here $w_{124}, w_{35}, \tilde{y}$ are affine coordinates on \mathbb{A}^3 and $x_1, x_2, x_3, x_4, x_5, z$ are parameters.

2.6 On S_n -covariance of higher compositions of kernels

If K is a multiplication kernel of birational type on variety X/k , then for any $n \geq 2$ and any planar binary rooted tree T with n leaves (or, equivalently, choice of bracketing on a product of n symbols) we get a variety Z_T which maps to $X^{n+1} \times A$ and is endowed with a volume element along fibers for the projection to the last factor X . This variety Z_T corresponds to the kernel of n -fold product with the chosen bracketing. Associativity implies that there exists a birational identification of varieties Z_T for different trees T . Commutativity of our kernel implies that one can lift each permutation $\sigma \in S_n$ of first n factors in X^{n+1} , to the birational identification of varieties Z_T . These identifications, however, are not compatible in general. It is desirable to lift these identifications to a S_n -action and construct just one variety Z_n (with an action of S_n) which maps to $X^{n+1} \times A$ in a S_n -covariant way.

One of the reasons why it is desirable is the following. Let k be a local field, and assume that the kernel is geometrically convergent. Then we expect that an appropriate space $\mathcal{F}(X(k))$ of complex-valued functions on $X(k)$ to be a commutative associative algebra with product $*$ given by our kernel K . For any finite extension $k' \supset k$ we get another algebra $(\mathcal{F}(X(k')), *)$. The existence of S_n -action on Z_n for $n = \deg(k'/k)$ gives rise to a homomorphism

$$(\mathcal{F}(X(k')), *) \rightarrow (\mathcal{F}(X(k)), *). \quad (2.14)$$

Namely, extension $k' \supset k$ gives a transitive action of $Gal(\bar{k}/k)$ on an n -elements set (the set of embeddings $k' \hookrightarrow \bar{k}$ over k), hence a homomorphism $\rho : Gal(\bar{k}/k) \rightarrow S_n$, up to conjugation. Using ρ we can define a twisted form $Z_{n,\rho}$ of Z_n . Recall that the set $Z_{n,\rho}(\bar{k})$ of \bar{k} -points of $Z_{n,\rho}$ coincides with $Z_n(\bar{k})$, but the action of $Gal(\bar{k}/k)$ is twisted by ρ . Variety $Z_{n,\rho}$ maps to the product $(X^n)_\rho \times X \times A$ where $(X^n)_\rho$ is the twisted forms of X^n , i.e. the Weil restriction $Res_{k'/k}(X)$. Recall that the latter is a variety over k such that its set of k -points is $X(k')$. The twisted multiplication kernel gives a map (2.14). One can check that it is a homomorphism of algebras.

In the case of varieties over finite fields (see setup 4 in Section 1.2) there are homomorphisms (2.14) for extensions $k' = \mathbb{F}_{q^m} \supset k = \mathbb{F}_q$ of finite fields, even without the action of S_n on Z_n . It looks plausible that in this case there is still an action of S_n on the corresponding motivic constructible sheaves. The tower of homomorphisms (2.14), or dually, maps (which are inclusions for finite fields)

$$Spec(\mathcal{F}(X(k)), *) \hookrightarrow Spec(\mathcal{F}(X(k')), *)$$

played an essential role in [7]. The inductive limit

$$\varinjlim_m Spec(\mathcal{F}(X(\mathbb{F}_{q^m}), *)_{\overline{\mathbb{Q}}})$$

is an infinite countable set endowed with commuting actions of two Galois groups: $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ and $Gal(\overline{\mathbb{Q}}^{CM}/\mathbb{Q})$.

Unfortunately, in general it seems to be impossible to lift S_n action from $X^n \times X \times A$ to Z_n . Let us describe this problem in the case of Example 2.2.2. Every binary rooted tree T with

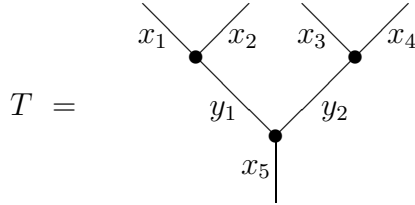
$n \geq 3$ leaves gives a family of Calabi-Yau varieties $V_{x_1, \dots, x_{n+1}, t}^T$ of dimension $n - 3$ depending on $n + 2$ parameters $(x_1, \dots, x_{n+1}, t) \in (\mathbb{G}_m)^{n+1} \times \mathbb{G}_a$ in the following way. Variety $V_{x_1, \dots, x_{n+1}, t}^T$ is a hypersurface in a toric variety, and it is given by equation in variables $(y_1, \dots, y_{n-2}) \in \mathbb{G}_m^{n-2}$

$$\sum_{v \in \{\text{vertices of } T\}} f_v = t$$

where parameters x_1, \dots, x_n are attached to leaves of T , parameter x_{n+1} is attached to the root of T , and parameters y_1, \dots, y_{n-2} are attached to inner edges of T . For each vertex v we define

$$f_v = f(z_1, z_2, z_3) = z_1 z_2 z_3 + \frac{z_1}{z_2 z_3} + \frac{z_2}{z_1 z_3} + \frac{z_3}{z_1 z_2}$$

where z_1, z_2, z_3 are variables attached to three edges adjacent to v . For example, if



then $V_{x_1, \dots, x_5, t}^T$ is an elliptic curve given by

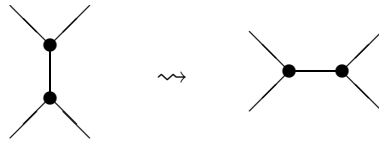
$$f(x_1, x_2, y_1) + f(x_3, x_4, y_2) + f(y_1, y_2, x_5) = t.$$

Variable t is the coordinate on $A = \mathbb{G}_a = \mathbb{A}^1$.

The volume element (up to a sign) on $V_{x_1, \dots, x_{n+1}, t}^T$ is given by

$$\frac{\wedge_{i=1}^{n-2} d \log y_i}{d(\sum_v f_v)}.$$

The integral operator corresponding to Z_T is given by a density on $X^{n+1} = (\mathbb{G}_m)^{n+1}$ with coordinates x_1, \dots, x_{n+1} . This density is the Fourier transform of function $t \mapsto \text{vol}(V_{x_1, \dots, x_{n+1}, t}^T)$ in variable t . One can construct a birational identification $V_{x_1, \dots, x_{n+1}, t}^T \sim V_{x_1, \dots, x_{n+1}, t}^{T'}$ where T' is obtained by a flip from T :



The composition of 5 flips corresponding to pentagon relation is a non-trivial automorphism of K3 surface $V_{x_1, \dots, x_5, t}^T$.

It might be possible to cure this problem by increasing the dimension of Z_n (or equivalently, adding auxiliary variables for the integration). This might require an extension of our formalism. One such possibility is discussed in the next Section.

2.7 Direct images of cyclic D -modules at a generic point

In this section we propose a mixed birational/ D -module type formalism using which one can speak about multiplication kernels. In this formalism, like in Section 2.1, one considers algebraic varieties only up to birational equivalence. On the other hand, we deal here with cyclic D -modules. Roughly speaking, we encode a “function” f on algebraic variety X by a system of algebraic linear differential equations satisfied by f at the generic point of X . The most interesting operation with these objects is the direct image (which we denote by π_*) defined below, which informally corresponds to the integration over an unspecified cycle. Almost all examples in our paper can be understood in this mixed formalism.

Let X be a smooth algebraic variety over field k of characteristic zero, endowed with a line bundle \mathcal{L} . Denote by $\text{Diff}_{\mathcal{L},rat}$ the algebra of differential operators in \mathcal{L} with coefficients in rational functions on X . Denote by $\mathcal{I}_{\mathcal{L}}$ the set of left ideals in $\text{Diff}_{\mathcal{L},rat}$. We can think about elements of $\mathcal{I}_{\mathcal{L}}$ as systems of differential equations on a “section” of \mathcal{L} (e.g. analytic germ of a section for $k = \mathbb{C}$). For any $I \in \mathcal{I}_{\mathcal{L}}$ the quotient $\text{Diff}_{\mathcal{L},rat}/I$ is a cyclic $\text{Diff}_{\mathcal{L},rat}$ -module. In this paper we deal only with examples where this cyclic module is holonomic. It is clear that both $\text{Diff}_{\mathcal{L},rat}$ and \mathcal{L} depends only on the birational type of X (and also on \mathcal{L}), i.e. only on the field $k(X)$ of rational functions, together with a 1-dimensional module over it.

There are three basic operations:

- 1) For a dominant map $\pi : Y \rightarrow X$, and \mathcal{L} on X , we have a pullback $\pi^* : \mathcal{I}_{\mathcal{L}} \rightarrow \mathcal{I}_{\pi^*\mathcal{L}}$.
- 2) In the same situation, we have pushforward $\pi_* : \mathcal{I}_{\pi^*\mathcal{L} \otimes K_{Y/X}} \rightarrow \mathcal{I}_{\mathcal{L}}$.
- 3) For a collection of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ on X we have product $\prod_i \mathcal{I}_{\mathcal{L}_i} \rightarrow \mathcal{I}_{\otimes_i \mathcal{L}_i}$.

In order to explain **1)** and **2)** it is convenient to have a non-linear flat connection on fibration $\pi : Y \rightarrow X$. We claim that there exists a Zariski open dense set $Y' \subset Y$ such that $\pi|_{Y'}$ is a submersion, and a non-linear flat connection ∇ on $\pi|_{Y'} : Y' \rightarrow X$, i.e. an integrable distribution on Y' transversal to fibers of π .

Indeed, passing to an open dense subset of Y we may assume that fibers of π are smooth locally closed subvarieties of \mathbb{A}^N for some N . Let us choose a generic affine projection $\mathbb{A}^N \rightarrow \mathbb{A}^n$ where $n = \dim Y - \dim X$. Then the fibers $\pi^{-1}(x)$ for $x \in X$ are ramified coverings over the *same* affine space \mathbb{A}^n , hence are locally identified in etale topology. This gives a non-linear flat connection outside of the ramification loci $\pi^{-1}(x) \rightarrow \mathbb{A}^n$, $x \in X$. Any non-linear flat connection ∇ gives rise to map of algebras $\nabla^* : \text{Diff}_{\mathcal{L},rat} \rightarrow \text{Diff}_{\pi^*\mathcal{L},rat}$. After making the choice of connection ∇ , the maps π^*, π_* in **1), 2)** are defined as follows. In the case **1)**, it is convenient to assume $\mathcal{L} = \mathcal{O}_X$ (this can be achieved by passing to an open dense set $X' \subset X$). For an ideal $I \in \mathcal{I}_{\mathcal{O}_X}$ we define π^*I to be the left ideal in $\mathcal{I}_{\mathcal{O}_X}$ generated by elements $\nabla^*(P)$ for $P \in I$, and by vector fields along fiber of π .

In the case **2)** it is convenient to assume $\mathcal{L} = K_X$, and the left ideals in $\text{Diff}_{\mathcal{L},rat}$ are the same as right ideals in $\text{Diff}_{\mathcal{O}_X,rat} = \text{Diff}_{\mathcal{L},rat}^{op}$. Let I be a right ideal in $\text{Diff}_{Y,rat}$ which can be thought of as an annihilator of a “volume form” on Y (e.g. analytic volume form in some domain for $k = \mathbb{C}$). We want to define π_*I to be the right ideal in $\text{Diff}_{X,rat}$ consisting of differential

operators annihilating formal integrals $\pi_*(\text{vol}_Y)$. By definition, $P \in \text{Diff}_{X,\text{rat}}$ belongs to π_*I iff $\nabla^*(P) \bmod \mathcal{I}_Y \in \mathcal{I}_Y \setminus \text{Diff}_{Y,\text{rat}}$ belongs to the image of the right submodule in $\text{Diff}_{Y,\text{rat}}$ generated by vector fields along fibers of π .

In both cases **1)**, **2)** the result of operations does not depend on the choice of flat connection ∇ .

In the case **3)**, we assume that all $\mathcal{L}_1, \dots, \mathcal{L}_n$ are trivialized. It is enough to consider the case $n = 2$. Define a linear map $\Delta : \text{Diff}_{\mathcal{O}_X,\text{rat}} \rightarrow \text{Diff}_{\mathcal{O}_X,\text{rat}} \otimes_{k(X)} \text{Diff}_{\mathcal{O}_X,\text{rat}}$ (here both factors on the right are understood as left $k(X)$ -modules), by the condition $\Delta P = \sum_{\alpha} P_{\alpha}^{(1)} \otimes P_{\alpha}^{(2)}$ if for any $f, g \in k(X)$ we have $P(fg) = \sum_{\alpha} P_{\alpha}^{(1)} f \cdot P_{\alpha}^{(2)} g$. For two left ideals $I_1, I_2 \in \mathcal{I}_{\mathcal{O}_X}$ we define their “product” J by $J = \{P \in \text{Diff}_{\mathcal{O}_X,\text{rat}}; \Delta P \in I_1 \otimes \text{Diff}_{\mathcal{O}_X,\text{rat}} + \text{Diff}_{\mathcal{O}_X,\text{rat}} \otimes I_2\}$. This is the ideal annihilating product $f_1 f_2$ where f_1, f_2 are general analytic functions annihilated by I_1, I_2 respectively.

3 Semi-classical kernels and their quantization

3.1 Semi-classical kernels associated with classical integrable systems

Definition 3.1.1. Let (M, ω) be a symplectic manifold. A semi-classical multiplication kernel is a tuple $((M, \omega), N, P)$ where $N \subset (M, \omega)$, $P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega)$ are Lagrangian submanifolds with the following properties:

1. Projections $\pi_i : P \rightarrow M$, $i = 1, 2, 3$ are submersions.
2. P is symmetric with respect to the interchanging of the first two components in M^3 .
3. The correspondence $M \times M \rightarrow M$ given by P is associative.
4. The map $(\pi_2 \times \pi_3)|_{\pi_1^{-1}(N) \cap P}$ is an inclusion whose image is open dense in the diagonal Lagrangian submanifold $M_{\text{diag}} = \{(m, m); m \in M\} \subset (M, -\omega) \times (M, \omega)$.

