Existence of oper structures

Goal: X/k sm proj. curve $\mathrm{char}(k)=0$
σ is a (dR) G-local system ← center is connected
on X $\Rightarrow$ there exists an oper structure on $\sigma|_U$ for some $\emptyset \neq U \subseteq X$.

Previous work:

- $G = GL_n$ Thm of Deligne
  (applies for open curves)

- If you replace X by $\overset{\circ}{D}$,
  theorem of Frenkel - Zhu.

- Beraldo - Kazhdan - Schlank
  showed G is classical (connected center)
  & σ is semisimple.

(prove a "contractibility" statement as well.)

- Arinkin: gen'l G (connected center) & σ is irreducible.
  (works for open curves as well).

Later in the semester: give some applications to automorphic sheaves.

<u>Rem</u>: This thm is a sort of $\kappa=\infty$ ~~[illegible]~~ analogue of thm I talked about last year w/ Faergeman (Whittaker coeffs are not lossy).

<u>Special cases</u>:

1) $G = GL_2$

A dR local system is a rk 2 v.b. w/ ∇. $(\mathcal{E}, \nabla)$.

Provisional definition;

(w/o singularities)

An oper structure is a line subbundle $\mathcal{L} \subseteq \mathcal{E}$ s.t. the map

$$\begin{tikzcd} \mathcal{L} \subseteq \mathcal{E} \arrow[r, "\nabla"] & \mathcal{E} \otimes \Omega^1 \arrow[d] \\ & \mathcal{E}/\mathcal{L} \otimes \Omega^1 \end{tikzcd}$$

is an isomorphism.

Note; if you're given $U \subseteq X$,

$\mathcal{L}_U \subseteq \mathcal{E}|_U$, $\exists!\ \mathcal{L} \subseteq \mathcal{E}$ extending $\mathcal{L}_U$, and here the condition looks like

$\mathcal{L} \longrightarrow \mathcal{E}/\mathcal{L} \otimes \Omega^1$ is non-zero.

Warmup: some simple proofs that generic oper strs exist:

Observation: for any fixed $\mathcal{L} \subseteq \mathcal{E}$, the enemy is when $\mathcal{L}$ is preserved by the connection.

a) There's at most a $\mathbb{P}^1$ of such line subbundles.
Can find some $\mathcal{L}$ very neg. degree that isn't on this list.

b) Find any line subbundle $\mathcal{L}$ w/ $\deg(\mathcal{L})<0$ $\Rightarrow$ doesn't support a ∇ so cannot be stable under ∇.

Generalizes to GL_n:

~~there's a finite-dimensional space~~

An oper w/ singularities is a complete flag

$$0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_n$$

$\mathrm{rk}(\mathcal{E}_i) = i$ $\qquad \mathcal{E}_i/\mathcal{E}_{i-1} \leftarrow$ line bundle

∇ on $\mathcal{E}$

$\nabla : \mathcal{E}_i \longrightarrow \mathcal{E}_{i+1} \otimes \Omega^1$ } Griffiths trans.

and maps $\mathcal{E}_i/\mathcal{E}_{i-1} \longrightarrow \mathcal{E}_{i+1}/\mathcal{E}_i \otimes \Omega^1$

should be non-zero (non-deg.).

Idea of construction of an oper str.:

there's a finite-dimensional space of ∇-stable subbundles $\hat{\mathcal{E}} \subsetneq \mathcal{E}$.

Choose some line subbundle $\mathcal{L} \subseteq \mathcal{E}$ that is not contained in any $\hat{\mathcal{E}} \leftarrow \nabla$-stable.

inductively: take $\mathcal{E}_1 = \mathcal{L}$

$$\mathcal{E}_2 = \left(\underbrace{\mathcal{L} \otimes \Omega^1}_{\mathcal{E}_1} + \nabla \underbrace{\mathcal{L}}_{\mathcal{E}_1}\right)^{\text{sat.}} \otimes (\Omega^1)^{-1}.$$

etc.

idea in gen'l:

want "many" $\underbrace{\text{degenerate}}_{\substack{\text{I have a flag} \\ \text{that satisfies Griffiths} \\ \text{but not non-deg.}}}$ oper structures

Argue you can find many that $\rightsquigarrow$ actual oper structures.
(by avoiding the "small" number of exceptions).

Gen'l G:

Def: A BD-oper for G is

$(\mathcal{P}_B, \nabla)$ where $\mathcal{P}_B$ is a B-bundle

∇ is a ∇ on the induced G-bundle

& $c(\nabla) \in \Gamma((\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B} \otimes \Omega^1)$ satisfies:

- $c(\nabla) \in \Gamma((\mathfrak{g}_{\geq -1}/\mathfrak{b})_{\mathcal{P}_B} \otimes \Omega^1)$

- $c(\nabla)_i \in \Gamma(\mathfrak{g}_{-\alpha_i} \otimes \Omega^1)$ is non-zero.
  $i \in I_{\mathfrak{g}}$ ← vertices of Dynkin diag.

