

# Outline of the proof of GIC

Motivation from Soergel theory:

Goal: understand  $\mathcal{D} \in \text{BAG category } \mathcal{D}$   
(derived)

"explicitly"

1) For  $C :=$  coinvariants algebra

( $\text{Spec } C =$  fiber of  $Z \rightarrow Z/W$   
at 0)

show  $\mathcal{D} \rightarrow \text{Spec } C$  "lives over"

$$C \rightarrow Z(\mathcal{D}).$$

$$\Leftrightarrow C\text{-mod} \simeq \mathcal{D}.$$

2) Define a functor  $V: \mathcal{D} \rightarrow \text{Vect}$

$$V(\mathcal{M}) = \mathcal{M} \otimes_{U(\mathfrak{n})} k_{\psi} \quad \psi: \mathfrak{n} \rightarrow k$$

is a non-deg. character.

3) Note: for formal reasons, the functor  $\mathbb{W}$  upgrades to a functor

$$\mathbb{W}^{\text{enh}}: \mathcal{D} \longrightarrow \text{C-mod.}$$

$$\begin{array}{ccc} & \mathbb{W} & \text{oblv} \\ & \downarrow & \downarrow \\ & \text{Vect} & \end{array}$$

4) Define  $\mathcal{D}^+ \subseteq \mathcal{D}$  subcategory gen'd by the big projective.

Prove:  $\mathbb{W}^{\text{enh}}: \mathcal{D}^+ \longrightarrow \text{C-mod}$  is an equivalence.

5) Homological algebra games.

(Simple minded for us.)

Pass to geometric Langlands:

Analogy:  $D(\text{Bun}_G) \leftarrow \mathcal{D}$

$\text{Spec } C \leftarrow \text{LocSys}_{\check{a}}$

Step 1 from before is:

Thm (Drinfeld-Gaitsgory):

$\mathcal{O}(\text{Ch}(\text{LocSys}_{\check{a}}))$  acts on  $D(\text{Bun}_G)$ ;

the action is uniquely characterized by a compatibility w/ Hecke functors.

Step 2: ~~Define~~ Define analogue of  $\mathcal{V}$ :

$\text{coeff} : D(\text{Bun}_G) \rightarrow \text{Vect}$

"first (zeroth?) Whittaker coefficient"

Explicitly:  $\text{Bun}_N^\Omega := \text{Bun}_B \times_{\text{Bun}_T} \check{\rho}(-\Omega_X)$

$\exists$  canonical character  $\psi: \text{Bun}_N^\Omega \rightarrow \mathbb{A}^1$ .

$\text{coeff}(F) := H_{\text{de}}^*(\text{Bun}_N^\Omega, \beta^!(F) \otimes \psi^!(\text{exp}))$ .

exponentiated  
on  $\mathbb{A}^1$ .

Example:  $G = \text{G}_m$ , this is taking  
the  $(!-)$  fiber at  $*$   $\xrightarrow{\text{triv.}}$   $\text{Bun}_{\text{G}_m}$ .

Step 3: Obtain for free:

$\exists \text{coeff}^{\text{enh}}: D(\text{Bun}_G) \rightarrow \text{QCh}(LS_{\tilde{q}})$

s.t.  $\Gamma(LS_{\tilde{q}}, \text{coeff}^{\text{enh}}(F)) = \text{coeff}(F)$ .

(+  $\text{QCh}(LS_{\tilde{q}})$ -linearity, this uniquely characterizes the functor.)

(5)

Step 4: Define  $D(\text{Ban}_q)^{\text{temp}} \begin{matrix} \xrightarrow{P^c} \\ \xleftarrow{P} \end{matrix} D(\text{Ban}_q)$ .

I discussed it last year. It involves derived Satake at a marked pt  $x \in X$ .

Thm (Faergeman-R.) 1) The definition is independent of the chosen point.

2)  $\text{coeff}^{\text{enh}} : D(\text{Ban}_q)^{\text{temp}} \longrightarrow \mathcal{QCoh}(S_{\tilde{q}})$  is conservative.

