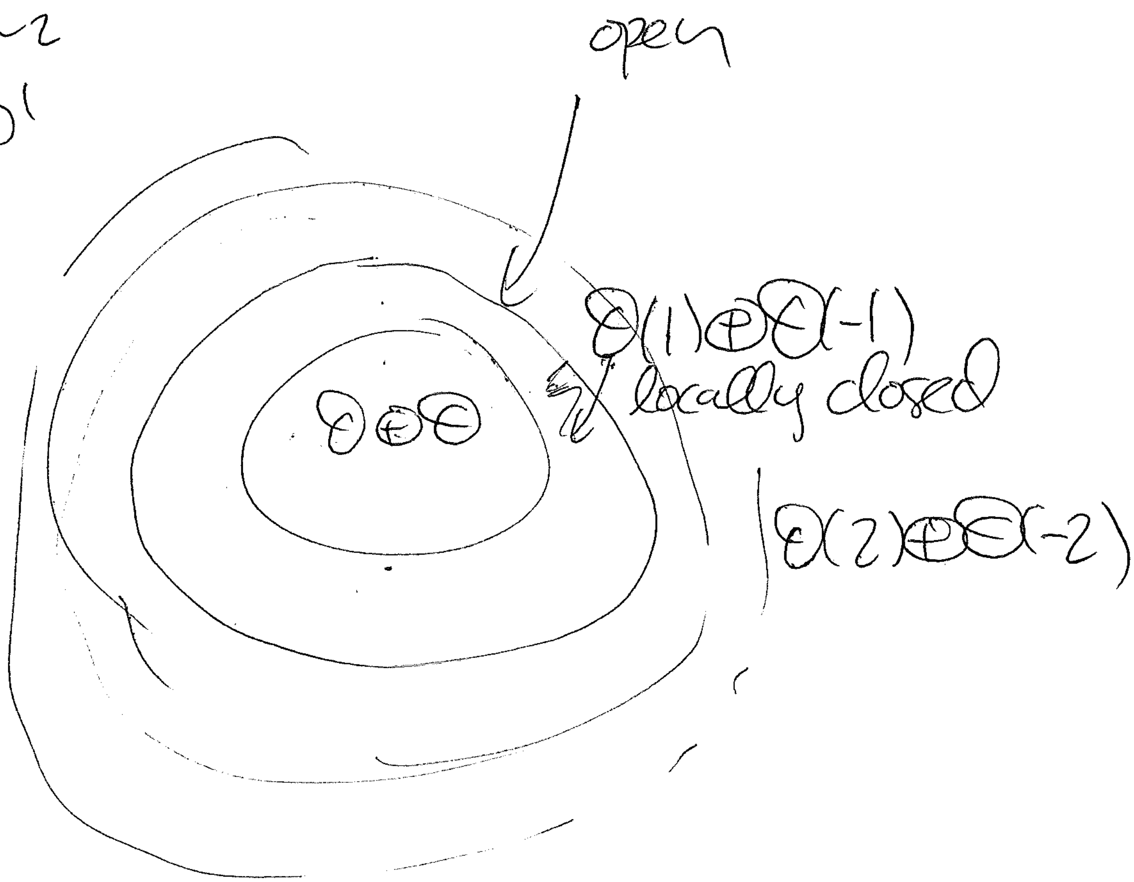
GL OH office hours

Basic background worth having in mind:

Picture of Bun_G

$G = SL_2$

$X = \mathbb{P}^1$



For gen'l X,

have the same picture

but the open is semistable bundles ($\{\mathcal{E}$ s.t. every line subbundle of $\mathcal{E}$ has deg $\leq 0\}$)

boundary strata are labeled by integers $d > 0$. parametrize

$\mathcal{E}$ that admits a (unique!) subbundle of deg $= d$.

stratum is isomorphic to

$$\{0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{\vee} \to 0 \mid \deg \mathcal{L} = d\}$$

$= \ldots\ldots$

$= \{ \mathcal{L} \in \mathrm{Pic}^d(X) + \omega \in \Gamma(\mathcal{L}^{\otimes 2})[1] \}.$

for $d >> 0$, note that $\Gamma(\mathcal{L}^{\otimes 2})[1]$ is purely in cohomological deg. -1.

Says: up to $\mathbb{B}$(unipotent gp), just get $\mathrm{Pic}^d(X)$.

(For more gen'l gps, see unipotent gerbes).

