Multiplication kernels, part IV

preview of talk by Maxim Kontsevich

Geom. Langlands seminar

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Let C be a compact smooth curve over $\mathbb{C}$ (the compactness assumption is not really necessary at the beginning) Let $S = T^*C$, symplectic variety.

We will define a certain partial compactification P of T^*C such that the
$T^*C \subset P$
complement $P \setminus T^*C$ is a disjoint

union of affine lines, $\coprod \mathbb{A}^1$, and the symplectic form ω on T^*C extends with first order poles on these $\mathbb{A}^1$ (so we have a Poisson structure on $\mathcal{P}$ with first order zeros on $\mathcal{P} - S$). These $\mathbb{A}^1$ components will correspond to pairs ($p \in C$, "irregular term"

$$\exp\left(\sum_{i=1}^{n} c_i x^{-r_i}\right))$$

where $c_i \in \mathbb{C}$ and $r_i > 0$ are rational numbers, $r_1 < r_2 < \cdots < r_n$. Here x is a local coordinate on C near p.

Recall that such irregular terms classify connections on the punctured formal disk with possibly irregular singularities. (Hukuhara - Levelt - Turritin theorem). Namely, such a connection which is semisimple is generated as a D-module by

$$\exp\Big(\sum_{\substack{i=1 \\ c_i \neq 0}}^{n} c_i x^{-z_i}\Big) x^{\lambda}, \quad \lambda \in \mathbb{C}/\mathbb{Z}$$

where the rank of the connection is the common denominator of z_i.

Example. Consider $f = e^{x^{-\frac{1}{2}}}$.

Then $\partial f = -\frac{1}{2} x^{-\frac{3}{2}} f = g$

Also $\partial g = \frac{3}{4} x^{-5/2} f + \frac{1}{4} x^{-3} f =$

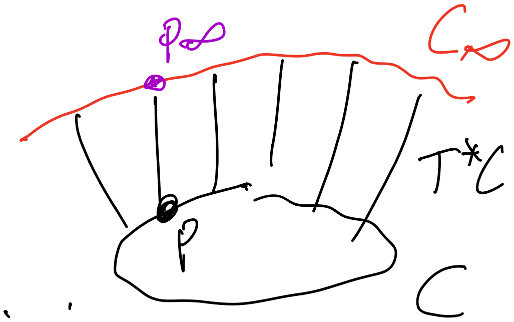
$-\frac{3}{2} x^{-1} g + \frac{1}{4} x^{-3} f.$

So we have

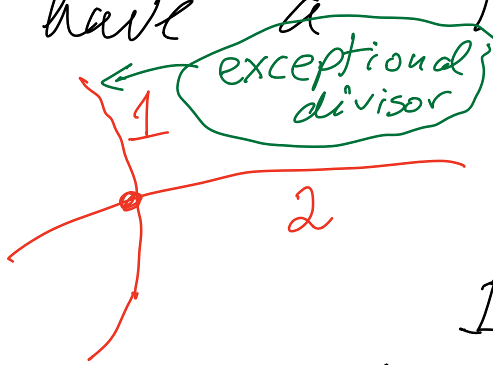
irregular connection

$$\partial \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4x^3} & -\frac{3}{2x} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

Geometrically this compactification is defined as follows. Consider $\overline{S} = \mathbb{P}(T^*C \oplus \mathcal{O}_C)$, a $\mathbb{P}^1$-bundle on C. We have the divisor at ∞, $C_\infty \subset \overline{S}$, and the form ω extends to $\overline{S}$ with 2-nd order pole at C_∞. Now given a point $p \in C$, let us blow up the corresponding point $p_\infty \in C_\infty$.

