PROOF OF THE GEOMETRIC LANGLANDS CONJECTURE IV: AMBIDEXTERITY

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Abstract. This paper performs the following steps toward the proof of GLC in the de Rham setting:
(i) We deduce GLC for $G = GL_n$;
(ii) We prove that the Langlands functor $\mathbb{L}_G$ constructed in [GLC1], when restricted to the cuspidal category, is *ambidextrous*;
(iii) We reduce GLC to the study of a *classical vector bundle with connection*, denoted $\mathcal{A}_{G,\mathrm{irred}}$, on the stack $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$ of irreducible local systems;
(iv) We prove that GLC is equivalent to the contractibility of the space of generic oper structures on irreducible local systems;
(v) Using [BKS], we deduce GLC for classical groups.

Contents

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Introduction

This paper is the fourth in the series of five papers, whose combined content will prove the geometric Langlands conjecture (GLC), as it was formulated in [GLC1, Conjecture 1.6.7].

0.1. **What is done in this paper?**

0.1.1. In the papers [GLC1, GLC2, GLC3] we constructed the *Langlands functor*

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}), \tag{0.1}$$

and GLC says that (0.1) is an equivalence.

0.1.2. The main result of [GLC3] says that (0.1) induces an equivalence

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})_{\mathrm{red}},$$

where:

- $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ is the full subcategory generated by Eisenstein series from proper Levi subgroups;
- $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})_{\mathrm{red}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ is the full subcategory consisting of objects, set-theoretically supported on the locus of reducible local systems.

0.1.3. As one of the first steps in this paper we will show that GLC is equivalent to the statement that the induced functor

$$\mathbb{L}_{G,\text{cusp}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \to \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}), \tag{0.2}$$

is an equivalence (see Corollary 1.3.10). (Note also that $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})$ is the same as the usual $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ category).

Thus, the proof of GLC amounts to the study of the functor $\mathbb{L}_{G,\text{cusp}}$.

0.1.4. Before we even begin the discussion of the main results of this paper, we observe (see Sect. 1.7) that the above considerations already allow us to deduce GLC for $G = GL_n$.

Namely, the fact that $\mathbb{L}_{G,\text{cusp}}$ is fully faithful for GL_n follows from [Ga2] (or, in a more modern language, from [Be1]).

We then show that its essential surjectivity is equivalent to the existence of (non-zero) Hecke eigensheaves attached to irreducible local systems, which was established in [FGV] using geometric methods (or, alternatively, in [BD1] using localization at the critical level).

0.1.5. The main result of this paper, Theorem 3.1.4, which we call the Ambidexterity Theorem, says that the left and right adjoints of the functor $\mathbb{L}_{G,\text{cusp}}$ are isomorphic.

This already gets us pretty close to the statement that $\mathbb{L}_{G,\text{cusp}}$ is an equivalence. Yet, we will need to "milk" the ambidexterity statement some more in order to obtain the actual proof. Some of this milking will be preformed in this paper, and some will be delegated to its sequel.

0.1.6. An additional crucial input comes from the paper [FR] (combined with [Be2]), which says that the functor $\mathbb{L}_{G,\text{cusp}}$ is conservative. This implies that in order to prove GLC, it is sufficient to show that the monad

$$\mathbb{L}_{G,\text{cusp}} \circ \mathbb{L}^L_{G,\text{cusp}} \tag{0.3}$$

acting on $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ is isomorphic to the identity functor.

We observe (see Sect. 1.6.2) that the monad (0.3) is given by tensor product with an associative algebra object

$$\mathcal{A}_{G,\text{irred}} \in \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}). \tag{0.4}$$

The monad (0.3) is an equivalence if and only if the unit map

$$\mathcal{O}_{\text{LS}^{\text{irred}}_{\check{G}}} \to \mathcal{A}_{G,\text{irred}} \tag{0.5}$$

is an isomorphism in $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$.

0.1.7. Now, the Ambidexterity Theorem tells us something about the structure of $\mathcal{A}_{G,\text{irred}}$. Namely, it implies that $\mathcal{A}_{G,\text{irred}}$ is self-dual as an object of $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$. In particular, it is perfect, and hence compact.

However, we prove more: we show (assuming that G is semi-simple) that $\mathcal{A}_{G,\text{irred}}$ is a *classical vector bundle*, equipped with a flat connection (see Theorem 3.1.8).

Thus, we can view $\mathcal{A}_{G,\text{irred}}$ as a classical local system on $\text{LS}^{\text{irred}}_{\check{G}}$. We also show that this local system has a finite monodromy (see Proposition 4.2.8); this latter statement will play a role in the final step of the proof of GLC in the next paper in this series.

0.1.8. The above additional pieces of information concerning $\mathcal{A}_{G,\text{irred}}$ result from Corollary 4.2.5, which says that the fiber of $\mathcal{A}_{G,\text{irred}}$ at a given irredicuble local system σ is isomorphic to the homology of the space of *generic oper structures* on σ.

We will explain the mechanism for this in Sect. 0.3.

0.2. How is ambidexterity proved?

0.2.1. The proof of the Ambidexterity Theorem is obtained by essentially staring at what we call the Fundamental Commutative Diagram (see [GLC2, Diagram (18.14)]):

$$
\begin{array}{ccc}
\text{Whit}^!(G)_{\text{Ran}} & \xrightarrow[\sim]{\text{CS}_G} & \text{Rep}(\check{G})_{\text{Ran}} \\
{\scriptstyle \text{coeff}_G}\uparrow & & \uparrow{\scriptstyle \Gamma^{\text{spec,IndCoh}}_{\check{G}}} \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\
{\scriptstyle \text{Loc}_G}\uparrow & & \uparrow{\scriptstyle \text{Poinc}^{\text{spec}}_{\check{G},*}} \\
\text{KL}(G)_{\text{crit,Ran}} & \xrightarrow[\sim]{\text{FLE}_{G,\text{crit}}} & \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}},
\end{array}
\tag{0.6}
$$

where we ignore some cohomological shifts and twists by constant lines.

Remark 0.2.2. In fact, (0.6) is a special case at levels (crit for G, ∞ for $\check{G}$) of an analogous diagram that is expected to exist in the quantum case:

$$
\begin{array}{ccc}
\text{Whit}^!(G)_{\kappa,\text{Ran}} & \xrightarrow[\sim]{\text{FLE}^{\vee}_{\check{G},\check{\kappa}}} & \text{KL}(\check{G})_{-\check{\kappa},\text{Ran}} \\
{\scriptstyle \text{coeff}_G}\uparrow & & \uparrow{\scriptstyle \Gamma_{\check{G},-\check{\kappa}}} \\
\text{D-mod}_{\kappa}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\kappa}} & \text{D-mod}_{-\check{\kappa}}(\text{Bun}_{\check{G}})_{\text{co}} \\
{\scriptstyle \text{Loc}_{G,\kappa}}\uparrow & & \uparrow{\scriptstyle \text{Poinc}_{\check{G},*}} \\
\text{KL}(G)_{\kappa,\text{Ran}} & \xrightarrow[\sim]{\text{FLE}_{G,\kappa}} & \text{Whit}_*(\check{G})_{-\check{\kappa},\text{Ran}}.
\end{array}
\tag{0.7}
$$

A remarkable feature of the quantum diagram is that it is *self-dual*: i.e., if we dualize all categories and arrows in (0.7) we obtain a similar diagram, but for $((G,\kappa),(\check{G},-\check{\kappa}))$ replaced by $((\check{G},\check{\kappa}),(G,-\kappa))$.

0.2.3. We break (0.6) into the upper and lower portions, i.e.,

$$
\begin{array}{ccc}
\text{Whit}^!(G)_{\text{Ran}} & \xrightarrow[\sim]{\text{CS}_G} & \text{Rep}(\check{G})_{\text{Ran}} \\
{\scriptstyle \text{coeff}_G}\uparrow & & \uparrow{\scriptstyle \Gamma^{\text{spec,IndCoh}}_{\check{G}}} \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})
\end{array}
\tag{0.8}
$$

and

$$
\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\
{\scriptstyle \text{Loc}_G}\uparrow & & \uparrow{\scriptstyle \text{Poinc}^{\text{spec}}_{\check{G},*}} \\
\text{KL}(G)_{\text{crit,Ran}} & \xrightarrow[\sim]{\text{FLE}_{G,\text{crit}}} & \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}},
\end{array}
\tag{0.9}
$$

and we combine (0.8) (resp., (0.9)) with the inclusion of (resp., projection to) the cuspidal subcategory:

$$
\begin{array}{ccc}
\text{Whit}^!(G)_{\text{Ran}} & \xrightarrow[\sim]{\text{CS}_G} & \text{Rep}(\check{G})_{\text{Ran}} \\
{\scriptstyle \text{coeff}_G}\uparrow & & \uparrow{\scriptstyle \Gamma^{\text{spec}}_{\check{G}}} \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\
{\scriptstyle \mathbf{e}}\uparrow & & \uparrow{\scriptstyle \jmath_*} \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xrightarrow{\mathbb{L}_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})
\end{array}
$$

and

$$
\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xrightarrow{\mathbb{L}_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \\
{\scriptstyle \mathbf{e}^L}\uparrow & & \uparrow{\scriptstyle \jmath^*} \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\
{\scriptstyle \text{Loc}_G}\uparrow & & \uparrow{\scriptstyle \text{Poinc}^{\text{spec}}_{\check{G},*}} \\
\text{KL}(G)_{\text{crit},\text{Ran}} & \underset{\sim}{\xrightarrow{\text{FLE}_{G,\text{crit}}}} & \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}},
\end{array}
$$

respectively.

0.2.4. Thus, we obtain the diagrams

$$
\begin{array}{ccc}
\text{Whit}^!(G)_{\text{Ran}} & \underset{\sim}{\xrightarrow{\text{CS}_G}} & \text{Rep}(\check{G})_{\text{Ran}} \\
{\scriptstyle \text{coeff}_G \circ \mathbf{e}}\uparrow & & \uparrow{\scriptstyle \Gamma^{\text{spec},\text{IndCoh}}_{\check{G}} \circ \jmath_*} \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xrightarrow{\mathbb{L}_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})
\end{array}
\tag{0.10}
$$

and

$$
\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xrightarrow{\mathbb{L}_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \\
{\scriptstyle \mathbf{e}^L \circ \text{Loc}_G}\uparrow & & \uparrow{\scriptstyle \jmath^* \circ \text{Poinc}^{\text{spec}}_{\check{G},*}} \\
\text{KL}(G)_{\text{crit},\text{Ran}} & \underset{\sim}{\xrightarrow{\text{FLE}_{G,\text{crit}}}} & \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}},
\end{array}
\tag{0.11}
$$

respectively.

The key feature of the latter diagrams is that in (0.10) the right vertical arrow is fully faithful, and in (0.11) the left vertical arrow is a *Verdier quotient* (a.k.a., is a localization).

0.2.5. Starting from diagrams (0.10) and (0.11), the ambidexterity assertion is proved as follows.

Consider the functor *dual* to $\mathbb{L}_{G,\text{cusp}}$ (with respect to the natural self-dualities of the two sides, see Sect. 2).

The point now is that the vertical arrows in (0.10) admit *left* adjoints, and these left adjoints are *essentially*[1] isomorphic to the duals of the original functors. Combined with the fact that the right vertical arrow is fully faithful, this implies that the dual of $\mathbb{L}_{G,\text{cusp}}$ is isomorphic to the left adjoint of $\mathbb{L}_{G,\text{cusp}}$.

Similarly, the vertical arrows in (0.11) admit *right* adjoints, and these right adjoints are *essentially* isomorphic to the duals of the original functors. Combined with the fact that the left vertical arrow is a Verdier quotient, this implies that the dual of $\mathbb{L}_{G,\text{cusp}}$ is isomorphic to the right adjoint of $\mathbb{L}_{G,\text{cusp}}$.

Thus, we have identified both the left and right adjoints of $\mathbb{L}_{G,\text{cusp}}$ with its dual.

0.3. **Relation to opers.** We now turn to the statements announced in Sect. 0.1.8 that relate the fiber of the object $\mathcal{A}_{G,\text{irrred}}$ at a given $\sigma \in \text{LS}^{\text{irred}}_{\check{G}}$ to the homology of the space $\text{Op}^{\text{gen}}_{\check{G},\sigma}$ of generic oper structures on σ.

---

[1]Essentially:=up to some twists.

