PROOF OF THE GEOMETRIC LANGLANDS CONJECTURE I: CONSTRUCTION OF THE FUNCTOR

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Abstract. In this paper we construct the geometric Langlands functor in one direction (from the automorphic to the spectral side) in characteristic zero settings (i.e., de Rham and Betti). We prove that various forms of the conjecture (de Rham vs Betti, restricted vs. non-restricted, tempered vs. non-tempered) are equivalent. We also discuss structural properties of Hecke eigensheaves.

First draft

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Introduction

This paper is the first in a series of five, in which the[1] geometric Langlands conjecture will be proved.

The entire project is joint work of D. Arinkin, D. Beraldo, L. Chen, J. Faegerman, D. Gaitsgory, K. Lin, S. Raskin and N. Rozenblyum.

The individual papers in the five-paper series will have different subsets of the above people as authors.

0.1. **The Langlands functor in the de Rham setting.** We first consider the de Rham version of the geometric Langlands conjecture (GLC).

0.1.1. Let k be a field of characteristic 0. Let X be a smooth and complete curve and G a connected reductive group over k. Let $\check{G}$ be the Langlands dual of G, viewed also as a reductive group over k.

The automorphic side of GLC is the category

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

of half-twisted D-modules on the moduli stack Bun_G of principal G-bundles on X, see Sect. 1.1.

The spectral side is the category

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$$

of ind-coherent sheaves with singular support in the nilpotent cone (see [AG, Sect. 11.1.5]) on the moduli stack $\text{LS}_{\check{G}}$ of de Rham $\check{G}$-local systems on X.

0.1.2. The Langlands functor, to be constructed in this paper, goes from the automorphic to the spectral side

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}). \tag{0.1}$$

The geometric Langlands conjecture (GLC) in the de Rham setting says that the functor (0.1) is an equivalence.

[1] Global and unramified.

0.1.3. However, there is an important piece of structure that enters the game before we attempt to construct $\mathbb{L}_G$, namely, the *spectral action.*

The assertion is that the combined action of the *Hecke functors* gives rise to a uniquely defined monoidal action of $\text{QCoh}(\text{LS}_{\check{G}})$ on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.

Such an action can be viewed as a "spectral decomposition" of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ along the stack $\text{LS}_{\check{G}}$.

The existence of the spectral action was established by V. Drinfeld and the first author (it is recorded in [Ga2, Corollary 4.5.5]), but its idea goes back to A. Beilinson, in the form of what is called the *Beilinson projector* (see [AGKRRV, Sect. 15] for a detailed discussion).

0.1.4. The construction of $\mathbb{L}_G$ proceeds in two steps: we first construct the *coarse* version of $\mathbb{L}_G$

$$\mathbb{L}_{G,\text{coarse}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{QCoh}(\text{LS}_{\check{G}}).$$

It is related to the desired functor $\mathbb{L}_G$ by

$$\Psi_{\text{Nilp},\{0\}} \circ \mathbb{L}_G \simeq \mathbb{L}_{G,\text{coarse}}, \tag{0.2}$$

where

$$\Psi_{\text{Nilp},\{0\}} : \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \to \text{QCoh}(\text{LS}_{\check{G}})$$

is the *coarsening* functor.

0.1.5. The functor $\mathbb{L}_{G,\text{coarse}}$ stems from the classical idea in the theory of automorphic functions that says that Langlands correspondence is normalized by the Whittaker model. In the geometric context this idea is incarnated by the fact that the functor $\mathbb{L}_{G,\text{coarse}}$ is uniquely determined by the following two requirements:

- The functor $\mathbb{L}_{G,\text{coarse}}$ is compatible with the Hecke action, i.e., it intertwines the action of $\text{QCoh}(\text{LS}_{\check{G}})$ on the two sides;
- It makes the triangle

$$\begin{array}{ccc} & \text{Vect} & \\ {\scriptstyle \text{coeff}^{\text{Vac,glob}}}\nearrow & & \nwarrow{\scriptstyle \Gamma(\text{LS}_{\check{G}},-)} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\text{coarse}}} & \text{QCoh}(\text{LS}_{\check{G}}) \end{array}$$

  commute, where $\text{coeff}^{\text{Vac,glob}}$ is the functor of *first Whittaker coefficient*, see Sect. 1.3.7.

0.1.6. Once the functor $\mathbb{L}_{G,\text{coarse}}$ is constructed, we lift it to the sought-for functor $\mathbb{L}_G$ using cohomological estimates provided by Theorem 1.6.2.

Namely, Theorem 1.6.2 says that the functor $\mathbb{L}_{G,\text{coarse}}$ sends compact objects of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ to *eventually coconnective* objects in $\text{QCoh}(\text{LS}_{\check{G}})$, i.e., objects that are cohomologically bounded on the left. As $\Psi_{\text{Nilp},\{0\}}$ is an equivalence on eventually coconnective subcategories, we obtain the functor $\mathbb{L}_G$, which is then uniquely characterized by (0.2) and the requirement that $\mathbb{L}_G$ send compacts to eventually coconnective objects.[2]

In fact, we show that the functor $\mathbb{L}_{G,\text{coarse}}$ has a cohomological amplitude bounded on the left. (Note, however, that this does *not* mean that the Langlands functor $\mathbb{L}_G$ has a cohomological amplitude bounded on the left; it rather has unbounded amplitude once G is non-abelian.)

Of less immediate practical importance, we also show that the functor $\mathbb{L}_{G,\text{coarse}}$ has a cohomological amplitude bounded on the *right*. This property is automatically inherited by $\mathbb{L}_G$.

[2]We remark that the latter requirement is necessarily satisfied by any putative geometric Langlands functor: such a functor ought to be an equivalence, so must preserve compacts; and compact objects are eventually coconnective in $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$.

0.1.7. One can thus say that the upgrade

$$\mathbb{L}_{G,\text{coarse}} \rightsquigarrow \mathbb{L}_G, \tag{0.3}$$

carried out in the present paper, is cohomological in nature. Its main ingredient, Theorem 1.6.2, uses non-trivial input: ultimately we deduce it from Theorem 2.3.8, proved in [FR] using methods developed in [AGKRRV] and [Lin].

An alternative approach to constructing $\mathbb{L}_G$ is being developed in separate work of D. Beraldo, L. Chen and K. Lin. Their work is based on "gluing" the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ from the categories $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)_{\text{temp}}$ (see Sect. 5.1.2) for Levi subgroups M of G, using the *Eisenstein series* functors, see [BeLi]. Then they construct the functor using a parallel procedure on the spectral side, i.e., gluing $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ from the categories $\text{QCoh}(\text{LS}_{\check{M}})$, using the *spectral Eisenstein series* functors, developed in [Be2]. Unlike the approach in the present paper, their work pursues (and better develops) the original proposal from [Ga2][3].

0.2. **The Betti setting.** We now consider GLC in the Betti setting.

0.2.1. In this case our algebraic geometry happens over the ground field $\mathbb{C}$ (so X and G are over $\mathbb{C}$), and we work with sheaves of e-vector spaces, where e is an arbitrary field of characteristic 0.

So, $\check{G}$ is an reductive group over e, and $\text{LS}_{\check{G}}$ is an algebraic stack over e.

0.2.2. A historical and conceptual difference between the de Rham and Betti situations is that while it has "always" (i.e., since the inception of the subject by A. Beilinson and V. Drinfeld) been understood what the automorphic side in the de Rham setting was (i.e., the entire category of half-twisted D-modules), in the Betti setting it was a relatively recent discovery, due to D. Ben-Zvi and D. Nadler, see [BZN].

Namely, the entire category $\text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G)$ of (twisted) Betti sheaves on Bun_G is too big to be equivalent to something on the spectral side. What Ben-Zvi and Nadler discovered was that there is a reasonable automorphic candidate: namely, this is the full category

$$\text{Shv}^{\text{Betti}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G)$$

of Betti sheaves with singular support in the global nilpotent cone.

0.2.3. The spectral side in the Betti setting must also be modified as compared to the de Rham setting, but that modification is evident: one should replace the stack $\text{LS}_{\check{G}}$ of de Rham local systems on X (i.e., the stack classifying $\check{G}$-bundles with a connection) by the stack $\text{LS}^{\text{Betti}}_{\check{G}}$ of Betti $\check{G}$-local systems (i.e., if we ignore the derived structure, this is the stack of homomorphisms $\pi_1(X) \to \check{G}$, divided by the action of $\check{G}$ by conjugation).

In retrospect, knowing the spectral side, one can convince oneself that $\text{Shv}^{\text{Betti}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)$ is the right candidate for the automorphic category as follows: this is the largest subcategory of $\text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G)$, on which the Hecke action gives rise to an action of $\text{QCoh}(\text{LS}^{\text{Betti}}_{\check{G}})$, see [AGKRRV, Theorem 18.1.4].

0.2.4. With the candidates for the automorphic and spectral sides in place, the construction of the Langlands functor in the Betti setting

$$\mathbb{L}^{\text{Betti}}_G : \text{Shv}^{\text{Betti}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{Betti}}_{\check{G}}) \tag{0.4}$$

proceeds along the same lines as in the de Rham setting, with one modification:

The functor $\text{coeff}^{\text{Vac,glob}}$ in the de Rham setting appealed to the exponential D-module on $\mathbb{G}_a$. There is (obviously) no Betti analog of the exponential D-module, but one finds a substitute using an additional group of symmetries (given by the torus T) acting in our situation, see Sect. 3.3.

[3] An additional merit of this approach is that, unlike using the definition of $\text{IndCoh}(-)$ from scratch, i.e., as the ind-completion of $\text{Coh}(-)$, it uses the formal properties of the IndCoh category.

0.2.5. The geometric Langlands conjecture (GLC) in the Betti setting says that the functor (0.4) is an equivalence.

0.3. **Comparison of the different versions of GLC.** In addition to the de Rham (resp., Betti) versions of GLC, which we will refer to as *full* GLC, in both settings there exist other versions, which we refer to as *restricted* and *tempered*, respectively. And there exist also the restricted tempered versions.

However, it turns out that all these versions are logically equivalent in the sense that any one implies the others.

0.3.1. The restricted version of GLC, in either de Rham or Betti versions, was introduced[4] in [AGKRRV, Conjecture 21.2.7].

On the automorphic side of the restricted GLC in the de Rham setting, we have the full subcategory

$$\text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

that consists of (twisted) D-modules with singular support in the nilpotent cone $\text{Nilp} \subset T^*(\text{Bun}_G)$. Objects of this category are (obviously) ind-holonomic. However, one can show that actually have regular singularities (this is a non-trivial result [AGKRRV, Corollary 16.5.6]).

On the automorphic side of the restricted GLC in the Betti setting, we have the full subcategory

$$\text{Shv}^{\text{Betti,constr}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}^{\text{Betti}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)$$

that consists of *ind-constructible* sheaves.

In fact, both categories

$$\text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \text{ and } \text{Shv}^{\text{Betti,constr}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)$$

fall into the paradigm of [AGKRRV, Sect. 14.1], where the restricted automorphic category is defined starting from a *constructible sheaf theory* $\text{Shv}(-)$ as in [AGKRRV, Sect. 1.1.1]. The two sheaf theories in question are

$$\text{Shv}(-) := \text{D-mod}^{\text{RS}}(-) \text{ and } \text{Shv}(-) := \text{Shv}^{\text{Betti,constr}}(-),$$

respectively.

0.3.2. On the spectral side of the restricted GLC we have the categories

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}}) \text{ and } \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{Betti,restr}}_{\check{G}}), \tag{0.5}$$

respectively, where $\text{LS}^{\text{restr}}_{\check{G}}$ and $\text{LS}^{\text{Betti,restr}}_{\check{G}}$ are the de Rham and Betti versions of the *stack of local systems with restricted variation* of [AGKRRV, Sect. 1.4].

0.3.3. Starting from the full geometric Langlands functors $\mathbb{L}_G$ and $\mathbb{L}^{\text{Betti}}_G$ one produces their restricted versions

$$\text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \overset{\mathbb{L}^{\text{restr}}_G}{\longrightarrow} \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}})$$

and

$$\text{Shv}^{\text{Betti,constr}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \overset{\mathbb{L}^{\text{Betti,restr}}_G}{\longrightarrow} \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{Betti,restr}}_{\check{G}}).$$

by applying the operations

$$- \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}})_{\text{restr}} \text{ and } - \underset{\text{QCoh}(\text{LS}^{\text{Betti}}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}^{\text{Betti}}_{\check{G}})_{\text{restr}},$$

where

$$\text{QCoh}(\text{LS}_{\check{G}})_{\text{restr}} \subset \text{QCoh}(\text{LS}_{\check{G}}) \text{ and } \text{QCoh}(\text{LS}^{\text{Betti}}_{\check{G}})_{\text{restr}} \subset \text{QCoh}(\text{LS}^{\text{Betti}}_{\check{G}})$$

are full subcategories that consist of objects set-theoretically supported on

$$\text{LS}^{\text{restr}}_{\check{G}} \subset \text{LS}_{\check{G}} \text{ and } \text{LS}^{\text{Betti,restr}}_{\check{G}} \subset \text{LS}^{\text{Betti}}_{\check{G}},$$

respectively.

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[4] Without specifying the functor in either direction.

It is clear that if $\mathbb{L}_G$ and $\mathbb{L}_G^{\mathrm{Betti}}$ are equivalences, then so are $\mathbb{L}_G^{\mathrm{restr}}$ and $\mathbb{L}_G^{\mathrm{Betti,restr}}$. However, it turns out that the converse implications also take place. This is proved in Sect. 5 of the present paper; see also Sect. 0.3.6 below. This material builds on [AGKRRV, Sect. 21.4].

0.3.4. Thus, we have:

$$\text{Full de Rham GLC} \Leftrightarrow \text{Restricted de Rham GLC}$$

and

$$\text{Full Betti GLC} \Leftrightarrow \text{Restricted Betti GLC.}$$

However, Riemann-Hilbert implies that we also have

$$\text{Restricted de Rham GLC} \Leftrightarrow \text{Restricted Betti GLC.}$$

As a result, we deduce that

$$\text{Full de Rham GLC} \Leftrightarrow \text{Full Betti GLC.} \tag{0.6}$$

0.3.5. In the rest of this series of papers we will focus on the de Rham version of GLC.

This is because it is in this context that one can use Kac-Moody localization, which provides a powerful tool in the study of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$. It would not be an exaggeration to say that the proofs of all the key results[5] in this series of papers ultimately appeal to this method.

However, thanks to (0.6), once we prove the de Rham version of GLC, we deduce the Betti case as well.

0.3.6. Let us now comment on how we deduce

$$\text{Restricted de Rham GLC} \Rightarrow \text{Full de Rham GLC}$$

(the Betti case is similar).

To do so, we introduce yet another version of GLC: the tempered one. Namely, the category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ admits a localization

$$\mathbf{u} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) : \mathbf{u}^R,$$

and similarly for its the restricted version

$$\mathbf{u} : \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \rightleftarrows \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) : \mathbf{u}^R.$$

The tempered versions of the Langlands functor map

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \overset{\mathbb{L}_{G,\mathrm{temp}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

and

$$\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \overset{\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{temp}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}),$$

respectively.

The tempered (resp., tempered restricted) version of GLC says that the functor $\mathbb{L}_{G,\mathrm{temp}}$ (resp., $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{temp}}$) is an equivalence.

0.3.7. It is clear that we have the implications

$$\text{Full GLC} \Rightarrow \text{Tempered GLC} \quad \text{and} \quad \text{Restricted GLC} \Rightarrow \text{Tempered restricted GLC}$$

In Sect. 5.2 we establish an inverse implication

$$\text{Tempered GLC} \Rightarrow \text{Full GLC.}$$

(A similar argument shows that Tempered restricted GLC $\Rightarrow$ Restricted GLC.)

[5] Including the existence of the spectral action.

0.3.8. It is also clear that

$$\text{Tempered GLC} \Rightarrow \text{Tempered restricted GLC}.$$

However, by mimicking the argument of [AGKRRV, Sect. 21.4] one proves that we in fact have

$$\text{Tempered GLC} \Leftrightarrow \text{Tempered restricted GLC}.$$

Remark 0.3.9. In fact, in [AGKRRV, Sect. 21.4] the equivalence

$$\text{Full GLC} \Leftrightarrow \text{Restricted GLC}$$

was proved directly, but under the assumption that the functor $\mathbb{L}_G$ admits a left adjoint.

The advantage of working with the tempered category is that the existence of the left adjoint of $\mathbb{L}_{G,\text{temp}}$ is automatic from the construction.

0.4. **Characteristic cycles of Hecke eigensheaves.** We prove an additional result the interaction between de Rham and Betti geometric Langlands.

Theorem 0.4.1. *Suppose that G has connected center, the genus of X is ≥ 2, and assume the geometric Langlands conjecture. Let σ be an irreducible $\check{G}$-local system and let $\mathcal{F}_\sigma$ be the corresponding Hecke eigensheaf.*

Then the characteristic cycle $\text{CC}(\mathcal{F}_\sigma)$ of $\mathcal{F}_\sigma$ equals the global nilpotent cone.

We prove this result via the interaction of Betti and de Rham geometric Langlands, and it is closely related to perversity properties of $\mathbb{L}_{G,\text{temp}}$ used elsewhere in the paper, which is why we consider it relevant to this paper.

We refer to Sect. 6 for more details, including a review of other known properties of $\mathcal{F}_\sigma$.

0.5. **Contents.** We now review the contents of the paper by section.

