Outline of the proof II

Recap:

Last time:
we defined $D(\mathrm{Bun}_G) \xrightarrow{\mathrm{coeff}^{\mathrm{enh}}} \mathrm{QCoh}(LS_{\check{G}})$

and let $\mathrm{Poinc}_!$ denote the image of $\mathcal{O}_{LS_{\check{G}}}$ under the left adjoint to this functor. We let

$$a_G := \mathrm{coeff}^{\mathrm{enh}}(\mathrm{Poinc}_!) \in \mathrm{QCoh}(LS_{\check{G}})$$

& reduced GLC to the statement

$$\mathcal{O}_{LS_{\check{G}}} \longrightarrow a_G \text{ is an } \simeq .$$

We asserted two thms to be shown later:

<u>Thm A</u>: $\mathcal{O} \longrightarrow \mathcal{a}_G$ becomes an iso after applying pullback to $\mathrm{LocSys}_{\check{P}}$ $\forall \check{P} \neq \check{G}$.

<u>Thm B</u>: $\mathcal{a}_G|_{LS^{irred}_{\check{G}}}$ is a <u>commutative</u> (connective) algebra that is finite étale over $\mathcal{O}_{LS^{irred}_{\check{G}}}$.

<u>Rem</u>: $\sigma \in LS^{irred}_{\check{G}}$, one can show

$$\mathrm{Spec}(\mathcal{a}_{G,\sigma}) = \left\{ \substack{\text{normalized} \\ \text{eigensheaves} \\ \text{for } \sigma} \right\} / \text{iso}.$$

Goal for today:

When G is adjoint (or connected center) and $g = g(X) > 1$, Thms A & B suffice to prove GLC.

Starting point: who is the enemy?

That there are too many eigensheaves, so $\mathcal{A}_G$ is too big.

Will give some result bounding the size of $\mathcal{A}_G$ in some sense.

<u>Prop</u>: $\Gamma(LS_{\check{\mathfrak{g}}}, \mathfrak{a}_G) = k$ (in particular: in degree 0).

~~[illegible]~~

<u>Unwinding</u>: Remind:

$\Gamma \circ \mathrm{coeff}^{\mathrm{enh}} = \mathrm{coeff} : D(\mathrm{Bun}_G) \to \mathrm{Vect}$

and $\mathfrak{a}_G = \mathrm{coeff}^{\mathrm{enh}}(\mathrm{Poinc}_!)$

So really claiming:

<u>Prop'</u>: $\mathrm{End}(\mathrm{Poinc}_!)$ $(= \mathrm{coeff}(\mathrm{Poinc}_!)) = k$.

We'll check this by contemplating $\mathrm{Poinc}_!$ for a few minutes.

~~Remark~~

Geometry of $\mathrm{Poinc}_!$:

$$
\begin{tikzcd}
\mathrm{Bun}_N^{\Omega} \arrow[r] \arrow[dd] & \mathrm{Bun}_N^{\Omega}/T \arrow[r] \arrow[d, phantom, "\text{closed}\ \cap"] \arrow[dd, bend right] & \mathrm{Bun}_G \\
& \mathrm{Bun}_B \arrow[ur] & \\
\bigoplus_{i \in \mathcal{I}_G} \mathbb{A}^1 \arrow[r] \arrow[d, "\mathrm{sum}"] & \left(\bigoplus \mathbb{A}^1\right)/T & \\
\mathbb{A}^1 & &
\end{tikzcd}
$$

<u>Basic claim:</u>

$$\mathrm{Bun}_N^{\Omega}/T \xrightarrow{j} \mathrm{Bun}_G$$ is a

locally closed embedding. $(g>1)$.

In fact: $\cancel{\mathcal{A}}$ Bun_B^d, $d = \deg(\check{\rho}(\Omega))$ is a Harder-Narasimhan stratum.

<u>Example:</u> $G = SL_2$, if $\mathcal{E}$ is an SL_2-bundle, it admits at most one line subbundle of positive degree.

$$\mathrm{Bun}_B^d = \{0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{\vee} \to 0\}$$

$$d = \deg \mathcal{L}$$

$d > 0$ $\mathrm{Bun}_B^d \longrightarrow \mathrm{Bun}_G$ is locally closed

Second basic claim:

if you take $\exp \in D(\mathbb{A}^1)$ and

!-pushforward to $\mathbb{A}^1/\mathbb{G}_m$, you

get $j_*(\mathrm{cnst}_{\mathring{\mathbb{A}}^1/\mathbb{G}_m})$ [shift?].

Similarly, if you !-pushforward

$\Psi \in D(\mathrm{Bun}_N^\Omega)$ to Bun_N^Ω/T,

we get $\tilde{j}_*(\mathrm{cnst}_{\mathring{\mathrm{Bun}}^\Omega_{N/T}})$

where $\mathring{\mathrm{Bun}}^\Omega_{N/T} = \{\psi_i \neq 0\}$ for $\psi_i : \mathrm{Bun}_N^\Omega \to \mathbb{A}^1$ for each $i \in I_G$

$$\begin{array}{c} \mathring{\mathrm{Bun}}^\Omega_{N/T} \\ \tilde{j} \downarrow \\ \mathrm{Bun}_N^\Omega/T \end{array}$$

third basic claim:

$(g>1)$

$\mathrm{Bun}_N^{\Omega} \longrightarrow (A')^{\vee}$ is a unipotent gerbe

(D-modules on them are the same).

