PROOF OF THE GEOMETRIC LANGLANDS CONJECTURE V: THE MULTIPLICITY ONE THEOREM

DENNIS GAITSGORY AND SAM RASKIN

To V. Drinfeld

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Contents

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Introduction

This paper is a conclusion of the series of papers [GLC1, GLC2, GLC3, GLC4], written jointly with D. Arinkin, D. Beraldo, L. Chen, J. Færgeman, K. Lin, and N. Rozenblyum.

In the preceding papers, we have constructed the Langlands functor

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}), \tag{0.1}$$

and established many of its properties.

In this paper we will prove the geometric Langlands conjecture (GLC) by showing that $\mathbb{L}_G$ is an equivalence.

This theorem confirms the original vision of Beilinson-Drinfeld, which was circulated by them in discussions of their work [BD]. A precise version of the conjecture was originally given in [AG, Conjecture 11.2.3]. The functor $\mathbb{L}_G$ was constructed in [GLC1], and a version of GLC with this precise functor included appears as [GLC1, Conjecture 1.6.7]. As discussed in [GLC1], the functor $\mathbb{L}_G$ is the unique possible equivalence that is compatible with the *Whittaker* and *Eisenstein* compatibilities of [Ga4].

0.1. **What was done in previous papers?** We begin with a short summary of the key points of [GLC1, GLC2, GLC3, GLC4]. This material is discussed in greater detail in Sect. 1.

0.1.1. Recall from [GLC1, Sect. 1.1] that $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ denotes the category of *half-twisted* D-modules on Bun_G. This is the category that appears on the automorphic side of the (de Rham) version of the geometric Langlands conjecture (GLC).

In [GLC1], we used the functor of Whittaker coefficient to construct the *geometric Langlands functor*

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}), \tag{0.2}$$

where the right hand side was defined in [AG]. This functor is linear with respect to the *spectral action* of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ (see [GLC1, Sect. 1.2]). The geometric Langlands conjecture states that $\mathbb{L}_G$ is an equivalence of categories.

0.1.2. We proved in [GLC3, Theorem 1.6.2] (building on [FR]) that the functor $\mathbb{L}_G$ is conservative. We also proved that $\mathbb{L}_G$ admits a left adjoint, to be denoted $\mathbb{L}_G^L$. Moreover, we proved that the composition functor $\mathbb{L}_G$ and its left adjoint $\mathbb{L}_G^L$, which is an endofunctor of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$, is given by tensoring by an object

$$\mathcal{A}_G \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}).$$

This object $\mathcal{A}_G$ is naturally an associative algebra object in $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$.

By Barr-Beck, we obtain an equivalence

$$\begin{tikzcd} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[r, dotted, "\simeq"] \arrow[dr, "\mathbb{L}_G"'] & \mathcal{A}_G\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})) \arrow[d] \\ & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \end{tikzcd}$$

between $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ and the category of $\mathcal{A}_G$-modules in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$.

Therefore, the geometric Langlands conjecture amounts to the assertion that the unit map

$$\mathcal{O}_{\mathrm{LS}_{\check{G}}} \to \mathcal{A}_G, \tag{0.3}$$

is an isomorphism in $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$.

0.1.3. The main theorem of [Lan2] asserts that the restriction of the map (0.3) to the locus of *reducible* local systems is an isomorphism.

Therefore, it remains to show that the map

$$\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}} \to \mathcal{A}_G|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}, \tag{0.4}$$

induced by (0.3), is an isomorphism.

0.1.4. The paper [GLC4] developed some structural features of $\mathcal{A}_{G,\mathrm{irred}}$.

Namely, this work proves that

$$\mathcal{A}_{G,\mathrm{irred}} := \mathcal{A}_G|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}$$

is a *classical* vector bundle[1] that carries a flat connection. Moreover, this connection has finite monodromy.

0.1.5. The above observations mark the starting point of the present paper.

0.2. What is done in this paper?

0.2.1. First, let us observe what needs to be done given the preliminaries above.

Let σ be an *irreducible* $\check{G}$-local system. We need to show that the fiber $\mathcal{A}_{G,\sigma}$ of $\mathcal{A}_G$ at σ is the unit associative algebra.

As $\mathcal{A}_{G,\sigma}$-mod is the category $\mathrm{Hecke}_\sigma(\mathrm{D\text{-}mod}(\mathrm{Bun}_G))$ of Hecke eigensheaves with eigenvalue σ, the above amounts to showing that there is a *unique* (up to tensoring with a vector space) Hecke eigensheaf for each such σ.

By the construction of the Langlands functor $\mathbb{L}_G$, this amounts to two assertions, (i) the existence of a Hecke eigensheaf for an irreducible spectral parameter σ, and (ii) a multiplicity one theorem: we need to show that cuspidal objects of $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ can be (uniquely) reconstructed from their Whittaker coefficients.

[1] Here we are for simplicity assuming that G is semi-simple, i.e., that $Z_G^0 = \{1\}$.

0.2.2. We now outline the argument presented in the paper.

For simplicity, we assume that G is simple (in particular, of adjoint type), that the genus g is at least 2; we also exclude the case when $g = 2$ and $G = PGL_2$. The proof is based on the three observations concerning the topology of $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$, the algebraic geometry of $\mathrm{LS}_{\check{G}}$, and the sheaf theory of Bun_G:

- $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is simply-connected (Theorem 4.3.2);
- $\mathrm{LS}_{\check{G}}$ is Cohen-Macaulay and the complement of $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ has codimension ≥ 2 (Corollary 5.3.3, Proposition 5.3.5);
- Endomorphisms of the vacuum Poincaré sheaf $\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!} \in \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ are just scalars (Theorem 5.2.3).

We remind here that the vacuum Poincaré sheaf appears in characterizing the Langlands functor $\mathbb{L}_G$, cf. [GLC1, Sect. 1.4].

The last of these three results is the simplest to prove. But as we will see below, it is ultimately the point most responsible for addressing the *multiplicity one* problem mentioned in Sect. 0.2.1.

0.2.3. The above three observations combine as follows.

The first observation combined with the features of $\mathcal{A}_{G,\mathrm{irred}}$ mentioned in Sect. 0.1.4 implies that $\mathcal{A}_{G,\mathrm{irred}}$ is isomorphic to the direct sum of several copies of the structure sheaf, i.e.,

$$\mathcal{A}_{G,\mathrm{irred}} \simeq \mathcal{O}^{\oplus n}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}.$$

As (0.4) is a morphism of algebras, it suffices to show that $n = 1$.

0.2.4. The second observation, combined with the fact that (0.3) is an isomorphism on the *reducible* locus, implies that

$$H^0\left(\Gamma(\mathrm{LS}_{\check{G}}, \mathcal{A}_G)\right) \to H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}, \mathcal{A}_{G,\mathrm{irred}})\right)$$

is an isomorphism (see Sect. 5.5.2 for more details).

So, we obtain that

$$\dim\left(H^0\left(\Gamma(\mathrm{LS}_{\check{G}}, \mathcal{A}_G)\right)\right) = \dim\left(H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}, \mathcal{A}_{G,\mathrm{irred}})\right)\right) = = \dim\left(H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}, \mathcal{O}^{\oplus n}_{\mathrm{LS}^{\mathrm{irred}}})\right)\right) \geq n. \tag{0.5}$$

0.2.5. Finally, by the construction of $\mathbb{L}_G$, we have

$$\Gamma(\mathrm{LS}_{\check{G}}, \mathcal{A}_G) \simeq \mathcal{E}nd_{\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)}(\mathrm{Poinc}^{\mathrm{Vac}}_{G,!}).$$

The third observation implies that H^0 of the right-hand side is one-dimensional. Combined with (0.5), this implies that $n = 1$, as desired.

Remark 0.2.6. The outline above uses a special feature of the de Rham setting. By contrast, Betti or étale moduli stacks of local systems have infinite dimensional algebras of global functions, but the de Rham moduli stack has very few global functions. For this reason, our strategy does not adapt to either Betti or étale settings. In particular, the dimension count above carries no meaning in those settings, and our overall strategy for controlling multiplicities does not adapt.

With that said, we remind that it was shown in [GLC1, Theorem 3.5.2] that the de Rham version of GLC proved in this paper implies the Betti version.

0.2.7. *Deficiencies.* An awkward aspect of this paper is that the argument outlined above does not apply in low genus. Namely, the three features mentioned in Sect. 0.2.2 break down when $g \leq 1$ (and the first two also for $g = 2$ when G is of type A_1).

Thus, the proof of GLC we give is *not* uniform across all reductive groups and genera.

0.2.8. We treat the outlying cases as follows (still assuming that G is simple):

First, when $g = 0$, there is nothing to prove, as $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is empty in this case.

Second, we recall from [GLC4, Theorem 1.8.2] that multiplicity one problems can be settled for arbitrary genus and groups of type A_n using features of the geometry of opers, which is much easier to understand in this case.

Thus, it remains to treat the case of $g = 1$ and a simple group of type different than A_n. However, in this case a theorem of [KS, BFM] asserts that $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is again empty.

0.3. **Complications stemming from a non-trivial center.** This subsection can be skipped on the first pass. That said, here we will point to a fun part of this paper: the 2-categorical Fourier-Mukai transform.

0.3.1. In the outline in Sect. 0.2, we have assumed that G is of adjoint type. However, this case is *insufficient* in order to deduce GLC for any reductive group G.

In fact, we prove (see Corollary 2.3.10) that GLC reduces to *almost simple simply-connected* groups.

In this subsection we indicate how the outline in Sect. 0.2 needs to me modified in order to treat this case.

0.3.2. Assume that G is semi-simple. Then if $\check{G}$ is not simply-connected, $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is not connected, nor are its connected components simply-connected. In fact

$$\pi_0(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \simeq \pi_0(\mathrm{Bun}_{\check{G}}) \simeq (Z_G)^\vee.$$

We denote this bijection by

$$(\alpha \in (Z_G)^\vee) \rightsquigarrow \mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha},$$

where $\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}$ is a connected component of $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$.

Moreover, each connected component of $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ has abelian fundamental group, and its characters are in bijection with Bun_{Z_G}. We denote this bijection by

$$(\mathcal{P}_{Z_G} \in \mathrm{Bun}_{Z_G}) \rightsquigarrow \mathcal{L}_{\mathcal{P}_{Z_G}},$$

where $\mathcal{L}_{\mathcal{P}_{Z_G}}$ is a 1-dimensional local system.

0.3.3. With this understood, it is not difficult to adapt the proof explained in Sect. 0.2, modulo the following issue:

We need to know that:

- The functor $\mathbb{L}_G$ sends the direct summand of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ on which the action of Z_G by automorphisms of the identity functor is given by α to the direct summand of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ consisting of sheaves supported on $\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}$.
- The functor $\mathbb{L}_G$ intertwines the automorphism of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ given by translation by $\mathcal{P}_{Z_G}$ with the automorphism of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ given by tensor product with $\mathcal{L}_{\mathcal{P}_{Z_G}}$.

0.3.4. The above two properties can be formulated purely on the geometric side, in terms of the spectral action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$, see Theorems 5.1.5 and 5.1.7, respectively.

The proofs of these two theorems involve a fun manipulation (that appears to be new) with the *2-categorical Fourier-Mukai transform.*

0.3.5. The 2-categorical Fourier-Mukai transform in the case at hand is an equivalence of 2-categories between *sheaves of categories* over the 2-stacks $\mathrm{Ge}_{Z_G}(X)$ and $\mathrm{Ge}_{\pi_1(\check{G})}(X)$, which classify gerbes on X with respect to Z_G and $\pi_1(\check{G})$, respectively.

In turns out that one can upgrade $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ to a sheaf of categories over $\mathrm{Ge}_{Z_G}(X)$ and over $\mathrm{Ge}_{\pi_1(\check{G})}(X)$, and the assertion is that the resulting two sheaves of categories map to one another under the 2-categorical Fourier-Mukai transform. Per Remark 8.1.5, the prestacks $\mathrm{Ge}_{Z_G}(X)$ and $\mathrm{Ge}_{\pi_1(\check{G})}(X)$ are *not* 1-affine, so it is essential to work with sheaves of categories here.

The above assertion implies the two properties mentioned in Sect. 0.3.3. But in fact it carries much more information. For example, it contains the answer to the following question:

What the spectral side of GLC, when on the automorphic side instead of the usual Bun_G, we consider the stack of bundles with respect to a non-pure inner form of G (i.e., a twist by a G_{ad}-torsor, which does not come from a G-torsor)?

It turns out that the answer is the twist of (the usual) $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ by a gerbe on $\mathrm{LS}_{\check{G}}$ that is attached to the G_{ad}-torsor in question.

0.4. What is not done in this paper?

0.4.1. First, there are a number of nearby problems that we do not consider and do not know how to solve. Here are some:

- Geometric Langlands with Iwahori ramification.
- Quantum geometric Langlands.
- Local geometric Langlands with wild ramification.
- Global geometric Langlands with wild ramification.
- Restricted geometric Langlands for ℓ-adic sheaves (for curves in positive characteristic).
- Geometric Langlands for Fargues-Fontaine curves.

0.4.2. Moreover, *even in the (global, untwisted, unramified, de Rham) setting of the present paper*, we feel there is more to understand. The tricks indicated in Sect. 0.2 analyze automorphic sheaves via their shadows (specifically, the core part of the argument uses a dimension count). In the remainder of this subsection, we indicate questions and projects that might let us observe them more directly.

0.4.3. As in Sect. 0.2.1, we need to be able to recover cuspidal D-modules – or equivalently: eigensheaves with irreducible spectral parameters – from their Whittaker coefficients. This can be stated more formally either as showing that the composition

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \xrightarrow{\mathrm{coeff}_G^{\mathrm{Vac,glob}}} \mathrm{Whit}(\mathrm{Gr}_{G,\mathrm{Ran}}) \tag{0.6}$$

is fully faithful, or showing that the functor of vacuum Whittaker coefficient

$$\mathrm{Hecke}_\sigma\left(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)\right) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \xrightarrow{\mathrm{coeff}_G^{\mathrm{Vac,glob}}} \mathrm{Vect}$$

is an equivalence for any irreducible local system σ.

For $G = GL_n$ (or a quotient thereof), the *mirabolic trick* suffices to prove this result; see [Be1] for a strong version of this assertion. We remind that a similar assertion holds at the level of automorphic functions, although in the arithmetic setup, such assertions are *not* valid for more general reductive groups.

A posteriori, the same statement for general G follows from the results of the present paper. But one can imagine tackling this statement more directly.

Below we record some possible strategies that we know have been discussed previously in the geometric Langlands community. We allow ourselves more informality in this discussion than elsewhere in this text.

0.4.4. *Proof via contractibility of opers?* Recall that $\mathcal{A}_G \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ was defined using sheaves on Bun_G, i.e., it crosses the barrier between G and $\check{G}$ – as should be no surprise.

However, one key point of [GLC4] is that there is an alternative construction of $\mathcal{A}_{G,\mathrm{irred}}$ purely in $\check{G}$ (i.e., spectral) terms: this sheaf is the relative homology of the space of rational opers (with irreducible underlying local system). Equivalently, for irreducible σ, the fiber $\mathcal{A}_{G,\sigma}$ can be identified with homology of the space of generic oper structures on σ.

Therefore, as recorded in [GLC4, Conjecture 4.5.7], it suffices to show that this space of generic oper structures is *contractible*. Note that one piece of this assertion – namely, that this space is non-empty – is a theorem of D. Arinkin, [Ari].

The contractibility assertion is easy for $G = GL_n$, and has recently been established in [BKS] for all classical groups. So this provides a uniform proof of GLC for classical groups, see [GLC4, Theorem 4.5.11].

The contractibility assertion for general G follows a posteriori from the results of this paper. But conceivably, one could find an a priori proof of this assertion for a general G that would yield a more satisfying conclusion to the geometric Langlands conjecture.

We remark that in [GLC4, Corollary 4.5.5], we showed that the space of generic oper structures on σ has vanishing homology in positive degrees, so it only remains to show that it is *connected*. In fact, as $\mathcal{A}_{G,\mathrm{irred}}$ is a vector bundle, it suffices to check this for a single irreducible local system σ (per connected component of $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$).

0.4.5. *Proof via microlocal sheaf theory?* To date, the most successful geometric approach to studying Whittaker coefficients of automorphic sheaves for general reductive groups is [FR], where it was shown that the functor (0.6) is *conservative*. One could try extending the techniques of [FR] to obtain a geometric proof of fully faithfulness. Here we briefly fantasize about one possible form this idea might take.

First, [FR] works primarily in the setting of sheaves *with nilpotent singular support* as in [AGKRRV1]. One key point of [FR] is the picture that the vacuum Whittaker coefficient is the microstalk at the basepoint of the global nilpotent cone Nilp, a picture that was then realized in a better and more precise form in [NT].

On the one hand, by [Wa], there is a category $\mu\,\mathrm{Perv}(T^*(\mathrm{Bun}_G))$ of *microsheaves supported on* $T^*(\mathrm{Bun}_G)$ and a subcategory $\mu\,\mathrm{Perv}_{\mathrm{Nilp}}(T^*(\mathrm{Bun}_G))$ of microsheaves *supported on the nilpotent cone.* Moreover, there is a fully faithful *microlocalization* functor $\mathrm{Perv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mu\,\mathrm{Perv}_{\mathrm{Nilp}}(T^*(\mathrm{Bun}_G))$.

For $\overset{\circ}{\mathrm{Nilp}} \subset \mathrm{Nilp}$ the open of *generically regular* nilpotent Higgs fields, [FR, Theorem G] morally says that the composition

$$\mathrm{Perv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \to \mathrm{Perv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mu\,\mathrm{Perv}_{\mathrm{Nilp}}(T^*(\mathrm{Bun}_G)) \to \mu\,\mathrm{Perv}_{\overset{\circ}{\mathrm{Nilp}}}(T^*(\mathrm{Bun}_G))$$

remains fully faithful.

On the other hand, one can dream of a strengthening of [NT] that expresses *all* Whittaker coefficients from the microlocalization to $\mu\,\mathrm{Perv}_{\overset{\circ}{\mathrm{Nilp}}}(T^*(\mathrm{Bun}_G))$, with this microlocal functor being fully faithful for some natural geometric reasons.

This would also yield another resolution to the geometric Langlands conjecture, and one that might teach us more than the present paper does.

0.4.6. *Proof via Verlinde gluing?* By [GLC1, Theorem 3.5.6], we can deduce the de Rham geometric Langlands conjecture from its Betti counterpart [BZN].

As in [BZN] Sect. 4.6, a better understanding of the automorphic side of Betti Langlands could allow one to glue $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ via degeneration to a nodal curve, using Bezrukavnikov-style equivalences as local input. As we understand, there is active and on-going work of D. Nadler and Z. Yun in this direction.

0.4.7. *Proof via arithmetic?* Suppose for simplicity that G is adjoint and the genus is at least 2, so that $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is connected.

Recall that $\mathcal{A}_G|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}$ is a vector bundle. As mentioned in Sect. 0.4.4, it follows that $\mathcal{A}_G|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}$ has one-dimensional fibers if it has a one-dimensional fiber at *any* point. This in turn should reduce by specialization to a statement in characteristic p. Using the intimate relationship between geometric and arithmetic Langlands established in [AGKRRV2], one should be able to reduce de Rham geometric Langlands to a suitable multiplicity one statement for essentially *any* class of unramified cusp forms (equipped with the action of V. Lafforgue's excursion operators).

With that said, we are not aware of any such arithmetic results for general reductive groups.

0.4.8. *Proof via... something else?* We do not intend to represent the above as an exhaustive summary of discussions that have occurred. More seriously, we anticipate future innovations whose form we do not yet know.

0.5. **Contents.**

0.5.1. Let us briefly review the actual contents of this paper.

In Sect. 1 we summarize the results of [GLC1, GLC2, GLC3, GLC4] that will be used in this paper.

In Sect. 2 we show that it is sufficient to prove GLC in the special case when the group G is almost simple and simply-connected.

In Sect. 3 we prove GLC when the genus g of our curve is either 0 or 1.

In Sect. 4 we express the fundamental groupoid of $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ in terms of Z_G. (In particular, we show that if G is simple, then $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is connected and simply-connected, under the assumption that $g \geq 2$, and excluding the case $g = 2$ and $G = PGL_2$.)

In Sect. 5 we prove GLC for curves of genus ≥ 2, essentially elaborating on the outline in Sect. 0.2. Here we also state some further results whose proofs are given in later sections.

In Sect. 6 we calculate (in the case $g \geq 2$) endomorphisms of the vacuum Poincaré object, and show that they consist essentially only of scalars.

In Sect. 7 we prove the properties concerning algebraic geometry and topology of the stacks $\mathrm{Bun}_{\check{G}}$ and $\mathrm{LS}_{\check{G}}$.

In Sect. 8 we introduce the 2-categorical Fourier-Mukai transform and use it establish the compatibility between the action of Bun_{Z_G} on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ and the spectral action. This material is used to adapt the discussion of Sect. 0.2 to the case when G is not of adjoint type.

0.5.2. *Conventions and notation.* This paper follows the conventions and notation of the [GLC1, GLC2, GLC3, GLC4] series.

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1. Summary of the preceding results

In this section we summarize the results from [GLC1, GLC2, GLC3, GLC4] that will be used in the present paper for the proof of GLC.

1.1. **The Langlands functor and its categorical properties.**

1.1.1. In [GLC1, Sect. 1], we have constructed a functor

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}).$$

It satisfies the following properties:

- The functor $\mathbb{L}_G$ admits a left adjoint $\mathbb{L}_G^L$, see [Lan2, Theorem 23.1.2];
- The functor $\mathbb{L}_G$ is conservative, see [GLC4, Theorem 1.6.2].

1.1.2. The geometric Langlands conjecture (GLC) says:

Conjecture 1.1.3. *The functor $\mathbb{L}_G$ is an equivalence.*

1.1.4. By the above, Conjecture 1.1.3 is equivalent to the fact that the unit of the adjunction

$$\text{Id}_{\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})} \to \mathbb{L}_G \circ \mathbb{L}_G^L \tag{1.1}$$

is an isomorphism.

1.2. **Spectral properties.**

1.2.1. Recall that the Hecke action gives rise to an action of the monoidal category $\text{QCoh}(\text{LS}_{\check{G}})$ on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, see [GLC1, Sect. 1.2].