0.5.1. In Sect. 1 we construct the Langlands functor $\mathbb{L}_G$ in the de Rham context.

We first collect the ingredients needed for the construction of the *coarse* Langlands functor (the spectral action, the vacuum Poincaré object).

We state Theorem 1.6.2, which is the tool that allows us to upgrade $\mathbb{L}_{G,\text{coarse}}$ to the Langlands functor $\mathbb{L}_G$.

We show that $\mathbb{L}_G$ is compatible with the actions of $\text{QCoh}(\text{LS}_{\check{G}})$ on the two sides.

0.5.2. In Sect. 2 we prove Theorem 1.6.2.

We show that compact objects of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ are bounded below (in fact, this is a general property of the category of D-modules on a *truncatable* algebraic stack with an affine diagonal), see Proposition 1.8.4. This reduces Theorem 1.6.2 to Theorem 2.1.2, which says that the functor $\mathbb{L}_{G,\text{coarse}}$ has a cohomological amplitude bounded on the left.

We express the condition on an object of $\text{QCoh}(\mathcal{Y})$ (here $\mathcal{Y}$ is an arbitrary eventually coconnective stack) to be bounded on the left in terms of its !-fibers, see Proposition 2.3.2.

The latter proposition allows us to reduce Theorem 2.1.2 to its version, where instead of $\mathbb{L}_{G,\text{coarse}}$, we are dealing with its restricted version, $\mathbb{L}^{\text{restr}}_{G,\text{coarse}}$. However, in the latter case, the required assertion has been already established in [FR].

By a similar manipulation, we prove Proposition 1.8.2, which says that the functor $\mathbb{L}_{G,\text{coarse}}$ (and, hence, $\mathbb{L}_G$) has a cohomological amplitude bounded on the right.

0.5.3. In Sect. 3 we construct the geometric Langlands functor $\mathbb{L}_G$ in the Betti setting.

We first review the basics of the category $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ (compact generation, relation to the entire category $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$).

We construct the Betti version of the vacuum Poincaré object, which requires a little bit of work, due to the fact that the exponential sheaf does not exist in the Betti context.

We state Theorem 3.4.4, which is a Betti counterpart of Theorem 1.6.2. Assuming this theorem, we construct $\mathbb{L}_G^{\mathrm{Betti}}$, the Betti version of the Langlands functor, by the same procedure as in the de Rham context.

We state Theorem 3.5.6, which says that the de Rham and Betti versions of GLC (along with their restricted variants) are all logically equivalent.

0.5.4. In Sect. 4, we prove Theorem 3.4.4.

We note that the proof of Theorem 3.4.4 does *not* mimic that of Theorem 1.6.2:

We do not a priori know that the compact generators of $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ are bounded below (although ultimately one can show that they are). Instead, we describe these compact generators explicitly and show directly that the functor $\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}}$ sends them to bounded below objects. While doing so, we study the interactions between the following functors:

- The *left* adjoint to $\mathbf{i}:\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)\hookrightarrow \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$;
- The *right* adjoint to $\iota^{\mathrm{Betti}}:\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)\hookrightarrow \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$;
- The *right* adjoint to $\mathbf{i}^{\mathrm{constr}}:\mathrm{Shv}^{\mathrm{Betti,contsr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)\hookrightarrow \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

In fact, it turns out that $(\mathbf{i}^{\mathrm{constr}})^R\simeq(\iota^{Betti})^R\circ\mathbf{i}\circ\mathbf{oblv}^{\mathrm{constr}}$, as functors

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G)\to \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G),$$

where $\mathbf{oblv}^{\mathrm{constr}}$ is the forgetful functor

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G)\to \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

0.5.5. In Sect. 5, we prove the implications

$$\text{Restricted de Rham GLC}\Rightarrow\text{Full de Rham GLC}\quad\text{and}\quad\text{Restricted Betti GLC}\Rightarrow\text{Full Betti GLC}$$

along the lines explained in Sect. 0.3.

We introduce the tempered GLC, in both full and restricted variants, and show, following [AGKRRV, Sect. 21.4] that these two variants are logically equivalent.

We then show that the tempered GLC is equivalent to the original full GLC.

0.5.6. In Sect. 6, we calculate characteristic cycles of Hecke eigensheaves (at least when G has connected center). We do this by proving constancy of characteristic cycles along the moduli of irreducible local systems and then appealing to [BD] to compute this constant value.

0.5.7. *Notations and conventions.* Notations and conventions in this paper largely follow those adopted in [AGKRRV].

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1. Construction of the Langlands functor (de Rham context)

In this section we will first review the ingredients that are needed for the construction of the Langlands functor, and then carry out the construction in question, all in the context of D-modules.

The "coarse" version of the functor will be automatic from the spectral action (see Sect. 1.4). To upgrade it to the actual Langlands functor, we will need some cohomological estimates, which are provided by Theorem 1.6.2.

1.1. **The automorphic category.** In this subsection we will introduce the main player on the automorphic side: the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ of (half-twisted) D-modules on the moduli space Bun_G of G-bundles on X.

1.1.1. Let $\det_{\text{Bun}_G}$ be the determinant line bundle on Bun_G, normalized so that it sends a G-bundle $\mathcal{P}_G$ to

$$\det\left(\Gamma(X, \mathfrak{g}_{\mathcal{P}_G})\right) \otimes \det\left(\Gamma(X, \mathfrak{g}_{\mathcal{P}^0_G})\right)^{\otimes -1},$$

where $\mathcal{P}^0_G$ is the trivial bundle.

Note also that up to the (constant) line $\det\left(\Gamma(X, \mathfrak{g}_{\mathcal{P}^0_G})\right)$, the line bundle $\det_{\text{Bun}_G}$ identifies with the canonical line bundle on Bun_G.

1.1.2. According to [BD, Sect. 4], the choice of $\omega_X^{\otimes \frac{1}{2}}$ gives rise to a square root of $\det_{\text{Bun}_G}$.

1.1.3. Let $\mathcal{Y}$ be a space, and let $\mathcal{G}$ be an étale μ_n-gerbe on $\mathcal{Y}$ for some $n \in \mathbb{N}$. Denote by

$$\text{D-mod}_{\mathcal{G}}(\mathcal{Y})$$

the corresponding category of twisted D-modules on $\mathcal{Y}$.

1.1.4. Let

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

be the twisted category of D-modules, corresponding to the μ_2-gerbe $\det^{\frac{1}{2}}_{\text{Bun}_G}$ of square roots of $\det_{\text{Bun}_G}$.

The category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ is the de Rham incarnation of the automorphic category. It is the primary object of study in the geometric Langlands theory.

1.1.5. Note, however, that by Sect. 1.1.2, we have an identification

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \simeq \text{D-mod}(\text{Bun}_G). \tag{1.1}$$

Such an identification can be convenient for the study of local properties of the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$. In particular, the results from [AGKRRV] pertaining to $\text{D-mod}_{\text{Nilp}}(\text{Bun}_G)$ apply to the corresponding category

$$\text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G).$$

Remark 1.1.6. It is crucial, however, to consider the twisted version, when we study the Hecke functors. Otherwise, instead of the usual category $\text{Rep}(\check{G})$ one has to consider its twist by a certain canonical $Z_{\check{G}}$-gerbe, see [GLys, Sect. 6.3].

1.2. **The spectral action.** An ingredient, whch is crucial for both for the construction of the Langlands functor and to the eventual proof that it is an equivalence, is the fact that the action of Hecke functors on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ gives rise to a monoidal action of the category $\text{QCoh}(\text{LS}_{\check{G}})$.

1.2.1. Let $\text{Rep}(\check{G})_{\text{Ran}}$ be the de Rham version of the category $\text{Rep}(\check{G})$ spread over the Ran space, see [Ga2, Sect. 4.2].

The (naive) geometric Satake functor and the Hecke action define an action of $\text{Rep}(\check{G})_{\text{Ran}}$ on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.

1.2.2. Recall now that we have a (symmetric) monoidal localization functor

$$\text{Loc}^{\text{spec}}_{\check{G}} : \text{Rep}(\check{G}) \to \text{QCoh}(\text{LS}_{\check{G}}), \tag{1.2}$$

see [Ga2, Sect. 4.3].

The category $\text{Rep}(\check{G})_{\text{Ran}}$ is compactly generated, and the functor $\text{Loc}^{\text{spec}}_{\check{G}}$ preserves compactness (being a symmetric monoidal functor between rigid symmetric monoidal categories). Hence, it admits a continuous right adjoint.

It is a basic fact that this right adjoint is fully faithful. I.e., the functor $\text{Loc}^{\text{spec}}_{\check{G}}$ is a localization.

1.2.3. We have the following result (see [Ga2, Corollary 4.5.5]):

Theorem 1.2.4. *The action of* $\text{Rep}(\check{G})_{\text{Ran}}$ *on* $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ *factors via the functor* (1.2).

1.2.5. Thanks to Theorem 1.2.4, we obtain a canonically defined action of the monoidal category $\text{QCoh}(\text{LS}_{\check{G}})$ on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.

We refer to it as the *spectral action*.

1.3. **The vacuum Poincaré sheaf.** The second ingredient for the construction of the Langlands functor $\mathbb{L}_G$ is the object of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, which is the image of $\mathcal{O}_{\text{LS}_{\check{G}}}$ under the would-be left adjoint $\mathbb{L}^L_G$ of $\mathbb{L}_G$.

This object is the *vacuum* Poincaré sheaf, introduced in this subsection. This particular choice for the image of $\mathcal{O}_{\text{LS}_{\check{G}}}$ reflects the idea, familiar from the classical theory of automorphic functions, that the Langlands correspondence is normalized by the Whittaker model.

1.3.1. Let $\mathcal{P}_T$ be any T-bundle. Consider the stack

$$\text{Bun}_{N,\mathcal{P}_T} \simeq \text{Bun}_B \underset{\text{Bun}_T}{\times} \text{pt},$$

where the map $\text{pt} \to \text{Bun}_T$ corresponds the point $\mathcal{P}_T$. Denote by $\mathfrak{p}$ the map

$$\text{Bun}_{N,\mathcal{P}_T} \to \text{Bun}_G\,.$$

Note that the pullback of $\det_{\text{Bun}_G}$ along this map is canonically constant. Denote the resulting line by

$$\mathfrak{l}_{G,N_{\mathcal{P}_T}}.$$

1.3.2. The Langlands functor depends on the choice of the square root $\omega_X^{\frac{1}{2}}$ of the canonical line bundle on X. Let $\rho(\omega_X)$ denote the T-bundle, induced by means of the homomorphism $2\rho : \mathbb{G}_m \to T$ from $\omega_X^{\frac{1}{2}}$, viewed as a $\mathbb{G}_m$-torsor. Consider the corresponding stack

$$\mathrm{Bun}_{N,\rho(\omega_X)}.$$

We claim:

Proposition 1.3.3. *The line $\mathfrak{l}_{G,N_{\rho(\omega_X)}}$ admits a canonical square root.*

Proof. Decompose $\mathfrak{g}$ with respect to the action of the principal SL_2

$$\mathfrak{g} \simeq \underset{e}{\oplus} V^e.$$

By definition, the line $\mathfrak{l}_{G,N_{\rho(\omega_X)}}$ is

$$\underset{e}{\otimes} \left(\det(\Gamma(X,(V^e)_{\rho(\omega_X)})) \otimes \det(\Gamma(X, V^e \otimes \mathcal{O}_X))^{\otimes -1}\right).$$

Decompose V^e into its weight spaces

$$V^e = \underset{n}{\oplus} V^e(n),$$

where each $V^e(n)$ is 1-dimensional.

We can write:

$$\det(\Gamma(X,(V^e)_{\rho(\omega_X)})) \otimes \det(\Gamma(X, V^e \otimes \mathcal{O}_X))^{\otimes -1} \simeq \tag{1.3}$$
$$\simeq \underset{n>0}{\otimes} \left(\det(\Gamma(X,\omega_X^{\otimes n})) \otimes \det(\Gamma(X,\omega_X^{\otimes -n})) \otimes \det(\Gamma(X,\mathcal{O}_X))^{\otimes -2}\right) \bigotimes$$
$$\bigotimes \underset{n>0}{\otimes} \left(V^e(n)^{\otimes n(2g-2)+(1-g)} \otimes V^e(-n)^{\otimes -n(2g-2)+(1-g)}\right).$$

We claim that each term of the form

$$\det(\Gamma(X,\omega_X^{\otimes n})) \otimes \det(\Gamma(X,\omega_X^{\otimes -n})) \otimes \det(\Gamma(X,\mathcal{O}_X))^{\otimes -2}. \tag{1.4}$$

admits a canonical square root.

Recall the formlula

$$\det(\Gamma(X,\mathcal{L}_1 \otimes \mathcal{L}_2)) \otimes \det(\Gamma(X,\mathcal{O}_X)) \simeq \det(\Gamma(X,\mathcal{L}_1)) \otimes \det(\Gamma(X,\mathcal{L}_2)) \otimes \mathrm{Weil}(\mathcal{L}_1,\mathcal{L}_2), \tag{1.5}$$

where $\mathrm{Weil}(-,-)$ is the Weil pairing.

We obtain that (1.4) is isomorphic to

$$\mathrm{Weil}(\omega_X^{\otimes n}, \omega_X^{\otimes n}). \tag{1.6}$$

Recall now that we have chosen a square root $\omega_X^{\otimes \frac{1}{2}}$ of ω_X. Then the expression in (1.6) is

$$\left(\mathrm{Weil}(\omega_X^{\otimes \frac{1}{2}}, \omega_X^{\otimes \frac{1}{2}})\right)^{\otimes 4n^2},$$

which is manifestly a tensor square.

We now claim that the tensor product

$$\underset{e}{\otimes} \underset{n>0}{\otimes} \left(V^e(n)^{\otimes n(2g-2)+(1-g)} \otimes V^e(-n)^{\otimes -n(2g-2)+(1-g)}\right) \tag{1.7}$$

admits a canonical square root.

Indeed, up to squares, the expression in (1.7) is isomorphic to

$$\left(\underset{e}{\otimes} \underset{n>0}{\otimes} (V^e(n) \otimes V^e(-n))\right)^{\otimes(1-g)} \simeq \left(\det(\mathfrak{n}) \otimes \det(\mathfrak{n}^-)\right)^{\otimes(1-g)}.$$

Now, the Killing form identifies $\mathfrak{n}^-$ with the dual of $\mathfrak{n}$, and hence trivializes the line $\det(\mathfrak{n}) \otimes \det(\mathfrak{n}^-)$.

□

1.3.4. Thanks to Proposition 1.3.3, we have a well-defined functor

$$\mathfrak{p}^! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{D-mod}(\text{Bun}_{N,\rho(\omega_X)}),$$

with a partially defined left adjoint, denoted $\mathfrak{p}_!$.

1.3.5. Note now that there is a canonical map

$$\chi : \text{Bun}_{N,\rho(\omega_X)} \to \mathbb{G}_a.$$

Indeed, for every vertex i of the Dynkin diagram of G, we have a map

$$\text{Bun}_{N,\rho(\omega_X)} \to \text{Bun}_{N_i,\rho_i(\omega_X)},$$

where (N_i, χ_i) is the corresponding pair for the subminimal Levi subgroup attached to i.

For every i, we have a canonical map

$$\chi_i : \text{Bun}_{N_i,\rho_i(\omega_X)} \to \mathbb{G}_a$$

that records the class of the extension.

We let χ be the map

$$\text{Bun}_{N,\rho(\omega_X)} \to \prod_i \text{Bun}_{N_i,\rho_i(\omega_X)} \overset{\prod_i \chi_i}{\longrightarrow} \prod_i \mathbb{G}_a \overset{\text{sum}}{\to} \mathbb{G}_a.$$

1.3.6. We let

$$\text{Poinc}^{\text{Vac,glob}}_{G,!} \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

be the object equal to

$$\mathfrak{p}_! \circ \chi^*(\exp),$$

where:

- $\exp \in \text{D-mod}(\mathbb{G}_a)$ is the exponential D-module, normalized so that
$$\text{add}^*(\exp) \simeq \exp \boxtimes \exp,$$
(i.e., $\exp \in \text{D-mod}(\mathbb{G}_a)^{\heartsuit}[-1]$);
- The functor $\mathfrak{p}_!$ is well-defined on $\chi^*(\exp)$, since the latter is holonomic.

1.3.7. Let

$$\text{coeff}^{\text{Vac,glob}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{Vect}$$

denote the functor co-represented by $\text{Poinc}^{\text{Vac,glob}}_{G,!}$.

Explicitly,

$$\text{coeff}^{\text{Vac,glob}} \simeq \text{C}^{\cdot}(\text{Bun}_{N,\rho(\omega_X)}, \mathfrak{p}^!(-) \overset{*}{\otimes} \exp_\chi),$$

where[6]

$$\exp_\chi := \chi^*(\exp) \in \text{D-mod}(\text{Bun}_{N,\rho(\omega_X)}).$$

1.4. **The coarse version of the functor.** In this subsection we will construct a *coarse* version of the Langlands functor $\mathbb{L}_G$, denoted $\mathbb{L}_{G,\text{coarse}}$.

The difference between the two versions is that $\mathbb{L}_G$ is supposed to take values in the category $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$, while $\mathbb{L}_{G,\text{coarse}}$ takes values in the usual category $\text{QCoh}(\text{LS}_{\check{G}})$.

1.4.1. Note that by Sect. 1.2.5, a choice of an object in $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ defines a $\text{QCoh}(\text{LS}_{\check{G}})$-linear functor $\text{QCoh}(\text{LS}_{\check{G}}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.

