

GL OH office hours III

Last time: we discussed chiral homology in various forms.

Today: want to explain how to use these ideas to calculate things in GL.

Need one more ingredient:

Suppose N is a unipotent algebraic gp,

\mathcal{A} is a factorization algebra w/

"Kac-Moody symmetry by N ":

$$L_n := \mathfrak{n} \otimes D_X \xrightarrow{i} \mathcal{A}$$

$$j_* \mathcal{A} \leftarrow \text{fibers at } x \in X$$
$$L^+ N = N(0)$$

$$\Rightarrow \mathcal{A} \in \text{KL}_N = \mathfrak{n} \otimes \mathfrak{u}(\mathfrak{gl}_1) \text{-mod } N(0)$$

Last time:

$$H_{\text{ch}}^{\text{enh}}(\mathcal{A}) \in \mathcal{D}(\text{Bun}_N).$$

Key property: fiber at the trivial bundle is $H_{\text{ch}}(\mathcal{A})$.

Another construction (of local nature):

$\text{BRST}(L_n, \mathcal{A}) \leftarrow$ new factorization algebra w/o any extra symmetries.

Idea of construction:

informally, ~~new~~ ~~factorization~~

given an $n(\mathfrak{h})$ -module \mathcal{M} ,

$$\text{BRST}(n(\mathfrak{h}), \mathcal{M}) \simeq \left(\mathcal{M}^{n(\mathfrak{h})} \right)_{n(\mathfrak{h})/n(\mathfrak{h})}.$$

(Makes sense for abelian n).

Rem: In this unipotent case, BRST is very explicit (cf. my paper on W-algebras Appendix A). More subtle in non-unipotent cases.

$\text{BRST}(L_n, \mathcal{A})$ is rigged to have fibers given by the construction I just said.

Examples: 1) $\mathcal{A} = \mathcal{A}_n := U^{\text{ch}}(L_n)$

("KM chiral algebra for n ")

then $\text{BRST}(L_n, \mathcal{A}) = \text{Fun}(\mathbb{B}^{\text{AN}})$.

i.e., group cohomology of AN .

Roughly, the idea is on fibers, $\mathcal{A}_{n,x} = \mathbb{V}_n$

$\mathbb{V}_n \leftarrow$ has a PBW filt. w/

$$\text{gr. } \mathbb{V}_n = \text{Sym}(\mathfrak{n}^{\text{ch}} / \mathfrak{n}[t])$$

properly: $\mathbb{V}_n = \text{and}_{\mathfrak{n}[t]}^{\mathfrak{n}^{\text{ch}}}(k)$.

$$BRST(V_n) = BRST(\text{ind}_{n\mathbb{Z}}^{n(\mathbb{H})}(k)) = \text{BRST.}$$

↙ same thing about

$$k^{n\mathbb{Z}} = k^{N(\sigma)}$$

↙ by unipotence.

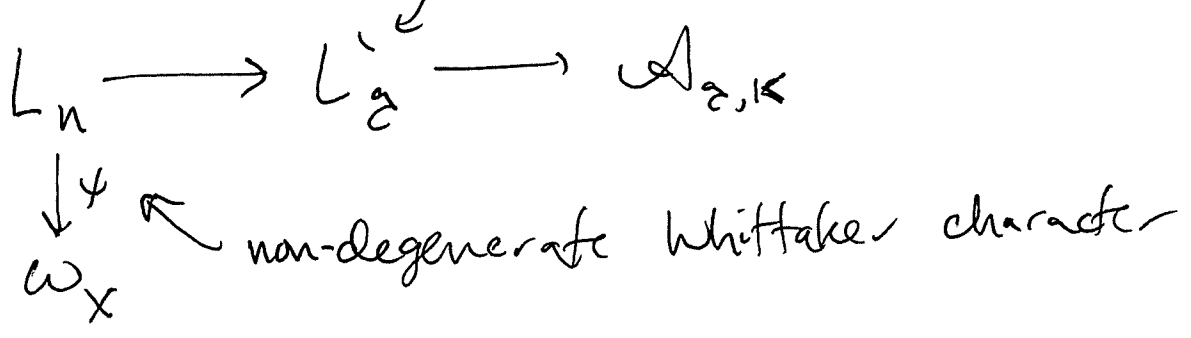
$$2) \mathcal{A} = \text{CDO}_N = \text{ind}_{n\mathbb{Z}}^{n(\mathbb{H})}(\text{Fun}(N(\sigma))).$$