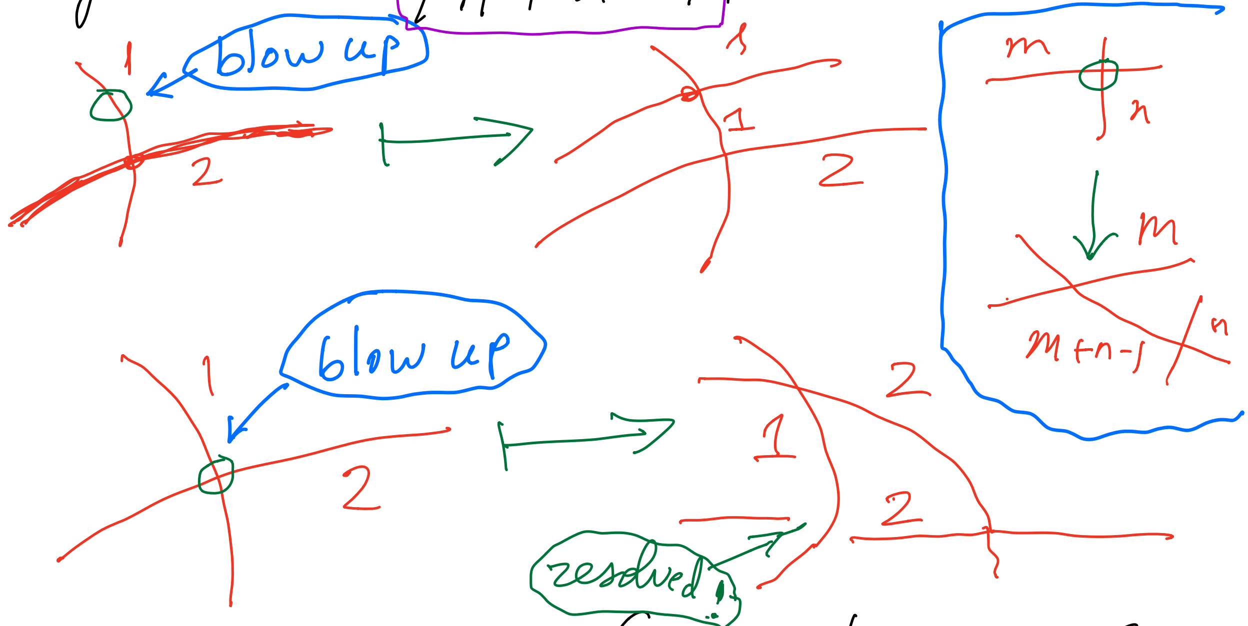


When we do,
we will obtain an exceptional divisor on which the form will have a 1st order pole:

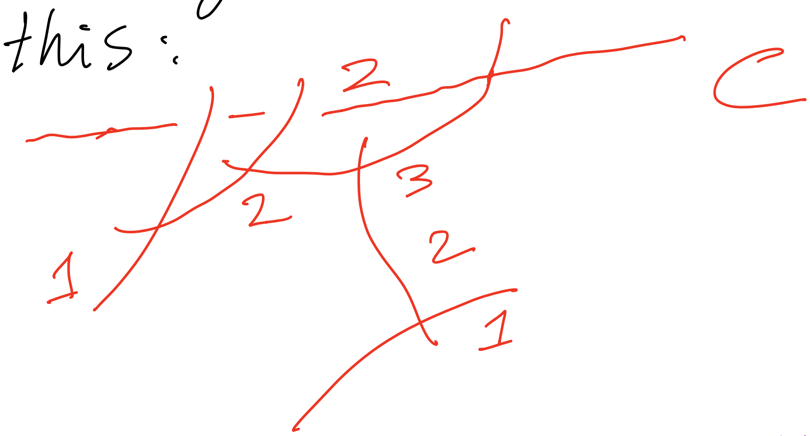


We now keep blowing up.

In doing so, we should note that when we blow up the intersection of two curves where ω has poles of order $\boxed{m, n}$, then on the exceptional divisor we get it will have pole of order $\boxed{m+n-1}$. So we have:

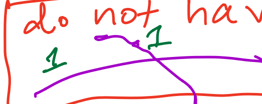


So after a few steps we can get a picture like this:



One can show that the "irregular terms" give rise to such a sequence of blow-ups. (such term is attached to every component with 1) (this will be explained below). Note that every component with first order pole is automatically isomorphic to $\mathbb{A}^1$; it can intersect only with one component. This is easy to see by induction, since we will not blow up points of such components which are not intersections with another component; otherwise we'll get a component with regular form ω, which we don't want. „T^*C"

Also we do not have



Definition $\Phi = S \cup \coprod \mathbb{A}^1$ is the union of S with these A'-s (having 1st order pole of ω). I.e., we throw away the closed components where ω has higher pole (including C_∞, where it has a pole of order 2).