[6] Strictly speaking, in the formula below one should have replaces χ by $-\chi$. However, it does not matter because the situation is equivariant with respect to the action of T.

1.4.2. We define the functor

$$\mathbb{L}^L_{G,\text{temp}} : \text{QCoh}(\text{LS}_{\check{G}}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G). \tag{1.8}$$

to correspond to the object

$$\text{Poinc}^{\text{Vac,glob}}_{G,!} \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G). \tag{1.9}$$

Since the object $\text{Poinc}^{\text{Vac,glob}}_{G,!}$ is compact and the monoidal category $\text{QCoh}(\text{LS}_{\check{G}})$ is compactly generated and rigid, the functor $\mathbb{L}^L_{G,\text{temp}}$ preserves compactness.

Remark 1.4.3. The notation $\mathbb{L}^L_{G,\text{temp}}$ has to do with the fact that the essential image of this functor lands in the subcategory

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G),$$

see Sect. 5.1.4.

Remark 1.4.4. Once GLC (Conjecture 1.6.7) is proved, it would follow that the functor $\mathbb{L}^L_{G,\text{temp}}$ is actually an equivalence

$$\text{QCoh}(\text{LS}_{\check{G}}) \overset{\sim}{\to} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}},$$

see Remark 5.1.7.

1.4.5. We let

$$\mathbb{L}_{G,\text{coarse}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{QCoh}(\text{LS}_{\check{G}})$$

be the right adjoint of $\mathbb{L}^L_{G,\text{temp}}$.

Since the category $\text{QCoh}(\text{LS}_{\check{G}})$ is compactly generated and the functor $\mathbb{L}^L_{G,\text{temp}}$ preserves compactness, the functor $\mathbb{L}_{G,\text{temp}}$ is continuous. By rigidity, $\mathbb{L}_{G,\text{temp}}$ is automatically $\text{QCoh}(\text{LS}_{\check{G}})$-linear.

1.4.6. One can equivalently describe $\mathbb{L}_{G,\text{coarse}}$ as follows. It is uniquely characterized by the following two pieces of structure:

- $\mathbb{L}_G$ is $\text{QCoh}(\text{LS}_{\check{G}})$-linear;
- $\Gamma(\text{LS}_{\check{G}}, \mathbb{L}_{G,\text{coarse}}(-)) \simeq \text{coeff}^{\text{Vac,glob}}$.

1.5. The case $G = T$.

1.5.1. Let $G = T$ be a torus. Consider the Fourier-Mukai equivalence

$$\text{FM} : \text{QCoh}(\text{Bun}_T) \to \text{QCoh}(\text{Bun}_{\check{T}}),$$

given by the Poincaré line bundle

$$\mathcal{L}_{\text{Poinc}} \in \text{QCoh}(\text{Bun}_T \times \text{Bun}_{\check{T}}),$$

as a kernel, where $\mathcal{L}_{\text{Poinc}}$, viewed as a map $\text{Bun}_T \times \text{Bun}_{\check{T}} \to B\mathbb{G}_m$, is given by the Weil pairing.

1.5.2. It is known that FM can be enhanced to an equivalence

$$\text{FM}^{\text{enh}} : \text{D-mod}(\text{Bun}_T) \to \text{QCoh}(\text{LS}_{\check{T}}),$$

that makes the following diagram commute:

$$\begin{array}{ccc}
\text{D-mod}(\text{Bun}_T) & \xrightarrow{\text{FM}^{\text{enh}}} & \text{QCoh}(\text{LS}_{\check{T}}) \\
\downarrow & & \downarrow \\
\text{QCoh}(\text{Bun}_T) & \xrightarrow{\text{FM}} & \text{QCoh}(\text{Bun}_{\check{T}}),
\end{array}$$

where:

- The functor $\text{D-mod}(\text{Bun}_T) \to \text{D-mod}(\text{Bun}_T)$ is $\textbf{oblv}^r$, the forgetful functor for "right" D-modules;
- The functor $\text{QCoh}(\text{LS}_{\check{T}}) \to \text{QCoh}(\text{Bun}_{\check{T}})$ is direct image along the projection

$$\text{LS}_{\check{T}} \to \text{Bun}_{\check{T}}.$$

1.5.3. Unwinding the definitions, we obtain that the functor $\mathbb{L}_{T,\text{coarse}}$ identifies with

$$\text{FM}^{\text{enh}}.$$

1.5.4. Note that in the case of $G = T$, the subset $\text{Nilp} \subset \text{Sing}(\text{LS}_{\check{T}})$ is the 0-section. Hence

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{T}}) = \text{QCoh}(\text{LS}_{\check{T}}).$$

We set

$$\mathbb{L}_T := \mathbb{L}_{T,\text{coarse}}.$$

So in this case there is no difference between the coarse Langlands functor and the sought-for Langlands functor $\mathbb{L}_T$.

1.6. **Statement of the main result.** In this subsection we will formulate Theorem 1.6.2, which would allow us to upgrade $\mathbb{L}_{G,\text{coarse}}$ to the actual Langlands functor $\mathbb{L}_G$.

1.6.1. The main technical result of this paper pertaining to the Langlands functor in the de Rham context reads:

Theorem 1.6.2. *The functor*

$$\mathbb{L}_{G,\text{coarse}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{QCoh}(\text{LS}_{\check{G}})$$

sends compact objects in $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ *to bounded below (i.e., eventually coconnective) objects in* $\text{QCoh}(\text{LS}_{\check{G}})$.

The proof will be given in Sect. 2. We now proceed to the consequences of Theorem 1.6.2 that pertain to the geometric Langlands functor.

1.6.3. Let $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ be the category of [AG, Sect. 11.1.5]. It is equipped with a forgetful functor

$$\Psi_{\text{Nilp},\{0\}} : \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \to \text{QCoh}(\text{LS}_{\check{G}}) \tag{1.10}$$

is t-exact and induces an equivalence

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})^{>-\infty} \to \text{QCoh}(\text{LS}_{\check{G}})^{>-\infty},$$

see [AG, Proposition 4.4.5].

1.6.4. Combining this with Theorem 1.6.2 and the fact that $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ is compactly generated (see [DG1, Theorem 0.1.2]), we obtain:

Corollary 1.6.5. *There exists a (colimit-preserving) functor*

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}),$$

uniquely characterized by the following properties:

- *The functor* $\mathbb{L}_G$ *sends compact objects in* $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ *to eventually coconnective objects in* $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$*, i.e., to* $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})^{>-\infty}$.
- $\Psi_{\text{Nilp},\{0\}} \circ \mathbb{L}_G \simeq \mathbb{L}_{G,\text{coarse}}$;

1.6.6. The functor $\mathbb{L}_G$ is *the* geometric Langlands functor in the de Rham context.

We can now state the geometric Langlands conjecture (GLC) in the de Rham context:

Conjecture 1.6.7. *The functor $\mathbb{L}_G$ is an equivalence.*

Remark 1.6.8. By the same logic as in Corollary 1.6.5, using Theorem 1.6.2, we can lift $\mathbb{L}_{G,\text{coarse}}$ to a functor

$$\mathbb{L}'_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}(\text{LS}_{\check{G}}),$$

so that

$$\Psi_{\text{all,Nilp}} \circ \mathbb{L}'_G \simeq \mathbb{L}_G,$$

where $\Psi_{\text{all,Nilp}}$ is the right adjoint of the inclusion

$$\Xi_{\text{Nilp,all}} : \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \to \text{IndCoh}(\text{LS}_{\check{G}}).$$

However, if we accept GLC, it follows from Lemma 5.4.6 below that the functor $\mathbb{L}'_G$ factors as

$$\Xi_{\text{Nilp,all}} \circ \mathbb{L}_G.$$

So the refinement $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \rightsquigarrow \text{IndCoh}(\text{LS}_{\check{G}})$ does not give us anything new.

Remark 1.6.9. That said, as was suggested by D. Arinkin, one can consider a *renormalized* version[7] $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{ren}}$ of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, equipped with a pair of adjoint functors

$$\text{ren} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{ren}} : \text{un-ren},$$

and one can refine $\mathbb{L}_G$ to a functor

$$\mathbb{L}_{G,\text{ren}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{ren}} \to \text{IndCoh}(\text{LS}_{\check{G}}),$$

so that the diagrams

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{ren}} & \xrightarrow{\mathbb{L}_{G,\text{ren}}} & \text{IndCoh}(\text{LS}_{\check{G}}) \\ {\scriptstyle\text{un-ren}}\downarrow & & \downarrow{\scriptstyle\Psi_{\text{all,Nilp}}} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \end{array}$$

and

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{ren}} & \xrightarrow{\mathbb{L}_{G,\text{ren}}} & \text{IndCoh}(\text{LS}_{\check{G}}) \\ {\scriptstyle\text{ren}}\uparrow & & \uparrow{\scriptstyle\Xi_{\text{all,Nilp}}} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \end{array}$$

commute.

One can then formulate a renormalized version of GLC, which says that $\mathbb{L}_{G,\text{ren}}$ is an equivalence. This will be addressed in a separate paper.

1.7. **Compatibility with the spectral action.** In this subsection we will assume the validity of Theorem 1.6.2, and hence of Corollary 1.6.5.

We will show that the functor $\mathbb{L}_G$ naturally upgrades to a functor between $\text{QCoh}(\text{LS}_{\check{G}})$-module categories.

[7]The compact generators of the renormalization are !-extension of *locally compact* objects on q.c. open substacks of Bun_G.

1.7.1. Note that the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ is naturally a module over $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$, and the functor $\Psi_{\mathrm{Nilp},\{0\}}$ is naturally $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$-linear.

Consider $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ also as a module over $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$, see Sect. 1.2. We claim:

Proposition 1.7.2. *The functor* $\mathbb{L}_G$ *carries a uniquely defined* $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$*-linear structure, so that the induced* $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$*-linear structure on*

$$\Psi_{\mathrm{Nilp},\{0\}} \circ \mathbb{L}_G \simeq \mathbb{L}_{G,\mathrm{coarse}}$$

is the natural $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$*-linear structure on* $\mathbb{L}_{G,\mathrm{coarse}}$.

Proof. Since $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ is compactly generated and rigid, in the statement of the proposition we can replace the monoidal category $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ by its full subcategory $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c$.

By rigidity, the action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ preserves compactness. Hence, it suffices to show that the functor

$$\mathbb{L}_G|_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$$

carries a uniquely defined $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c$-linear structure, so that its composition with $\Psi_{\mathrm{Nilp},\{0\}}$ reproduces the natural $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c$-linear structure on

$$\mathbb{L}_{G,\mathrm{coarse}}|_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}).$$

By the construction of $\mathbb{L}_G$, the restriction $\mathbb{L}_G|_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c}$ factors via a full subcategory

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).$$

Hence, it suffices to show that

$$\mathbb{L}_G|_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty}$$

carries a uniquely defined $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c$-linear structure, so that its composition with $\Psi_{\mathrm{Nilp},\{0\}}$ reproduces the natural $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c$-linear structure on

$$\mathbb{L}_{G,\mathrm{coarse}}|_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}})^{>-\infty}.$$

However, the latter is automatic since $\Psi_{\mathrm{Nilp},\{0\}}$ is an equivalence on the eventually coconnective subcategories.

□

1.8. **The bound on the right.** The contents of this subsection are tangential to rest of this paper (and will not be needed either for the construction of $\mathbb{L}_G$, or for the proof that it is an equivalence).

1.8.1. In Sect. 2.5 we will prove:

Proposition 1.8.2. *The functor* $\mathbb{L}_{G,\mathrm{coarse}}$ *has a bounded cohomological amplitude on the right, i.e., there exists an integer* d *so that* $\mathbb{L}_{G,\mathrm{coarse}}[d]$ *is right t-exact.*

1.8.3. Since the functor $\Psi_{\mathrm{Nilp},\{0\}}$ is t-exact and induces an equivalence on eventually coconnective subcategories, from Proposition 1.8.2 we obtain:

Corollary 1.8.4. *The functor* $\mathbb{L}_G$ *has a bounded cohomological amplitude on the right.*

2. Proof of Theorem 1.6.2

In this section we will prove Theorem 1.6.2, which was used in order to bootstrap the coarse Langlands functor $\mathbb{L}_{G,\mathrm{coarse}}$ to the actual Langlands functor $\mathbb{L}_G$.

The main input is a result of [FR], stated here as Theorem 2.3.8, which says that the *restricted* version of the coarse Langlands functor is t-exact (up to a cohomological shift).

2.1. **Strategy of the proof.** The statement of Theorem 1.6.2 follows immediately from the combination of the following two results:

Proposition 2.1.1. *Compact objects of* $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ *are bounded below (i.e., are eventually coconnective).*

Theorem 2.1.2. *The functor* $\mathbb{L}_{G,\text{coarse}}$ *has a bounded cohomological amplitude on the left, i.e., there exists an integer* d *so that* $\mathbb{L}_{G,\text{coarse}}[-d]$ *is left t-exact.*

Remark 2.1.3. In the course of the proof we will see that the integer d in Theorem 2.1.2 can be taken to be the dimension of the classical algebraic stack underlying $\text{LS}_{\check{G}}$ plus $\dim(\text{Bun}_{N,\rho(\omega_X)})$ minus $\dim(\text{Bun}_G)$.

Note that when the genus g of our curve is > 1 and G is semi-simple, the stack $\text{LS}_{\check{G}}$ is a classical l.c.i., so its dimension equals the virtual dimension, i.e., $(2g-2)\cdot\dim(G)$.

When G has a connected center (but g is still > 1) the dimension of the classical prestack underlying $\text{LS}_{\check{G}}$ is $(2g-2)\cdot\dim(G)+\dim(Z_G)$.

2.1.4. It is likely that the integer d in Remark 2.1.3 is not the sharp bound. For example, when $G=T$ is a torus, the functor $\mathbb{L}_{G,\text{coarse}}$ is left t-exact as-is. We propose:

Question 2.1.5. *What is the actual bound on the left amplitude of* $\mathbb{L}_{G,\text{coarse}}$*?*

Remark 2.1.6. Note that Theorem 2.1.2 does *not* imply that the functor $\mathbb{L}_G$ has a bounded cohomological amplitude on the left. In fact, this amplitude *is* unbounded, unless G is a torus.

Namely, we claim the "constant sheaf"" $\underline{k}_{\text{Bun}_G}\in\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ is sent by $\mathbb{L}_G$ to an *infinitely connective* object, i.e., an object that lies in $\text{IndCoh}_{\text{Nilp}}(\text{LS}_G)^{\leq -n}$ for any n.

Indeed, the statement is equivalent to $\Psi_{\text{Nilp},\{0\}}\circ\mathbb{L}_G(\underline{k}_{\text{Bun}_G})=0$, which is a valid statement, since $\mathbb{L}_{G,\text{coarse}}(\underline{k}_{\text{Bun}_G})$ is indeed zero.

2.2. **Proof of Proposition 2.1.1.** Before we launch the proof, let us explain what we are battling against: the stack Bun_G is not quasi-compact, and we have to estimate the cohomological amplitude of the !-direct image functors for open embeddings $U\hookrightarrow U'$, for a fixed quasi-compact open substack U and variable U'.

The idea of the proof is that (since Bun_G has an affine diagonal), these amplitudes only depend on U and not on U'.

2.2.1. The proof given below applies for the category $\text{D-mod}(\mathcal{Y})$, where $\mathcal{Y}$ is any *truncatable* algebraic stack (see [DG1, Sect. 0.2.3] for what this means), such that each of its quasi-compact open substacks is a global quotient[8] (i.e., is of the form Z/H, where Z is a (separated) scheme and H as affine algebraic group).

The stack Bun_G is truncatable by [DG1, Theorem 0.2.5]. And it is a standard fact that each of its quasi-compact open substacks is a global quotient.

2.2.2. First, we remark that the assertion is clear when $\mathcal{Y}$ is quasi-compact. Indeed, in this case the category $\text{D-mod}(\mathcal{Y})$ is compactly generated by objects of the form

$$\mathbf{ind}_r(\mathcal{M}),\quad \mathcal{M}\in\text{Coh}(\mathcal{Y}),$$

and such objects are bounded on both sides.

In the above formula,

$$\mathbf{ind}_r:\text{IndCoh}(\mathcal{Y})\to\text{D-mod}(\mathcal{Y})$$

is the left adjoint of

$$\mathbf{oblv}_r:\text{D-mod}(\mathcal{Y})\to\text{IndCoh}(\mathcal{Y}),$$

see [DG2, Sect. 5.1.5].

[8]It is likely that the latter assumption is not needed, and it suffices to assume that $\mathcal{Y}$ has an affine diagonal.

2.2.3. Let now $\mathcal{Y}$ be an arbitrary truncatable algebraic stack. Write $\mathcal{Y}$ as a union of *co-truncative* open substacks

$$U \overset{j}{\hookrightarrow} \mathcal{Y},$$

so that the functor

$$j_! : \text{D-mod}(U) \to \text{D-mod}(\mathcal{Y}),$$

left adjoint to the restriction functor $j^! = j^*$ is defined.

Then compact generators of D-mod($\mathcal{Y}$) are of the form $j_!(\mathcal{F}_U)$, where $\mathcal{F}_U$ are compact generators of D-mod(U).