The functor $\mathbb{L}_G$ has the following features with respect to this action:

- The functor $\mathbb{L}_G$ is $\text{QCoh}(\text{LS}_{\check{G}})$-linear, i.e., it intertwines the actions of $\text{QCoh}(\text{LS}_{\check{G}})$ on the two sides, see [GLC1, Sect. 1.7]. As $\text{QCoh}(\text{LS}_{\check{G}})$ is rigid monoidal, this implies that the functor $\mathbb{L}_G^L$ is also $\text{QCoh}(\text{LS}_{\check{G}})$-linear;
- The $\text{QCoh}(\text{LS}_{\check{G}})$-linear monad $\mathbb{L}_G \circ \mathbb{L}_G^L$ on $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ is given by a (uniquely defined) associative algebra object

$$\mathcal{A}_G \in \text{QCoh}(\text{LS}_{\check{G}}),$$

  see [Lan2, Theorem 23.6.2].

1.2.2. The unit map

$$\mathcal{O}_{\text{LS}_{\check{G}}} \to \mathcal{A}_G \tag{1.2}$$

for the associative algebra $\mathcal{A}_G$ in $\text{QCoh}(\text{LS}_{\check{G}})$ gives rise to the map (1.1). Therefore, to see that (1.1) is an isomorphism of functors, it suffices to show that (1.2) is an isomorphism of quasi-coherent sheaves.

1.3. **Restriction to the irreducible locus.**

1.3.1. Let
$$\mathrm{LS}^{\mathrm{red}}_{\check{G}} \subset \mathrm{LS}_{\check{G}}$$
be the closed substack consisting of *reducible* local systems, i.e., the union of the images of the (proper) maps
$$\mathrm{LS}_{\check{P}} \to \mathrm{LS}_{\check{G}},$$
where $\check{P} \subsetneq \check{G}$ are proper parabolics of $\check{G}$.

Let
$$\mathrm{LS}^{\mathrm{irred}}_{\check{G}} \overset{\jmath}{\hookrightarrow} \mathrm{LS}_{\check{G}}$$
denote the embedding of the complement to $\mathrm{LS}^{\mathrm{red}}_{\check{G}}$, i.e., $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is the open substack of *irreducible* local systems.

1.3.2. Let
$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{red}} \subset \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \tag{1.3}$$
be the full subcategory consisting of objects set-theoretically supported on $\mathrm{LS}^{\mathrm{red}}_{\check{G}}$, i.e.,
$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{red}} = \ker\left(\jmath^* : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})\right).$$

The above faithful embedding (1.3) admits a right adjoint, denoted $\widehat{\iota}^!$. Explicitly,
$$\widehat{\iota}^! \simeq \mathrm{Fib}(\mathrm{Id} \to \jmath_* \circ \jmath^*(-)).$$

1.3.3. The following is the main result of [Lan2] (it is equivalent to Theorem 24.1.2 in *loc. cit.*, see Sect. 24.3):

- The map $\widehat{\iota}^!(\mathcal{O}_{\mathrm{LS}_{\check{G}}}) \to \widehat{\iota}^!(\mathcal{A}_G)$ induced by (1.2) is an isomorphism.

1.3.4. Set
$$\mathcal{A}_{G,\mathrm{irred}} := \jmath^*(\mathcal{A}_G).$$

Given the isomorphism in Sect. 1.3.3, we obtain that the fact that (1.2) is an isomorphism (and hence, the statement of GLC) is equivalent to the fact that the map
$$\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}} \to \mathcal{A}_{G,\mathrm{irred}}, \tag{1.4}$$
induced by (1.2) is an isomorphism.

1.4. Properties of $\mathcal{A}_{G,\mathrm{irred}}$.

1.4.1. In this subsection we will assume that G is semi-simple. In this case $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is a classical smooth algebraic stack.

1.4.2. The following property of $\mathcal{A}_{G,\mathrm{irred}}$ is one of the two main results of the paper [GLC4], see Theorem 3.1.8 in *loc. cit.*:

- The object $\mathcal{A}_{G,\mathrm{irred}} \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$ is a vector bundle (in particular, it is concentrated in cohomological degree 0);

1.4.3. In addition, we have:

- The object $\mathcal{A}_{G,\mathrm{irred}}$ is canonically of the form $\mathbf{oblv}^l(\mathcal{F})$, where $\mathcal{F} \in \mathrm{D\text{-}mod}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$ and
$$\mathbf{oblv}^l : \mathrm{D\text{-}mod}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$$
denotes the "left" forgetful functor. This is [GLC4, Corollary 4.2.5].
- The above object $\mathcal{F}$ is a local system with a finite monodromy (in particular, it is concentrated in cohomological degree 0). This is [GLC4, Proposition 4.2.8].

2. Reduction to the case when G is almost simple and simply-connected

For a number of technical reasons,[2] in the main argument in the proof of GLC, we will need to assume that the group G is almost simple and simply-connected. In this section we will perform a reduction to this case.

2.1. **Compatibility between Langlands functors.**

2.1.1. Let G_1 and G_2 be a pair of reductive groups, and let

$$\phi : G_1 \to G_2$$

be an *almost isogeny*, i.e., a map that induces an isogeny of their derived groups.

By a slight abuse of notation we will denote by the same symbol ϕ the induced map

$$\mathrm{Bun}_{G_1} \to \mathrm{Bun}_{G_2}.$$

2.1.2. The map ϕ induces a map

$$\phi^\vee : \check{G}_2 \to \check{G}_1.$$

By a slight abuse of notation, we will denote by the same symbol $\phi^\vee$ the induced map

$$\mathrm{LS}_{\check{G}_2} \to \mathrm{LS}_{\check{G}_1}.$$

2.1.3. Note that the functor

$$\phi^! : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) \tag{2.1}$$

is linear with respect to the action of $\mathrm{Rep}(\check{G}_1)_{\mathrm{Ran}}$, where the action on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2})$ is via the functor

$$\mathrm{Rep}(\check{G}_1)_{\mathrm{Ran}} \to \mathrm{Rep}(\check{G}_2)_{\mathrm{Ran}},$$

given by restriction along $\phi^\vee$.

Hence, (2.1) is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}_1})$-linear, where the action on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2})$ is via

$$(\phi^\vee)^* : \mathrm{QCoh}(\mathrm{LS}_{\check{G}_1}) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}_2}).$$

2.1.4. Note also that the functor

$$(\phi^\vee)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}_2}) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}_1})$$

sends

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}_2}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}_1}).$$

This follows, e.g., from [AG, Proposition 7.1.3(b)].

2.1.5. We will prove:

Proposition 2.1.6. *The following diagram of functors commutes*

$$\begin{tikzcd} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) \arrow[r, "\mathbb{L}_{G_1}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}_1}) \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2}) \arrow[u, "\phi^!"] \arrow[r, "\mathbb{L}_{G_2}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}_2}). \arrow[u, "(\phi^\vee)_*^{\mathrm{IndCoh}}"'] \end{tikzcd} \tag{2.2}$$

Moreover, this datum of commutativity is compatible with the action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}_1})$.

2.2. **Proof of Proposition 2.1.6.**

[2]One reason is not very serious: if G has a center of positive dimension, the stack $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$ is not smooth, which is a silly annoyance. The more serious reason is that when $g = 2$ we will need to assume that G has no factors of A_1 in its Dynkin diagram, see the preamble to Sect. 5.5.

2.2.1. First, we can factor ϕ as

$$G_1 \overset{\phi'}{\to} G_1 \times Z^0_{G_2} \overset{\phi''}{\to} G_2.$$

The proposition evidently holds for ϕ'. Hence, it is sufficient to prove it for ϕ''. Thus, we can assume that our ϕ is surjective.

2.2.2. When ϕ is surjective, we factor it as

$$G_1 \overset{\phi'}{\to} G_1/\ker(\phi)^0 \overset{\phi''}{\to} G_2,$$

so that the kernel of ϕ' is a connected torus, and ϕ'' is an actual isogeny.

The case of a surjection with kernel being a connected torus will be analyzed explicitly in Sect. 2.4.2.

Hence, it is enough to prove Proposition 2.1.6 for ϕ being an isogeny (this assumption will be used in one place in the course of the proof, namely, in Sect. 2.2.4).

2.2.3. First, we claim that it is sufficient show that the outer square in the following diagram commutes:

$$\begin{tikzcd}
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1}) \arrow[r, "\mathbb{L}_{G_1}"] & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_1}) \arrow[r] & \text{QCoh}(\text{LS}_{\check{G}_1}) \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}) \arrow[r, "\mathbb{L}_{G_2}"] \arrow[u, "\phi^!"] & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_2}) \arrow[r] \arrow[u, "(\phi^\vee)_*"] & \text{QCoh}(\text{LS}_{\check{G}_2}), \arrow[u, "(\phi^\vee)_*"]
\end{tikzcd} \tag{2.3}$$

where the horizontal composite functors are

$$\mathbb{L}_{G_1,\text{coarse}} \text{ and } \mathbb{L}_{G_2,\text{coarse}},$$

respectively (see [GLC1, Sect. 1.4]).

Indeed, suppose that (2.3) commutes, and let us show that this uniquely upgrades to the commutation of (2.2).

2.2.4. It is enough to show that the two circuits in (2.2) are isomorphic when restricted to the subcategory of compact objects in $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2})$. Since the functor

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_1}) \to \text{QCoh}(\text{LS}_{\check{G}_1})$$

is fully faithful on

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_1})^{>-\infty} \subset \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_1}),$$

it suffices to show that both circuits in (2.2), when restricted to

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2})^c \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2})$$

map to $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_1})^{>-\infty}$.

By [GLC1, Theorem 1.6.2], the functors $\mathbb{L}_{G_i,\text{coarse}}$ send

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_i})^c \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_i})^{>-\infty},$$

$i = 1, 2$.

The statement about the counterclockwise circuit follows now because the functor $(\phi^\vee)_*$ has finite cohomological amplitude.

The statement for the clockwise circuit follows similarly: indeed since $\phi : G_1 \to G_2$ is an isogeny, the map

$$\phi : \text{Bun}_{G_1} \to \text{Bun}_{G_2}$$

is étale, and hence the functor

$$\phi^! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2})$$

preserves compactness.

2.2.5. The same argument implies the following:

Once we prove that the outer circuit in (2.3) commutes as a diagram of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}_1})$-module categories, it would follow that its restriction to compact objects commutes as a diagram of $\mathrm{Perf}(\mathrm{LS}_{\check{G}_1})$-module categories, so the resulting left commuting square in (2.3) is one of $\mathrm{Perf}(\mathrm{LS}_{\check{G}_1})$-module categories, and hence of $\mathrm{Ind}(\mathrm{Perf}(\mathrm{LS}_{\check{G}_1})) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}_1})$-module categories.

2.2.6. Let us now prove the commutativity of

$$\begin{tikzcd} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) \arrow[r, "\mathbb{L}_{G_1,\mathrm{coarse}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}_1}) \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2}) \arrow[u, "\phi^!"] \arrow[r, "\mathbb{L}_{G_2,\mathrm{coarse}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}_2}). \arrow[u, "(\phi^\vee)_*"'] \end{tikzcd} \tag{2.4}$$

It will follow from the construction that the data of commutativity is compatible with the Hecke action of $\mathrm{Rep}(\check{G}_1)_{\mathrm{Ran}}$, and hence also of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}_1})$.

2.2.7. Since the functor

$$\Gamma^{\mathrm{spec}}_{\check{G}_1} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}_1}) \to \mathrm{Rep}(\check{G}_1)_{\mathrm{Ran}}$$

is fully faithful, it suffices to show that the outer square in the concatenation of (2.4) with the commutative diagram

$$\begin{tikzcd} \mathrm{QCoh}(\mathrm{LS}_{\check{G}_1}) \arrow[r] & \mathrm{Rep}(\check{G}_1)_{\mathrm{Ran}} \\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}_2}) \arrow[u, "(\phi^\vee)_*"] \arrow[r] & \mathrm{Rep}(\check{G}_2)_{\mathrm{Ran}} \arrow[u, "\mathrm{coInd}^{\phi^\vee}"'] \end{tikzcd} \tag{2.5}$$

commutes, where

$$\mathrm{coInd}^{\phi^\vee} : \mathrm{Rep}(\check{G}_2) \to \mathrm{Rep}(\check{G}_1)$$

is the functor of co-induction, i.e., the right adjoint to the functor

$$\mathrm{Res}^{\phi^\vee} : \mathrm{Rep}(\check{G}_1) \to \mathrm{Rep}(\check{G}_2).$$

2.2.8. Note that pullback with respect to the map

$$\phi : \mathrm{Gr}_{G_1} \to \mathrm{Gr}_{G_2}$$

induces a functor

$$\phi^! : \mathrm{Whit}(G_2) \to \mathrm{Whit}(G_2).$$

It is easy to see that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Whit}(G_1) \arrow[r, "\mathrm{CS}_{G_1}"] & \mathrm{Rep}(\check{G}_1) \\ \mathrm{Whit}(G_2) \arrow[u, "\phi^!"] \arrow[r, "\mathrm{CS}_{G_2}"] & \mathrm{Rep}(\check{G}_2). \arrow[u, "\mathrm{coInd}^{\phi^\vee}"'] \end{tikzcd}$$

Combining this with the commutative diagrams [Lan2, Equation (20.10)]

$$\begin{tikzcd} \mathrm{Whit}(G_i)_{\mathrm{Ran}} \arrow[r] & \mathrm{Rep}(\check{G}_i)_{\mathrm{Ran}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_i}) \arrow[u, "{\mathrm{coeff}_{G_i}[2\delta_{N_{\rho(\omega_X)}}]}"] \arrow[r, "\mathbb{L}_{G_i,\mathrm{course}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}_i}) \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G}_i}"'] \end{tikzcd}$$

for $i=1,2$, we obtain that the required commutativity is equivalent to the commutativity of

$$\begin{tikzcd} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1}) \arrow[r, "\text{coeff}_{G_1}"] & \text{Whit}(G_1)_{\text{Ran}} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}) \arrow[r, "\text{coeff}_{G_2}"] \arrow[u, "\phi^!"] & \text{Whit}(G_2)_{\text{Ran}}. \arrow[u, "\phi^!"'] \end{tikzcd} \tag{2.6}$$

2.2.9. Passing to left adjoints, the commutativity of (2.6) is equivalent to the commutativity of

$$\begin{tikzcd} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1}) \arrow[d, "\phi_!"'] & \text{Whit}(G_1)_{\text{Ran}} \arrow[l, "\text{Poinc}_{G_1,!}"'] \arrow[d, "\phi_!"] \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}) & \text{Whit}(G_2)_{\text{Ran}}, \arrow[l, "\text{Poinc}_{G_2,!}"'] \end{tikzcd}$$

while the latter is tautological.

□[Proposition 2.1.6]

2.3. **Changing the group.**

2.3.1. Passing to left adjoint functors in (2.2), we obtain a commutative diagram

$$\begin{tikzcd} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1}) \arrow[d, "\phi_!"'] & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_1}) \arrow[l, "\mathbb{L}^L_{\check{G}_1}"'] \arrow[d, "(\phi^\vee)^*"] \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}) & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_2}), \arrow[l, "\mathbb{L}^L_{\check{G}_2}"'] \end{tikzcd}$$

compatible with the action of $\text{QCoh}(\text{LS}_{\check{G}_1})$. Tensoring up, we obtain a commutative diagram

$$\begin{tikzcd} \text{QCoh}(\text{LS}_{\check{G}_2}) \underset{\text{QCoh}(\text{LS}_{\check{G}_1})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1}) \arrow[d] & \text{QCoh}(\text{LS}_{\check{G}_2}) \underset{\text{QCoh}(\text{LS}_{\check{G}_1})}{\otimes} \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_1}) \arrow[l, "\mathbb{L}^L_{\check{G}_1}"'] \arrow[d] \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}) & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_2}). \arrow[l, "\mathbb{L}^L_{\check{G}_2}"'] \end{tikzcd} \tag{2.7}$$

2.3.2. We claim:

Proposition 2.3.3. *The functor*

$$\text{QCoh}(\text{LS}_{\check{G}_2}) \underset{\text{QCoh}(\text{LS}_{\check{G}_1})}{\otimes} \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_1}) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}_2})$$

is an equivalence.

Indeed, this is particular case (extended to stacks in a straightforward way) of [AG, Corollary 7.6.2]. In Sect. 2.4, we will prove:

Theorem 2.3.4. *The functor*

$$\text{QCoh}(\text{LS}_{\check{G}_2}) \underset{\text{QCoh}(\text{LS}_{\check{G}_1})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}) \tag{2.8}$$

is an equivalence.

2.3.5. Combining Proposition 2.3.3 and Theorem 2.3.4, we obtain:

Corollary 2.3.6. *Suppose that the functor $\mathbb{L}_{G_1}$ is an equivalence. Then so is $\mathbb{L}_{G_2}$.*

2.3.7. For a given reductive group G, take $G_2 = G$, and let $G_1 := G_{\mathrm{sc}}$ be the simply-connected cover of its derived group. Let ϕ be the canonical map

$$G_{\mathrm{sc}} \to G.$$

As a particular case of Corollary 2.3.6, we obtain:

Corollary 2.3.8. *If GLC holds for G_{sc}, then it also holds for G.*

2.3.9. Let $G = G_1 \times G_2$. We have

$$\mathrm{Bun}_G \simeq \mathrm{Bun}_{G_1} \times \mathrm{Bun}_{G_2},$$

and hence

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2}).$$

We also have:

$$\check{G} \simeq \check{G}_1 \times \check{G}_2,$$

and hence

$$\mathrm{LS}_{\check{G}} \simeq \mathrm{LS}_{\check{G}_1} \times \mathrm{LS}_{\check{G}_2},$$

so that

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}_1}) \otimes \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}_2}).$$

It is clear that under the above identifications,

$$\mathbb{L}_G \simeq \mathbb{L}_{G_1} \otimes \mathbb{L}_{G_2}.$$

Combining with Corollary 2.3.8, we obtain:

Corollary 2.3.10. *If GLC holds for all G that are almost simple and simply-connected, then it holds for any reductive G.*

2.4. **Proof of Theorem 2.3.4.**

2.4.1. We will distinguish two special types of homomorphisms ϕ:

Type A: ϕ is injective;

Type B: ϕ is surjective with a connected kernel.

Note that any ϕ can be factored as a composition

$$G_1 \to G_1' \to G_2' \to G_2,$$

where:

- The homomorphisms $G_1 \to G_1'$ and $G_2' \to G_2$ are of type B;
- $G_1' \to G_2'$ is of type A.

So, it is enough to prove Theorem 2.3.4 for ϕ of each of the above two types separately.

2.4.2. *Proof for type B.* Set

$$T := \ker(\phi).$$

(In this subsection T is just a torus, i.e., it is *not* the Cartan subgroup of either G_1 or G_2.)

We have an action of Bun_T on Bun_{G_1}, and

$$\mathrm{Bun}_{G_2} \simeq \mathrm{Bun}_{G_1} / \mathrm{Bun}_T. \tag{2.9}$$

We also have a projection

$$\mathrm{LS}_{\check{G}_1} \to \mathrm{LS}_{\check{T}},$$

and

$$\mathrm{LS}_{\check{G}_2} \simeq \mathrm{pt} \underset{\mathrm{LS}_{\check{T}}}{\times} \mathrm{LS}_{\check{G}_1} \tag{2.10}$$

where $\mathrm{pt} \to \mathrm{LS}_{\check{T}}$ is the unit point.

From (2.9) we obtain that the naturally defined functor

$$\mathrm{Vect} \underset{\text{D-mod}(\mathrm{Bun}_T)}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2}) \tag{2.11}$$

is an equivalence, where:

- $\text{D-mod}(\mathrm{Bun}_T)$ acts on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1})$ by !-convolution;
- The functor $\text{D-mod}(\mathrm{Bun}_T) \to \mathrm{Vect}$ is cohomology with compact supports.

From (2.10) we obtain:

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}_2}) \simeq \mathrm{Vect} \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{T}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check{G}_1}),$$

and hence

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}_2}) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}_1})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1})$$

can be rewritten as

$$\mathrm{Vect} \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{T}})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}).$$

However, by Fourier-Mukai (i.e., GLC for tori)

$$\mathrm{QCoh}(\mathrm{LS}_{\check{T}}) \overset{\mathrm{FM}}{\simeq} \text{D-mod}(\mathrm{Bun}_T).$$

Hence, we obtain

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}_2}) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}_1})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) \simeq \mathrm{Vect} \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{T}})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) \overset{\mathrm{FM}}{\simeq}$$

$$\simeq \mathrm{Vect} \underset{\text{D-mod}(\mathrm{Bun}_T)}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) \overset{(2.11)}{\simeq} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2}).$$

This is the desired equivalence (2.8).

2.4.3. *Proof for type A.* First, replacing G_1 by its derived group, we obtain that it is enough to consider the case when G_1 is semi-simple, which we will from now on assume.

Denote by T the cokernel of ϕ, and denote by ψ the projection

$$\mathrm{Bun}_{G_2} \to \mathrm{Bun}_T\,.$$

Consider $\text{D-mod}(\mathrm{Bun}_T)$ as a (symmetric) monoidal category with respect to the pointwise $\overset{!}{\otimes}$ tensor product, and let it act on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2})$ via $\psi^!(-) \overset{!}{\otimes} (-)$.

Denote

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1})' := \mathrm{Vect} \underset{\text{D-mod}(\mathrm{Bun}_T)}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2}), \tag{2.12}$$

where $\text{D-mod}(\mathrm{Bun}_T) \to \mathrm{Vect}$ is the functor of !-fiber at the unit point.

The functor $\phi^!$ naturally factors as

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_2}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1})' \overset{(\phi^!)'}{\to} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}),$$

and it is easy to see that the functor $(\phi^!)'$ is fully faithful. In fact, its essential image is a *direct summand* in $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1})$, described as follows:

Note that the group Z_{G_1} (which is finite, due to the assumption that G_1 is semi-simple) acts by automorphisms of the identity functor of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1})$. Hence, the category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1})$ splits as a direct sum according to characters of Z_{G_1}:

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1}) = \underset{\alpha \in (Z_{G_1})^\vee}{\oplus} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_1})_\alpha.$$

Now,

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1})' = \underset{\alpha}{\oplus} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1})_\alpha,$$

where α runs over the subset consisting of those characters that vanish on the subgroup

$$\ker(Z_{G_1} \to Z_{G_2}/Z^0_{G_2}).$$

The functor $\phi_!$ factors as

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1})' \overset{(\phi_!)'}{\to} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}),$$

where the first arrow is the corresponding orthogonal projection.