Remark 3.1.1. In other words, (M, ω) is a commutative monoid in the symplectic monoidal category introduced by A. Weinstein. The product in this commutative monoid is defined by P and the neutral element is defined by N . In general, the objects of this category are symplectic manifolds (in algebraic or C^∞ sense) and morphisms $\text{Hom}((M_1, \omega_1), (M_2, \omega_2))$ are defined as Lagrangian submanifolds in $(M_1, -\omega_1) \times (M_2, \omega_2)$. The tensor product is given by the usual product of symplectic manifolds. Notice that there are transversality problems in the definition of the composition as the composition of correspondences. Strictly speaking, the above definition is a rough sketch, a first approximation to a not yet found more satisfactory and rigorous notion.

Remark 3.1.2. The constraint on Lagrangian subvariety P to be smooth (i.e. to be a manifold) seems to be not completely natural. The closure of P in M^3 is usually singular.

It seems plausible that semi-classical multiplication kernels are essentially the same as Li-

ouville integrable systems endowed with a Lagrangian section.

Let (M, ω) be a symplectic manifold with a structure of Liouville integrable system. In other words, we have a fibration

$$f : M \rightarrow B$$

where $\dim B = \frac{1}{2} \dim M$ and such that pullback of functions on B Poisson commute. Then $L_b = f^{-1}(b) \subset M$ is a Lagrangian submanifold of M for generic point $b \in B$, endowed with a natural affine structure (a torsion-free flat connection on the tangent bundle TL_b). Let us assume for simplicity that the generic fiber L_b is connected and compact, and moreover a Lagrangian section $\sigma : B \rightarrow M$, $f \circ \sigma = id$ is chosen. Then L_b will have a structure of a commutative group with the identity element given by $\sigma(b) \in L_b$. In fact, L_b is an abelian variety¹⁵. Define $P_b \subset L_b \times L_b \times L_b \subset M \times M \times M$ by $P_b = \{(u, v, u + v); u, v \in L_b = f^{-1}(b)\}$. Then one can see that $P = \cup_{b \in B} P_b$ is a Lagrangian submanifold in $(M, -\omega) \times (M, -\omega) \times (M, \omega)$. In this way we obtain a semi-classical multiplication kernel given by the correspondence P , with the neutral element given by $N = \sigma(B) \subset M$.

Remark 3.1.3. If we do not choose section σ (or submanifold N), then we will have naturally only a structure of an abelian torsor on $L_b, b \in B$. Instead of multiplication $P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega)$ we will have a manifold $\{(u_1, u_2, u_3, u_4) \in M^4; f(u_1) = f(u_2) = f(u_3) = f(u_4), u_1 + u_2 = u_3 + u_4\}$ which is a Lagrangian submanifold in $(M, -\omega) \times (M, -\omega) \times (M, \omega) \times (M, \omega)$. This is a structure similar to one in Example 2, Section 1.5 where we consider commuting operators with simple joint spectrum and without the choice of normalization for the eigenfunctions.

Let $M = T^*X$ be cotangent bundle with the canonical symplectic structure where X is an algebraic variety. Furthermore, to simplify notations we assume that $X = \mathbb{A}^1$ with an affine coordinate x . In this case $M \cong \mathbb{A}^2$ with canonical coordinates x, p and $\omega = dx \wedge dp$. Let $f : M \rightarrow B$ is given by the formula $(x, p) \mapsto f(x, p)$ where $f(x, p)$ is a meromorphic function. In this case the graph of our commutative associative multiplication is defined by the system of equations of the form

$$f(x_1, p_1) = f(x_2, p_2) = f(x_3, p_3), \quad g(x_1, p_1, x_2, p_2, x_3, p_3) = 0 \quad (3.15)$$

where $(x_1, p_1, x_2, p_2, x_3, p_3)$ are coordinates on $M \times M \times M$, and the first two equations mean that $f(m_1) = f(m_2) = f(m_3) = b$ for some $b \in B$. Here $(m_1, m_2, m_3) \in M \times M \times M$ and (x_i, p_i) are coordinates of point m_i , $i = 1, 2, 3$.

Notice that commutativity of our semi-classical multiplication kernel means that¹⁶

$$g(x_1, p_1, x_2, p_2, x_3, p_3) = g(x_2, p_2, x_1, p_1, x_3, p_3)$$

¹⁵In practice, fibers of f are often noncompact and admit compactification to abelian varieties. On the other hand, in the case $\dim M = 2$, there are many integrable systems for which the generic fiber of f is a punctured curve of genus $g > 1$, e.g. one can take $M = T^*\mathbb{A}^1 = \mathbb{A}^2$ and $B = \mathbb{A}^1$, with the map f given by a polynomial in 2 variables of sufficiently high degree.

¹⁶Strictly speaking, we need only the equivalence $g(x_1, p_1, x_2, p_2, x_3, p_3) = 0 \iff g(x_2, p_2, x_1, p_1, x_3, p_3) = 0$.

and associativity means that if we take the system of equations

$$\begin{aligned} f(x_1, p_1) &= f(x_2, p_2) = f(x_3, p_3) = f(x_4, p_4) = f(x_5, p_5), \\ g(x_1, p_1, x_2, p_2, x_3, p_3) &= 0, \quad g(x_3, p_3, x_4, p_4, x_5, p_5) = 0 \end{aligned}$$

and exclude x_3, p_3 from it, the resulting affine variety will be symmetric with respect to interchanging of pairs (x_2, p_2) and (x_4, p_4) .

Recall also that $P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega)$ is a Lagrangian submanifold. This means that if I is the ideal generated by the equations (3.15), then

$$\{I, I\} \subset I$$

where $\{, \}$ are Poisson brackets defined by $\{p_1, x_1\} = \{p_2, x_2\} = -1$, $\{p_3, x_3\} = 1$ and other brackets between coordinates are equal to zero.

More generally, let $M = T^*\mathbb{A}^n$ with canonical coordinates $x_1, \dots, x_n, p_1, \dots, p_n$ be a phase space of an integrable system. Let $f_1(x_1, \dots, p_n), \dots, f_n(x_1, \dots, p_n)$ be commuting integrals of this integrable system. In this case Lagrangian submanifold¹⁷ $P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega)$ is defined by a system of equations in the variables $x_{1,i}, \dots, x_{n,i}, p_{1,i}, \dots, p_{n,i}$, $i = 1, 2, 3$ of the form

$$\begin{aligned} f_i(x_{1,1}, \dots, x_{n,1}, p_{1,1}, \dots, p_{n,1}) &= f_i(x_{1,2}, \dots, x_{n,2}, p_{1,2}, \dots, p_{n,2}), \\ f_i(x_{1,2}, \dots, x_{n,2}, p_{1,2}, \dots, p_{n,2}) &= f_i(x_{1,3}, \dots, x_{n,3}, p_{1,3}, \dots, p_{n,3}), \\ g_i(x_{1,1}, \dots, p_{n,3}) &= 0, \quad i = 1, \dots, n \end{aligned} \tag{3.16}$$

for some functions g_1, \dots, g_n . This system should satisfy the following properties:

1. Let I^{123} be the ideal generated by the equations (3.16). Then $\{I^{123}, I^{123}\} \subset I^{123}$ where $\{, \}$ is the canonical Poisson structure on the symplectic manifold $(M, -\omega) \times (M, -\omega) \times (M, \omega)$. This property means that P is a Lagrangian submanifold in $(M, -\omega) \times (M, -\omega) \times (M, \omega)$.

2. Let I^{213} be the ideal obtained from I^{123} by interchanging of variables $x_{i,1}, p_{i,1}$ and $x_{i,2}, p_{i,2}$, $i = 1, \dots, n$. Then $I^{213} = I^{123}$. This property means commutativity of our semi-classical kernel.

3. Let I^{345} be the ideal obtained from I^{123} by replacing of variables $x_{i,1}, p_{i,1}, x_{i,2}, p_{i,2}, x_{i,3}, p_{i,3}$ by the variables $x_{i,3}, p_{i,3}, x_{i,4}, p_{i,4}, x_{i,5}, p_{i,5}$ for $i = 1, \dots, n$. Let I^{12345} be the ideal generated by I^{123} and I^{345} . Then the ideal obtained from I^{12345} by excluding the variables $x_{i,3}, p_{i,3}$, $i = 1, \dots, n$ should be symmetric with respect to interchanging of variables $x_{i,2}, p_{i,2}$ and $x_{i,4}, p_{i,4}$, $i = 1, \dots, n$. This property means associativity of our semi-classical kernel.

3.2 Quantization of semi-classical kernels and quantum integrable systems

Let us discuss quantization of the picture above. Let us start with the case $M = T^*\mathbb{A}^1$. As usual, we replace the Poisson algebra in the variables $x_1, x_2, x_3, p_1, p_2, p_3$ by the ring of

¹⁷Here as usual $\omega = dx_1 \wedge dp_1 + \dots + dx_n \wedge dp_n$.

differential operators¹⁸ in $x_1, x_2, x_3, \frac{d}{dx_1}, \frac{d}{dx_2}, \frac{d}{dx_3}$. The system (3.15) should be replaced by a system of differential equations for a quantum multiplication kernel $K(x_1, x_2, x_3)$

$$\begin{aligned} \hat{f}\left(x_1, \frac{d}{dx_1}\right)^* K(x_1, x_2, x_3) &= \hat{f}\left(x_2, \frac{d}{dx_2}\right)^* K(x_1, x_2, x_3) = \hat{f}\left(x_3, \frac{d}{dx_3}\right)^* K(x_1, x_2, x_3), \\ \hat{g}\left(x_1, \frac{d}{dx_1}, x_2, \frac{d}{dx_2}, x_3, \frac{d}{dx_3}\right) K(x_1, x_2, x_3) &= 0 \end{aligned} \quad (3.17)$$

Here \hat{f}, \hat{g} are differential operators which are quantizations of f, g and $*$ is the anti-involution of the algebra of differential operators defined by $x_i^* = x_i, \left(\frac{d}{dx_i}\right)^* = -\frac{d}{dx_i}, (AB)^* = B^*A^*$.

We assume that the left ideal \hat{I} in the algebra of differential operators in $x_1, x_2, x_3, \frac{d}{dx_1}, \frac{d}{dx_2}, \frac{d}{dx_3}$ generated by the equations (3.17), satisfies the property

$$[\hat{I}, \hat{I}] \subset \hat{I}.$$

This is a quantization of the property $\{I, I\} \subset I$ in the semi-classical case.

We also assume that $K(x_1, x_2, x_3) = K(x_2, x_1, x_3)$ and the differential operator

$$\hat{g}\left(x_1, \frac{d}{dx_1}, x_2, \frac{d}{dx_2}, x_3, \frac{d}{dx_3}\right)$$

is symmetric with respect to interchanging of x_1, x_2 . This means that our quantum multiplication is commutative.

It is less clear however how to lift associativity to the quantum case. We see the following two possibility:

P1. There exists a non-zero solution $K(x_1, x_2, x_3)$ of the system (3.17) such that K is a multiplication kernel of birational type.

P2. Denote by $\hat{I}^{(123)}$ the left ideal in the algebra of differential operators generated by equations (3.17). Let $\hat{I}^{(345)}$ be obtained from $\hat{I}^{(123)}$ by replacing x_1, x_2, x_3 by x_3, x_4, x_5 . Let

$$\hat{I}^{(1245)} = (\hat{I}^{(123)} + \hat{I}^{(345)} + J) \cap R_{1245}$$

where J is the *right* ideal in the ring of differential operators in variables x_1, \dots, x_5 generated by $\frac{d}{dx_3}$ and R_{1245} is the ring of differential operators in x_1, x_2, x_4, x_5 . We assume that $\hat{I}^{(1245)}$ is symmetric with respect to interchanging of x_2 and x_4 . Notice that $\hat{I}^{(1245)}$ is the left ideal in the ring R_{1245} .

It is not clear how these two properties are related in general. Informally, if **P1** holds, then

$$\int_{\Gamma} K(x_1, x_2, x_3)K(x_3, x_4, x_5)dx_3 = \int_{\Gamma} K(x_1, x_4, x_3)K(x_3, x_2, x_5)dx_3 \quad (3.18)$$

¹⁸Informally, we replace $p_1 \mapsto \frac{d}{dx_1}, p_2 \mapsto \frac{d}{dx_2}, p_3 \mapsto -\frac{d}{dx_3}$ and choose a “correct” ordering of operators. Here we set Planck constant to one. The sign difference between derivatives by x_1, x_2 and x_3 reflects the sign difference in ω in the semi-classical case.

for any cycle Γ . This means that the l.h.s. and the r.h.s. of (3.18) should satisfy the same system of differential equations as a function in x_1, x_2, x_4, x_5 . Therefore, **P2** looks feasible but it is not clear how to make these considerations rigorous.

On the other hand, if **P2** holds, then the l.h.s. and the r.h.s of (3.18) should satisfy the same system of differential equations. In general this does not mean that the r.h.s. is obtained from the l.h.s. by a birational mapping, but it would be natural to expect this in examples.

The advantage of **P2** is that it can in principle be verified algorithmically. In this paper, however, we concentrate on **P1**.

Let us discuss quantization of semi-classical kernels in a more general case. Consider a quantum integrable system defined by commuting differential operators D_1, \dots, D_n in the variables x_1, \dots, x_n . A quantization of the system (3.16) has a form

$$\begin{aligned} D_{i,1}^* K &= D_{i,2}^* K = D_{i,3} K, \quad i = 1, \dots, n, \\ G_i K &= 0, \quad i = 1, \dots, n \end{aligned} \tag{3.19}$$

for some differential operators G_i in the variables $x_{i,j}$, $i = 1, \dots, n, j = 1, 2, 3$. Here $D_{i,j}$, $i = 1, \dots, n, j = 1, 2, 3$ are obtained from D_i by replacing the variable x_1, \dots, x_n by $x_{1,j}, \dots, x_{n,j}$, and K is a function in variables $x_{i,j}$. This system should satisfy the following properties:

1. Let \hat{I}^{123} be the left ideal generated by the equations (3.19). Then $[\hat{I}^{123}, \hat{I}^{123}] \subset \hat{I}^{123}$ where $[A, B] = AB - BA$ is the usual commutator of differential operators. This property means that the system (3.19) is holonomic.