2.2.4. Write $U = Z/H$, and let f denote the map $Z \to U$. Consider the *partially defined* left adjoint $f_!$ of the functor

$$f^! : \text{D-mod}(U) \to \text{D-mod}(Z).$$

As in [AGKRRV, Sect. F.3.5], we can take as compact generators of D-mod(U) objects of the form

$$f_!(\mathcal{F}_Z), \quad \mathcal{F}_Z \in \text{D-mod}(Z)^c,$$

where $\mathcal{F}_Z$ is such that the partially defined functor $f_!$ *is* defined on $\mathcal{F}_Z$.

It suffices to show that each of the objects

$$j_!(f_!(\mathcal{F}_Z)),$$

for $\mathcal{F}_Z$ as above, is bounded below.

Suppose that Z can be covered by n affines. We will show that if $\mathcal{F}_Z \in \text{D-mod}(Z)^{\geq n}$, then $j_!(f_!(\mathcal{F}_Z)) \in \text{D-mod}(\mathcal{Y})^{\geq 0}$. The assertion is equivalent to the fact that for any quasi-compact open $U' \supset U$, we have

$$j_!(f_!(\mathcal{F}_Z))|_{U'} \in \text{D-mod}(U')^{\geq 0}.$$

Since the specified bound does not depend on U', we can assume that $\mathcal{Y}$ itself is quasi-compact.

2.2.5. Denote $\widetilde{f} := j \circ f$. Note that $j_!(f_!(\mathcal{F}_Z))$ is isomorphic to the value on $\mathcal{F}_Z$ of the partially defined left adjoint $\widetilde{f}_!$ of $\widetilde{f}^!$.

Let $g : Y' \to \mathcal{Y}$ be a smooth cover, where Y' is an affine scheme. Set

$$Z' := Z \underset{\mathcal{Y}}{\times} Y'.$$

Denote by g_Z the map $Z' \to Z$. Let $\widetilde{f}'$ denote the map $Z' \to Y'$. Since g is smooth, the functors

$$g_Z^* : \text{D-mod}(Z) \to \text{D-mod}(Z') \text{ and } g^* : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(Y')$$

are well defined.

We obtain that the partially defined left adjoint $\widetilde{f}'_!$ of $\widetilde{f}'^!$ is well-defined on $g_Z^*(\mathcal{F}_Z)$, and we have

$$g^*(\widetilde{f}_!(\mathcal{F}_Z) \simeq \widetilde{f}'_!(g_Z^*(\mathcal{F}_Z)).$$

Hence, it suffices to show that for any

$$\mathcal{F}_{Z'} \in \text{D-mod}(Z')^c \cap \text{D-mod}(Z')^{\geq n},$$

on which the partially defined functor $\widetilde{f}'_!$ is defined, we have

$$\widetilde{f}'_!(\mathcal{F}_{Z'}) \in \text{D-mod}(Y')^{\geq 0}.$$

2.2.6. Note that the functor $\widetilde{f}'_!$ is always defined as a functor

$$\text{D-mod}(Z')^c \to \text{Pro}(\text{D-mod}(Y')^c),$$

and it is enough to show that this functor, shifted by $[-n]$, is left t-exact.

Let Z_i be the n affine schemes that cover Z, and let Z'_i denote their preimages in Z. For a subset $\underline{i}$ of indices, let $j'_{\underline{i}}$ denote the open embedding

$$\underset{i\in\underline{i}}{\cap} Z'_i =: Z'_{\underline{i}} \hookrightarrow Z'.$$

By Cousin, for $\mathcal{F} \in \text{D-mod}(Z')^c$, the object $\widetilde{f}'_!(\mathcal{F}) \in \text{Pro}(\text{D-mod}(Y')^c)$ admits a canonical filtration (of length n), whose m's subquotient is

$$\underset{|\underline{i}|=m}{\oplus} (\widetilde{f}' \circ j'_{\underline{i}})_!(\mathcal{F}|_{Z_{\underline{i}}})[m].$$

Hence, it is enough to show that each of the functors

$$(\widetilde{f}' \circ j'_{\underline{i}})_! : \text{D-mod}(Z'_{\underline{i}}) \to \text{Pro}(\text{D-mod}(Y')^c)$$

is left t-exact.

2.2.7. Note that $\mathcal{Y}$ has an affine diagonal. Hence, the morphism $Y' \to \mathcal{Y}$ is affine, and hence so is the projection $Z' \to Z$. We obtain that each $Z'_{\underline{i}}$ is an affine schenme, and hence so is the morphism $\widetilde{f}' \circ j'_{\underline{i}}$.

Now the desired left t-exactness assertion follows from [Ra, Theorem 3.4.1].

$\square$[Proposition 2.1.1]

2.3. **Proof of Theorem 2.1.2.** The idea of the proof is to reduce Theorem 2.1.2 to its restricted version, namely, Theorem 2.3.8, using a Cousin-type argument. The latter is encapsulated by Proposition 2.3.2 below.

2.3.1. We have the following assertion, proved in Sect. 2.4:

Proposition 2.3.2. *Let $\mathcal{Y}$ be an eventually coconnective algebraic stack. Let $\mathcal{M}$ be an object of* $\text{QCoh}(\mathcal{Y})$ *such that for every field extension $k \subset k'$ and a k'-point y of $\mathcal{Y}$, we have:*

$$i^!_y(\mathcal{M}') \in \text{Vect}^{\geq 0}_{k'},$$

where:

- $\mathcal{M}'$ *denotes the pullback of* $\mathcal{M}$ *along* $\mathcal{Y} \underset{\text{Spec}(k)}{\times} \text{Spec}(k') =: \mathcal{Y}' \to \mathcal{Y}$;
- i_y *denotes the map* $\text{Spec}(k') \to \mathcal{Y}'$ *corresponding to* y;
- $i^!_y$ *denotes the (not necessarily continuous) right adjoint of* $(i_y)_*$,

then $\mathcal{M} \in \text{QCoh}(\mathcal{Y})^{\geq -d}$, *where d is the dimension of the classical algebraic stack underlying $\mathcal{Y}$.*

Remark 2.3.3. In practice, the fields k' that will appear are fields of rational functions on irreducible subschemes of an affine scheme Y that smoothly covers $\mathcal{Y}$.

2.3.4. Let us prove Theorem 2.1.2.

Let d be the integer from Proposition 2.3.2 for the stack $\text{LS}_{\check{G}}$. We will show that the functor $\mathbb{L}_{G,\text{coarse}}$ has an amplitude bounded on the left by $d - \dim(\text{Bun}_G) + \dim(\text{Bun}_{N,\rho(\omega_X)})$.

Let $\mathcal{F}$ be an object of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, which is cohomologically $\geq \dim(\text{Bun}_{N,\rho(\omega_X)}) - \dim(\text{Bun}_G)$. Applying Proposition 2.3.2, we need to show that for every algebraically closed field extension $k \subset k'$ and a k'-point σ of

$$\text{LS}'_{\check{G}} := \text{LS}_{\check{G}} \underset{\text{Spec}(k)}{\times} \text{Spec}(k'),$$

the object

$$i^!_\sigma(\mathbb{L}_{G,\text{coarse}}(\mathcal{F})') \in \text{Vect}_{k'}$$

belongs to $\text{Vect}^{\geq 0}_{k'}$, where $\mathbb{L}_{G,\text{coarse}}(\mathcal{F})'$ denotes the pullback of $\mathbb{L}_{G,\text{coarse}}(\mathcal{F})$ to $\text{LS}'_{\check{G}}$.

Base changing everything along $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, we can assume that $k' = k$, so that σ is a rational point of $\mathrm{LS}_{\check{G}}$.

2.3.5. Consider the sub prestack

$$\mathrm{LS}_{\check{G}}^{\mathrm{restr}} \subset \mathrm{LS}_{\check{G}},$$

see [AGKRRV, Sect. 4.1].

Let

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{restr}} \overset{\iota^{\mathrm{spec}}}{\hookrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

be the (fully faithful) embedding of the subcategory consisting of objects in with set-theoretic support on $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$. Let $(\iota^{\mathrm{spec}})^R$ denote the right adjoint functor.

2.3.6. Note that for a k-rational point σ of $\mathrm{LS}_{\check{G}}$, the morphism i_σ factors as

$$\mathrm{Spec}(k) \to \mathrm{LS}_{\check{G}}^{\mathrm{restr}} \to \mathrm{LS}_{\check{G}},$$

where the second arrow is the map of [AGKRRV, Equation (4.2)].

Hence, the functor $(i_\sigma)_*$ factors via the subcategory $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{restr}}$. Hence, the functor $\iota_\sigma^!$ factors as

$$(i_\sigma^!|_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{restr}}}) \circ (\iota^{\mathrm{spec}})^R.$$

Since the functor $i_\sigma^!|_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{restr}}}$ is right t-exact, it suffices to show that

$$(\iota^{\mathrm{spec}})^R(\mathbb{L}_{G,\mathrm{coarse}}(\mathcal{F})) \in \mathrm{Vect}^{\geq 0}.$$

2.3.7. Let

$$\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\iota}{\hookrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

be the full subcategory consisting of D-modules with singular support in the global nilpotent cone.

According to [AGKRRV, Proposition 14.5.3], we have

$$\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{restr}}. \tag{2.1}$$

Hence, the restriction of $\mathbb{L}_{G,\mathrm{coarse}}$ to $\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ is a functor

$$\mathbb{L}_{G,\mathrm{coarse}}^{\mathrm{restr}} : \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{restr}}.$$

Furthermore, we have,

$$\mathbb{L}_{G,\mathrm{coarse}}^{\mathrm{restr}} \circ \iota^R \simeq (\iota^{\mathrm{spec}})^R \circ \mathbb{L}_{G,\mathrm{coarse}}.$$

Since the functor ι is t-exact, the functor ι^R is left t-exact. Hence, it suffices to show that show that the functor $\mathbb{L}_{G,\mathrm{coarse}}^{\mathrm{restr}}[\dim(\mathrm{Bun}_G) - \dim(\mathrm{Bun}_{N,\rho(\omega_X)})]$ is left t-exact.

However, this is the "left t-exactness half" of the following result of [FR] (see Sect. 1.6.2 in *loc. cit.*):

Theorem 2.3.8. *The functor*

$$\mathbb{L}_{G,\mathrm{coarse}}^{\mathrm{restr}}[\dim(\mathrm{Bun}_G) - \dim(\mathrm{Bun}_{N,\rho(\omega_X)})] : \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}).$$

is t-exact.

□[Theorem 2.1.2]

Remark 2.3.9. We emphasize that, unlike $\mathbb{L}_{G,\mathrm{coarse}}^{\mathrm{restr}}$, the functor $\mathbb{L}_{G,\mathrm{coarse}}$ (shifted cohomologically by $[\dim(\mathrm{Bun}_G) - \dim(\mathrm{Bun}_{N,\rho(\omega_X)})]$), is *not* right t-exact. One can see this already when $G = \mathbb{G}_m$, in which case $\mathbb{L}_{G,\mathrm{coarse}}$ is the Fourier-Laumon transform.

Note, however, that Proposition 1.8.2 says that its cohomological amplitude on the right is bounded.

2.4. **Proof of Proposition 2.3.2.**

2.4.1. Let $f : Y \to \mathcal{Y}$ be a smooth cover of $\mathcal{Y}$ by an affine scheme. Since

$$\mathcal{M} \in \mathrm{QCoh}(\mathcal{Y})^{\geq 0} \Leftrightarrow f^!(\mathcal{M})[-\dim(Y/\mathcal{Y})] \in \mathrm{QCoh}(Y)^{\geq 0},$$

we can assume that $\mathcal{Y} = Y$ is an affine scheme.

2.4.2. Let $Z \overset{i_Z}{\hookrightarrow} Y$ be an irreducible subvariety and let

$$\eta_Z \overset{j_Z}{\hookrightarrow} Z$$

be the embedding of its generic point.

Let

$$\mathrm{QCoh}(Y)_Z \overset{(\widehat{i}_Z)_!}{\hookrightarrow} \mathrm{QCoh}(Y)$$

be the embedding of the full subcategory consisting of objects set-theoretically supported on Z. Let $(\widehat{i}_Z)^!$ denote the right adjoint of $(\widehat{i}_Z)_!$.

Let

$$j_Z^* : \mathrm{QCoh}(Y)_Z \rightleftarrows \mathrm{QCoh}(Y)_{\eta_Z} : (j_Z)_*$$

be the localization of $\mathrm{QCoh}(Y)_Z$ at the generic point.

Applying Cousin decomposition, we obtain that $\mathcal{M} \in \mathrm{QCoh}(Y)^{\geq -d}$ if and only if for every Z as above,

$$j_Z^* \circ (\widehat{i}_Z)^!(\mathcal{M}) \in \mathrm{QCoh}(Y)^{\geq -d}_{\eta_Z}. \tag{2.2}$$

2.4.3. We can represent the formal completion of Z in Y as

$$\underset{n}{\mathrm{colim}}\, Z_n,$$

where $Z_n \overset{i_{Z_n}}{\hookrightarrow} Y$ are regularly embedded closed subschemes of Y (see. e.g., [GaRo1, Proposition 6.7.4]).

We have

$$(\widehat{i}_Z)^!(\mathcal{M}) \simeq \underset{n}{\mathrm{colim}}\, (i_{Z_n})_* \circ (i_{Z_n})^*(\mathcal{M}).$$

Hence, in order to establish (2.2), it suffices to show that for every n above, we have

$$j_Z^* \circ (i_{Z_n})^!(\mathcal{M}) \in \mathrm{QCoh}(\eta_{Z_n})^{\geq -d}, \tag{2.3}$$

where η_{Z_n} is the localization of Z_n at η_Z.

2.4.4. Let K be the field of rational functions on Z, and let $\eta_{Z_n} = \mathrm{Spec}(A_{Z_n})$. Note, however, that since Y is eventually, coconnective, A_{Z_n} is an extension of finitely many copies of K with non-positive shifts.

Hence, in order to establish (2.3), it suffices to show that

$$j_Z^* \circ i_Z^!(\mathcal{M}) \in \mathrm{Vect}_K^{\geq -d}. \tag{2.4}$$

2.4.5. Denote

$$Y' := Y \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(K_Z),\ Z' := Z \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(K_Z);$$

let i'_Z denote the embedding $Z' \to Y'$, obtained by base-changing i_Z. Let $\mathcal{M}'$ denote the pullback of $\mathcal{M}$ to Y'.

Note that Z' has a canonical K-rational point z_{can}. Denote by i_{can} the corresponding morphism $\mathrm{Spec}(K) \to Z'$.

The morphism j_Z factors as

$$\mathrm{Spec}(K) \overset{i_{\mathrm{can}}}{\longrightarrow} Z' \to Z.$$

Hence,

$$j_Z^* \circ i_Z^!(\mathcal{M}) \simeq (i_{\mathrm{can}})^* \circ (i'_Z)^!(\mathcal{M}').$$

2.4.6. Note, however, that z_{can} is a smooth point of Z'. Hence,

$$(i_{\text{can}})^* \simeq (i_{\text{can}})^![\dim(Z)]$$

(up to a determinant line, which we ignore).

Hence, we can further rewrite

$$j_Z^* \circ i_Z^!(\mathcal{M}) \simeq (i_{\text{can}})^! \circ (i'_Z)^!(\mathcal{M}')[\dim(Z)] = (i'_Z \circ i_{\text{can}})^!(\mathcal{M})[\dim(Z)].$$

2.4.7. Now, the condition of the proposition implies that

$$(i'_Z \circ i_{\text{can}})^!(\mathcal{M}) \in \text{Vect}_K^{\geq 0}.$$

Hence,

$$(i'_Z \circ i_{\text{can}})^!(\mathcal{M})[\dim(Z)] \in \text{Vect}_K^{\geq -d},$$

as desired.

□[Proposition 2.3.2]

2.5. **Proof of Proposition 1.8.2.**

2.5.1. We have the following counterpart of Proposition 2.3.2 for right t-exactness:

Proposition 2.5.2. *Let $\mathcal{Y}$ be an eventually coconnective algebraic stack. There exists an integer d such that the following holds:*

If $\mathcal{M}$ be an object of $\text{QCoh}(\mathcal{Y})$ such that for every field extension $k \subset k'$ and a k'-point y of $\mathcal{Y}$, we have:

$$i_y^*(\mathcal{M}') \in \text{Vect}_{k'}^{\leq 0}.$$

Then $\mathcal{M} \in \text{QCoh}(\mathcal{Y})^{\leq d}$.

We take d to be the dimension of the classical scheme underlying an affine smooth cover $Y \to \mathcal{Y}$.

The proof is parallel to that of Proposition 2.3.2. The only difference is that we use the fact that

$$j_Z^* \circ (i_{Z_n})^! \simeq j_Z^* \circ (i_{Z_n})^*[-\operatorname{codim}(Z)].$$

2.5.3. We proceed to the proof of Proposition 1.8.2. As a main ingredient we will use the fact that the functor

$$\iota^R : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)$$

has a bounded cohomological dimension on the right, see [AGKRRV, Corollary 14.5.5 and Proposition 17.3.10]. Let d' denote this bound.

We will show that the functor $\mathbb{L}_{G,\text{coarse}}$ has a cohomological amplitude bounded on the right by $d + d' + \dim(\text{Bun}_G) - \dim(\text{Bun}_{N,\rho(\omega_X)})$, where d is the integer from Proposition 2.5.2 for the stack $\text{LS}_{\check{G}}$.

2.5.4. Applying Proposition 2.5.2 and arguing as in Sect. 2.3.4, it suffices to show that for a k-rational point σ of $\text{LS}_{\check{G}}$, the functor

$$i_\sigma^* \circ \mathbb{L}_{G,\text{coarse}}[d' + \dim(\text{Bun}_G) - \dim(\text{Bun}_{N,\rho(\omega_X)})]$$

is right t-exact.

