Plan:

1. Present explicit examples of (quantum) Langlands dual categories

$\Longrightarrow$ Con; some categories are equivalent (contains FLE as a special case)

Initiated by D. Gaiotto

2. What cases are known

3. Discuss the proof in the simplest new case.

G - reductive $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$.

κ - invariant symm. form on $\mathfrak{g}$,

Quantum local GLC: categories with strong $G(\mathcal{K})$ action of level $\kappa + \kappa_{crit}$

$(\kappa_{crit} = -\frac{1}{2}\,\text{Killing})$

$$\mathcal{C} \longleftrightarrow \mathcal{C}^\vee$$

$\Updownarrow$

Cat. with $G^\vee(\mathcal{K})$-action of level $-\kappa^\vee - \kappa_{crit}$

$\kappa = 0$. RHS $\rightsquigarrow$ categories over $\mathrm{LocSys}_{G^\vee}(\mathcal{D}^*)$

$$\mathcal{C}^{G(\mathcal{O})} \simeq (\mathcal{C}^\vee)^{G^\vee(\mathcal{O})}$$

κ - non-deg.

$\kappa = 0$ $\quad \mathcal{C}^{G(\mathcal{O})}$ = fiber of $\mathcal{C}^\vee$ over the trivial local system

<u>Question</u> What kind of explicit examples of $(\mathcal{C}, \mathcal{C}^\vee)$ do we know?

$$\text{D-mod}_{\kappa + \kappa_{crit}}(Gr_G)^\vee = \text{D}\text{mod}_{-\kappa^\vee}(Gr_{G^\vee})$$

$$\mathcal{C} = \hat{\mathfrak{g}}_{k+k_{crit}}\text{-mod} \qquad \mathcal{C}^\vee = D\text{-mod}_{-\check{k}-\check{k}_{crit}}\left(\check{G}(K)\middle/(\check{N}(K),\chi)\right)$$

$$KL_{k+k_{crit}}(\mathfrak{g}) \simeq D\text{-mod}_{k^\vee}\left((\check{N}(K),\chi)\backslash Gr_{G^\vee}\right)$$

<u>Setup</u> <u>for</u> <u>generalization</u>:

Let G^s be an algebraic super-group s.t.

1) $G^{even} = G$ 2) $\mathfrak{g}^s$ has an invariant non-degenerate symmetric bilinear form.

$M \leq N$

$G^s = GL(M|N)$ $\qquad$ $G^{even} = GL(M) \times GL(N)$

$\mathfrak{g}^{odd} = \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N) \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^M)$.

$K(x,y) = c \cdot s\mathrm{Tr}(xy)$
$c \in \mathbb{C}$

Note: $\mathfrak{g}^{odd}$ is always a symplectic representation of G^{even}

K defines a central ext. of $\mathfrak{g}^s((K))$

$\mathcal{C} = \mathfrak{g}^s_{K+K^s_{crit}}((K))\text{-mod}$ $\circlearrowleft$ $G(K)$ acts on level $K + K_{crit}$

$\mathcal{C}^\vee$?

Roughly speaking we'll have $H \subset G^\vee$, $\chi : H \to \mathbb{G}_a$

$$\mathcal{C}^\vee = D\text{-mod}_{k^\vee}\left(\check{G}(K) / (H(K), \chi) \right).$$

