Multiplication Kernels

M. Kontsevich, Feb 24, 2021,
part II.
(notes by Pasha Etingof).

Let us discuss what one may mean by an "explicit formula."

1. Integration over local fields.

F a local field,
$X/_F$ algebraic variety
$Y \xrightarrow[\pi]{} X$ dominant map
$vol \in \Gamma_{alg}(Y, K_{Y/X})$
fiberwise algebraic top form

$|\mathrm{vol}|$ fiberwise density

Suppose $\int_{p^{-1}(y)} |\mathrm{vol}|$ converges

for generic y. Then get

a measure

$\pi_*(|\mathrm{vol}|)$ on $X(F)$.

<u>Example</u>: $X = \mathrm{Bun}_G(C)^3$.

To have a kernel

acting on half-densities,

we need vol to

be a section of

$K_{Y/X} \otimes K_X^{1/2}$.

Then we can take

$K(x,y,z) = \pi_*(|\mathrm{vol}|)$,

it is then a half-density in each variable.

Variations:

If $h : Y \to \mathbb{G}_a$ then can twist by $\psi(h)$ where ψ is an additive character.

If $h : Y \to \mathbb{G}_m$ then can twist by $\chi(h)$ χ multiplicative character.

② char$(k) = 0$

Take an explicit holonomic $\mathcal{D}$-module $\mathcal{M}$ on Y (a 1-dim. local system),

take direct image to X.
But need to choose
a cyclic vector
(solution of a diff.
equation defining the
D-module).

$\pi_* M$

But then we lose a
constant since e^x and
$2e^x$ satisfy the same ODE.

3 Use motivic constructible sheaves
instead of D-modules.
this would also give
result over $\mathbb{F}_q$.

4 characteristic p:
$\overline{\mathbb{Q}}^{CM}$-function on $X(\mathbb{F}_{q^n})$

We can pass from 3 to 4 by taking trace of Frobenius.

Now consider the quantum Hitchin system.

This is an integrable system on $Bun_G(C)$ where C is a smooth projective curve and G a semisimple group.

These are commuting differential operators on n-dimensional variety:

$$H_1, \ldots, H_n.$$

Let $\lambda_1, \ldots, \lambda_n$

Be the solution of holonomic system $H_i f = \lambda_i f$
(over $\mathbb{C}$ we should take single-valued sections).
And normalize by
$f_\lambda(\text{special point}) = 1$
in an appropriate sense
(probably need to consider the asymptotics since the special point likely to be singular)

We can consider the semiclassical limit of this picture:
We have

$\pi : T^*\mathrm{Bun}_G \xrightarrow{\text{Hitchin map}}$ Hitchin Base

$\| 2$ Birationally

$\mathcal{M} \leftarrow$ Moduli of Higgs bundles

Fibers of π are abelian varieties.

Let us fix a section S of this bundle π.

(make each fiber an algebraic group by choosing an origin).

Then we can define the Lagrangian subvariety

$\mathcal{Y} \subset T^*\mathrm{Bun} \times T^*\mathrm{Bun} \times T^*\mathrm{Bun}$

opposite sympl. form

cut out by the equations

$$\pi(x) = \pi(y) = \pi(z)$$

$$x + y = z.$$

(graph of addition in the fibers).

We expect that the kernel K comes from quantization of $\mathcal{Y}$.

This picture can be understood in terms of the Weinstein category $\mathcal{W}$. The objects are symplectic varieties,

$\mathrm{Hom}(X, Y) =$ Lagrangian subvarieties in $\overline{X} \times Y$.

Composition = convolution

of Lagrangian correspondences. This category is somewhat ill-defined similarly to Fukaya category since convolution of Lagrangians is not always a Lagrangian, but generically it is. Then this category is monoidal under $\times$, and an integrable system with a section is a commutative algebra in W.

Example. Let $X = \mathbb{A}^1 \setminus 0$.

