GL OH office hours III

Last time: we discussed chiral homology in various forms.

Today: Want to explain how to use these ideas to calculate things in GL.

Need one more ingredient:

Suppose N is a unipotent algebraic gp, & $\mathcal{A}$ is a factorization algebra w/ "Kac-Moody symmetry by N":

$$L_n := n \otimes D_X \xrightarrow{i} \mathcal{A}$$

$\circlearrowright$

$\mathfrak{L}N$ ← fibers at $x \in X$

$L^+N = N(O)$

$\Leftrightarrow$ $\mathcal{A} \in KL_N = n((t))\text{-mod}^{N(O)}$

Last time:

$\mathrm{Hch}^{\mathrm{enh}}(\mathcal{A}) \in D(\mathrm{Bun}_N)$.

Key property: fiber at the trivial bundle is $\mathrm{Hch}(\mathcal{A})$.

Another construction (of local nature):

$\mathrm{BRST}(L_n, \mathcal{A}) \leftarrow$ new factorization algebra w/o any extra symmetries.

Idea of construction:

informally, ~~...~~ given an $n((t))$-module M,

$$\mathrm{BRST}(n((t)), M) \simeq \left(M^{n[[t]]}\right)_{n((t))/n[[t]]}.$$

(Makes sense for abelian n).

Rem: In this unipotent case, BRST is very explicit (cf. my paper on W-algebras Appendix A). More subtle in non-unipotent cases.

BRST$(L_n, \mathcal{A})$ is rigged to have fibers given by the construction I just said.

Examples: 1) $\mathcal{A} = \mathcal{A}_n := U^{ch}(L_n)$
("KM chiral algebra for n")

then $\mathrm{BRST}(L_n, \mathcal{A}) = \mathrm{Fun}(\mathbb{B}\mathfrak{L}N)$.

i.e., group cohomology of $\mathfrak{L}N$.

Roughly, the idea: on fibers, $\mathcal{A}_{n,x} = \mathbb{V}_n$

$\mathbb{V}_n$ ← has a PBW filt. w/

$$\mathrm{gr}.\,\mathbb{V}_n = \mathrm{Sym}\left(n((t))/n[[t]]\right)$$

properly: $\mathbb{V}_n = \mathrm{ind}_{n[[t]]}^{n((t))}(k)$.

$$\mathrm{BRST}(\mathbb{V}_n) = \mathrm{BRST}\left(\mathrm{ind}_{n[[t]]}^{n((t))}(k)\right) = $$ general thing about BRST.

$$k^{n[[t]]} = k^{N(\mathcal{O})}$$ by unipotence.

2) $\mathcal{A} = \mathrm{CDO}_N = \mathrm{ind}_{n[[t]]}^{n((t))}(\mathrm{Fun}(N(\mathcal{O})))$.

on fibers

$$\mathrm{BRST}(L_n, \mathcal{A}) = \mathrm{Fun}(N(\mathcal{O}))^{n[[t]]} = \mathrm{Fun}(N(\mathcal{O}))^{N(\mathcal{O})}$$ unipotence

$= k$.

3) ~~$\mathcal{A}$~~ $\mathcal{A} = \mathcal{A}_{\mathfrak{g},\kappa}$ ← Kac-Moody chiral algebra for G

($N = \mathrm{Rad}(B)$)

(For $\kappa = 0$, $\mathcal{A}_{\mathfrak{g},0} = U^{ch}(L_{\mathfrak{g}})$).

~~$\mathcal{A}$~~ $\mathfrak{L}G \curvearrowright \mathcal{A}_{\mathfrak{g},\kappa} \Rightarrow \mathfrak{L}N$ acts.

central ext'n involving κ

$$L_n \longrightarrow L'_{\mathfrak{g}} \longrightarrow \mathcal{A}_{\mathfrak{g},\kappa}$$

$$L_n \xrightarrow{\psi} \omega_X$$ non-degenerate Whittaker character

And I'll take $L_n \longrightarrow \mathcal{A}_{\mathfrak{g},\kappa}$ to be the difference of these ($\omega_X \in \mathcal{A}_{\mathfrak{g},\kappa}$ as the unit).

Rem: Properly, there are twists by differential forms on X (e.g. $\check{\rho}(\Omega_X)$) on $L_{\mathfrak{g}}, L_n, \dots$. I'm ignoring them here.

