n

$$D(\mathrm{Bun}_G) \overset{?}{\simeq} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

Automorphic	Spectral
$D(\mathrm{Bun}_G)_{\mathrm{temp}}$	$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$
$D(\mathrm{Bun}_G)_{\mathrm{co}}$	$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$
$\mathrm{KL}(G)_{\mathrm{crit}}$	$\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$
$\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}})$	$\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$

$$\begin{array}{c} \mathrm{IndCoh}_{\mathrm{Nilp}} \\ \uparrow\downarrow \\ \mathrm{QCoh} \end{array}$$

$$D(\mathrm{Bun}_G)_{co} \xrightarrow[\sim]{\mathrm{PsId}_!} D(\mathrm{Bun}_G)$$

$$D(\mathrm{Bun}_G)_{co} \xrightarrow{\mathrm{PsId}^{\mathrm{naive}}} D(\mathrm{Bun}_G)$$

The two functors agree on the cuspidal subcategory.

Kevin Lin

$$\begin{tikzcd} & \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[dl, "\mathrm{Poinc}_*"'] \arrow[dr, "\mathrm{Poinc}_!"] & \\ D(\mathrm{Bun}_G)_{co} \arrow[rr, "\mathrm{PsId}_!"] & & D(\mathrm{Bun}_G) \end{tikzcd}$$

$$\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}})^\vee \simeq \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}})$$

$$(\mathrm{Poinc}_*)^\vee = (\mathrm{Poinc}_!)^R$$

$$\begin{tikzcd} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}"] \\ \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \end{tikzcd}$$

$$(\mathrm{Loc}^{\mathrm{spec}}_{G})^{\vee} = (\mathrm{Loc}^{\mathrm{spec}}_{G})^{R} = \Gamma^{\mathrm{spec.}}_{\check{G}}$$

$$\begin{tikzcd} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[d, "\mathrm{Loc}_G"] & \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \\ D(\mathrm{Bun}_G) & D(\mathrm{Bun}_G)_{\mathrm{co}} \arrow[u, "\Gamma_G"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}) \arrow[d, "\mathrm{Poinc}^{\mathrm{spec}}"] \\ \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \end{tikzcd}$$

doesn't have an adjoint

Goal:

$$D(\mathrm{Bun}_G) \xleftarrow{\ \Phi_G\ } \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check G})$$

$$\begin{tikzcd}
\mathrm{Rep}(\check G)_{\mathrm{Ran}} \arrow[rr, bend left] \arrow[d, shift right, "\mathrm{loc}"'] & & D(\mathrm{Bun}_G) \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check G}) \arrow[u, shift right] \arrow[urr, dashed, bend left] & &
\end{tikzcd}$$

$$D(\mathrm{Bun}_G) \xleftarrow{\ \mathring{\Phi}_G\ } \mathrm{QCoh}(\mathrm{LocSys}_{\check G})$$

$\mathring{\Phi}_G$ act on $\mathcal{W}_! = \mathring{\Phi}_G(\mathcal{O}_{\mathrm{LocSys}_{\check G}})$

$\mathcal{W}_! = \mathrm{Poinc}_!(\mathrm{Vac}_{\mathrm{Whit}})$

$$\begin{tikzcd} \mathrm{Bun}_N^{g.w} \arrow[r, "ev"] \arrow[d, "p"] & \mathbb{A}^1 \\ \mathrm{Bun}_G & \end{tikzcd}$$

$W_! = p_! \circ ev^* (A\text{-}Sch)$

Observe: $\mathring{\Phi}_G$ preserves compactness.

$$\begin{tikzcd} \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{Poinc}_!"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "\mathrm{Loc}^{spec}"] \\ D(\mathrm{Bun}_G) & \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[l, "\mathring{\Phi}_G"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, phantom, "\simeq"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ D(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}"] \arrow[r, "\mathring{\Psi}_G"] & \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[u, "\Gamma^{spec}"'] \end{tikzcd}$$

$$\mathring{\Psi}_G := (\mathring{\Phi}_G)^\vee \simeq (\mathring{\Phi}_G)^R$$

$$\begin{tikzcd}
D(\mathrm{Bun}_G) & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \\
D(\mathrm{Bun}_M) \arrow[u, "\mathrm{Eis}_{P,!}"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{M}}) \arrow[u, "\mathrm{Eis}^{\mathrm{spec}}_{P}"']
\end{tikzcd}$$

<u>Thm</u> (Beraldo, Chen) $\exists$

$$D(\mathrm{Bun}_G) \xrightarrow{\Psi_G} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

with the following properties:

a)
$$\begin{tikzcd}
 & \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \\
D(\mathrm{Bun}_G) \arrow[ur, "\mathring{\Psi}_G"] \arrow[r, "\Psi_G"'] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}) \arrow[u]
\end{tikzcd}$$