Let us now explain how these blowups are related to the functions

$$e^{\sum c_i x^{-r_i}} x^{\lambda}.$$

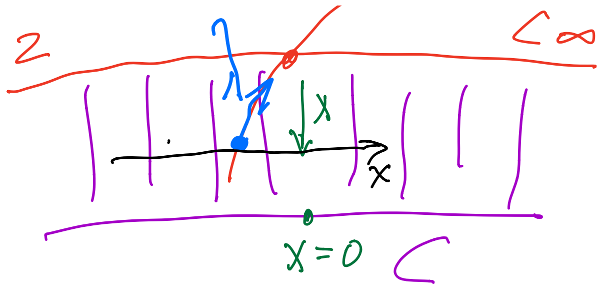
mentioned above.

To this end

consider the symplectic form ω around such A', and let us note that this A' is given locally by the equation

1/

$\boxed{x=0}$. Let λ be the second coordinate



such that the symplectic form locally has the form

$$\omega = d\lambda \wedge \frac{dx}{x}.$$

However, the map $x : \Sigma \to C$ may be ramified around our $\mathbb{A}^1$ with some ramification index $\boxed{r}$. In this case the coordinate x should be replaced by $\boxed{x^{1/r} = u}$. So we have

$$\boxed{\omega = d\lambda \wedge \frac{dx}{x} = \frac{1}{r}\, d\lambda \wedge \frac{du}{u}}.$$

So the canonical 1-form

$\eta = p\,dx$ on T^*C such that $d\eta = \omega$ extends as

$$\eta = p\,dx = \left(\lambda + f\left(x^{\frac{1}{r}}\right)\right)\frac{dx}{x},$$

where f is a meromorphic function near 0. Note that we can replace (as $d\eta = d\lambda \wedge \frac{dx}{x}$). λ by $\lambda + g(u)$ for any holomorphic function of u, so we may uniquely specify λ by the condition that

$f(u) \in \mathbb{C}[u^{-1}]\cdot u^{-1}$ (only singular part).

Thus we have

$$p = \lambda + f\left(x^{\frac{1}{r}}\right),$$

exponential

So the "area" function (exponential indefinite integral)

$$\exp\int^{\lambda=\lambda_0} p\,dx = \exp\left(\lambda\log x + \int f(x^{\frac{1}{r}})\frac{dx}{x}\right)$$
$$= \exp\left(\int f(x^{\frac{1}{r}})\frac{dx}{x}\right)\cdot x^{\lambda}$$
$$= \exp\left(\sum c_i x^{-r_i}\right)x^{\lambda}.$$

function from before

Exercise: Describe how to go back from $f(u)$ to the blow-up algorithm.

So we see that λ is, in fact, a natural coordinate on our $\mathbb{A}^1$-component.

This concludes the discussion of the local picture.

Now let us consider **Spectral curves**. This is a global problem so it is now important that C is compact.

For this purpose fix points $z_1, \ldots, z_d \in C - S$
$$C - S = \coprod \mathbb{A}^1.$$

Def. A spectral curve associated to this data is a compact curve

Σ which lies entirely in P and intersects the divisor $P \setminus S$ transversally at the points z_i only. (if some z_i coincide, the intersection index should equal the multiplicity)

Let g be the genus of Σ.

<u>Proposition.</u> Such curves Σ of genus g are parametrized by an affine g-dimensional space A^g. (we'll consider smooth irreducible ones which will form a dense open set in this A^g)

Note that Σ carries a 1-form η with singularities at the points z_i. Fix a line bundle L on Σ. of some degree d.

We have projection $\pi : \Sigma \to C$ of degree r. Consider the rank r vector bundle $E = E_{\Sigma,L}$ on C, $E_{\Sigma,L} = \pi_* L$. (of some degree d').

This bundle carries a Higgs field $\Omega_{\Sigma,L}$ obtained from η, whose spectral curve is Σ. The map $(\Sigma, L) \longmapsto (E, \Omega)$ is generically a bijection.

Thus we can think of pairs (Σ, L) as points on an appropriate

Hitchin moduli space

$$\widetilde{\mathcal{M}}_{Higgs} \overset{\text{Birationally}}{\cong} T^* \widetilde{Bun}_{GL_r}(C).$$