0.3.1. Let

$$\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}} \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}),$$

be the object obtained by applying the *left* forgetful functor

$$\mathbf{oblv}^l : \mathrm{D\text{-}mod}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$$

to the object

$$(\pi^{\mathrm{irred}}_{\mathrm{Ran}})_!(\omega_{\mathrm{Op}^{\mathrm{mon\text{-}free,irred}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}}),$$

where:

- $\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}$ is the space of pairs (σ, o), where $\sigma \in \mathrm{LS}_{\check{G}}$, and o is a generic oper structure on it;
- $\pi_{\mathrm{Ran}} : \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}} \to \mathrm{LS}_{\check{G}}$ is the tautological map $(\sigma, o) \mapsto \sigma$;
- $\mathrm{Op}^{\mathrm{mon\text{-}free,irred}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}$ and $\pi^{\mathrm{irred}}_{\mathrm{Ran}}$ is the base change of the above objects along the inclusion $\mathrm{LS}^{\mathrm{irred}}_{\check{G}} \to \mathrm{LS}_{\check{G}}$.

By construction, $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ is naturally a co-commutative coalgebra in $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$.

Note that since the map $\pi^{\mathrm{irred}}_{\mathrm{Ran}}$ is pseudo-proper (see Sect. 3.3.11), the fiber of $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ at a given $\sigma \in \mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is indeed given by the homology of the space

$$\mathrm{Op}^{\mathrm{gen}}_{\check{G},\sigma} := \{\sigma\} \underset{\mathrm{LS}_{\check{G}}}{\times} \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(X^{\mathrm{gen}})$$

of generic oper structures on σ.

0.3.2. The point of departure is Theorem 4.6.3, which says that the comonad on

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$$

given by

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R, \quad \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} := \jmath^* \circ \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}$$

is isomorphic to the comonad given by tensoring with $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$.

0.3.3. Consider the comonad

$$\mathbb{L}_G \circ \mathbb{L}_G^R$$

acting on $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$. Since it is $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$-linear, it is given by tensor product with a co-associative coalgebra object, denoted

$$\mathcal{B}_{G,\mathrm{irred}} \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}).$$

The fact that the left vertical arrow in (0.11) is a Verdier quotient implies that we have an isomorphism of comonads

$$\mathbb{L}_G \circ \mathbb{L}_G^R \simeq \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R.$$

Combining with Theorem 4.6.3, we obtain an isomorphism[2]

$$\mathcal{B}_{G,\mathrm{irred}} \simeq \mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}.$$

However, the Ambidexterity Theorem implies that $\mathcal{A}_{G,\mathrm{irred}} \simeq \mathcal{B}_{G,\mathrm{irred}}$ (as plain objects of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$). Combining, we obtain an isomorphism

$$\mathcal{A}_{G,\mathrm{irred}} \simeq \mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}, \tag{0.12}$$

also as plain objects of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$.

From here we obtain the desired statements relating the fiber of $\mathcal{A}_{G,\mathrm{irred}}$ at a given irreducible local system σ with the homology of $\mathrm{Op}^{\mathrm{gen}}_{\check{G},\sigma}$.

[2]One can show that this isomorphism respects the co-associative coalgebra structures, but we will neither prove nor use this fact.

0.3.4. Note, however, that the isomorphism (0.12) gives us more. Namely, since we already know that $\mathcal{A}_{G,\text{irred}}$ is concentrated in cohomological degree 0, we obtain that the connected components of $\text{Op}^{\text{gen}}_{\check{G},\sigma}$ are *homologically contractible.*

And since GLC is equivalent to the fact that the map (0.5) is an isomorphism, we obtain that it is equivalent to either of the following:

- For every irreducible σ, the space $\text{Op}^{\text{gen}}_{\check{G},\sigma}$ is homologically contractible;
- For every irreducible σ, the space $\text{Op}^{\text{gen}}_{\check{G},\sigma}$ is connected.

0.3.5. Recall now that a recent result of [BKS] proves the homological contractibility of the spaces $\text{Op}^{\text{gen}}_{\check{G},\sigma}$, whenever G (and hence $\check{G}$) is classical.

Hence, we obtain that GLC is a theorem for classical G.

0.4. **Contents.** We now briefly review the contents of this paper section-by-section.

0.4.1. In Sect. 1 we review the contents of [GLC1, GLC2, GLC3] relevant for this paper, and draw some consequences. In particular:

- We show that the functor $\mathbb{L}_G$ is an equivalence if and only if the corresponding functor $\mathbb{L}_{G,\text{cusp}}$ is;
- We show that $\mathbb{L}_{G,\text{cusp}}$ is an equivalence if and only if the object $\mathcal{A}_{G,\text{irred}}$ is isomorphic to the structure sheaf;
- We deduce GLC for GL_n.

0.4.2. In Sect. 2 we review the self-duality identifications on the two sides of (0.2), and we show that the *left* adjoint of $\mathbb{L}_{G,\text{cusp}}$ identifies with its dual, up to a twist. This uses the compatibility of the functor $\mathbb{L}_G$ with the Whittaker model, i.e., the upper portion of (0.6).

In Sect. 3, we show that the *right* adjoint of $\mathbb{L}_{G,\text{cusp}}$ also identifies with its dual (up to the same twist). This uses the compatibility of the functor $\mathbb{L}_G$ with localization at the critical level, i.e., the lower portion of (0.6).

Combining, we deduce the Ambidexterity Theorem, which says that the left and right adjoints of $\mathbb{L}_{G,\text{cusp}}$ are isomorphic.

From here, we deduce that the object $\mathcal{A}_{G,\text{irred}} \in \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ is self-dual, and hence compact.

0.4.3. In Sect. 4 we express $\mathcal{A}_{G,\text{irred}}$ via the space of generic oper structures.

As a result, we prove that $\mathcal{A}_{G,\text{irred}}$ is a classical vector bundle (when G is semi-simple).

And we deduce GLC for classical groups.

0.4.4. In Sect. 5 we reduce Theorem 4.1.5 stated in the previous section to the combination of two general assertions about the space of rational maps. These assertions are proved in Sect. A and Sects. B+C, respectively.

0.4.5. *Conventions and notation.* The conventions and notation in this paper follow those in [GLC3].

0.5. **Acknowledgements.** We wish to thank Justin Campbell for collaboration on other papers constituting this project, namely, [GLC2, GLC3].

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1. Summary of the Langlands functor

In the papers [GLC1], a functor

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$$

was constructed.

The geometric Langlands conjecture, a.k.a. GLC ([GLC1, Conjecture 1.6.7]), says that the functor $\mathbb{L}_G$ is an equivalence. For the duration of this paper, we will assume the validity of GLC for proper Levi subgroups of G.

In this section we will summarize the properties of $\mathbb{L}_G$ relevant for this paper, and draw some consequences.

1.1. **The functor $\mathbb{L}_G$ via the Whittaker model.** In this subsection we will recall the "main" feature of the functor $\mathbb{L}_G$; its compatibility with the Whittaker model.

1.1.1. Recall (see [GLC2, Sects. 9.3 and 9.4]) that the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ is related to the *Whittaker category* by a pair of adjoint functors

$$\text{Poinc}_{G,!} : \text{Whit}^!(G)_{\text{Ran}} \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) : \text{coeff}_G\,.$$

1.1.2. Recall also that the category $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ is related to the category $\text{Rep}(\check{G})_{\text{Ran}}$ by a pair of adjoint functors

$$\text{Loc}^{\text{spec}}_{\check{G}} : \text{Rep}(\check{G})_{\text{Ran}} \rightleftarrows \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) : \Gamma^{\text{spec},\text{IndCoh}}_{\check{G}},$$

see [GLC2, Sect. 17.6].

Note, however, that the functor $\text{Loc}^{\text{spec}}_{\check{G}}$ factors as

$$\text{Rep}(\check{G})_{\text{Ran}} \to \text{QCoh}(\text{LS}_{\check{G}}) \overset{\Xi_{\{0\},\text{Nilp}}}{\hookrightarrow} \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}),$$

and the functor $\Gamma^{\text{spec},\text{IndCoh}}_{\check{G}}$ factors as

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \overset{\Psi_{\{0\},\text{Nilp}}}{\twoheadrightarrow} \text{QCoh}(\text{LS}_{\check{G}}) \xrightarrow{\Gamma^{\text{spec}}_{\check{G}}} \text{Rep}(\check{G})_{\text{Ran}},$$

where

$$\Xi_{\{0\},\text{Nilp}} : \text{QCoh}(\text{LS}_{\check{G}}) \rightleftarrows \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) : \Psi_{\{0\},\text{Nilp}}$$

are the natural embedding and projection, respectively.

1.1.3. We record the following (see Sect. C.1.9 for the proof):

Proposition 1.1.4. *The functor*

$$\Gamma^{\text{spec}}_{\check{G}} : \text{QCoh}(\text{LS}_{\check{G}}) \to \text{Rep}(\check{G})_{\text{Ran}}$$

is fully faithful.

1.1.5. The Langlands functor $\mathbb{L}_G$ is *essentially*[3] determined by the property that it makes the diagram

$$\begin{array}{ccc} \text{Whit}^!(G)_{\text{Ran}} & \xrightarrow[\sim]{\text{CS}_G} & \text{Rep}(\check{G})_{\text{Ran}} \\ {\scriptstyle \text{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \Gamma^{\text{spec},\text{IndCoh}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \end{array} \tag{1.1}$$

commute, where CS_G is the geometric Casselman-Shalika equivalence (see [GLC2, Sect. 1.4]), and

$$\delta_{N_{\rho(\omega_X)}} = \dim(\text{Bun}_{N_{\rho(\omega_X)}}).$$

Remark 1.1.6. The commutation of (1.1) is the point of departure for any of the constructions of the Langlands functor.

[3]See [GLC1, Sects. 1.4 and 1.6] for what the word "essentially" refers to.

1.1.7. It is shown in [GLC3, Theorem 16.1.2] that the functor $\mathbb{L}_G$ admits a left adjoint, to be denoted $\mathbb{L}_G^L$. Passing to the left adjoints in (1.1), we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xleftarrow[\sim]{\mathrm{CS}_G^{-1}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
{\scriptstyle \mathrm{Poinc}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}_G^L} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).
\end{array}
\tag{1.2}
$$

1.2. **Langlands functor and Eisenstein series.** In this subsection we will summarize the properties of $\mathbb{L}_G$ relevant for this paper that have to do with the Eisenstein series and constant term functors.

1.2.1. A key property of the functor $\mathbb{L}_G$ is that it commutes with the Eisenstein functors, i.e., for a parabolic P with Levi quotient M, the diagram

$$
\begin{array}{ccc}
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \\
{\scriptstyle \mathrm{Eis}_{!,\mathrm{twk}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{spec}}} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})
\end{array}
\tag{1.3}
$$

commutes, where $\mathrm{Eis}_{!,\mathrm{twk}}$ is a "tweaked" !-Eisenstein functor, where the tweak involves a translation along Bun_M and a cohomological shift (the precise details of the tweak are irrelevant for this paper), see [GLC3, Theorem 14.2.2].

1.2.2. Passing to the right adjoints along the vertical arrows in (1.3), we obtain a diagram

$$
\begin{array}{ccc}
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \\
{\scriptstyle \mathrm{CT}_{*,\mathrm{twk}}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{\mathrm{spec}}} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}),
\end{array}
\tag{1.4}
$$

which *a priori* commutes up to a natural transformation (in the above diagram, $\mathrm{CT}_{*,\mathrm{twk}}$ is a "tweaked" Constant Term functor, set to be the right adjoint of $\mathrm{Eis}_{!,\mathrm{twk}}$).

However, one of the main results of the paper [GLC3], namely, Theorem 15.1.2 in *loc. cit.*, says that the natural transformation in (1.4) is an isomorphism. I.e., the diagram (1.4) commutes.

1.2.3. Note that by passing to left adjoints along all arrows in (1.4), we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xleftarrow{\mathbb{L}_M^L} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}) \\
{\scriptstyle \mathrm{Eis}_{!,\mathrm{twk}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{spec}}} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}_G^L} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).
\end{array}
\tag{1.5}
$$

1.2.4. Let

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}} \subset \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

be the full subcategory generated by the essential images of the functors $\mathrm{Eis}_!$ (equivalently, $\mathrm{Eis}_{!,\mathrm{twk}}$) for proper parabolics,

Let

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})_{\mathrm{red}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$$

be the full subcategory consisting of objects, set-theoretically supported on the locus

$$\mathrm{LS}^{\mathrm{red}}_{\check{G}} \subset \mathrm{LS}_{\check{G}}$$

consisting of *reducible* local systems, i.e.., the union of the images of the (proper) maps

$$\mathrm{LS}_{\check{P}} \to \mathrm{LS}_{\check{G}}$$

for proper parabolic subgroups.

Combining diagrams (1.3) and (1.5) we obtain that the functors $\mathbb{L}_G$ and $\mathbb{L}_G^L$ send the subcatergories

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \text{ and } \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})_{\text{red}}$$

to one another, thereby inducing a pair of adjoint functors

$$\mathbb{L}_{G,\text{Eis}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \rightleftarrows \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})_{\text{red}} : \mathbb{L}^L_{G,\text{Eis}}. \tag{1.6}$$

The main result of [GLC3], namely, Theorem 17.1.2 in *loc. cit.* says:

Theorem 1.2.5. *The adjoint functors in* (1.6) *are (mutually inverse) equivalences.*

1.3. **The Langlands functor on the cuspidal part.** In this subsection we will study the restriction of $\mathbb{L}_G$ to the cuspidal subcategory. We will show that GLC is equivalent to the statement that the resulting functor

$$\mathbb{L}_{G,\text{cusp}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \to \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \simeq \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$$

is an equivalence.