2. Let \hat{I}^{213} be the left ideal obtained from \hat{I}^{123} by interchanging of variables $x_{i,1}$ and $x_{i,2}$, $i = 1, \dots, n$. Then $\hat{I}^{213} = \hat{I}^{123}$. This property means commutativity of our kernel.

3. Let \hat{I}^{345} be the left ideal obtained from \hat{I}^{123} by replacing of variables $x_{i,1}, x_{i,2}, x_{i,3}$ by the variables $x_{i,3}, x_{i,4}, x_{i,5}$ for $i = 1, \dots, n$. Let

$$\hat{I}^{1245} = (\hat{I}^{123} + \hat{I}^{345} + J) \cap R_{1245}$$

where J is the *right* ideal generated by $\frac{d}{dx_{i,3}}$, $i = 1, \dots, n$ and R_{1245} is the ring of differential operators in $x_{i,1}, x_{i,2}, x_{i,4}, x_{i,5}$, $i = 1, \dots, n$. Then \hat{I}^{1245} should be symmetric with respect to interchanging of the variables¹⁹ $x_{i,2}$ and $x_{i,4}$, $i = 1, \dots, n$. This property means associativity of our kernel.

3'. There exists a non-zero solution K of the system (3.19) is a multiplication kernel of birational type.

Notice that the informal discussion about the properties **P1** and **P2** of the system (3.17) is also applicable here to the properties **3** and **3'**.

Remark 3.2.1. Weinstein category (in C^∞ setting) discussed in Section 3.1 can be thought of as a semi-classical limit of the symmetric monoidal category of complex Hilbert spaces, infinite-dimensional in general. Informally, for a real symplectic manifold (M, ω) and a small

¹⁹Notice that \hat{I}^{1245} is a left ideal in the ring R_{1245} .

positive Planck constant $\hbar \ll 1$ one constructs the corresponding Hilbert space $\mathcal{H}_\hbar(M, \omega)$, the quantization of M . The dimension of this space $\dim \mathcal{H}_\hbar(M, \omega) \approx \frac{1}{(2\pi\hbar)^n} \int_M \frac{\omega^n}{n!}$ where $n = \frac{1}{2} \dim M$. Notice that $\dim \mathcal{H}_\hbar(M, \omega)$ is infinite if M has infinite volume, for example if $M = T^*X$ is a cotangent bundle. The Hilbert space $\mathcal{H}_\hbar(M, -\omega)$ is the dual to $\mathcal{H}_\hbar(M, \omega)$. The product of symplectic manifolds corresponds to the tensor product of Hilbert spaces. A Lagrangian submanifold $L \subset (M, \omega)$ corresponds approximately to a vector $\psi_L \in \mathcal{H}_\hbar(M, \omega)$ defined up to multiplication by a phase. This picture relates collections of commuting operators provided by quantum integrable systems with multiplication kernels (see the beginning of Introduction).

There is another type of quantization of real symplectic manifolds. Namely, with (M, ω) one can associate an A_∞ -category depending on small parameter $e^{-\frac{1}{\hbar}}$, the Fukaya category $\mathcal{F}(M, \omega)$. The passing to the opposite manifold $(M, -\omega)$ corresponds to the passing to the opposite category, and the product of manifolds corresponds to the tensor product of A_∞ -categories. Hence, a semi-classical multiplication kernel (i.e. the structure of an integrable system with a Lagrangian section) corresponds to a commutative monoids in symmetric monoidal $(\infty, 1)$ -category of small A_∞ -categories. In other words, we get a symmetric monoidal A_∞ -category. The basic example of such a category is $\text{Perf}(Y)$ where Y is an algebraic variety endowed with the tensor product over \mathcal{O}_Y . In this way one gets a homological mirror symmetry equivalence $\mathcal{F}(M, \omega) \sim \text{Perf}(Y)$.

3.3 Example corresponding to Hitchin systems for rank 2 bundles on the projective line with 4 regular singular points

In this section we describe the semi-classical and quantum multiplication kernels corresponding to Example 2.3.1 in Section 2.3.

Fix an affine coordinate x on \mathbb{P}^1 , and choose a point on \mathbb{P}^1 with coordinate $t \neq 0, 1, \infty$. Define $M_0 = T^*(\mathbb{P}^1 \setminus \{0, 1, t, \infty\})$, it is a subset of $\mathbb{A}^2 = T^*\mathbb{A}^1$ with coordinates x, p .

Let $B = \mathbb{A}^1$ and define a fibration $f : M_0 \rightarrow B$ by

$$f(x, p) = x(x-1)(x-t)p^2 - s^2x - \frac{k_1^2 t}{x} + \frac{k_2^2(t-1)}{x-1} - \frac{k_3^2 t(t-1)}{x-t}.$$

Any fiber of f is an elliptic curve (maybe degenerate) with 4 punctures. We define $M \supset M_0$ as a partial compactification obtained by adding 4 missing points on $f^{-1}(b)$ for each $b \in B = \mathbb{A}^1$, i.e. adding 4 copies of \mathbb{A}^1 .

Define a Lagrangian submanifold $P \subset (M, -\omega) \times (M, -\omega) \times (M, \omega)$ by

$$\begin{aligned} f(x_1, p_1) &= f(x_2, p_2) = f(x_3, p_3), \\ \frac{x_1(x_1-1)(x_1-t)}{(x_1-x_2)(x_1-x_3)}p_1 + \frac{x_2(x_2-1)(x_2-t)}{(x_2-x_1)(x_2-x_3)}p_2 - \frac{x_3(x_3-1)(x_3-t)}{(x_3-x_1)(x_3-x_2)}p_3 - s &= 0 \end{aligned} \quad (3.20)$$

where t, k_1, k_2, k_3, s are arbitrary parameters.

Define Lagrangian submanifold $N \subset M$ as the glued copy of $B = \mathbb{A}^1$ corresponding to the puncture $x = \infty$ on the generic fiber of the original map $f : M_0 = T^*\mathbb{A}^1 \rightarrow \mathbb{P}^1$.

Theorem 3.3.1. The tuple $((M, \omega), N, P)$ constructed above in this Subsection is a semi-classical multiplication kernel.

Proof. All properties of a semi-classical multiplication kernel listed in the beginning of this Section can be verified by direct computation. In particular, the last equation in the system (3.20) corresponds to addition on the elliptic curve embedded into \mathbb{A}^2 with coordinates x, p and defined by the equation

$$x(x-1)(x-t)p^2 - s^2x - \frac{k_1^2 t}{x} + \frac{k_2^2(t-1)}{x-1} - \frac{k_3^2 t(t-1)}{x-t} = b.$$

□

To quantize the above system introduce the differential operators

$$D_x = \frac{d}{dx} \cdot x(x-1)(x-t) \cdot \frac{d}{dx} - s(s+2)x - \frac{k_1^2 t}{x} + \frac{k_2^2(t-1)}{x-1} - \frac{k_3^2 t(t-1)}{x-t}, \quad (3.21)$$

$$L = \frac{x_1(x_1-1)(x_1-t)}{(x_1-x_2)(x_1-x_3)} \cdot \frac{d}{dx_1} + \frac{x_2(x_2-1)(x_2-t)}{(x_2-x_1)(x_2-x_3)} \cdot \frac{d}{dx_2} + \frac{x_3(x_3-1)(x_3-t)}{(x_3-x_1)(x_3-x_2)} \cdot \frac{d}{dx_3} - s.$$

Consider the following system of equations

$$D_{x_1}K(x_1, x_2, x_3) = D_{x_2}K(x_1, x_2, x_3) = D_{x_3}K(x_1, x_2, x_3), \quad (3.22)$$

$$LK(x_1, x_2, x_3) = 0.$$

Theorem 3.3.2. The system (3.22) satisfies the properties **1**, **2**, **3'** listed after the system (3.19). Moreover, let

$$K(x_1, x_2, x_3) = (x_1 x_2 x_3)^s F(u, v)$$

where

$$u = \frac{t^2(x_1-1)(x_2-1)(x_3-1)}{(t-1)^2 x_1 x_2 x_3}, \quad v = \frac{(x_1-t)(x_2-t)(x_3-t)}{(t-1)^2 x_1 x_2 x_3}, \quad (3.23)$$

and

$$F(u, v) = u^{-k_2} v^{-k_3} \int_{\Gamma} q_1^{2k_2} q_2^{2k_3} \left(1 + q_1 + q_2\right)^{s-k_1-k_2-k_3} \left(1 + \frac{u}{q_1} + \frac{v}{q_2}\right)^{s+k_1+k_2+k_3} \frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2}$$

where Γ is a cycle. Then $K(x_1, x_2, x_3)$ satisfies the system (3.22), and if we write

$$K(x_1, x_2, x_3) = \int_{\Gamma} \tilde{K}(x_1, x_2, x_3, q_1, q_2) dq_1 \wedge dq_2$$

where \tilde{K} is defined by (3.23), then

$$\tilde{K}(x_1, x_2, x_3, q_1, q_2) \tilde{K}(x_3, x_4, x_5, q_3, q_4) dx_3 \wedge dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4 =$$

$$\tilde{K}(x_1, x_4, \tilde{x}_3, \tilde{q}_1, \tilde{q}_2) \tilde{K}(\tilde{x}_3, x_2, x_5, \tilde{q}_3, \tilde{q}_4) d\tilde{x}_3 \wedge d\tilde{q}_1 \wedge d\tilde{q}_2 \wedge d\tilde{q}_3 \wedge d\tilde{q}_4$$

for some birational mapping $(x_3, q_1, q_2, q_3, q_4) \rightarrow (\tilde{x}_3, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4)$.

Proof. The Property **1** can be verified by direct computation²⁰ and the Property **2** is clear

²⁰In fact, the system (3.22) was obtained as a quantization of the system (3.20) with the Properties **1**, **2**.

because L is symmetric with respect to interchanging of x_1 and x_2 .

Substituting $K(x_1, x_2, x_3)$ in the form (3.23) in the system (3.22) one can check that the last equation $LK = 0$ holds for an arbitrary function $F(u, v)$ and the first two equations are equivalent to the following system for the function $F(u, v)$

$$\frac{\partial^2 G}{\partial u_1 \partial v_1} = \frac{\partial^2 G}{\partial u_2 \partial v_2} = \frac{\partial^2 G}{\partial u_3 \partial v_3} \quad (3.24)$$

where²¹

$$G = F\left(\frac{u_1 v_1}{u_3 v_3}, \frac{u_2 v_2}{u_3 v_3}\right) \left(\frac{v_1}{u_1}\right)^{k_2} \left(\frac{v_2}{u_2}\right)^{k_3} \left(\frac{v_3}{u_3}\right)^{k_1} (u_3 v_3)^s. \quad (3.25)$$

An Euler type integral representation for a solution of this system can be obtained using the theory of generalized hypergeometric functions. Let us write

$$G = \int_{\Gamma} t_1^{2k_2} t_2^{2k_3} t_3^{2k_1} (u_1 t_1 + u_2 t_2 + u_3 t_3)^{s-k_1-k_2-k_3} \left(\frac{v_1}{t_1} + \frac{v_2}{t_2} + \frac{v_3}{t_3}\right)^{s+k_1+k_2+k_3} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \frac{dt_3}{t_3}.$$

It is clear that this expression satisfies (3.24). On the other hand, after change of variables $t_1 = \frac{q_1 q_3}{u_1}$, $t_2 = \frac{q_2 q_3}{u_2}$, $t_3 = \frac{q_3}{u_3}$ and integrating out q_3 we obtain the representation (3.25) where $F(u, v)$ are given by (3.23).

Finally, notice that the family of kernels $\tilde{K}(x_1, x_2, x_3, q_1, q_2)$ is essentially the same as the family of kernels $K(x_1, x_2, y, q_1, q_2)$ in the Example 2.3.1, Section 2.3 (see also Theorem 2.3.1). These kernels are related by a gauge transformation of the form $\tilde{K}(x_1, x_2, x_3, q_1, q_2) \mapsto K(x_1, x_2, x_3, q_1, q_2) \frac{q(x_1)q(x_2)}{q(x_3)}$, a change of variables $(x_1, x_2, x_3, t) \mapsto (\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{y}, \frac{1}{t})$, and redefinition of other parameters. \square

Remark 3.3.1. Differential operators D_x and L in the system (3.22) can be written in the form

$$D_x = \frac{d}{dx} \cdot (x^3 + g_2 x + g_3) \cdot \frac{d}{dx} - s(s+2)x + \frac{k_0 + k_1 x + k_2 x^2}{x^3 + g_2 x + g_3},$$

$$L = \frac{x_1^3 + g_2 x_1 + g_3}{(x_1 - x_2)(x_1 - x_3)} \cdot \frac{d}{dx_1} + \frac{x_2^3 + g_2 x_2 + g_3}{(x_2 - x_1)(x_2 - x_3)} \cdot \frac{d}{dx_2} + \frac{x_3^3 + g_2 x_3 + g_3}{(x_3 - x_1)(x_3 - x_2)} \cdot \frac{d}{dx_3} - s$$

where $g_2, g_3, k_0, k_1, k_2, s$ are arbitrary constants. This form is convenient to study degenerations of this family. Moreover, if $s = -1$, $k_0 + k_1 x + k_2 x^2 = (q_1 + q_2 x)^2$ for some constants q_1, q_2 , then a solution of the system (3.20) can be written is a form

$$K(x_1, x_2, x_3) = \exp\left(q_1 f_1(x_1, x_2, x_3) + q_2 f_2(x_1, x_2, x_3)\right) \frac{1}{P(x_1, x_2, x_3)^{-1/2}}$$

where

$$P(x_1, x_2, x_3) = 2x_1 x_2 x_3 (x_1 + x_2 + x_3) - x_1^2 x_2^2 - x_1^2 x_3^2 - x_2^2 x_3^2 + 2g_2 (x_1 x_2 + x_1 x_3 + x_2 x_3) +$$

²¹In this formula k_1, k_2, k_3 are defined up to sign because the differential operator D_x depends on k_1^2, k_2^2, k_3^2 only. Different choice may give different solutions of the system (3.22) and any solution can be obtained in this way.

$$4g_3(x_1 + x_2 + x_3) - g_2^2$$

and $f(x_1, x_2, x_3) = q_1 f_1(x_1, x_2, x_3) + q_2 f_2(x_1, x_2, x_3)$ satisfies the system of differential equations

$$\frac{\partial f}{\partial x_3} = \frac{1}{(x_3^3 + g_2 x_3 + g_3) P(x_1, x_2, x_3)^{1/2}} \left(q_1 (2x_3^2 + x_1 x_3 + x_2 x_3 - x_1 x_2 + g_2) - \right. \\ \left. q_2 (x_1 x_2 x_3 - x_1 x_3^2 - x_2 x_3^2 + g_2 x_3 + 2g_3) \right)$$

and other two equations are obtained from this by a cyclic permutation of x_1, x_2, x_3 . This family of multiplication kernels coincides, up to a gauge transformation and redefining constants, with a family from the Example 2.2.3, Section 2.2.