2.4.4. Consider the map

$$\phi^\vee : \text{LS}_{\check{G}_2} \to \text{LS}_{\check{G}_1}.$$

Let

$$\text{LS}'_{\check{G}_1} \subset \text{LS}_{\check{G}_1} \tag{2.13}$$

be the union of connected components that lie in the essential image of $\phi^\vee$. The following assertion is a particular case of Theorem 5.1.5 below:

Proposition 2.4.5. *The full subcategory*

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1})' \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1})$$

equals

$$\text{QCoh}(\text{LS}'_{\check{G}_1}) \underset{\text{QCoh}(\text{LS}_{\check{G}_1})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1}).$$

Let us assume this proposition and proceed with the proof of Case A of Theorem 2.3.4. We obtain that the functor (2.8) is an equivalence if and only if the functor

$$\text{QCoh}(\text{LS}_{\check{G}_2}) \underset{\text{QCoh}(\text{LS}'_{\check{G}_1})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1})' \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}), \tag{2.14}$$

induced by $(\phi_!)'$, is an equivalence.

2.4.6. Consider $\text{QCoh}(\text{LS}_{\check{T}})$ as a monoidal category with respect to *convolution*. As such, it acts on $\text{QCoh}(\text{LS}_{\check{G}_2})$ via the map

$$\psi^\vee : \text{LS}_{\check{T}} \to \text{LS}_{\check{G}_2}.$$

Note that using the Fourier-Mukai equivalence (i.e., GLC for tori)

$$\text{QCoh}(\text{LS}_{\check{T}}) \overset{\text{FM}}{\simeq} \text{D-mod}(\text{Bun}_T),$$

we can rewrite $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_1})'$ as

$$\text{Vect} \underset{\text{QCoh}(\text{LS}_{\check{T}})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2}),$$

where the functor $\text{QCoh}(\text{LS}_{\check{T}}) \to \text{Vect}$ is $\Gamma(\text{LS}_{\check{T}}, -)$.

2.4.7. Note the action of $\text{QCoh}(\text{LS}_{\check{G}_2})$ on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2})$ is compatible with the actions of $\text{QCoh}(\text{LS}_{\check{T}})$ on both. Thus, (2.14) can be rewritten as the special case (for $\mathbf{C} := \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_2})$) of the functor

$$\text{QCoh}(\text{LS}_{\check{G}_2}) \underset{\text{QCoh}(\text{LS}'_{\check{G}_1})}{\otimes} \left(\text{Vect} \underset{\text{QCoh}(\text{LS}_{\check{T}})}{\otimes} \mathbf{C}\right) \to \mathbf{C}, \tag{2.15}$$

defined for a DG category $\mathbf{C}$, equipped with an action of $\text{QCoh}(\text{LS}_{\check{G}_2})$ and a compatible action of $\text{QCoh}(\text{LS}_{\check{T}})$.

2.4.8. We claim that (2.15) is an equivalence for any such $\mathbf{C}$. Indeed, let $\mathcal{Y}$ be an algebraic stack with an affine diagonal, and let

$$\widetilde{\mathcal{Y}} \to \mathcal{Y}$$

be a torsor with respect to a group-stack $\mathcal{T}$, also with an affine diagonal.

Assume that both $\mathcal{Y}$ and $\mathcal{T}$ are quasi-compact, locally almost of finite type and eventually coconnective (so that [Ga3, Theorem 2.2.6] is applicable).

Then the 2-category of DG categories tensored over $\mathrm{QCoh}(\widetilde{\mathcal{Y}})$ and equipped with a compatible action of $\mathrm{QCoh}(\mathcal{T})$ is equivalent to the 2-category of DG categories tensored over $\mathrm{QCoh}(\mathcal{Y})$, with the mutually inverse equivalences being

$$\mathbf{C} \mapsto \mathrm{Vect} \underset{\mathrm{QCoh}(\mathcal{T})}{\otimes} \mathbf{C}$$

and

$$\mathbf{D} \mapsto \mathrm{QCoh}(\widetilde{\mathcal{Y}}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{D}.$$

□[Theorem 2.3.4]

3. Low genus cases

The device that we to prove the GLC breaks down when X has genus 0 or 1. In this section, we treat these cases separately. We highlight the key role played by the main results of [GLC3] and [GLC4] in this material.

3.1. **What do we need to prove?** According to Sect. 1.3.4, in order to prove GLC, we need to prove that the map (1.4) is an isomorphism.

We will show that this is automatic when X has low genus.

3.2. **The case of $g = 0$.** Note that for a curve of genus 0, we have $\mathrm{LS}^{\mathrm{irred}}(X) = \emptyset$, so that (1.4) holds trivially.

3.3. **The case $g = 1$.**

3.3.1. According to Corollary 2.3.10, we can assume that G is almost simple and simply-connected. We will separate two cases:

(a) $G = SL_n$;

(b) $G \neq SL_n$.

3.3.2. In case (a), the dual group $\check{G}$ is isomorphic to PGL_n. In this case, [GLC4, Conjecture 4.5.7] is known (in fact, it is a trivial particular case of [BKS])[3].

This implies GLC by [GLC4, Corollary 4.5.5].

3.3.3. Note that this proof covers the case of $G = SL_n$ for *any* genus.

3.3.4. We now consider case (b).

Proposition 3.3.5 ([KS], [BFM])**.** *Let G be an adjoint group different from PGL_n. Then for a curve of genus 1, we have $\mathrm{LS}^{\mathrm{irred}}_{\mathsf{G}} = \emptyset$.*

From the proposition, we obtain that (1.4) holds trivially in this case.

[3] Fix an irreducible PGL_n local system σ, and choose its generic lift to an SL_n-local system; denote the underlying vector bundle by $\mathcal{E}$. Then the space of generic oper structures on σ is isomorphic to the space of generically defined line subbundles in $\mathcal{E}$, and this space is known to be homologically contractible by [Ga5].

3.3.6. *Proof of Proposition 3.3.5.* Appealing to Riemann-Hilbert, it is enough to show that a Riemann surface X of genus 1 does not admit irreducible (Betti) $\mathsf{G}_{\mathbb{C}}$-local systems.

A $\mathsf{G}_{\mathbb{C}}$-local system σ on X is given by a pair of commuting elements

$$g_1, g_2 \in \mathsf{G}_{\mathbb{C}}.$$

Consider the subgroup

$$Z_{\mathsf{G}}(g_1, g_2) \simeq \mathrm{Aut}_\sigma .$$

A standard argument shows that if σ is irreducible then (in any genus) the Lie algebra of Aut_σ is zero. Hence, $Z_{\mathsf{G}}(g_1, g_2)$ is finite.

Since g_1 and g_2 commute, the subgroup that they generate is contained in $Z_{\mathsf{G}}(g_1, g_2)$, and hence is itself finite. Hence, the pair (g_1, g_2) is contained in a compact form K of $\mathsf{G}_{\mathbb{C}}$.

However, now [BFM, Proposition 4.1.1][4] implies that $K \simeq PSU_n$ for some n; hence $\mathsf{G} \simeq PGL_n$.
$\square$[Proposition 3.3.5]

4. Calculation of the fundamental group

In this section we let G be a semi-simple group.

One of the key parts of the argument in the proof of GLC is that[5] the fundamental group of the stack $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is small (outside a few exceptional cases).

For example, if G is adjoint (in which case $\check{G}$ is simply-connected), the stack $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is also simply-connected. The reader may choose to focus on this case on the first pass.

For a general G, we will show that the fundamental group of $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is controlled by the finite group Z_G.

We remark that the arguments in this section are of de Rham nature. It would be nice to also have a direct topological proof of Theorem 4.3.2 in its Betti incarnation.

4.1. **The fundamental groupoid of** $\mathrm{Bun}_{\check{G}}$.

4.1.1. Let $\mathcal{S}$ be a connective spectrum. We can regard it as a constant prestack, and we let $\mathcal{S}_{\mathrm{et}}$ be its étale sheafification.

For example, if $\mathcal{S} = B(\Gamma)$, where Γ is a finite abelian group, then $B(\Gamma)_{\mathrm{et}}$ is the étale stack pt/Γ.

4.1.2. Consider the tautological map

$$\check{G} \to \mathrm{pt}/\pi_1(\check{G}),$$

where $\pi_1(\check{G})$ denotes the étale fundamental group of $\check{G}$.

The above map induces a map

$$\mathrm{pt}/\check{G} \to B^2(\pi_1(\check{G}))_{\mathrm{et}},$$

and hence to a map

$$\mathrm{Bun}_{\check{G}} = \mathrm{Maps}(X, \mathrm{pt}/\check{G}) \to \mathrm{Maps}(X, B^2(\pi_1(\check{G}))_{\mathrm{et}}) =: \mathrm{Ge}_{\pi_1(\check{G})}(X), \tag{4.1}$$

where $\mathrm{Maps}(-,-)$ denoted the prestack of maps.

Remark 4.1.3. The map (4.1) means that to a $\check{G}$-bundle we can canonically associate an étale $\pi_1(\check{G})$-gerbe. Namely, this is the gerbe of étale-local lifts of our bundle to the simply-connected cover of $\check{G}$.

[4] A related result is established also in [KS].
[5] Under the assumption that $g \geq 2$ (and if $g = 2$, the Dynkin diagram of G has no A_1 factors).

4.1.4. Note that we can think of $\mathrm{Ge}_{\pi_1(\check{G})}(X)$ also as

$$B^2(\mathrm{C}^\cdot(X,\pi_1(\check{G})))_{\mathrm{et}},$$

where $\mathrm{C}^\cdot(X,\pi_1(\check{G}))$ is the spectrum of étale cochains on X with coefficients in $\pi_1(\check{G})$.

Accordingly, the (2)-stack $\mathrm{Ge}_{\pi_1(\check{G})}(X)$ splits into connected components indexed by $H^2(X,\pi_1(\check{G}))$. The neutral connected component is canonically isomorphic to

$$B(\mathrm{Bun}_{\pi_1(\check{G})})_{\mathrm{et}}.$$

4.1.5. We will prove:

Proposition 4.1.6. *The map* (4.1) *defines an isomorphism of* $\tau_{\leq 1}$ *truncations of étale homotopy types.*

The concrete meaning of this proposition is that the map (4.1) defines a bijection on the sets of connected components, and on each connected component an isomorphism of étale fundamental groups.

The proof will be given in Sect. 7.1.

4.2. **Line bundles on** $\mathrm{Bun}_{\check{G}}$**.** We will now use the map (4.1) to construct line bundles on $\mathrm{Bun}_{\check{G}}$ starting from Z_G-torsors. This will be part of a more general construction, which will be extensively used in Sect. 8.

4.2.1. Note that we have a canonical identification

$$\pi_1(\check{G}) \simeq (Z_G)^\vee(1), \tag{4.2}$$

where $(-)^\vee$ denotes Cartier duality and (1) denotes the Tate twist.

4.2.2. Combining (4.2) with Verdier duality

$$\mathrm{C}^\cdot(X,Z_G)^\vee \simeq B^2\left(\mathrm{C}^\cdot(X,(Z_G)^\vee(1))\right),$$

we obtain an identification

$$\mathrm{C}^\cdot(X,Z_G)^\vee \simeq B^2(\mathrm{C}^\cdot(X,\pi_1(\check{G}))), \tag{4.3}$$

and in particular a bilinear pairing

$$B^2(\mathrm{C}^\cdot(X,Z_G)) \times B^2(\mathrm{C}^\cdot(X,\pi_1(\check{G}))) \to B^2(\mu_\infty). \tag{4.4}$$

4.2.3. After étale sheafification, from (4.4) we obtain a bilinear pairing

$$\mathrm{Ge}_{Z_G}(X) \times \mathrm{Ge}_{\pi_1(\check{G})}(X) \to \mathrm{Ge}_{\mu_\infty}(\mathrm{pt}), \tag{4.5}$$

where

$$\mathrm{Ge}_{\mu_\infty}(\mathrm{pt}) := B^2(\mu_\infty)_{\mathrm{et}}.$$

4.2.4. Looping (4.5) along the first factor, we obtain a pairing[6]

$$\mathrm{Bun}_{Z_G} \times \mathrm{Ge}_{\pi_1(\check{G})}(X) \to B(\mu_\infty)_{\mathrm{et}} \to \mathrm{pt}/\mathbb{G}_m. \tag{4.6}$$

In particular, we obtain that a point

$$\mathcal{P}_{Z_G} \in \mathrm{Bun}_{Z_G}$$

gives rise to a canonically defined μ_∞-torsor, to be denoted

$$\mathcal{L}_{\mathcal{P}_{Z_G}},$$

on $\mathrm{Ge}_{\pi_1(\check{G})}(X)$.

The fact that (4.4) is a perfect pairing implies that every μ_∞-torsor on every connected component of $\mathrm{Ge}_{\pi_1(\check{G})}(X)$ is the restriction of $\mathcal{L}_{\mathcal{P}_{Z_G}}$ for some $\mathcal{P}_{Z_G}$.

[6] We will return to the untruncated pairing (4.5) in Sect. 8, where it will play a fundamental role.

4.2.5. We will denote by the same symbol $\mathcal{L}_{\mathcal{P}_{Z_G}}$ the pullback of the above μ_∞-torsor along the map (4.1). By a slight abuse of notation, we will continue to use the same symbol $\mathcal{L}_{\mathcal{P}_{Z_G}}$ the corresponding étale local system of k-vector spaces on $\mathrm{Bun}_{\check{G}}$.

By Proposition 4.1.6, every étale local system of k-vector spaces on a given component of $\mathrm{Bun}_{\check{G}}$ splits as a direct sum of 1-dimensional ones, and each of the latter is isomorphic to the restriction of $\mathcal{L}_{\mathcal{P}_{Z_G}}$ for some $\mathcal{P}_{Z_G}$.

4.2.6. Since $\mathcal{L}_{\mathcal{P}_{Z_G}}$, viewed as an étale local system of k-vector spaces comes from a μ_∞-torsor, we can canonically associate to it a de Rham local system, which we will still denote by the same character $\mathcal{L}_{\mathcal{P}_{Z_G}}$.

We will also use the same symbol $\mathcal{L}_{\mathcal{P}_{Z_G}}$ to denote the corresponding line bundle (viewed either as the line bundle underlying the corresponding de Rham local system, or equivalently, as a $\mathcal{O}^\times$-torsor induced by the map $\mu_\infty \to \mathcal{O}^\times$).

By a further abuse of notation, we will use the same symbol $\mathcal{L}_{\mathcal{P}_{Z_G}}$ to denote its pullback (in all three incarnations: étale, de Rham, coherent) along the map

$$\mathrm{LS}_{\check{G}} \to \mathrm{Bun}_{\check{G}}.$$

4.3. The fundamental group of $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$.

4.3.1. Consider the map

$$\mathrm{LS}_{\check{G}}^{\mathrm{irred}} \to \mathrm{LS}_{\check{G}} \to \mathrm{Bun}_{\check{G}}. \tag{4.7}$$

The main result of this subsection is the following assertion:

Theorem 4.3.2. *Assume that $g \geq 2$, and if $g = 2$, then its root system does not have A_1 factors. Then the map* (4.7) *induces an isomorphism on the $\tau_{\leq 1}$ truncations of étale homotopy types.*

Before we proceed to the proof, we record the following corollary, obtained by combining Theorem 4.3.2 and Proposition 4.1.6:

Corollary 4.3.3. *For every connected component of $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$, every étale local system of k-vector spaces on it splits as a direct sum of 1-dimensional ones. Each of the latter is isomorphic to the restriction of $\mathcal{L}_{\mathcal{P}_{Z_G}}$ for some $\mathcal{P}_{Z_G} \in \mathrm{Bun}_{Z_G}$.*

The rest of this subsection is devoted to the proof of Theorem 4.3.2.

4.3.4. Let

$$\mathrm{Bun}_{\check{G}}^{\mathrm{stbl}} \subset \mathrm{Bun}_{\check{G}}$$

be the stable locus. We will prove:

Proposition 4.3.5. *Under the assumptions on G and g specified in Theorem 4.3.2, the complement of $\mathrm{Bun}_{\check{G}}^{\mathrm{stbl}}$ in $\mathrm{Bun}_{\check{G}}$ has codimension ≥ 2.*

Remark 4.3.6. Statements of this type are classical in the literature of Bun_G; they begin with [NR] Sect. 9. The literature we found concerned coarse moduli spaces instead of moduli stacks, so we include the argument for Proposition 4.3.5 in Sect. 7.2. There are no significant differences between our argument and those in the existing literature.

4.3.7. Combining Propositions 4.3.5 and 4.1.6, we obtain:

Corollary 4.3.8. *Under the assumptions on G and g specified in Theorem 4.3.2, for every connected component of $\mathrm{Bun}_{\check{G}}^{\mathrm{stbl}}$, every étale local system of k-vector spaces on it splits as a direct sum of 1-dimensional ones. Each of the latter is isomorphic to the restriction of $\mathcal{L}_{\mathcal{P}_{Z_G}}$ for some $\mathcal{P}_{Z_G} \in \mathrm{Bun}_{Z_G}$.*

4.3.9. Denote

$$\mathrm{LS}_{\check{G}}^{\mathrm{stbl}} := \mathrm{LS}_{\check{G}} \underset{\mathrm{Bun}_{\check{G}}}{\times} \mathrm{Bun}_{\check{G}}^{\mathrm{stbl}}.$$

The following is well-known:

Proposition 4.3.10. *The map*

$$\mathrm{LS}_{\check{G}}^{\mathrm{stbl}} \to \mathrm{Bun}_{\check{G}}^{\mathrm{stbl}}$$

is smooth and surjective. The fibers of this map are affine spaces.

For the sake of completeness, we will supply a proof in Sect. 7.4. Combining Proposition 4.3.10 and Corollary 4.3.8, we obtain:

Corollary 4.3.11. *Under the assumptions on G and g specified in Theorem 4.3.2, for every connected component of $\mathrm{LS}_{\check{G}}^{\mathrm{stbl}}$, every étale local system of k-vector spaces on it splits as a direct sum of 1-dimensional ones. Each of the latter is isomorphic to the restriction of $\mathcal{L}_{\mathcal{P}_{Z_G}}$ for some $\mathcal{P}_{Z_G} \in \mathrm{Bun}_{Z_G}$.*

4.3.12. *Proof of Theorem 4.3.2.* Note that we have an inclusion

$$\mathrm{LS}_{\check{G}}^{\mathrm{stbl}} \subset \mathrm{LS}_{\check{G}}^{\mathrm{irred}}$$

with both spaces being smooth Deligne-Mumford stacks, and both are open inside $\mathrm{LS}_{\check{G}}$.

It is enough to establish the inclusion at the level of k-points. Suppose $\sigma \in \mathrm{LS}_{\check{G}}^{\mathrm{stbl}}$; we need to show it does not admit a reduction $\sigma_{\check{P}}$ to any proper parabolic $\check{P} \subset \check{G}$. Suppose we had such a reduction. Then, by the definition of stability (see Appendix A), we would have $\langle 2\check{\rho}_{\check{P}}, \deg(\sigma_P)\rangle < 0$. However, the above integer is the degree of the line bundle induced from $\sigma_{\check{P}}$ using $2\check{\rho}_{\check{P}}$, viewed as a homomorphism $\check{P} \to \check{M} \to \mathbb{G}_m$. Since the line bundle is endowed with a connection, its degree must be zero, which is a contradiction.

Thus, given Corollary 4.3.11, in order to prove Theorem 4.3.2, it suffices to show that $\mathrm{LS}_{\check{G}}^{\mathrm{stbl}}$ is dense in $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$. This is equivalent to saying that $\mathrm{LS}_{\check{G}}^{\mathrm{stbl}}$ has a non-empty intersection with every irreducible component of $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$. Since $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$ is smooth, its irreducible components are the same as connected components.

According to Corollary 5.3.7 below, the embedding

$$\mathrm{LS}_{\check{G}}^{\mathrm{irred}} \hookrightarrow \mathrm{LS}_{\check{G}}$$

induces a bijection on the sets of connected components.

Hence, we obtain that it is sufficient to show that the embedding

$$\mathrm{LS}_{\check{G}}^{\mathrm{stbl}} \hookrightarrow \mathrm{LS}_{\check{G}}$$

induces a bijection on the sets of connected components.

We have a commutative square

$$\begin{tikzcd} \mathrm{LS}_{\check{G}}^{\mathrm{stbl}} \arrow[r] \arrow[d] & \mathrm{LS}_{\check{G}} \arrow[d] \\ \mathrm{Bun}_{\check{G}}^{\mathrm{stbl}} \arrow[r] & \mathrm{Bun}_{\check{G}}. \end{tikzcd}$$

In it, the lower horizontal arrow and the left vertical arrow induce bijections on the sets of connected components, by Propositions 4.3.5 and 4.3.10, respectively.

Hence, the desired assertion follows from the fact that the map

$$\mathrm{LS}_{\check{G}} \to \mathrm{Bun}_{\check{G}}$$

induces a bijection on π_0 (see [BD, Proposition 2.11.4]).

□[Theorem 4.3.2]

5. The core of the proof

Throughout this section, we will assume that G is semi-simple.

After some preparations, in this section we will give a proof of GLC for curves of genus ≥ 2.

We note that the proof is particularly simple when G is adjoint, so that $\check{G}$ is simply-connected (in this case, one needs neither Sect. 5.1 nor Theorem 5.2.7). The reader may choose to focus on this case on the first pass.

5.1. **Action of the center.**

5.1.1. Note that the (abelian) group-stack Bun_{Z_G} acts on Bun_G, and this action lifts to an action of Bun_{Z_G} on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

For $\mathcal{P}_{Z_G} \in \mathrm{Bun}_{Z_G}$, we will denote the corresponding automorphism of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ by

$$\mathcal{P}_{Z_G} \cdot -. \tag{5.1}$$

5.1.2. Note that the group Z_G acts on the unit point $\mathcal{P}^0_{Z_G} \in \mathrm{Bun}_{Z_G}$. Since

$$\mathcal{P}^0_{Z_G} \cdot (-)$$

is the identity functor, we obtain that Z_G acts by automorphisms of the identity endofunctor of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

Let

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \simeq \underset{\alpha}{\oplus} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_\alpha, \quad \alpha \in (Z_G)^\vee$$

denote the corresponding decomposition. Denote by P_α the idempotent on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$, corresponding to $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_\alpha$.

5.1.3. Let $\pi_{1,\mathrm{alg}}(\check{G})$ denote the *algebraic fundamental group* of $\check{G}$, i.e., the quotient of the coweight lattice by the coroot lattice. We have

$$\pi_{1,\mathrm{alg}}(\check{G}) \simeq \pi_0(\mathrm{Bun}_{\check{G}}) \simeq \pi_0(\mathrm{LS}_{\check{G}}).$$

Note that we have

$$\pi_{1,\mathrm{alg}}(\check{G}) \simeq \pi_1(\check{G})(-1),$$

so that we have a canonical identification

$$(Z_G)^\vee \simeq \pi_{1,\mathrm{alg}}(\check{G}).$$

5.1.4. For a given $\alpha \in (Z_G)^\vee$, let $\mathrm{LS}_{\check{G},\alpha}$ denote the corresponding connected component of $\mathrm{LS}_{\check{G}}$. Consider the corresponding idempotent

$$\mathcal{O}_{\mathrm{LS}_{\check{G},\alpha}} \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}).$$

We will prove (see Sect. 8.4.10):

Theorem 5.1.5. *For $\alpha \in (Z_G)^\vee$ as above, the idempotent*

$$\mathcal{O}_{\mathrm{LS}_{\check{G},\alpha}} \otimes (-) : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

where $\otimes$ denotes the spectral action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$, identifies canonically with P_α.

