GL OH office hours II

Goal: Global aspects of the theory of chiral/factorization algebras.

Last time: black boxed the existence of a theory, and we discussed some local aspects.

Today: Assume X is smooth & projective.

"Bare bones" chiral homology:

Reminder: There's something called

$\mathrm{Ran}_X \leftarrow$ moduli of finite subsets of X.

A factorization algebra $\mathcal{A}$ on X is

a $\mathcal{A}_{\mathrm{Ran}_X} \in D(\mathrm{Ran}_X)$ w/ extra structure.

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Def: $H_\nabla(X, \mathcal{A}) := \underbrace{C_{dR}}(\mathrm{Ran}_X, \mathcal{A}_{\mathrm{Ran}_X})$.

dR cohomology as a chain cplx ($*$-push to a point)

Vect.

Rem: Morally, if X were not proper, you would want compactly supported cohomology, which is problematic for non-holonomic D-modules.

Fact: This yields useful invariants in practice.

Examples: 1) L a Lie-$*$ algebra, I remind $\exists\, U^{ch}(L)$ a factorization alg. associated to L.

$C_{dR}(X, L) \longleftarrow$ tautologically a (dg) Lie algebra

& $H_\nabla(\mathcal{U}^{ch}(L)) \simeq \underbrace{C_\bullet}_{\substack{\text{homological} \\ \text{Chevalley complex.}}}(C_{dR}(L))$

This iso. is also a kind of tautology.

(Ref: Francis - Gaitsgory paper).

Aside: Geometric interpretation, maybe:

At any point $x \in X$, the x-fiber L_x of L is a pro-vector space w/ a Lie algebra structure.

Eg.: $L = L_{\mathfrak{g}} = \mathfrak{g} \otimes D_X$ ("Kac-Moody Lie-* alg.")

fibers are $\mathfrak{g}[[t_x]]$.

$\mathbb{B}\exp(L)$

$\rightsquigarrow$ fml stack over X_{dR} w/ fibers

$\mathbb{B}(\underbrace{\exp}_{\text{fml gp attached to the Lie alg.}}(L_x))$.

In our example, $\mathbb{B}(JG_1^\wedge)$ ~~if you know~~

for JG = jets into G.

There's a natural formal moduli space

$\mathrm{Sect}(\mathbb{B}\exp(L)) \leftarrow$ sections over X_{dR}
AKA "flat sections".

E.g.: The way jets work, $\mathrm{Sect}(JG) = \mathrm{Maps}(X, G)$

& $\mathrm{Sect}(\mathbb{B}JG) = \mathrm{Bun}_G = \mathrm{Maps}(X, \mathbb{B}G)$

& $\mathrm{Sect}(\mathbb{B}JG_1^\wedge)$ = fml cplt. of Bun_G at the trivial bundle.

Then $C_{dR}(L) \longleftrightarrow \mathrm{Sect}(\mathbb{B}G^{\exp(L)})$

$$\Rightarrow H_\nabla(U^{ch}(L)) = \Gamma^{\mathrm{IndCoh}}(\mathrm{Sect}(\mathbb{B}\exp(L)), \omega).$$

dualizing sheaf.

Rem: If I use $\mathfrak{g}\otimes k_X$, similar picture but LocSys replaces Bun.

Rem: Every chiral algebra can be "resolved" by chiral enveloping algebras, so this is useful for calculating H_∇ in general.

Example: Say $\mathcal{A}$ is a commutative connective (can be weakened) factorization algebra. Remind:

$$\mathrm{Spec}_{X_{dR}}(\mathcal{A}) = \underbrace{\mathcal{Y} \longrightarrow}_{\text{affine}} X_{dR}.$$

~~Obs~~ Then $H_\nabla(\mathcal{A})$ is a commutative algebra and:

$$\operatorname{Spec} H_\nabla(\mathcal{A}) = \underbrace{\operatorname{Sect}}(\mathcal{Y}).$$

as before: sections of $\mathcal{Y} \to X_{dR}$.

<u>Idea of proof</u>:

Any commutative algebra can be resolved by "polynomial" algebras. So we're reduced to $\mathcal{A} = \operatorname{Sym}(\mathcal{F})$ $\mathcal{F} \in D(X)$.

Then $\mathcal{A} = U^{ch}(\mathcal{F})$ $\mathcal{F}$ is regarded as an abelian Lie-* algebra and this becomes any easy check from earlier formula.

Example: There's a commutative chiral algebra w/ fibers $\mathrm{Fun}(\mathrm{Op}_{\check{\mathfrak{g}}}(D_x))$.

Its chiral homology $\simeq \mathrm{Fun}(\mathrm{Op}_{\check{\mathfrak{g}}}(X))$.

There are variants on this:

1) $H_\nabla(X, \mathcal{A}; -) : \mathcal{A}\text{-mod}_x^{fact} \longrightarrow \mathrm{Vect}$

so that $H_\nabla(X, \mathcal{A}) = H_\nabla(X, \mathcal{A}; \mathcal{A}_x)$.