Remark 3.3.2. Let D_x be differential operator given by (3.21). Define a function $K_4(x_1, x_2, x_3, x_4)$ by

$$K_4 = F \left(\frac{x_1 x_2 x_3 x_4}{t^2}, \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)(x_4 - 1)}{(t - 1)^2}, \frac{(x_1 - t)(x_2 - t)(x_3 - t)(x_4 - t)}{t^2(t - 1)^2} \right)$$

where $F(u, v, w)$ satisfies a system of differential equations

$$\frac{\partial^2 G}{\partial u_1 \partial v_1} = \frac{\partial^2 G}{\partial u_2 \partial v_2} = \frac{\partial^2 G}{\partial u_3 \partial v_3} = \frac{\partial^2 G}{\partial u_4 \partial v_4}.$$

Here G is given by

$$G = F \left(\frac{u_2 v_2}{u_1 v_1}, \frac{u_3 v_3}{u_1 v_1}, \frac{u_4 v_4}{u_1 v_1} \right) \cdot \left(\frac{v_2}{u_2} \right)^{k_1} \left(\frac{v_3}{u_3} \right)^{k_2} \left(\frac{v_4}{u_4} \right)^{k_3} \left(\frac{v_1}{u_1} \right)^{s+1} \cdot \frac{1}{u_1 v_1}.$$

Then K_4 satisfies the equations

$$D_{x_1} K_4 = D_{x_2} K_4 = D_{x_3} K_4 = D_{x_4} K_4.$$

One can obtain an Euler type integral representation (similar to the one in Theorem 3.3.2) using the theory of generalized hypergeometric functions [3]. It looks feasible that K_4 gives an example of a structure discussed in Section 1.6, problem 2. See also Remark 3.4.2 for the generalization of the kernel K_4 .

3.4 Example corresponding to Hitchin systems for rank 2 bundles on the projective line with more than 4 regular singular points

Fix an affine coordinate x on \mathbb{P}^1 , and choose $n > 1$ pairwise distinct points \mathbb{P}^1 with coordinates $t_1, \dots, t_n \neq 0, 1, \infty$. Introduce a differential operator

$$D_x = \frac{\partial}{\partial x} \cdot x(x - 1)(x - t_1) \dots (x - t_n) \frac{\partial}{\partial x} - s(s + n + 1)x^n - \sum_{i=-1}^n \frac{k_{i+2}^2 \prod_{j \neq i} (t_i - t_j)}{x - t_i}$$

where $t_{-1} = 0, t_0 = 1$ and s, k_1, \dots, k_{n+2} are generic parameters. Define furthermore

$$L_{x_1, \dots, x_{n+1}} = \sum_{i=1}^{n+1} \frac{1}{\prod_{j \neq i} (x_i - x_j)} D_{x_i},$$

$$M_{x_1, \dots, x_{n+1}} = \sum_{i=1}^{n+1} \frac{x_i(x_i - 1)(x_i - t_1) \dots (x_i - t_n)}{\prod_{j \neq i} (x_i - x_j)} \cdot \frac{\partial}{\partial x_i} - s.$$

Theorem 3.4.1. Define the kernel

$$K_n(x_1, \dots, x_{2n+1}) = \int K(x_1, \dots, x_{2n+1}, q_1, \dots, q_{n+1}) dq_1 \dots dq_{n+1}$$

where the kernel in the r.h.s. is given by (2.11). Then the kernel K_n satisfies the following system of holonomic differential equations

$$L_{x_{i_1}, \dots, x_{i_{n+1}}} K_n = 0, \quad (3.26)$$

$$M_{x_{i_1}, \dots, x_{i_{n+1}}} K_n = 0 \quad (3.27)$$

where $1 \leq i_1 < \dots < i_{n+1} \leq 2n + 1$.

Proof. It is similar to proof of Theorem 3.3.2 from the previous Section. The system (3.26) has the following general solution in terms of an arbitrary function $F(w_1, \dots, w_{n+1})$

$$K_n = (x_1 \dots x_{2n+1})^s F(w_0, \dots, w_n)$$

where

$$w_0 = \frac{(x_1 - 1) \dots (x_{2n+1} - 1) t_1^2 \dots t_n^2}{x_1 \dots x_{2n+1} (t_1 - 1)^2 \dots (t_n - 1)^2}, \text{ and } w_i = \frac{(x_1 - t_i) \dots (x_{2n+1} - t_i) \prod_{j \neq i} t_j^2}{x_1 \dots x_{2n+1} (t_i - 1)^2 \prod_{j \neq i} (t_i - t_j)^2}, \quad i = 1, \dots, n.$$

The system (3.27) can be written in terms of F as

$$\frac{\partial^2 G}{\partial u_1 \partial v_1} = \frac{\partial^2 G}{\partial u_2 \partial v_2} = \dots = \frac{\partial^2 G}{\partial u_{n+2} \partial v_{n+2}}$$

where

$$G = F\left(\frac{u_2 v_2}{u_1 v_1}, \dots, \frac{u_{n+2} v_{n+2}}{u_1 v_1}\right) \cdot \left(\frac{u_1}{v_1}\right)^{k_1} \dots \left(\frac{u_{n+2}}{v_{n+2}}\right)^{k_{n+2}} \cdot (u_1 v_1)^s.$$

Our integral representation for the kernel K is the Euler type representation in the theory of generalized hypergeometric functions [3]. See also the proof of Theorem 3.3.2 and the system (2.12). \square

Remark 3.4.1. The operator D_x can be also written in the form

$$D_x = \frac{\partial}{\partial x} \cdot P_{n+2}(x) \frac{\partial}{\partial x} - s(s + n + 1)x^n + \frac{Q_{n+1}(x)}{P_{n+2}(x)}$$

where $P_{n+2}(x)$, $Q_{n+1}(x)$ are arbitrary polynomials in x of degrees $n+2$ and $n+1$ respectively. In this case the operator $L_{x_1, \dots, x_{n+1}}$ is given by the same formula and

$$M_{x_1, \dots, x_{n+1}} = \sum_{i=1}^{n+1} \frac{P_{n+2}(x_i)}{\prod_{j \neq i} (x_i - x_j)} \cdot \frac{\partial}{\partial x_i} - s.$$

Conjecture 3.4.1 Holonomic cyclic D -module given by (3.26), (3.27) defines a generalized product in the sense of Section 4.1. This D -module seems to have the semi-classical limit described in abstract terms in Section 4.2, and corresponds to Hitchin integrable system for rank 2 bundles with regular singular points at $0, 1, t_1, \dots, t_n, \infty$. The fixed point u_{i_0} in notations of Section 4.2 is point ∞ .

Remark 3.4.2. Define a family of differential operators by

$$D_x = \prod_{i=1}^n (x - t_i) \cdot \left(\frac{\partial^2}{\partial x^2} - \sum_{j=1}^n \frac{b_{j,1} + b_{j,2} - 1}{x - t_j} \cdot \frac{\partial}{\partial x} \right) + \sum_{i=1}^n \frac{b_{i,1} b_{i,2} \prod_{j \neq i} (t_i - t_j)}{x - t_i}$$

where we assume $\sum_{i=1}^n (b_{i,1} + b_{i,2}) = n - 2$.

The operator D_x has n regular singular points at $x = t_1, \dots, t_n$ and solutions of the equation $D_x f(x) = 0$ near $x = t_i$ have a form $f(x) = (x - t_i)^{b_{i,j}} (1 + O(x - t_i))$ for $i = 1, \dots, n$, $j = 1, 2$. Moreover, any differential operator with these properties and the same symbol as D_x has form $D_x + \lambda_1 + \lambda_2 x + \dots + \lambda_{n-3} x^{n-4}$ where $\lambda_1, \dots, \lambda_{n-3}$ are arbitrary parameters.

Fix an integer l such that $2 \leq l \leq n$ and construct a function $K_{n,l}(x_1, \dots, x_{l+n-4})$ in $n+l-4$ variables as follows

$$K_{n,l} = F \left(q_1 \frac{(x_1 - t_1) \dots (x_{l+n-4} - t_1)}{(x_1 - t_l) \dots (x_{l+n-4} - t_l)}, \dots, q_{l-1} \frac{(x_1 - t_{l-1}) \dots (x_{l+n-4} - t_{l-1})}{(x_1 - t_l) \dots (x_{l+n-4} - t_l)} \right) \times \\ ((x_1 - t_l) \dots (x_{l+n-4} - t_l))^{-b_{l+1,1} - \dots - b_{n,1}} \cdot \prod_{l+1 \leq k \leq n} ((x_1 - t_k) \dots (x_{l+n-4} - t_k))^{b_{k,1}}$$

where

$$q_i = \prod_{\substack{1 \leq j \leq l-1 \\ j \neq i}} \frac{(t_l - t_j)^2}{(t_i - t_j)^2} \cdot \prod_{l+1 \leq j \leq n} \frac{t_l - t_j}{t_i - t_j}$$

and F satisfies the system of differential equations²²

$$\frac{\partial^2 G}{\partial u_{1,1} \partial u_{1,2}} = \frac{\partial^2 G}{\partial u_{2,1} \partial u_{2,2}} = \dots = \frac{\partial^2 G}{\partial u_{l,1} \partial u_{l,2}}.$$

Here

$$G = F \left(\frac{u_{1,1} u_{1,2}}{u_{l,1} u_{l,2}}, \dots, \frac{u_{l-1,1} u_{l-1,2}}{u_{l,1} u_{l,2}} \right) \cdot \prod_{\substack{1 \leq \alpha \leq l \\ 1 \leq \beta \leq 2}} u_{\alpha, \beta}^{-b_{\alpha, \beta}} \cdot (u_{l,1} u_{l,2})^{-b_{l+1,1} - \dots - b_{n,1}}.$$

²²One can write solutions of this system in terms of Euler type integrals [3].

Notice that if $l = n$, then in the formulas above we have $-b_{l+1,1} - \dots - b_{n,1} = 0$ and $\prod_{l+1 \leq j \leq n} = 1$. Moreover, in this case the kernel $K_{n,l}$ is a function of $2g+2$ variables (here $g = n-3$ is the genus of the spectral curve), and is covariant with respect to the full symmetry group $S_n \times (\mathbb{Z}/2)^n$ of the problem.

Define furthermore

$$L_{x_1, \dots, x_{n-2}} = \sum_{i=1}^{n-2} \frac{1}{\prod_{j \neq i} (x_i - x_j)} D_{x_i}.$$

Then function $K_{n,l}$ satisfies the following system of holonomic differential equations

$$L_{x_{i_1}, \dots, x_{i_{n-2}}} K_{n,l} = 0 \tag{3.28}$$

where $1 \leq i_1 < \dots < i_{n-2} \leq l + n - 4$.

It will be interesting to understand for which n, l the kernel $K_{n,l}$ gives an example of structures discussed in Section 1.6.

4 Separation of variables

4.1 Families of functions in one variable, and generalized products

This section can be considered as a continuation of Section 1.4. We will describe the general framework for the Sklyanin method of separation of variables in a broad and informal way.

Recall that in Section 1.4 we consider a situation when we have a collection of functions $\psi_{\lambda_1, \dots, \lambda_n}(x_1, \dots, x_n)$ in n variables x_1, \dots, x_n depending on n parameters $\lambda_1, \dots, \lambda_n$. These functions are the normalized eigenfunctions of a family of commuting operators. Moreover, these functions form in a sense a continuous basis of the algebra of functions in x_1, \dots, x_n , thus giving a new product $*$ on functions.

Now let us consider a different situation: suppose we have a collection $\phi_{\lambda_1, \dots, \lambda_n}(x)$ of functions in one variable x depending on n parameters $\lambda_1, \dots, \lambda_n$. Then we can construct a collection of functions in n variables x_1, \dots, x_n (symmetric under permutations) and depending again on n parameters by

$$\psi_{\lambda_1, \dots, \lambda_n}(x_1, \dots, x_n) = \phi_{\lambda_1, \dots, \lambda_n}(x_1) \dots \phi_{\lambda_1, \dots, \lambda_n}(x_n).$$

Hence one can expect that the functions $\psi_{\lambda_1, \dots, \lambda_n}$ form a continuous basis of space of S_n -invariant functions in x_1, \dots, x_n .

Conversely, functions $\psi_{x_1, \dots, x_n}^*(\lambda_1, \dots, \lambda_n) = \psi_{\lambda_1, \dots, \lambda_n}(x_1, \dots, x_n)$ form a continuous basis of the space of functions in $\lambda_1, \dots, \lambda_n$, this is an analogue of the inverse Fourier transform.

Example 4.1.1. Let $\phi_{\lambda_1, \dots, \lambda_n}(x) = e^{\lambda_1 x + \lambda_2 x^2 + \dots + \lambda_n x^n}$. Then

$$\psi_{\lambda_1, \dots, \lambda_n}(x_1, \dots, x_n) = e^{\lambda_1(x_1 + \dots + x_n)} e^{\lambda_2(x_1^2 + \dots + x_n^2)} \dots e^{\lambda_n(x_1^n + \dots + x_n^n)}.$$

Notice that $h_1 = x_1 + \dots + x_n, \dots, h_n = x_1^n + \dots + x_n^n$ are coordinates on $Sym^n \mathbb{A}^1 = \mathbb{A}^n$. Hence, we see that the functions $\psi_{\lambda_1, \dots, \lambda_n}$ forms the continuous basis of Fourier modes.