In any case: D-mod (stratum) $\simeq$ D-mod$(\mathrm{Bun}^d_{\mathbb{G}_m})$. Φ

D-mod

Given $\mathcal{F} \in D(\mathrm{Bun}_{SL_2})$, $CT^d_*(\mathcal{F}) \in D(\mathrm{Bun}^d_{\mathbb{G}_m})$

Φ is just the restriction of $\mathcal{F}$ to this stratum for $d >> 0$.

Cor: If $\mathcal{F}$ is cuspidal (all $CT_*(\mathcal{F})=0$), then $\mathcal{F}$ is $*$-extended from a qcpt open in Bun_{SL_2}. In fact, can show $\mathcal{F}$ is also $!$-extended from this open.

Rem: This generalizes to G, but you see all Levis/parabolics appearing (no surprise). ("Harder–Narasimhan" = keyword.)

Other thing I want to highlight: generators / basic objs of D-mod(Bun_G).

Three types:

1) Kac-Moody localized objects.

Examples: $x_1, \dots, x_n \in X$ $\quad \lambda_1, \dots, \lambda_n \in \Lambda^+$

$\mathrm{ind}(\underbrace{\mathcal{E}^{\sum \lambda_i x_i}}_{\text{vector bundle on } \mathrm{Bun}_{\check{G}}}) = \mathrm{Loc}(\otimes \mathbb{V}^{\lambda_i}_{x_i})$

$\uparrow$ D-mod induction.

Thm: $KL \xrightarrow{\mathrm{Loc}} D(\mathrm{Bun}_G)$
image contains all objects $*$-extended from a qcpt open.

Conj: Image is exactly the tempered category.

2) Poincaré series

$\mathrm{Poinc}_! = \mathcal{W}_{vac}$ "vacuum Whittaker sheaf"
& what you get by applying Hecke functors to it.

Thm (Faergeman-R.): The subcat. you obtain in this way is exactly $D(\mathrm{Bun}_G)_{temp}$.

3) Eisenstein series.

These are exactly designed to handle "behavior around ∞" in Bun_G.

Basic yoga:

it's "easy" (defined below) to compute Homs (or better, pairings) between two objects on $D(\mathrm{Bun}_G)$ of different classes.

Drinfeld-Gaitsgory

there:

- Pairing: $D(\mathrm{Bun}_G) \otimes D(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$.

- Example: it's supposed to be "easy" to compute

$\mathrm{Hom}(\mathrm{Poinc}_!, \mathrm{Loc}(\mathcal{M}))$, $\mathcal{M} \in KL$.

LHS = $\mathrm{coeff}_0(\mathrm{Loc}(\mathcal{M}))$, and that's what Dennis explained how to do last week.

- What's "easy"?

~~[illegible]~~ For <u>example</u>: you calculate something locally. (Using FLE's, maybe some functors coinciding under FLE's, ...)

- Then you produce a D-module on Ran space as some "local Homs" & take its dR cohomology (AKA: chiral homology).

Okay, on to factorization algebras:

Factorization algs have local / global components to the story.

Start w/ local (& driving at the FLE at critical level).

<u>Formal mechanics</u>: $\mathcal{A}$ } factorization / chiral alg. on X

$x \in X \rightsquigarrow \mathcal{A}\text{-mod}_x^{fact} \leftarrow$ DG category of modules at the point $x \in X$.

<u>Examples</u>: 1) $\mathcal{A} = \mathcal{A}_{\mathfrak{g},\kappa}$ "Kac-Moody chiral algebra"

$$\mathcal{A}_{\mathfrak{g},\kappa}\text{-mod}_x^{fact} = \hat{\mathfrak{g}}_\kappa\text{-mod}, \quad \hat{\mathfrak{g}} \simeq \mathfrak{g}((t_x)).$$

2) $\mathcal{A}$ is commutative (& let's say connective).

~~Aff~~ Commutative connective chiral algebras

$$\Longleftrightarrow \quad \underbrace{\mathcal{Y} \xrightarrow{\quad} X_{dR}}_{\text{affine}} \qquad \mathcal{Y} = \mathrm{Spec}_{X_{dR}}(\mathcal{A})$$

"affine D_X-schemes"

Given such an affine D_X-scheme,

standard ~~data~~ constructions:

global

$$\mathrm{Sect}^{\nabla}(X, \mathcal{Y}) = \{ X_{dR} \to \mathcal{Y} \text{ splitting the projection} \}.$$

$$\mathrm{Sect}^{\nabla}(\hat{D}_x, \mathcal{Y}) = \text{''}$$

$$\mathrm{Sect}^{\nabla}(\overset{\circ}{D}_x, \mathcal{Y}) = \text{''} .$$

local players.