Take diff. equation

$$\left(\left(x\frac{d}{dx}\right)^2 - \frac{1}{2}\left(x+\frac{1}{x}\right) - \lambda^2\right)\psi_\lambda(x)=0.$$

This is obtained from the example of $\mathbb{P}^1 \setminus (0,1,t,\infty)$ by sending two points to 0 and two points to ∞.

(Mathieu equation:

$$x = e^{i\theta} \to \left(\frac{d^2}{d\theta^2} + \cos\theta + \lambda^2\right)\psi = 0)$$

Then we have

$$K(x,y,z) = \exp(\phi(x,y,z))$$

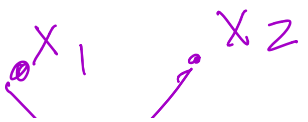
$$\phi(x,y,z) = \frac{x+y+z+xyz}{\sqrt{xyz}}.$$

and we may try to compute higher kernels

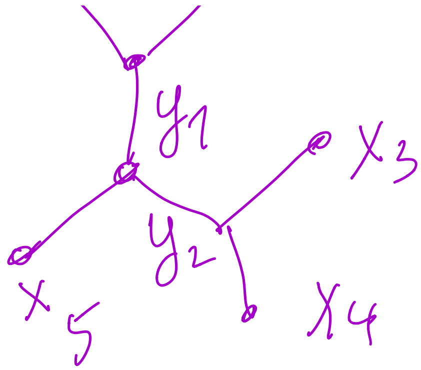
$$K(x_1,\dots,x_n) = \int\!\!\int\!\!\int \exp(\Sigma\phi) \prod_{i=1}^{n-3} \frac{dy_i}{y_i}$$

where $\exp(\Sigma\phi)$ has the following structure:

e.g. for $n=5$



$$\Sigma\phi = \phi(x_1,x_2,y_1)$$

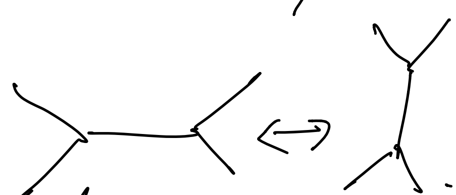


$+\phi(y_1, y_2, x_5)$

$+\phi(y_2, x_3, x_4)$

(sum over internal vertices)

Note that this will be independent on the way of connecting because of the Σ_4-symmetry:



of the 4-point function.

Question. Consider upper half-plane $\mathcal{H}$ and quotient $\mathcal{H}/\Gamma$,

$\Gamma = PSL_2(\mathbb{Z})$.

$\forall$ weight n consider Hecke eigenforms (parabolic holomorphic)

$f_{\alpha,n}(q) = q + \cdots$ here $\mathbb{Q}$ coefficients. $(\alpha = 1, \ldots, [\frac{n}{12}])$

normalized to have lead. coeff. 1. CM

Consider the sum

$$\sum_{\substack{\alpha \\ n \geq 1}} f_{\alpha,n}(q_1) f_{\alpha,n}(q_2) f_{\alpha,n}(q_3) t^n \in \mathbb{Q}[[q_1, q_2, q_3, t]].$$

Can we say anything about this function?

Example. $X = [0,1]$

$K : X^3 \to \mathbb{R}$

characteristic function of the simplex with vertices $(0,0,0)$ $(0,1,1)$, $(1,0,1)$, $(1,1,0)$

$\forall n \geq 1$ $\quad A_n = \mathbb{Q}^{X \cap \frac{1}{n}\mathbb{Z}}$

$= \mathbb{Q}^{\{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}\}} = \mathbb{Q}^{n+1}$

It has basis e_x
$\quad x \in X \cap \frac{1}{n}\mathbb{Z}$

$e_{x_1} \cdot e_{x_2} = K(x_1, x_2, x_3)\, e_{x_3}$

Commutative associative product. The proof is uniform with respect to n.

This is nothing but the Verlinde algebra for $\mathfrak{sl}_2$ at level n.

We expect that there is a similar realization of the Verlinde algebra for $\mathfrak{sl}_N$.