Note that the functor i_σ^* also factors as

$$(i_\sigma^*|_{\text{QCoh}(\text{LS}_{\check{G}})_{\text{restr}}}) \circ (\iota^{\text{spec}})^R,$$

while the functor $i_\sigma^*|_{\text{QCoh}(\text{LS}_{\check{G}})_{\text{restr}}}$ is right t-exact.

Hence, it is enough to show that the functor

$$(\iota^{\text{spec}})^R \circ \mathbb{L}_{G,\text{coarse}}[d' + \dim(\text{Bun}_G) - \dim(\text{Bun}_{N,\rho(\omega_X)})]$$

is right t-exact.

2.5.5. We rewrite $(\iota^{\mathrm{spec}})^R \circ \mathbb{L}_{G,\mathrm{coarse}}$ as

$$\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{course}} \circ \iota^R.$$

Now, the desired assertion follows from the fact that $\iota^R[d']$ is right t-exact and the fact that $\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}[\dim(\mathrm{Bun}_G) - \dim(\mathrm{Bun}_{N,\rho(\omega_X)})]$ is right t-exact (this is the "right t-exactness half" of Theorem 2.3.8).

$\square$[Proposition 1.8.2]

3. Geometric Langlands functor in the Betti context

In this section we take our ground field k to be $\mathbb{C}$. Throughout this section we will work with the "big" category of Betti sheaves of e-vector spaces (see [AGKRRV, Appendix G]), where e is an arbitrary field of coefficients (assumed of characteristic zero).

We will construct the Langlands functor in the Betti setting.

Once the functor is constructed, we will formulate a theorem to the effect that this functor is an equivalence if and only if its de Rham counterpart is.

3.1. **The category with nilpotent singular support.** In this subsection we will introduce (following [BZN]) the automorphic category in the Betti context.

3.1.1. Note that since $\det^{\frac{1}{2}}_{\mathrm{Bun}_G}$ is a gerbe with respect to $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$, it gives rise to an étale $\mathsf{e}^\times$-gerbe for any field e of characteristic different from 2.

Hence, it makes sense to consider the category $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$ of sheaves of e-vector spaces in the classical topology on Bun_G, twisted by $\det^{\frac{1}{2}}_{\mathrm{Bun}_G}$.

Note, however, that by Sect. 1.1.2, we can identify

$$\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}^{\mathrm{Betti}}(\mathrm{Bun}_G).$$

3.1.2. Let

$$\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbf{i}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

be the full subcategory consisting of sheaves with singular support contained in Nilp.

According to [AGKRRV, Sect. 18.2.6], the functor $\mathbf{i}$ admits a left adjoint, to be denoted $\mathbf{i}^L$.

3.1.3. The functor $\mathbf{i}^L$ allows us to construct compact objects in $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

For a stack $\mathcal{Y}$ and a rational point y, let $\delta_y \in \mathrm{Shv}^{\mathrm{Betti}}(\mathcal{Y})$ be the corresponding δ-function object, i.e., $(i_y)_!(\mathsf{e})$, where i_y be the morphism

$$\mathrm{pt} \to \mathcal{Y}$$

corresponding to y.

According to [AGKRRV, Proposition G.3.5], objects of the form

$$\mathbf{i}^L(\delta_y), \quad y \in \mathrm{Bun}_G$$

form a set of compact generators of $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Remark 3.1.4. Note, however, that objects $\delta_y \in \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$ are *not* compact, and that the category $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$ itself is *not* compactly generated.

3.2. **The Hecke action in the Betti context.**

3.2.1. Let $\operatorname{Rep}(\check{G})^{\mathrm{Betti}}_{\mathrm{Ran}}$ be the Betti version of the category $\operatorname{Rep}(\check{G})_{\mathrm{Ran}}$, defined as in [AGKRRV, Remark 11.1.10].

As in the de Rham setting, we have a localization functor

$$\operatorname{Loc}^{\mathrm{spec}}_{\check{G}} : \operatorname{Rep}(\check{G})^{\mathrm{Betti}}_{\mathrm{Ran}} \to \operatorname{QCoh}(\operatorname{LS}^{\mathrm{Betti}}_{\check{G}}), \tag{3.1}$$

which admits a fully faithful continuous right adjoint.

3.2.2. We have a canonically defined monoidal action of $\operatorname{Rep}(\check{G})^{\mathrm{Betti}}_{\mathrm{Ran}}$ on $\operatorname{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\operatorname{Bun}_G)$, which preserves the subcategory $\operatorname{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\operatorname{Bun}_G)$.

The following is a key observation from [NY] (see also [AGKRRV, Theorem 18.1.4]):

Proposition 3.2.3. *The action of* $\operatorname{Rep}(\check{G})^{\mathrm{Betti}}_{\mathrm{Ran}}$ *on* $\operatorname{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\operatorname{Bun}_G)$ *factors via the localization* (3.1).

As a result, we obtain that the category $\operatorname{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\operatorname{Bun}_G)$ carries a canonically defined action of the monoidal category $\operatorname{QCoh}(\operatorname{LS}^{\mathrm{Betti}}_{\check{G}})$.

3.3. **The vacuum Poincaré sheaf in the Betti context.** Our current goal is to construct the object

$$\operatorname{Poinc}^{\mathrm{Vac,glob}}_{!,\mathrm{Nilp}} \in \operatorname{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\operatorname{Bun}_G). \tag{3.2}$$

The slight hiccup is that the exponential sheaf is not defined as an object of $\operatorname{Shv}^{\mathrm{Betti}}(\mathbb{G}_a)$. In order to circumvent this, we will first apply the procedure of $\mathbb{G}_m$-averaging.

3.3.1. Define the object

$$\exp/\mathbb{G}_m \in \operatorname{Shv}^{\mathrm{Betti}}(\mathbb{G}_a/\mathbb{G}_m)$$

to be the *-extension of

$$\mathsf{e}[-1] \in \operatorname{Vect} = \operatorname{Shv}^{\mathrm{Betti}}(\mathrm{pt})$$

along the open embedding

$$\mathrm{pt} \simeq (\mathbb{G}_a - \{0\})/\mathbb{G}_m \hookrightarrow \mathbb{G}_a/\mathbb{G}_m.$$

Remark 3.3.2. Note that when $\mathsf{e} = \mathbb{C}$, under the Riemann-Hilbert equivalence, the object $\exp/\mathbb{G}_m$ corresponds to the !-direct image of exp along the projection

$$\mathbb{G}_a \to \mathbb{G}_a/\mathbb{G}_m.$$

3.3.3. We will now consider a variant of this construction. Consider the stack

$$\prod_i \mathbb{G}_a/T,$$

where T acts on $\prod_i \mathbb{G}_a$ by means of

$$T \to T/Z_G \overset{\text{simple roots}}{\simeq} \prod_i \mathbb{G}_m.$$

Consider the open embedding

$$\mathrm{pt}/Z_G \simeq \left(\prod_i (\mathbb{G}_a - 0)\right)/T \hookrightarrow (\prod_i \mathbb{G}_a)/T. \tag{3.3}$$

Let

$$(\boxtimes_i \exp)/T \in \operatorname{Shv}^{\mathrm{Betti}}\left((\prod_i \mathbb{G}_a)/T\right)$$

be the *-direct image along (3.3) of

$$R_{Z_G}[-r] \in \operatorname{Shv}^{\mathrm{Betti}}(\mathrm{pt}/Z_G),$$

where:

- R_{Z_G} is the !-direct image of $\mathsf{e} \in \operatorname{Vect} = \operatorname{Shv}^{\mathrm{Betti}}(\mathrm{pt})$ along $\mathrm{pt} \to \mathrm{pt}/Z_G$;
- r is the semi-simple rank of G.

Remark 3.3.4. It follows from Remark 3.3.2 and the projection formula that when $\mathsf{e} = \mathbb{C}$, under the Riemann-Hilbert equivalence, the object $(\underset{i}{\boxtimes} \exp)/T$ is the !-direct image along the projection

$$\prod_i \mathbb{G}_a \to (\prod_i \mathbb{G}_a)/T$$

of

$$\text{add}^*(\exp) \simeq \underset{i}{\boxtimes} \exp .$$

3.3.5. Note that the map

$$\mathfrak{p} : \text{Bun}_{N,\rho(\omega_X)} \to \text{Bun}_G$$

factors as

$$\text{Bun}_{N,\rho(\omega_X)} \to \text{Bun}_{N,\rho(\omega_X)}/T \overset{\mathfrak{p}/T}{\to} \text{Bun}_G .$$

By construction, the map $\text{Bun}_{N,\rho(\omega_X)} \overset{\chi}{\to} \mathbb{G}_a$ factors as

$$\text{Bun}_{N,\rho(\omega_X)} \overset{\underline{\chi}}{\to} \prod_i \mathbb{G}_a \overset{\text{sum}}{\to} \mathbb{G}_a.$$

Consider the resulting map

$$\text{Bun}_{N,\rho(\omega_X)}/T \overset{\underline{\chi}/T}{\longrightarrow} (\prod_i \mathbb{G}_a)/T.$$

Let

$$\exp_\chi/T \in \text{Shv}^{\text{Betti}}(\text{Bun}_{N,\rho(\omega_X)}/T)$$

be the pullback of the object $(\underset{i}{\boxtimes} \exp)/T$ along the map $\underline{\chi}/T$.

Remark 3.3.6. It follows from Remark 3.3.4 that when $\mathsf{e} = \mathbb{C}$, under the Riemann-Hilbert equivalence, the object $\exp_\chi/T$ corresponds to the !-direct image along the projection

$$\text{Bun}_{N,\rho(\omega_X)} \to \text{Bun}_{N,\rho(\omega_X)}/T$$

of the object

$$\exp_\chi := \chi^*(\exp) \in \text{D-mod}(\text{Bun}_{N,\rho(\omega_X)}).$$

3.3.7. Let

$$\text{Poinc}_!^{\text{Vac,glob}} := (\mathfrak{p}/T)_!(\exp_\chi/T) \in \text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G).$$

Finally, set

$$\text{Poinc}_{!,\text{Nilp}}^{\text{Vac,glob}} := \mathbf{i}^L(\text{Poinc}_!^{\text{Vac,glob}}).$$

3.3.8. We claim:

Proposition 3.3.9. *The object* $\text{Poinc}_{!,\text{Nilp}}^{\text{Vac,glob}} \in \text{Shv}^{\text{Betti}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)$ *is compact.*

Proof. Note that the object $(\underset{i}{\boxtimes} \exp)/T$ is a finite extension of δ-functions.

The map

$$\underline{\chi} : \text{Bun}_{N,\rho(\omega_X)} \to \prod_i \mathbb{G}_a)$$

is a unipotent gerbe. Hence, the map

$$\underline{\chi}/T : \text{Bun}_{N,\rho(\omega_X)}/T \to (\prod_i \mathbb{G}_a)/T$$

has a similar property.

It follows that the object $\exp_\chi/T \in \text{Shv}^{\text{Betti}}(\text{Bun}_{N,\rho(\omega_X)}/T)$ is a finite extension of δ-functions. Hence, so is $\text{Poinc}_!^{\text{Vac,glob}}$.

The assertion of the proposition follows now from Sect. 3.1.3.

□

3.4. **Construction of the functor.** Having the object $\text{Poinc}_{!,\text{Nilp}}^{\text{Vac,glob}}$ at our disposal, the construction of the Langlands functor in the Betti setting mimics its de Rham counterpart.

3.4.1. As in the de Rham context, we define the functor

$$\mathbb{L}^{\mathrm{Betti},L}_{G,\mathrm{temp}} : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}}) \to \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$$

to be given by acting on the object

$$\mathrm{Poinc}^{\mathrm{Vac,glob}}_{!,\mathrm{Nilp}} \in \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Since the object $\mathrm{Poinc}^{\mathrm{Vac,glob}}_{!,\mathrm{Nilp}}$ is compact and $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$ is compactly generated and rigid, the functor $\mathbb{L}^{\mathrm{Betti},L}_{G,\mathrm{temp}}$ preserves compactness.

3.4.2. Let

$$\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}} : \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$$

denote the right adjoint of $\mathbb{L}^{\mathrm{Betti},L}_{G,\mathrm{temp}}$.

Since $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$ is compactly generated and $\mathbb{L}^{\mathrm{Betti},L}_{G,\mathrm{temp}}$ preserves compactness, the functor $\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}}$ is continuous. By rigidity, it is automatically $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$-linear.

3.4.3. We have the following assertion, which is parallel to Theorem 1.6.2:

Theorem 3.4.4. *The functor*

$$\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}} : \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$$

sends compact objects in $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ *to bounded below objects in* $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$.

The proof will be given in Sect. 4.

3.4.5. Assuming Theorem 3.4.4, as in the de Rham context, we obtain that there exists a uniquely defined continuous functor

$$\mathbb{L}^{\mathrm{Betti}}_{G} : \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}}), \tag{3.4}$$

characterized by the requirements that:

- The functor $\mathbb{L}^{\mathrm{Betti}}_{G}$ sends compact objects in $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ to eventually coconnective objects in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$;
- $\Psi_{\mathrm{Nilp},\{0\}} \circ \mathbb{L}_G \simeq \mathbb{L}_{G,\mathrm{coarse}}$.

Moreover, the functor $\mathbb{L}^{\mathrm{Betti}}_{G}$ is automatically $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$-linear.

3.4.6. The geometric Langlands conjecture in the Betti context, originally formulated by D. Ben-Zvi and D. Nadler in [BZN], says:

Conjecture 3.4.7. *The functor* $\mathbb{L}^{\mathrm{Betti}}_{G}$ *of* (3.4) *is an equivalence.*

3.5. **Comparsion of de Rham and Betti versions of GLC.** Although the Betti and de Rham versions of the automorphic category (i.e., $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ and $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$) look very different, it turns out that the respective versions of GLC in the two contexts are logically equivalent.

The passage is realized by showing that either version is equivalent to its *restricted* variant, and the latter can be compared via Riemann-Hilbert.

The material in this subsection relies heavily on that of [AGKRRV].

3.5.1. We are going to prove:

Theorem 3.5.2. *The de Rham version of GLC implies the Betti version.*

In fact, we will prove a more precise version of Theorem 3.5.2, see Theorem 3.5.6 below.

3.5.3. Let

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$$

be the ind-constructible category, as defined in [AGKRRV, Sect. E.5].

Recall that according to [AGKRRV, Proposition 18.3.2], the forgetful functor

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is fully faithful.

Moreover, by [AGKRRV, Theorem 18.3.6], the subcategory

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{3.5}$$

equals

$$\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})_{\mathrm{restr}}.$$

Note also that we have an equivalence

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}}) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})_{\mathrm{restr}} \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}). \tag{3.6}$$

Hence, the functor $\mathbb{L}^{\mathrm{Betti}}_G$ induces a functor, to be denoted

$$\mathbb{L}^{\mathrm{Betti,restr}}_G : \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}).$$

3.5.4. Similarly, we have

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{restr}} \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}). \tag{3.7}$$

Hence, by (2.1), the functor $\mathbb{L}_G$ induces a functor

$$\mathbb{L}^{\mathrm{restr}}_G : \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}).$$

3.5.5. We will prove:

Theorem 3.5.6. *The following statements are logically equivalent:*

(i) *The functor $\mathbb{L}_G$ is an equivalence.*

(i') *The functor $\mathbb{L}^{\mathrm{restr}}_G$ is an equivalence;*

(ii) *The functor $\mathbb{L}^{\mathrm{Betti}}_G$ is an equivalence.*

(ii') *The functor $\mathbb{L}^{\mathrm{Betti,restr}}_G$ is an equivalence;*

Note that the implications (i) $\Rightarrow$ (i') and (ii) $\Rightarrow$ (ii') in Theorem 3.5.6 are immediate from the equivalences (3.7) and (3.6), respectively.

The equivalence of (i') and (ii') will be proved in Sect. 4.2.4.

The implications (i') $\Rightarrow$ (i) and (ii') $\Rightarrow$ (ii) will be proved in Sect. 5.

4. Proof of Theorem 3.4.4

The proof of Theorem 3.4.4 follows ideas, similar to its de Rham counterpart.

The main difference is that in the Betti setting, the embedding of the category with nilpotent singular support has a *left* adjoint, while in the de Rham setting it admitted a right adjoint.

4.1. **Reduction steps.** The idea of the proof is to reduce the assertion of Theorem 3.4.4 to statements that concern the restricted version of the functor $\mathbb{L}^{\mathrm{Betti}}_G$. Those are formulated as Propositions 4.1.5 and 4.1.6 at the end of this subsection.

4.1.1. Recall that the left adjoint $\mathbf{i}^L$ to the tautological embedding

$$\mathbf{i} : \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is well-defined.

Recall also (see Sect. 3.1.3) that objects of the form

$$\mathbf{i}^L(\delta_y), \quad y \in \mathrm{Bun}_G$$

form a set of compact generators of $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Thus, in order to prove Theorem 3.4.4, it suffices to show that the objects

$$\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}} \circ \mathbf{i}^L(\delta_y) \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$$

are bounded below.

4.1.2. Note that for an extension of fields of coefficients $\mathsf{e} \subset \mathsf{e}'$, the base change

$$\mathrm{LS}^{\mathrm{Betti}}_{\check{G}} \underset{\mathrm{Spec}(\mathsf{e})}{\times} \mathrm{Spec}(\mathsf{e}')$$

identifies with the stack of Betti local systems with e'-coefficients.

