Outline of the proof of GLC

Motivation from Soergel theory:

Goal: understand $\mathcal{O} \leftarrow$ BGG category $\mathcal{O}$
(derived)

"explicitly":

1) For $C :=$ coinvariants algebra
($\operatorname{Spec} C =$ fiber of $\mathfrak{t} \to \mathfrak{t}/\!/W$ at 0)

show $\mathcal{O} \longrightarrow \operatorname{Spec} C$ "lives over"

$C \to Z(\mathcal{O})$.

$\Leftrightarrow$ C-mod $\curvearrowright \mathcal{O}$.

2) Define a functor $\mathbb{V} : \mathcal{O} \longrightarrow \mathrm{Vect}$

$\mathbb{V}(M) = M \otimes_{U(\mathfrak{n}^-)} k_\psi$ $\quad \psi : \mathfrak{n}^- \to k$ is a non-deg. character.

3) Note: for formal reasons, the functor $\mathbb{V}$ upgrades to a functor

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\mathbb{V}^{enh}"] \arrow[dr, "\mathbb{V}"'] & C\text{-mod} \arrow[d, "oblv"] \\ & \mathrm{Vect} \end{tikzcd}$$

4) Define $\mathcal{O}^{+} \subseteq \mathcal{O}$ subcategory gen'd by the big projective.

<u>Prove</u>: $\mathbb{V}^{enh} : \mathcal{O}^{+} \longrightarrow C\text{-mod}$ is an equivalence.

5) Homological algebra games.

(Simple minded for us.)

Pass to geometric Langlands:

<u>Analogy</u>: $D(\mathrm{Bun}_G) \leftarrow\!\!\shortmid\ \theta$.

$\mathrm{Spec}\, C \leftarrow\!\!\shortmid\ \mathrm{LocSys}_{\check{G}}$.

<u>Step 1</u> from before is:

<u>Thm</u> (Drinfeld-Gaitsgory):

$QCoh(\mathrm{LocSys}_{\check{G}})$ acts on $D(\mathrm{Bun}_G)$;

the action is uniquely characterized by a compatibility w/ Hecke functors.

<u>Step 2</u>: Define analogue of W:

$\mathrm{coeff} : D(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$

"first (~~zeroth~~?) Whittaker coefficient"

Explicitly: $\mathrm{Bun}_N^{\Omega} := \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \check{\rho}(-\Omega_X)$

$\exists$ canonical "character" $\psi : \mathrm{Bun}_N^{\Omega} \longrightarrow \mathbb{A}^1$.

$$\mathrm{coeff}(\mathcal{F}) := H^*_{dR}\big(\mathrm{Bun}_N^{\Omega}, \mathfrak{p}^!(\mathcal{F}) \overset{!}{\otimes} \psi^!\underbrace{(\exp)}_{\text{exponential on } \mathbb{A}^1}\big).$$

Example: $G = \mathbb{G}_m$, this is taking the (!-)fiber at $* \xrightarrow{\text{triv}} \mathrm{Bun}_{\mathbb{G}_m}$.

Step 3: Obtain for free:

$$\exists \ \mathrm{coeff}^{\mathrm{enh}} : D(\mathrm{Bun}_G) \longrightarrow \mathrm{QCoh}(LS_{\check{G}})$$

s.t. $\Gamma(LS_{\check{G}}, \mathrm{coeff}^{\mathrm{enh}}(\mathcal{F})) =$

$\mathrm{coeff}(\mathcal{F})$.

(+ $\mathrm{QCoh}(LS_{\check{G}})$-linearity, this uniquely characterizes the functor.)

Step 4: Define $D(\mathrm{Bun}_G)^{temp} \underset{P}{\overset{P^L}{\rightleftarrows}} D(\mathrm{Bun}_G)$.

I discussed it last year. It involves derived Satake at a marked pt $x \in X$.

Thm (Faergeman - R.) 1) The definition is independant of the chosen point.

2) $\mathrm{coeff}^{enh} : D(\mathrm{Bun}_G)^{temp} \longrightarrow QCh(LS_{\check{G}})$

is conservative.

Tempered GLC: The functor coeff^{enh} gives ~~have~~ an equivalence $D(\mathrm{Bun}_G)^{temp} \xrightarrow{\simeq} QCh(LS_{\check{G}})$.

Mostly today, I want to focus on this statement.

Before going on:

<u>Step 5</u>) Assume tGLC.

<u>Want</u>: deduce "usual" GLC.

<u>Prop</u>: Given $\mathcal{F}, \mathcal{G}$ on $D(\mathrm{Bun}_G)$ s.t. $p(\mathcal{F}) = 0$ (i.e. "$\mathcal{F}$ is anti-tempered") and s.t. $\mathcal{G}$ is compact,

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0.$$

<u>Pf</u> (sketch): Dennis defined functors (following Drinfeld):

$$\begin{tikzcd} D(\mathrm{Bun}_G)^\vee \arrow[r, "\mathrm{Ps\text{-}id}_!", "\sim"'] \arrow[d, "\mathrm{Ps\text{-}id}_*"] & D(\mathrm{Bun}_G) \\ D(\mathrm{Bun}_G) & \end{tikzcd}$$

$\rightsquigarrow$ a certain ~~endo~~ endo functor $T = (P_S\text{-}id)_*(P_S\text{-}id_!)^{-1}$

of $D(Bun_G)$.