1.3.1. Let

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} := \Big(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}}\Big)^{\perp}$$

be the cuspidal subcategory.

Tautologically, $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$ is the intersection of the kernels of the functors CT_* (equivalently, $\text{CT}_{*,\text{twk}}$) for all proper parabolics.

1.3.2. Denote by $\mathbf{e}$ the tautological embedding

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G).$$

Since $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}}$ is generated by objects that are compact in $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, the functor $\mathbf{e}$ admits a left adjoint, to be denoted $\mathbf{e}^L$.

1.3.3. Let

$$\text{LS}^{\text{irred}}_{\check{G}} \overset{\jmath}{\hookrightarrow} \text{LS}_{\check{G}}$$

be the embedding of the irreducible locus. We can regard $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})$ as a full subcategory of $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ of $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ via $\jmath_*$, and as such it identifies with

$$(\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})_{\text{red}})^{\perp}.$$

Tautologically,

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})_{\text{red}} = \ker(\jmath^*).$$

1.3.4. From the commutation of (1.4), we obtain that the functor $\mathbb{L}_G$ sends $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$ to $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})$. Denote the resulting functor by

$$\mathbb{L}_{G,\text{cusp}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \to \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}). \tag{1.7}$$

I.e., we have a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\ {\scriptstyle \mathbf{e}}\uparrow & & \uparrow{\scriptstyle \jmath_*} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xrightarrow{\mathbb{L}_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}). \end{array} \tag{1.8}$$

Note that from the commutation of (1.3) we obtain a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\ {\scriptstyle \mathbf{e}^L}\downarrow & & \downarrow{\scriptstyle \jmath^*} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xrightarrow{\mathbb{L}_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}). \end{array} \tag{1.9}$$

1.3.5. Note also that when we view $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$ and $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})$ as quotient categories of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ and $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ via $\mathbf{e}^L$ and $\jmath^*$, respectively, from the commutation of (1.5), we obtain that there exists a well-defined functor

$$\mathbb{L}^L_{G,\text{cusp}} : \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$$

that makes the diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xleftarrow{\mathbb{L}^L_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\ {\scriptstyle \mathbf{e}^L}\downarrow & & \downarrow{\scriptstyle \jmath^*} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xleftarrow{\mathbb{L}^L_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}). \end{array} \tag{1.10}$$

commute.

Taking into account (1.9), we obtain that the functors

$$\mathbb{L}_{G,\text{cusp}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \rightleftarrows \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})_{\text{red}} : \mathbb{L}^L_{G,\text{cusp}} \tag{1.11}$$

are mutually adjoint, i.e.,

$$\mathbb{L}^L_{G,\text{cusp}} \simeq (\mathbb{L}_{G,\text{cusp}})^L.$$

1.3.6. We claim, however:

Proposition 1.3.7. *The functor* $\mathbb{L}^L_G$ *sends* $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})$ *to* $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$.

Proof. Follows from [GLC2, Proposition 17.2.2]. We will supply an alternative argument in Sect. 1.4.8. □

Corollary 1.3.8. *The following diagram commutes:*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xleftarrow{\mathbb{L}^L_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\ {\scriptstyle \mathbf{e}}\uparrow & & \uparrow{\scriptstyle \jmath_*} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xleftarrow{\mathbb{L}^L_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}). \end{array} \tag{1.12}$$

1.3.9. Taking into account Theorem 1.2.5, we obtain:

Corollary 1.3.10. *The functor* $\mathbb{L}_G$ *is an equivalence if and only if so is the functor* $\mathbb{L}_{G,\text{cusp}}$.

1.4. **Spectral action.** In this subsection we will recall another crucial feature of the functor $\mathbb{L}_G$: its compatibility with the $\text{QCoh}(\text{LS}_{\check{G}})$-actions on the two sides.

1.4.1. Recall (see e.g., [GLC1, Theorem 1.2.4]) that the Hecke action gives rise to an action of the monoidal category $\text{QCoh}(\text{LS}_{\check{G}})$ on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.

1.4.2. We have:

Proposition 1.4.3. *With respect to the* $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$*-action on* $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$*, the full subcategory*

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is set-theoretically supported on $\mathrm{LS}^{\mathrm{red}}_{\check{G}}$*, i.e.,*

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}} \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) = 0. \tag{1.13}$$

This proposition is probably well-known. We will supply a proof for completeness in Sect. 1.4.9.

1.4.4. As a formal consequence of Proposition 1.4.3, we obtain:

Corollary 1.4.5. *We have an inclusion*

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \tag{1.14}$$

as full subcategories of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

In fact, a stronger assertion is true (to be proved in Sect. 1.5.6):

Theorem 1.4.6. *The inclusion* (1.14) *is an equality.*

1.4.7. By the construction of the functor $\mathbb{L}_G$, it is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$-linear, where $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ acts on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ naturally.

Since the symmetric monoidal category $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ is rigid, we obtain that the functor $\mathbb{L}^L_G$ is also equipped with a natural $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$-linear structure.

Moreover, the monad

$$\mathbb{L}^L_G \circ \mathbb{L}_G,$$

acting on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$, is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$-linear.

1.4.8. *Second proof of Proposition 1.3.7.* By $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$-linearity, the functor $\mathbb{L}^L_G$ sends

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}),$$

viewed as a full subcategory of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$, to

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}),$$

viewed as a full subcategory of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

However, the former is tautologically the same as $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$, while the latter is contained in $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}$ by Corollary 1.4.5.

□[Proposition 1.3.7]

1.4.9. *Proof of Proposition 1.4.3.* According to [BG][4], the functor $\mathrm{Eis}_!$ can be factored as a composition

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{M}})}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check{P}}) \overset{\mathrm{Eis}^{\mathrm{part.enh}}_!}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

where the functor $\mathrm{Eis}^{\mathrm{part.enh}}_!$ is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$-linear.

This implies the required assertion.

□[Proposition 1.4.3]

[4]The paper [BG] only treats the case of $P = B$. The general case will be treated in a forthcoming paper of J. Fægerman and A. Hayash.

1.5. **Conservativity.** In this subsection we recall a crucial result from [FR], which says that the functor $\mathbb{L}_{G,\text{cusp}}$ is conservative.

In a sense, this unveils the main reason why GLC holds: that the functor $\mathbb{L}_G$ does not lose information (in a very coarse sense, by sending some objects to zero).

1.5.1. We now import the following result from [FR, Theorem A] (which is a combination of [FR, Theorem B] and [Be2, Theorem A]):

Theorem 1.5.2. *The functor $\mathbb{L}_{G,\text{cusp}}$ is conservative.*

Remark 1.5.3. Note that in the case when $G = GL_n$, the assertion of Theorem 1.5.2 follows immediately from (the much more elementary) Theorem 1.7.2 below.

1.5.4. We now claim:

Theorem 1.5.5. *The functor $\mathbb{L}_G$ is conservative.*

Proof. The follows immediately by combining Theorems 1.5.2 and 1.2.5. □

1.5.6. *Proof of Theorem 1.4.6.* The assertion of the theorem is equivalent to

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}})_{\text{red}} = 0, \tag{1.15}$$

where $\text{QCoh}(\text{LS}_{\check{G}})_{\text{red}} = \ker(\jmath^*)$.

By Theorem 1.5.2, it suffices to show that the functor $\mathbb{L}_{G,\text{cusp}}$ annihilates the subcategory (1.15). However, $\mathbb{L}_{G,\text{cusp}}$ sends this category to

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}})_{\text{red}},$$

and the latter is obviously 0.

□[Theorem 1.4.6]

1.6. **The algebra $\mathcal{A}_{G,\text{irred}}$.** In this subsection we will introduce an object

$$\mathcal{A}_{G,\text{irred}} \in \text{AssocAlg}(\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}),$$

which encodes the monad $\mathbb{L}_{G,\text{cusp}} \circ \mathbb{L}^L_{G,\text{cusp}}$.

We will show that the validity of GLC is equivalent to the fact that the unit map (1.16) is an isomorphism.

The proof of GLC that will be presented in the sequel to this paper will amount to the showing that the algebraic geometry and topology of $\text{LS}^{\text{irred}}_{\check{G}}$ essentially force this map to be an isomorphism (modulo a certain computation on the automorphic side).

1.6.1. Consider the monad $\mathbb{L}_{G,\text{cusp}} \circ \mathbb{L}^L_{G,\text{cusp}}$ on $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})$ corresponding to the adjoint functors (1.11). By Sect. 1.4.7, this monad is $\text{QCoh}(\text{LS}_{\check{G}})$-linear.

Since the action of $\text{QCoh}(\text{LS}_{\check{G}})$ on $\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}})$ factors through

$$\jmath^* : \text{QCoh}(\text{LS}_{\check{G}}) \to \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}),$$

we obtain that $\mathbb{L}_{G,\text{cusp}} \circ \mathbb{L}^L_{G,\text{cusp}}$ is $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$-linear.

1.6.2. Note that $\text{Nilp}|_{\text{LS}^{\text{irred}}_{\check{G}}} = \{0\}$, so

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) = \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}).$$

Hence, the monad $\mathbb{L}_{G,\text{cusp}} \circ \mathbb{L}^L_{G,\text{cusp}}$ is a $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$-linear monad on $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ itself, and thus corresponds to a unital associative algebra, to be denoted

$$\mathcal{A}_{G,\text{irred}} \in \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}).$$

1.6.3. The unit of the $(\mathbb{L}^L_{G,\mathrm{cusp}}, \mathbb{L}_{G,\mathrm{cusp}})$-adjunction corresponds to the unit map

$$\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}} \to \mathcal{A}_{G,\mathrm{irred}}. \tag{1.16}$$

Tautologically, the functor $\mathbb{L}^L_{G,\mathrm{cusp}}$ is fully faithful if and only if the map (1.16) is an isomorphism.

1.6.4. Given Theorem 1.5.2 and Corollary 1.3.10, we obtain:

Corollary 1.6.5. *The functor $\mathbb{L}_G$ is an equivalence if and only if the map* (1.16) *is an isomorphism in* $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$.

1.7. **Proof of GLC for $G = GL_n$.** In this subsection, we will show how Theorem 1.2.5 allows us to prove GLC in the case when $G = GL_n$.

1.7.1. The point of departure is the following result, established in [Be1] (or, in a slightly different language, in [Ga2]):

Theorem 1.7.2. *The restriction of the functor coeff_G to the subcategory*

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is fully faithful.

1.7.3. From Theorem 1.7.2 we will now deduce:

Corollary 1.7.4. *The functor $\mathbb{L}_{G,\mathrm{cusp}}$ is fully faithful.*

Proof. From (1.1) and (1.8), we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xrightarrow[\sim]{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
{\scriptstyle \mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \\
{\scriptstyle \mathbf{e}}\uparrow & & \uparrow{\scriptstyle \jmath_*} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} & \xrightarrow{\mathbb{L}_{G,\mathrm{cusp}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}).
\end{array}$$

It is sufficient to show that the composite right vertical arrow in the above diagram is fully faithful. Indeed, this would imply that the fully-faithfulness of the functors

$$\mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}] \circ \mathbf{e} \text{ and } \mathbb{L}_{G,\mathrm{cusp}}$$

are logically equivalent.

Note that since

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) = \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}),$$

the right vertical arrow in the above diagram can be identified with

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \overset{\jmath_*}{\hookrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{\Gamma^{\mathrm{spec}}_{\check{G}}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}. \tag{1.17}$$

Now, in the composition (1.17) both arrows are fully faithful: this is obvious for $\jmath_*$, and for $\Gamma^{\mathrm{spec}}_{\check{G}}$ this is the content of Proposition 1.1.4.

□

Remark 1.7.5. Note that the proof of Corollary 1.7.4 shows that it is actually logically equivalent to Theorem 1.7.2. Hence, once we establish GLC, we will know that Theorem 1.7.2 also holds for any G.

1.7.6. By Corollary 1.6.5, in order to prove GLC, we need to show that the map (1.16) is an isomorphism in $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$. Since $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is eventually coconnective, it is sufficient to show that for any field-valued point

$$\sigma : \mathrm{Spec}(K) \to \mathrm{LS}^{\mathrm{irred}}_{\check{G}},$$

the resulting map

$$K \to \mathcal{A}_{G,\sigma} \tag{1.18}$$

is an isomorphism, where $\mathcal{A}_{G,\sigma}$ denotes the fiber of $\mathcal{A}_{G,\mathrm{irred}}$ at σ.