For any x_1, \dots, x_{n+1} consider the function in $\lambda_1, \dots, \lambda_n$ given by

$$(\lambda_1, \dots, \lambda_n) \mapsto \phi_{\lambda_1, \dots, \lambda_n}(x_1) \dots \phi_{\lambda_1, \dots, \lambda_n}(x_{n+1}).$$

Let us expand this function in the continuous basis $\psi_{x_1, \dots, x_n}^*(\lambda_1, \dots, \lambda_n)$. Then we get the generalization of multiplication formulas in Section 1.3.

$$\phi_{\lambda_1, \dots, \lambda_n}(x_1) \dots \phi_{\lambda_1, \dots, \lambda_n}(x_{n+1}) = \int K_{n+1, n}(x_1, \dots, x_{n+1}, y_1, \dots, y_n) \phi_{\lambda_1, \dots, \lambda_n}(y_1) \dots \phi_{\lambda_1, \dots, \lambda_n}(y_n) dy_1 \dots dy_n$$

for some kernel $K_{n+1, n}$ depending on $2n + 1$ variables. This kernel satisfies a generalized associativity condition (see also Section 1.5):

$$\int K_{n+1, n}(x_1, \dots, x_{n+1}, z_1, \dots, z_n) K_{n+1, n}(z_1, \dots, z_n, x_{n+2}, y_1, \dots, y_n) dz_1 \dots dz_n$$

is symmetric with respect of permutations of x_1, \dots, x_{n+2} . Formally, this property implies that the kernel in $3n$ variables given by

$$K_{2n, n}(x_1, \dots, x_n, x'_1, \dots, x'_n, y_1, \dots, y_n) = \int K_{n+1, n}(x_1, \dots, x_n, x'_1, z_1^{(1)}, \dots, z_n^{(1)}) \cdot K_{n+1, n}(z_1^{(1)}, \dots, z_n^{(1)}, x'_2, z_1^{(2)}, \dots, z_n^{(2)}) \dots K_{n+1, n}(z_1^{(n-1)}, \dots, z_n^{(n-1)}, x'_n, y_1, \dots, y_n) \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} dz_j^{(i)}$$

gives a commutative associative product on the space of S_n -invariant functions in n variables. In particular, $K_{2n, n}$ is symmetric with respect of permutations of x'_1, \dots, x'_n .

The above informal considerations can be done formally in the linear algebra framework.

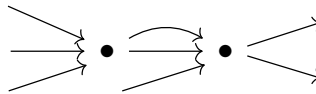
Definition 4.1.1. For $n \geq 1$ the generalized commutative associative product on a vector space V is a linear map

$$\mu_{n+1, n} : V^{\otimes(n+1)} \rightarrow V^{\otimes n}$$

which is invariant under $S_{n+1} \times S_n$ -action, i.e. it factorises as

$$V^{\otimes(n+1)} \rightarrow (V^{\otimes(n+1)})_{S_{n+1}} \rightarrow (V^{\otimes n})_{S_n} \hookrightarrow V^{\otimes n},$$

and the map $\mu_{n+2, n} : V^{\otimes(n+2)} \rightarrow V^{\otimes n}$ given by $\mu_{n+1, n} \circ (\mu_{n+1, n} \otimes id_V)$ is $S_{n+2} \times S_n$ -invariant. For $n = 2$ this map can be represented by the picture

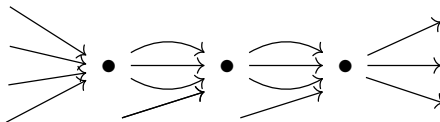


This definition of a generalized commutative associative product makes sense in arbitrary symmetric monoidal category.

Proposition 4.1.1. Let us define $\mu_{2n,n} : V^{\otimes 2n} \rightarrow V^{\otimes n}$ as the composition of $\mu_{n+1,n} \otimes id_V^{\otimes k}$ for $k = n - 1, \dots, 1, 0$. For example, for $n = 3$ we have

$$\mu_{6,3} = \mu_{4,3} \circ (\mu_{4,3} \otimes id_V) \circ (\mu_{4,3} \otimes id_V^{\otimes 2}).$$

This map can be also represented by the picture



Then $\mu_{2n,n}$ defines a structure of a commutative associative (possibly non-unital) algebra on $Sym^n V$.

4.2 Semi-classical generalized products

In this section we describe a class of Poisson compactifications (introduced in Section 8.3 in [9]) of cotangent bundles to curves, and a geometric construction of semi-classical generalized products.

Let C be a smooth, not necessarily compact, algebraic curve over \mathbb{C} . Write $C = \overline{C} \setminus S$ where $S = \{s_1, \dots, s_m\}$ is a finite subset of the compact curve \overline{C} . Denote by \mathcal{P}_0 the compact surface which is the total space of \mathbb{P}^1 -bundle over \overline{C} , given by $\mathbb{P}(\mathcal{O}_{\overline{C}} \oplus T_{\overline{C}, \log S}^*)$. Surface \mathcal{P}_0 carries a natural Poisson structure γ_0 with the symplectic leaf isomorphic to T^*C . Poisson structure γ_0 vanishes on the complement $\mathcal{P}_0 \setminus T^*C$, which is the union of smooth divisors $\overline{C}_\infty \cong \overline{C}$ and $T_{s_i}^* \overline{C} \cong \mathbb{A}^1$, $i = 1, \dots, m$ intersecting transversally.

Symplectic form $\omega = \gamma_0^{-1}$ has poles of order 2 along the horizontal divisor \overline{C}_∞ , and of order one along vertical divisors $T_{s_i}^* \overline{C}$. Starting with Poisson surface \mathcal{P}_0 , let us construct a sequence of Poisson surfaces $(\mathcal{P}_i, \gamma_i)$, $i = 0, 1, \dots, N$ for some $N \geq 0$ recursively by applying a sequence of blowups of the following type. Let $p = p_i$ be a point²³ in \mathcal{P}_i such that Poisson tensor γ_i vanishes at p , and there are local coordinates x, y near p such that $\gamma_i = x^a y^b (1 + O(x, y)) \partial_x \partial_y$, point p has coordinates $x = y = 0$, and $a + b \geq 2$ for $a, b \in \mathbb{Z}_{\geq 0}$. Then we make a blowup at p and obtain a new Poisson surface $\mathcal{P}_{i+1} = Bl_p \mathcal{P}_i$ with Poisson tensor γ_{i+1} . The Poisson tensor γ_{i+1} vanishes with order $a + b - 1$ at the exceptional divisor.

Let $\mathcal{P} = \mathcal{P}_N$ be the final term of our sequence. The divisor D of zeros of the Poisson structure $\gamma = \gamma_N$ on \mathcal{P} has simple normal crossing. The open dense symplectic leaf $\mathcal{P} \setminus D$ is equal to T^*C .

Denote by $\{D_\alpha\}$ the set of irreducible components of D at which γ vanishes with multiplicity one (or, equivalently, the meromorphic symplectic form $\omega = \gamma^{-1}$ has pole of order one)²⁴. It follows by induction from the construction that divisors D_α do not intersect each other.

²³Our sequence \mathcal{P}_i depends on choices of these points.

²⁴We will not use other components of D in our considerations.

Moreover, each D_α contains exactly one double point of D , and the complement D_α^0 to this point is isomorphic to \mathbb{A}^1 . In fact, there is a canonical coordinate on D_α^0 given by the residue of the restriction of the Liouville 1-form on T^*C to a small disc in \mathcal{P} transversally intersecting D_α^0 .

Remark 4.2.1. One can associate with each component D_α as above, a point $v \in \overline{C}$ and a Puiseux series $f(x) \in \cup_{n \geq 1} \mathbb{C}((x^{\frac{1}{n}})) = \overline{\mathbb{C}((x))}$ where x is a local coordinate at v , up to certain identification. First, we identify two series differ by $\cup_{n \geq 1} \mathbb{C}[[x^{\frac{1}{n}}]]$. In other words, we can keep only terms with strictly negative exponents. Second, we identify series which differ by the action of the Galois group $\widehat{\mathbb{Z}}$ of $\overline{\mathbb{C}((x))}$, with the topological generator $x^{\frac{1}{n}} \mapsto e^{\frac{2\pi i}{n}} x^{\frac{1}{n}}$.

For a given component D_α the corresponding Puiseux series is defined as follows. Let us choose a germ Σ_α of smooth curve in \mathcal{P} intersecting D_α^0 transversely at one point, and such that Σ_α projects to \overline{C} non-trivially, i.e. not to a point. Then the punctured disc $\Sigma_\alpha \setminus \Sigma_\alpha \cap D_\alpha^0$ contained in T^*C can be considered as the graph of a meromorphic multivalued 1-form at a point $v \in \overline{C}$. This form can be written in local coordinate x at v as $df(x) + \lambda d \log x$ where $\lambda \in \mathbb{C}$, $f(x) \in \overline{\mathbb{C}((x))}$. One can show that different choices of germ Σ_α give equivalent series $f(x)$ in the above sense. Incidentally, the equivalence classes of such series correspond to all the possible irregular terms for formal meromorphic connections on $\mathbb{C}((x))$.

Let us choose a collection of points $\sum_i n_i u_i$, $n_i \geq 1$ in $\coprod_\alpha D_\alpha^0 = \coprod_\alpha \mathbb{A}_\alpha^1$ with multiplicities, and an integer $r \geq 1$. With the tuple $(\mathcal{P}, \sum_i n_i u_i, r)$ we associate a classical integrable system. The base B of this system will be the set of smooth connected curves $\Sigma \subset \mathcal{P}$ (spectral curves) such that $\Sigma \not\subset D$ and $\Sigma \cap D = \sum_i n_i u_i$, and the projection $\Sigma \rightarrow \overline{C}$ has degree r .

Let us assume that $B \neq \emptyset$ (this assumption considered as a property of $(\mathcal{P}, \sum_i n_i u_i, r)$ is related with the additive Deligne-Simpson problem [10]). One can show that B is an open dense subset of \mathbb{A}^g where g is the genus of any spectral curve Σ . The tangent space $T_\Sigma B$ is canonically identified with $\Gamma(\Sigma, \Omega_\Sigma^1)$.

Let us fix an integer $d \in \mathbb{Z}$. Define M_d to be the space of pairs $(\Sigma, [L])$ where $[L] \in \text{Pic}_d(\Sigma)$ is a class of a line bundle of degree d .

There is a natural symplectic form ω_{M_d} on M_d , and the natural projection $M_d \rightarrow B$ is a Liouville integrable system.

The above construction gives an alternative description of a Zariski open dense part of Hitchin integrable systems for group GL_r on \overline{C} with possibly irregular singularities.

Notice that among integrable systems $M_d \rightarrow B$ only one, corresponding to $d = 0$, has an obvious Lagrangian section, which is the zero section.

On the other hand, we can define a birational symplectomorphism²⁵ $M_g \sim \text{Sym}^g T^*C \sim T^* \text{Sym}^g C$. Indeed, we have $\text{Pic}_g(\Sigma) \sim \text{Sym}^g \Sigma$. Hence, a generic point in M_g is a spectral curve Σ and a collection of g points (t_1, \dots, t_g) of Σ up to a permutation. Generically all points t_i will be distinct and do not belong to $\Sigma \cap D$, hence give g points in T^*C . Conversely, given g generic points in T^*C , there exists a unique spectral curve Σ passing through them, and therefore we

²⁵We use the notation \sim for birational equivalence.

obtain a point in M_g .

This is a geometric version of method of separation of variables. This is different from the usual Hitchin systems, where the total space is identified birationally with the cotangent bundle to the moduli space $Bun_G(\overline{C})$ of G -bundles on \overline{C} , or its version associated with marked points and irregular singularities.

Now we can define a semi-classical generalized product. Let us pick a point u_{i_0} among the collection $\{u_i\}$. Notice that $u_{i_0} \in \Sigma$ for any spectral curve $\Sigma \in B$. Using u_{i_0} we can identify M_d for all $d \in \mathbb{Z}$ by adding multiples of u_{i_0} . Also, we define a Lagrangian subvariety

$$L_{g+1,g} \subset Sym^{g+1}(T^*C, -\omega) \times Sym^g(T^*C, \omega)$$

by the formula

$$L_{g+1,g} = \{(a_1, \dots, a_{g+1}, b_1, \dots, b_g); \text{ there exists } \Sigma \in B \text{ such that} \\ a_1, \dots, a_{g+1}, b_1, \dots, b_g \in \Sigma, \quad a_1 + \dots + a_{g+1} = b_1 + \dots + b_g + u_{i_0} \in Pic(\Sigma)\}.$$

This variety might be singular, so we treat it only as a first rough approximation.

It is clear that $L_{g+1,g}$ considered as a morphism in Weinstein category, is a generalized product.

Similarly, one can define Lagrangian correspondences $L_{g+k,g}$ for any $k \geq 1$. In the case $k = g$ we obtain a semi-classical multiplication kernel for the symplectic variety $M_0 = M_g$.

Remark 4.2.2. The choice of a point u_{i_0} corresponds in a sense to the choice of a normalization of functions $\phi_{\lambda_1, \dots, \lambda_n}(x)$ in Section 4.1, and therefore, to the choice of a normalization of functions $\psi_{\lambda_1, \dots, \lambda_n}(x_1, \dots, x_n)$.

4.3 Towards quantized generalized product

In this section we will propose a hypothetical construction of the quantization of the semi-classical generalized product introduced in Section 4.2, understood as a holonomic D -module on C^{2g+1} together with a cyclic vector.