5.1.6. We will also prove (see Sect. 8.4.10):

Theorem 5.1.7. *For $\mathcal{P}_{Z_G} \in \mathrm{Bun}_{Z_G}$ and the corresponding $\mathcal{L}_{\mathcal{P}_{Z_G}} \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$, the functor*

$$\mathcal{L}_{\mathcal{P}_{Z_G}} \otimes (-) : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

where $\otimes$ denotes the spectral action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$, identifies canonically with (5.1).

5.2. **Endomorphisms of the vacuum Poincaré object.**

In this subsection, we will assume that $g \geq 2$.

5.2.1. Recall the object
$$\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$
see [Lan2, Sect. 12.5.4]. For $\alpha \in (Z_G)^\vee$, let $\mathrm{Poinc}^{\mathrm{Vac}}_{G,!,\alpha}$ denote the corresponding direct summand.

5.2.2. We will prove (see Sect. 6.2):

Theorem 5.2.3. *For every α, the map*
$$k \to H^0(\mathcal{E}nd(\mathrm{Poinc}^{\mathrm{Vac}}_{G,!,\alpha}))$$
is an isomorphism.

In fact, we will prove a more precise result, but only Theorem 5.2.3 will be needed for the proof of GLC:

Theorem 5.2.4. $H^i(\mathcal{E}nd(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!})) = 0$ *for* $i \neq 0$.

As a corollary of Theorem 5.2.3, we obtain:

Corollary 5.2.5. $\dim\left(H^0(\mathcal{E}nd(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}))\right) = |Z_G|$.

5.2.6. We will also prove (see Sect. 6.3):

Theorem 5.2.7. *For a non-trivial* $\mathcal{P}_{Z_G} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$,
$$H^0\left(\mathcal{H}om(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}, \mathcal{P}_{Z_G} \cdot \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!})\right) = 0.$$

As in the case of Theorem 5.2.3, we will actually prove a more precise result (but only Theorem 5.2.7 will be needed for the proof of GLC):

Theorem 5.2.8. *For a non-trivial* $\mathcal{P}_{Z_G} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$,
$$\mathcal{H}om(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}, \mathcal{P}_{Z_G} \cdot \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}) = 0.$$

5.3. **Algebraic geometry of** $\mathrm{LS}_{\check{G}}$**.** In this subsection, we continue to assume that $g \geq 2$.

5.3.1. First, we recall (see [BD, Proposition 2.11.2]):

Theorem 5.3.2. *The stack $\mathrm{LS}_{\check{G}}$ is a classical locally complete intersection of dimension*
$$\dim(\mathfrak{g}) \cdot 2(g-1).$$

Corollary 5.3.3. *The stack $\mathrm{LS}_{\check{G}}$ is Cohen-Macaulay of dimension* $\dim(\mathfrak{g}) \cdot 2(g-1)$.

5.3.4. Next we claim:

Proposition 5.3.5. *Excluding the case of $g = 2$ with the root system of G containing an A_1 factor, the complement to $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ in $\mathrm{LS}_{\check{G}}$ has codimension ≥ 2.*

This proposition will be proved in Sect. 7.3.

5.3.6. Note that from Corollary 5.3.3 and Proposition 5.3.5, we obtain:

Corollary 5.3.7. *The embedding $\mathrm{LS}^{\mathrm{irred}}_{\check{G}} \hookrightarrow \mathrm{LS}_{\check{G}}$ induces a bijection between the sets of connected components.*

5.4. **Structure of** $\mathcal{A}_{G,\mathrm{irred}}$**.** In this subsection we continue to assume that $g \geq 2$, and we will exclude the case that $g = 2$ and the root system of G contains a factor of A_1 (as in Proposition 5.3.5).

5.4.1. Recall that $\mathcal{A}_{G,\mathrm{irred}}$ is a vector bundle on $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ (see Sect. 1.4.2). The next proposition provides an explicit description of the shape that $\mathcal{A}_{G,\mathrm{irred}}$ can have. This description will play a crucial role in the proof of GLC given below.

Proposition 5.4.2. *The restriction of $\mathcal{A}_{G,\mathrm{irred}}$ to every connected component is isomorphic to a direct sum of lines bundles, each of which is a restriction of some $\mathcal{L}_{P_{Z_G}}$ (see Sect. 4.2.6).*

5.4.3. *Proof of Proposition 5.4.2.* According to Sect. 1.4, the object $\mathcal{A}_{G,\mathrm{irred}}$ is the vector bundle underlying a local system $\mathcal{F}$ with finite monodromy (in particular, it has regular singularities).

Thanks to the finite monodromy property, we can think of $\mathcal{F}$ as an étale local system of k-vector spaces. The assertion of the proposition follows now from Corollary 4.3.3.

□[Proposition 5.4.2]

5.5. **Proof of GLC.** Let g and G be as in Sect. 5.4. Note that it is sufficient to prove GLC under these assumptions:

Indeed, by Sect. 3, we can assume that $g \geq 2$. By Corollary 2.3.10, we may assume that G is almost simple and simply-connected. By Sect. 3.3.3 we can assume that it is not isomorphic to SL_2.

5.5.1. *Step 0.* Fix a connected component $\mathrm{LS}_{\check{G},\alpha}$ of $\mathrm{LS}_{\check{G}}$. Denote

$$\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha} := \mathrm{LS}_{\check{G},\alpha} \cap \mathrm{LS}^{\mathrm{irred}}_{\check{G}}.$$

Using Proposition 5.4.2, we can write

$$\mathcal{A}_{G,\mathrm{irred}}|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}} \simeq \underset{\mathcal{P}_{Z_G} \in \mathrm{Bun}_{Z_G}}{\oplus} \mathcal{L}_{\mathcal{P}_{Z_G}}^{\oplus n_{\mathcal{P}_{Z_G},\alpha}}|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}} \tag{5.2}$$

for some integers $n_{\mathcal{P}_{Z_G},\alpha}$.

We wish to show that for every α

$$n_{\mathcal{P}_{Z_G},\alpha} = \begin{cases} 1 \text{ if } \mathcal{P}_{Z_G} \text{ is trivial;} \\ 0 \text{ if } \mathcal{P}_{Z_G} \text{ is non-trivial.} \end{cases}$$

Since each $\mathcal{A}_{G,\mathrm{irred}}|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}}$ is a unital associative algebra in $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha})$, the latter will automatically imply (see [Lan2, Proposition 24.3.8]) that the unit map

$$\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}} \to \mathcal{A}_{G,\mathrm{irred}}|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}} \tag{5.3}$$

is an isomorphism, i.e., (1.4) is an isomorphism.

5.5.2. *Step 1.* Fix a particular $\mathcal{P}_{Z_G}$. We are going to prove that the map

$$\mathcal{A}_G \otimes \mathcal{L}_{\mathcal{P}_{Z_G}} \to \jmath_* \circ \jmath^*(\mathcal{A}_G \otimes \mathcal{L}_{\mathcal{P}_{Z_G}}) = \jmath_*(\mathcal{A}_{G,\mathrm{irred}}) \otimes \mathcal{L}_{\mathcal{P}_{Z_G}}$$

induces an isomorphism at the level of $H^0(\Gamma(\mathrm{LS}_{\check{G}}, -))$.

Recall that $\widehat{\iota}^!$ denotes the right adjoint to the embedding

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{red}} \hookrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}),$$

so that for every $\mathcal{F} \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ we have a fiber sequence

$$\widehat{\iota}^!(\mathcal{F}) \to \mathcal{F} \to \jmath \circ \jmath^*(\mathcal{F}).$$

Thus, it is sufficient to prove that

$$\widehat{\iota}^!(\mathcal{A}_G \otimes \mathcal{L}_{P_{Z_G}})$$

is concentrated in cohomological degrees ≥ 2.

We have

$$\widehat{\iota}^!(\mathcal{A}_G \otimes \mathcal{L}_{P_{Z_G}}) \simeq \widehat{\iota}^!(\mathcal{A}_G) \otimes \mathcal{L}_{P_{Z_G}},$$

where $\mathcal{L}_{P_{Z_G}}$ is a line bundle. So it is enough to show that

$$\widehat{\iota}^!(\mathcal{A}_G)$$

is concentrated in cohomological degres ≥ 2.

Recall (see Sect. 1.3.3) that the unit map

$$\mathcal{O}_{\mathrm{LS}_{\check{G}}} \to \mathcal{A}_G$$

induces an isomorphism

$$\hat{\iota}^!(\mathcal{O}_{\mathrm{LS}_{\check{G}}}) \to \hat{\iota}^!(\mathcal{A}_G).$$

Thus, it remains to show that

$$\hat{\iota}^!(\mathcal{O}_{\mathrm{LS}_{\check{G}}})$$

is concentrated in cohomological degrees ≥ 2.

However, this follows from Proposition 5.3.5 and Corollary 5.3.3.

Remark 5.5.3. Note that the last step in the above argument shows that the map

$$\mathcal{O}_{\mathrm{LS}_{\check{G}}} \to \jmath_*(\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}})$$

also induces an isomorphism at the level of $H^0(\Gamma(\mathrm{LS}_{\check{G}}, -))$.

5.5.4. *Step 2.* We will now show that $n_{\mathcal{P}_{Z_G},\alpha} = 0$ if $\mathcal{P}_{Z_G}$ is non-trivial.

We begin by showing that

$$\Gamma(\mathrm{LS}_{\check{G}}, \mathcal{A}_G \otimes \mathcal{L}_{P_{Z_G}}) = 0. \tag{5.4}$$

Recall that

$$\mathcal{A}_G = \mathbb{L}_G \circ \mathbb{L}_G^L(\mathcal{O}_{\mathrm{LS}_{\check{G}}}).$$

Therefore, we have

$$\Gamma(\mathrm{LS}_{\check{G}}, \mathcal{A}_G \otimes \mathcal{L}_{P_{Z_G}}) \simeq \mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)}(\mathbb{L}_G^L(\mathcal{O}_{\mathrm{LS}_{\check{G}}}), \mathbb{L}_G^L(\mathcal{O}_{\mathrm{LS}_{\check{G}}} \otimes \mathcal{L}_{P_{Z_G}})). \tag{5.5}$$

Recall that

$$\mathbb{L}_G^L(\mathcal{O}_{\mathrm{LS}_{\check{G}}}) \simeq \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}.$$

Hence, since the functor $\mathbb{L}_{\check{G}}^L$ is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$-linear, and taking into account Theorem 5.1.7, we can rewrite the expression in (5.5) as

$$\mathcal{H}om(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}, \mathcal{P}_{Z_G} \cdot \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}).$$

By Theorem 5.2.7, this expression vanishes, so the same is true of (5.4).

Applying Step 1, we find that

$$H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}, \mathcal{A}_{G,\mathrm{irred}} \otimes \mathcal{L}_{P_{Z_G}})\right) = H^0\left(\Gamma(\mathrm{LS}_{\check{G}}, \mathcal{A}_G \otimes \mathcal{L}_{P_{Z_G}})\right) = 0.$$

However, by definition, $\mathcal{A}_{G,\mathrm{irred}} \otimes \mathcal{L}_{P_{Z_G}}$ carries $n_{\mathcal{P}_{Z_G},\alpha}$ direct summands isomorphic to $\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}}$. Therefore, we see that

$$0 = \dim H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}, \mathcal{A}_{G,\mathrm{irred}} \otimes \mathcal{L}_{P_{Z_G}})\right) \geq H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}, \mathcal{O}^{\oplus n_{\mathcal{P}_{Z_G},\alpha}}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}})\right) \geq n_{\mathcal{P}_{Z_G},\alpha}$$

meaning that $n_{\mathcal{P}_{Z_G},\alpha} = 0$, as was desired.

5.5.5. *Step 3.* Thus, we obtain that the decomposition (5.2) is in fact of the form

$$\mathcal{A}_{G,\mathrm{irred}}|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}} \simeq \mathcal{O}^{\oplus n_\alpha}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}}, \tag{5.6}$$

for some integers n_α, and we wish to show that they are all equal to 1.

By Step 1, for every α, we have

$$H^0\left(\Gamma(\mathrm{LS}_{\check{G},\alpha}, \mathcal{A}_G|_{\mathrm{LS}_{\check{G},\alpha}})\right) \simeq H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}, \mathcal{A}_{G,\mathrm{irred}}|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}})\right).$$

Hence,

$$\dim H^0\left(\Gamma(\mathrm{LS}_{\check{G},\alpha}, \mathcal{A}_G|_{\mathrm{LS}_{\check{G},\alpha}})\right) = \dim H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}, \mathcal{O}^{\oplus n_\alpha}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}})\right) \geq n_\alpha.$$

Therefore,

$$\dim H^0\left(\Gamma(\mathrm{LS}_{\check{G}}, \mathcal{A}_G)\right) \geq \sum_\alpha n_\alpha.$$

Now,

$$\text{(5.7)}\quad \Gamma(\mathrm{LS}_{\check{G}},\mathcal{A}_G)\simeq \mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)}(\mathbb{L}_G^L(\mathcal{O}_{\mathrm{LS}_{\check{G}}}),\mathbb{L}_G^L(\mathcal{O}_{\mathrm{LS}_{\check{G}}}))\simeq$$
$$\simeq \mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)}(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!},\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}).$$

Applying Corollary 5.2.5, we obtain

$$\sum_\alpha 1=|Z_G|=|\pi_0(\mathrm{LS}_{\check{G}})|\geq \sum_\alpha n_\alpha.$$

Hence, in order to prove the desired equality, it suffices to show that $n_\alpha\neq 0$ for all α. I.e., we have to show that $\mathcal{A}_{G,\mathrm{irred}}$ does not vanish on any connected component $\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}$.

5.5.6. *Step 4.* By Theorem 5.1.5, we have:

$$\Gamma(\mathrm{LS}_{\check{G},\alpha},\mathcal{A}_G|_{\mathrm{LS}_{\check{G},\alpha}})\simeq \mathcal{E}nd_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)}(\mathrm{Poinc}^{\mathrm{Vac}}_{G,!,\alpha}).$$

As $\mathrm{Poinc}^{\mathrm{Vac}}_{G,!,-\alpha}\neq 0$ by Theorem 5.2.3, we must have:

$$0\neq \mathrm{id}\in H^0\left(\mathcal{E}nd(\mathrm{Poinc}^{\mathrm{Vac}}_{G,!,\alpha})\right)\simeq H^0\left(\Gamma(\mathrm{LS}_{\check{G},\alpha},\mathcal{A}_G|_{\mathrm{LS}_{\check{G},\alpha}})\right)\simeq H^0\left(\Gamma(\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha},\mathcal{A}_{G,\mathrm{irred}}|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}})\right).$$

□[GLC]

5.6. **Additional remarks.**

5.6.1. For G simply-connected, the assertion that all $\mathcal{A}_{G,\mathrm{irred}}|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}}$ are non-zero can be also deduced from the main result of [Ari]:

Let σ be a point of $\mathrm{LS}^{\mathrm{irred}}_{\check{G},\alpha}$. Recall that according to [GLC4, Theorem 3.1.5], the fiber of $\mathcal{A}_G$ at σ is isomorphic to the homology of the space of generic oper structures on σ.

Thus, it is sufficient to know that the latter space is non-empty. However, this is precisely the result[7] of [Ari].

5.6.2. Note that taking into account Theorem 5.2.4, and knowing that $\mathcal{O}_{\mathrm{LS}_{\check{G}}}\simeq\mathcal{A}_G$, from (5.7) we obtain:

Corollary 5.6.3. *For $g\geq 2$, for every connected component α of $\mathrm{LS}_{\check{G}}$, the map*

$$k\to\Gamma(\mathrm{LS}_{\check{G},\alpha},\mathcal{O}_{\mathrm{LS}_{\check{G},\alpha}})$$

is an isomorphism.

Remark 5.6.4. One can prove Corollary 5.6.3 directly by a deformation argument using [FT, Theorem 4.2].

Remark 5.6.5. We remark that Corollary 5.6.3 is a special feature of the de Rham setting; there are many more global functions on the Betti moduli stack.

5.6.6. Similarly, from Theorem 5.2.8, we obtain:

Corollary 5.6.7. *For $g\geq 2$, for a non-trivial $\mathcal{P}_{Z_G}$, we have* $\Gamma(\mathrm{LS}_{\check{G}},\mathcal{L}_{\mathcal{P}_{Z_G}})=0$.

6. The vacuum Poincaré object

In this section we will prove Theorems 5.2.3 and 5.2.7, along with their strengthenings, given by Theorems 5.2.4 and 5.2.8, respectively.

6.1. **How to calculate those endomorphisms?**

[7] Note that the result of [Ari] is about $\check{\mathfrak{g}}$-opers, which are different from $\check{G}$-opers, unless $\check{G}$ is adjoint. However, as we have seen earlier, it is sufficient to prove GLC in the latter case.

6.1.1. Recall that the object $\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}$ is the !-direct image under the map

$$\mathfrak{p} : \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathrm{Bun}_G$$

of $(\chi^{\mathrm{glob}})^*(\exp)$, see [GLC1, Sect. 1.3.6].

We factor the above map $\mathfrak{p}$ as a composition

$$\mathrm{Bun}_{N,\rho(\omega_X)} \overset{\mathsf{f}}{\to} \mathrm{Bun}_{N,\rho(\omega_X)}/T \hookrightarrow \mathrm{Bun}_B^{(g-1)\cdot 2\rho} \to \mathrm{Bun}_G\,. \tag{6.1}$$

6.1.2. Here is the crucial observation:

Since $g \geq 2$, the coweight $(g-1)\cdot 2\rho$ belongs to Λ^{++}, and hence the map

$$\mathrm{Bun}_B^{(g-1)\cdot 2\rho} \to \mathrm{Bun}_G$$

is a locally closed embedding (see [DG, Theorem 7.4.3(1')]). Hence, the !-direct image with respect to it is fully faithful.

Hence, for the proofs of the theorems of in this section, we can perform the calculations on $\mathrm{Bun}_B^{(g-1)\cdot 2\rho}$. Here we remark that because $Z_G \subset B$, the action of Bun_{Z_G} on Bun_G lifts to an action on $\mathrm{Bun}_B^{(g-1)\cdot 2\rho}$

6.2. **Proof of Theorems 5.2.3 and 5.2.4.**

6.2.1. The map

$$\mathrm{Bun}_{N,\rho(\omega_X)}/T \hookrightarrow \mathrm{Bun}_B^{(g-1)\cdot 2\rho} \tag{6.2}$$

is a closed embedding as it comes via base-change from the closed embedding $B(T) \overset{\rho(\omega_X)}{\to} \mathrm{Bun}_T$.

Note that the action of

$$B(Z_G) \subset \mathrm{Bun}_{Z_G}$$

on $\mathrm{Bun}_B^{(g-1)\cdot 2\rho}$ preserves the above locally closed embedding.

Hence, in order to prove the theorems in this subsection, we can perform the calculations on $\mathrm{Bun}_{N,\rho(\omega_X)}/T$.

6.2.2. Let r be the semi-simple rank of G. For each vertex of the Dynkin diagram we have a canonically defined map

$$\mathrm{Bun}_{N,\rho(\omega_X)} \to \mathbb{A}^1.$$

The resulting map

$$\mathrm{Bun}_{N,\rho(\omega_X)} \to (\mathbb{A}^1)^r = \mathbb{A}^r \overset{\mathrm{sum}}{\to} \mathbb{A}^1$$

is the map χ^{glob}.

We have a Cartesian square

$$\begin{CD} \mathrm{Bun}_{N,\rho(\omega_X)} @>{\chi^{\mathrm{glob}}}>> \mathbb{A}^r \\ @V{\mathsf{f}}VV @VV{\mathsf{f}'}V \\ \mathrm{Bun}_{N,\rho(\omega_X)}/T @>{\chi^{\mathrm{glob}}/T}>> \mathbb{A}^r/T, \end{CD} \tag{6.3}$$

where T acts on $\mathbb{A}^r$ via

$$T \to T_{\mathrm{ad}} \simeq (\mathbb{G}_m)^r.$$

6.2.3. Since

$$Z_G = \ker(T \to T_{\mathrm{ad}}),$$

we have an action of $B(Z_G)$ on $\mathbb{A}^r/T$, and the above map

$$\chi^{\mathrm{glob}}/T : \mathrm{Bun}_{N,\rho(\omega_X)}/T \to \mathbb{A}^r/T$$

is $B(Z_G)$-equivariant.

Since the horizontal arrows in (6.3) are smooth with affine space fibers (and hence, the corresponding pullback functors are fully faithful), we obtain that in order to prove Theorems 5.2.3 and 5.2.4, it suffices to perform the corresponding calculation on $\mathbb{A}^r/T$.

6.2.4. Note that the map

$$\mathbb{A}^r \to \mathbb{A}^r/T$$

naturally factors as

$$\mathbb{A}^r = \mathbb{A}^r \times \mathrm{pt} \to \mathbb{A}^r \times B(Z_G) \simeq \mathbb{A}^r/Z_G \to \mathbb{A}^r/T,$$

and we have a Cartesian square

$$\begin{tikzcd} \mathbb{A}^r \times B(Z_G) \arrow[r] \arrow[d, "\mathsf{f}'"'] & \mathbb{A}^r \arrow[d, "\mathsf{f}''"] \\ \mathbb{A}^r/T \arrow[r] & \mathbb{A}^r/T_{\mathrm{ad}} \simeq (\mathbb{A}^1/\mathbb{G}_m)^r. \end{tikzcd}$$

This reduces the assertions of Theorems 5.2.3 and 5.2.4 to the claim that the map

$$k \to \mathcal{E}nd(\mathsf{f}''_! \circ \mathrm{sum}^*(\exp))$$

is an isomorphism.

6.2.5. Since

$$\mathrm{sum}^*(\exp) \simeq \exp^{\boxtimes r},$$

by Künneth formula, we are reduced to showing that the map

$$k \to \mathcal{E}nd(\mathsf{f}'''_!(\exp))$$

is an isomorphism, where

$$\mathsf{f}''' : \mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m.$$

However, the latter is well-known: the object

$$\mathsf{f}'''_!(\exp) \in \mathrm{D\text{-}mod}(\mathbb{A}^1/\mathbb{G}_m)$$

is the $*$-direct image of $k \in \mathrm{D\text{-}mod}(\mathrm{pt})$ along

$$\mathrm{pt} \simeq \mathbb{A}^1 - 0/\mathbb{G}_m \hookrightarrow \mathbb{A}^1/\mathbb{G}_m.$$

$\square$[Theorems 5.2.3 and 5.2.4]

6.3. **Proof of Theorems 5.2.7 and 5.2.8.**

6.3.1. To prove Theorems 5.2.7 and 5.2.8, it suffices to show that the image of the closed embedding (6.2) and its translate by means of a non-trivial $\mathcal{P}_{Z_G}$ are disjoint.