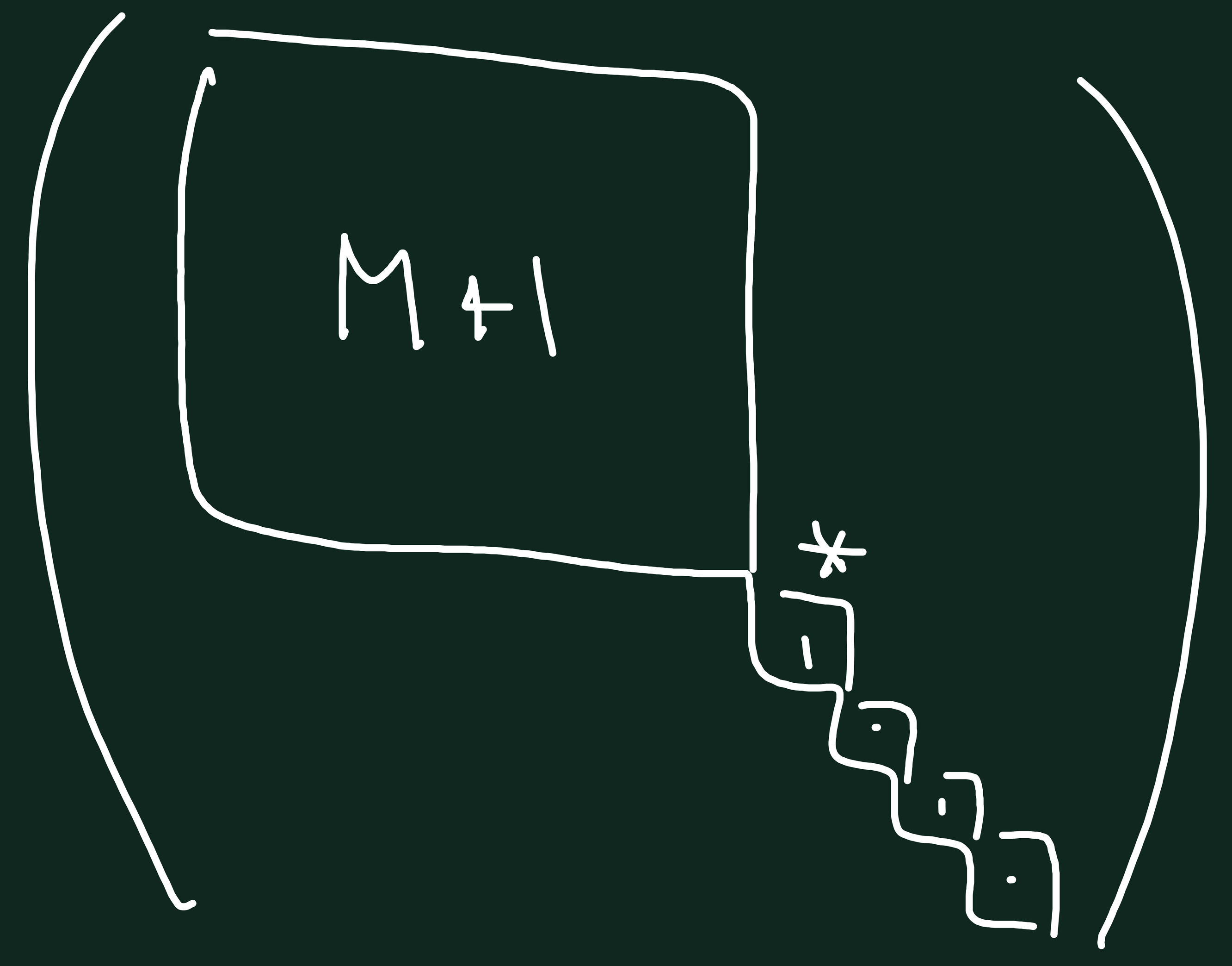
<u>Example</u> $G^s = GL(M|N)$ $M < N$. $G = G^\vee = GL(M) \times GL(N)$

$GL(N) \supset H = GL(M) \ltimes U_{M,N}$ $U_{M,N} \subset GL(N)$

$M = 0$ 1) $H = U_N \subset GL(N)$ 2) $M = N-1$ $H = GL(N-1)$

max. unip.

In general $U_{M,N}$ = unip. radical



$$\chi : U_{M,N} \to \mathbb{G}_a$$

Take restriction of standard character of U_N.

$$H_{M,N} \subset GL(M)\times GL(N) \qquad G^{even}/H = GL(N)/U_{M,N}$$

$$KL_{\kappa+\kappa^s_{crit}}\left(\widehat{\mathfrak{gl}(M|N)}\right) \simeq D\text{-mod}_{\kappa^\vee}\left((U_{M,N},\chi)\ltimes GL(M,\mathcal{O})\backslash Gr_{GL(N)}\right)$$

$M = N-1$

$$\mathcal{D}\text{-mod}_{\kappa}\left({}_{GL(N-1,\mathcal{O})}\backslash \mathrm{Gr}_{GL(N)} \right) \simeq \mathrm{Rep}_q\left(GL(N|N-1) \right)$$

$$q = e^{\pi i/\kappa}$$

For $q=1$ we need $\mathrm{Rep}(\bar{G}^s)$

$\bar{\mathfrak{g}}^s$ $\quad$ $\mathfrak{g}^{even}$ – the same $\quad$ $[\cdot,\cdot] : \mathfrak{g}^{odd} \times \mathfrak{g}^{odd} \to \mathfrak{g}^{even}$ is 0.