In this case, $\mathrm{BRST}(L_n, \mathcal{A}_{\mathfrak{g},\kappa}) =: W_{\mathfrak{g},\kappa}$.
"W-algebra" for $(\mathfrak{g},\kappa)$.

Basic facts to know:

- $W_{\mathfrak{g},\kappa} \simeq W_{\check{\mathfrak{g}},\check{\kappa}}$. (Feigin-Frenkel duality)
- $W_{\check{\mathfrak{g}},\underbrace{\check{crit}}_{\infty}} = \mathrm{Fun}(\mathrm{Op}_{\check{\mathfrak{g}}}(D))$.

Comparison of these notions:

Thm (Chen-Fu): For $\mathcal{A}$ w/ KM sym. by N as before, there is a natural iso:

$$H_{dR}(\mathrm{Bun}_N, H^{\mathrm{enh}}_{ch}(\mathcal{A})) = H_{ch}(\mathrm{BRST}(\mathcal{A})).$$

Examples: 1) $\mathcal{A} = \mathcal{A}_n = \mathcal{U}^{ch}(\mathcal{L}_n) (= \mathbb{V}_n)$.

$$H^{\mathrm{enh}}_{ch}(\mathcal{A}_n) = \mathrm{Diff}_{\mathrm{Bun}_N} \in D(\mathrm{Bun}_N).$$

true for any gp.

$\Rightarrow$ LHS $= H_{dR}(\mathrm{Diff}_{\mathrm{Bun}_N}) = \mathrm{Fun}(\mathrm{Bun}_N)$

$(= \Gamma(\mathrm{Bun}_N, \mathcal{O}))$.

OTOH: $\mathrm{BRST}(\mathcal{A}_n) = \mathrm{Fun}(\mathbb{B}\mathfrak{g}N)$.

Reminder from last time:

$$\mathcal{Y} \xrightarrow{\text{affine}} X_{dR} \qquad \mathrm{Fun}(\mathcal{Y}) \leftarrow \text{commutative chiral alg.}$$

$$H_{ch}(\mathrm{Fun}(\mathcal{Y})) = \mathrm{Fun}(\mathrm{Sect}_{/X_{dR}}(\mathcal{Y})).$$

<u>Idea</u>: $\mathbb{B}$(unipotent gp) is not so far from being affine ("coaffine").

So this applies to $\mathbb{B}\mathfrak{g}N$ as well,

and $\mathrm{Sect}_{/X_{dR}}(\mathbb{B}\mathfrak{g}N) = \mathrm{Maps}(X, \mathbb{B}N)$
$= \mathrm{Bun}_N$.

(Unipotence is ~~key~~ crucial on this second calculation).

2) $\mathcal{A} = CDO_N$

$BRST(\mathcal{A}) = \omega_X$ (or: k).

$\underline{\text{LHS}}$: $H^{enh}_{ch}(CDO_N) = \underbrace{\delta_{triv,*}}$

take $\operatorname{Spec} k \xrightarrow{triv} Bun_N$ and *-push forward.

(This is a variant of what Dennis said:

$H^{enh}_{ch}(N \times N, CDO_N) = \Delta_* e_{Bun_N} \in D(Bun_N \times Bun_N)$).

Unipotence is not important here.

Therefore, LHS $= H_{dR}(\delta_{triv,*}) = k$.

$\underline{\text{RHS}}$: $H_{ch}(BRST(\mathcal{A})) = H_{ch}(\omega_X) = k$.

Actually unipotence was not needed at all here.

3) Just want to say the upshot here.

$\mathcal{A} = \mathcal{A}_{\mathfrak{g},\kappa}$

$\mathrm{BRST}(L_n, \mathcal{A}_{\mathfrak{g},\kappa}, \psi) = \mathcal{W}_{\mathfrak{g},\kappa}$.

One side is $H_{ch}(\mathcal{W}_{\mathfrak{g},\kappa})$.

Other side is: $H^{enh}_{ch}(N, \mathcal{A}_{\mathfrak{g},\kappa})$

$=$!-pullback of $H^{enh}_{ch}(G, \mathcal{A}_{\mathfrak{g},\kappa}) = \mathrm{Diff}_{Bun_G, \kappa}$

along $Bun_N \to Bun_G$.

then tensor w/ $\psi \leftarrow$ character sheaf on Bun_N.