Here: $\mathfrak{g}_{\geq n}$ ← filtration by $\check{\rho}$-eigenspaces.

For $\mathfrak{gl}_n$: $\mathfrak{g}_{\geq -1} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix}$

Rem; $\forall \alpha_i \in$ simple root

$c(\nabla)_{\alpha_i}$ has zeros of some order.

Form $D \in \Lambda^{pos}$-valued divisor

encodes these ~~zeroes~~.

"discrepancy divisor."

The above is the right notion for

G adjoint, not quite in general.

Want a little more in general:

A "true" oper structure w/ no discrepancy also includes the

data of an isomorphism

$$P_T \simeq \underbrace{\check{\rho}(\Omega)}_{2\check{\rho}(\Omega^{1/2})}$$

← depends on a fixed choice of $\Omega_X^{1/2}$.

~~[illegible]~~ Variant w/ prescribed singularities:

$D \leftarrow \check{\Lambda}^+$ dominant coweight-valued divisor

~~[illegible]~~ "true" oper w/ singularities D

is a B-bundle $\mathcal{P}_B$, w/ ∇ on $\mathcal{P}_G$,

~~[illegible]~~ + an isomorphism

$\mathcal{P}_T \simeq \check{\rho}(\Omega)(-D)$ and:

$$c(\nabla)_i \in \Gamma\left((\mathfrak{g}/\mathfrak{b})_{-\alpha_i} \otimes \Omega^1\right) =$$

$$\Gamma\left(\Omega^{-1}(D,\alpha_i) \otimes \Omega^1\right)$$

$$= \Gamma\left(\mathcal{O}(D,\alpha_i)\right).$$

should be 1

Easy to see: when Z is connected, any BD-oper lifts to a true oper. (for some D)

Let's dig in on the proof.

Let's take G adjoint for simplicity.

Also will assume $g > 1$, $g = g(X)$.

Let's take $Op_G^{\check{\lambda}}$ ← all opers w/ divisor having degree $\check{\lambda} \in \check{\Lambda}^+$.

First claim: for all $\check{\lambda} >> 0$ ($\langle \check{\lambda}, \alpha_i \rangle >> 0$ for all i)

the map $Op^{\check{\lambda}} \longrightarrow LS_{\check{G}}$

is dominant onto a fixed irred. cpt of $LS_{\check{G}}$.

Specifically;

<u>Prop</u> (Beilinson-Drinfeld):

$LS_{\check{G}} \longrightarrow Bun_G$ is an isomorphism on sets of irreducible cpts.

For RHS, well-known to be $\pi_1(G) = \check{\Lambda}/\text{coroots}$.

Idea of pf:

- Goal: Show that there exists a point in $Op^{\check{\lambda}}$ where the map $Op^{\check{\lambda}} \longrightarrow LS_{\check{G}}$ is smooth.

  $\Rightarrow$ dominant over its image.

- <u>Claim</u>: Can reduce this to an assertion about Higgs bundles.

Work over $\mathbb{A}^1_\hbar$.

A $\hbar$-connection on a vector bundle $\mathcal{E}$ is a map $\nabla_\hbar : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1$

s.t. $\nabla_\hbar(fs) = f\nabla_\hbar s + \hbar\, df \otimes s$.

$\hbar \neq 0 \iff$ usual connection

$\hbar = 0 \iff$ a Higgs bundle.

$$\begin{tikzcd}
LS_{G,\hbar} \arrow[r] & \mathbb{A}^1_\hbar \\
Op_{G,\hbar} \arrow[u, "\check{\lambda}"] \arrow[ur] &
\end{tikzcd}$$

<u>Assume</u>: $\exists$ a point $(\mathcal{P}_B, \underbrace{\varphi}_{\substack{\text{Higgs field} \\ \text{on } \mathcal{P}_G}})\in \mathrm{Op}^{\check\lambda}_{G,\hbar=0}$

s.t. $\mathrm{Op}^{\check\lambda}_{G,\hbar=0} \longrightarrow \mathrm{Higgs}$ is smooth

at this point. <u>Then</u>: $\exists$ such a point in $\mathrm{Op}^{\check\lambda}_G$.

<u>Pf</u>: $\exists$ an open nbhd of $(\mathcal{P}_B, \varphi)$ in $\mathrm{Op}^{\check\lambda}_{G,\hbar}$ on which the map to $LS_\hbar$ is smooth. <u>NTS</u>: any such open nbhd contains a point w/ $\hbar \neq 0$.