Hence, as in Sect. 2.3.4, it suffices to show that the objects

$$i^!_\sigma\left(\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}} \circ \mathbf{i}^L(\delta_y)\right) \in \mathrm{Vect}_{\mathsf{e}}$$

belong to $\mathrm{Vect}^{\geq n}_{\mathsf{e}}$, where n is an integer that only depends on y (but not on the choice of e or σ).

4.1.3. Let

$$\iota^{\mathrm{Betti,spec}} : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})_{\mathrm{restr}} \rightleftarrows \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}}) : (\iota^{\mathrm{Betti,spec}})^R$$

be the corresponding pair of adjoint functors.

As in Sect. 2.3.6, it suffices to show that

$$(\iota^{\mathrm{Betti,spec}})^R \circ \mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}} \circ \mathbf{i}^L(\delta_y) \in \mathrm{Vect}^{\geq n}_{\mathsf{e}},$$

where n only depends on y.

4.1.4. Consider the embedding

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\iota^{\mathrm{Betti}}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G),$$

see Sect. 3.5.3.

By (3.5), this embedding admits a continuous right adjoint, to be denoted $(\iota^{\mathrm{Betti}})^R$. Moreover, the functor $\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}}$ induces a functor, to be denoted

$$\mathbb{L}^{\mathrm{Betti,restr}}_{G,\mathrm{coarse}} : \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}),$$

and we have

$$(\iota^{\mathrm{Betti,spec}})^R \circ \mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}} \simeq \mathbb{L}^{\mathrm{Betti,restr}}_{G,\mathrm{coarse}} \circ (\iota^{\mathrm{Betti}})^R.$$

Hence, it suffices to show that

$$\mathbb{L}^{\mathrm{Betti,restr}}_{G,\mathrm{coarse}} \circ (\iota^{\mathrm{Betti}})^R \circ \mathbf{i}^L(\delta_y) \in \mathrm{Vect}^{\geq n}_{\mathsf{e}}.$$

This is obtained by combining the following two assertions:

Proposition 4.1.5. *Objects of the form*

$$(\iota^{\mathrm{Betti}})^R \circ \mathbf{i}^L(\delta_y) \in \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$$

are bounded below.

Proposition 4.1.6. *The functor* $\mathbb{L}^{\mathrm{Betti,restr}}_{G,\mathrm{coarse}}[\dim(\mathrm{Bun}_G) - \dim(\mathrm{Bun}_{N,\rho(\omega_X)})]$ *is t-exact.*

Remark 4.1.7. When it comes to t-exactness properties, there is a substantial difference between the de Rham and Betti settings: one can show that the entire functor

$$\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{coarse}}[\dim(\mathrm{Bun}_G) - \dim(\mathrm{Bun}_{N,\rho(\omega_X)})] : \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$$

is t-exact.

4.2. **Applications of Riemann-Hilbert.** In this subsection we will exploit the fact that Riemann-Hilbert allows us to compare directly the categories $\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ and $\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$, and also the prestacks $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}$ and $\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}$.

4.2.1. *Proof of Proposition 4.1.6.* With no restriction of generality, we can assume that $\mathsf{e} = \mathbb{C}$.

Consider the de Rham context with $k = \mathbb{C}$. Recall that according to [AGKRRV, Corollary 16.5.6], the embedding

$$\text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is an equality, where the superscript "RS" denotes to the sheaf-theoretic context of holonomic D-modules with regular singularities in the sense of [AGKRRV, Sect. 1.1.1][9].

Now, Riemann-Hilbert defines a t-exact equivalence

$$\text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$$

and an isomorphism of stacks

$$\mathrm{LS}^{\mathrm{restr}}_{\check{G}} \simeq \mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}.$$

Indeed, the above stacks in either context are defined by Tannakian formalism, which takes as an input the corresponding symmetric monoidal category of local systems $\mathrm{Lisse}(X)$ (see [AGKRRV, Sect. 1.2]), and these categories are identified by Riemann-Hilbert.

We claim now that Theorem 2.3.8 implies Proposition 4.1.6. Indeed, follows from the next assertion:

Lemma 4.2.2. *The diagram*

$$\begin{array}{ccc} \text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}^{\mathrm{restr}}_{G,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}^{\mathrm{Betti,restr}}_{G,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}) \end{array} \tag{4.1}$$

commutes.

Proof. We only have to show that the Riemann-Hilbert equivalence sents

$$\iota(\mathrm{Poinc}^{\mathrm{Vac.glob}}_{!}) \in \text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \text{ and } \mathrm{Poinc}^{\mathrm{Vac.glob}}_{!,\mathrm{Nilp}} \in \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Let us show that the functors that there objects co-represent get identified. However, in both contexts, the functor in question is

$$\mathcal{F} \mapsto \mathcal{H}om(\exp_\chi /T, (\mathfrak{p}/T)^!(\mathcal{F})),$$

see Sect. 3.3.5 and Remark 3.3.6

□

□[Proposition 4.1.6]

Remark 4.2.3. There was no actual need to apply to Riemann-Hilbert in order to prove Proposition 4.1.6: one could repeat the argument in [FR] verbatim in the Betti setting.

[9] Recall the formation of $\text{D-mod}^{\mathrm{RS}}(-)$ on a stack involves the operation of in-completion on affine schemes, and the inverse limit over affine schemes mapping to the given stack.

4.2.4. In the rest of this subsection we will assume Theorem 3.4.4, so that the functor $\mathbb{L}_G^{\mathrm{Betti}}$ is defined, and we will prove the equivalence of statements (i') and (ii') in Theorem 3.5.6.

By Lefschetz principle, we can assume that in (i') the ground field k is $\mathbb{C}$ and in (ii') the field e of coefficients is also $\mathbb{C}$.

Applying Riemann-Hilbert, it suffices to show that the diagram

$$\begin{array}{ccc} \text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G^{\mathrm{restr}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G^{\mathrm{Betti,restr}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}) \end{array} \tag{4.2}$$

commutes.

In order to prove the commutativity of (4.2), it suffices to show that the diagram

$$\begin{array}{ccc} \text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)^c & \xrightarrow{\mathbb{L}_G^{\mathrm{restr}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)^c & \xrightarrow{\mathbb{L}_G^{\mathrm{Betti,restr}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}) \end{array} \tag{4.3}$$

commutes.

4.2.5. Recall (see [AGKRRV, Theorem 16.1.1]) that the category

$$\text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is compactly generated generated, and since the right adjoint ι^R to the embedding

$$\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\iota}{\hookrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is continuous, compact objects of $\text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ are compact as objects of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

In particular, by Proposition 2.1.1, we obtain that the compact generators of $\text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ are bounded below. By Theorem 1.6.2 we obtain that the top horizontal arrow in (4.3) takes values in

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})^{>-\infty} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}).$$

Recall (see [AGKRRV, Sect. 18.3.2]) that the embedding ι^{Betti} preserves compactness. By Theorem 3.4.4 we obtain that the bottom horizontal arrow in (4.3) takes values in

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}})^{>-\infty} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}).$$

Hence, we can replace (4.3) by

$$\begin{array}{ccc} \text{D-mod}^{\mathrm{RS}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)^c & \xrightarrow{\mathbb{L}_G^{\mathrm{restr}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})^{>-\infty} \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)^c & \xrightarrow{\mathbb{L}_G^{\mathrm{Betti,restr}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}})^{>-\infty}. \end{array}$$

However, the commutativity of the latter diagram follows formally from the commutativity of (4.1). □[Equivalence of (i') and (ii')]

4.3. **Proof of Proposition 4.1.5.** The idea of the proof is to express the functor $(\iota^{\mathrm{Betti}})^R \circ \mathbf{i}^L$ via a particular Hecke functor, known as the *Beilinson projector*.

This will allow us to replace the left adjoint $\mathbf{i}^L$ by a right adjoint for some other functor, which has an evident left-exactness property, 4.3.1. Let $\mathsf{P}_{\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}}$ be the Beilinson projector from [AGKRRV, Sect. 15.4.5], viewed as an endofunctor of $\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

We claim:

Lemma 4.3.2. *The functor* $(\iota^{\mathrm{Betti}})^R \circ \mathbf{i}^L$*, followed by*

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\iota^{\mathrm{Betti}}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbf{i}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

identifies canonically with $\mathsf{P}_{\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}}$.

Let us assume this lemma and prove Proposition 4.1.5.

4.3.3. Denote by $\mathbf{i}^{\mathrm{constr}}$ the embedding

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

By [AGKRRV, Theorem 16.4.10 and Proposition 17.2.3], the functor $\mathbf{i}^{\mathrm{constr}}$ admits a continuous right adjoint. Moreover, the comonad $\mathbf{i}^{\mathrm{constr}} \circ (\mathbf{i}^{\mathrm{constr}})^R$ is given by the Beilinson projector $\mathsf{P}_{\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}}$, viewed as an endofunctor of $\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

Note also that we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbf{oblv}^{\mathrm{constr}}} & \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G) \\
{\scriptstyle \mathsf{P}_{\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}}}\downarrow & & \downarrow{\scriptstyle \mathsf{P}_{\mathrm{LS}^{\mathrm{Betti,restr}}_{\check{G}}}} \\
\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbf{oblv}^{\mathrm{constr}}} & \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G),
\end{array}$$

in which the horizontal arrows are the tautological forgetful functor.

Remark 4.3.4. Note that the functor

$$\mathbf{oblv}^{\mathrm{constr}} : \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G) \tag{4.4}$$

is not fully faithful. However, the composition

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbf{i}^{\mathrm{constr}}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathbf{oblv}^{\mathrm{constr}}}{\longrightarrow} \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is fully faithful, since it can be rewritten as

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\iota^{\mathrm{Betti}}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathbf{i}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

4.3.5. Applying Lemma 4.3.2, we obtain that we have a canonical isomorphism

$$(\iota^{\mathrm{Betti}})^R \circ \mathbf{i}^L \circ \mathbf{oblv}^{\mathrm{constr}} \simeq (\mathbf{i}^{\mathrm{constr}})^R$$

as functors

$$\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G).$$

In particular, since the functor $(\mathbf{i}^{\mathrm{constr}})^R$ is left t-exact, we obtain that if $\mathcal{F} \in \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G)$ is eventually coconnective, then so is $(\iota^{\mathrm{Betti}})^R \circ \mathbf{i}^L \circ \mathbf{oblv}^{\mathrm{constr}}(\mathcal{F})$.

4.3.6. Hence, it remains to show that the objects

$$\delta_y \in \mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

are eventually coconnective.

This follows by Riemann-Hilbert from Proposition 2.1.1 (or can be reproved directly within $\mathrm{Shv}^{\mathrm{Betti,constr}}_{\frac{1}{2}}(\mathrm{Bun}_G)$ by the same argument).

□[Proposition 4.1.5]

4.4. **Proof of Lemma 4.3.2.** The idea of the proof is that all functors in sight are given by the various versions of the Beilinson projector.

4.4.1. Recall (see [AGKRRV, Corollary 18.2.9(a)]) that the functor $\mathbf{i}\circ\mathbf{i}^L$ is given by the action of a *Hecke functor*, denoted $\mathsf{P}^{\text{enh}}_{\text{LS}^{\text{Betti}}_{\check{G}}}$.

Similarly, the functor $\iota^{\text{Betti}}\circ(\iota^{\text{Betti}})^R$ is given by the restriction to

$$\text{Shv}^{\text{Betti}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)\subset\text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G)$$

of the Hecke functor $\mathsf{P}_{\text{LS}^{\text{Betti,restr}}_{\check{G}}}$.

Now, any two Hecke functors commute, so we have

$$\mathbf{i}\circ\iota^{\text{Betti}}\circ(\iota^{\text{Betti}})^R\circ\mathbf{i}^L\simeq\mathsf{P}_{\text{LS}^{\text{Betti,restr}}_{\check{G}}}\circ\mathsf{P}_{\text{LS}^{\text{Betti}}_{\check{G}}}\simeq\mathsf{P}^{\text{enh}}_{\text{LS}^{\text{Betti}}_{\check{G}}}\circ\mathsf{P}_{\text{LS}^{\text{Betti,restr}}_{\check{G}}}.$$

4.4.2. Hence, in order to prove Lemma 4.3.2, it remains to show that

$$\mathsf{P}^{\text{enh}}_{\text{LS}^{\text{Betti}}_{\check{G}}}\circ\mathsf{P}_{\text{LS}^{\text{Betti,restr}}_{\check{G}}}\simeq\mathsf{P}_{\text{LS}^{\text{Betti,restr}}_{\check{G}}}.$$

I.e., we need to show that the functor $\mathsf{P}^{\text{enh}}_{\text{LS}^{\text{Betti}}_{\check{G}}}$ acts as identity on objects lying in the essential image of the functor $\mathsf{P}_{\text{LS}^{\text{Betti,restr}}_{\check{G}}}$.

By [AGKRRV, Corollary 18.2.9(a)], it suffices to show that the essential image of the endofunctor $\mathsf{P}_{\text{LS}^{\text{Betti,restr}}_{\check{G}}}$ is contained in $\text{Shv}^{\text{Betti}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)$.

4.4.3. Note that the essential image of the functor $\mathsf{P}_{\text{LS}^{\text{Betti,restr}}_{\check{G}}}$ lies in

$$\text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G)^{\text{Hecke-fin.mon.}}\subset\text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G)$$

(see [AGKRRV, Sect. 18.3]).

In particular, this essential image lies in

$$\text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G)^{\text{Hecke-loc.const.}}\subset\text{Shv}^{\text{Betti}}_{\frac{1}{2}}(\text{Bun}_G).$$

Hence, by [AGKRRV, Theorem 18.1.6], it belongs to $\text{Shv}^{\text{Betti}}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)$, as required.

□[Lemma 4.3.2]

5. Restricted vs full GLC

In this section we will be prove the implications (i') ⇒ (i) and (ii') ⇒ (ii) in Theorem 3.5.6.

While doing so, we will encounter yet another version of GLC (in both de Rham and Betti settings), namely, the *tempered* GLC. Ultimately, the logic of the proof (in either context) will be:

Full GLC ⇔ Full tempered GLC ⇔ Restricted tempered GLC ⇔ Restricted GLC.

5.1. **The de Rham context.** In this subsection we will introduce the tempered version of the Langlands functor. We will show that the full tempered GLC is equivalent to its restricted version.

Remark 5.1.1. The implication (i') ⇒ (i) was proved in [AGKRRV, Sect. 21.4], under the assumption that the functor $\mathbb{L}_G$ admits a left adjoint.

We will prove that this is the case in one of the subsequent papers in this series[10]. In this subsection we will give a different proof of this implication, which can then be adapted to the de Betti context.

[10] In fact, we will ultimately prove that $\mathbb{L}_G$ is an equivalence, so that (i) holds unconditionally.

5.1.2. Let
$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \overset{\mathbf{u}}{\hookrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$
denote the *tempered* subcategory, defined as in [AG, Sect. 17.8.4].

It follows easily from the definition that the above embedding admits a right adjoint, to be denoted $\mathbf{u}^R$. Thus, we obtain an adjunction
$$\mathbf{u} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) : \mathbf{u}^R,$$
and for most practical purposes, it is convenient to view $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$ is a localization of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.

Remark 5.1.3. The definition of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$ in *loc. cit.* depended on the choice of a point $x \in X$, and independence of the point was conjectural. This conjecture was established by Beraldo (unpublished) and later by Faegerman-Raskin by a different method (see [FR, Sect. 2.6.2]).

However, for the purposes of this paper, one can work with $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$ without knowing that it is point-independent.

5.1.4. We claim that the object $\text{Poinc}_!^{\text{Vac,glob}}$ belongs to $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$. This can be shown by re-interpreting $\text{Poinc}_!^{\text{Vac,glob}}$ via a local-to-global construction[11]. This will be discussed in detail in the sequel to this paper.

Since the Hecke action preserves the tempered subcategory, we obtain that the essential image of the functor $\mathbb{L}^L_{G,\text{temp}}$ is contained in $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$.

By adjunction, we obtain that the functor $\mathbb{L}_{G,\text{coarse}}$ factors as
$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \overset{\mathbf{u}^R}{\twoheadrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \overset{\mathbb{L}_{G,\text{temp}}}{\longrightarrow} \text{QCoh}(\text{LS}_G)$$
for a uniquely defined functor
$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \overset{\mathbb{L}_{G,\text{temp}}}{\longrightarrow} \text{QCoh}(\text{LS}_G).$$

Since
$$\Psi_{\text{Nilp},\{0\}} \circ \mathbb{L}_G \simeq \mathbb{L}_{G,\text{coarse}},$$
we have a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\ \mathbf{u}^R \big\downarrow & & \big\downarrow \Psi_{\text{Nilp},\{0\}} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} & \xrightarrow{\mathbb{L}_{G,\text{temp}}} & \text{QCoh}(\text{LS}_{\check{G}}), \end{array} \tag{5.1}$$

where the functor $\mathbb{L}_{G,\text{temp}}$ is the right adjoint of the functor $\mathbb{L}^L_{G,\text{temp}}$ of (1.8).

5.1.5. Set
$$\text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)_{\text{temp}} := \text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \cap \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G).$$

It follows from (2.1) that
$$\text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)_{\text{temp}} = \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}})_{\text{restr}},$$
as subcategories of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$.