Upshot: LHS of theorem

is $\underbrace{\mathrm{coeff}}_{\text{basic Whittaker coefficient}}(\mathrm{Diff}_{Bun_G,\kappa})$.

Say $\kappa = \text{crit}$.

We're saying

$$\text{coeff}(\text{Diff}_{\text{Bun}_G,\text{crit}}) = \text{Hch}(\mathcal{W}_{\mathfrak{g},\text{crit}})$$
$$= \text{Hch}(\mathcal{W}_{\check{\mathfrak{g}},\text{crit}})$$
$$= \text{Hch}(\text{Fun}(\text{Op}_{\check{\mathfrak{g}}}(D)))$$
$$= \text{Fun}(\text{Op}_{\check{\mathfrak{g}}}(X)).$$

Moving on:

There's a generalization of this involving modules for $\mathcal{A}$.

<u>Reminder</u>: $\mathcal{A}\text{-mod}_x^{\text{fact}}(KL_\nu) \xrightarrow{\text{Hch}^{\text{enh}}} D(\text{Bun}_N)$.

<u>Exercise</u>: figure out the statement.

The following diagram commutes:

Cor:
$$\begin{tikzcd}
KL_{G,k,x} \arrow[r, "DS = \psi\text{-twisted BRST}"] \arrow[d, "H_{ch}^{enh} = Loc"'] & \mathcal{W}_{\check{g},k}\text{-mod}^{fact}_x \arrow[d, "H_{ch}(\mathcal{W}_{\check{g},k}, -)"] \\
D_k(Bun_G) \arrow[r, "coeff"] & Vect
\end{tikzcd}$$

Explicitly (Drinfeld-style):

$\mathbb{V}_k^\lambda \in KL_{G,k,x}$ $\qquad D^{\lambda,x} = loc(\mathbb{V}_k^\lambda) = \underbrace{ind}_{\text{D-module induction}} \big(\underbrace{\mathcal{E}^{\lambda,x}}_{\text{v.b. on } Bun_G \text{ from } Bun_G \xrightarrow{ev_x} BG} \big)$

This theorem computes $coeff(D^{\lambda,x})$ as chiral homology for $\mathcal{W}_{\check{g},k}$.

If $\kappa = \mathrm{crit}$:

$$\begin{tikzcd}
KL_{G,\mathrm{crit}} \arrow[r, "DS"] \arrow[d, "(*)"'] & QCoh(Op_{\check{G}}(\mathring{D})) \arrow[d, "\substack{\text{pullback to } Op_{\check{G}}(X\setminus x) \\ \text{and take global} \\ \text{sections}}"] \\
D(Bun_G) \arrow[r, "coeff"] & Vect
\end{tikzcd}$$

<u>Variant</u>:

$$\begin{tikzcd}
KL_{G,\mathrm{crit}} \arrow[r, "DS^{enh}"] \arrow[d, "Loc"'] & QCoh(Op^{mf}_{\check{G}}(\mathring{D})) \arrow[d, "\substack{\text{pullback to } Op^{mf}_{\check{G}}(X) \\ \& \text{ \sout{then take } \Gamma} \\ \text{pushforward.}}"] \\
D(Bun_G) \arrow[r, "coeff^{enh}"] & QCoh(LocSys_{\check{G}})
\end{tikzcd}$$

$\ni \mathrm{Poinc}^{spec}$ at x

<u>Reminder</u>: $coeff^{enh}$ is uniquely characterized by $\mathrm{Rep}\,\check{G}_{Ran}$-linearity +

$$\Gamma(LS_{\check{G}}, coeff^{enh}(-)) = coeff.$$

DS^{enh} is also characterized ~~xx~~ by $\mathrm{Rep}\,\check{G}_{Ran}$ linearity + "forgetful" condition.

In more detail:

$\mathrm{coeff}^{enh} \circ \mathrm{Loc}$ is characterized by:

- $\mathrm{Rep}\,\check{G}_{Ran}$-linearity
- $\Gamma\, \mathrm{coeff}^{enh} \circ \mathrm{Loc} = \mathrm{coeff} \circ \mathrm{Loc}$.

~~Then~~ Other leg of the diagram is also $\mathrm{Rep}\,\check{G}_{Ran}$-linear, and satisfies the second condition by bare bones thing we discussed (diagram (*)).