Let $(\mathcal{P}, \sum_i n_i u_i, r)$ be a tuple as in Section 4.2. We now assume that for any two different points $u_i \neq u_j$ belonging to the same component $D_\alpha^0 = \mathbb{A}^1$ the difference $\mu_i - \mu_j$ between their canonical coordinates is not an integer. Starting with such a tuple we construct a family of cyclic D -modules on \overline{C} depending on g parameters $\lambda_1, \dots, \lambda_g$. Namely, consider meromorphic differential operators L of order r acting from the trivial line bundle $\mathcal{O}_{\overline{C}}$ to $K_{\overline{C}}^{\otimes r}$, with symbol $\left(\frac{\partial}{\partial x}\right)^r$ in local coordinates, and such that singularities of solutions correspond to $\sum_i n_i u_i$. To explain the latter condition more precisely, recall that a point $u_i \in D$ belongs to a component D_{α_i} of the divisor D . In Section 4.2 we explained that divisor D_{α_i} gives a Puiseux series $f_i(x)$ at point $v_i = pr_{D \rightarrow \overline{C}}(u_i) \in \overline{C}$. Denote by $x = x_i$ the local coordinate at v_i , and by $\mu_i \in \mathbb{C}$ the position of point $u_i \in D$ on $D_{\alpha_i}^0 = \mathbb{A}^1$. Then we say that differential equation $L\phi = 0$ has

singularity at v_i corresponding to u_i and $n_i \geq 1$ if this equation has solutions with asymptotic behaviour

$$\phi(x) = x^{\mu_i} e^{f_i(x)} \log(x)^k (1 + \dots), \quad k = 0, \dots, n_i - 1.$$

One can show that differential operators L satisfying these properties form an affine space of dimension g (a version of variety of opers for the Hitchin system for group GL_r), and can be written as $L = L_0 + \sum_{i=1}^g \lambda_i L_i$ where $\deg L_0 = r$, $\deg L_i < r$, $i = 1, \dots, g$. Here $(\lambda_1, \dots, \lambda_g)$ are parameters (coordinates on the space of opers).

Recall that in Section 4.2 we made a choice u_{i_0} of one point of $\Sigma \cap D$. Let us assume that the corresponding Puiseux series f_{i_0} is unramified²⁶, i.e. belong to $\mathbb{C}((x)) \subset \cup_{n \geq 1} \mathbb{C}((x^{\frac{1}{n}}))$. We define the normalized solution corresponding to u_{i_0} as the unique formal solution at v_{i_0} of the form

$$\phi_{\lambda_1, \dots, \lambda_g}(x) = x^{\mu_{i_0}} e^{f_{i_0}(x)} (1 + \sum_{j \geq 1} P_j x^j)$$

where P_1, P_2, \dots depend on $\lambda_1, \dots, \lambda_g$. One can see that P_j are polynomials in $\lambda_1, \dots, \lambda_g$, we set $P_0 = 1$.

Without loss of generality, in order to simplify the exposition, we assume that $\mu_{i_0} = 0$, $f_{i_0}(x) = 0$. This can be achieved by the conjugation of L .

Examples suggest that the following is true:

The set $\{P_{i_1} \dots P_{i_g}; i_1 \leq \dots \leq i_g\}$ forms a linear basis of $\mathbb{C}[\lambda_1, \dots, \lambda_g]$.

Assuming this property, we can define coefficients

$$C_{i_1, \dots, i_{g+1}}^{j_1, \dots, j_g} = \text{Coeff}_{P_{j_1} \dots P_{j_g}}(P_{i_1} \dots P_{i_{g+1}}).$$

These coefficients are structure constants of a generalized commutative associative product on an infinite-dimensional space with a basis e_0, e_1, \dots where e_j corresponds to P_j .

Remark 4.3.1. In general, suppose that A is a commutative associative algebra over a field k of characteristic zero, and $V \subset A$ is a vector subspace such that the composition $\text{Sym}^g V \rightarrow V^{\otimes g} \rightarrow A^{\otimes g} \rightarrow A$ is an isomorphism of vector spaces, where the last map is induced by multiplication in A . Then V carries a structure of commutative associative generalised product $\text{Sym}^{g+1} V \rightarrow \text{Sym}^g V = A$.

Let us consider generating series

$$K_{g+1, g}(x_1, \dots, x_{g+1}, y_1, \dots, y_g) = \sum_{i_1, \dots, i_g \geq 0} C_{i_1, \dots, i_{g+1}}^{j_1, \dots, j_g} x_1^{i_1} \dots x_{g+1}^{i_{g+1}} y_1^{-j_1} \dots y_g^{-j_g} \prod_{k=1}^g \frac{dy_k}{y_k}. \quad (4.29)$$

We consider $K_{g+1, g}$ as an element of W/W_+ where $W = \mathbb{C}[[y_1, \dots, y_g]][y_1^{-1}, \dots, y_g^{-1}][[x_1, \dots, x_{g+1}]]$ and $W_+ \subset W$ is the subspace consisting of series which do not have poles in variable y_i for some $i = 1, \dots, g$.

²⁶We do not know how to extend our construction to the case of ramified f_{i_0} .

Consider algebraic variety $X = \overline{C}^{2g+1}$ with the line bundle $\mathcal{L} = \otimes_{i=g+2}^{2g+1} pr_i^* T_{\overline{C}}^*$ and a point $p = \underbrace{(v_{i_0}, \dots, v_{i_0})}_{2g+1 \text{ times}}$. Then we have an algebra Dif_{rat} of differential operators on \mathcal{L} with coefficients

in rational functions on X (i.e. differential operators at the generic point of X). It has a subalgebra $\text{Dif}_{rat,p}$ of differential operators without poles at p . Vector spaces W, W_+ and hence W/W_+ are $\text{Dif}_{rat,p}$ -modules.

Conjecture 4.3.1. The cyclic $\text{Dif}_{rat,p}$ -module $\mathbf{K}_{g+1,g}$ generated by $K_{g+1,g}$ is holonomic. Moreover, $\text{Dif}_{rat} \otimes_{\text{Dif}_{rat,p}} \mathbf{K}_{g+1,g}$ is the kernel of the generalized product localized at the generic point of $X = \overline{C}^{2g+1}$.

Examples suggest that this kernel can be written in an explicit form, see equations (1.4), (1.6) in Introduction where the integration is understood as the direct image of holonomic D -modules.

Remark 4.3.2. The example of a family of functions $\phi_{\lambda_1, \dots, \lambda_g}(x) = e^{\lambda_1 x + \dots + \lambda_g x^g}$ does not fit to the above scheme. Here $\phi_{\lambda_1, \dots, \lambda_g}(x)$ is the unique solution of the equation $(L_0 + \lambda_1 L_1 + \dots + \lambda_g L_g)\phi(x) = 0$ in $\mathbb{C}[[x]]$ with constant term 1. Here $L_0 = \frac{\partial}{\partial x}$, $L_i = -ix^{i-1}$, $i = 1, \dots, g$. It will be interesting to find a general framework which includes this example.

Conjecture 4.3.1 is formulated in terms of the quotient W/W_+ which is a complicated object not suitable for explicit computations. It is more convenient to see $K_{g+1,g}$ in (4.29) literally as a function of x_i, y_i . For example, we can assume that $\overline{C} = \mathbb{P}^1$ and x is a global coordinate.

The examples studied in the next section indicate that the cyclic D -module for the lifted kernel is still holonomic, and it maps epimorphically to the expected generalized multiplication kernel.

Remark 4.3.3. The quantum generalized multiplication kernel should produce isomorphisms of the following type. For all $\vec{\lambda} = (\lambda_1, \dots, \lambda_g)$ we expect that

$$(C^{2g+1} \rightarrow C^{g+1})_* (\mathbf{K}_{g+1,g} \otimes \underbrace{(\mathcal{E}_{\vec{\lambda}} \boxtimes \dots \boxtimes \mathcal{E}_{\vec{\lambda}})}_{g \text{ times}}) \simeq \underbrace{\mathcal{E}_{\vec{\lambda}} \boxtimes \dots \boxtimes \mathcal{E}_{\vec{\lambda}}}_{g+1 \text{ times}} \quad (4.30)$$

where $\mathcal{E}_{\vec{\lambda}}$ is a cyclic holonomic D -module (oper) parameterized by $\vec{\lambda}$. It is known in the theory of integrable systems that the locus ofopers is a Lagrangian subvariety in the symplectic manifold parameterizing all (non-cyclic) holonomic D -modules on C with given singularities (de Rham moduli space). We claim that the equation (4.30) holds also for more general modules. The rough reason is that (4.30) makes sense in Betti realization, in which the locus ofopers is Zariski dense.

4.4 A generalization to other Poisson surfaces

In Section 4.2 we made a sequence of blowups of the Poisson surface $\mathcal{P}_0 = \mathbb{P}(\mathbb{O}_{\overline{C}} \oplus T_{\overline{C}, \log S}^*)$. Assume that $S \neq \emptyset$, hence C is affine. The quantization of the open dense symplectic leaf of \mathcal{P}_0 is the algebra of differential operators on C . Let us refer to this class of Poisson surfaces

as to “rational”. There are two other classes of compact Poisson surfaces, which we will call “trigonometric” and “elliptic”, with an open dense symplectic leaf M . In all three cases the open dense symplectic leaf M is an affine variety.

In the trigonometric case \mathcal{P} is any toric compactification of its symplectic leaf $M = \mathbb{C}^* \times \mathbb{C}^*$ with $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$. The quantization of M is quantum torus $A_q = \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1} \rangle / XY = qYX$.

In the elliptic case $\mathcal{P} = \mathbb{C}P^2$ with the symplectic leaf $M = \mathcal{P} \setminus \{\text{cubic curve}\}$. The quantization is an inhomogeneous version of Sklyanin algebra with three generators.

In the elliptic case, there is also a version with non-affine M similar to T^*C for compact curve C . Namely, let $\mathcal{P} = \mathbb{P}^1 \times E$ where E is an elliptic curve and $M = \mathbb{C}^* \times E$. In this case instead of algebras one should consider abelian or triangulated categories (analogs of categories of modules).

In our considerations in Section 4.2 the principal role was played by divisors on the blowup at which ω has pole of order one.

In the trigonometric case such divisors do not have continuous parameters and correspond to pairs of coprime integers (a, b) , i.e. these divisors are toric divisors. Each D_α^0 is isomorphic to \mathbb{C}^* .

In the elliptic case there is only one such a divisor, the initial cubic curve E (and there are no non-trivial blowups).

Similarly to Section 4.2, we choose a sequence of blowups (trivial in the elliptic case) and a collection of points with multiplicities on $\coprod_\alpha D_\alpha^0$.

In this way we obtain a classical integrable system. Choosing one point u_{i_0} we get a semi-classical kernel $K_{g+1,g}$. The analog of opers will be cyclic holonomic A -modules where A is the quantization of M . The quantum kernel should be a cyclic holonomic $A^{\otimes(g+1)} \otimes (A^{op})^{\otimes g}$ -module.

Notice that the rational case $M = T^*C$ is related to the geometric Langlands correspondence for groups GL_r . In the trigonometric (resp. elliptic) cases the multiplication kernels $K_{g+1,g}$ are also expected to be a kind of motivic in trigonometric (resp. elliptic) sense. For example, the q -analog of motivic holonomic modules are discussed in Section 6.1 of [8]. Roughly, these modules are build from the basic A_q -modules with cyclic vector $A_q/A_q \cdot (X + Y - 1)$ by external tensor products, actions of $Sp(2n, \mathbb{Z})$ and pushforwards.

5 Multiplication kernels associated with differential operators

In this section we always work in the global coordinate on \mathbb{A}^1 .

5.1 General setup

Let $P_0(\lambda), P_1(\lambda), \dots \in \mathbb{C}[\lambda]$ be a basis of the vector space $\mathbb{C}[\lambda]$ such that $\deg P_i = i$ and $P_0 = 1$. We have

$$P_i(\lambda)P_j(\lambda) = \sum_{k=0}^{\infty} C_{i,j}^k P_k(\lambda) \quad (5.31)$$

where $C_{i,j}^k$ are structure constants of polynomial multiplication in the basis $P_i(\lambda)$. Here we assume that $C_{i,j}^k$ are independent of λ .

Construct generating functions:

$$f_\lambda(x) = \sum_{i=0}^{\infty} P_i(\lambda)x^i, \quad (5.32)$$

$$K(x_1, x_2, y) = \sum_{i,j,k \geq 0} C_{i,j}^k \frac{x_1^i x_2^j}{y^{k+1}}. \quad (5.33)$$

Notice that $f_\lambda(x) = 1 + O(x)$ and $K(x, 0, y) = \frac{1}{y-x}$.

We have by construction

$$f_\lambda(x_1)f_\lambda(x_2) = \frac{1}{2\pi i} \oint K(x_1, x_2, y)f_\lambda(y)dy \quad (5.34)$$

where integral is taken by a small circle around zero. This means that the associativity condition holds

$$\oint K(x_1, x_2, y)K(y, x_3, z)dy = \oint K(x_1, x_3, y)K(y, x_2, z)dy. \quad (5.35)$$

Let D_x be a differential operator in x . Assume that our generating function is a solution of the differential equation

$$D_x f_\lambda(x) = \lambda f_\lambda(x).$$

In this case the kernel $K(x_1, x_2, y)$ satisfies the equations

$$D_{x_1} K(x_1, x_2, y) = D_{x_2} K(x_1, x_2, y)$$

and

$$D_{x_1} K(x_1, x_2, y) - D_y K(x_1, x_2, y) \in \mathbb{C}[[y]].$$

Example 5.1.1. Let $D_x = \frac{d}{dx}$ and $f_\lambda(x) = e^{\lambda x}$. In this case we have

$$K(x_1, x_2, y) = \frac{1}{y - x_1 - x_2}.$$

The construction above can be generalized to the case of polynomials $P_i(\lambda_1, \dots, \lambda_g)$, $i = 0, 1, \dots$ such that $P_0(\lambda_1, \dots, \lambda_g) = 1$ and $\{P_{i_1} \dots P_{i_g}, 0 \leq i_1 \leq i_2 \leq \dots \leq i_g\}$ is a basis in the vector space $\mathbb{C}[\lambda_1, \dots, \lambda_g]$. We have

$$P_{i_1} \dots P_{i_{g+1}} = \sum_{j_1, \dots, j_g \geq 0} C_{i_1, \dots, i_{g+1}}^{j_1, \dots, j_g} P_{j_1} \dots P_{j_g} \quad (5.36)$$

where we assume that $C_{i_1, \dots, i_{g+1}}^{j_1, \dots, j_g}$ are independent of $\lambda_1, \dots, \lambda_g$ and symmetric with respect to indexes j_1, \dots, j_g .