Tempered GLC: The functor  $\text{coeff}^{\text{enh}}$  gives an equivalence  $D(\text{Ban}_q)^{\text{temp}} \xrightarrow{\cong} \mathcal{QCoh}(S_{\tilde{q}})$ .

Mostly today, I want to focus on this statement.

Before going on:

Step 5) Assume ~~⊗~~  $\neq$  GLC.

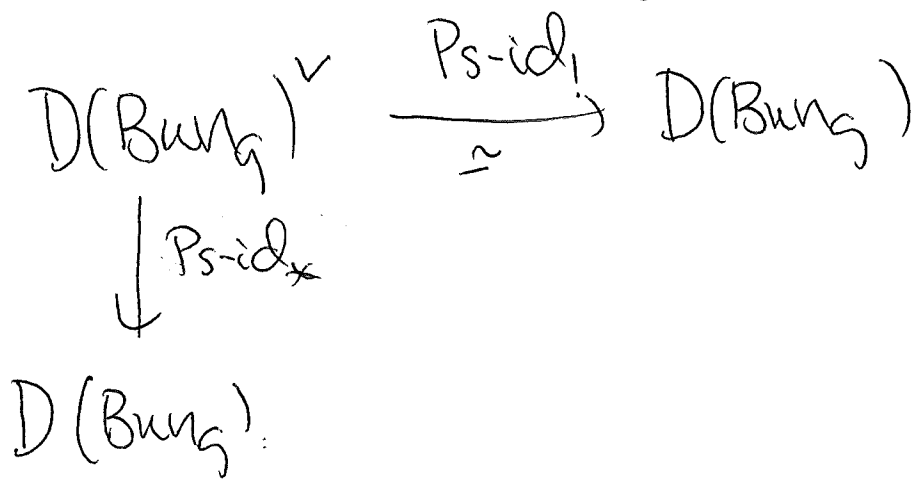
Want: deduce "usual" GLC.

~~⊗~~ Prop: Given  $F, g$  on  $D(Bun_g)$   
s.t.  $p(F) = 0$  (i.e. " $F$  is anti-tempered")  
and s.t.  $g$  is compact,

$$\text{Hom}(F, g) = 0.$$

~~⊗~~ PF (sketch): Dennis defined

~~the~~ functors (following Drinfeld):



~> a certain <sup>endo</sup> functor  $T = (P_5 - id)_* (P_5 - id_!)^{-1}$   
of  $D(Bun_g)$ .

Fact: a)  $T$  is fully-faithful on compacts

$$\Rightarrow Hom(F, g) \xrightarrow{\cong} Hom(T(F), T(g))$$

whenever  $g$  is compact.

b)  $T$  annihilates anti-tempered objects,  
 $\uparrow$  ( $P_5 - id_{**}$  already does this.)

Thm (Beraldo.)

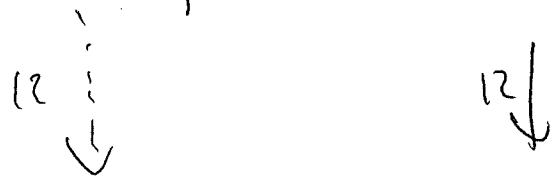
Cor:  $D(Bun_g)^c \subseteq D(Bun_g) \xrightarrow{P} D(Bun_g)^{temp}$

cpt objs  
in  $D(Bun_g)$

this composition is fully-faithful.

~~show~~ If we believe tGLC, need to

$$\text{show: } D(\text{Bun}_g)^c \hookrightarrow D(\text{Bun}_g)^{\text{temp}}$$



$$\text{Coh}_{\text{Nilp}}(LS_{\check{g}}) \subseteq \text{QGr}(LS_{\check{g}})$$

defined by Arinkin-Gaitsgory.

Idea:  $D(\text{Bun}_g)^c$  is Karoubi-generated

by tempered compact objects and Eisenstein objects ( $\text{Eis}_i$  of a compact from a Levi).