Applying base change to the functor $\mathbb{L}_{G,\mathrm{cusp}}$ along σ, we obtain a functor

$$\mathbb{L}_{G,\sigma} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})}{\otimes} \mathrm{Vect}_K \to \mathrm{Vect}_K .$$

Since the functor $\mathbb{L}_{G,\mathrm{cusp}}$ is fully faithful and admits a left adjoint, we obtain that $\mathbb{L}_{G,\sigma}$ is also fully faithful. In particular, $\mathbb{L}_{G,\sigma}$ is conservative. Hence, by Barr-Beck, it can be identified with the forgetful functor

$$\mathcal{A}_{G,\sigma}\text{-mod} \to \mathrm{Vect}_K .$$

Such a functor can be fully faithful either when (1.18) is an isomorphism (which is what we want to show), or when $\mathcal{A}_{G,\sigma} = 0$. Thus, it remains to rule out the latter possibility.

1.7.7. We need to show that the category

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_\sigma := \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})}{\otimes} \mathrm{Vect}_K$$

is non-zero. For this, we can further replace K by its algebraic closure.

Performing base change $k \rightsquigarrow K$, we can assume that $K = k$. Then the category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_\sigma$ is, by definition, the category of *Hecke eigen-sheaves* with respect to σ.

However, it was shown in [FGV] that for $G = GL_n$ and σ irreducible, the category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_\sigma$ contains a non-zero object.

Remark 1.7.8. Alternatively, the proof of the existence of a non-zero Hecke eigensheaf for a given irreducible local system, valid for any G, follows by combining the [BD1] construction of Hecke eigensheaves via localization at the critical level and the result of [Ari], which says that any irreducible local system carries a generic oper structure.

$\square$[GLC for $G = GL_n$]

2. Left adjoint as the dual

In this section we will establish the "first half" of the Ambidexterity Theorem, namely that the functor *left adjoint* to $\mathbb{L}_{G,\mathrm{cusp}}$ is, up to a certain twist, is canonically isomorphic to its dual.

In order to do so, we will first have to show that the source and the target of $\mathbb{L}_{G,\mathrm{cusp}}$ are canonically self-dual.

2.1. **The dual automorphic category.** In this subsection we recall, following [DG] or [Ga1], the description of the dual of the category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

2.1.1. Recall the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$. It is defined as the *colimit*

$$\underset{U}{\text{colim}}\, \text{D-mod}_{\frac{1}{2}}(U),$$

where U runs over the poset of quasi-compact open substacks of Bun_G, and for $U_1 \overset{j_{1,2}}{\to} U_2$ the corresponding functor

$$\text{D-mod}_{\frac{1}{2}}(U_1) \to \text{D-mod}_{\frac{1}{2}}(U_2)$$

is $(j_{1,2})_*$.

For a given quasi-compact open

$$U \overset{j}{\hookrightarrow} \text{Bun}_G, \tag{2.1}$$

we let

$$j_{\text{co},*} : \text{D-mod}_{\frac{1}{2}}(U) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

denote the tautological functor.

2.1.2. The category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$ is endowed with a tautologically defined functor

$$\text{Ps-Id}^{\text{nv}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G),$$

characterized by the following property:

For a quasi-compact open as in (2.1), we have

$$\text{Ps-Id}^{\text{nv}} \circ j_{\text{co},*} \simeq j_*, \tag{2.2}$$

as functors

$$\text{D-mod}_{\frac{1}{2}}(U) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G).$$

2.1.3. Note that Verdier duality on Bun_G gives rise to a canonical identification

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^\vee \overset{\mathbf{D}^{\text{Verdier}}}{\simeq} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}. \tag{2.3}$$

It is characterized by the requirement that for (2.1), we have

$$j_{co,*} \simeq (j^*)^\vee,$$

where we identify

$$\text{D-mod}_{\frac{1}{2}}(U)^\vee \simeq \text{D-mod}_{\frac{1}{2}}(U)$$

via usual Verdier duality, also denoted $\mathbf{D}^{\text{Verdier}}$.

2.2. **The dual of the cuspidal category.** In this subsection we will use Sect. 2.1 to show that the cuspidal automorphic category is canonically self-dual.

2.2.1. Let

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

be the full subcategory, generated by the essential images of the functors

$$\text{Eis}_{\text{co},*} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)_{\text{co}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

(see [Ga1, Sect. 1.4]) for *proper* parabolic subgroups.

Set

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}} := \left(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,Eis}}\right)^\perp.$$

Let

$$\mathbf{e}_{\text{co}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

denote the tautological embedding. It admits a *left adjoint*, making $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}}$ into a *localization* of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$.

2.2.2. The identification (2.3) gives rise to an identification

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^{\vee}_{\text{cusp}} \overset{\mathbf{D}^{\text{Verdier}}_{\text{cusp}}}{\simeq} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}}, \tag{2.4}$$

so that

$$\mathbf{e}_{\text{co}} \simeq (\mathbf{e}^L)^{\vee} \text{ and } \mathbf{e}^L_{\text{co}} \simeq \mathbf{e}^{\vee}.$$

2.2.3. Recall now that the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}}$ has the following property (see [Ga1, Proposition 2.3.4]): there exists a quasi-compact open substack

$$U_0 \overset{j_0}{\hookrightarrow} \text{Bun}_G,$$

such that the functor

$$\mathbf{e}_{\text{co}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

factors as

$$(j_0)_{\text{co},*} \circ \mathbf{e}_{U_0,\text{co}},$$

for (an automatically uniquely defined fully faithful functor)

$$\mathbf{e}_{U_0,\text{co}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}} \to \text{D-mod}_{\frac{1}{2}}(U_0).$$

2.2.4. Furthermore, according to [Ga1, Theorem 2.2.7], the functor Ps-Id^{nv} sends $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}}$ to $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$, and the resulting functor

$$\text{Ps-Id}^{\text{nv}}_{\text{cusp}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co,cusp}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$$

is an equivalence.

We automatically have

$$\mathbf{e} \circ \text{Ps-Id}^{\text{nv}}_{\text{cusp}} \simeq (j_0)_* \circ \mathbf{e}_{U_0,\text{co}}. \tag{2.5}$$

2.2.5. Thus, combining (2.4) with the equivalence $\text{Ps-Id}^{\text{nv}}_{\text{cusp}}$ we obtain a self-duality

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)^{\vee}_{\text{cusp}} \overset{\text{Ps-Id}^{\text{nv}}_{\text{cusp}} \circ \mathbf{D}^{\text{Verdier}}_{\text{cusp}}}{\simeq} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}. \tag{2.6}$$

2.2.6. For later use we introduce the following notation:

Let

$$\mathbf{e}_{U_0} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \to \text{D-mod}_{\frac{1}{2}}(U_0) \tag{2.7}$$

be the uniquely defined functor, so that

$$\mathbf{e}_{U_0,\text{co}} \simeq \mathbf{e}_{U_0} \circ \text{Ps-Id}^{\text{nv}}_{\text{cusp}}.$$

The functor $\mathbf{e}_{U_0}$ is automatically fully faithful and

$$\mathbf{e} \simeq (j_0)_* \circ \mathbf{e}_{U_0}.$$

Let $\mathbf{e}^L_{U_0}$ denote the left adjoint of $\mathbf{e}_{U_0}$. We have:

$$\mathbf{e}^L \simeq \mathbf{e}^L_{U_0} \circ j_0^*. \tag{2.8}$$

2.3. **Duality and the Poincaré functors.** In this section we will recall the result of [Lin] that says that the !- and *-versions of the geometric Poincaré functor become isomorphic, once we project to the cuspidal automorphic category.

2.3.1. Recall (see [GLC2, Sect. 9.5]) that in addition to the functor

$$\text{Poinc}_{G,!} : \text{Whit}^!(G)_{\text{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G),$$

there exists a naturally defined functor

$$\text{Poinc}_{G,*} : \text{Whit}_*(G)_{\text{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}.$$

2.3.2. By construction, with respect the duality (2.3) and the canonical duality

$$(\mathrm{Whit}^!(G)_{\mathrm{Ran}})^\vee \simeq \mathrm{Whit}_*(G)_{\mathrm{Ran}}, \tag{2.9}$$

we have

$$(\mathrm{coeff}_G)^\vee \simeq \mathrm{Poinc}_{G,*}. \tag{2.10}$$

2.3.3. Recall now (see [GLC2, Theorem 1.3.13]) that, in addition to the duality (2.9), there exists a canonical equivalence

$$\Theta_{\mathrm{Whit}(G)} : \mathrm{Whit}_*(G)_{\mathrm{Ran}} \simeq \mathrm{Whit}^!(G)_{\mathrm{Ran}}. \tag{2.11}$$

The following assertion is a "baby" version of the main result of [Lin] (see Lemma 1.3.4 in *loc. cit.*):

Theorem 2.3.4. *The functors*

$$\mathbf{e}^L \circ \mathrm{Poinc}_{G,!} \circ \Theta_{\mathrm{Whit}(G)} \text{ and } \mathbf{e}^L \circ \text{Ps-Id}^{\mathrm{nv}} \circ \mathrm{Poinc}_{G,*}[2\delta_{N_{\rho(\omega_X)}}],$$

$$\mathrm{Whit}_*(G)_{\mathrm{Ran}} \rightrightarrows \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}$$

are canonically isomorphic.

2.4. **Self-duality on the spectral side.** In this short subsection we set up our conventions regarding the self-duality of the category $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$.

2.4.1. Let us identify $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ with its own dual via the *naive duality*

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^\vee \overset{\mathbf{D}^{\mathrm{naive}}}{\simeq} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}). \tag{2.12}$$

I.e., the corresponding anti-self equivalence of

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}})^c = \mathrm{QCoh}(\mathrm{LS}_{\check{G}})^{\mathrm{perf}}$$

is given by monoidal dualization.

2.4.2. The self-duality (2.12) induces a self-duality

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})^\vee \overset{\mathbf{D}^{\mathrm{naive}}}{\simeq} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}). \tag{2.13}$$

2.4.3. Recall now (see, e.g., [AGKRRV1, Sect. 11.3]) that the canonical self-duality on $\mathrm{Rep}(\check{G})$ gives rise to a self-duality of the category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$.

2.4.4. We have:

Lemma 2.4.5. *With respect to the above self-dualities of* $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ *and* $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$*, the functors*

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \leftrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) : \Gamma^{\mathrm{spec}}_{\check{G}}$$

identify with each other's duals:

$$(\mathrm{Loc}^{\mathrm{spec}}_{\check{G}})^\vee \simeq \Gamma^{\mathrm{spec}}_{\check{G}} \text{ and } (\Gamma^{\mathrm{spec}}_{\check{G}})^\vee \simeq \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}.$$

2.5. **Left adjoint vs dual.** In this subsection we finally formulate and prove the main result of this section, Theorem 2.5.3, which says that the left adjoint and the dual of $\mathbb{L}_{G,\mathrm{cusp}}$ are isomorphic, up to a twist.

2.5.1. Consider the functor *dual* to $\mathbb{L}_{G,\mathrm{cusp}}$

$$\mathbb{L}^\vee_{G,\mathrm{cusp}} : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})^\vee \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^\vee_{\mathrm{cusp}}.$$

Using the identifications (2.13) and (2.6), we can think of $\mathbb{L}^\vee_{G,\mathrm{cusp}}$ as a functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}. \tag{2.14}$$

Let

$$\Phi_{G,\mathrm{cusp}} : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}$$

denote the composition of the functor (2.14) with the Chevalley involution and the shift $[-2\delta_{N_{\rho(\omega_X)}}]$, i.e.,

$$\Phi_{G,\mathrm{cusp}} := \tau_G \circ \mathbb{L}^\vee_{G,\mathrm{cusp}}[-2\delta_{N_{\rho(\omega_X)}}].$$

2.5.2. We are going to prove:

Theorem 2.5.3. *The functor $\Phi_{G,\mathrm{cusp}}$ identifies canonically with $\mathbb{L}^L_{G,\mathrm{cusp}}$.*

This theorem is a particular case of [GLC3, Theorem 16.2.3], and its proof is much simpler in that it only uses Theorem 2.3.4, rather than the full force of the result from [Lin]. We will supply a proof for the sake of completeness, and it occupies the rest of this subsection.

2.5.4. By Proposition 1.1.4, it suffices to establish an isomorphism

$$\mathbb{L}^L_{G,\mathrm{cusp}} \circ \jmath^* \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G}} \simeq \Phi_{G,\mathrm{cusp}} \circ \jmath^* \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$$

as functors

$$\mathrm{Rep}(\check{G}) \rightrightarrows \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}.$$

We will do so by showing that both diagrams

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xleftarrow[\sim]{\mathrm{CS}_G^{-1}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{Poinc}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\ {\scriptstyle \mathbf{e}^L}\downarrow & & \downarrow{\scriptstyle \jmath^*} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} & \xleftarrow{\mathbb{L}^L_{G,\mathrm{cusp}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \end{array} \tag{2.15}$$

and

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xleftarrow{\mathrm{CS}_G^{-1}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{Poinc}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\ {\scriptstyle \mathbf{e}^L}\downarrow & & \downarrow{\scriptstyle \jmath^*} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} & \xleftarrow{\Phi_{G,\mathrm{cusp}}} & \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \end{array} \tag{2.16}$$

commute.