It is easy to see that the functor $\mathbf{u}^R$ sends
$$\text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G) \to \text{D-mod}_{\frac{1}{2},\text{Nilp}}(\text{Bun}_G)_{\text{temp}}.$$

[11] Namely, $\text{Poinc}_!^{\text{Vac,glob}}$ is the image of the (vacuum) object in the *local Whittaker category* $\text{Whit}(G)_x$ under the Poincaré functor $\text{Poinc}_{!,x} : \text{Whit}(G)_x \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, while all of $\text{Whit}(G)_x$ is tempered.

Applying the functor

$$- \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})_{\mathrm{restr}},$$

from (5.1), we obtain a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G^{\mathrm{restr}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \\ {\scriptstyle \mathbf{u}^R}\downarrow & & \downarrow{\scriptstyle \Psi_{\mathrm{Nilp},\{0\}}} \\ \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} & \xrightarrow{\mathbb{L}_{G,\mathrm{temp}}^{\mathrm{restr}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}). \end{array} \tag{5.2}$$

5.1.6. Suppose that the functor $\mathbb{L}_G^{\mathrm{restr}}$ is an equivalence. We claim that this implies that $\mathbb{L}_{G,\mathrm{temp}}^{\mathrm{restr}}$ is also an equivalence.

Indeed, the fact that $\mathbb{L}_G^{\mathrm{restr}}$ is an equivalence implies that $\mathbb{L}_{G,\mathrm{temp}}^{\mathrm{restr}}$ is a Verdier quotient. However, we also know that the functor $\mathbb{L}_{G,\mathrm{temp}}^{\mathrm{restr}}$ is conservative: this is the main result of [FR].

Remark 5.1.7. One can avoid appealing to [FR] for the proof of "$\mathbb{L}_G^{\mathrm{restr}}$ is an equivalence" $\Rightarrow$ "$\mathbb{L}_{G,\mathrm{temp}}^{\mathrm{restr}}$ is an equivalence".

Indeed, it is not difficult to show that the functor $\mathbb{L}_G$ respects the action of the full Satake category (at a given point $x \in X$) on the two sides[12]. Now,

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \underset{\mathrm{Sat}_G}{\otimes} \mathrm{Sat}_{G,\mathrm{temp}}$$

and

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \underset{\mathrm{Sat}_G}{\otimes} \mathrm{Sat}_{G,\mathrm{temp}},$$

and also

$$\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \simeq \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{Sat}_G}{\otimes} \mathrm{Sat}_{G,\mathrm{temp}}$$

and

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \underset{\mathrm{Sat}_G}{\otimes} \mathrm{Sat}_{G,\mathrm{temp}}.$$

By a similar logic, if $\mathbb{L}_G$ is an equivalence, then it formally follows that $\mathbb{L}_{G,\mathrm{temp}}$ is also an equivalence.

Remark 5.1.8. In Sect. 5.2 we will prove an inverse implication (which is far less trivial): the fact that $\mathbb{L}_{G,\mathrm{temp}}$ is an equivalence implies that $\mathbb{L}_G$ is an equivalence.

The same argument will show that if $\mathbb{L}_{G,\mathrm{temp}}^{\mathrm{restr}}$ is an equivalence then $\mathbb{L}_G^{\mathrm{restr}}$ is an equivalence.

5.1.9. We now claim that *if* $\mathbb{L}_{G,\mathrm{temp}}^{\mathrm{restr}}$ is an equivalence (for all extensions of the ground field $k \subset k'$), *then* $\mathbb{L}_{G,\mathrm{temp}}$ is an equivalence.

This follows by repeating verbatim the argument of [AGKRRV, Sect. 21.4].

Remark 5.1.10. Both deductions:

$$\text{Full tempered GLC} \Rightarrow \text{Restricted tempered GLC},$$

carried out above, and

$$\text{Full GLC} \Rightarrow \text{Restricted GLC},$$

carried out in [AGKRRV, Sect. 21.4], follow the same logic, and both need that the corresponding version of the Langlands functor admit an adjoint. Now, the latter is immediate from the construction for $\mathbb{L}_{G,\mathrm{temp}}$, but requires more work for the original Langlands functor $\mathbb{L}_G$.

5.1.11. Thus, we obtain that in order to prove the implication (i') $\Rightarrow$ (i) in Theorem 3.5.6, it suffices to show *if* $\mathbb{L}_{G,\mathrm{temp}}$ is an equivalence *then* $\mathbb{L}_G$ is an equivalence.

This will be done on the next section.

[12]This will be elaborated on in the subsequent paper in this series.

5.2. **Full vs tempered Langlands.** In this subsection we will assume that $\mathbb{L}_{G,\text{temp}}$ is an equivalence and deduce that $\mathbb{L}_G$ is an equivalence.

The idea of the proof is that the behavior of $\mathbb{L}_{G,\text{temp}}$ recovers the behavior of $\mathbb{L}_G$ on *compact objects*.

5.2.1. Recall that the category $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}})$ is compactly generated (see [AG, Sect. 11.1.6]); its subcategory of compact objects is $\text{Coh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}})$, the category of *coherent* sheaves with nilpotent singular support.

Note that the functor

$$\Psi : \text{IndCoh}(\text{LS}_{\check{G}}) \to \text{QCoh}(\text{LS}_{\check{G}})$$

is fully faithful, when restricted to the subcategory of compact objects. Indeed, this restriction is the natural embedding

$$\text{Coh}(\text{LS}_{\check{G}}) \hookrightarrow \text{QCoh}(\text{LS}_{\check{G}})$$

.

In particular, we obtain that the functor $\Psi_{\text{Nilp},\{0\}}$ is also fully faithful when restricted to

$$\text{Coh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}}) = \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{restr}}_{\check{G}})^c.$$

5.2.2. The proof of the desired implication is a formal consequence of the combination of the following two assertions:

Proposition 5.2.3. *The functor*

$$\mathbf{u}^R|_{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^c} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^c \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$$

is fully faithful.

Proposition 5.2.4. *Assume that* $\mathbb{L}_{G,\text{temp}}$ *is an equivalence (for the group G and all its Levi subgroups). Then the essential image of* $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^c$ *in* $\text{QCoh}(\text{LS}_{\check{G}})$ *of* $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^c$ *under*

$$\mathbb{L}_{G,\text{temp}} \circ \mathbf{u}^R \simeq \mathbb{L}_{G,\text{coarse}}$$

equals $\text{Coh}_{\text{Nilp}}(\text{LS}_{\check{G}})$.

5.3. **Proof of Proposition 5.2.3.** The proof will be obtained by combining the following two ingredients. One is the *miraculous functor* on Bun_G. The other is a result of [Be1], which says that objects $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ that are *-extended from quasi-compact open substacks are tempered.

5.3.1. For an object $\mathcal{F} \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, let

$$\mathcal{F}_{\text{temp}} \to \mathcal{F} \to \mathcal{F}_{\text{anti-temp}}$$

be the fiber sequence associated with

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G).$$

Explicitly,

$$\mathcal{F} \simeq \mathbf{u} \circ \mathbf{u}^R(\mathcal{F}).$$

We have to show that for a pair of compact object $\mathcal{F}_1, \mathcal{F}_2 \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, the map

$$\text{Hom}_{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)}(\mathcal{F}_1, \mathcal{F}_2) \to \text{Hom}_{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)}(\mathcal{F}_{1,\text{temp}}, \mathcal{F}_{2,\text{temp}}) \tag{5.3}$$

is an isomorphism. As we shall see, just the assumption that $\mathcal{F}_2$ be compact will suffice.

The map (5.3) is tautologically an isomorphism if $\mathcal{F}_1$ is tempered (for any $\mathcal{F}_2$). Hence, it suffices to show that if $\mathcal{F}_2$ is compact and $\mathcal{F}_1$ is *anti-tempered*, i.e., if $\mathcal{F}_{1,\text{temp}} = 0$ or, equivalently, if

$$\mathcal{F}_1 \to \mathcal{F}_{1,\text{anti-temp}}$$

is an isomorphism, then

$$\text{Hom}_{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)}(\mathcal{F}_1, \mathcal{F}_2) = 0.$$

5.3.2. We will now use the *miraculous functor*

$$\mathrm{Mir}_{\mathrm{Bun}_G} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

see [Ga1, Sect. 3.1.1].

According to [Ga1, Theorem 3.1.5]), the functor $\mathrm{Mir}_{\mathrm{Bun}_G}$ is an equivalence. Hence, it is enough to show that for $\mathcal{F}_2$ compact and $\mathcal{F}_1$ anti-tempered,

$$\mathrm{Hom}_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}}(\mathrm{Mir}^{-1}_{\mathrm{Bun}_G}(\mathcal{F}_1), \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}(\mathcal{F}_2)) = 0. \tag{5.4}$$

5.3.3. Recall that compact objects in $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ are of the form

$$(j_U)_!(\mathcal{F}_U),$$

where

$$U \overset{j_U}{\hookrightarrow} \mathrm{Bun}_G$$

is the embedding of a quasi-compact open, and $\mathcal{F}_U \in \text{D-mod}(U)^c$.

With no restriction of generality, we can assume that U is *co-truncative* (see [DG1, Sect. 3.1] for what this means). In this case, the functor

$$\mathrm{Mir}_{\mathrm{Bun}_U} : \text{D-mod}_{\frac{1}{2}}(U) \to \text{D-mod}_{\frac{1}{2}}(U)$$

is an equivalence (see [DG1, Lemma 4.5.7]) and we have

$$\mathrm{Mir}^{-1}_{\mathrm{Bun}_G} \circ (j_U)_! \simeq (j_U)_{*,\mathrm{co}} \circ \mathrm{Mir}_U^{-1},$$

where

$$(j_U)_{*,\mathrm{co}} : \text{D-mod}(U) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}$$

is the tautological functor, see [DG1, Lemma 4.4.12].

Hence, in order the prove (5.4), it suffices to show that if $\mathcal{F} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ is anti-tempered, then

$$(j_U)^*_{\mathrm{co}} \circ \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}(\mathcal{F}) = 0,$$

where $(j_U)^*_{\mathrm{co}}$ is the left adjoint of $(j_U)_{*,\mathrm{co}}$.

5.3.4. Let

$$\mathrm{Id}^{\mathrm{nv}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

be the naive functor (see [Ga1, Sect. 2.1]). Recall that

$$(j_U)^*_{\mathrm{co}} \simeq (j_U)^* \circ \mathrm{Id}^{\mathrm{nv}}$$

(see [Ga1, Corollary 2.1.5]).

Hence, it suffices to show that the composite functor

$$\mathrm{Id}^{\mathrm{nv}} \circ \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}$$

annihilates the anti-tempered subcategory.

It is easy to see that both functors $\mathrm{Id}^{\mathrm{nv}}$ and $\mathrm{Mir}_{\mathrm{Bun}_G}$ commute with the Hecke action. Hence, it suffices to show that for any $\mathcal{F} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$, we have

$$(\mathrm{Id}^{\mathrm{nv}} \circ \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}(\mathcal{F}))_{\text{anti-temp}} = 0.$$

However, this follows follows from the fact the essential image of the functor $\mathrm{Id}^{\mathrm{nv}}$ is contained in $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}}$, see [Be1, Theorem B].

$\square$[Proposition 5.2.3]

5.4. **Proof of Proposition 5.2.4.** The idea of the proof is that compact generators of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ can be assembled from tempered compact objects, and *Eisenstein series.*

The behavior of the former is controlled by the assumption that $\mathbb{L}_{G,\text{temp}}$ is an equivalence. The behavior of the latter is an expression of a basic compatibility between Eisenstein series and the Langlands functor.

5.4.1. Let

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

be the full subcategory, generated by the Eisenstein functors

$$\text{Eis}_! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

for *proper* parabolic subgroups of G.

5.4.2. We claim:

Lemma 5.4.3. *The objects in*

$$\left(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}\right)^c \text{ and } \left(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}}\right)^c$$

generate $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.

Proof. Let

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} := \left(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}}\right)^{\perp}.$$

Denote by $\mathbf{e}$ the embedding

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G),$$

and let $\mathbf{e}^L$ denote its right adjoint.

Since the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$ is compactly generated[13], it suffices to show that the essential image of the composite functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \overset{\mathbf{u}}{\hookrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \overset{\mathbf{e}^L}{\twoheadrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$$

generates the target category. We will show that this functor is in fact a localization (i.e., its right adjoint is fully faithful).

Indeed, the right adjoint to the above functor is the composition

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}} \overset{\mathbf{u}^R}{\hookleftarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \overset{\mathbf{e}}{\twoheadleftarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}.$$

Now, according to [Be1, Theorem A], the essential image of the functor $\mathbf{e}$ is contained in $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$, i.e., $\mathbf{e}$ factors as

$$\mathbf{u} \circ \mathbf{e}_{\text{temp}}$$

for a uniquely defined (fully faithful) functor

$$\mathbf{e}_{\text{temp}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}.$$

Hence, the above right adjoint identifies with

$$\mathbf{u}^R \circ \mathbf{u} \circ \mathbf{e}_{\text{temp}} \simeq \mathbf{e}_{\text{temp}},$$

as required.

□

[13]On the one hand, since we have assumed that $\mathbb{L}_{G,\text{coarse}}$ is an equivalence, the compact generation of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$ follows from that of $\text{QCoh}(\text{LS}_{\check{G}})$. On the other hand, this compact generation can be proved unconditionally: in [FR] it is shown that the objects obtained by acting by $\text{Rep}(\check{G})_{\text{Ran}}$ on $\text{Poinc}_!^{\text{Vac,glob}}$ generate $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{temp}}$.

5.4.4. Let

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})_{\mathrm{Eis}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$$

be the full subcategory generated by the essential images of the *spectral Eisenstein functors*

$$\mathrm{Eis}^{\mathrm{spec}} : \mathrm{QCoh}(\mathrm{LS}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$$

for proper parabolic subgroups.

According to [AG, Theorem 13.3.6], the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ is compactly generated by

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c \text{ and } (\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})_{\mathrm{Eis}})^c,$$

where we view $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ as a subcategory of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ via the functor $\Xi_{\{0\},\mathrm{Nilp}}$, the left adjoint of $\Psi_{\mathrm{Nilp},\{0\}}$.

Hence, given Lemma 5.4.3, in order to prove Proposition 5.2.4, it suffices to show that:

(1) The essential image of $\left(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}}\right)^c$ in $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ under the functor $\mathbb{L}_{G,\mathrm{coarse}}$ equals $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c$;

(2) The essential image of $\left(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}}\right)^c$ under $\mathbb{L}_{G,\mathrm{coarse}}$ equals $(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})_{\mathrm{Eis}})^c$.

5.4.5. Point (1) is immediate from the fact that $\mathbb{L}_{G,\mathrm{temp}}$ is an equivalence.

Point (2) follows by induction on the semi-simple rank from the fact that $\mathbb{L}_{G,\mathrm{temp}}$ is fully faithful from the next assertion (which is well-known, and will be discussed in detail in a subsequent paper in the series):

Lemma 5.4.6. *For a given parabolic subgroup P with Levi quotient M, the diagram*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_{M,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}) \\ {\scriptstyle \mathrm{Eis}_!}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{spec}}_{\mathrm{coarse}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \end{array}$$

commutes up to cohomological shifts and tensoring by line bundles, where

$$\mathrm{Eis}^{\mathrm{spec}}_{\mathrm{coarse}} := \Psi_{\mathrm{Nilp},\{0\}} \circ \mathrm{Eis}^{\mathrm{spec}}.$$

□[Proposition 5.2.4]

5.5. **The Betti context.** The goal of this subsection is to prove the implication (ii') ⇒ (ii) in Theorem 3.5.6. The argument will be parallel to the proof of the implication (i') ⇒ (i) given above.

5.5.1. Define the full subcategory

$$\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \subset \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$$

as in [AG, Sect. 12.8.4].

Then the material in Sects. 5.1.4-5.1.11 applies as is. I.e., it suffices to show that if the functor

$$\mathbb{L}^{\mathrm{Betti}}_{G,\mathrm{temp}} : \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}} \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$$

is an equivalence, then so is $\mathbb{L}^{\mathrm{Betti}}_G$.

The latter in turn reduces to the combination of the following two propositions:

Proposition 5.5.2. *The functor*

$$\mathbf{u}^R|_{\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)^c} : \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)^c \to \mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{temp}}$$

is fully faithful.

Proposition 5.5.3. *Assume that* $\mathbb{L}_{G,\text{temp}}^{\text{Betti}}$ *is an equivalence (for the group G and all its Levi subgroups). Then the essential image of* $\text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)^c$ *in* $\text{QCoh}(\text{LS}_{\check{G}}^{\text{Betti}})$ *under*

$$\mathbb{L}_{G,\text{temp}}^{\text{Betti}} \circ \mathbf{u}^R \simeq \mathbb{L}_{G,\text{coarse}}^{\text{Betti}}$$

equals $\text{Coh}_{\text{Nilp}}(\text{LS}_{\check{G}}^{\text{Betti}})$.

5.5.4. Proposition 5.5.2 is proved along the same lines as Proposition 5.2.3 using the following ingredients:

- The category $\text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)_{\text{co}}$ is well-defined (see [AGKRRV, Theorem 14.1.5]), and the functor
$$\text{Mir}_{\text{Bun}_G} : \text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)_{\text{co}} \to \text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)$$
is an equivalence;
- The essential image of the functor
$$\text{Id}^{\text{nv}} : \text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)_{\text{co}} \to \text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)$$
lies in $\text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)_{\text{temp}}$.

Both these assertions are proved in the same way as in the de Rham setting.