Construct generating functions:

$$f_{\vec{\lambda}}(x) = \sum_{i=0}^{\infty} P_i(\lambda_1, \dots, \lambda_g) x^i, \quad \text{where } \vec{\lambda} = (\lambda_1, \dots, \lambda_g), \quad (5.37)$$

$$K(x_1, \dots, x_{g+1}, y_1, \dots, y_g) = \sum_{i_1, \dots, i_g \geq 0} C_{i_1, \dots, i_{g+1}}^{j_1, \dots, j_g} x_1^{i_1} \dots x_{g+1}^{i_{g+1}} y_1^{-j_1-1} \dots y_g^{-j_g-1}. \quad (5.38)$$

Notice that $f_{\vec{\lambda}}(x) = 1 + O(x)$ and

$$K(x_1, \dots, x_n, 0, y_1, \dots, y_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{(y_{\sigma_1} - x_1) \dots (y_{\sigma_n} - x_n)}.$$

We have by construction

$$f_{\vec{\lambda}}(x_1) \dots f_{\vec{\lambda}}(x_{n+1}) = \frac{1}{(2\pi i)^n} \oint \dots \oint K(x_1, \dots, x_{n+1}, y_1, \dots, y_n) f_{\vec{\lambda}}(y_1) \dots f_{\vec{\lambda}}(y_n) dy_1 \dots dy_n \quad (5.39)$$

where we integrate over small circles around zero with respect to each y_i . We also have an associativity condition: the expression

$$\oint \dots \oint K(x_1, \dots, x_{n+1}, y_1, \dots, y_n) K(x_{n+2}, y_1, \dots, y_n, z_1, \dots, z_n) dy_1 \dots dy_n \quad (5.40)$$

is symmetric with respect to x_1, \dots, x_{n+2} .

Assume that²⁷

$$D_x f_{\vec{\lambda}}(x) = (\lambda_1 + \lambda_2 x + \dots + \lambda_g x^{g-1}) f_{\vec{\lambda}}(x)$$

for a differential operator D_x . In this case the kernel $K(x_1, \dots, x_{n+1}, y_1, \dots, y_n)$ satisfies the equation

$$\sum_{i=1}^{n+1} \frac{1}{(x_1 - x_i) \dots \hat{i} \dots (x_{n+1} - x_i)} D_{x_i} K(x_1, \dots, x_{n+1}, y_1, \dots, y_n) = 0.$$

5.2 A kernel associated with first order differential operators

Here we return to the basic example introduced in Section 4.1.

Define polynomials $P_i(\lambda_1, \dots, \lambda_g)$ by

$$e^{\lambda_1 x + \lambda_2 x^2 + \dots + \lambda_g x^g} = \sum_{i \geq 0} P_i(\lambda_1, \dots, \lambda_g) x^i.$$

²⁷In the notations of Section 4.3 we have $L_0 = D_x$, $L_i = -x^{i-1}$, $i = 1, \dots, g$.

Notice that if

$$D_x = \frac{d}{dx} - (\lambda_1 + 2\lambda_2 x + \dots + g\lambda_g x^{g-1}),$$

then $D_x e^{\lambda_1 x + \lambda_2 x^2 + \dots + \lambda_g x^g} = 0$.

Define structure constants $C_{i_1, \dots, i_{g+1}}^{j_1, \dots, j_g} \in \mathbb{Q}$ by (5.36).

Define kernel $K(x_1, \dots, x_{g+1}, y_1, \dots, y_g)$ as the generating function for these structure constants by (5.38).

Theorem 5.2.1.

$$K(x_1, \dots, x_{g+1}, y_1, \dots, y_g) = \frac{1}{g!} \sum_{\sigma \in S_g} \frac{1}{(y_{\sigma_1} - q_1) \dots (y_{\sigma_g} - q_g)}$$

where q_1, \dots, q_g are roots of a polynomials

$$Q(t) = t^g - (x_1 + \dots + x_{g+1})t^{g-1} + (x_1 x_2 + \dots + x_g x_{g+1})t^{g-2} + \dots \pm (x_1 \dots x_g + \dots + x_2 \dots x_{g+1}).$$

Coefficients of this polynomial in t are elementary symmetric polynomials in x_1, \dots, x_{g+1} .

Proof. To simplify notations let $g = 2$, the general case is similar. We have

$$\begin{aligned} e^{\lambda_1(x_1+x_2+x_3)+\lambda_2(x_1^2+x_2^2+x_3^2)} &= \sum_{i_1, i_2, i_3 \geq 0} P_{i_1} P_{i_2} P_{i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} = \sum_{i_1, \dots, j_2 \geq 0} C_{i_1 i_2 i_3}^{j_1 j_2} x_1^{i_1} x_2^{i_2} x_3^{i_3} P_{j_1} P_{j_2} = \\ &= \frac{1}{(2\pi i)^2} \sum_{i_1, \dots, j_2 \geq 0} C_{i_1 i_2 i_3}^{j_1 j_2} x_1^{i_1} x_2^{i_2} x_3^{i_3} \oint \oint \frac{e^{\lambda_1(y_1+y_2)+\lambda_2(y_1^2+y_2^2)} dy_1 dy_2}{y_1^{j_1+1} y_2^{j_2+1}} = \\ &= \frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2) e^{\lambda_1(y_1+y_2)+\lambda_2(y_1^2+y_2^2)} dy_1 dy_2 \end{aligned}$$

where integrals are taken over small circles around 0. Expanding in power series in λ_1, λ_2 and equating coefficients at $\lambda_1^{n_1} \lambda_2^{n_2}$ we get

$$(x_1 + x_2 + x_3)^{n_1} (x_1^2 + x_2^2 + x_3^2)^{n_2} = \frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2) (y_1 + y_2)^{n_1} (y_1^2 + y_2^2)^{n_2} dy_1 dy_2.$$

It follows that

$$\phi(x_1 + x_2 + x_3, x_1^2 + x_2^2 + x_3^2) = \frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2) \phi(y_1 + y_2, y_1^2 + y_2^2) dy_1 dy_2$$

where $\phi(u, v)$ is an arbitrary function analytic near $u = v = 0$. We can write this condition as

$$\phi(x_1 + x_2 + x_3, x_1 x_2 + x_2 x_3 + x_1 x_3) = \frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2) \phi(y_1 + y_2, y_1 y_2) dy_1 dy_2$$

by changing variables in ϕ (because elementary symmetric functions can be written in terms of sums of powers).

Let $\phi = \phi_{m_1, m_2}$ where $\phi_{m_1, m_2}(y_1 + y_2, y_1 y_2) = y_1^{m_1} y_2^{m_2} + y_2^{m_1} y_1^{m_2}$, it is clear that $\phi_{m_1, m_2}(u, v) = s_1^{m_1} s_2^{m_2} + s_2^{m_1} s_1^{m_2}$ where s_1, s_2 are roots of polynomial $t^2 - ut + v$. With this choice of ϕ we obtain

$$\frac{1}{(2\pi i)^2} \oint \oint K(x_1, x_2, x_3, y_1, y_2)(y_1^{m_1} y_2^{m_2} + y_2^{m_1} y_1^{m_2}) dy_1 dy_2 = q_1^{m_1} q_2^{m_2} + q_2^{m_1} q_1^{m_2}$$

where q_1, q_2 are roots of polynomial $t^2 - (x_1 + x_2 + x_3)t + x_1 x_2 + x_2 x_3 + x_1 x_3$. It follows

$$K(x_1, x_2, x_3, y_1, y_2) = \frac{1}{2} \sum_{m_1, m_2 \geq 0} (q_1^{m_1} q_2^{m_2} y_1^{-m_1-1} y_2^{-m_2-1} + q_2^{m_1} q_1^{m_2} y_1^{-m_1-1} y_2^{-m_2-1})$$

and summing up geometric series we obtain the statement of the Theorem. \square

5.3 The case of second order differential operators with 4 regular singular points

Let

$$D_x = x(x-1)(x-t) \frac{d^2}{dx^2} + x(x-1)(x-t) \left(\frac{s_1}{x} + \frac{s_2}{x-1} + \frac{s_3}{x-t} \right) \frac{d}{dx} + r_1 r_2 x + \lambda$$

where $t, s_1, s_2, s_3, r_1, r_2, \lambda$ are parameters such that $t \neq 0, 1$ and

$$r_1 + r_2 = s_1 + s_2 + s_3 - 1.$$

This is the most general second order differential operators with regular singularities at $x = 0, 1, t, \infty$ and with analytic solutions near $x = 0, 1, t$.

There exists a unique solution $f_\lambda(x)$ of the differential equation

$$D_x f_\lambda(x) = 0$$

such that $f_\lambda(x)$ is analytic near $x = 0$ and $f_\lambda(0) = 1$. We have

$$f_\lambda(x) = \sum_{i=0}^{\infty} P_i(\lambda) x^i$$

where $P_i(\lambda)$ are polynomials in λ of degree i and $P_0(\lambda) = 1$.

Since $P_i(\lambda), i = 0, 1, \dots$ is a basis of the vector space $\mathbb{C}[\lambda]$ we can define structure constants of polynomial multiplication in this basis by (5.31).

Define kernel $K(x_1, x_2, y)$ as a generating function of these structure constants by (5.33).

Recall that the Gauss hypergeometric function is given by

$$F(a, b, c, u) = \sum_{n=0}^{\infty} \frac{a(a+1)\dots(a+n-1) \cdot b(b+1)\dots(b+n-1)}{c(c+1)\dots(c+n-1)} \cdot \frac{u^n}{n!} =$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 p^{b-1}(1-p)^{c-b-1}(1-pu)^{-a} dp.$$

Theorem 5.3.1. The kernel $K(x_1, x_2, y)$ is given by

$$\begin{aligned} K(x_1, x_2, y) &= \frac{1}{y-1} \sum_{i=0}^{\infty} \frac{(r_1 + r_2 - s_1 - s_3 + 1) \dots (r_1 + r_2 - s_1 - s_3 + i)}{(r_1 + r_2 - s_1 + i) \dots (r_1 + r_2 - s_1 + 2i - 1)} \times \\ &F\left(i + 1, r_1 + r_2 - s_1 - s_3 + i + 1, r_1 + r_2 - s_1 + 2i + 1, \frac{t-1}{y-1}\right) \times \\ &F\left(-i, r_1 + r_2 - s_1 + i, r_1 + r_2 - s_1 - s_3 + 1, \frac{t(x_1-1)(x_2-1)}{(t-1)(x_1x_2-t)}\right) \times \\ &F\left(r_1 + i, r_2 + i, s_1, \frac{x_1x_2}{t}\right) \left(\frac{x_1x_2}{t} - 1\right)^i \frac{(t-1)^i}{(y-1)^i}. \end{aligned}$$

The proof is based on the following

Lemma 5.3.1. The function $K(x_1, x_2, y)$ is the unique function characterized by the following properties:

1. $K(x_1, x_2, y) = K(x_2, x_1, y)$.
2. $K(x_1, x_2, y)$ has Laurent series expansion by non negative powers of x_1, x_2 and negative powers of y .
3. $D_{x_1}K(x_1, x_2, y) = D_{x_2}K(x_1, x_2, y)$.
4. $K(x_1, 0, y) = \frac{1}{y-x_1}$.

Proof. These properties of $K(x_1, x_2, y)$ follow from definition, see the discussion in Section 5.1. The proof of uniqueness of the solution of the differential equation $(D_{x_1} - D_{x_2})K = 0$ with the properties **1, 2, 4** is omitted. \square

Based on this explicit series representation for the kernel $K(x_1, x_2, y)$ one can derive the following holonomic system of differential equations.

Theorem 5.3.2. The function $K(x_1, x_2, y)$ satisfies the following differential equations:

$$\begin{aligned} D_{x_1}K - D_{x_2}K &= 0, \\ D_y^*K - D_{x_1}K &= (r_1 - 1)(r_2 - 1)F\left(r_1, r_2, s_1, \frac{x_1x_2}{t}\right) \\ LK(x_1, x_2, y) &= \frac{(s_1 - 1)t}{x_1x_2(x_1 - y)(x_2 - y)}F\left(r_1 - 1, r_2 - 1, s_1 - 1, \frac{x_1x_2}{t}\right) \end{aligned}$$

where

$$D_y^* = \frac{d^2}{dy^2} \cdot y(y-1)(y-t) - \frac{d}{dy} \cdot y(y-1)(y-t) \left(\frac{s_1}{y} + \frac{s_2}{y-1} + \frac{s_3}{y-t} \right) + r_1r_2y + \lambda$$

is the differential operator conjugate to D_y and

$$L = \frac{(x_1 - 1)(x_1 - t)}{(x_1 - x_2)(x_1 - y)} \cdot \frac{d}{dx_1} + \frac{(x_2 - 1)(x_2 - t)}{(x_2 - x_1)(x_2 - y)} \cdot \frac{d}{dx_2} + \frac{(y - 1)(y - t)}{(y - x_1)(y - x_2)} \cdot \frac{d}{dy} - \frac{(r_1 + r_2 - 2)x_1x_2y + (s_2 + 1 - r_1 - r_2)x_1x_2 - (s_1 + s_2 - 2)tx_1x_2 + (s_1 - 1)t(x_1 + x_2 - y)}{x_1x_2(x_1 - y)(x_2 - y)}.$$

Theorem 5.3.3. The kernel $K(x_1, x_2, y)$ admits the following integral representation

$$K(x_1, x_2, y) = \frac{s_1 - 1}{1 - t} \left(\frac{2(x_1 - 1)(x_2 - 1)}{1 - t} \right)^{1-s_2} \left(\frac{2(x_1 - t)(x_2 - t)}{t(t-1)} \right)^{1-s_3} \times \int_0^1 F(r_1 - 1, r_2 - 1, s_1 - 1, uq) q^{s_1 - 2} Q^{-1/2} \left(-uq - v + w + 1 + Q^{1/2} \right)^{s_2 - 1} \left(-uq + v - w + 1 - Q^{1/2} \right)^{s_3 - 1} dq$$

where

$$u = \frac{x_1x_2}{t}, \quad v = \frac{(x_1 - 1)(x_2 - 1)(y - 1)}{(t - 1)^2}, \quad w = \frac{(x_1 - t)(x_2 - t)(y - t)}{t(t - 1)^2},$$

$$Q = (v - w)^2 - 2(v + w)(uq - 1) + (uq - 1)^2,$$

and we assume that $Q^{1/2} \sim v - w$ for $y \rightarrow \infty$.