↑ on fibers

$$BRST(\mathcal{A}) = \text{Fun}(N(\sigma))^{n\mathbb{Z}} \stackrel{\text{unipotence}}{=} \text{Fun}(N(\sigma))^{N(\sigma)} = k.$$

3) ~~\mathcal{A}~~ $\mathcal{A} = \mathcal{A}_{\mathfrak{g}, \kappa} \leftarrow$ Kac-Moody chiral algebra for \mathfrak{g}
 ($N = \text{Rad}(\mathfrak{B})$)
 (For $\kappa=0$, $\mathcal{A}_{\mathfrak{g}, 0} = U^{ch}(\mathfrak{L}_{\mathfrak{g}})$).

~~\mathfrak{g}~~ $\mathfrak{g} \supset \mathcal{A}_{\mathfrak{g}, \kappa} \Rightarrow \mathfrak{N}$ acts.



And I'll take $L_n \rightarrow \mathcal{A}_{g,k}$ to be
 the difference of these ($\omega_x \in \mathcal{A}_{g,k}$
 as the unit).

Rem: Properly, there are twists by differential
 forms on X (e.g. $\check{\rho}(\Omega_X)$) on L_g, L_n, \dots .
 I'm ignoring them here.

In this case, $BZST(L_n, \mathcal{A}_{g,k}) =: W_{g,k}$.

"W-algebra" for (g,k) .

Basic facts to know:

- $W_{g,k} \cong W_{g^v,k^v}$. (Fermion-Frenkel duality)
- $W_{g,k} \cong \text{Fun}(\check{\rho}_g(D))$.

Comparison of these notions:

Thm (Cheer-Fu): For \mathcal{A} w/ KM sym. by \mathcal{N} as before, there is a natural iso:

$$H_{\text{dR}}(\text{Bun}_{\mathcal{N}}, H_{\text{ch}}^{\text{enh}}(\mathcal{A})) = H_{\text{ch}}(\text{BRST}(\mathcal{A})).$$

Examples: 1) $\mathcal{A} = \mathcal{A}_n = U^{\text{ch}}(\mathcal{L}_n)(= \mathcal{V}_n)$.

$$H_{\text{ch}}^{\text{enh}}(\mathcal{A}_n) = \text{Diff}_{\text{Bun}_{\mathcal{N}}} \in \mathcal{D}(\text{Bun}_{\mathcal{N}}).$$

↑
true for any gp.

$$\Rightarrow \text{LHS} = H_{\text{dR}}(\text{Diff}_{\text{Bun}_{\mathcal{N}}}) = \text{Fun}(\text{Bun}_{\mathcal{N}}) \\ (= \Gamma(\text{Bun}_{\mathcal{N}}, \mathcal{O})).$$

$$\text{OTOH: } \text{BRST}(\mathcal{A}_n) = \text{Fun}(\text{IB}g_N).$$

Reminder from last time:

$$Y \xrightarrow{\text{affine}} X_{\text{dR}} \quad \text{Fun}(Y) \leftarrow \text{commutative dvr-alg.}$$

$$\text{Hoch}(\text{Fun}(Y)) = \text{Fun}(\text{Sect}_{X_{\text{dR}}}(Y)).$$

Idea: $\text{B}(\text{unipotent gp})$ is not so far from being affine ("coaffine").

So this applies to $\text{IB}g_N$ as well,

$$\begin{aligned} \text{and } \text{Sect}_{X_{\text{dR}}}(\text{IB}g_N) &= \text{Maps}(X, \text{IB}N) \\ &= \text{Bun}_N. \end{aligned}$$

(Unipotency is ^{crucial} ~~key~~ on this second calculation).