$\mathfrak{g}^{odd}$ – " –

$$\mathrm{Rep}(\check{G}^s) = \left(\Lambda(\mathfrak{g}^{odd})\text{-mod}\right)^{G^{even}}$$

Theorem $\text{D-mod}\left((U_{M,N}(K),\chi) \times_{GL(M|0)} \backslash^{GR}_{GL(N)} \right) \simeq$

$$\simeq \Lambda\left(\mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N) \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^M)\text{-mod} \right)^{GL(M)\times GL(N)}$$

E.g. $\text{D-mod}\left({}_{GL(N-1|0)} \backslash^{GK}_{GL(N)} \right)$ – derived cat. is the derived cat. of perverse sheaves

$M = N$

$$\mathcal{C}^\vee \qquad D\text{-mod}\left(GL(N,K) \times K^N \right)$$

$$\begin{array}{c} \circlearrowleft \\ GL(N,K) \times GL(N,K) \end{array}$$

$G^\vee \supset (H, \chi)$ $\qquad$ $D\text{-mod}\left(G^\vee/(H,\chi) \right) =$ quant. of

$$T^* G^\vee /\!/ (H, \chi)$$

What is $T^*G^\vee /\!/ (H, \chi)$ – ?

G^{aux} – red. group

$e \in \mathfrak{g}^*$ nilpotent element

$G^{aux} \times Z_{red}(e)$ acts.

$\parallel$

$G^\vee$

$G^{aux} \times S_e$ – symplectic

Slodowy slice

$N > M$

$G^{aux} = GL(N)$

$GL(M|N)$

$$e = \begin{pmatrix} \begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{matrix} & \cancel{N}-M \\ 0 & 0 \end{pmatrix}$$

Question

What kind of equiv. Slodowy slices appear?

$G^\vee \supset (H, \chi)$

$$D\text{-mod}\left({}_{(H(K),\chi)}\backslash \mathrm{Gr}_{G^\vee} \right) \simeq \mathrm{Rep}(\bar{G}^s)$$

E.g. $\chi = 0$ "countable" $H(K)$ has countably many orbits on $\mathrm{Gr}_{G^\vee}$.

$\iff$ $G^\vee/H$ is a spherical variety

Let Y be a symplectic affine $G^\vee$-variety

<u>Want</u>: condition s.t. if $Y = T^*X$ then X is spherical

<u>Hyperspherical</u>: Generic $G^\vee$-orbit is coisotropic

$\Updownarrow$

$\mathbb{C}[Y]^{G^\vee}$ - Poisson commutative

When is $G^{aux} \times S_e$ hyper-spherical for $G^{aux} \times Z_{red}(e)$?

N large

E.g. if $G = GL(N) \iff e = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{matrix}} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

Simple $\mathfrak{g}^s$ with k-non-degenerate

"$\mathfrak{gl}(M|N)$"

$SOSp(k, 2n)$ $\qquad G^{even} = SO(k) \times Sp(2n)$

$\qquad \mathfrak{g}^{odd} = \mathbb{C}^k \otimes \mathbb{C}^{2n}$

$G(3)$ $\qquad SL(2) \times G(2)$

$F(4)$ $\qquad SL(2) \times Spin(7)$

Lie super-group with invariant form

$$\left(G^{\mathrm{even}}\right)^{\vee}$$

metaplectic dual

$$=\!=$$

Hyper-spherical eq. Slodowy slices

$$G^{\mathrm{aux}} \times \mathcal{Z}_{\mathrm{red}}(e)$$

$\mathfrak{g}^{\mathrm{odd}}$ – symp. rep. of $G^{\mathrm{even}} \longrightarrow (\cdot\,,\,\cdot)$ on $\mathfrak{g}^{\mathrm{even}}$ integral, even

$$\longrightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z}$$

$$\mathrm{Rep}_q(SO(2n+1)) \overset{\mathrm{FLE}}{\simeq} \mathrm{Whit}_k(\mathrm{Gr}_{Sp(2n)})$$

$$\wr\!\wr$$

$$\mathrm{Rep}_{q'}(SOSp(1|2n))$$

$$G^{even} = Sp(2n)$$

$$q' = q\sqrt{-1}$$