Exercise: Guess generalizations of our earlier formulae for chiral homology.

Towards the def'n:

A factorization module $\mathcal{M}$ for $\mathcal{A}$ at $x \in X$ is an object $\mathcal{M}_{\mathrm{Ran}_{X,x}} \in D(\mathrm{Ran}_{X,x})$ where $\mathrm{Ran}_{X,x}$ is "marked Ran space" $= \{$finite subsets of X containing $x\}$.

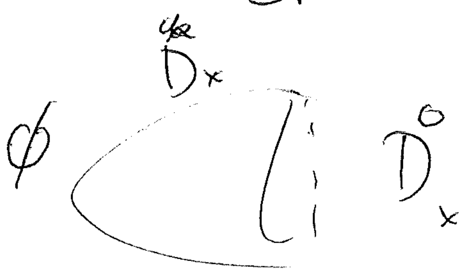
Then $H_\nabla(X, \mathcal{A}; \mathcal{M}) := C_{dR}(\mathrm{Ran}_{X,x}, \mathcal{M}_{\mathrm{Ran}_{X,x}})$.

2) Rem: You can imagine $\underset{\text{fact. algebra}}{\mathcal{A}} \rightsquigarrow$

"3d TFT $Z_{\mathcal{A}}$ defined on 1 & 2-mflds"

where $Z_{\mathcal{A}}(\mathring{D}_x) = \mathcal{A}\text{-mod}^{fact}_x$

$$Z_{\mathcal{A}}(X) = H_\nabla(X, \mathcal{A})$$



$\rightsquigarrow \mathcal{A}_x \in \mathcal{A}\text{-mod}^{fact}_x$

"vacuum rep'n"

$\rightsquigarrow H_\nabla(X, \mathcal{A}; -) : \mathcal{A}\text{-mod}^{fact}_x \longrightarrow \mathrm{Vect}$.

<u>Variants on this:</u>

1) Suppose $\mathcal{A}$ has "Kac-Moody symmetry" by G:

$$\begin{array}{c} JG \curvearrowright \mathcal{A} \\ \uparrow \\ L_{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{D}_X \end{array}$$

fibers of JG are $G(O_x)$ and $L_{\mathfrak{g}} = \mathrm{Lie}(JG)$ in a suitable sense.

HC datum.

(Key example: $\mathcal{A} = U^{ch}(L_{\mathfrak{g}})$).

Then $\exists$ an "enhanced" object

$$H^{enh}_{\nabla}(X, \mathcal{A}) \in D(\mathrm{Bun}_G)$$

whose !-fiber at the trivial bundle is $H_\nabla(X, \mathcal{A})$.

(Detailed ref: Chen-Fu, also great for many related things.)

Example: $\mathcal{A} = U^{ch}(L_{\mathfrak{g},C})$, $H^{enh}_{\nabla}(\mathcal{A}) = \underbrace{D_{Bun_G}}_{\text{sheaf of diff'l operators}}$.

Similarly: $\mathcal{A}\text{-mod}_x^{G(O_x)} \xrightarrow{H^{enh}_{\nabla}} D(Bun_G)$.

2) Suppose $\mathcal{A}$ has "gauge symmetry" by G: $\underbrace{G \times X_{dR}}_{\text{constant D-sch.}}$ acts on $\mathcal{A}$

$\Leftarrow$ $\mathcal{A}$ upgrades to a fact. alg. in $\mathrm{Rep}\,G_{Ran}$.

$\Rightarrow$ $H^{enh}_{\nabla}(\mathcal{A}) \in QCoh(LocSys_G)$

w/ $\Gamma(H^{enh}_{\nabla}(\mathcal{A})) = H^{\cdot}_{\nabla}(\mathcal{A})$.

Example: $\mathrm{Fun}\,\mathrm{Op}_{\check{G}}(D) \leftarrow$ commutative chiral algebra
$\quad\; \mathbin{\rotatebox{90}{=}}$
$\quad\; \mathfrak{z}_G$

has $\mathfrak{z}_{\check{G}}^{\mathrm{enh}} \circledast \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ corresponding

to $\mathrm{Op}_{\check{G}}(D) \longrightarrow \mathrm{LocSys}_{\check{G}}(D) (= B\check{G})$.

Corresponds to the fact that

$\Gamma(\mathrm{Op}(X), \mathcal{O}_{\mathrm{Op}(X)})$ is $\Gamma(\mathrm{LocSys}_{\check{G}}, \pi_* \mathcal{O}_{\mathrm{Op}})$

$\pi: \mathrm{Op}(X) \longrightarrow \mathrm{LocSys}(X)$.

Exercise: Generalize the earlier formulae about chiral homology to this "enhanced" setting.

Similarly, $\mathcal{A}\text{-mod}^{\mathrm{enh}\,\mathrm{fact}}_{*}(\mathrm{Rep}\,\check{G}^{\otimes}) \xrightarrow{H_\nabla^{\mathrm{enh}}} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$.