I.e., we need $\mathrm{Op}^{\check\lambda}_\hbar \longrightarrow \mathbb{A}^1$ is flat at our point.

For this: use Prop (Beilinson-Drinfeld):

~~$\mathfrak{B}$~~ $Higgs_G$ is classical (as a DG stack).

$\Rightarrow Op^{\check{\lambda}}_{\mathfrak{g},\hbar=0}$ is classical at $(\mathcal{P}_B, \varphi)$

$\Rightarrow Op^{\check{\lambda}}_{\mathfrak{g},\hbar} \longrightarrow \mathbb{A}^1$ was flat at this point. //

~~Construction~~ Explicit construction of a "good" point in classical locus.

$\omega \in \Gamma(\Omega^1_X)$ non-zero

$x \in X$ where $\omega|_x \neq 0$.

Will define: $\forall \check\lambda \in \check\Lambda^+$

a certain $(\mathcal{P}_{B,n}, \varphi_n)$.

$\check\lambda = 0$:

$\mathcal{P} = \check\rho(\Omega)$ ← T-bundle ⇝ B, G-bundles.

$$\text{Higgs}_G(\mathcal{P}) = \begin{pmatrix} \Omega & \Omega^2 & \Omega^3 \\ \mathcal{O} & \Omega & \Omega^2 \\ \Omega^{-1} & \mathcal{O} & \Omega \end{pmatrix}$$

$$\varphi_0 = f + \check\rho \cdot \omega$$

$\underbrace{f = \sum_i f_i}_{\text{for princ. } sl_2}$.

$$= \begin{pmatrix} \omega & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -\omega \end{pmatrix}$$

f $\quad$ $\check\rho \cdot \omega$

For gen'l $\check\lambda$:

$\mathcal{P}_n = \check\rho(\Omega)(-\check\lambda x)$

$$\text{Higgs}(\mathcal{P}_n) = \begin{pmatrix} \Omega & & \\ \mathcal{O}(\check\lambda,\alpha_1)\Omega & & \\ & \mathcal{O}(\check\lambda,\alpha_2)\Omega & \end{pmatrix} \ni f + \check\rho \cdot \omega.$$

Claim: $\check{\lambda} \gg 0$, the map is smooth at this point.

Pf: Calculation; relative tangent cplx to $Op^{\check{\lambda}}_{\hbar=0} \longrightarrow$ Higgs at a point is:

$$\Gamma\Big(\underbrace{(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}}_{\text{coh. }0} \xrightarrow{[\varphi,-]} \underbrace{(\mathfrak{g}/\mathfrak{g}_{\geq -1})_{\mathcal{P}_B}}_{\text{coh. }1} \otimes \Omega^1\Big).$$

For us: claim 1: $\forall \check{\lambda}$ the map $[\varphi_n, -]$ is a surjective map of vector bundles.

Away from $x \in X$ this is usual Kostant theory.

At x, the Higgs field looks like

$$\underbrace{\begin{pmatrix} & \ddots & \\ \times & & \end{pmatrix}}_{\check\rho\cdot\omega_x} \Rightarrow \text{ win here also.}$$

Let $\mathcal{E}_{\check\lambda} := \mathrm{Ker}\left((\mathfrak{g}/\mathfrak{b})_{P_B} \xrightarrow{[\varphi_{\check\lambda},-]} (\mathfrak{g}/\mathfrak{g}_{\geq -1})_{P_B} \otimes \Omega^1 \right)$

when $\check\lambda > \check\mu$, we have a natural map $\mathcal{E}_{\check\lambda} \longleftrightarrow \mathcal{E}_{\check\mu}$. compatible w/ φ's.

$$(\mathfrak{g}/\mathfrak{b})_{P_{B,\check\lambda}} = \begin{pmatrix} & \times & \\ \mathcal{O}(\langle\check\lambda,\alpha_1\rangle x) & & \\ & \mathcal{O}(\langle\check\lambda,\alpha_2\rangle x) & \end{pmatrix}$$

These are injective maps of vector bundles of the same rank.

In particular: $\mathcal{E}_0 \subseteq \mathcal{E}_{\check{\lambda}}$.

Also: better:

$(\check{\lambda}, \alpha_i) \geq n \quad \forall n.$

$\Rightarrow \mathcal{E}_0(nx) \subseteq \mathcal{E}_{\check{\lambda}}$.

$\mathcal{E}_{\check{\lambda}} / \mathcal{E}_0(nx)$ is torsion.

Also: for $n \gg 0$ $\quad H^1(X, \mathcal{E}_0(nx)) = 0$.

$\Rightarrow H^1(\mathcal{E}_{\check{\lambda}}) = 0$

$R\Gamma(\mathcal{E}_{\check{\lambda}}) \leftarrow$ concentrated in deg. 0.

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