2.5.5. The commutation of (2.15) is immediate from (1.2) and (1.10). Thus, it remains to deal with (2.16).

First, according to [GLC2, Lemma 1.4.12], we have

$$\tau_G \circ \mathrm{CS}_G^{-1} \simeq \Theta_{\mathrm{Whit}(G)} \circ \mathrm{CS}_G^{\vee}.$$

Combining with Theorem 2.3.4, this allows us to replace (2.16) by

$$\begin{CD}
\mathrm{Whit}_*(G)_{\mathrm{Ran}} @<{\mathrm{CS}_G^{\vee}}<< \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
@V{\mathrm{Poinc}_{G,*}}VV @VV{\mathrm{Loc}_{\check{G}}^{\mathrm{spec}}}V \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} @. \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\
@V{\mathbf{e}^L \circ \text{Ps-Id}^{\mathrm{nv}}}VV @VV{\jmath^*}V \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} @<{\mathbb{L}^{\vee}_{G,\mathrm{irred}}[-2\delta_{N_{\rho(\omega_X)}}]}<< \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}).
\end{CD} \tag{2.17}$$

2.5.6. Using

$$\mathbf{e}^L \circ \text{Ps-Id}^{\mathrm{nv}} \simeq \text{Ps-Id}^{\mathrm{nv}}_{\mathrm{cusp}} \circ \mathbf{e}^{\vee},$$

we can rewrite (2.17) as

$$\begin{CD}
\mathrm{Whit}_*(G)_{\mathrm{Ran}} @<{\mathrm{CS}_G^{\vee}}<< \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
@V{\mathrm{Poinc}_{G,*}}VV @VV{\mathrm{Loc}_{\check{G}}^{\mathrm{spec}}}V \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} @. \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\
@V{\text{Ps-Id}^{\mathrm{nv}}_{\mathrm{cusp}} \circ \mathbf{e}^{\vee}}VV @VV{\jmath^*}V \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} @<{\mathbb{L}^{\vee}_{G,\mathrm{irred}}[-2\delta_{N_{\rho(\omega_X)}}]}<< \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}),
\end{CD}$$

and further by

$$\begin{CD}
\mathrm{Whit}_*(G)_{\mathrm{Ran}} @<{\mathrm{CS}_G^{\vee}}<< \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
@V{\mathrm{Poinc}_{G,*}}VV @VV{\mathrm{Loc}_{\check{G}}^{\mathrm{spec}}}V \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} @. \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\
@V{\mathbf{e}^{\vee}}VV @VV{\jmath^*}V \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co,cusp}} @<{\mathbb{L}^{\vee}_{G,\mathrm{irred}}[-2\delta_{N_{\rho(\omega_X)}}]}<< \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}),
\end{CD} \tag{2.18}$$

where we now think of $\mathbb{L}^{\vee}_{G,\mathrm{irred}}$ as a functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co,cusp}}$$

via (2.13) and (2.4).

2.5.7. Passing to the dual functors in (2.18), we obtain that it is equivalent to

$$\begin{array}{ccc} \text{Whit}_*(G)_{\text{Ran}} & \xrightarrow{\text{CS}_G} & \text{Rep}(\check{G})_{\text{Ran}} \\ {\scriptstyle \text{coeff}_G}\uparrow & & \uparrow{\scriptstyle \Gamma^{\text{spec}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & & \text{QCoh}(\text{LS}_{\check{G}}) \\ {\scriptstyle \mathbf{e}}\uparrow & & \uparrow{\scriptstyle \jmath_*} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xrightarrow{\mathbb{L}_{G,\text{irred}}[-2\delta_{N_{\rho(\omega_X)}}]} & \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}). \end{array}$$

However, the commutativity of the latter diagram follows from (1.1) and (1.8).

□[Theorem 2.5.3]

3. Right adjoint as the dual

In this section we will assume that G is semi-simple[5].

We will prove the "second half" of the ambidexterity theorem, namely, that the functor *right adjoint* to $\mathbb{L}_{G,\text{cusp}}$ is isomorphic to the (twist of) the dual of $\mathbb{L}_{G,\text{cusp}}$.

The full Ambidexterity Theorem says that the left and right adjoints of $\mathbb{L}_{G,\text{cusp}}$ are isomorphic. Of course, this statement *a posteriori* follows from GLC, but in the current strategy, it constitutes a step in its proof.

3.1. **The ambidexterity statement.**

3.1.1. We continue to regard the categories $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$ and $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ as self-dual via the identifications (2.13) and (2.6), respectively.

Recall the functor $\Phi_{G,\text{cusp}}$, see Sect. 2.5.1. We will prove:

Theorem 3.1.2. *The functor $\Phi_{G,\text{cusp}}$ identifies canonically with the right adjoint of $\mathbb{L}_{G,\text{cusp}}$.*

3.1.3. Before we prove the theorem, let us draw some consequences. First, by combining Theorems 2.5.3 and 3.1.2, we obtain what we call the Ambidexterity Theorem:

Main Theorem 3.1.4. *The left and right adjoints of $\mathbb{L}_{G,\text{cusp}}$ are isomorphic.*

Corollary 3.1.5. *The endofunctor*

$$\mathbb{L}_{G,\text{cusp}} \circ \mathbb{L}^L_{G,\text{cusp}}$$

of $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ *is isomorphic to its own left and right adjoint.*

3.1.6. Recall (see Sect. 1.6.2) that the functor $\mathbb{L}_{G,\text{cusp}} \circ \mathbb{L}^L_{G,\text{cusp}}$ is given by tensoring with the associative algebra object

$$\mathcal{A}_{G,\text{irred}} \in \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}).$$

From Corollary 3.1.5 we obtain:

Corollary 3.1.7. *The object $\mathcal{A}_{G,\text{irred}} \in \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ is self-dual. In particular, it belongs to $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})^{\text{perf}}$, i.e., it is compact.*

Eventually we will prove an even more precise version of the second part of the above corollary (see Sect. 4.4):

Main Theorem 3.1.8. *The object $\mathcal{A}_{G,\text{irred}} \in \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ is a classical vector bundle, which is equipped with a naturally defined flat connection*[6].

Remark 3.1.9. Note that it makes sense to talk about classical vector bundles on $\text{LS}^{\text{irred}}_{\check{G}}$, since, under the assumption that $\check{G}$ is semi-simple, the stack $\text{LS}^{\text{irred}}_{\check{G}}$ is classical and smooth.

[5]This assumption is just a convenience. The statement holds for any reductive G, just the proof would involve slightly more notation.

[6]Note that Corollary 4.2.5 and Proposition 4.2.8 imply that the resulting local system on $\text{LS}^{\text{irred}}_{\check{G}}$ has a finite monodromy

3.1.10. The rest of this section is devoted to the proof of Theorem 3.1.2.

3.2. **Critical localization.** In this subsection, we will show that the right adjoint of the critical localization functor (functor Loc_G below) is *essentially* isomorphic to its dual, once we restrict to the cuspidal automorphic category.

3.2.1. Let Loc_G be the functor

$$\mathrm{KL}(G)_{\mathrm{crit,Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

of [GLC2, Sect. 14.1.4].

3.2.2. Denote by $\mathrm{Loc}_{G,\mathrm{cusp}}$ the composite functor

$$\mathrm{KL}(G)_{\mathrm{crit,Ran}} \overset{\mathrm{Loc}_G}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathbf{e}^L}{\to} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}.$$

We have the following counterpart of Proposition 1.1.4:

Proposition 3.2.3. *The functor* $\mathrm{Loc}_{G,\mathrm{cusp}}$ *is Verdier quotient.*

Proof. Let U_0 be as in Sect. 2.2.3. Denote

$$\mathrm{Loc}_{G,U_0} := j_0^* \circ \mathrm{Loc}_G, \quad \mathrm{KL}(G)_{\mathrm{crit,Ran}} \to \text{D-mod}_{\frac{1}{2}}(U_0),$$

so that

$$\mathrm{Loc}_{G,\mathrm{cusp}} \simeq \mathbf{e}^L_{U_0} \circ \mathrm{Loc}_{G,U_0} \tag{3.1}$$

where $\mathbf{e}^L_{U_0}$ is the left adjoint of the functor $\mathbf{e}_{U_0}$ of (2.7).

It is known that for any quasi-compact U, the corresponding functor $\mathrm{Loc}_{G,U}$ is a Verdier quotient (see [GLC2, Theorem 13.4.2]). Now, the assertion follows from (3.1), since $\mathbf{e}^L_{U_0}$ is also a Verdier quotient. □

3.2.4. Recall (see [GLC2, Sect. 2.2.4]) that the category $\mathrm{KL}(G)_{\mathrm{crit,Ran}}$ is also canonically self-dual. Thus, using the self-duality of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}$ given by (2.6), we can regard the dual of $\mathrm{Loc}_{G,\mathrm{cusp}}$ as a functor

$$\mathrm{Loc}^\vee_{G,\mathrm{cusp}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \to \mathrm{KL}(G)_{\mathrm{crit,Ran}}. \tag{3.2}$$

The following assertion is a counterpart of Lemma 2.4.5:

Proposition 3.2.5. *We have a canonical identification between* $\mathrm{Loc}^\vee_{G,\mathrm{cusp}}$ *and*

$$\mathrm{Loc}^R_{G,\mathrm{cusp}} \otimes \det(\Gamma(X, \mathcal{O}_X) \otimes \mathfrak{g})[\delta_G],$$

where $\delta_G = \dim(\mathrm{Bun}_G)$.

Proof. Let Loc_{G,U_0} be as in the proof of Proposition 3.2.3. It follows formally that we have the identifications

$$\mathrm{Loc}^\vee_{G,\mathrm{cusp}} \simeq \mathrm{Loc}^\vee_{G,U_0} \circ \mathbf{e}_{U_0} \text{ and } \mathrm{Loc}^R_{G,\mathrm{cusp}} \simeq \mathrm{Loc}^R_{G,U_0} \circ \mathbf{e}_{U_0}.$$

Thus, in order to prove Proposition 3.2.5, it suffices to show that for a quasi-compact $U \subset \mathrm{Bun}_G$, with respect to the Verdier self-duality of $\text{D-mod}_{\frac{1}{2}}(U)$, we have

$$\mathrm{Loc}^\vee_{G,U} \simeq \mathrm{Loc}^R_{G,U} \otimes \det(\Gamma(X, \mathcal{O}_X) \otimes \mathfrak{g})[\delta_G] \tag{3.3}$$

as functors

$$\text{D-mod}_{\frac{1}{2}}(U) \to \mathrm{KL}(G)_{\mathrm{crit,Ran}}.$$

Denote by $\mathrm{Loc}_{G,\mathrm{crit}}$ the composition of Loc_G with the identification

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \simeq \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

of [GLC2, Equation (10.2)], and similarly for Bun_G replaced by U.

In terms of the self-duality of $\text{D-mod}_{\text{crit}}(U)$ specified in [GLC2, Sect. 10.5.2], we can reformulate (3.3) as

$$\text{Loc}^{\vee}_{G,\text{crit},U} \simeq \text{Loc}^{R}_{G,\text{crit},U}. \tag{3.4}$$

Note that by the definition of the functor $\text{Loc}_{G,\text{crit}}$ in [GLC2, Sect. 11.3.3-11.3.4], the functor $\text{Loc}^{R}_{G,\text{crit},U}$ is the functor $\Gamma_{G,\text{crit}} \circ j_{\text{co},*}$, where

$$\Gamma_{G,\text{crit}} : \text{D-mod}_{\text{crit}}(\text{Bun}_G)_{\text{co}} \to \text{KL}(G)_{\text{crit},\text{Ran}}$$

is the functor of [GLC2, Sect. 10.2.5].

We have:

$$\text{Loc}^{\vee}_{G,\text{crit},U} \simeq \text{Loc}^{\vee}_{G,\text{crit}} \circ (j_*)^{\vee} = \text{Loc}^{\vee}_{G,\text{crit}} \circ j_{\text{co},*}.$$

Hence, in order to prove (3.4) it remains to identify

$$\text{Loc}^{\vee}_{G,\text{crit}} \simeq \Gamma_{G,\text{crit}}.$$

However, the latter is the statement of [GLC2, Proposition 10.5.7(b)].

□

3.3. **The spectral Poincaré functor.** In this section we will show that the right adjoint of the spectral Poincaré functor (functor $\text{Poinc}^{\text{spec}}_{\check{G},*}$ below) is *essentially* isomorphic to its dual, once we restrict to the locus of irreducible local systems.

NB: it is in this subsection that the assumption that G (rather $\check{G}$) is semi-simple is used[7].

