Multiplication kernels
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Last time we discussed the multiplication kernel $K(a,b,c)$ where a,b,c belong to some n-dimensional variety X. In our main example $X = Bun_G$ is the moduli space of G-bundles on a curve C. In this example, we also have Hecke operators $H_x : L^2(X) \to L^2(X)$ for $x \in C$ which commute with each other and with the integral operators $K_a : L^2(X) \to L^2(X)$ defined by the kernel $K_a(b,c) = K(a,b,c)$.

For $G = PGL_2$, $C = \mathbb{P}^1$ and 4 parabolic points, by accident we have $X^0_{ss} \cong C$ (where X^0_{ss} is the semistable part of degree 0), and $H_x = K_x$. But in general this cannot be so since H_x is parametrized by $x \in C$ while K_a by $a \in X$.

So how are these pictures related?

This question is answered by a procedure called Separation of variables due to E. Sklyanin.

The key geometric fact behind separation of variables is that we have a birational isomorphism $T^*\mathrm{Bun}^d_G \cong T^*\mathrm{Sym}^n C$ (d - fixed degree) for $G = GL_r$. (here $n = \dim \mathrm{Bun}_G$ = genus of the spectral curve)

This isomorphism can be quantized, which defines an isomorphism $S: L^2(\mathrm{Bun}^d_G) \simeq L^2(\mathrm{Sym}^n C)$.

<u>Note</u>: The measure on $\mathrm{Sym}^n C$ is <u>**NOT**</u> the product of same measures on factors, it includes a discriminant-like factor.

which is a certain integral transform. This operator has the property that it maps eigenfunctions of Hecke operators $\psi_\lambda(a)$ such that

$$H_x \psi_\lambda = \beta_\lambda(x)\, \psi_\lambda$$

to the functions

$$\beta_\lambda(y_1) \cdots \beta_\lambda(y_n):$$

$$(Sf)(y_1, \dots, y_n) = \int_X S(y_1, \dots, y_n, a)\, f(a)\, da$$

$y_i \in C$

and

$$(S\psi_\lambda)(y_1, \dots, y_n) = \beta_\lambda(y_1) \cdots \beta_\lambda(y_n).$$

So this integral transform maps complicated functions of several variables to $\boxtimes$-powers of a single function of one variable (eigenvalue of the Hecke operator). In the genus 0 case, $G = PGL_2$, this kernel $S(\vec{y}, a)$ is known explicitly, and one can also write a formula for the kernel of the inverse operator S^{-1}, for which we have

$$S^{-1}(\beta_\lambda \boxtimes \cdots \boxtimes \beta_\lambda) = \psi_\lambda.$$

Recall that ψ_λ are eigenfunctions of the Gaudin hamiltonians.
So this allows us to "separate variables" in the Gaudin equations, which is why S is called the (quantum) separation of variables map.

Example. $G = GL(1)$.

Let $g = \text{genus}(C)$. Then $Bun_G^d(C) = Pic_d(C)$.

we identify $Pic_0 \cong Pic_d$ by choosing a line bdle of deg=1.

In this case $n = g$, and the birational isomorphism
$$T^* Sym^g C \longrightarrow T^* Pic_g(C)$$
is actually induced

by a map of the bases – the Abel–Jacobi map $AJ: \mathrm{Sym}^g C \longrightarrow \mathrm{Pic}_d(C)$

$$(x_1, \ldots, x_g) \to \mathcal{O}(x_1) \otimes \cdots \otimes \mathcal{O}(x_g),$$

which is a birational isomorphism. As a result, the "integral transform" S is simply the change of variable,

$$(S\psi)(y_1, \ldots, y_g) = \psi(AJ(y_1, \ldots, y_g)).$$

The Hecke operators in this situations are just shifts, and their eigenfunctions are Fourier harmonics, which are characters ψ of $\mathrm{Pic}_0(C)(F)$, and we have

$$\varphi_\lambda(AJ(y_1,\dots,y_g)) =$$

$$= \varphi_\lambda(\mathcal{O}(y_1)\otimes\cdots\otimes\mathcal{O}(y_g)) =$$

$$= \varphi_\lambda(\mathcal{O}(y_1))\cdots\varphi_\lambda(\mathcal{O}(y_g)) =$$

$$\beta_\lambda(y_1)\cdots\beta_\lambda(y_g)$$

where $\beta_\lambda(y) = \varphi_\lambda(\mathcal{O}(y))$.