After substitution the integral representation of Gauss hypergeometric function and the change of variables

$$q_1 = \frac{-uq - v + w + 1 + Q^{1/2}}{2(1 - pq_u)}, \quad q_2 = \frac{-uq + v - w + 1 - Q^{1/2}}{2(1 - pq_u)} \quad (5.41)$$

we obtain another integral formula for the kernel

$$K(x_1, x_2, y) = \left(\frac{x_1x_2}{t} \right)^{1-s_1} \left(\frac{(x_1 - 1)(x_2 - 1)}{1 - t} \right)^{1-s_2} \left(\frac{(x_1 - t)(x_2 - t)}{t(t-1)} \right)^{1-s_3} \times \frac{\Gamma(s_1)}{\Gamma(r_1 - 1)\Gamma(s_1 - r_1)(1 - t)} \int_D q_1^{s_2 - 1} q_2^{s_3 - 1} (1 - q_1 - q_2)^{s_1 - r_1 - 1} \left(1 + \frac{v}{q_1} + \frac{w}{q_2} \right)^{r_1 - 2} \frac{dq_1}{q_1} \cdot \frac{dq_2}{q_2}$$

where $D = \{(q_1, q_2), 0 \leq p, q \leq 1\}$ and q_1, q_2 are parameterized by (5.41).

Theorem 5.3.4. Let $s_1 = r_1 = 1$. In this case the kernel $K(x_1, x_2, y)$ is given by

$$K(x_1, x_2, y) = \left(\frac{2t - t(x_1 + x_2 + y) + x_1x_2y + tyP^{1/2}}{2t(1 - x_1)(1 - x_2)} \right)^{s_2 - 1} \times$$

$$\left(\frac{2t^2 - t(x_1 + x_2 + y) + x_1x_2y + tyP^{1/2}}{2(t-x_1)(t-x_2)} \right)^{s_3-1} \frac{1}{yP^{1/2}}$$

where

$$P = 1 - \frac{2x_1}{y} - \frac{2x_2}{y} + \frac{x_1^2}{y^2} + \frac{x_2^2}{y^2} - \frac{2x_1^2x_2}{ty} - \frac{2x_1x_2^2}{ty} + \frac{x_1^2x_2^2}{t^2} + \frac{2(2ty - y^2 - t + 2y)x_1x_2}{ty^2}.$$

Proof. Set $r_1 = 1$ in the integral formula for the kernel above. After that write $(s_1 - 1)q^{s_1-2}dq = d(q^{s_1-1})$, integrate by parts and set $s_1 = 1$. \square

5.4 The case of second order differential operators with more than 4 regular singular points

Fix a natural number $n \geq 1$, pairwise distinct points $t_1, \dots, t_n \in \mathbb{C}$ such that $t_i \neq 0, 1$ for $i = 1, \dots, n$ and parameters $s_1, \dots, s_{n+2}, r_1, r_2 \in \mathbb{C}$ such that

$$s_1 + \dots + s_{n+2} = r_1 + r_2 + 1.$$

Let

$$D_x = x(x-1)(x-t_1)\dots(x-t_n) \left(\frac{d^2}{dx^2} + \left(\frac{s_1}{x} + \frac{s_2}{x-1} + \frac{s_3}{x-t_1} + \dots + \frac{s_{n+2}}{x-t_n} \right) \frac{d}{dx} \right) + \lambda_1 + \lambda_2x + \dots + \lambda_nx^{n-1} + r_1r_2x^n.$$

This is the most general second order differential operator with regular singularities at $x = 0, 1, t_1, \dots, t_n, \infty$ and having analytic solutions near $x = 0, 1, t_1, \dots, t_n$.

Let $f_{\vec{\lambda}}(x)$ be the unique solution of the equation $D_x f_{\vec{\lambda}}(x) = 0$ analytic at $x = 0$ and such that $f_{\vec{\lambda}}(x) = 1 + O(x)$. Write

$$f_{\vec{\lambda}}(x) = \sum_{i=0}^{\infty} P_i x^i$$

where P_i , $i = 0, 1, \dots$ are polynomials in $\lambda_1, \dots, \lambda_n$ and $P_0 = 0$. One can show that the products $\{P_{i_1} \dots P_{i_n}, 0 \leq i_1 \leq \dots \leq i_n\}$ form a basis in the vector space $\mathbb{C}[\lambda_1, \dots, \lambda_n]$.

Define structure constants $C_{i_1 \dots i_{n+1}}^{j_1 \dots j_n}$ by (5.36).

Define the kernel $K(x_1, \dots, x_{n+1}, y_1, \dots, y_n)$ as the generating function by (5.38).

Lemma 5.4.1. The kernel $K(x_1, \dots, x_{n+1}, y_1, \dots, y_n)$ is the unique function characterized by the following properties:

1. It is symmetric with respect to x_1, \dots, x_{n+1} and with respect to y_1, \dots, y_n .
2. It has Laurent series expansion by non-negative powers of x_1, \dots, x_{n+1} and by negative powers of y_1, \dots, y_n .

3. The following differential equation holds

$$\sum_{i=1}^{n+1} \frac{1}{(x_1 - x_i) \dots \hat{i} \dots (x_{n+1} - x_i)} D_{x_i} K(x_1, \dots, x_{n+1}, y_1, \dots, y_n) = 0$$

4. $K(x_1, \dots, x_n, 0, y_1, \dots, y_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{(y_{\sigma_1} - x_1) \dots (y_{\sigma_n} - x_n)}$.

Theorem 5.4.1. The following equation holds

$$K(x_1, \dots, x_{n+1}, y_1, \dots, y_n) = \sum_{i_1, \dots, i_n \geq 0} F\left(r_1 + i_1 + \dots + i_n, r_2 + i_1 + \dots + i_n, s_1, \frac{x_1 \dots x_{n+1}}{t_1 \dots t_n}\right) \times \\ u_0^{i_1 + \dots + i_n} P_{i_1, \dots, i_n}\left(\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0}\right) Q_{i_1, \dots, i_n}(y_1, \dots, y_n)$$

where F is the Gauss hypergeometric function,

$$P_{i_1, \dots, i_n}(v_1, \dots, v_n) = \sum_{j_1, \dots, j_n \geq 0} \frac{\prod_{l=1}^{j_1 + \dots + j_n} (s_2 + i_1 + \dots + i_n - l) \prod_{l=1}^{j_1} (i_1 - l + 1) \dots \prod_{l=1}^{j_n} (i_n - l + 1)}{\prod_{l=1}^{j_1} (s_3 + l - 1) \dots \prod_{l=1}^{j_n} (s_{n+2} + l - 1)} \frac{v_1^{j_1} \dots v_n^{j_n}}{j_1! \dots j_n!} \\ u_i = \frac{(x_1 - t_i) \dots (x_{n+1} - t_i)}{t_i(t_i - 1)(t_1 - t_i) \dots \hat{i} \dots (t_n - t_i)}, \quad i = 1, \dots, n, \quad u_0 = \frac{(x_1 - 1) \dots (x_{n+1} - 1)}{(t_1 - 1) \dots (t_n - 1)}.$$

Note that $u_0^{i_1 + \dots + i_n} P_{i_1, \dots, i_n}\left(\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0}\right)$ are polynomials in u_0, u_1, \dots, u_n for non-negative $i_1, \dots, i_n \in \mathbb{Z}$ and the sum in the definition of P_{i_1, \dots, i_n} is finite in this case.

The functions $Q_{i_1, \dots, i_n}(y_1, \dots, y_n)$ are determined by the system of equations

$$\sum_{i_1, \dots, i_n \geq 0} \tilde{u}_0^{i_1 + \dots + i_n} P_{i_1, \dots, i_n}\left(\frac{\tilde{u}_1}{\tilde{u}_0}, \dots, \frac{\tilde{u}_n}{\tilde{u}_0}\right) Q_{i_1, \dots, i_n}(y_1, \dots, y_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{(y_{\sigma_1} - x_1) \dots (y_{\sigma_n} - x_n)}$$

where $\tilde{u}_i = u_i|_{x_{n+1}=0}$.

We expect that the holonomic D -module generated by K maps epimorphically at the generic point to the one described in Section 3.4 with a possible change of the cyclic vector.

5.5 The case of third order differential operators with 3 regular singular points

Let

$$D_x = x^2(x-1)^2 \frac{d^3}{dx^3} + x(x-1)(a_1 + a_2x) \frac{d^2}{dx^2} + (a_3 + a_4x + a_5x^2) \frac{d}{dx} + a_6x + \lambda$$

where a_1, \dots, a_6, λ are parameters. This is the most general third order differential operator with regular singularities at $x = 0, 1, \infty$ and with analytic solutions near $x = 0, 1$.

There exists a unique solution $f_\lambda(x)$ of the differential equation

$$D_x f_\lambda(x) = 0$$

such that $f_\lambda(x)$ is analytic near $x = 0$ and $f_\lambda(0) = 1$. We have

$$f_\lambda(x) = \sum_{i=0}^{\infty} P_i(\lambda) x^i$$

where $P_i(\lambda)$ are polynomials in λ of degree i and $P_0(\lambda) = 1$.

Since $P_i(\lambda)$, $i = 0, 1, \dots$ is a basis of the vector space $\mathbb{C}[\lambda]$, we can define structure constants of polynomial multiplication in this basis by (5.31).

Define kernel $K(x_1, x_2, y)$ as the generating function of these structure constants by (5.33)

Introduce new parameters b_1, b_2, c_1, c_2, c_3 by

$$b_1 + b_2 = -a_1 - 1, \quad b_1 b_2 = a_3, \quad c_1 + c_2 + c_3 = a_2 - 3,$$

$$c_1 c_2 + c_1 c_3 + c_2 c_3 = a_5 - a_2 + 2, \quad c_1 c_2 c_3 = a_6.$$

Lemma 5.5.1. The kernel $K(x_1, x_2, y)$ satisfies the following differential equations

$$D_{x_1} K = D_{x_2} K,$$

$$D_y^* - D_{x_1} K = (c_1 - 1)(c_2 - 1)(c_3 - 1) \sum_{i=0}^{\infty} \prod_{l=0}^{i-1} \frac{(c_1 + l)(c_2 + l)(c_3 + l)}{(b_1 + l)(b_2 + l)} \cdot \frac{(x_1 x_2)^i}{i!}$$

where D_y^* is the conjugate differential operator given by

$$D_y^* = -\frac{d^3}{dy^3} \cdot y^2(y-1)^2 + \frac{d^2}{dy^2} \cdot y(y-1)(a_1 + a_2 y) - \frac{d}{dy} \cdot (a_3 + a_4 y + a_5 y^2) + a_6 y + \lambda$$

Theorem 5.5.1. The kernel $K(x_1, x_2, y)$ is given by

$$K(x_1, x_2, y) = \frac{1}{y-1} \sum_{\substack{0 \leq j \leq k \leq i+j, \\ 0 \leq i}} \frac{(-1)^j k!}{j!(k-j)!(i+j-k)!} \cdot \frac{(x_1 x_2)^i \left((x_1 - 1)(x_2 - 1) \right)^j}{(y-1)^k} \times$$

$$\prod_{l=j+1}^k (b_1 b_2 + c_1 c_2 + c_1 c_3 + c_2 c_3 + a_4 + l(c_1 + c_2 + c_3) + (1-l)(b_1 + b_2) + l^2 - l + 1) \times$$

$$\frac{\prod_{l=k}^{i+j-1} (c_1 + l)(c_2 + l)(c_3 + l)}{\prod_{l=0}^{i-1} (b_1 + l)(b_2 + l)}.$$

Remark 5.5.1. Let D_x be third order differential operator with symbol $(x - t_1)^2(x - t_2)^2(x - t_3)^2 \frac{\partial^3}{\partial x^3}$, with regular singular points at $x = t_1, t_2, t_3$, and solutions near these points of the form $f(x) = (x - t_i)^{b_{i,j}}(1 + O(x - t_i))$ for $i, j = 1, 2, 3$. Here $t_i, b_{i,j}$ are generic parameters such that $\sum_{1 \leq i, j \leq 3} b_{i,j} = 3$. Notice that any two differential operators with these properties are differ by an arbitrary constant.

Define a function $K_3(x, y, z)$ as

$$K_3 = F \left(q_1 \frac{(x - t_1)(y - t_1)(z - t_1)}{(x - t_3)(y - t_3)(z - t_3)}, q_2 \frac{(x - t_2)(y - t_2)(z - t_2)}{(x - t_3)(y - t_3)(z - t_3)} \right)$$

where $q_1 = \frac{(t_2 - t_3)^3}{(t_1 - t_2)^3}$, $q_2 = \frac{(t_1 - t_3)^3}{(t_2 - t_1)^3}$ and F satisfies the system of differential equations

$$\frac{\partial^3 G}{\partial u_{11} \partial u_{12} \partial u_{13}} = \frac{\partial^3 G}{\partial u_{21} \partial u_{22} \partial u_{23}} = \frac{\partial^3 G}{\partial u_{31} \partial u_{32} \partial u_{33}}.$$

Here

$$G = F \left(\frac{u_{11}u_{12}u_{13}}{u_{31}u_{32}u_{33}}, \frac{u_{21}u_{22}u_{23}}{u_{31}u_{32}u_{33}} \right) \cdot \prod_{1 \leq i, j \leq 3} u_{i,j}^{-b_{i,j}}.$$

Then $K_3(x, y, z)$ satisfies the differential equations

$$\frac{\partial K_3}{\partial x} = \frac{\partial K_3}{\partial y} = \frac{\partial K_3}{\partial z}.$$

It will be interesting to understand if K_3 is somehow connected with the kernel K constructed in Theorem 5.5.1. It looks feasible that K_3 also satisfies to some kind of associativity condition. The kernel K_3 is similar to kernels constructed in Remarks 3.4.2 for $l = n$, and 3.3.2.

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