3.3.1. Let

$$\text{Poinc}^{\text{spec}}_{\check{G},*} : \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}} \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$$

be the functor of [GLC2, Sect. 17.4.2].

Denote by $\text{Poinc}^{\text{spec}}_{\check{G},*,\text{irred}}$ the composite functor

$$\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}} \overset{\text{Poinc}^{\text{spec}}_{\check{G},*}}{\longrightarrow} \text{QCoh}(\text{LS}_{\check{G}}) \overset{\jmath^*}{\to} \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}).$$

3.3.2. Recall now (see [GLC2, Sect. 3.2]) that in addition to $\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}$, we can consider the category

$$\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}},$$

and we have a canonical identification

$$(\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}})^{\vee} \overset{\mathbf{D}^{\text{Serre}}}{\simeq} \text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}. \tag{3.5}$$

.

In addition, we have an identification

$$\Theta_{\text{Op}^{\text{mon-free}}(\check{G})} : \text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}} \to \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}},$$

see [GLC2, Equation (3.21)].

Composing we obtain a datum of self-duality:

$$\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})^{\vee}_{\text{Ran}} \overset{\Theta_{\text{Op}^{\text{mon-free}}(\check{G})} \circ \mathbf{D}^{\text{Serre}}}{\simeq} \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}. \tag{3.6}$$

[7] In fact, it is used twice, and one can show that the two usages cancel each other out. We just chose not to go through this exercise.

3.3.3. We are going to prove:

Proposition 3.3.4. *With respect to the self-dualities* (3.6) *and* (2.13)*, we have a canonical identification between the functor* dual *to* $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}}$ *and*

$$(\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R \otimes \mathfrak{l}_{\mathrm{Kost}}[-\delta_G],$$

where $\mathfrak{l}_{\mathrm{Kost}}$ *is the line of* [GLC2, Sect. 17.2.2].

The rest of this subsection is devoted to the proof of Proposition 3.3.4.

3.3.5. Recall (see [GLC2, Sect. 17.4.1]) that in addition to the functor $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}$, there exists a functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})_{\mathrm{Ran}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).$$

Denote

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\mathrm{irred}} := \jmath^* \circ \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}\,.$$

According to [GLC2, Theorem 17.4.7], we have:

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!} \otimes \mathfrak{l}_{\mathrm{Kost}}[-\delta_G] \simeq \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*} \circ \Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}(\check{G})}.$$

Hence, the assertion of the proposition can be reformulated as an isomorphism

$$(\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\mathrm{irred}})^\vee \simeq (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R[2\delta_G], \tag{3.7}$$

as functors

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \rightrightarrows \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})_{\mathrm{Ran}},$$

where we regard $(\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\mathrm{irred}})^\vee$ as a functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \overset{(2.13)}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})^\vee \to \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})^\vee_{\mathrm{Ran}} \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})_{\mathrm{Ran}}.$$

To simplify the notation, we will prove a variant of (3.7), where instead of the entire Ran we worked at a fixed point $\underline{x} \in \mathrm{Ran}$. I.e., we will prove

$$(\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x},\mathrm{irred}})^\vee \simeq (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x},\mathrm{irred}})^R[2\delta_G] \tag{3.8}$$

as functors

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \rightrightarrows \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}),$$

3.3.6. Recall that

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x}} \text{ and } \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}}$$

are given by

$$(\pi_{\underline{x}})^{\mathrm{IndCoh}}_* \circ (s_{\underline{x}})^{\mathrm{IndCoh},*} \text{ and } (\pi_{\underline{x}})^{\mathrm{IndCoh}}_* \circ (s_{\underline{x}})^!,$$

respectively, for the morphisms

$$\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}} \overset{s_{\underline{x}}}{\leftarrow} \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(X-\underline{x}) \overset{\pi_{\underline{x}}}{\to} \mathrm{LS}_{\check{G}},$$

see [GLC2, Sects. 17.4.1 and 17.4.2].

Let

$$\mathrm{coeff}^{\mathrm{spec}}_{\check{G},\underline{x}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}})$$

denote the functor

$$(s_{\underline{x}})^{\mathrm{IndCoh}}_* \circ (\pi_{\underline{x}})^!.$$

For future use, we also introduce the notation for the Ran version of this functor

$$\mathrm{coeff}^{\mathrm{spec}}_{\check{G}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})_{\mathrm{Ran}}.$$

3.3.7. Let

$$\mathrm{IndCoh}(\mathrm{LS}_{\check{G}})^\vee \overset{\mathbf{D}^{\mathrm{Serre}}}{\to} \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \tag{3.9}$$

be the identification, given by *Serre duality*.

Note that with respect to identifications (3.9) and (3.5), we have

$$(\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}})^\vee \simeq \mathrm{coeff}^{\mathrm{spec}}_{\check{G},\underline{x}}. \tag{3.10}$$

3.3.8. Due to the assumption that G (and hence $\check{G}$) is semi-simple, the stack $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is smooth, so the natural embedding

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \hookrightarrow \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$$

is an equivalence.

In particular, the identification (3.9) induces an identification

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})^\vee \overset{\mathbf{D}^{\mathrm{Serre}}}{\to} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}). \tag{3.11}$$

From (3.10) we obtain

$$(\jmath^* \circ \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}})^\vee \simeq \mathrm{coeff}^{\mathrm{spec}}_{\check{G},\underline{x}} \circ \jmath_* =: \mathrm{coeff}^{\mathrm{spec}}_{\check{G},\underline{x},\mathrm{irred}},$$

as functors

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \rightrightarrows \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}),$$

where we use (3.11) to identify $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$ with its own dual.

3.3.9. Note now that since $\check{G}$ is semi-simple, the Killing form on $\check{\mathfrak{g}}$ defines a canonical symplectic structure on $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$. Hence,

$$\mathbf{D}^{\mathrm{Serre}} \simeq \mathbf{D}^{\mathrm{naive}}[\dim(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X))] = \mathbf{D}^{\mathrm{naive}}[2\delta_G].$$

Hence, (3.8) becomes equivalent to an isomorphism

$$\mathrm{coeff}^{\mathrm{spec}}_{\check{G},\underline{x},\mathrm{irred}} \simeq (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x},\mathrm{irred}})^R. \tag{3.12}$$

3.3.10. Thus, we have to establish an adjunction between

$$\jmath^* \circ (\pi_{\underline{x}})^{\mathrm{IndCoh}}_* \circ (s_{\underline{x}})^* \text{ and } (s_{\underline{x}})^{\mathrm{IndCoh}}_* \circ (\pi_{\underline{x}})^! \circ \jmath_*.$$

Since the functors $((s_{\underline{x}})^{\mathrm{IndCoh},*}, (s_{\underline{x}})^{\mathrm{IndCoh}}_*)$ form an adjoint pair, it suffices to establish an adjunction between

$$\jmath^* \circ (\pi_{\underline{x}})^{\mathrm{IndCoh}}_* \text{ and } (\pi_{\underline{x}})^! \circ \jmath_*.$$

3.3.11. Set

$$\mathrm{Op}^{\mathrm{mon\text{-}free,irred}}_{\check{G}}(X-\underline{x}) := \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(X-\underline{x}) \underset{\mathrm{LS}_{\check{G}}}{\times} \mathrm{LS}^{\mathrm{irred}}_{\check{G}}.$$

Let $\pi^{\mathrm{irred}}_{\underline{x}}$ denote the resulting morphism

$$\mathrm{Op}^{\mathrm{mon\text{-}free,irred}}_{\check{G}}(X-\underline{x}) \to \mathrm{LS}^{\mathrm{irred}}_{\check{G}}.$$

By base change, the required adjunction is equivalent to an adjunction between

$$\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free,irred}}_{\check{G}}(X-\underline{x})) \overset{(\pi^{\mathrm{irred}}_{\underline{x}})^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$$

and

$$\mathrm{IndCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \overset{(\pi^{\mathrm{irred}}_{\underline{x}})^!}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free,irred}}_{\check{G}}(X-\underline{x})).$$

However, this follows from the fact that, under the assumption that $\check{G}$ is semi-simple, the morphism $\pi^{\mathrm{irred}}_{\underline{x}}$ is ind-proper. Indeed, the generic non-degeneracy condition for opers is automatic, once the underlying local system is irreducible.

□[Proposition 3.3.4]

3.4. **Proof of Theorem 3.1.2.** This proof will amount to comparing the commutative diagrams obtained from the diagram expressing the compatibility of $\mathbb{L}_G$ with critical localization (diagram (3.14) below) by passage to right adjoint and dual functors, respectively.

3.4.1. By Proposition 3.2.3, in order to construct an isomorphism

$$(\mathbb{L}_{G,\mathrm{cusp}})^R \simeq \Phi_{G,\mathrm{cusp}},$$

it suffices to establish an isomorphism

$$(\mathrm{Loc}_{G,\mathrm{cusp}})^\vee \circ (\mathbb{L}_{G,\mathrm{cusp}})^R \simeq (\mathrm{Loc}_{G,\mathrm{cusp}})^\vee \circ \Phi_{G,\mathrm{cusp}}. \tag{3.13}$$

3.4.2. Recall that according to [GLC2, Theorem 18.5.2], we have the following commutative diagram:

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}}\uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}, \end{array} \tag{3.14}$$

where :

- $\mathrm{FLE}_{G,\mathrm{crit}}$ is the *critical FLE functor* of [GLC2, Equation (6.7)];
- $\mathfrak{l}$ is the comologically graded line

$$\mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}],$$

  where:
  - $\mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}}$ is the (non-graded) line of [GLC2, Equation (9.7)];
  - $\mathfrak{l}_{N_{\rho(\omega_X)}}$ is the (non-graded) line of [GLC2, Equation (14.2)];
  - $\delta_{N_{\rho(\omega_X)}} = \dim(\mathrm{Bun}_{N_{\rho(\omega_X)}})$.

3.4.3. Concatenating diagrams (3.14) and (1.9), we obtain a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} & \xrightarrow{\mathbb{L}_{G,\mathrm{cusp}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \\ {\scriptstyle \mathrm{Loc}_{G,\mathrm{cusp}} \otimes \mathfrak{l}}\uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}. \end{array} \tag{3.15}$$

Passing to the right adjoints in (3.15) we obtain a diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} & \xleftarrow{(\mathbb{L}_{G,\mathrm{cusp}})^R} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \\ {\scriptstyle \mathrm{Loc}^R_{G,\mathrm{cusp}}}\downarrow & & \downarrow{\scriptstyle (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R \otimes \mathfrak{l}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xleftarrow{\mathrm{FLE}^{-1}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}. \end{array} \tag{3.16}$$

We will establish (3.13) by showing that the diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} & \xleftarrow{\Phi_{G,\mathrm{cusp}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \\ {\scriptstyle \mathrm{Loc}^R_{G,\mathrm{cusp}}}\downarrow & & \downarrow{\scriptstyle (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R \otimes \mathfrak{l}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xleftarrow{\mathrm{FLE}^{-1}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \end{array} \tag{3.17}$$

commutes as well.