This was extended to GL_2 by V. Drinfeld where S is much more complicated but still can be constructed using certain Radon transforms.

Now recall that

$$K(a,b,c)=\sum_\lambda \psi_\lambda(a)\psi_\lambda(b)\psi_\lambda(c).$$

Applying S with respect to a, we get

$$(S\,K)(x_1,\dots,x_n,b,c)$$

$$=\sum_\lambda \beta_\lambda(x_1)\cdots\beta_\lambda(x_n)\,\psi_\lambda(b)\,\psi_\lambda(c)$$

which is the kernel for the operator

$H_{x_1}\cdots H_{x_n}$.

$$(S\,K_a)(x_1,\dots,x_n)=H_{x_1}\cdots H_{x_n}.$$

So separation of variables relates the multiplication kernel with Hecke operators.

We may also apply S to all 3 components of $K(a,b,c)$.
Then we get

$$(S^{\otimes 3}K)(\vec{x},\vec{y},\vec{z}) = \widetilde{K}_{n,n,n}(\vec{x},\vec{y},\vec{z})$$

$$\sum_{\lambda} \beta_{\lambda}(x_1)\cdots\beta_{\lambda}(x_n)\beta_{\lambda}(y_1)\cdots\beta_{\lambda}(y_n)\beta_{\lambda}(z_1)\cdots\beta_{\lambda}(z_n).$$

If we can get an explicit formula for this function $\widetilde{K}_{n,n,n}$ of $3n$ variables, we are still in business (have our multiplication kernel), although now written in terms of the curve (i.e. on the spectral side of the Langlands correspondence). For $F = \mathbb{R}, \mathbb{C}$

The function $\beta_\lambda(x)$ is going to be a solution of the corresponding oper for G^L.

One benefit of this is that this function of $3n$ variables can be expressed in terms of a simpler function of $2n+1$ variables:

$$\tilde{K}_{1,n,n}(x,\vec{y},\vec{z}) =$$

$$\sum_{\lambda} \beta_\lambda(x)\,\beta_\lambda(y_1)\cdots\beta_\lambda(y_n)\,\beta_\lambda(z_1)\cdots\beta_\lambda(z_n).$$

Namely, $\left(S_y^{-1}\otimes S_z^{-1}\right)\tilde{K}_{1,n,n}(x,\vec{y},\vec{z})$

$= H(x,a,b)$ is the kernel of the Hecke operator H_x.

The function $\tilde{K}_{1,n,n}$ has the following meaning:

We have

$$\beta_\lambda(x)\,\beta_\lambda(y_1)\cdots\beta_\lambda(y_n) =$$

$$= \int \tilde{K}_{1,n,n}(x,y_1,\dots,y_n,z_1,\dots,z_n)\,\beta_\lambda(z_1)\cdots\beta_\lambda(z_n)\,dz.$$

In other words,
while for 4 points on $\mathbb{P}^1$
($\dim \mathrm{Bun} = 1$) we had

$$H_x H_y = \int K(x,y,z) H_z \, dz$$

So the product of two
Hecke operators expresses
as a "linear combination"
of Hecke operators,
if $\dim \mathrm{Bun} > 1$ this

is no longer so.
But if $\dim \mathrm{Bun} = n$
then the product of
$n+1$ Hecke operators
already expresses
via products of n

Hecke operators:

$$H_x H_{y_1} \cdots H_{y_n} = \int \widetilde{K}_{1,n,n}(x, y_1, \ldots, y_n, z_1, \ldots, z_n) H_{z_1} \cdots H_{z_n} dz$$

There is again a notion in symmetric monoidal categories $\mathcal{C}$ that corresponds to this.

<u>Definition</u> An $(n+1,n)$-commutative algebra in $\mathcal{C}$ is an object V with a map

$$\mu : S^{n+1} V \longrightarrow S^n V$$

Such that $\mu(\mu \otimes 1) : V^{\otimes n+2} \longrightarrow S^n V$ is Σ_{n+2}-symmetric.

Proposition. If V is an $(n+1,n)$-algebra then $S^n V$ is a usual commutative associative algebra.

In our example, V is the space of functions on C, and $\tilde{K}_{1,m,n}$ defines a structure of a commutative $(n+1,n)$-algebra on V.

This gives a usual commutative associative algebra structure on $S^n V = Fun(Sym^n C)$, which we discussed last time (given by the multiplication kernel K).

(Remark: this is all modulo analytic issues that need to be addressed in each case).