5)
$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\Psi_G"] & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check G}) \\ D(\mathrm{Bun}_M) \arrow[u, "\mathrm{Eis}_{P,!}"] \arrow[r, "\Psi_M"'] & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check M}) \arrow[u, "\mathrm{Eis}_P^{spec}"'] \end{tikzcd}$$

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\Psi_G"] \arrow[d, "\mathrm{CT}_{P,*}"'] & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check G}) \arrow[d, "\mathrm{CT}_P^{spec}"] \\ D(\mathrm{Bun}_M) \arrow[r, "\Psi_M"'] \arrow[ur, phantom, "\Nearrow"] & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check M}) \end{tikzcd}$$

__Q-Thm__ (G has connected center)

This natural transformation is an isomorphism.

__Cor__ $D(\mathrm{Bun}_G)_{\mathrm{Eis}} \subseteq D(\mathrm{Bun}_G)$

Ψ_G induces an equivalence

$$D(\mathrm{Bun}_G)_{Eis} \xrightarrow{\sim} \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check{G}})_{red}$$

$\Phi_G := \Psi_G^\vee$

$$\begin{tikzcd}
D(\mathrm{Bun}_G) & & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check{G}}) \arrow[ll, "\Phi_G"'] \\
D(\mathrm{Bun}_M) \arrow[u, "\mathrm{Eis}_{P,!}"'] & & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check{M}}) \arrow[ll, "\Phi_M"] \arrow[u, "\mathrm{Eis}^{spec}_{P}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\mathrm{Poinc}_!"'] & \simeq & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, "\mathrm{Loc}"] \\
 & & \mathrm{QCoh}(\;) \arrow[d, hook] \\
D(\mathrm{Bun}_G) & & \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_{\check{G}}) \arrow[ll, "\Phi_G"']
\end{tikzcd}$$

<u>Thm</u> (Faergeman – Raskin)
The essential image of Φ_G generates

the target category

Cor

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})}{\otimes} D(\mathrm{Bun}_G) \subseteq D(\mathrm{Bun}_G)$$

$$\cup\!\!\!\!| \quad D(\mathrm{Bun}_G)_{\mathrm{cusp}}$$

The containment is an equality.

Remains to prove:

$$D(\mathrm{Bun}_G)_{\mathrm{cusp}} \xleftarrow{\Phi_{\mathrm{cusp}}} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}})$$

is an equivalence.

Thm Φ_{cusp} is ambidexterous, i.e.

$$\Phi^R_{\mathrm{cusp}} \simeq \Phi^L_{\mathrm{cusp}}.$$

$$\| \quad //$$

$$\Big\Downarrow \wr$$

$$(\Phi_{\mathrm{cusp}})^\vee$$

1) Different strategy to construct the functor.
2) Proof of $\mathbb{Q}$-thm
3) Proof of Ambidexterity
4) Ambidexterity $\Longrightarrow$ GLC.

The following two classes of objects generate $D(\mathrm{Bun}_G)$:

i) $\mathrm{Im}(\mathrm{Eis}_{P,!})$

ii) $\mathrm{Im}(\mathrm{PsId}^{\mathrm{naive}}) \subseteq \mathrm{Im}(KL)$

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}^{\mathrm{spec}}} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

$$K \in \ker(\mathrm{Loc}^{\mathrm{spec}})$$

$$\chi * \mathcal{F} = 0 \qquad \text{for } \mathcal{F} \in D(\mathrm{Bun}_G)$$

i) $\mathcal{F} = \mathrm{Eis}_{P,!}(\mathcal{F}_M)$; $\mathcal{F}_M \in D(\mathrm{Bun}_M)$

ii) $\mathcal{F} = \mathrm{Loc}(M)$; $M \in KL(G)_{\mathrm{Ren}}$.

Betti: $\qquad$ X - space

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes C \longrightarrow C \otimes \mathrm{Lisse}^{\mathrm{Betti}}(X)^{\otimes I}$$

$$\Updownarrow$$

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X)) \circlearrowright C.$$

Restricted: $\qquad$ H - Tannakian cat

(ex $H = \mathrm{Lisse}(X)$)

$$\mathrm{Rep}(\check{G})^{I} \otimes C \longrightarrow C \otimes H^{\otimes I}$$

$$\Updownarrow$$

$$\mathrm{QCoh}(\mathrm{Funct}^{\otimes}(\mathrm{Rep}(\check{G}), H)) \circlearrowright C.$$

de Rham

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathrm{Dmod}(X^I)$$

$$\mathrm{QCoh}(\mathrm{LocSys}^{dR}_{\check{G}}(X)) \circlearrowright \mathcal{C}$$