2) $\mathcal{A} = \text{CDO}_N$

$\text{BRST}(\mathcal{A}) = \omega_x \text{ (or: } k \text{)}$

LHS: $H_{\text{ch}}^{\text{enh}}(\text{CDO}_N) = \underbrace{\mathcal{F}_{\text{triv},x}}$

take $\text{Spec } k \xrightarrow{\text{triv}} \text{Bun}_N$
and $*$ -push forward.

(This is a variant of what Dennis said:

$$H_{\text{ch}}^{\text{enh}}(N \times N, \text{CDO}_N) = \Delta_{\text{Bun}_N} \in \mathcal{D}(\text{Bun}_N \times \text{Bun}_N)$$

Unipotency is not important here.

Therefore, $\text{LHS} = H_{\text{ch}}(\mathcal{F}_{\text{triv},x}) = k$.

RHS: $H_{\text{ch}}(\text{BRST}(\mathcal{A})) = H_{\text{ch}}(\omega_x) = k$.

Actually unipotency was not needed at all here.

3) Just want to say the upshot here.

$$\mathcal{A} = \mathcal{A}_{2,k}$$

$$\text{BRST}(L_{\mathcal{A}_{2,k}}) = W_{2,k}.$$

one side is $H_{\text{ch}}(W_{2,k})$.

Other side is: $H_{\text{ch}}^{\text{enh}}(N, \mathcal{A}_{2,k})$

\mathbb{G} = !-pullback of $\mathbb{P}_{2,k}$

$$H_{\text{ch}}^{\text{enh}}(\mathbb{G}, \mathcal{A}_{2,k}) = \text{Diff}_{B_{\text{un}}^{\mathbb{G}}, k}$$

along $B_{\text{un}}^{\mathbb{G}} \rightarrow B_{\text{un}}^{\mathbb{G}}$.

then tensor w/ $\psi \in$ character sheaf on $B_{\text{un}}^{\mathbb{G}}$.

Upshot: LHS of theorem

is $\underbrace{\text{coeff}}_{\text{basic Whittaker-coefficient}}(\text{Diff}_{B_{\text{un}}^{\mathbb{G}}, k})$.

basic Whittaker-coefficient.

⊗ Say $K = \text{crit}$.

We're saying

$$\begin{aligned}
\text{coeff}(\text{Diff}_{\text{Bun}_{\mathbb{A}^1, \text{crit}}}) &= \text{Hch}(W_{\mathbb{A}^1, \text{crit}}) \\
&= \text{Hch}(W_{\mathbb{A}^1, \text{crit}}) \\
&= \text{Hch}(\text{Fun}(\text{Op}_{\mathbb{A}^1}(D))) \\
&= \text{Fun}(\text{Op}_{\mathbb{A}^1}(X)).
\end{aligned}$$

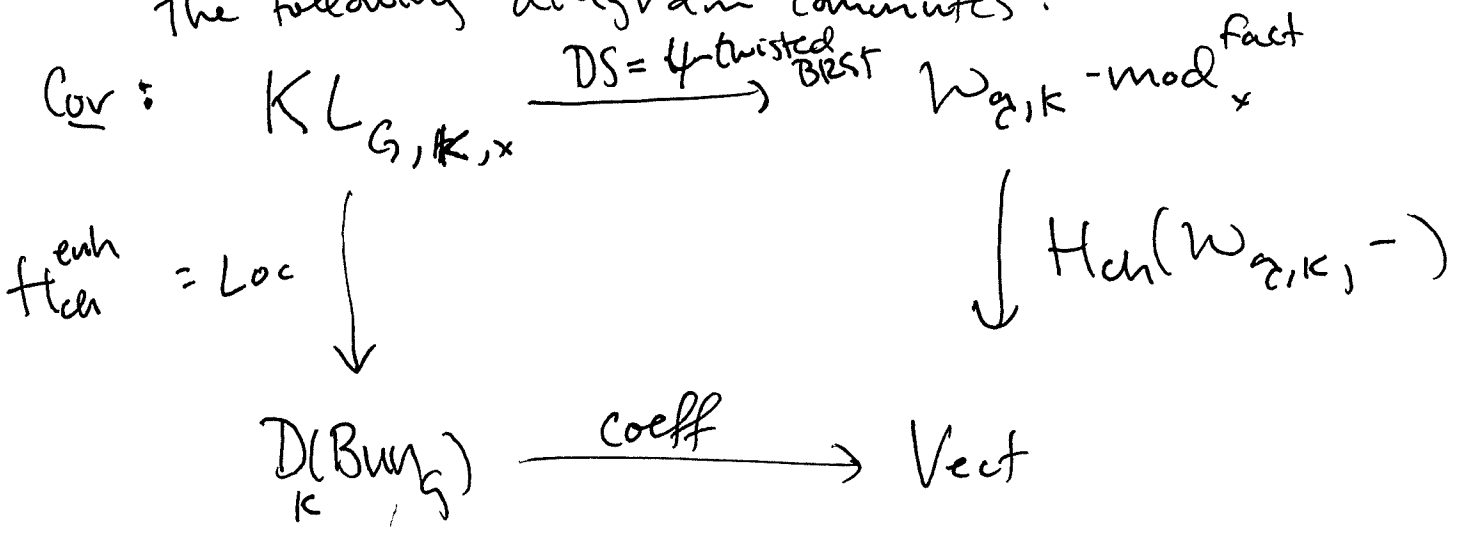
Moving on:

There's a generalization of this involving modules for \mathcal{A} .

Reminder: $\mathcal{A}\text{-mod}_*^{\text{fact}}(KL_i) \xrightarrow{\text{Hch}^{\text{enh}}} D(\text{Bun}_W).$

Exercise: figure out the statement.

The following diagram commutes:



~~Corollary~~

Explicitly (Drinfeld-style):

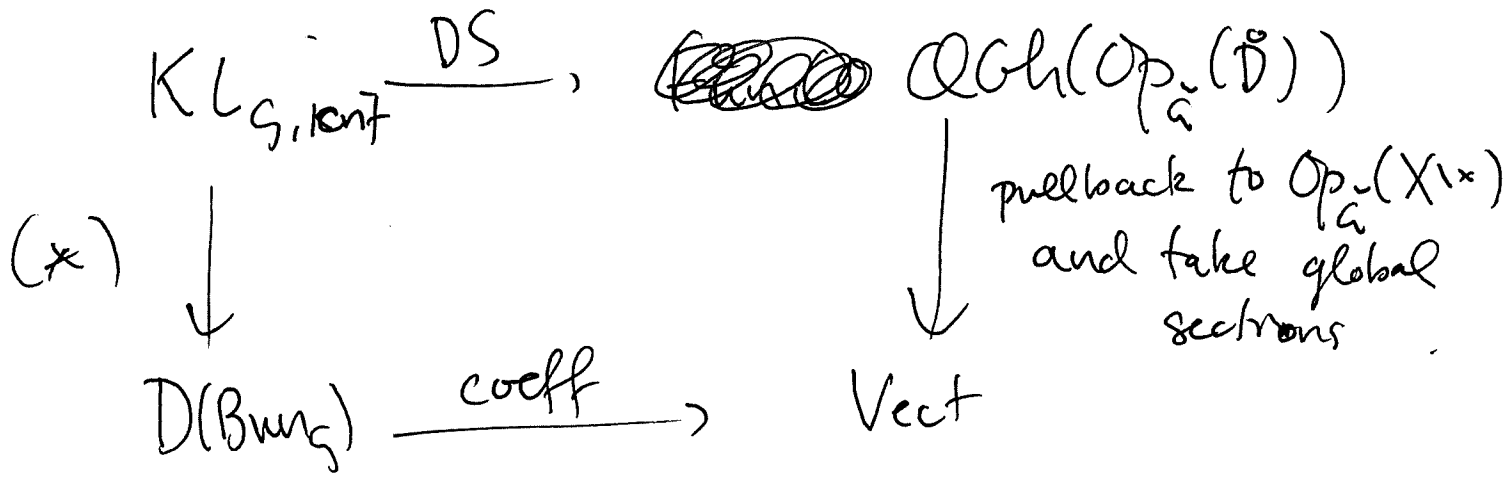
$$W_{\mathbb{K}}^{\lambda} \in KL_{G, \mathbb{K}, x} \quad \text{Loc}(W_{\mathbb{K}}^{\lambda}) = \underbrace{\text{ind}}_{\substack{\text{D-module} \\ \text{induction}}} \left(\underbrace{E^{\lambda, x}}_{\substack{\text{v.b. on} \\ \text{Bun}_g}} \right)$$