<u>Upshot</u>: It is interesting to find kernels

$$M(x, y_1, \dots, y_n, z_1, \dots, z_n)$$

which define a structure of an $(n+1, n)$-commutative algebra. For this they must satisfy a quadratic equation (associativity-commutativity axioms)

<u>Example</u>: Twisted

Langlands on $\mathbb{P}^1$ with $n+3$ parabolic points. $(G=PGL_2)$. $t_1,\dots,t_{n+2},t_{n+3}=\infty$

$n\geq 1$

I.e. we consider $\widetilde{Bun}_G$ = the space of bundles with trivializations at $t_1,\dots,t_{n+3}$.

Let $B=\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\in PGL_2\right\}$.

Then B^{n+3} acts on $\widetilde{Bun}_G$ freely,

$$\widetilde{Bun}_G/B^{n+3}=Bun_G.$$

And we fix

$x_1,\dots,x_{n+2},s=x_{n+3}\in i\mathbb{R}$

and define

$$L^2(\mathrm{Bun}_G)_{\chi}$$

to be the space of "functions" on $\widetilde{\mathrm{Bun}}_G$ which change according to the character

$$\bigotimes_{j=1}^{n+3} |b_j|^{\chi_j} \quad \text{of } B.$$

In this case we have the usual story with Hecke operators, quantum Hitchin, etc.

and the corresponding opers (with monodromy having eigenvalues $\exp(i 2\pi \chi_j)$ near t_j

are as follows:

$$\mathbb{D}_{x,\lambda} = \frac{d}{dx} P \frac{d}{dx} - s(s+n+1)x^n$$

$$- \frac{Q}{P} + (\lambda_1 x^{n-1} + \dots + \lambda_n)$$

where $P(x) = \prod_{i=1}^{n+2} (x - t_i)$,

$Q \in \mathbb{C}[x]$ $\deg Q \le n+1$

interpolation polynomial:

$$Q(t_i) = x_i^2 P'(t_i)^2$$

We have

$$\mathbb{D}_{x,\lambda} \beta_\lambda(x) = 0$$

For $F = \mathbb{C}$ we need

β_λ single-valued

(with leading coeff.

of asymptotics 1 at ∞).

For $F = \mathbb{R}$ we need β_λ to have appropriate asymptotics at t_i (assuming $t_i \in \mathbb{R}$).

e.g. $\beta_\lambda \sim (x - t_i)^{\pm \alpha_i}$ $x \to t_i$

$\sim x^s$ or x^{-s-n-1}, $x \to \infty$.

(this can be said more precisely but there was no time in the talk).

In some sense this means "trivial monodromy around the real locus ($\mathbb{R}$)" (but this needs to be taken with a big grain of salt).

Theorem. For this case

$K(x_1, \dots, x_{2n+1})$ ← symmetric in all variables

$$= \int_{\substack{n+2 \\ \dim}} \prod_{i=1}^{n+2} \left(w_{i,+}^{x_i} w_{i,-}^{-x_i} \right) \left(\sum_i (w_{i+} + w_{i-}) \right)^{2s} \wedge \frac{dw_{i+}}{w_{i+}}$$

with constraint

$$\forall i \quad w_{i,+} w_{i,-} = \frac{\prod_{\alpha=1}^{2n+1} (x_\alpha - t_i)}{P'(t_i)^2}$$

Trick: This can be reduced to an $n+1$-dimensional integral, by replacing

$$\left(\sum_i (w_{i,+} + w_{i,-}) \right)^{2s}$$ by

$$\left(\sum_i w_{i,+} \right)^{s - x_1 - \dots - x_{n+2}} \left(\sum_i w_{i,-} \right)^{s + x_1 + \dots + x_{n+2}}$$

(after integration this replacement becomes legitimate).

Example. $n=1$ (4 points but with twisting by $\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4$; the oper is the Darboux operator on $E=\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$

$$\partial_z^2 - \sum_{\substack{p\in E \\ \text{of order } 2}} k_p(k_p-1)\wp(z-p,\tau) - \Lambda$$

descended to $E/(z\mapsto -z)$.

If all $\varkappa_i$ are 0, get $K(x,y,z)=\frac{1}{\sqrt{f_t(x,y,z)}}$

for an explicit

polynomial f_t, but this is a happy accident. In general this is a double integral that does not seem to simplify.

(i.e. the answer is still motivic but direct image under a map with fibers of dimension $n+1$).