For that, it suffices to show that their images under the projection

$$\mathrm{Bun}_B \to \mathrm{Bun}_T$$

are disjoint.

6.3.2. For the latter it is sufficient to show that under the further projection

$$\mathrm{Bun}_T \to \mathrm{Bun}_T/B(T)$$

(where $\mathrm{Bun}_T/B(T)$ is the coarse moduli scheme), the above two images correspond to two distinct points:

$$\rho(\omega_X) \text{ and } \mathcal{P}_{Z_G} \cdot \rho(\omega_X).$$

6.3.3. However, the latter follows from the fact that $\mathcal{P}_{Z_G}$ is *non-trivial* as a T-bundle.
$\square$[Theorems 5.2.7 and 5.2.8]

7. Geometry of Bun_G

The goal of this section is to prove Propositions 4.1.6, 4.3.5, 5.3.5 and 4.3.10.

For the duration of this section, we will change the notation from $\check{G}$ to G.

7.1. Proof of Proposition 4.1.6.

7.1.1. Let G' be a reductive group equipped with a surjective map

$$\phi : G' \to G,$$

such that

- The kernel T_0 of ϕ is a (connected) torus (which automatically lies in the center of G');
- The derived group of G' is simply-connected.

7.1.2. Denote

$$T_1 := G'_{\mathrm{ab}}.$$

We obtain an isogeny

$$\phi' := T_0 \to T_1,$$

and it is easy to see that we have a canonical isomorphism

$$\ker(\phi') \simeq \pi_1(G). \tag{7.1}$$

7.1.3. *Example.* One can take G' be the *dual* group of

$$\check{G} \times \check{T}/Z_{\check{G}},$$

where $\check{T}$ is the (abstract) Cartan of $\check{G}$.

Then

$$T_0 = (\check{T}/Z_{\check{G}})^\vee \simeq T_{\mathrm{sc}},$$

where T_{sc} is the (abstract) Cartan of the simply-connected cover G_{sc} of G. We also have $T_1 = T$, so (7.1) becomes the isomorphism

$$\pi_1(G) \simeq \ker(G_{sc} \to G) \simeq \ker(T_{sc} \to T).$$

7.1.4. *Example.* Let $G = PGL_n$. In this case we can take $G' := GL_n$. We have

$$T_0 \simeq \mathbb{G}_m \text{ and } T_0 \simeq \mathbb{G}_m,$$

and the map ϕ' is raising to the power n.

Then (7.1) becomes the identification

$$\pi_1(PGL_n) \simeq \mu_n.$$

7.1.5. Note that the map

$$\mathrm{Bun}_{G'} \to \mathrm{Bun}_{T_1} \tag{7.2}$$

is smooth with fibers that are connected and simply-connected:

Indeed, the fibers are isomorphic to Bun_{G_1}, where G_1 is a twisted form of $[G', G']$, the derived group of G', and the moduli stack of bundles for a simply-connected group is connected and simply-connected.

Hence, we obtain that the map (7.2) induces an isomorphism of the $\tau_{\leq 1}$-truncations of étale homotopy types.

7.1.6. The map ϕ induces an isomorphism

$$\mathrm{Bun}_{G'} / \mathrm{Bun}_{T_0} \simeq \mathrm{Bun}_G .$$

Hence, we obtain that the map

$$\mathrm{Bun}_G \simeq \mathrm{Bun}_{G'} / \mathrm{Bun}_{T_0} \to \mathrm{Bun}_{T_1} / \mathrm{Bun}_{T_0}$$

induces an isomorphism of the $\tau_{\leq 1}$-truncations of étale homotopy types.

7.1.7. Finally, we note that the isomorphism (7.1) induces am identification

$$\mathrm{Bun}_{T_1} / \mathrm{Bun}_{T_0} \simeq B^2(\mathrm{C}^{\cdot}(X, \pi_1(G)))_{\mathrm{et}},$$

and the resulting map

$$\mathrm{Bun}_G \to B^2(\mathrm{C}^{\cdot}(X, \pi_1(G)))_{\mathrm{et}}$$

is the same as (4.1).

□[Proposition 4.1.6]

7.2. **Proof of Proposition 4.3.5.**

7.2.1. At this point, we refer the reader to Appendix A for background on stable bundles and some relevant notation.

Let $\mathrm{Bun}_G^{\mathrm{unstbl}} \subset \mathrm{Bun}_G$ be the closed substack of unstable[8] bundles. We need to show $\mathrm{Bun}_G^{\mathrm{unstbl}}$ has codimension ≥ 2 under our hypotheses on X.

By definition of stability for G-bundles, every point of $\mathrm{Bun}_G^{\mathrm{unstbl}}$ is in the image of some map Bun_P^{λ} for $P \subsetneq G$ a maximal parabolic with Levi quotient M and $\lambda \in \pi_{1,\mathrm{alg}}(M)$ satisfying $\langle 2\check{\rho}_P, \lambda\rangle \geq 0$.

Therefore, it suffices to show

$$\dim(\mathrm{Bun}_P^{\lambda}) \leq \dim(\mathrm{Bun}_G) - 2$$

for such λ.

7.2.2. By Riemann-Roch,

$$\dim(\mathrm{Bun}_G) = \dim(\mathfrak{g}) \cdot (g-1) = \dim(\mathfrak{m}) \cdot (g-1) + 2\dim(\mathfrak{n}(P)) \cdot (g-1)$$

and

$$\dim(\mathrm{Bun}_P^{\lambda}) = \dim(\mathfrak{m}) \cdot (g-1) + \dim(\mathfrak{n}(P)) \cdot (g-1) - \langle 2\check{\rho}_P, \lambda\rangle,$$

where $2\check{\rho}_P$ is as in Appendix A.

Therefore, we have to show

$$2 - \langle 2\check{\rho}_P, \lambda\rangle \leq \dim(\mathfrak{n}(P)) \cdot (g-1).$$

By assumption on λ, the left hand side is at most 2. As P is a proper parabolic and $g \geq 2$, this inequality obviously holds outside the exceptional case where $\dim(\mathfrak{n}(P)) = (g-1) = 1$, which only happens if $g = 2$ and G_{ad} contains a PGL_2 factor.

□[Proposition 4.3.5]

Remark 7.2.3. Note that the assertion of Proposition 4.3.5 is false for $G = SL_2$ and $g = 2$: in this case the dimension of the semi-stable but unstable locus is 2, which is $>$ than

$$1 = 3 - 2 = \dim(\mathrm{Bun}_G) - 2.$$

7.3. **Proof of Proposition 5.3.5.** It is enough to show that for every maximal parabolic subgroup $P \subsetneq G$, we have

$$\dim(\mathrm{LS}_P) \leq \dim(\mathrm{LS}_G) - 2 = \dim(\mathfrak{g}) \cdot (2g-2) - 2. \tag{7.3}$$

[8]Here"unstable" means "not stable", rather than "not semi-stable".

7.3.1. Consider the stack LS_M. It is quasi-smooth of virtual dimension

$$\dim(\mathfrak{m}) \cdot (2g-2),$$

and if $g \geq 2$, by Theorem 5.3.2, its underlying classical stack is a locally complete intersection of dimension

$$\dim(\mathfrak{m}) \cdot (2g-2) + \dim(\mathfrak{z}_M),$$

where $\mathfrak{z}_M := \mathrm{Lie}(Z_M)$.

Indeed, this follows by considering the fibration $\mathrm{LS}_M \to \mathrm{LS}_{M/[M,M]}$, applying Theorem 5.3.2 for the derived group $[M,M]$, and noting that $\mathrm{LS}_{\mathbb{G}_m}$ has dimension one more than its virtual dimension by explicit analysis.

7.3.2. Consider the map

$$\mathsf{q} : \mathrm{LS}_P \to \mathrm{LS}_M . \tag{7.4}$$

It is quasi-smooth of virtual relative dimension

$$\dim(\mathfrak{n}(P)) \cdot (2g-2).$$

Lemma 7.3.3.
(a) *Each fiber of the map* q *has dimension* $\leq \dim(\mathfrak{n}(P)) \cdot (2g-1)$.
(b) *There exists a dense open substack of* LS_M *over which* q *is smooth.*

Let us assume this lemma for a moment and proceed with the proof of (7.3).

It follows from point (b) of the lemma that the generic fiber of q has dimension $\dim(\mathfrak{n}(P)) \cdot (2g-2)$, so the substack of LS_M over which q has fibers of larger dimension has codimension at least one. We obtain:

Corollary 7.3.4. $\dim(\mathrm{LS}_P) \leq \dim(\mathrm{LS}_M) + \dim(\mathfrak{n}(P)) \cdot (2g-1) - 1$.

7.3.5. From Corollary 7.3.4, we obtain:

$$\dim(\mathrm{LS}_P) \leq \dim(\mathfrak{m}) \cdot (2g-2) + \dim(\mathfrak{z}_M) + \dim(\mathfrak{n}(P)) \cdot (2g-1) - 1.$$

Thus it remains to show that, under the assumptions of Proposition 5.3.5,

$$\begin{aligned} \dim(\mathfrak{m}) \cdot (2g-2) + \dim(\mathfrak{z}_M) + \dim(\mathfrak{n}(P)) \cdot (2g-1) - 1 &\leq \dim(\mathfrak{g}) \cdot (2g-2) - 2 \\ &= \dim(\mathfrak{m}) \cdot (2g-2) + 2\dim(\mathfrak{n}(P)) \cdot (2g-2) - 2. \end{aligned} \tag{7.5}$$

This is equivalent to

$$\dim(\mathfrak{z}_M) + 1 \leq \dim(\mathfrak{n}(P)) \cdot (2g-3). \tag{7.6}$$

7.3.6. We now use the assumption that G is semi-simple and that the corank of P is one, so that $\dim(\mathfrak{z}_M) = 1$, i.e., (7.6) becomes

$$2 \leq \dim(\mathfrak{n}(P)) \cdot (2g-3). \tag{7.7}$$

This holds automatically if $g \geq 3$. If $g = 2$, the above inequality can only be violated if $\dim(\mathfrak{n}(P)) = 1$, but this only happens if the Dynkin diagram of G has an A_1 factor.

$\square$[Proposition 5.3.5]

Remark 7.3.7. Note that the assertion of Proposition 5.3.5 is false for $G = SL_2$ and $g = 2$: in this case the dimension of the reducible locus is 5, which is greater than

$$4 = 6 - 2 = \dim(\mathrm{LS}_G) - 2.$$

7.3.8. *Proof of Lemma 7.3.3(b).* It suffices to show that for every $\sigma_M \in \mathrm{LS}_M$, there exists a point σ'_M that lies in the same irreducible component, over which the fiber of (7.4) is smooth.

Note that for $\sigma_M \in \mathrm{LS}_M$ and

$$\sigma_P \in \mathfrak{q}^{-1}(\sigma_M) \simeq \mathrm{LS}_{N(P)_{\sigma_M}},$$

the obstruction to the smoothness of the fiber is

$$H^2(X, \mathfrak{n}(P)_{\sigma_P}).$$

The latter is non-zero if for some subquotient V of $\mathfrak{n}(P)$ as a M-representation, the local system V_{σ_M} admits a trivial quotient.

Let Z^0_M denote the neutral connected component of Z_M and consider its action on $\mathfrak{n}(P)$. It acts on every V as above by a *non-trivial* character. Hence, for a generic point $\sigma_Z \in \mathrm{LS}_{Z_M}$ and

$$\sigma'_M := \sigma_Z \otimes \sigma_M,$$

the local system $V_{\sigma'_M}$ will not have trivial quotients.

Since $\mathrm{LS}_{Z^0_M}$ is irreducible, its action on LS_M preserves irreducible components, i.e., σ' lies in the same irreducible component as σ.

□[Lemma 7.3.3(b)]

7.3.9. *Proof of Lemma 7.3.3(a).* We will use the following assertion:

Lemma 7.3.10. *Let Y be a quasi-smooth scheme of virtual dimension d. Suppose that m is an integer such that for all field-valued points $y \in Y$ we have*

$$\dim(H^{-1}(T^*_y(Y))) \leq m.$$

Then $\dim(Y) \leq d + m$.

Proof. It is enough to show that for every field-valued point $y \in Y$, the dimension of the classical cotangent space to Y at y is $\leq d + m$. However, the classical cotangent space is just $H^0(T^*_y(Y))$. We have

$$\dim(H^0(T^*_y(Y))) = \dim(H^0(T^*_y(Y)) - \dim(H^1(T^*_y(Y))) + \dim(H^1(T^*_y(Y))),$$

where $\dim(H^0(T^*_y(Y)) - \dim(H^1(T^*_y(Y)))$ is the virtual dimension of Y.

□

The assertion of the lemma automatically extends to algebraic stacks. We apply it to the fibers of the map (7.4), i.e., to the stacks

$$\mathrm{LS}_{N(P)_{\sigma_M}}, \quad \sigma_M \in \mathrm{LS}_M.$$

It remains to show that

$$\dim(H^{-1}(T^*_{\sigma_P}(\mathrm{LS}_{N(P)_{\sigma_M}})) \leq \dim(\mathfrak{n}(P)), \quad \sigma_P \in \mathrm{LS}_{N(P)_\sigma}.$$

We have:

$$T^*_{\sigma_P}(\mathrm{LS}_{N(P)_{\sigma_M}}) \simeq \mathrm{C}^\cdot(X, \mathfrak{n}(P)_{\sigma_P}[1])^\vee,$$

so

$$H^{-1}(T^*_{\sigma_P}(\mathrm{LS}_{N(P)_{\sigma_M}})) \simeq H^2(X, \mathfrak{n}(P)_{\sigma_P})^\vee,$$

which identifies with

$$H^0\left(X, (\mathfrak{n}(P))^\vee_{\sigma_P}\right)$$

by Verdier duality.

We clearly have

$$\dim(H^0(X, (\mathfrak{n}(P))^\vee_{\sigma_P}) \leq \dim(\mathfrak{n}(P)).$$

□[Lemma 7.3.3(a)]

7.4. **Proof of Proposition 4.3.10.**

7.4.1. The fact that the non-empty fibers of the map

$$\mathrm{LS}_G \to \mathrm{Bun}_G$$

are affine spaces is completely general:

For a given $\mathcal{P}_G \in \mathrm{Bun}_G$, the fiber in question is a torsor for the (derived) vector space

$$\Gamma(X, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X). \tag{7.8}$$

Warning: In the above formula ω_X stands for the canonical line bundle on X, and *not* the dualizing sheaf of X, which is the [1] shift of that. This deviates from the conventions adopted in this series, according to which for a prestack $\mathcal{Y}$, we denote by $\omega_{\mathcal{Y}}$ the dualizing sheaf on $\mathcal{Y}$. So, the curve X itself is the only exception for this convention.

7.4.2. Let us show that the map in question is smooth over the stable locus. This is equivalent to the fibers being smooth as derived schemes.

By the above torsor property, it suffices to show that if $\mathcal{P}_G \in \mathrm{Bun}_G$ is stable, then the derived vector space (7.8) is classical, i.e., that

$$H^1(X, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X) = 0.$$

By Serre duality (and using the Killing form on $\mathfrak{g}$), this is equivalent to

$$H^0(X, \mathfrak{g}_{\mathcal{P}_G}) = 0.$$

I.e., we need to show that stable bundles do not admit infinitesimal automorphisms. This is standard; we supply a proof for completeness.

7.4.3. Suppose the contrary. Let A be an infinitesimal automorphism of $\mathcal{P}_G$. First, we show that A is nilpotent.

Consider the characteristic polynomial of the $\mathcal{O}_X$-valued Higgs field A, i.e., the map

$$X \to \mathfrak{t}/\!/W =: \mathfrak{a}$$

coming from A.

This map is necessarily constant; denote its image by a.

Let $t \in \mathfrak{t}$ be a semi-simple element that maps under $\mathfrak{t} \to \mathfrak{a}$ to a. In this case, $\mathcal{P}_G$ admits a reduction to $Z_G(t)$, which is a Levi subgroup.

If a were not nilpotent, we would have $t \neq 0$, and $Z_G(t)$ is a *proper* Levi subgroup. However, the existence of such a reduction contradicts the assumption that $\mathcal{P}_G$ is stable.

7.4.4. Thus, A is (non-zero) nilpotent. The Jacobson-Morosov theory supplies a (decreasing) filtration on the vector bundle $\mathfrak{g}_{\mathcal{P}_G}$ so that

$$(\mathfrak{g}_{\mathcal{P}_G})^{\geq 1} \subset \mathfrak{g}_{\mathcal{P}_G}$$

is the unipotent radical of a parabolic reduction canonically associated to A, and the operator $(\mathrm{ad}_A)^n$ defines a map

$$\mathrm{gr}^{-n}(\mathfrak{g}_{\mathcal{P}_G}) \to \mathrm{gr}^n(\mathfrak{g}_{\mathcal{P}_G}), \tag{7.9}$$

which is an isomorphism at the generic point of X.

Since $\mathcal{P}_G$ was assumed stable, $\deg((\mathfrak{g}_{\mathcal{P}_G})^{\geq 1}) < 0$. Hence, for some $n \geq 1$, we have $\deg(\mathrm{gr}^n(\mathfrak{g}_{\mathcal{P}_G})) < 0$. However,

$$\deg(\mathrm{gr}^{-n}(\mathfrak{g}_{\mathcal{P}_G})) = -\deg(\mathrm{gr}^n(\mathfrak{g}_{\mathcal{P}_G})),$$

and this contradicts the existence of (7.9).

7.4.5. It remains show that every stable G-bundle *admits* a connection. For a general $\mathcal{P}_G \in \mathrm{Bun}_G$, the obstruction to having a connection is given by its *Atiyah class*, which is an element of

$$H^1(X, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X).$$

However, we have just proved that this group is zero for stable $\mathcal{P}_G$.

$\square$[Proposition 4.3.10]

Remark 7.4.6. The above argument can be refined to prove the following criterion (originally due to A. Weil) for a G-bundle $\mathcal{P}_G$ to admit a connection:

This happens if and only if, for every reduction of $\mathcal{P}_G$ to a *Levi subgroup* M, this reduction, viewed as an M-bundle, has degree 0. See [AB] for more details.

8. 2-Fourier-Mukai transform of the automorphic category

The goal of this section is to prove Theorems 5.1.5 and 5.1.7. We will do so by considering a more general picture that involves twisting the constant group-scheme with fiber G by Z_G-gerbes.

8.1. **The notion of 2-Fourier-Mukai transform.** Recall that the usual Fourier-Mukai transform is a functor between categories of quasi-coherent sheaves on a pair of prestacks.

In this subsection we introduce the notion of *2-Fourier-Mukai transform*, which is a (2-)functor between 2-categories of *sheaves of categories* on a pair of prestacks.

8.1.1. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be a pair of prestacks equipped with a map

$$\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathrm{pt}/\mathbb{G}_m, \tag{8.1}$$

i.e., a line bundle, denoted $\mathcal{L}_{1,2}$ on $\mathcal{Y}_1 \times \mathcal{Y}_2$.

Assume that the functor $(p_2)_* : \mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_2)$ is continuous (this happens, e.g., when $\mathcal{Y}_1$ is quasi-compact with an affine diagonal).

Consider the functor

$$\mathrm{FM}_{\mathcal{Y}_1 \to \mathcal{Y}_2} : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2), \quad \mathcal{F} \mapsto (p_2)_*(\mathcal{L}_{1,2} \otimes p_1^*(\mathcal{F})). \tag{8.2}$$

We shall say that the map (8.1) is of Fourier-Mukai type if the functor (8.2) is an equivalence.

8.1.2. Let us now be given a map

$$\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathrm{Ge}_{\mathbb{G}_m}(\mathrm{pt}), \tag{8.3}$$

where $\mathrm{Ge}_{\mathbb{G}_m}(\mathrm{pt}) = B^2(\mathbb{G}_m)_{\mathrm{et}}$ is the (2-algebraic) stack classifying $\mathbb{G}_m$-gerbes. Let $\mathcal{G}_{1,2}$ denote the corresponding gerbe on $\mathcal{Y}_1 \times \mathcal{Y}_2$.

Recall the notion of a *sheaf of categories*, see [Ga3, Sect. 1.1]. Consider the 2-functor

$$2\text{-}\mathrm{FM}_{\mathcal{Y}_1 \to \mathcal{Y}_2} : \mathrm{ShvCat}(\mathcal{Y}_1) \to \mathrm{ShvCat}(\mathcal{Y}_2), \quad \mathbf{C} \mapsto (p_2)_*((p_1^*(\mathbf{C})_{\mathcal{G}_{1,2}}), \tag{8.4}$$

where:

- For a morphism $f : \mathcal{Y}' \to \mathcal{Y}''$ between prestacks, f^* denotes the pullback functor
$$\mathrm{ShvCat}(\mathcal{Y}'') \to \mathrm{ShvCat}(\mathcal{Y}'),$$
see [Ga3, Sect. 3.1.2] (in *loc. cit.* it is denoted $\mathbf{cores}_f$);
- For a morphism $f_* : \mathcal{Y}' \to \mathcal{Y}''$ between prestacks, f_* denotes the pushforward functor
$$\mathrm{ShvCat}(\mathcal{Y}') \to \mathrm{ShvCat}(\mathcal{Y}''),$$
see [Ga3, Sect. 3.1.3] (in *loc. cit.* it is denoted $\mathbf{ind}_f$);
- For a prestack $\mathcal{Y}$, and $\mathbf{C} \in \mathrm{ShvCat}(\mathcal{Y})$ and a $\mathbb{G}_m$-gerbe $\mathcal{G}$ on $\mathcal{Y}$, we denote by $\mathbf{C}_{\mathcal{G}}$ the twist of $\mathbf{C}$ by $\mathcal{G}$, see [GLys, Sect. 1.7.2].

We shall say that the map (8.3) is of 2-Fourier-Mukai type if the functor (8.4) is an equivalence.

Note that the notion of being of 2-Fourier-Mukai type is in general asymmetric.