3.4.4. Consider the diagram obtained by passing to the duals in (3.15):

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xleftarrow{(\mathbb{L}_{G,\text{cusp}})^\vee} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \\ {\scriptstyle \text{Loc}^\vee_{G,\text{cusp}} \otimes \mathfrak{l}} \Big\downarrow & & \Big\downarrow {\scriptstyle (\text{Poinc}^{\text{spec}}_{\check{G},*,\text{irred}})^\vee} \\ \text{KL}(G)_{\text{crit},\text{Ran}} & \xleftarrow{\text{FLE}^\vee_{G,\text{crit}}} & \text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}} \end{array} \tag{3.18}$$

Recall now that according to [GLC2, Theorem 8.1.4], we have a canonical identification

$$\text{FLE}^\vee_{G,\text{crit}} \simeq \tau_G \circ \text{FLE}^{-1}_{G,\text{crit}} \circ \Theta_{\text{Op}^{\text{mon-free}}(\check{G})}$$

as functors

$$\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}} \to \text{KL}(G)_{\text{crit},\text{Ran}}.$$

Combining with Proposition 3.3.4, we can rewrite (3.18) as

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xleftarrow{\tau_G \circ (\mathbb{L}_{G,\text{cusp}})^\vee} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \\ {\scriptstyle \text{Loc}^\vee_{G,\text{cusp}} \otimes \mathfrak{l}} \Big\downarrow & & \Big\downarrow {\scriptstyle (\text{Poinc}^{\text{spec}}_{\check{G},*,\text{irred}})^R \otimes \mathfrak{l}^{\otimes -1}_{\text{Kost}}[\delta_G]} \\ \text{KL}(G)_{\text{crit},\text{Ran}} & \xleftarrow{\text{FLE}^{-1}_{G,\text{crit}}} & \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}, \end{array} \tag{3.19}$$

and further as

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xleftarrow{\Phi_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \\ {\scriptstyle \text{Loc}^\vee_{G,\text{cusp}}[2\delta_{N_{\rho(\omega_X)}}]} \Big\downarrow & & \Big\downarrow {\scriptstyle (\text{Poinc}^{\text{spec}}_{\check{G},*,\text{irred}})^R \otimes \mathfrak{l}^{\otimes -1} \otimes \mathfrak{l}^{\otimes -1}_{\text{Kost}}[\delta_G]} \\ \text{KL}(G)_{\text{crit},\text{Ran}} & \xleftarrow{\text{FLE}^{-1}_{G,\text{crit}}} & \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}. \end{array} \tag{3.20}$$

Taking into account Proposition 3.2.5, we can further rewrite (3.20) as

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} & \xleftarrow{\Phi_{G,\text{cusp}}} & \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}) \\ {\scriptstyle \text{Loc}^R_{G,\text{cusp}}[2\delta_{N_{\rho(\omega_X)}}]} \Big\downarrow & & \Big\downarrow {\scriptstyle (\text{Poinc}^{\text{spec}}_{\check{G},*,\text{irred}})^R \otimes \mathfrak{l}^{\otimes -1} \otimes \mathfrak{l}^{\otimes -1}_{\text{Kost}} \otimes \det(\Gamma(X,\mathcal{O}_X) \otimes \mathfrak{g})^{\otimes -1}} \\ \text{KL}(G)_{\text{crit},\text{Ran}} & \xleftarrow{\text{FLE}^{-1}_{G,\text{crit}}} & \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}. \end{array} \tag{3.21}$$

3.4.5. Comparing (3.21) with the desired diagram (3.17), we conclude that it establish the isomorphism

$$\mathfrak{l}_{\text{Kost}} \otimes \det(\Gamma(X, \mathcal{O}_X) \otimes \mathfrak{g})^{\otimes -1} \simeq (\mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}})^{\otimes 2} \otimes \mathfrak{l}^{\otimes -2}_{N_{\rho(\omega_X)}}.$$

However, the latter is given by [GLC3, Proposition 15.1.10].

$\square$[Theorem 3.1.2]

3.5. **An addendum: ambidexterity for eigensheaves.** The contents of this subsection will not be used elsewhere in the paper. Here we will explain another approach to ambidexterity, albeit so far working only for Hecke eigensheaves (or more generally D-modules with nilpotent singular support, see Remark 3.5.9).

3.5.1. Fix a point $\sigma \in \text{LS}^{\text{irred}}_{\check{G}}$, and let

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma := \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{Vect}$$

be the corresponding category of Hecke eigensheaves, where

$$\text{QCoh}(\text{LS}_{\check{G}}) \to \text{Vect}$$

is the functor of *-fiber of σ, to be denoted $(i_\sigma)^*$.

Note that since σ was assumed irreducible, the forgetful functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma \overset{\mathbf{oblv}_\sigma}{\to} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \tag{3.22}$$

lands in $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}$, see Corollary 1.4.5.

3.5.2. The functor $\mathbb{L}_G$ induces a functor

$$\mathbb{L}_{G,\sigma} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma \to \text{Vect}.$$

According to Theorem 3.1.4, the left and rights adjoints of $\mathbb{L}_{G,\sigma}$ are (canonically) isomorphic. In this subsection we will exhibit another way of constructing such an isomorphism[8].

3.5.3. Note that by construction, the functor $\mathbb{L}_{G,\sigma}$ is isomorphic to the composition

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma \overset{\mathbf{oblv}_\sigma}{\to} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \overset{\text{coeff}_G^{\text{Vac,glob}}}{\longrightarrow} \text{Vect},$$

where $\text{coeff}_G^{\text{Vac,glob}}$ is as in [GLC2, Sect. 9.6.3].

The left adjoint of $\mathbb{L}_{G,\sigma}$, denoted $\mathbb{L}^L_{G,\sigma}$ sends the generator $k \in \text{Vect}$ to the object

$$(i_\sigma)^*(\text{Poinc}^{\text{Vac,glob}}_{G,!}),$$

where:

- $(i_\sigma)^*$ denotes the functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}}) \overset{\text{id} \otimes (i_\sigma)^*}{\longrightarrow}$$
$$\to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{Vect} = \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma,$$

  left adjoint to the forgetful functor $\mathbf{oblv}_\sigma$;
- $\text{Poinc}^{\text{Vac,glob}}_{G,!} \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ is the object from [GLC1, Sect. 1.3].

3.5.4. Thus, we wish to construct a canonical isomorphism

$$\mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma}(\mathcal{F}, \mathbb{L}^L_{G,\sigma}(\mathsf{V})) \simeq \mathcal{H}om_{\text{Vect}}(\text{coeff}_G^{\text{Vac,glob}} \circ \mathbf{oblv}_\sigma(\mathcal{F}), \mathsf{V}) \tag{3.23}$$

for $\mathcal{F} \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma$ and $\mathsf{V} \in \text{Vect}$.

We will rewrite both sides of (3.23) and show that they are canonically isomorphic.

3.5.5. Using Theorem 2.3.4, we rewrite $\mathbb{L}^L_{G,\sigma}(\mathsf{V})$ as

$$(i_\sigma)^*(\text{Ps-Id}^{\text{nv}}(\text{Poinc}^{\text{Vac,glob}}_{G,*})) \otimes \mathsf{V}, \tag{3.24}$$

where

$$\text{Poinc}^{\text{Vac,glob}}_{G,*} := \mathbb{D}^{\text{Verdier}}(\text{Poinc}^{\text{Vac,glob}}_{G,!})[-2\delta_{N_{\rho(\omega_X)}}] \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}.$$

Since σ is a smooth point of $\text{LS}^{\text{irred}}_{\check{G}}$, we have

$$(i_\sigma)^* \simeq (i_\sigma)^![\dim(\text{LS}^{\text{irred}}_{\check{G}})] \otimes \det(T^*_\sigma(\text{LS}_{\check{G}})),$$

where i_σ : $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}) \to \text{Vect}$ is the functor of !-pullback, which is defined for maps of finite Tor-dimension, and is the the *right* adjoint of $(i_\sigma)_*$, since the map $i_\sigma : \text{pt} \to \text{LS}^{\text{irred}}_{\check{G}}$ is proper.

In particular, the functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{QCoh}(\text{LS}_{\check{G}}) \overset{\text{id} \otimes (i_\sigma)^!}{\longrightarrow}$$
$$\to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LS}_{\check{G}})}{\otimes} \text{Vect} = \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma$$

is the right adjoint of $\mathbf{oblv}_\sigma$.

[8]However, it is is not obvious that the isomorphism we will construct in this subsection is the same as one from Theorem 3.1.4.

Note also that using the symplectic structure on $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$, we can trivialize the line $\det(T^*_\sigma(\mathrm{LS}_{\check{G}}))$, and we note that $\dim(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) = 2\dim(\mathrm{Bun}_G)$.

Combining, we obtain that the left-hand side in (3.23) identifies with

$$\mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)}(\mathbf{oblv}_\sigma(\mathcal{F}), \text{Ps-Id}^{\mathrm{nv}}(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,*}) \otimes \mathsf{V})[2\dim(\mathrm{Bun}_G)]. \tag{3.25}$$

3.5.6. Denote

$$\mathcal{F}' := \mathbf{oblv}_\sigma(\mathcal{F}).$$

We rewrite (3.25) using Verdier duality as

$$\mathcal{H}om_{\mathrm{Vect}}(\mathrm{C}^\cdot_c(\mathrm{Bun}_G, \mathcal{F}' \overset{*}{\otimes} \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}), \mathsf{V})[2\dim(\mathrm{Bun}_G) + 2\delta_{N_{\rho(\omega_X)}}]. \tag{3.26}$$

And we rewrite the right-hand side of (3.23) as

$$\mathcal{H}om_{\mathrm{Vect}}(\mathrm{C}^\cdot(\mathrm{Bun}_G, \mathcal{F}' \overset{!}{\otimes} \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,*}), \mathsf{V})[2\delta_{N_{\rho(\omega_X)}}]. \tag{3.27}$$

Hence, in order to establish (3.23), we need to construct an isomorphism

$$\mathrm{C}^\cdot_c(\mathrm{Bun}_G, \mathcal{F}' \overset{*}{\otimes} \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!})[-2\dim(\mathrm{Bun}_G)] \simeq \mathrm{C}^\cdot(\mathrm{Bun}_G, \mathcal{F}' \overset{!}{\otimes} \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,*}). \tag{3.28}$$

3.5.7. By the main theorem of [Lin], we have

$$\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}[-2\dim(\mathrm{Bun}_G)] \simeq \mathrm{Mir}_{\mathrm{Bun}_G}(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,*}).$$

Now (3.28) follows from the fact that $\mathcal{F}'$ has nilpotent singular support (see [AGKRRV1, Corollary 14.4.10]) combined with the next general assertion from [AGKRRV2, Theorem 3.4.2]:

Theorem 3.5.8. *For any $\mathcal{F}' \in \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)$ and any $\mathcal{F}'' \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}$, there is a canonical isomorphism*

$$\mathrm{C}^\cdot_c(\mathrm{Bun}_G, \mathcal{F}' \overset{*}{\otimes} \mathrm{Mir}_{\mathrm{Bun}_G}(\mathcal{F}'')) \simeq \mathrm{C}^\cdot(\mathrm{Bun}_G, \mathcal{F}' \overset{!}{\otimes} \mathcal{F}'').$$

Remark 3.5.9. The above argument can be generalized so that it proves ambidexterity for the functor induced by $\mathbb{L}_G$

$$\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \to \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{irred,restr}}_{\check{G}}),$$

where

$$\text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G)_{\mathrm{cusp}} := \text{D-mod}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_G) \cap \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}$$

and

$$\mathrm{LS}^{\mathrm{irred,restr}}_{\check{G}} := \mathrm{LS}^{\mathrm{restr}}_{\check{G}} \cap \mathrm{LS}^{\mathrm{irred}}_{\check{G}}.$$

Note that $\mathrm{LS}^{\mathrm{irred,restr}}_{\check{G}}$ is a disjoint union of formal schemes, each of which is isomorphic to the formal completion of a point in a smooth symplectic scheme of dimension $2\dim(\mathrm{Bun}_G)$.

4. The expression for $\mathcal{A}_{G,\mathrm{irred}}$ via opers

In this section we will prove that the object $\mathcal{A}_{G,\mathrm{irred}} \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$ can be expressed via opers.

This will lead to a number of structural results concerning $\mathcal{A}_{G,\mathrm{irred}}$, as well as the space of generic oper structures on irreducible local systems.

Furthermore, given the recent result of [BKS], we will deduce GLC for classical groups.

4.1. Statement of the result.

4.1.1. Consider the space $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}$ fibered over Ran, whose fiber over $\underline{x} \in \mathrm{Ran}$ is

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X - \underline{x}).$$

Let π_{Ran} denote the resulting map

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}} \to \mathrm{LS}_{\check{G}}\,.$$

Set

$$\mathrm{Op}^{\text{mon-free,irred}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}} := \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}} \underset{\mathrm{LS}_{\check{G}}}{\times} \mathrm{LS}^{\mathrm{irred}}_{\check{G}}\,.$$

Let $\pi^{\mathrm{irred}}_{\mathrm{Ran}}$ denote the resulting map

$$\mathrm{Op}^{\text{mon-free,irred}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}} \to \mathrm{LS}^{\mathrm{irred}}_{\check{G}}\,.$$

Note that the morphism $\pi^{\mathrm{irred}}_{\mathrm{Ran}}$ is pseudo-proper, i.e., a (not necessarily filtered) colimit of proper maps, see Sect. 3.3.11.

4.1.2. Consider the object[9]

$$\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}} := \mathbf{oblv}^l\left((\pi^{\mathrm{irred}}_{\mathrm{Ran}})_!(\omega_{\mathrm{Op}^{\text{mon-free,irred}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}})\right) \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}),$$

where

$$\mathbf{oblv}^l : \text{D-mod}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$$

is the "left" forgetful functor, see [GaRo1, Equation (5.3)].

By construction, $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ is a co-commutative coalgebra in $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$.

4.1.3. Denote by

$$\mathbb{L}^R_{G,\mathrm{cusp}} : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}$$

the functor *right adjoint* to $\mathbb{L}_{G,\mathrm{cusp}}$.

Since the monoidal category $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$ is rigid and the functor $\mathbb{L}_{G,\mathrm{cusp}}$ is $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$-linear, the functor $\mathbb{L}^R_{G,\mathrm{cusp}}$ is also naturally $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$-linear.

Hence, the comonad

$$\mathbb{L}_{G,\mathrm{cusp}} \circ \mathbb{L}^R_{G,\mathrm{cusp}}$$

acting on $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$ is given by tensoring by a co-associative coalgebra object, to be denoted $\mathcal{B}_{G,\mathrm{irred}}$.