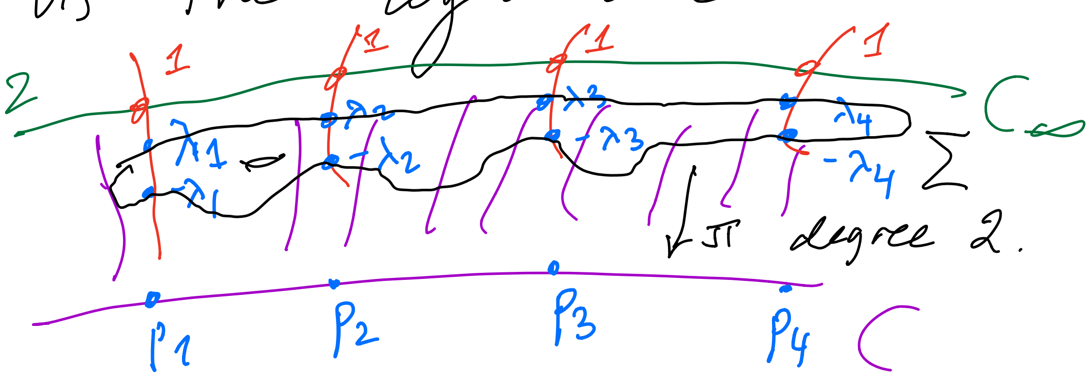
the moduli of bundles with appropriate level structure at a finite collection of points of C, defined by our irregular terms.

Moreover, the map

$$(\Sigma, L) \longmapsto \Sigma$$

is the Hitchin integrable system. Indeed, the set of possible Σ is the Hitchin base (describing the spectrum at each point of C). The fiber is the set of all bundles L of degree d, which is the abelian variety $\mathrm{Pic}_d(\Sigma)$. This is thus a complex integrable system. (an irregular Hitchin system).

Remark. The simplest case is the regular case:



This is related to the (for genus $(C)=0$) Deligne-Simpson problem: describe n-tuples of $N\times N$ matrices A_i

$i=1,\dots,n$ such that

$A_1+\dots+A_n=0$

and each A_i belongs to fixed conjugacy class

$C_i \subset \mathfrak{g} \subset \mathfrak{gl}_n$. (adjoint orbit)

Let us now assume that one of the points z_i is distinguished (call it z_0).

Then we can identify

$\mathrm{Pic}_d(\Sigma) \cong \mathrm{Pic}_0(\Sigma)$

by tensoring with $\mathcal{O}(z_0)^{\otimes d}$.

Recall now that

$\mathrm{Pic}_g \Sigma \overset{\text{Birationally}}{\cong} \mathrm{Sym}^g \Sigma$

by the Abel-Jacobi map

$\mathcal{O}(p_1)\otimes\cdots\otimes\mathcal{O}(p_g) \longleftrightarrow (p_1,\dots,p_g)$

We claim that we have a birational symplectomorphism

$$\mathcal{M}^{\mathrm{sing}}_{\mathrm{Higgs}} = \{(\Sigma, L)\} \xrightarrow[\xi]{\sim} \mathrm{Sym}^g T^*C$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad T^*\mathrm{Sym}^g C$$

defined as follows.

Given $(\Sigma, \underset{\deg = g}{L})$, write L as

$$L = \mathcal{O}(p_1) \otimes \cdots \otimes \mathcal{O}(p_g)$$

for $p_i \in \Sigma$, then $p_i \in T^*C$ as $\Sigma \subset T^*C$,

so $\xi(\Sigma, L) = (p_1, \ldots, p_g)$.

Now we can construct the quasiclassical "shift kernel" in

$(T^*C)^{g+1} \times (\overline{T^*C})^g \supset Z_{g+1,g}$
(a Lagrangian subvariety).

Namely, $Z_{1,g,g}$
is the set of $2g+1$
-tuples
$(p_0, p_1, \ldots, p_g, q_1, \ldots, q_g)$
of points in T^*C such
that they belong to
the same (unique)
spectral curve Σ,
and

distinguish. point.

$\mathcal{O}(p_0) \otimes \cdots \otimes \mathcal{O}(p_g) \simeq$
$\mathcal{O}(q_1) \otimes \cdots \otimes \mathcal{O}(q_g) \otimes \mathcal{O}(z_0)$

We may also define the

quasiclassical "multiplication kernel", Lagrangian $Z_{g,g,g}$ by convolving $Z_{1,g,g}$ with itself g times.

Remark. In fact, we never really used that S was the cotangent bundle to C in an essential way.