The former maps to  $\text{Perf}(LS_{\check{g}})$  by tGLC.

The latter maps to

$$\begin{aligned} & \text{coeff}_g^{\text{enh}} \text{Eis}_i(\text{D}(\text{Bun}_g)^c) \\ &= \text{Eis}^{\text{spec}} \text{coeff}_u^{\text{enh}} (\text{D}(\text{Bun}_u)^{\text{temp},c}) \end{aligned}$$



$$= \text{Eis spec}(\text{Perf}(\text{LS}_{\tilde{u}})).$$

~~By defn (2)~~

The key property of Coh $_{W|p}$  is that it is the category obtained this way.

(End of Step 5).

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Q: What should we do to prove tGLC?

~~Define~~ Define  $P_{\text{omc}} \in D(\text{Bun}_q)$  as the object corepresenting coeff.

Explicitly:  $\mathbb{P}_1^*(\text{exp})$  (up to normalizing shift).

The functor  $\text{coeff}^{\text{enh}}$  has a left adjoint:

$$\text{Qcoh}(LS_{\tilde{a}}) \longrightarrow D(\text{Bun}_g)$$

$$g \longmapsto g \times \text{Poinc}_1$$

"spectral action"

FWIW; It maps into  $D(\text{Bun}_g)^{\text{temp}}$

We now have adjoint functors:

$$\text{Qcoh}(LS_{\tilde{a}}) \begin{array}{c} \xrightarrow{\times \text{Poinc}_1} \\ \xleftarrow{\text{coeff}^{\text{enh}}} \end{array} D(\text{Bun}_g)^{\text{temp}}$$

The right adjoint is conservative, so it suffices to show that the left adjoint is ~~not~~ fully faithful.

$$\Leftrightarrow \text{coeff}^{\text{enh}}(g \times \text{Poinc}_1) \xrightarrow{\sim} g$$

linearity  $\rightarrow$  "

$$g \otimes_{\text{Qcoh}(LS_{\tilde{a}})} \text{coeff}^{\text{enh}}(\text{Poinc}_1)$$

Notation:  $\mathcal{A}_G := \text{coeff}^{\text{enh}}(\text{Poinc}_G)$ .

Summary:

$\text{tGLC} \Leftrightarrow$  the natural map

$$\mathcal{D} \xrightarrow{\text{LS}_G} \mathcal{A}_G$$

is an isomorphism.

Yoga: All problems have something to do  
w/ understanding  $\mathcal{A}_G$ .

Aside:  $\mathcal{A}_G$  is naturally an associative algebra  
in  $\mathcal{QCh}(\text{LS}_G)$ .

What we get for free:

$$D(\text{Bun}_G)^{\text{temp}} \xrightarrow{\cong} \mathcal{A}_G\text{-mod}(\mathcal{QCh}(\text{LS}_G))$$

What do we know about  $A_G$ ?

(As a preview).

1) ~~is~~ "Ambidexterity" + more } ops stuff.

$A_G / \text{LocSys}_{\check{a}}^{\text{red}}$  is a commutative algebra, that is finite étale over  $\mathcal{D}_{\text{LocSys}_{\check{a}}^{\text{red}}}$ .

2)  $\text{CT}^{\text{spec}}(A_G) \leftarrow \text{CT}^{\text{spec}}(\mathcal{D}_{\text{LS}_{\check{a}}})$  in  $\text{QCoh}(\text{LS}_{\check{a}})$ .

Calculation of  $\text{CT} \circ \text{Eis}$ , ~~is~~ tells you this is an isomorphism.

Highly non-trivial. ~~unless ops exist~~. Maybe not forever (Glu okay).

(Lin Chen & Kevin Lin - in progress).

NB: 2)  $\Rightarrow \mathcal{O} \rightarrow \mathcal{A}_G$  is an isomorphism on the reducible locus.

3) Trick: if  $G$  has connected center,  
1) + 2)  $\Rightarrow$  +GLC. (Using some known facts about geometry of  $\text{LocSys}_G$ ).