5.5.5. Proposition 5.5.3 is proved along the same lines as Proposition 5.2.4 using the following ingredients:

- The subcategory
$$\text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)_{\text{cusp}} \subset \text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G)$$
is contained in the essential image of the functor Id^{nv}.
- The diagram
$$\begin{array}{ccc} \text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_M) & \xrightarrow{\mathbb{L}_{M,\text{coarse}}^{\text{Betti}}} & \text{QCoh}(\text{LS}_{\check{M}}^{\text{Betti}}) \\ {\scriptstyle \text{Eis}_!}\downarrow & & \downarrow{\scriptstyle \text{Eis}_{\text{coarse}}^{\text{spec}}} \\ \text{Shv}_{\frac{1}{2},\text{Nilp}}^{\text{Betti}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\text{coarse}}^{\text{Betti}}} & \text{QCoh}(\text{LS}_{\check{G}}^{\text{Betti}}) \end{array}$$
commutes up to cohomological shifts and tensoring by line bundles.

Both these assertions are proved in the same way as in the de Rham setting.

6. The structure of Hecke eigensheaves

In this section, we discuss the structure of Hecke eigensheaves for irreducible spectral parameters, building on recent results from [AGKRRV] and [FR].

Below, we will assume GLC in the de Rham setting, and hence in the Betti setting.

6.1. **Statement of the main result.**

6.1.1. *The Hecke eigensheaf.* Let $\sigma \in \text{LS}_{\check{G}}$ be a k-point of $\text{LS}_{\check{G}}$.

We define the category

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Hecke},\sigma} := \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{Vect}$$

of *Hecke eigensheaves* for σ. Here $\text{QCoh}(\text{LS}_{\check{G}})$ acts on Vect by pullback along $\sigma : \text{Spec}(k) \to \text{LS}_{\check{G}}$.

Using the pushforward functor $\sigma_* : \text{Vect} \to \text{QCoh}(\text{LS}_{\check{G}})$, we obtain a functor functor

$$\mathbf{oblv}_{\text{Hecke},\sigma} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Hecke},\sigma} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G).$$

Now assume that σ is an *irreducible* local system, i.e., σ does not admit a reduction to any proper parabolic $\check{P} \subsetneq \check{G}$. By [AG, Prop. 13.3.3] and our running assumption that GLC holds, the functor $\mathbb{L}_G$ induces an equivalence:

$$\mathbb{L}_{G,\sigma} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Hecke},\sigma} \simeq \text{Vect}\,.$$

Throughout this section, we let

$$\widetilde{\mathcal{F}}_\sigma \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Hecke},\sigma}$$

denote the unique object with

$$\mathbb{L}_{G,\sigma}(\widetilde{\mathcal{F}}_\sigma) = k[-\dim(\text{Bun}_G) + \dim(\text{Bun}_{N,\rho(\omega_X)})].$$

The shift here is motivated by Theorem 2.3.8.

We then let

$$\mathcal{F}_\sigma := \mathbf{oblv}_{\text{Hecke},\sigma}(\widetilde{\mathcal{F}}_\sigma) \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

denote the underlying D-module of the canonical Hecke eigensheaf.

6.1.2. We will prove the following result. Most parts of it are not original to this work; we will discuss attributions after stating the result.

Theorem 6.1.3. *As above, let σ be an irreducible local system and assume the geometric Langlands conjecture.*

(1) *$\mathcal{F}_\sigma$ is holonomic and has regular singularities.*
(2) *The singular support of $\mathcal{F}_\sigma$ lies in the nilpotent cone* $\text{Nilp} \subset T^*(\text{Bun}_G)$.
(3) *$\mathcal{F}_\sigma$ lies in the heart* $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^\heartsuit$ *of* $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.
(4) *Let S_σ denote the (non-derived) group of automorphisms of σ as a local system. Then $\mathcal{F}_\sigma$ has a decomposition:*

$$\mathcal{F}_\sigma \simeq \oplus_{\rho \in \text{Irrep}(S_\sigma)} \mathcal{F}_{\sigma,\rho}^{\dim \rho}$$

where each object

$$\mathcal{F}_{\sigma,\rho} \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^\heartsuit$$

is a simple *(holonomic) D-module. Moreover, these simple objects are distinct: $\mathcal{F}_{\sigma,\rho_1} \simeq \mathcal{F}_{\sigma,\rho_2}$ if and only if $\rho_1 = \rho_2$.*

In particular, $\mathcal{F}_\sigma \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^\heartsuit$ is semi-simple.

(5) *Assume the genus of X is at least two and G has connected center.*

Let [Nilp] *denote the cycle on $T^*(\text{Bun}_G)$ defined by the nilpotent cone. In other words, the multiplicities of* [Nilp] *are the multiplicities of irreducible components of the closed substack*[14] $\text{Nilp} \subset T^*(\text{Bun}_G)$. *Then the characteristic cycle* $\text{CC}(\mathcal{F}_\sigma)$ *equals* [Nilp].

In the above, (1) and (2) are due to [AGKRRV] and (3) is due to [FR]. Also, [FR] observed that its results imply (4) assuming GLC, and proved a special case of this result without GLC. The last result (5) is new.

The above results have old folklore status as conjectures. Most appear in [Laum, Conj. 6.3.2]. We expect (5) to hold without the restriction on G, and our methods below prove something in this direction in general (e.g., we prove that (5) holds for a non-trivial class of local systems σ).

Remark 6.1.4. We note that one can check directly that the multiplicity of the zero section in [Nilp] is $\prod_{d_i \geq 2} d_i^{\dim \Gamma(\Omega_X^{1,\otimes d_i})} = \prod d_i^{(2d_i-1)(g-1)}$ for d_i running over the exponents of G; it follows that this is the rank of $\mathcal{F}_\sigma$ at a generic point of Bun_G.

[14] We recall that Nilp is defined scheme-theoretically as the preimage of 0 under the Hitchin map.

Proof of Theorem 6.1.3, (1)-(4). First, (2) is [AGKRRV, Cor. 14.4.10]. As Nilp is Lagrangian, we deduce that Hecke eigensheaves are holonomic. They have regular singularities by [AGKRRV, Cor. 16.5.7]. Then (3) is [FR, Thm. 11.2.1.2].

We now turn to (4). Recall that S_σ is an extension of a finite group by the center $Z_{\check{G}}$ of $\check{G}$; in particular, S_σ is a (disconnected) reductive group.

Let $\mathrm{LS}^{\mathrm{restr}}_{\check{G},\sigma} \subset \mathrm{LS}^{\mathrm{restr}}_{\check{G},\sigma}$ denote the connected component containing σ. Define:

$$\text{D-mod}_{\mathrm{Nilp},\frac{1}{2}}(\mathrm{Bun}_G)_\sigma := \text{D-mod}_{\mathrm{Nilp},\frac{1}{2}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\sigma}).$$

As in [AGKRRV, Cor. 14.3.5], this category is a direct summand of $\text{D-mod}_{\mathrm{Nilp},\frac{1}{2}}(\mathrm{Bun}_G)$.

By Lemma 5.4.6, every object of $\text{D-mod}_{\mathrm{Nilp},\frac{1}{2}}(\mathrm{Bun}_G)_\sigma$ is cuspidal. Therefore, by [Be1], every object of $\text{D-mod}_{\mathrm{Nilp},\frac{1}{2}}(\mathrm{Bun}_G)_\sigma$ is tempered. Therefore, $\mathbb{L}_G$ induces an equivalence

$$\mathbb{L}_{G,\widehat{\sigma}} : \text{D-mod}_{\mathrm{Nilp},\frac{1}{2}}(\mathrm{Bun}_G)_\sigma \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\sigma}).$$

We then have a diagram

$$\begin{array}{ccc} & \mathrm{Vect} & \\ {\scriptstyle \mathcal{F}_\sigma}\swarrow & & \searrow{\scriptstyle \sigma_*} \\ \mathrm{Shv}_{\mathrm{Nilp},\frac{1}{2}}(\mathrm{Bun}_G)_\sigma & \xrightarrow{\mathbb{L}_{G,\widehat{\sigma}}[C]} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\sigma}) \end{array}$$

that commutes by definition of $\mathcal{F}_\sigma$. Here $C := \dim(\mathrm{Bun}_G) - \dim(\mathrm{Bun}_{N,\rho(\omega_X)})$ and σ_* denotes pushforward along $\sigma : \mathrm{Spec}(k) \to \mathrm{LS}^{\mathrm{restr}}_{\check{G}}$.

The bottom arrow in the above diagram is a t-exact equivalence by GLC and Theorem 2.3.8. Therefore, Jordan-Hölder questions for $\mathcal{F}_\sigma$ are the same as for $\sigma_*(k)$.

We now remind ([AGKRRV, Prop. 4.3.5]) that σ factors as $\mathrm{Spec}(k) \to \mathrm{Spec}(k)/S_\sigma \to \mathrm{LS}^{\mathrm{restr}}_{\check{G},\sigma}$ with the latter map being a closed embedding. Therefore, there is a fully faithful embedding $\mathrm{Rep}(S_\sigma)^\heartsuit \subset \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G},\sigma})^\heartsuit$ with $\sigma_*(k)$ corresponding to the regular representation of S_σ.

Now the result follows from the fact that S_σ is reductive, so its (left) regular representation has the form $R_{S_\sigma} \simeq \oplus_{\rho\in \mathrm{Irrep}(S_\sigma)} \rho^{\dim \rho}$.

□

6.2. **Characteristic cycles for eigensheaves.** We now turn to the proof of Theorem 6.1.3 (5).

6.2.1. *Idea of the proof.* Our main new result is the following. We always assume genus ≥ 2 in what follows, but G is allowed to be arbitrary.

Theorem 6.2.2. *For every pair of irreducible local systems $\sigma_1, \sigma_2 \in \mathrm{LS}_{\check{G}}$ lying in the same irreducible component of $\mathrm{LS}_{\check{G}}$, $\mathrm{CC}(\mathcal{F}_{\sigma_1}) = \mathrm{CC}(\mathcal{F}_{\sigma_2})$.*

The following is [BD, Prop. 5.1.2]:

Theorem 6.2.3 (Beilinson-Drinfeld)**.** *Let $\mathrm{LS}_{\check{G},\mathrm{neut}} \subset \mathrm{LS}_{\check{G}}$ be the irreducible component of $\mathrm{LS}_{\check{G}}$ containing the trivial local system. Then there exists an irreducible local system $\sigma \in \mathrm{LS}_{\check{G},\mathrm{neut}}$ such that* $\mathrm{CC}(\mathcal{F}_\sigma) = [\mathrm{Nilp}]$.

We now recall that $\mathrm{LS}_{\check{G}}$ is irreducible when $\check{G}$ has simply-connected derived group (and the genus is ≥ 2), see [BD, Prop. 2.11.4]. Therefore, the above two results imply the claim.

Remark 6.2.4. Theorem 6.2.3 is proved by de Rham methods, but Theorem 6.2.2 will be proved by Betti methods.

6.3. **Proof of Theorem 6.2.2.** We will prove this result using some theory of Betti constructible sheaves.

By the Lefschetz principle, note that Theorem 6.2.2 reduces to its Betti version. Therefore, we put ourselves in this setting in what follows.

6.3.1. *Review of microstalks.* Let $k = \mathbb{C}$, let $\mathcal{Y}$ be a smooth algebraic stack of finite type over $\mathbb{C}$, and recall that $\mathrm{Shv}^{\mathrm{Betti}}(\mathcal{Y})$ is the category of all Betti sheaves on $\mathcal{Y}$. Let $\Lambda \subset T^*(\mathcal{Y})$ be a conical (algebraic) Lagrangian and consider $\mathrm{Shv}^{\mathrm{Betti}}_{\Lambda}(\mathcal{Y}) \subset \mathrm{Shv}^{\mathrm{Betti}}(\mathcal{Y})$, which we recall is the subcategory of sheaves with singular support in Λ and is defined in [AGKRRV, Sect. F.6] for stacks.

Let $\xi \in \Lambda$ be a point lying in the smooth locus of Λ. The following result summarizes the microstalk theory for our needs.

Theorem 6.3.2. *There is a functor* $\mu_\xi : \mathrm{Shv}_{\Lambda}(\mathcal{Y}) \to \mathrm{Vect}$ *with the following properties:*

1. μ_ξ *is t-exact.*
2. μ_ξ *admits a left adjoint, i.e.,* μ_ξ *is corepresented by an object* $\mu\delta_\xi \in \mathrm{Shv}_{\Lambda}(\mathcal{Y})$*.*
3. *For a constructible sheaf* $\mathcal{F} \in \mathrm{Shv}^{\mathrm{Betti}}_{\Lambda}(\mathcal{Y})$*,* $\dim(\mu_\xi(\mathcal{F}))$ *equals the order of the characteristic cycle of* $\mathcal{F}$ *at* ξ*.*

Proof. First, assume $\mathcal{Y}$ is a smooth scheme, see e.g. [GPS, Sect. 4.4] for a review of the construction and the corepresentability. Exactness (up to a normalizing shift) follows from [KS, Cor. 10.3.13]. The relation to the characteristic cycle is [KS, Examples 9.5.7].

In general, choose $\pi : Y \to \mathcal{Y}$ a smooth cover by a smooth scheme, and we assume Let $y \in \mathcal{Y}$ be the point so $\xi \in T^*_y(\mathcal{Y})$; let $\widetilde{y} \in \pi^{-1}(y)$ be a lift of y and let $\widetilde{\xi} := \pi^*(\xi) \in T^*_{\widetilde{y}}(Y)$. Define:

$$\mu_\xi(\mathcal{F}) := \mu_{\widetilde{\xi}} \circ \pi^!(\mathcal{F})[-\dim Y + \dim \mathcal{Y}] = \mu_{\widetilde{\xi}} \circ \pi^*(\mathcal{F})[\dim Y - \dim \mathcal{Y}].$$

Finally, recall from [AGKRRV, Cor. G.7.6] that the embedding $\iota : \mathrm{Shv}^{\mathrm{Betti}}_{\Lambda}(\mathcal{Y}) \to \mathrm{Shv}^{\mathrm{Betti}}(\mathcal{Y})$ admits a left adjoint ι^L. Then:

$$\mu\delta_\xi := \iota^L(\pi_!(\mu\delta_{\widetilde{\xi}}))[\dim Y - \dim \mathcal{Y}].$$

evidently corepresents the microstalk at ξ.

□

6.3.3. *Microstalks and the Langlands equivalence.* Let $\mathrm{Nilp}^\alpha \subset \mathrm{Nilp}$ be an irreducible component and let $\xi \in \mathrm{Nilp}^\alpha$ be a generic point lying in the smooth locus. We have a microstalk functor:

$$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp},\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Vect}$$

and a corepresenting object $\mu\delta_\xi$.

Define $\mathcal{E}_\xi \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$ as:

$$\mathcal{E}_\xi := \mathbb{L}^{\mathrm{Betti}}_G(\mu\delta_\xi)[-\dim(\mathrm{Bun}_G) + \dim(\mathrm{Bun}_{N,\rho(\omega_X)})].$$

We let $\mathcal{E}^{\mathrm{irred}}_\xi \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti,irred}}_{\check{G}})$ be the restriction of $\mathcal{E}_\xi$ to the irreducible locus.

6.3.4. Observe that $\mathcal{E}_\xi$ is compact, i.e., it lies in $\mathrm{Coh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})$: this follows as it corresponds to the compact object $\mu\delta_\xi$ under the Betti Langlands equivalence.

In particular, $\mathcal{E}^{\mathrm{irred}}_\xi$ is perfect, so the Euler characteristic of its fibers form a locally constant function on $\mathrm{LS}^{\mathrm{Betti,irred}}_{\check{G}}$.

We compute the dual:

$$\mathrm{Hom}_{\mathrm{Vect}}(\sigma^*(\mathcal{E}_\xi), \mathbb{C})$$

to this fiber at $\sigma \in \mathrm{LS}^{\mathrm{Betti,irred}}_{\check{G}}(\mathbb{C})$ as:

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}})}(\mathcal{E}_\xi, \sigma_*(\mathbb{C})) \simeq \mathrm{Hom}_{\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{Nilp},\frac{1}{2}}}(\mu\delta_\xi, \mathcal{F}_\sigma) = \mu_\xi(\mathcal{F}_\sigma)$$

for $\mathcal{F}_\sigma$ the (Betti) eigensheaf corresponding to σ, including the shift as in Sect. 6.1.1.

Applying Theorem 6.3.2, we see (i) $\sigma^*(\mathcal{E}_\xi)$ is actually concentrated in one degree, so $\mathcal{E}_\xi^{\mathrm{irred}}$ is actually a vector bundle, and (ii) the fiber of $\mathcal{E}_\xi^{\mathrm{irred}}$ at σ has the same dimension as the order of the characteristic cycle of $\mathcal{F}_\sigma$ at ξ.

Therefore, the function $\sigma \mapsto \mathrm{ord}_\xi(\mathrm{CC}(\mathcal{F}_\sigma))$ is a locally constant function on $\mathrm{LS}_{\check{G}}^{\mathrm{Betti,irred}}$. It now suffices to observe that $\mathrm{LS}_{\check{G}}^{\mathrm{Betti,irred}}$ is smooth and dense in $\mathrm{LS}_{\check{G}}^{\mathrm{Betti}}$, so connected components of $\mathrm{LS}_{\check{G}}^{\mathrm{Betti,irred}}$ are in bijection with irreducible components of $\mathrm{LS}_{\check{G}}^{\mathrm{Betti}}$.

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