For example, one can take $S = \mathbb{C}^\times \times \mathbb{C}^\times$ and partially compactify with a disjoint union of copies of $\mathbb{C}^*$

on which the symplectic form has first order poles. We can still define spectral curves $\Sigma \subset \mathcal{P}$ (where $\mathcal{P}$ is a compactification of $\mathcal{S}$) and define shift and multiplication kernels is

$$S^{g+1} \times \overline{S}^{g} \supset Z_{g+1,g}$$

$$S^{2g} \times \overline{S}^{g} \supset Z_{2g,g}$$

in the same way as before: e.g.

$$(p_0, p_1, \ldots, p_g, q_1, \ldots, q_g) \in Z_{g+1,g}$$

$\Longleftrightarrow$ they belong to the same spectral curve Σ and $\mathcal{O}(p_0) \otimes \cdots \otimes \mathcal{O}(p_g) =$

$$\mathcal{O}(q_1) \otimes \cdots \otimes \mathcal{O}(q_g) \otimes \mathcal{O}(z_0).$$

One can also consider more general (rational) surfaces S.

Finally, let us discuss the quantization of this picture. In fact, there is a canonical one.

To our configuration of irregular term corresponds an irregular oper:

e.g. for $r=2$

$$(x-x_1)\cdots(x-x_m)\frac{d^2}{dx^2}+\cdots$$

where the λ-coordinates of points z_i occur linearly as parameters.

One then needs to find at each $\mathbb{A}^1$-component the solution ψ_λ of this Oper which is our irregular term times an element ψ_λ^0 of $1+x^{\frac{1}{2}}\mathbb{C}[[x^{\frac{1}{2}}]]$

Example.

$$\left(-\frac{d^2}{dx^2}+x-\lambda\right)\psi_\lambda(x)=0$$

(Airy equation).

Solution $\psi_\lambda(x)=Ai(x-\lambda)$

$$\mathrm{Ai}(x-\lambda) \underset{x\to\infty}{=}$$

$$= e^{-\frac{2}{3}x^{\frac{3}{2}}} \cdot x^{-\frac{1}{4}} \sum_{n=0}^{\infty} P_n(\lambda)\, x^{-n}$$

where $P_n(\lambda)$ are polynomials of degree n.

(There is a similar formula for a general differential equation).

i.e. $\psi^0_\lambda(x) = \sum_{n=0}^{\infty} P_n(\lambda)\, x^{n/r}$ near $x=0$

where $P_0(\lambda)=1$, $P_n(\lambda)$ has degree n.

$$\psi_\lambda(x) = e^{\sum c_i x^{-r_i}}\, x^{\lambda}\, \psi^0_\lambda(x).$$

Now define structure constants of $\mathbb{C}[\lambda]$

in the basis $P_i(\lambda)$:

$$P_i(\lambda)\, P_j(\lambda) = \sum_{k \le i+j} C_{ij}^{k}\, P_k(\lambda).$$

Now let K be the generating function of these structure constants:

$$K = \sum c_{ij}^{k}\, x^i y^j z^{-k} \frac{dz}{z}.$$

This is an element of

$$\mathbb{C}[[x]] \hat{\otimes} \mathbb{C}[[y]] \hat{\otimes} \mathbb{C}[[z]]^*$$

where $\mathbb{C}[[z]]^* = \dfrac{\mathbb{C}((z))}{\mathbb{C}[[z]]} \dfrac{dz}{z}$

Claim. This Kernel satisfies a holonomic differential equation, so generates a holonomic $\mathcal{D}$-module. This should be the quantum addition kernel.

There is a similar formula for multidimensional case, where the space of opers has $\dim = g$ and parameters $\lambda_1, \dots, \lambda_g$. In this case we get some bases of $\mathbb{C}[\lambda_1, \dots, \lambda_g]$ out of asymptotic expansions as above, and define structure constants and their generating function, which should generate a holonomic $\mathcal{D}$-module. Thus we have quantized our Lagrangians to a specific holonomic $\mathcal{D}$-module, and moreover with a cyclic vector (defined up to scaling).

This story extends to more general rational symplectic surfaces S (e.g. $\mathbb{C}^\times \times \mathbb{C}^\times$). In this case instead of a holonomic D-module we get a holonomic module over a quantization of S.

For example, for $\mathbb{C}^\times \times \mathbb{C}^\times$ we will get holonomic q-D-modules, i.e. modules over tensor powers of the algebra of q-difference operators,

$$A = \mathbb{C}\langle T, X\rangle / TX = qXT$$

(quantum torus). I.e. for shift kernel,

$(A^{\otimes g+1}, A^{\otimes g})$ - bimodule,

holonomic and with a cyclic vector.