Summary:

$$\overset{\circ}{\mathrm{Bun}}{}^{\Omega}_{N/T} \xrightarrow[\text{open}]{\tilde{j}} \mathrm{Bun}_{N/T} \xrightarrow[\text{locally closed}]{j} \mathrm{Bun}_G$$

$\overset{\circ}{\mathrm{Bun}}{}^{\Omega}_{N/T} = \mathbb{B}(\text{unipotent})$

$\tilde{j}$: $*$-extend

j: $!$-extend.

(Ends pf of Prop.)

Second main idea to handle

$\mathfrak{a}_G$:

<u>Thm</u>: $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$ is simply-connected

($\check{G}$ = simply connected, $g > 1$).

except in the case $g=2$ and $\underline{\check{G} = SL_2}$.

really: $\check{G} \xrightarrow{\text{non-triv}} PGL_2$

<u>Lemma 1</u>: $\mathrm{Bun}_{\check{G}}$ is simply-connected.

("Standard." see J. Feergeman's paper for lots of details.)

<u>Lemma 2</u>: Except in the exceptional case,

$\mathrm{Bun}_{\check{G}}^{\mathrm{unstable}} \overset{\text{closed}}{\subseteq} \mathrm{Bun}_{\check{G}}$ has codimension ≥ 2.

(Essentially due to Narasimhan - Ramanan '69.)

<u>Pf</u>: find the dimension of unstable bundles in the coarsest way.

Lemma 3: Every stable bundle admits a connection.

(In gen'l: variant of "Weil criterion" for admitting a connection for any $P_{\check{G}} \in \mathrm{Bun}_{\check{G}}$:

$\forall \check{M}$ and character $\check{M} \xrightarrow{\lambda} \mathbb{G}_m$ and every red'n of $P_{\check{G}}$ to $P_{\check{M}}$,

$\deg(\lambda(P_{\check{M}})) = 0$. Ref: Azad-Biswas 2002

irred. by [BD].

Picture:

$$\begin{array}{ccc} \pi^{-1}(\mathrm{Bun}_{\check{G}}^{\text{stable}}) \subseteq LS_{\check{G}}^{\text{irred}} & \subseteq & LS_{\check{G}} \\ \Big\downarrow {\scriptstyle \text{torsor for } T^*\mathrm{Bun}_{\check{G}}^{\text{stable}}} & & \Big\downarrow \pi \\ \underbrace{\mathrm{Bun}_{\check{G}}^{\text{stable}}}_{\text{DM}} & \subseteq & \mathrm{Bun}_{\check{G}} \end{array}$$

$\mathrm{Bun}_{\check{G}}$ s.c. $\Rightarrow$ $\mathrm{Bun}_{\check{G}}^{\text{stable}}$ s.c. $\Rightarrow$ $\underbrace{\pi^{-1}(\mathrm{Bun}_{\check{G}}^{\text{stable}})}^{\neq \emptyset}$ s.c.

$\Rightarrow LS_{\check{G}}^{\text{irred}}$ is s.c.

Third input (avoidable):

Thm (Frenkel-Teleman): $\Gamma(LS_{\check{a}}, \mathcal{O}) = k$
(under our hypotheses).

$\Rightarrow$ $\mathcal{O} \longrightarrow a_G$ is an isomorphism on Γ.

Let's put everything together:

$$\mathcal{O} \longrightarrow a_G$$

is an isomorphism on Γ.

Also, it is an isomorphism on $*$-restriction to $\mathrm{LocSys}^{\mathrm{red'ble}}$ (Thm A).

It follows:

$$\mathcal{O}_{LS^{irred}} \longrightarrow a_G|_{LS^{irred}}$$

$\rightsquigarrow$ an isomorphism on Γ.

But RHS $= \mathcal{O}^{\oplus r}$ for some r

by Thm B & s.c. of LocSys.

Clearly $\Rightarrow r=1$ & map is an iso.

Alternative to using Frenkel-Teleman:

$$i : LS^{\text{red'ble}}_{\check{G}} \hookrightarrow LS_{\check{G}} \hookleftarrow LS^{\text{irred}}_{\check{G}} : j$$

$$\underbrace{\hat{i}_* \hat{i}^! \mathfrak{a}_G}_{\substack{\| \text{ Thm A} \\ \hat{i}_* \hat{i}^! \mathcal{O}_{\mathrm{LocSys}_{\check{G}}}}} \longrightarrow \mathfrak{a}_G \longrightarrow \underbrace{j_* j^* \mathfrak{a}_G}_{j_* \mathcal{O}^{\oplus r}_{LS^{\text{irred}}_{\check{G}}}}$$

Claim: in coh. degs ≥ 2

Pf: $\mathrm{LocSys}_{\check{G}}$ is ~~genly~~ smooth away from codim 2 & lci (+classical)

$\Rightarrow$ normal

So Serre $\Rightarrow$ claim.

<u>Upshot</u>: for the t-structure

$$H^0(\mathcal{A}_G) \xrightarrow{\sim} H^0\left(j_* \mathcal{O}_{LS_{\check{G}}^{irred}}^{\oplus r}\right)$$
$$\| $$
$$\mathcal{O}_{LS_{\check{G}}}^{\oplus r}.$$

$\Rightarrow$ $r = 1$ (by calculation of $\Gamma(\mathcal{A}_G)$.)

$\Rightarrow$ $\mathcal{O} \longrightarrow \mathcal{A}_G$

is an iso on $LS_{\check{G}}^{irred}$ also. //