from $\text{Bun}_g \xrightarrow{\text{ev}_x} \text{BG}$

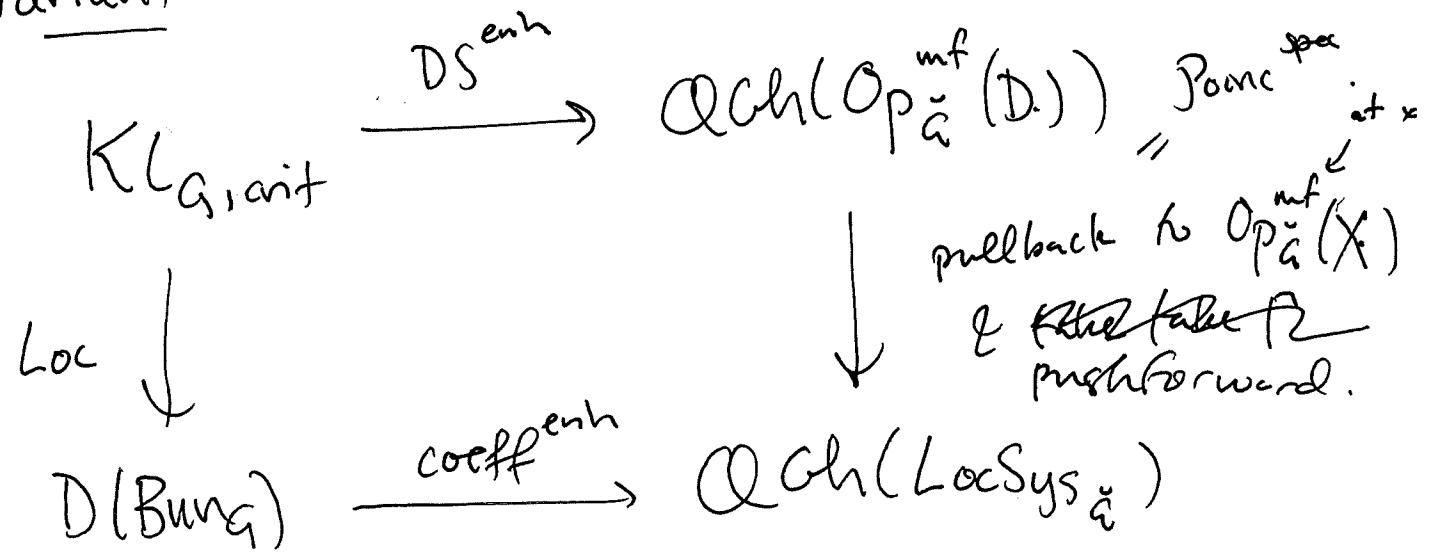
This theorem computes

$\text{coeff}(D^{\lambda, x})$ as chiral homology for $W_{g, \mathbb{K}}$.

If $K = \text{crit}$:



Variant:



Reminder: $\text{coeff}^{\text{enh}}$ is uniquely characterized
 by $\text{Rep}_{\check{a}}^{\text{ran}}$ -linearity +

$$\Gamma(LS_{\check{a}}, \text{coeff}^{\text{enh}}(-)) = \text{coeff}.$$

DS^{enh} is also characterized ~~by~~ by $Rep \tilde{G}_{ran}$ linearity + "forgetful" condition.

In more detail:

$coeff^{enh} \circ Loc$ is characterized by:

- $Rep \tilde{G}_{ran}$ -linearity
- $\Gamma^{coeff^{enh}} \circ Loc = coeff \circ Loc$.

~~The~~ Other leg of the diagram is also $Rep \tilde{G}_{ran}$ -linear, and satisfies the second condition by ~~some~~ some basic thing we discussed (diagram (*)).