<u>Fact</u>: a) T is fully-faithful on compacts

$\Rightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(T(\mathcal{F}), T(\mathcal{G}))$

whenever $\mathcal{G}$ is compact.

b) T annihilates anti-tempered objects,

(${P_S\text{-}id}_*$ already does this.)

Thm (Beraldo.)

<u>Cor</u>: $\underbrace{D(Bun_G)^c}_{\text{cpt objs in } D(Bun_G)} \subseteq D(Bun_G) \xrightarrow{p} D(Bun_G)^{temp}$

this composition is fully-faithful.

If we believe tGLC, need to show:

$$\begin{tikzcd} D(\mathrm{Bun}_G)^c \arrow[r, hook] \arrow[d, dashed, "\wr"'] & D(\mathrm{Bun}_G)^{temp} \arrow[d, "\wr"'] \\ \mathrm{Coh}_{\underbrace{\mathcal{N}ilp}_{\text{defined by Arinkin-Gaitsgory}}}(LS_{\check{G}}) \arrow[r, phantom, "\subseteq"] & \mathrm{QCoh}(LS_{\check{G}}) \end{tikzcd}$$

Idea: $D(\mathrm{Bun}_G)^c$ is Karoubi-generated by tempered compact objects and Eisenstein objects ($\mathrm{Eis}_!$ of a compact from a Levi).

The former maps to $\mathrm{Perf}(LS_{\check{G}})$ by tGLC.

The latter maps to

$$\mathrm{coeff}_G^{enh}\, \mathrm{Eis}_!\left(D(\mathrm{Bun}_M)^{c,temp}\right)$$

$$= \mathrm{Eis}^{spec}\, \mathrm{coeff}_M^{enh}\left(D(\mathrm{Bun}_M)^{temp,c}\right)$$

$$= \mathrm{Eis}^{\mathrm{spec}}(\mathrm{Perf}(LS_{\check{\mathfrak{u}}})).$$

~~By defn (2)~~

The key property of $\mathrm{Coh}_{\mathrm{Nilp}}$ is that it is the category obtained this way.

(End of Step 5).

<u>Q</u>: What should we do to prove tGLC?

- Define $\mathrm{Poinc}_! \in D(\mathrm{Bun}_G)$ as the object corepresenting coeff.

<u>Explicitly</u>: $\mathfrak{p}_!(\mathrm{ev})^*(\exp^{\otimes I})$ (up to normalizing shift).

The functor $\mathrm{coeff}^{\mathrm{enh}}$ has a left adjoint:

$$\mathrm{QCoh}(LS_{\check{G}}) \longrightarrow D(\mathrm{Bun}_G)$$

$$\mathcal{G} \longmapsto \mathcal{G} \underset{\uparrow}{\star} \mathrm{Poinc}_! .$$

"spectral action".

FWIW; It maps onto $D(\mathrm{Bun}_G)^{\mathrm{temp}}$.

We now have adjoint functors:

$$\mathrm{QCoh}(LS_{\check{G}}) \underset{\mathrm{coeff}^{\mathrm{enh}}}{\overset{\star \mathrm{Poinc}_!}{\rightleftarrows}} D(\mathrm{Bun}_G)^{\mathrm{temp}}$$

The right adjoint is conservative, so it suffices to show that the left adjoint is ~~...~~ fully faithful.

$$\Leftrightarrow \mathrm{coeff}^{\mathrm{enh}}(\mathcal{G} \star \mathrm{Poinc}_!) \xleftarrow{\simeq} \mathcal{G}.$$

linearity $\longrightarrow$ ||

$$\mathcal{G} \underset{\mathcal{O}_{LS_{\check{G}}}}{\otimes} \mathrm{coeff}^{\mathrm{enh}}(\mathrm{Poinc}_!)$$

<u>Notation</u>: $\mathcal{a}_G := \mathrm{coeff}^{\mathrm{enh}}(\mathrm{Poinc}_!)$.

Summary:

tGLC $\Leftrightarrow$ the natural map

$$\mathcal{O}_{LS_{\check{G}}} \longrightarrow \mathcal{a}_G$$

is an isomorphism.

<u>Yoga</u>: All problems have something to do w/ understanding $\mathcal{a}_G$.

<u>Aside</u>: $\mathcal{a}_G$ is naturally an associative algebra in $\mathrm{QCoh}(LS_{\check{G}})$.

What we get for free:

$$D(\mathrm{Bun}_G)^{\mathrm{temp}} \xrightarrow{\simeq} \mathcal{a}_G\text{-mod}(\mathrm{QCoh}(LS_{\check{G}})).$$

What do we know about $\mathcal{A}_G$?
(As a preview).

1) "Ambidexterity" + more $\}$ opers stuff.

$\mathcal{A}_G|_{\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}}$ is a

<u>commutative</u> algebra, that is finite étale over $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}}$.

2) $CT^{\mathrm{spec}}(\mathcal{A}_G) \leftarrow CT^{\mathrm{spec}}(\mathcal{O}_{LS_{\check{G}}})$

in $QCoh(LS_{\check{T}})$.

Calculation of $CT \circ \mathrm{Eis}_!$ tells you this is an isomorphism.

Highly non-trivial. ~~gluless opers~~

Maybe not forever (Gln okay).
(Lin Chen & Kevin Lin — in progress).

NB: 2) $\Rightarrow \mathcal{O} \to \mathfrak{a}_{\check{G}}$ is an isomorphism on the reducible locus.

3) Trick: if G has connected center, 1) + 2) $\Rightarrow$ FGLC. (using some known facts about geometry of $\mathrm{LocSys}_{\check{G}}$).