8.1.3. *Example.* Let Γ be a finite abelian group, and let $\Gamma^\vee$ be its Cartier dual. Take

$$\mathcal{Y}_1 = B^2(\Gamma)_{\mathrm{et}} =: \mathrm{Ge}_\Gamma(\mathrm{pt})$$

and

$$\mathcal{Y}_2 = \Gamma^\vee.$$

Then $\mathrm{ShvCat}(\mathcal{Y}_1)$ is the 2-category of DG categories acted on by pt/Γ. In other words, these are categories $\mathbf{C}$ equipped with an action of Γ on $\mathrm{Id}_{\mathbf{C}}$. Decomposing with respect to the characters of Γ, we obtain that a datum of such $\mathbf{C}$ is equivalent to the datum of a category graded by $\Gamma^\vee$

$$\mathbf{C} \mapsto \{\mathbf{C}_\chi,\, \chi \in \Gamma^\vee\}. \tag{8.5}$$

Evaluation defines a map

$$\mathrm{Ge}_\Gamma(\mathrm{pt}) \times \Gamma^\vee \to \mathrm{Ge}_{\mathbb{G}_m}(\mathrm{pt}). \tag{8.6}$$

We claim that (8.6) is of 2-Fourier-Mukai type.

Indeed, unwinding the definitions, we obtain that the functor $2\text{-}\mathrm{FM}_{\mathrm{Ge}_\Gamma \to \Gamma^\vee}$ is given exactly by (8.5), and hence is an equivalence.

8.1.4. Swapping the factors in (8.6) we obtain a pairing

$$\Gamma^\vee \times \mathrm{Ge}_\Gamma(\mathrm{pt}) \to \mathrm{Ge}_{\mathbb{G}_m}(\mathrm{pt}), \tag{8.7}$$

and it is easy to see that it is also of 2-Fourier-Mukai type.

Indeed, the corresponding functor $2\text{-}\mathrm{FM}_{\Gamma^\vee \to \mathrm{Ge}_\Gamma(\mathrm{pt})}$ is the inverse of $2\text{-}\mathrm{FM}_{\mathrm{Ge}_\Gamma(\mathrm{pt}) \to \Gamma^\vee}$ up to the inversion on Γ.

Remark 8.1.5. The central players in the paper [Ga3] are prestacks that are *1-affine*, i.e., those for each the functor of *enhanced* global sections

$$\mathrm{ShvCat}(\mathcal{Y}) \overset{\Gamma^{\mathrm{enh}}(\mathcal{Y},-)}{\longrightarrow} \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod} \tag{8.8}$$

is an equivalence.

Note that the prestack $\mathrm{Ge}_\Gamma(\mathrm{pt})$ is *not* 1-affine. Namely $\mathrm{QCoh}(\mathrm{Ge}_\Gamma(\mathrm{pt})) \simeq \mathrm{Vect}$, and the functor (8.8) sends $\mathbf{C}$ as above to $\mathbf{C}_0$, i.e., the fiber of $2\text{-}\mathrm{FM}_{\mathrm{Ge}_\Gamma(\mathrm{pt}) \to \Gamma^\vee}(\mathbf{C})$ at the point $0 \in \Gamma^\vee$.

8.1.6. *Example.* For Γ as above, take

$$\mathcal{Y}_1 := \mathrm{pt}/\Gamma \text{ and } \mathcal{Y}_2 := \mathrm{pt}/\Gamma^\vee.$$

Cup product defines a map

$$\mathrm{pt}/\Gamma \times \mathrm{pt}/\Gamma^\vee \to \mathrm{Ge}_{\mathbb{G}_m}(\mathrm{pt}). \tag{8.9}$$

We claim that (8.9) is of 2-Fourier-Mukai type.

Note that $\mathrm{ShvCat}(\mathrm{pt}/\Gamma)$ (resp., $\mathrm{ShvCat}(\mathrm{pt}/\Gamma^\vee)$) identifies with the 2-category of DG categories equipped with an action of $\mathrm{QCoh}(\Gamma)$ (resp., $\mathrm{QCoh}(\Gamma^\vee)$), viewed as a monoidal category with respect to *convolution*. Note also that pt/Γ is 1-affine, and

$$\mathrm{QCoh}(\mathrm{pt}/\Gamma) \simeq \mathrm{Rep}(\Gamma).$$

Unwinding the definitions, we obtain that the functor $2\text{-}\mathrm{FM}_{\mathrm{pt}/\Gamma \to \mathrm{pt}/\Gamma^\vee}$ identifies with

$$\mathrm{ShvCat}(\mathrm{pt}/\Gamma) \overset{\Gamma^{\mathrm{enh}}(\mathrm{pt}/\Gamma,-)}{\longrightarrow} \mathrm{Rep}(\Gamma)\text{-}\mathbf{mod} \simeq \mathrm{QCoh}(\Gamma^\vee)\text{-}\mathbf{mod},$$

where we identify

$$\mathrm{Rep}(\Gamma) \simeq \mathrm{QCoh}(\Gamma^\vee)$$

by Fourier transform.

Note also that the composition

$$2\text{-}\mathrm{FM}_{\mathrm{pt}/\Gamma^\vee \to \mathrm{pt}/\Gamma} \circ 2\text{-}\mathrm{FM}_{\mathrm{pt}/\Gamma \to \mathrm{pt}/\Gamma^\vee}$$

is the identity endofunctor of $\mathrm{ShvCat}(\mathrm{pt}/\Gamma)$ up to the inversion involution on Γ.

8.2. **2-Fourier-Mukai transform and Poincaré duality.** In this subsection we consider a particular pair of prestacks that are 2-Fourier-Mukai dual to each other.

Both sides have to do with gerbes for a finite abelian group on a smooth and complete curve X.

8.2.1. Let Γ be a finite abelian group as above. Take

$$\mathcal{Y}_1 := \mathrm{Ge}_\Gamma(X) \text{ and } \mathcal{Y}_2 := \mathrm{Ge}_{\Gamma^\vee(1)}(X),$$

where $(-)(1)$ denotes the Tate twist, so that

$$\Gamma^\vee \simeq \mathrm{Hom}(\Gamma, \mathbb{Z}/n\mathbb{Z})(1)$$

for $n \cdot \Gamma = 0$.

Verdier duality defines a pairing

$$B^2(\mathrm{C}^\cdot(X, \Gamma)) \times B^2(\mathrm{C}^\cdot(X, \Gamma^\vee(1))) \to B^2(\mu_\infty).$$

Applying étale sheafification, we obtain a pairing

$$\mathrm{Ge}_\Gamma(X) \times \mathrm{Ge}_{\Gamma^\vee(1)}(X) \to \mathrm{Ge}_{\mathbb{G}_m}(\mathrm{pt}). \tag{8.10}$$

8.2.2. We claim:

Theorem 8.2.3.
(a) *The pairing* (8.10) *is of 2-Fourier-Mukai type.*
(b) *The composition*

$$\text{2-FM}_{\mathrm{Ge}_{\Gamma^\vee(1)}(X) \to \mathrm{Ge}_\Gamma(X)} \circ \text{2-FM}_{\mathrm{Ge}_\Gamma(X) \to \mathrm{Ge}_{\Gamma^\vee(1)}(X)}$$

is the involution of $\mathrm{ShvCat}(\mathrm{Ge}_\Gamma(X))$ *coming from the inversion on* Γ,

$$(\Gamma^\vee(1))^\vee(1) \simeq \Gamma.$$

8.2.4. *Proof of Theorem 8.2.3.* Choose a point $x \in X$. Then we can split

$$\mathrm{Ge}_\Gamma(X) \simeq \mathrm{Ge}_\Gamma(\mathrm{pt}) \times \mathrm{pt}/H^1_{\mathrm{et}}(X, \Gamma) \times H^2_{\mathrm{et}}(X, \Gamma)$$

and

$$\mathrm{Ge}_{\Gamma^\vee(1)}(X) \simeq \mathrm{Ge}_{\Gamma^\vee(1)}(\mathrm{pt}) \times \mathrm{pt}/H^1_{\mathrm{et}}(X, \Gamma^\vee(1)) \times H^2_{\mathrm{et}}(X, \Gamma^\vee(1)).$$

The pairing (8.10) splits as a product of:

- The pairing (8.6), where we identify $H^2(X, \Gamma^\vee(1)) \simeq \Gamma^\vee$;
- The pairing (8.6) with the two sides swapped, where we identify $H^2_{\mathrm{et}}(X, \Gamma) \simeq (\Gamma^\vee(1))^\vee$;
- The pairing (8.9), where we identify $H^1_{\mathrm{et}}(X, \Gamma^\vee(1)) \simeq H^1_{\mathrm{et}}(X, \Gamma)^\vee$.

Now the assertion of the theorem follows by combining the examples from Sects. 8.1.3, 8.1.4 and 8.1.6.

$\square$[Theorem 8.2.3]

8.2.5. For a $\Gamma^\vee(1)$-gerbe $\mathfrak{G}_{\Gamma^\vee(1)}$ on X, let $\mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}$ be the $\mathbb{G}_m$-gerbe on $\mathrm{Ge}_\Gamma(X)$, corresponding to the restriction of the map (8.10) along

$$\mathrm{Ge}_\Gamma(X) \times \{\mathfrak{G}_{\Gamma^\vee(1)}\} \to \mathrm{Ge}_\Gamma(X) \times \mathrm{Ge}_{\Gamma^\vee(1)}(X).$$

We obtain that for an object

$$\underline{\mathbf{C}}_\Gamma \in \mathrm{ShvCat}(\mathrm{Ge}_\Gamma(X))$$

and the corresponding object

$$\underline{\mathbf{C}}_{\Gamma^\vee(1)} := \text{2-FM}_{\mathrm{Ge}_\Gamma(X) \to \mathrm{Ge}_{\Gamma^\vee(1)}(X)}(\underline{\mathbf{C}}_\Gamma) \in \mathrm{ShvCat}(\mathrm{Ge}_{\Gamma^\vee(1)}(X)),$$

we have

$$\underline{\mathbf{C}}_{\Gamma^\vee(1)}|_{\mathfrak{G}_{\Gamma^\vee(1)}} \simeq \Gamma\left(\mathrm{Ge}_\Gamma(X), (\underline{\mathbf{C}}_\Gamma)_{\mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}}\right), \tag{8.11}$$

where:

- $(-)|_{\mathfrak{G}_{\Gamma^\vee(1)}}$ denotes the fiber of a given sheaf of categories at the point $\mathfrak{G}_{\Gamma^\vee(1)} \in \mathrm{Ge}_{\Gamma^\vee(1)}(X)$;

- $(-)_{\mathcal{G}}$ denotes the twist of a given sheaf of categories over some prestack by a $\mathbb{G}_m$-gerbe $\mathcal{G}$ on that prestack.

By the involutivity assertion in Theorem 8.2.3, we also obtain that for a Γ-gerbe $\mathfrak{G}_\Gamma$ on X, and the corresponding $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathfrak{G}_\Gamma}$ on $\mathrm{Ge}_{\Gamma^\vee(1)}(X)$, we have

$$\underline{\mathbf{C}}_\Gamma|_{\mathfrak{G}_\Gamma^{-1}} \simeq \Gamma\left(\Gamma^\vee(1), (\underline{\mathbf{C}}_{\Gamma^\vee(1)})_{\mathcal{G}_{\mathfrak{G}_\Gamma}}\right). \tag{8.12}$$

8.2.6. For $\underline{\mathbf{C}}_\Gamma$ as above, denote

$$\mathbf{C}_\Gamma := \Gamma(\mathrm{Ge}_\Gamma(X), \underline{\mathbf{C}}_\Gamma) \text{ and } \mathbf{C}_{\Gamma^\vee(1)} := \Gamma(\mathrm{Ge}_{\Gamma^\vee(1)}(X), \underline{\mathbf{C}}_{\Gamma^\vee(1)}).$$

Let $\mathfrak{G}^0_\Gamma$ (resp., $\mathfrak{G}^0_{\Gamma^\vee(1)}$) denote the trivial Γ-gerbe (resp., $\Gamma^\vee(1)$)-gerbe on X. As a particular case of (8.11), we obtain an equivalence

$$\underline{\mathbf{C}}_{\Gamma^\vee(1)}|_{\mathfrak{G}^0_{\Gamma^\vee(1)}} \simeq \mathbf{C}_\Gamma, \tag{8.13}$$

and as a particular case of (8.12) we obtain an equivalence

$$\underline{\mathbf{C}}_\Gamma|_{\mathfrak{G}^0_\Gamma} \simeq \mathbf{C}_{\Gamma^\vee(1)}. \tag{8.14}$$

8.2.7. Note that $\mathbf{C}_\Gamma$ is a category acted on by $\mathrm{QCoh}(\mathrm{Ge}_\Gamma(X))$. For

$$\alpha \in \Gamma(-1) \simeq H^2_{\mathrm{et}}(X, \Gamma) \simeq \pi_0(\mathrm{Ge}_\Gamma(X))),$$

consider the corresponding idempotent

$$\mathcal{O}_{\mathrm{Ge}_\Gamma(X),\alpha} \in \mathrm{QCoh}(\mathrm{Ge}_\Gamma(X))$$

as acting on $\mathbf{C}_\Gamma$.

Note also that $\Gamma^\vee(1)$ acts by automorphisms of the identity functor on $\mathfrak{G}^0_{\Gamma^\vee(1)}$, and hence also by the automorphisms of the identity functor of $\underline{\mathbf{C}}_{\Gamma^\vee(1)}|_{\mathfrak{G}^0_{\Gamma^\vee(1)}}$. For

$$\alpha \in \Gamma(-1) \simeq (\Gamma^\vee(1))^\vee$$

let P_α denote the corresponding idempotent on $\underline{\mathbf{C}}_{\Gamma^\vee(1)}|_{\mathfrak{G}^0_{\Gamma^\vee(1)}}$.

Unwinding the construction, we obtain:

Lemma 8.2.8. *Under the identification* (8.13)*, the action of* $\mathcal{O}_{\mathrm{Ge}_\Gamma(X),\alpha}$ *on* $\mathbf{C}_\Gamma$ *corresponds to the action of* P_α *on* $\underline{\mathbf{C}}_{\Gamma^\vee(1)}|_{\mathfrak{G}^0_{\Gamma^\vee(1)}}$.

8.2.9. As in Sect. 4.2.4, a point

$$\mathcal{P}_{\Gamma^\vee(1)} \in \mathrm{Bun}_{\Gamma^\vee(1)}$$

gives rise to a line bundle denoted $\mathcal{L}_{\mathcal{P}_{\Gamma^\vee(1)}}$ on $\mathrm{Ge}_\Gamma(X)$.

In particular, we consider the endofunctor

$$\mathcal{L}_{\mathcal{P}_{\Gamma^\vee(1)}} \otimes (-)$$

of $\mathbf{C}_\Gamma$.

We can view $\mathcal{P}_{\Gamma^\vee(1)}$ itself as an automorphism of $\mathfrak{G}^0_{\Gamma^\vee(1)}$. And as such, it induces an autoequivalence of $\underline{\mathbf{C}}_{\Gamma^\vee(1)}|_{\mathfrak{G}^0_{\Gamma^\vee(1)}}$.

Unwinding the construction, we obtain:

Lemma 8.2.10. *Under the identification* (8.13)*, the action of* $\mathcal{L}_{\mathcal{P}_{\Gamma^\vee(1)}}$ *on* $\mathbf{C}_\Gamma$ *corresponds to the action of* $\mathcal{P}_{\Gamma^\vee(1)}$ *on* $\underline{\mathbf{C}}_{\Gamma^\vee(1)}|_{\mathfrak{G}^0_{\Gamma^\vee(1)}}$.

8.3. **Some further compatibilities.** In this subsection we review some further properties of the equivalence of Theorem 8.2.3 that will be needed for the proof of Theorem 8.4.8 below.

This subsection could be skipped on the first pass and returned to when necessary.

8.3.1. Note that for any $\mathfrak{G}_{\Gamma^\vee(1)} \in \mathrm{Ge}_{\Gamma^\vee(1)}(X)$ we have the evaluation functor

$$\mathrm{ev}\,|_{\mathfrak{G}_{\Gamma^\vee(1)}} : \mathbf{C}_{\Gamma^\vee(1)} \to (\underline{\mathbf{C}}_{\Gamma^\vee(1)})|_{\mathfrak{G}_{\Gamma^\vee(1)}}.$$

More generally, for a $\mathbb{G}_m$-gerbe $\mathcal{G}$ on $\mathrm{Ge}_{\Gamma^\vee(1)}(X)$, we have the evaluation functor

$$\mathrm{ev}_{\mathcal{G}}\,|_{\mathfrak{G}_{\Gamma^\vee(1)}} : \Gamma\left(\mathrm{Ge}_{\Gamma^\vee(1)}(X), (\underline{\mathbf{C}}_{\Gamma^\vee(1)})_{\mathcal{G}}\right) \to (\underline{\mathbf{C}}_{\Gamma^\vee(1)})_{\mathcal{G}}|_{\mathfrak{G}_{\Gamma^\vee(1)}}.$$

8.3.2. For a given $\mathfrak{G}_{\Gamma^\vee(1)} \in \mathrm{Ge}_{\Gamma^\vee(1)}(X)$, let

$$(\mathrm{ev}_{\mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}}\,|_{\mathfrak{G}^0_\Gamma})^L : \underline{\mathbf{C}}_\Gamma|_{\mathfrak{G}^0_\Gamma} \to \Gamma\left(\mathrm{Ge}_\Gamma(X), (\underline{\mathbf{C}}_\Gamma)_{\mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}}\right) \tag{8.15}$$

be the left adjoint of the evaluation functor

$$\Gamma\left(\mathrm{Ge}_\Gamma(X), (\underline{\mathbf{C}}_\Gamma)_{\mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}}\right) \overset{\mathrm{ev}_{\mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}}|_{\mathfrak{G}^0_\Gamma}}{\longrightarrow} (\underline{\mathbf{C}}_\Gamma)_{\mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}}|_{\mathfrak{G}^0_\Gamma} \simeq \underline{\mathbf{C}}_\Gamma|_{\mathfrak{G}^0_\Gamma}.$$

8.3.3. Unwinding the constructions we obtain:

Lemma 8.3.4. *The following diagram commutes:*

$$\begin{tikzcd}
\mathbf{C}_{\Gamma^\vee(1)} \arrow[r, "\mathrm{ev}\,|_{\mathfrak{G}_{\Gamma^\vee(1)}}"] & (\underline{\mathbf{C}}_{\Gamma^\vee(1)})|_{\mathfrak{G}_{\Gamma^\vee(1)}} \\
\underline{\mathbf{C}}_\Gamma|_{\mathfrak{G}^0_\Gamma} \arrow[u, "(8.14)", "\sim"'] \arrow[r, "(8.15)"] & \Gamma(\mathrm{Ge}_\Gamma(X), (\underline{\mathbf{C}}_\Gamma)_{\mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}}). \arrow[u, "\sim", "(8.11)"']
\end{tikzcd}$$

8.4. **Two sheaves associated with automorphic category.** Let G be a semi-simple group and consider the category

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

We will upgrade it to two objects

$$\underline{\mathrm{D\text{-}mod}}^{Z_G}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}}) \in \mathrm{ShvCat}(\mathrm{Ge}_{Z_G}(X)) \text{ and } \underline{\mathrm{D\text{-}mod}}^{\pi_1(\check{G})}_{\frac{1}{2}}(\mathrm{Bun}_G) \in \mathrm{ShvCat}(\mathrm{Ge}_{\pi_1(G)}(X)).$$

We will state Theorem 8.4.8 that says that the above two objects correspond to one another under the 2-Fourier-Mukai transform. The nature of the two constructions will then immediately yield Theorems 5.1.5 and 5.1.7.

8.4.1. The short exact sequence of groups

$$1 \to Z_G \to G \to G_{\mathrm{ad}} \to 1$$

gives rise to a map

$$\mathsf{p}_{Z_G} : \mathrm{Bun}_{G_{\mathrm{ad}}} \to \mathrm{Ge}_{Z_G}(X). \tag{8.16}$$

Consider the induced map

$$(\mathsf{p}_{Z_G})_{\mathrm{dR}} : (\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathrm{dR}} \to (\mathrm{Ge}_{Z_G}(X))_{\mathrm{dR}}.$$

8.4.2. Note, however, that (by nil-invariance of étale cohomology) for a finite abelian group Γ, the map of prestacks

$$\mathrm{Ge}_\Gamma(X) \to (\mathrm{Ge}_\Gamma(X))_{\mathrm{dR}}$$

is an isomorphism.

Hence, we can regard $(\mathsf{p}_{Z_G})_{\mathrm{dR}}$ as a map

$$(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathrm{dR}} \to \mathrm{Ge}_{Z_G}(X). \tag{8.17}$$

8.4.3. The category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}})$ is (tautologically) the category of global sections of a sheaf of categories, denoted

$$\underline{\text{D-mod}}_{\frac{1}{2}}^{G_{\text{ad}}}(\text{Bun}_{G_{\text{ad}}})$$

over $(\text{Bun}_{G_{\text{ad}}})_{\text{dR}}$.

Set

$$\underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\text{Bun}_{G_{\text{ad}}}) := ((\mathsf{p}_{Z_G})_{\text{dR}})_*(\underline{\text{D-mod}}_{\frac{1}{2}}^{G_{\text{ad}}}(\text{Bun}_{G_{\text{ad}}})) \in \text{ShvCat}(\text{Ge}_{Z_G}(X)).$$

8.4.4. Tautologically, we have

$$\Gamma\left(\text{Ge}_{Z_G}(X), \underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\text{Bun}_{G_{\text{ad}}})\right) \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}}). \tag{8.18}$$

In addition,

$$\left(\underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\text{Bun}_{G_{\text{ad}}})\right)|_{\mathfrak{G}^0_{Z_G}} \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G), \tag{8.19}$$

see the notations in Sect. 8.2.5.

More generally, for a given $\mathfrak{G}_{Z_G} \in \text{Ge}_{Z_G}(X)$, we have

$$\left(\underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\text{Bun}_{G_{\text{ad}}})\right)|_{\mathfrak{G}_{Z_G}} \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G,\mathfrak{G}_{Z_G}}), \tag{8.20}$$

where

$$\text{Bun}_{G,\mathfrak{G}_{Z_G}} := \text{Bun}_G \underset{\text{Ge}_{Z_G}(X)}{\times} \{\mathfrak{G}_{Z_G}\}.$$

Remark 8.4.5. We can think $\text{Bun}_{G,\mathfrak{G}_{Z_G}}$ as follows: pick a G_{ad}-torsor $\mathcal{P}_{G_{\text{ad}}}$ that maps to $\mathfrak{G}_{Z_G}$, and let

$$G_{\mathcal{P}_{G_{\text{ad}}}}$$

be the corresponding (non-pure) inner form of the constant group-scheme with fiber G.