In this case, $\mathcal{A}\text{-mod}_x^{fact}$ "=" $\mathrm{QCoh}(\mathrm{Sect}^{\nabla}(\overset{\circ}{D}_x, Y))$

(more like the dual of the RHS .....).

__Rem__: In practice, $\mathcal{A}\text{-mod}_x^{fact}$ is almost always "bad" and needs to be renormalized. It's bdded below category behaves as expected, but the equalities above are overly optimistic ~~as~~ as stated.

One ~~[illegible]~~ relevant commutative chiral algebra for us is one I'll denote $\mathfrak{Z}_{\check{G}}$ $(= W_{\check{\mathfrak{g}},\infty})$.

By construction, this factorization algebra is related to opers.

Fibers are $\mathrm{Fun}(\mathrm{Op}_{\check{G}}(D))$,

$$\Rightarrow \quad \mathrm{Sect}^{\nabla}(X_{dR}, \mathrm{Spec}\, \mathfrak{z}) = \mathrm{Op}_{\check{G}}(X)$$

$$\mathrm{Sect}^{\nabla}(\mathring{D}_{dR}, \mathrm{Spec}\, \mathfrak{z}) = \mathrm{Op}_{\check{G}}(\mathring{D}).$$

In particular, $\mathfrak{z}\text{-mod}^{fact}_x = \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathring{D}))$.

Reminder: RHS is not what appeared in FLE_{crit}, that was $\mathrm{QCoh}(\mathrm{Op}^{mf}_{\check{G}})$

$$\mathrm{Op}^{mf}_{\check{G}} := \mathrm{Op}_{\check{G}}(\mathring{D}) \underset{LS_{\check{G}}(\mathring{D})}{\times} LS_{\check{G}}(D)^{I\check{B}}.$$

Here's the recipe:

- $\exists$ an object $\mathfrak{z}^{enh}_{\check{G}} \in \mathrm{Rep}\,\check{G}_{Ran}$ that's a commutative factorization algebra.

its fiber at a point is

$\pi_* \mathcal{O}_{Op_{\check{G}}(D)} \in Rep\, \check{G}$ ~~where~~ where

$\pi : Op_{\check{G}}(D) \longrightarrow \mathbb{B}\check{G} = LocSys_{\check{G}}(D)_{\$}$

Its image under $Rep\, \check{G} \xrightarrow{(-)^{\check{G}}} Vect$

is exactly $\mathfrak{z}_{\check{G}}$.

The key point:

$$\mathfrak{z}^{enh}_{\check{G}}\text{-mod}^{fact}(Rep\, \check{G}) = QCh(Op^{mf}_{\check{G}}).$$

Construction of FLE_{crit}:

Dennis explained that there is a nice functor:

$$\underbrace{\Psi}_{\text{Drinfeld-Sokolov}} : KL_{crit} \longrightarrow Vect. \quad \text{map of factorization categories.}$$

Basic property (Feigin-Frenkel):

$$\Psi(\mathbb{V}_{crit}) = \mathfrak{z}_{\check{\mathfrak{g}}} \quad \text{as factorization algebras.}$$

Souped up version:

$\mathrm{Rep}\,\check{G} \circlearrowright KL_{crit}$ by Hecke functors.

$$\mathrm{Rep}\,\check{G} \otimes KL_{crit} \xrightarrow{act} KL_{crit} \xrightarrow{\Psi} Vect$$

or by duality:

$$\begin{tikzcd}
KL_{crit} \arrow[r, "\Psi^{enh}"] \arrow[dr, "\Psi"'] & (\mathrm{Rep}\,\check{G})^{\vee} = \mathrm{Rep}\,\check{G} \arrow[d, "(-)^{\check{G}}"] \\
& Vect
\end{tikzcd}$$

Fact: $\Psi^{enh}(\mathbb{V}_{crit}) = \mathfrak{z}^{enh}_{\check{\mathfrak{g}}}$.

(Beilinson-Drinfeld)

Therefore: we obtain:

$$\begin{tikzcd}
KL_{crit} \arrow[r, "FLE_{crit}"] \arrow[dr, "\Psi^{enh}"'] & \mathfrak{z}^{enh}_{\check{\mathfrak{g}}}\text{-mod}^{fact}(\mathrm{Rep}\,\check{G}) = \mathrm{QCoh}(\mathrm{Op}^{mf}_{\check{\mathfrak{g}}}) \arrow[d, "oblv"] \\
& \mathrm{Rep}\,\check{G}
\end{tikzcd}$$