Then

$$\text{Bun}_{G,\mathfrak{G}_{Z_G}} \simeq \text{Bun}_{G_{\mathcal{P}_{G_{\text{ad}}}}}.$$

(Note that different choices for $\mathcal{P}_{G_{\text{ad}}}$ differ by G-torsors, and hence the corresponding moduli spaces $\text{Bun}_{G_{\mathcal{P}_{G_{\text{ad}}}}}$ are a priori canonically isomorphic.)

8.4.6. We now consider $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ as equipped with the spectral action of $\text{QCoh}(\text{LS}_{\check{G}})$. Since the stack $\text{LS}_{\check{G}}$ is 1-affine, we can canonically attach to $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ an object

$$\underline{\text{D-mod}}_{\frac{1}{2}}^{\check{G}}(\text{Bun}_G) \in \text{ShvCat}(\text{LS}_{\check{G}}).$$

The short exact sequence of groups[9]

$$1 \to \pi_1(\check{G}) \to \check{G}_{\text{sc}} \to \check{G} \to 1$$

give rise to a map

$$\mathsf{p}_{\pi_1(\check{G})} : \text{LS}_{\check{G}} \to \text{Ge}_{\pi_1(\check{G})}(X). \tag{8.21}$$

Denote

$$\underline{\text{D-mod}}_{\frac{1}{2}}^{\pi_1(\check{G})}(\text{Bun}_G) := (\mathsf{p}_{\pi_1(\check{G})})_*(\underline{\text{D-mod}}_{\frac{1}{2}}^{\check{G}}(\text{Bun}_G)) \in \text{ShvCat}(\text{Ge}_{\pi_1(\check{G})}(X)).$$

Note that tautologically,

$$\Gamma\left(\text{Ge}_{\pi_1(\check{G})}(X), \underline{\text{D-mod}}_{\frac{1}{2}}^{\pi_1(\check{G})}(\text{Bun}_G)\right) \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G). \tag{8.22}$$

[9]Note that $\check{G}_{\text{sc}}$ is the Langlands dual of G_{ad}.

8.4.7. We will prove:

Theorem 8.4.8. *Under the identification $\pi_1(\check{G}) \simeq (Z_G)^\vee(1)$, we have*

$$\text{2-FM}_{\text{Ge}_{Z_G}(X)\to\text{Ge}_{\pi_1(\check{G})}(X)}\left(\underline{\text{D-mod}}^{Z_G}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}})\right) \simeq \underline{\text{D-mod}}^{\pi_1(\check{G})}_{\frac{1}{2}}(\text{Bun}_G).$$

Combining Theorem 8.4.8 with the involutivity assertion in Theorem 8.2.3, we obtain:

Corollary 8.4.9. *Under the identification $Z_G \simeq (\pi_1(\check{G})^\vee)(1)$, we have:*

$$\text{2-FM}_{\text{Ge}_{\pi_1(\check{G})}(X)\to\text{Ge}_{Z_G}(X)}\left(\underline{\text{D-mod}}^{\pi_1(\check{G})}_{\frac{1}{2}}(\text{Bun}_G)\right) \simeq \underline{\text{D-mod}}^{Z_G}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}}),$$

up to the inversion involution on Z_G.

8.4.10. Let us show how Corollary 8.4.9 implies Theorems 5.1.5 and 5.1.7.

Indeed, the two theorems follow immediately from Lemma 8.2.8 and 8.2.10, respectively.

□[Theorems 5.1.5 and 5.1.7]

Remark 8.4.11. Note that, in view of Remark 8.4.5, from Corollary 8.4.9 we obtain an expression of the twisted categories $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G,\mathfrak{G}_{Z_G}})$ in terms of the usual $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ and the spectral action.

Namely, using (8.11) and the fact that $\text{LS}_{\check{G}}$ is 1-affine, we obtain:

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G,\mathfrak{G}^{-1}_{Z_G}}) \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}})_{\mathcal{G}_{\mathfrak{G}_{Z_G}}}, \tag{8.23}$$

where $\text{QCoh}(\text{LS}_{\check{G}})_{\mathcal{G}_{\mathfrak{G}_{Z_G}}}$ is the twist of $\text{QCoh}(\text{LS}_{\check{G}})$ by the pullback of the gerbe $\mathcal{G}_{\mathfrak{G}_{Z_G}}$ on $\text{Ge}_{\pi_1(\check{G})}(X)$ along the map $\mathsf{p}_{\pi_1(\check{G})}$ of (8.21).

8.5. Proof of Theorem 8.4.8.

8.5.1. We start by constructing a functor

$$\underline{\text{D-mod}}^{\pi_1(\check{G})}_{\frac{1}{2}}(\text{Bun}_G) \to \text{2-FM}_{\text{Ge}_{Z_G}(X)\to\text{Ge}_{\pi_1(\check{G})}(X)}\left(\underline{\text{D-mod}}^{Z_G}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}})\right) \tag{8.24}$$

Since $\text{Ge}_{\pi_1(\check{G})}(X)$ is algebro-geometrically discrete, the datum of (8.24) consists of the data of functors

$$\underline{\text{D-mod}}^{\pi_1(\check{G})}_{\frac{1}{2}}(\text{Bun}_G)|_{\mathfrak{G}_{\pi_1(\check{G})}} \to \text{2-FM}_{\text{Ge}_{Z_G}(X)\to\text{Ge}_{\pi_1(\check{G})}(X)}\left(\underline{\text{D-mod}}^{Z_G}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}})\right)|_{\mathfrak{G}_{\pi_1(\check{G})}} \tag{8.25}$$

that depend functorially on $\mathfrak{G}_{\pi_1(\check{G})} \in \text{Ge}_{\pi_1(\check{G})}(X)$.

Applying (8.11), the datum of (8.25) is equivalent to that of a functor

$$\underline{\text{D-mod}}^{\pi_1(\check{G})}_{\frac{1}{2}}(\text{Bun}_G)|_{\mathfrak{G}_{\pi_1(\check{G})}} \to \Gamma\left(\text{Ge}_{Z_G}(X), \underline{\text{D-mod}}^{Z_G}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}\right). \tag{8.26}$$

We rewrite the right-hand side in (8.26) as

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}, \tag{8.27}$$

where by a slight abuse of notation we regard $\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}$ as a $\mathbb{G}_m$-gerbe on $(\text{Bun}_{G_{\text{ad}}})_{\text{dR}}$.

8.5.2. We rewrite the left-hand side in (8.26) as

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}_{\text{sc}},\mathfrak{G}_{\pi_1(\check{G})}}), \tag{8.28}$$

where

$$\text{LS}_{\check{G}_{\text{sc}},\mathfrak{G}_{\pi_1(\check{G})}} := \text{LS}_{\check{G}} \underset{\text{Ge}_{\pi_1(\check{G})}(X)}{\times} \{\mathfrak{G}_{\pi_1(\check{G})}\}.$$

The notation $\text{LS}_{\check{G}_{\text{sc}},\mathfrak{G}_{\pi_1(\check{G})}}$ expresses the fact that this stack is a twisted form of $\text{LS}_{\check{G}_{\text{sc}}}$. Namely, for $\mathfrak{G}_{\pi_1(\check{G})} = \mathfrak{G}^0_{\pi_1(\check{G})}$ we have

$$\text{LS}_{\check{G}_{\text{sc}},\mathfrak{G}^0_{\pi_1(\check{G})}} = \text{LS}_{\check{G}_{\text{sc}}}.$$

8.5.3. Let $\operatorname{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran},\mathfrak{G}_{\pi_1(\check{G})}}$ be the $\mathfrak{G}_{\pi_1(\check{G})}$-twist of $\operatorname{Rep}(\check{G}_{\mathrm{sc}})$, i.e., this is the factorization category that associates to a point $\underline{x} \in \mathrm{Ran}$ the category

$$\operatorname{Rep}(\check{G}_{\mathrm{sc}})_{\underline{x},\mathfrak{G}_{\pi_1(\check{G})}} := \operatorname{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}(\mathcal{D}_{\underline{x}})),$$

where:

- $\mathcal{D}_{\underline{x}}$ is the formal disc around $\underline{x}$;
- $\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}(\mathcal{D}_{\underline{x}}) := \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}) \underset{\mathrm{Ge}_{\pi_1(\check{G})}(\mathcal{D}_{\underline{x}})}{\times} \{\mathfrak{G}_{\pi_1(\check{G})}|_{\mathcal{D}_{\underline{x}}}\}$.

8.5.4. Recall now that we have the (symmetric) monoidal localization functors

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \operatorname{Rep}(\check{G})_{\mathrm{Ran}} \to \operatorname{QCoh}(\mathrm{LS}_{\check{G}}) \text{ and } \mathrm{Loc}^{\mathrm{spec}}_{\check{G}_{\mathrm{sc}}} : \operatorname{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran}} \to \operatorname{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}}}).$$

We have the corresponding twisted version:

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}} : \operatorname{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran},\mathfrak{G}_{\pi_1(\check{G})}} \to \operatorname{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}),$$

and as in the case of $\mathrm{Loc}^{\mathrm{spec}}_{\check{G}_{\mathrm{sc}}}$, one shows that the functor $\mathrm{Loc}^{\mathrm{spec}}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}$ is a *localization*, i.e., its right adjoint is fully faithful. See [GLC4, Prop. C.1.7] for a general result encompassing these statements.

8.5.5. Let ϕ denote the tautological map $G \to G_{\mathrm{ad}}$, and also the map

$$\mathrm{Bun}_G \to \mathrm{Bun}_{G_{\mathrm{ad}}}.$$

Note that the pullback along ϕ of $\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}$, viewed as a gerbe on $(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathrm{dR}}$ canonically trivializes. Hence, we obtain a functor

$$\phi_! : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}. \tag{8.29}$$

8.5.6. We will construct the following gerbe-twisted version of the Hecke action:

Proposition 8.5.7. *There is a canonically defined action of the monoidal category* $\operatorname{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran},\mathfrak{G}_{\pi_1(\check{G})}}$ *on* $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}$. *Moreover, the following properties hold:*

(a) *The functor $\phi_!$ of* (8.29) *is equivariant with respect to the* $\operatorname{Rep}(\check{G})_{\mathrm{Ran}}$*-action on* $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ *via the tautological functor*

$$\operatorname{Rep}(\check{G})_{\mathrm{Ran}} \to \operatorname{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran},\mathfrak{G}_{\pi_1(\check{G})}}.$$

(b) *The resulting* $\operatorname{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran},\mathfrak{G}_{\pi_1(\check{G})}}$*-action on* $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}$ *factors canonically via the localization functor*

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}} : \operatorname{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran},\mathfrak{G}_{\pi_1(\check{G})}} \to \operatorname{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}).$$

The proposition will be proved in Sects. 8.6 and 8.7.

8.5.8. We now return to the sought-for functor (8.26), thought of as a functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \underset{\operatorname{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \operatorname{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}) \to (\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}. \tag{8.30}$$

Its construction follows from Proposition 8.5.7 by considering the adjoint pair

$$\operatorname{QCoh}(\mathrm{LS}_{\check{G}})\textbf{-mod} \rightleftarrows \operatorname{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}),$$

where the right adjoint is the forgetful functor, and the left adjoint is the tensoring up functor with respect to

$$\operatorname{QCoh}(\mathrm{LS}_{\check{G}}) \to \operatorname{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}).$$

8.5.9. The next assertion results from Lemma 8.3.4:

Lemma 8.5.10. *The map*

$$\Gamma\left(\mathrm{Ge}_{\pi_1(\check{G})}(X), \underline{\text{D-mod}}_{\frac{1}{2}}^{\pi_1(\check{G})}(\mathrm{Bun}_G)\right) \to \tag{8.31}$$
$$\to \Gamma\left(\mathrm{Ge}_{\pi_1(\check{G})}(X), \text{2-FM}_{\mathrm{Ge}_{Z_G}(X)\to\mathrm{Ge}_{\pi_1(\check{G})}(X)}\left(\underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\mathrm{Bun}_{G_{\mathrm{ad}}})\right)\right)$$

induced by (8.24)*, identifies via*

$$\Gamma\left(\mathrm{Ge}_{\pi_1(\check{G})}(X), \underline{\text{D-mod}}_{\frac{1}{2}}^{\pi_1(\check{G})}(\mathrm{Bun}_G)\right) \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

and

$$\Gamma\left(\mathrm{Ge}_{\pi_1(\check{G})}(X), \text{2-FM}_{\mathrm{Ge}_{Z_G}(X)\to\mathrm{Ge}_{\pi_1(\check{G})}(X)}\left(\underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\mathrm{Bun}_{G_{\mathrm{ad}}})\right)\right) \overset{(8.12)}{\simeq} \tag{8.32}$$
$$\simeq \left(\underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\mathrm{Bun}_{G_{\mathrm{ad}}})\right)|_{\mathfrak{G}^0_{Z_G}} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

with the identity endofunctor on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

8.5.11. We are now ready to prove that the map (8.24) is an equivalence.

Note that by point (b) of Proposition 8.5.7, for every $\mathfrak{G}_{\pi_1(\check{G})} \in \mathrm{Ge}_{\pi_1(\check{G})}(X)$, the category

$$\text{2-FM}_{\mathrm{Ge}_{Z_G}(X)\to\mathrm{Ge}_{\pi_1(\check{G})}(X)}\left(\underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\mathrm{Bun}_{G_{\mathrm{ad}}})\right)|_{\mathfrak{G}_{\pi_1(\check{G})}} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}$$

is a module over $\mathrm{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}})$, while $\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}$ is 1-affine. Hence, the object

$$\text{2-FM}_{\mathrm{Ge}_{Z_G}(X)\to\mathrm{Ge}_{\pi_1(\check{G})}(X)}\left(\underline{\text{D-mod}}_{\frac{1}{2}}^{Z_G}(\mathrm{Bun}_{G_{\mathrm{ad}}})\right) \in \mathrm{ShvCat}(\mathrm{Ge}_{\pi_1(\check{G})}(X))$$

is canonically of the form

$$(\mathsf{p}_{\pi_1(\check{G})})_*\left(\underline{\text{D-mod}}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathrm{univ}}}\right),$$

where

$$\underline{\text{D-mod}}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathrm{univ}}} \in \mathrm{ShvCat}(\mathrm{LS}_{\check{G}}).$$

Moreover, by construction, the map (8.24) comes from a map

$$\underline{\text{D-mod}}_{\frac{1}{2}}^{\check{G}}(\mathrm{Bun}_G) \to \underline{\text{D-mod}}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathrm{univ}}} \tag{8.33}$$

in $\mathrm{ShvCat}(\mathrm{LS}_{\check{G}})$.

It is sufficient to show that the map (8.33) is an equivalence. However, the stack $\mathrm{LS}_{\check{G}}$ is 1-affine, and hence, the functor

$$\Gamma(\mathrm{LS}_{\check{G}}, -) : \mathrm{ShvCat}(\mathrm{LS}_{\check{G}}) \to \mathrm{DGCat}$$

is conservative.

Hence, it is sufficient to show that the resulting functor

$$\Gamma(\mathrm{LS}_{\check{G}}, \underline{\text{D-mod}}_{\frac{1}{2}}^{\check{G}}(\mathrm{Bun}_G)) \to \Gamma(\mathrm{LS}_{\check{G}}, \underline{\text{D-mod}}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathrm{univ}}})$$

is an equivalence.

However, the latter functor identifies with the functor (8.31), and hence is an equivalence by Lemma 8.5.10.

□[Theorem 8.4.8]

Remark 8.5.12. As a particular case of Theorem 8.4.8, we obtain that the functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}_{\text{sc}},\mathfrak{G}_{\pi_1(\check{G})}}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}$$

of (8.30) is an equivalence.

With appropriate modifications, the above proof applies also in the setting of [AGKRRV1] when we work with constructible sheaves on Bun_G (resp., $\text{Bun}_{G_{ad}}$) with *nilpotent singular support* in any suitable sheaf theory. In particular, this applies for the ground field $k = \overline{\mathbb{F}}_q$ and ℓ-adic sheaves.

We expect similar phenomena in the setting of the Fargues-Scholze theory, see [FS]. Note that *loc. cit.* considers non-trivially twisted group schemes, but not spaces like $\text{LS}_{\check{G}_{\text{sc}},\mathfrak{G}_{\pi_1(\check{G})}}$ or categories like $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{G_{\text{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}$. We expect that there is some additional arithmetic content to be found by such considerations.

8.6. **Proof of Proposition 8.5.7(a).**

8.6.1. Let us return to the setting of Sect. 8.2. Let $\underline{x}$ be a point of Ran. Consider the spaces

$$\text{Ge}_\Gamma(X)_{\underline{x}} := \text{Fib}(\text{Ge}_\Gamma(X) \to \text{Ge}_\Gamma(X - \underline{x}))$$

and

$$\text{Ge}_\Gamma(\mathcal{D}_{\underline{x}})_{\underline{x}} := \text{Fib}(\text{Ge}_\Gamma(\mathcal{D}_{\underline{x}}) \to \text{Ge}_\Gamma(\mathcal{D}^\times_{\underline{x}}),$$

where

$$\mathcal{D}^\times_{\underline{x}} := \mathcal{D}_{\underline{x}} - \underline{x}.$$

Restriction along $\mathcal{D}_{\underline{x}} \to X$ defines an *isomorphism*

$$\text{Ge}_\Gamma(X)_{\underline{x}} \to \text{Ge}_\Gamma(\mathcal{D}_{\underline{x}})_{\underline{x}}.$$

Remark 8.6.2. Here is an explicit description of the spaces $\text{Ge}_\Gamma(\mathcal{D}_{\underline{x}})$, $\text{Ge}_\Gamma(\mathcal{D}^\times_{\underline{x}})$ and $\text{Ge}_\Gamma(\mathcal{D}_{\underline{x}})_{\underline{x}}$:

Write Γ as the kernel of a homomorphism of two tori $T_0 \to T_1$. Then

$$\text{Ge}_\Gamma(\mathcal{D}_{\underline{x}}) = B^2(\ker(\mathfrak{L}^+(T_0)_{\underline{x}} \to \mathfrak{L}^+(T_1)_{\underline{x}}))_{\text{et}}.$$

When $\underline{x}$ is a singleton, the above space is just $\text{Ge}_\Gamma(\text{pt})$.

Further,

$$\text{Ge}_\Gamma(\mathcal{D}^\times_{\underline{x}}) \simeq B^2(\text{Fib}(\mathfrak{L}(T_0)_{\underline{x}} \to \mathfrak{L}(T_1)_{\underline{x}}))_{\text{et}} \simeq B^1(\text{coFib}(\mathfrak{L}(T_0)_{\underline{x}} \to \mathfrak{L}(T_1)_{\underline{x}}))_{\text{et}}.$$

Finally,

$$\text{Ge}_\Gamma(\mathcal{D}_{\underline{x}})_{\underline{x}} \simeq \text{Gr}_{T_1,\underline{x}}/\text{Gr}_{T_0,\underline{x}}.$$

8.6.3. A local variant of (8.10) is a pairing

$$\text{Ge}_\Gamma(\mathcal{D}_{\underline{x}})_{\underline{x}} \times \text{Ge}_{\Gamma^\vee(1)}(\mathcal{D}_{\underline{x}}) \to \text{Ge}_{\mathbb{G}_m}(\text{pt}), \tag{8.34}$$

and it is easy to show that (8.34) is also of Fourier-Mukai type.

In particular, to a $\Gamma^\vee(1)$-gerbe $\mathfrak{G}^{\text{loc}}_{\Gamma^\vee(1)}$ on $\mathcal{D}_{\underline{x}}$ we can canonically associate a $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathfrak{G}^{\text{loc}}_{\Gamma^\vee(1)}}$ on $\text{Ge}_\Gamma(\mathcal{D}_{\underline{x}})_{\underline{x}}$.

8.6.4. Note that we have the following commutative diagram of pairings

$$
\begin{tikzcd}
\mathrm{Ge}_\Gamma(X)_{\underline{x}} \times \mathrm{Ge}_{\Gamma^\vee(1)}(X) \arrow[r] & \mathrm{Ge}_\Gamma(X) \times \mathrm{Ge}_{\Gamma^\vee(1)}(X) \arrow[dd, "(8.10)"] \\
\mathrm{Ge}_\Gamma(\mathcal{D}_{\underline{x}})_{\underline{x}} \times \mathrm{Ge}_{\Gamma^\vee(1)}(X) \arrow[u, "\simeq"] \arrow[d] & \\
\mathrm{Ge}_\Gamma(\mathcal{D}_{\underline{x}})_{\underline{x}} \times \mathrm{Ge}_{\Gamma^\vee(1)}(\mathcal{D}_{\underline{x}}) \arrow[r, "(8.34)"] & \mathrm{Ge}_{\mathbb{G}_m}(\mathrm{pt}).
\end{tikzcd}
\tag{8.35}
$$

In particular, for a $\Gamma^\vee(1)$-gerbe $\mathfrak{G}_{\Gamma^\vee(1)}$ on X and

$$\mathfrak{G}^{\mathrm{loc}}_{\Gamma^\vee(1)} := \mathfrak{G}_{\Gamma^\vee(1)}|_{\mathcal{D}_{\underline{x}}}$$

we have

$$\mathcal{G}_{\mathfrak{G}^{\mathrm{loc}}_{\Gamma^\vee(1)}}|_{\mathrm{Ge}_\Gamma(X)_{\underline{x}}} \simeq \mathcal{G}_{\mathfrak{G}_{\Gamma^\vee(1)}}|_{\mathrm{Ge}_\Gamma(X)_{\underline{x}}}.$$

8.6.5. Let

$$\mathrm{Hecke}^{\mathrm{loc}}_{G_{\mathrm{ad}},\underline{x}} := \mathfrak{L}^+(G_{\mathrm{ad}})_{\underline{x}} \backslash \mathfrak{L}(G_{\mathrm{ad}})_{\underline{x}} / \mathfrak{L}^+(G_{\mathrm{ad}})_{\underline{x}}$$

be the local Hecke stack for G_{ad} at x.

We have a natural projection

$$\mathrm{Hecke}^{\mathrm{loc}}_{G_{\mathrm{ad}},\underline{x}} \to \mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}}) \underset{\mathrm{Ge}_{Z_G}(\mathcal{D}^\times_{\underline{x}})}{\times} \mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}}).$$

The commutative group structure on the space of gerbes gives rise to a map

$$\mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}}) \underset{\mathrm{Ge}_{Z_G}(\mathcal{D}^\times_{\underline{x}})}{\times} \mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}}) \to \mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}})_{\underline{x}}.$$

Composing we obtain a map

$$\mathrm{Hecke}^{\mathrm{loc}}_{G_{\mathrm{ad}},\underline{x}} \to \mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}})_{\underline{x}}. \tag{8.36}$$

Remark 8.6.6. Note that the map (8.36) induces a bijection on the sets of connected components when G is simply-connected.

8.6.7. Let $\mathcal{G}^{\mathrm{loc}}$ be a $\mathbb{G}_m$-gerbe on $\mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}})_{\underline{x}}$. By a slight abuse of notation, we will denote by

$$\mathrm{Sph}(G_{\mathrm{ad}})_{\underline{x},\mathcal{G}^{\mathrm{loc}}}$$

the corresponding twisted version of the category

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Hecke}^{\mathrm{loc}}_{G_{\mathrm{ad}},\underline{x}}),$$

obtained by pulling back the gerbe $\mathcal{G}^{\mathrm{loc}}$ along (8.36).

Assume now that $\mathcal{G}^{\mathrm{loc}}$ is *multiplicative* (with respect to the group structure on $\mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}})_{\underline{x}}$). Then the category $\mathrm{Sph}(G_{\mathrm{ad}})_{\underline{x},\mathcal{G}^{\mathrm{loc}}}$ acquires a natural monoidal structure.

8.6.8. Let $\mathfrak{G}^{\mathrm{loc}}_{\pi_1(\check{G})}$ be a $\pi_1(\check{G})$-gerbe on $\mathcal{D}_{\underline{x}}$. Note that since the pairing (8.34) is bilinear, the corresponding $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathfrak{G}^{\mathrm{loc}}_{\pi_1(\check{G})}}$ on $\mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}})_{\underline{x}}$ has a natural multiplicative structure.

The following is a twisted version of the (naive) geometric Satake functor:

Lemma 8.6.9. *There exists a symmetric monoidal functor*

$$\mathrm{Sat}^{\mathrm{nv}}_{G_{\mathrm{ad}},\mathfrak{G}^{\mathrm{loc}}_{\pi_1(\check{G})}} : \mathrm{Rep}(\check{G}_{\mathrm{sc}})_{\underline{x},\mathfrak{G}_{\pi_1(\check{G})}} \to \mathrm{Sph}(G_{\mathrm{ad}})_{\underline{x},\mathcal{G}_{\mathfrak{G}^{\mathrm{loc}}_{\pi_1(\check{G})}}}.$$

Proof. The pairing (8.34) induces a bijection between:

- The set of characters of the (finite) group $\mathrm{Maps}(\mathcal{D}_{\underline{x}}, \pi_1(\check{G}))$, which is a subgroup of

$$\mathrm{Maps}(\mathcal{D}_{\underline{x}}, \pi_1(\check{G}_{\mathrm{ad}})) \simeq \mathrm{Maps}(\mathcal{D}_{\underline{x}}, Z_{\check{G}_{\mathrm{sc}}}) \simeq Z_{(\check{G}_{\mathrm{sc}})_{\underline{x}}};$$

- The set $\pi_0(\mathrm{Ge}_{Z_G}(\mathcal{D}_{\underline{x}})_{\underline{x}})$, which is a quotient of the set

$$\pi_0(\mathrm{Hecke}^{\mathrm{loc}}_{G_{\mathrm{ad}},\underline{x}}) \simeq \pi_0(\mathrm{Ge}_{Z_{G_{\mathrm{sc}}}}(\mathcal{D}_{\underline{x}})_{\underline{x}}).$$

The assertion of the lemma follows from the fact that under the usual (naive) geometric Satake functor

$$\mathrm{Sat}^{\mathrm{nv}}_{G_{\mathrm{ad}}} : \mathrm{Rep}(\check{G}_{\mathrm{sc}})_{\underline{x}} \to \mathrm{Sph}(G_{\mathrm{ad}})_{\underline{x}},$$

the decomposition of $\mathrm{Rep}(\check{G}_{\mathrm{sc}})_{\underline{x}}$ according to central characters corresponds to the decomposition of $\mathrm{Sph}(G_{\mathrm{ad}})_{\underline{x}}$ along the connected components of $\mathrm{Hecke}^{\mathrm{loc}}_{G_{\mathrm{ad}},\underline{x}}$, according to support. Indeed, this observation implies that $\mathrm{Sat}^{\mathrm{nv}}_{G_{\mathrm{ad}}} : \mathrm{Rep}(\check{G}_{\mathrm{sc}})_{\underline{x}} \to \mathrm{Sph}(G_{\mathrm{ad}})_{\underline{x}}$ is equivariant for the action of $B(Z_{(\check{G}_{\mathrm{sc}})_{\underline{x}}})$ on both sides, and therefore we can twist $\mathrm{Sat}^{\mathrm{nv}}_{G_{\mathrm{ad}}}$ by $Z_{(\check{G}_{\mathrm{sc}})_{\underline{x}}}$-gerbes.

□

8.6.10. The constructions in Sect. 8.6.5 have immediate counterparts for the global Hecke stack

$$\mathrm{Hecke}^{\mathrm{glob}}_{G_{\mathrm{ad}},\underline{x}} := \mathrm{Bun}_{G_{\mathrm{ad}}} \underset{\mathrm{Bun}_{G_{\mathrm{ad}}}(X-\underline{x})}{\times} \mathrm{Bun}_{G_{\mathrm{ad}}}.$$

In particular, a multiplicative $\mathbb{G}_m$-gerbe $\mathcal{G}^{\mathrm{loc}}$ on $\mathrm{Ge}_{Z_G}(X)_{\underline{x}}$ gives rise to a monoidal category

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Hecke}^{\mathrm{glob}}_{G_{\mathrm{ad}},\underline{x}})_{\mathcal{G}^{\mathrm{loc}}}. \tag{8.37}$$

Note that pullback defines a monoidal functor

$$\mathrm{Sph}(G_{\mathrm{ad}})_{\underline{x},\mathcal{G}^{\mathrm{loc}}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Hecke}^{\mathrm{glob}}_{G_{\mathrm{ad}},\underline{x}})_{\mathcal{G}^{\mathrm{loc}}}.$$

Assume now that $\mathcal{G}^{\mathrm{loc}}$ is obtained by restriction along

$$\mathrm{Ge}_{Z_G}(X)_{\underline{x}} \to \mathrm{Ge}_{Z_G}(X)$$

of a multiplicative $\mathbb{G}_m$-gerbe $\mathcal{G}$ on $\mathrm{Ge}_{Z_G}(X)$.

Then we have a natural monoidal action of (8.37) on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}}$.

8.6.11. Combining the above ingredients, we obtain that for a $\pi_1(\check{G})$-gerbe $\mathfrak{G}_{\pi_1(\check{G})}$ on X, we have a monoidal action of $\mathrm{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran},\mathfrak{G}_{\pi_1(\check{G})}}$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}$.

This construction makes sense in families as $\underline{x}$ moves over the Ran space, thereby giving rise to the sought-for action of $\mathrm{Rep}(\check{G}_{\mathrm{sc}})_{\mathrm{Ran},\mathfrak{G}_{\pi_1(\check{G})}}$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}$.

The compatibility in point (a) of Proposition 8.5.7 follows by construction.

8.7. **Proof of Proposition 8.5.7(b).** We will show how to adapt the proof of [Ga4, Theorem 4.5.2] to apply in the current gerbe-twisted situation.

8.7.1. Choose a point $x \in X$. Since $H^2_{\mathrm{et}}(X - x, \pi_1(\check{G})) = 0$, we can choose a trivialization of $\mathfrak{G}_{\pi_1(\check{G})}$ over $X - x$. I.e., we can assume that $\mathfrak{G}_{\pi_1(\check{G})}$ comes from an object

$$\mathfrak{G}^{\mathrm{loc}}_{\pi_1(\check{G})} \in \mathrm{Ge}_{\pi_1(\check{G})}(X)_x.$$

Note that the space $\mathrm{Ge}_{\pi_1(\check{G})}(X)_x$ is canonically the discrete set $\pi_1(\check{G})(-1)$, which is in bijection with the set of characters of Z_G. Denote the element corresponding to our gerbe by χ.

8.7.2. We claim now that the proof of the spectral decomposition theorem in [Ga4, Sect. 11.1] applies in the current context. Recall that in *loc. cit.* the proof was based on considering the localization functor

$$\mathrm{Loc}_{G_{\mathrm{ad}}} : \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}}).$$

In the present twisted situation we will need to make the following modifications.

8.7.3. We replace Ran by its relative version Ran_x that classifies finite subsets of X that contain the point x.

For further discussion, in order to simplify the notation, we will work with a fixed element $\underline{x} \in \mathrm{Ran}_x$. Write $\underline{x} = \underline{x}' \sqcup \{x\}$.

8.7.4. We replace the category

$$\mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x}} \simeq \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x}'} \otimes \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},x}$$

by

$$\mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x},\chi} := \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x}'} \otimes \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},x,\chi},$$

where $\mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},x,\chi}$ is the full subcategory of $\mathrm{KL}(G)_{\mathrm{crit},x}$, consisting of objects, on which

$$Z_G \subset \mathfrak{L}^+(G)_x$$

acts by the character χ.

8.7.5. Note that by (8.35), the $\mathbb{G}_m$-gerbe $\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}$ on $\mathrm{Ge}_{Z_G}(X)$ is obtained by pullback via the map

$$\mathrm{Ge}_{Z_G}(X) \to \mathrm{Ge}_{Z_G}(\mathcal{D}_x) \simeq \mathrm{Ge}_{Z_G}(\mathrm{pt}) \simeq B^2(Z_G)_{\mathrm{et}}$$

from the $\mathbb{G}_m$-gerbe on $B^2(Z_G)_{\mathrm{et}}$ corresponding to the character χ.

The pullback of $\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}$ to $\mathrm{Bun}_{G_{\mathrm{ad}}}$ trivializes over the cover

$$\mathrm{Bun}_{G_{\mathrm{ad}}} \underset{\mathrm{pt}/\mathfrak{L}^+(G_{\mathrm{ad}})_x}{\times} \mathrm{pt}/\mathfrak{L}^+(G)_x,$$

and corresponds to the multiplicative line bundle on pt/Z_G given by χ.

From here we obtain that we have a well-defined localization functor

$$\mathrm{Loc}_{G_{\mathrm{ad}},\underline{x},\chi} : \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x},\chi} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}}.$$

8.7.6. Recall that the category $\mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x}}$ is acted on by

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{\underline{x}})).$$

The key point in the proof of [Ga4, Theorem 4.5.2] is the fact (going back to [BD]) that the functor

$$\mathrm{Loc}_{G_{\mathrm{ad}},\underline{x}} : \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})$$

factors as

$$\mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x}} \to \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{\underline{x}}))}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(X-x)) \xrightarrow{\mathrm{Loc}^{\mathrm{glob}}_{G_{\mathrm{ad}},\underline{x}}}$$
$$\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}}),$$

where the functor $\mathrm{Loc}^{\mathrm{glob}}_{G_{\mathrm{ad}},\underline{x}}$ is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}}})$-linear with respect to the tautological projection $\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(X-x) \to \mathrm{LS}_{\check{G}_{\mathrm{sc}}}$.

We will now explain the modification of this construction.

8.7.7. Consider the space $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}_x^\times)$. We claim that there is a canonically defined map

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Ge}_{\pi_1(\check{G})}(\mathcal{D}_x)_x. \tag{8.38}$$

To construct it, it suffices to show that the composition

$$\mathrm{Op}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{LS}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Ge}_{\pi_1(\check{G})}(\mathcal{D}_x^\times) \tag{8.39}$$

factors through the point of $\mathrm{Ge}_{\pi_1(\check{G})}(\mathcal{D}_x^\times)$, corresponding to the trivial $\pi_1(\check{G})$-gerbe,

Note that the map (8.39) factors as

$$\mathrm{Op}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{LS}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Bun}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Ge}_{\pi_1(\check{G})}(\mathcal{D}_x^\times),$$

while the map

$$\mathrm{Op}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{LS}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Bun}_{\check{G}}(\mathcal{D}_x^\times)$$

in turn factors as

$$\mathrm{Op}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Bun}_{\check{B}}(\mathcal{D}_x^\times) \to \mathrm{Bun}_{\check{G}}(\mathcal{D}_x^\times).$$

Hence, (8.39) factors as

$$\mathrm{Op}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Bun}_{\check{B}}(\mathcal{D}_x^\times) \to \mathrm{Bun}_{\check{T}}(\mathcal{D}_x^\times) \to \mathrm{Ge}_{\pi_1(\check{G})}(\mathcal{D}_x^\times), \tag{8.40}$$

where we think of $\pi_1(\check{G})$ as $\ker(\check{T} \to \check{T}_{\mathrm{sc}})$.

However, the map

$$\mathrm{Op}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Bun}_{\check{B}}(\mathcal{D}_x^\times) \to \mathrm{Bun}_{\check{T}}(\mathcal{D}_x^\times)$$

corresponds to the point $2\check{\rho}(\omega^{\otimes \frac{1}{2}}) \in \mathrm{Bun}_{\check{T}}(\mathcal{D}_x^\times)$, and that point lifts to a point of $\mathrm{Bun}_{\check{T}_{\mathrm{sc}}}(\mathcal{D}_x^\times)$. This implies that the map (8.40) factors via the trivial gerbe.

Remark 8.7.8. We claim that $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}_x^\times)$ maps in fact to a twisted form of the affine Grassmannian of the group $\check{G}$, so that (8.38) factors via this map[10].

Indeed, recall that for a fixed curve X, we can think of $\check{G}$-opers as connections of the standard form on a fixed $\check{G}$-bundle $\mathcal{P}^{\mathrm{Op}}_{\check{G}}$, induced from a particular Borel bundle for a principal SL_2-triple.

Consider the twisted affine Grassmannian $\mathrm{Gr}_{\check{G},\mathcal{P}^{\mathrm{Op}}_{\check{G}},x}$ (see [Lan2, Sect. 1.2]), i.e., the moduli space of pairs

$$(\mathcal{P}_{\check{G}}, \alpha),$$

where $\mathcal{P}_{\check{G}}$ is a $\check{G}$-bundle on $\mathcal{D}_x$, and α is an isomorphism $\mathcal{P}_{\check{G}} \simeq \mathcal{P}^{\mathrm{Op}}_{\check{G}}$ over $\mathcal{D}_x^\times$.

Note that we can think of a point of $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}_x^\times)$ as a triple

$$(A, \mathcal{P}_{\check{G}}, \alpha),$$

where:

- A is a connection of the standard oper form on $\mathcal{P}^{\mathrm{Op}}_{\check{G}}$ over $\mathcal{D}_x^\times$;
- $\mathcal{P}_{\check{G}}$ is a $\check{G}$-bundle on $\mathcal{D}_x$;
- α is an isomorphism $\mathcal{P}_{\check{G}} \simeq \mathcal{P}^{\mathrm{Op}}_{\check{G}}$ over $\mathcal{D}_x^\times$, so that the *a priori meromorphic* connection on $\mathcal{P}_{\check{G}}$, induced by A via α is *regular*.

The assignment

$$(A, \mathcal{P}_{\check{G}}, \alpha) \mapsto (\mathcal{P}_{\check{G}}, \alpha)$$

is a map

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}_x^\times) \to \mathrm{Gr}_{\check{G},\mathcal{P}^{\mathrm{Op}}_{\check{G}},x}$$

mentioned above.

[10]This remark is inessential for the sequel and the reader may choose to skip it.

8.7.9. In the twisted situation we replace

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{\underline{x}}) \simeq \mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{\underline{x}'}) \times \mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{x})$$

by

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{\underline{x}})_\chi := \mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{\underline{x}'}) \times \mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{x})_\chi,$$

where $\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{x})_\chi$ is the preimage of the point χ under the projection (8.38).

Note that with respect to the $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{x}))$-action on $\mathrm{KL}(G)_{\mathrm{crit},x}$, the subcategory $\mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},x,\chi}$ is exactly the direct summand that is supported over

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{x})_\chi \subset \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{x}).$$

8.7.10. Consider the space

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(X-x)_\chi := \mathrm{Op}_{\check{G}_{\mathrm{sc}}}(X-x) \underset{\mathrm{LS}_{\check{G}_{\mathrm{sc}}}(X-x)}{\times} \mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}}.$$

Note that we have a naturally defined map

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(X-x)_\chi \to \mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{\underline{x}})_\chi.$$

Now, by the same principle as in [Ga4, Theorem 10.3.4], we obtain that the functor $\mathrm{Loc}_{G_{\mathrm{ad}},\underline{x},\chi}$ factors as

$$\mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x},\chi} \to \mathrm{KL}(G_{\mathrm{ad}})_{\mathrm{crit},\underline{x},\chi} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(\mathcal{D}^\times_{\underline{x}}))_\chi}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}_{\mathrm{sc}}}(X-x)_\chi) \overset{\mathrm{Loc}^{\mathrm{glob}}_{G_{\mathrm{ad}},\underline{x},\chi}}{\longrightarrow}$$
$$\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{G_{\mathrm{ad}}})_{\mathcal{G}_{\mathfrak{G}_{\pi_1(\check{G})}}},$$

where the functor $\mathrm{Loc}^{\mathrm{glob}}_{G_{\mathrm{ad}},\underline{x},\chi}$ is linear with respect to $\mathrm{QCoh}(\mathrm{LS}_{\check{G}_{\mathrm{sc}},\mathfrak{G}_{\pi_1(\check{G})}})$.

Now the argument parallel to that in [Ga4, Sect. 11.1] establishes the factorization of the action stated in Proposition 8.5.7(b).

Appendix A. Review of (semi-)stability for G-bundles

We briefly review the basic notions from the theory of (semi-)stable G-bundles following the original source [Ra].

A.1. **Some characterizations of (semi-)stability.**

In what follows, we only consider parabolic subgroups P containing our fixed Borel.

A.1.1. *Notation related to root data.* Recall that Λ denotes the coweight lattice of G and $\check{\Lambda}$ denotes the weight lattice. As is standard, $2\check{\rho}_G \in \check{\Lambda}$ denotes the sum of the positive roots.

Let P be a parabolic subgroup of G with Levi quotient M. As the weight lattices of M and G coincide, we also have the weight $2\check{\rho}_M \in \check{\Lambda}$. We set $2\check{\rho}_P := 2\check{\rho}_G - 2\check{\rho}_M$, which is the sum of the roots occurring in $\mathfrak{n}(P)$.

We remark that $\langle 2\check{\rho}_P, \alpha_i\rangle = 0$ for every vertex i in the Dynkin diagram I_M of M, i.e., for each simple coroot α_i whose $\mathfrak{sl}_2$ maps into $\mathfrak{m}$. Indeed, we have $\langle 2\check{\rho}_G, \alpha_i\rangle = \langle 2\check{\rho}_M, \alpha_i\rangle = 2$ for such α_i.

It follows that for a coweight $\lambda \in \Lambda$, the value of $\langle 2\check{\rho}_P, \lambda\rangle$ only depends on the class of λ in $\Lambda/\operatorname{Span}\{\alpha_i\}_{i\in I_M} =: \pi_{1,\mathrm{alg}}(M)$.

A.1.2. We have the following definition (cf. [Ra]):

Definition A.1.3. A G-bundle $\mathcal{P}_G$ on X is *semi-stable* (resp. *stable*) if for every maximal (proper) parabolic subgroup $P \subsetneq G$ and every reduction $\mathcal{P}_P$ of $\mathcal{P}_G$ to P, we have

$$\langle 2\check{\rho}_P, \deg(\mathcal{P}_P)\rangle \leq 0 \quad (\text{resp. } < 0).$$

Here we remind that $\deg(\mathcal{P}_P)$ is an element of $\pi_{1,\mathrm{alg}}(M)$.

We remark that the integer $\langle 2\check{\rho}_P, \deg(\mathcal{P}_P)\rangle$ appearing above is the degree of the vector bundle $\mathfrak{n}(P)_{\mathcal{P}_P}$ on X.

Example A.1.4. This definition is rigged to recover the usual one for $G = GL_n$.

Indeed, suppose $\mathcal{E}$ has rank n and P is the maximal parabolic whose reductions correspond to subbundles $\mathcal{E}_0 \subset \mathcal{E}$ of rank m. Then a straightforward calculation yields

$$\langle 2\check{\rho}_P, \deg(\mathcal{P}_P)\rangle = \mathrm{rank}(\mathcal{E}) \cdot \deg(\mathcal{E}_0) - \mathrm{rank}(\mathcal{E}_0) \cdot \deg(\mathcal{E}).$$

A.1.5. We have the following basic result.

Proposition A.1.6. *For a G-bundle $\mathcal{P}_G$ on X, the following conditions are equivalent.*

(a) *$\mathcal{P}_G$ is semi-stable (resp. stable).*
(b) *For every proper parabolic subgroup $P \subsetneq G$ (possibly not of corank 1) and every reduction $\mathcal{P}_P$ of $\mathcal{P}_G$ to P, we have*

$$\langle 2\check{\rho}_P, \deg(\mathcal{P}_P)\rangle \leq 0 \quad (\textit{resp. } < 0)$$

(c) *For every reduction $\mathcal{P}_B$ of $\mathcal{P}_G$ to the Borel, we have:*

$$\deg(\mathcal{P}_B) = \sum_{i \in I_G} n_i \alpha_i + \varepsilon, \;\; n_i \in \mathbb{Q}^{\leq 0}, \; \varepsilon \in \mathbb{Q} \cdot \Lambda_{Z_G} \, (\textit{resp. } n_i \in \mathbb{Q}^{<0}). \tag{A.1}$$

Here Λ_{Z_G} is the set of coweights mapping into the center of G.

Proof. The key point is to observe

$$\begin{cases} \langle 2\check{\rho}_P, \varepsilon\rangle = 0, & \\ \langle 2\check{\rho}_P, \alpha_i\rangle = 0 & \text{if } i \in I_M \\ \langle 2\check{\rho}_P, \alpha_i\rangle \geq 2 & \text{if } i \notin I_M \end{cases} \tag{A.2}$$

where the last expression follows as $\langle 2\check{\rho}_G, \alpha_i\rangle = 2$ and $2\check{\rho}_M$ is a sum of roots $\check{\alpha}_j$ with $j \in I_M$ (so $j \neq i$). Then (b) tautologically implies (a), (c) implies (b) by (A.2), and (a) implies (c) again by noting that for P_i the maximal parabolic corresponding to $i \in I_G$, we have $\langle 2\check{\rho}_{P_i}, \deg(\mathcal{P}_B)\rangle = n_i \langle 2\check{\rho}_P, \alpha_i\rangle \in n_i \cdot \mathbb{Z}^{>0}$ by (A.2).

□

In the above, the condition (c) immediately matches the notion of semi-stability used in [DG] (see *loc. cit.* Lemma 7.3.2), and it matches the notion of stability implicitly suggested in *loc. cit.*

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[11] This paper is currently being split into [GLC2] and [GLC3]. Once this process is completed, the references to [Lan2] will be replaced appropriately.