4.1.4. The main result of this section reads:

Theorem 4.1.5. *There exists a canonical isomorphism between*

$$\mathcal{B}_{G,\mathrm{irred}} \simeq \mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$$

as plain objects of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$.

Remark 4.1.6. One can show that the isomorphism of Theorem 4.1.5 respects the co-associative coalgebra structures on the two sides. However, we will neither prove[10] nor use this.

4.2. **Combining with ambidexterity.** Prior to proving Theorem 4.1.5 we will derive some consequences.

4.2.1. Note that *a priori*, the comonad

$$\mathbb{L}_{G,\mathrm{cusp}} \circ \mathbb{L}^R_{G,\mathrm{cusp}}$$

is the right adjoint of the monad

$$\mathbb{L}_{G,\mathrm{cusp}} \circ \mathbb{L}^L_{G,\mathrm{cusp}}.$$

Hence, the *coalgebra* $\mathcal{B}_{G,\mathrm{irred}}$ identifies a priori with the monoidal dual of the *algebra* $\mathcal{A}_{G,\mathrm{irred}}$.

[9]The superscript "Op" in the notation below refers to opers and *not* to the opposite algebra structure.
[10]See, however, Remark 4.6.7.

4.2.2. Combining with Corollary 3.1.7 we obtain:

Corollary 4.2.3. *There is a canonical isomorphism*

$$\mathcal{A}_{G,\mathrm{irred}} \simeq \mathcal{B}_{G,\mathrm{irred}}$$

as objects of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}})$.

4.2.4. Combing further with Theorem 4.1.5, we obtain:

Corollary 4.2.5. *There is a canonical isomorphism*

$$\mathcal{A}_{G,\mathrm{irred}} \simeq \mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$$

as objects of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}})$.

And as a result:

Corollary 4.2.6. *The object* $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}} \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}})$ *is compact.*

4.2.7. Consider the object

$$(\pi^{\mathrm{irred}}_{\mathrm{Ran}})_!(\omega_{\mathrm{Op}_{\check{G}}^{\text{mon-free,irred}}(X^{\mathrm{gen}})_{\mathrm{Ran}}}) \in \text{D-mod}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}}). \tag{4.1}$$

Assuming Theorem 3.1.8 for a moment, we obtain that the object (4.1) has the form

$$\omega_{\mathrm{LS}_{\check{G}}^{\mathrm{irred}}} \otimes \underline{\mathcal{B}}^{\mathrm{Op}}_{G,\mathrm{irred}},$$

where $\underline{\mathcal{B}}^{\mathrm{Op}}_{G,\mathrm{irred}}$ is a *classical local system of finite rank* on $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$.

We will prove the following assertion, which would be needed for the final step in the proof of GLC:

Proposition 4.2.8. *The local system* $\underline{\mathcal{B}}^{\mathrm{Op}}_{G,\mathrm{irred}}$ *has a finite monodromy, i.e., it trivializes over a finite étale cover of* $\mathrm{LS}_{\check{G}}^{\mathrm{irred}}$.

4.3. **Proof of Proposition 4.2.8.** We will deduce Proposition 4.2.8 from Theorem 4.1.5.

4.3.1. Denote

$$\mathcal{F} := (\pi^{\mathrm{irred}}_{\mathrm{Ran}})_!(\omega_{\mathrm{Op}_{\check{G}}^{\text{mon-free,irred}}(X^{\mathrm{gen}})_{\mathrm{Ran}}})[-n],$$

where[11] $n = \dim(\mathrm{LS}_{\check{G}})$.

Since the map $\pi^{\mathrm{irred}}_{\mathrm{Ran}}$ is pseudo-proper, this object can be written as

$$\operatorname*{colim}_{i\in I} \mathcal{F}_i \quad \mathcal{F}_i \in \text{D-mod}(\mathrm{LS}_{\check{G}}^{\mathrm{irred}}),$$

over some diagram I, where each $\mathcal{F}_i$ is of the form

$$(f_i)_{*,\mathrm{dR}}(\omega_{\mathcal{Y}_i})[-n],$$

where $f_i : \mathcal{Y}_i \to \mathrm{LS}_{\check{G}}^{\mathrm{irred}}$ is a proper map of algebraic stacks.

For each index i, consider the Stein factorization of the map f_i

$$\mathcal{Y}_i \to \mathcal{Y}_i^0 \xrightarrow{f_i^0} \mathrm{LS}_{\check{G}}^{\mathrm{irred}},$$

so that f_i^0 is a finite map.

Denote

$$\mathcal{F}_i^0 := (f_i^0)_{*,\mathrm{dR}}(\omega_{\mathcal{Y}_i^0})[-n].$$

[11]The cohomological shift is introduced for the sake of perverse normalization.

4.3.2. Let η be the generic point of a connected component of $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$. It is enough to show that $\mathcal{F}|_\eta$ has a finite monodromy.

We have

$$\mathcal{F}_i|_\eta \in \text{D-mod}(\eta)^{\leq 0},\ \mathcal{F}^0_i|_\eta \in \text{D-mod}(\eta)^{\heartsuit}$$

and the map

$$\mathcal{F}_i \to \mathcal{F}^0_i$$

induces an isomorphism

$$H^0(\mathcal{F}_i|_\eta) \to \mathcal{F}^0_i|_\eta.$$

By Theorem 3.1.8 combined with Corollary 4.2.5, the object $\mathcal{F}|_\eta$ is concentrated in cohomological degree 0. Hence, we obtain that

$$\mathcal{F}|_\eta \simeq \underset{i\in I}{\mathrm{colim}}\, \mathcal{F}^0_i|_\eta.$$

Since $\mathcal{F}|_\eta$ is finite-dimensional, we obtain that I contains a finite sub-diagram $I^f \subset I$ such that the map

$$\underset{i\in I^f}{\oplus} \mathcal{F}^0_i|_\eta \to \mathcal{F}|_\eta$$

is surjective.

Since each $\mathcal{F}^0_i|_\eta$ has a finite monodromy, we obtain that so does $\mathcal{F}|_\eta$.

$\square$[Proposition 4.2.8]

4.4. **Proof of Theorem 3.1.8.** In this subsection we will continue to assume Theorem 4.1.5, and deduce Theorem 3.1.8.

4.4.1. Since $\mathcal{A}_{G,\mathrm{irred}}$ is perfect, in order to prove that it is a classical vector bundle, it suffices to show that the *-fibers of $\mathcal{A}_{G,\mathrm{irred}}$ at k-points of $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ are concentrated in cohomological degree 0.

By the self-duality assertion in Corollary 3.1.7, it suffices to show that the *-fibers of $\mathcal{A}_{G,\mathrm{irred}}$ are concentrated in *non-positive* cohomological degrees.

4.4.2. By Corollary 4.2.5, it suffices to show that the *-fibers of $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ are concentrated in *non-positive* cohomological degrees.

However, the *-fiber of $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ at a k-point $\sigma \in \mathrm{LS}^{\mathrm{irred}}_{\check{G}}$ is isomorphic to $\mathrm{C}_\cdot(\mathrm{Op}^{\mathrm{gen}}_{\check{G},\sigma})$, where

$$\mathrm{Op}^{\mathrm{gen}}_{\check{G},\sigma} := \{\sigma\} \underset{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}{\times} \mathrm{Op}^{\mathrm{mon\text{-}free,irred}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}.$$

It is automatically concentrated in *non-positive* cohomological degrees, being the *homology* of a prestack.

4.4.3. The D-module structure on $\mathcal{A}_{G,\mathrm{irred}}$ comes from the isomorphism of Corollary 4.2.5.

$\square$[Theorem 3.1.8]

4.5. **Contractibility of opers.** In this subsection we will continue to assume Theorem 4.1.5. We will show that the validity of GLC is equivalent to the contractibility (and, in fact, just connectedness) of the space of generic oper structures on irreducible local systems.

4.5.1. Note that in the course of the proof of Theorem 3.1.8 above we have established:

Corollary 4.5.2. *The homology of the fibers of the map* $\pi^{\mathrm{irred}}_{\mathrm{Ran}}$ *is acyclic off degree* 0.

This can be equivalently reformulated as follows:

Corollary 4.5.3. *The connected components of the fibers of the map* $\pi^{\mathrm{irred}}_{\mathrm{Ran}}$ *are homologically contractible.*

4.5.4. Applying Corollary 1.6.5, we obtain:

Corollary 4.5.5. *The following assertions are equivalent:*

(i) *The functor* $\mathbb{L}_G$ *is equivalence.*

(ii) *The fibers of the map* $\pi^{\text{irred}}_{\text{Ran}}$ *are connected.*

(iii) *The fibers of the map* $\pi^{\text{irred}}_{\text{Ran}}$ *are* homologically contractible.

4.5.6. In particular, we obtain that GLC is equivalent to the following conjecture:

Conjecture 4.5.7. *The space of* generic oper structures *on a given irreducible local system is homologically contractible.*

Remark 4.5.8. Note that the "bottom" layer of Conjecture 4.5.7 says that the space of generic oper structures on a given irreducible local system is *non-empty*. This statement is actually a theorem, thanks to [Ari].

Remark 4.5.9. The assertion of Conjecture 4.5.7 is easy for $G = GL_n$. In particular, in this way we obtain another proof of GLC in this case (i.e., one that is logically different from that in Sect. 1.7)[12].

4.5.10. Recall now that thanks to [BKS], Conjecture 4.5.7 is actually a theorem whenever G is a classical group[13]. Hence, we obtain:

Main Theorem 4.5.11. *The geometric Langlands conjecture holds when G is a classical group.*

Remark 4.5.12. Formally speaking, the main theorem of [BKS] establishes Conjecture 4.5.7 for a slightly different notion of oper, namely, for $\check{\mathfrak{g}}$-opers, rather than $\check{G}$-opers (and it is the latter that appears in Conjecture 4.5.7). In other words, [BKS] implies Conjecture 4.5.7 not for $\check{G}$ itself but rather for its adjoint quotient.

However, as we shall see in the sequel to this paper, the statement of GLC for a given pair $(G, \check{G})$ formally follows from the case when $\check{G}$ is replaced by its adjoint quotient (resp., G is replaced by the simply-connected cover of its derived group).

4.6. **Proof of Theorem 4.1.5.** As we shall presently see, the proof of the theorem follows almost immediately from diagram (3.15), once we combine the following pieces of information:

- The functor $\text{FLE}_{G,\text{crit}}$ is an equivalence;
- The functor $\mathbb{L}_{G,\text{cusp}}$ is a Verdier quotient.

4.6.1. Consider the adjoint pair

$$\text{Poinc}^{\text{spec}}_{\check{G},*,\text{irred}} : \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}} \rightleftarrows \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}) : \text{coeff}^{\text{spec}}_{\check{G},\text{irred}}, \tag{4.2}$$

see (3.12), where $\text{coeff}^{\text{spec}}_{\check{G},\text{irred}}$ is the version of the functor $\text{coeff}^{\text{spec}}_{\check{G},\underline{x},\text{irred}}$ when $\underline{x}$ varies in families along Ran.

4.6.2. We will deduce Theorem 4.1.5 from the following assertion, which takes place purely on the spectral side.

Theorem 4.6.3. *The comonad on* $\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}})$ *corresponding to* (4.2) *is given by tensor product with* $\mathcal{B}^{\text{Op}}_{G,\text{irred}}$.

This theorem will be proved in Sect. 5. Let us assume it, and proceed with the proof of Theorem 4.1.5.

[12]The difference between the two arguments is that one uses a fully faithfulness assertion on the automorphic side, another on the spectral side.

[13]Here by a classical group we mean a reductive group whose root datum is of type A, B, C or D.

4.6.4. Since we only want to identify

$$\mathcal{B}_{G,\mathrm{irred}} \simeq \mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}} \tag{4.3}$$

as objects of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$ (and not as co-algebras), it suffices to construct an isomorphism of comonads

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R \simeq \mathbb{L}_{G,\mathrm{cusp}} \circ (\mathbb{L}_{G,\mathrm{cusp}})^R \tag{4.4}$$

acting on $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$. Indeed, each side of (4.3) is obtained by applying the corresponding side of (4.4) to $\mathcal{O}_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}$.

4.6.5. Since $\mathrm{FLE}_{G,\mathrm{crit}}$ is an equivalence, the left-hand side in (4.4) identifies with

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ \mathrm{FLE}_{G,\mathrm{crit}} \circ \mathrm{FLE}^R_{G,\mathrm{crit}} \circ (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R.$$

Since $\mathrm{Loc}_{G,\mathrm{cusp}}$ is a Verdier quotient, the right-hand side in (4.4) identifies with

$$\mathbb{L}_{G,\mathrm{irred}} \circ (\mathrm{Loc}_{G,\mathrm{cusp}} \otimes \mathfrak{l}) \circ ((\mathrm{Loc}_{G,\mathrm{cusp}} \otimes \mathfrak{l})^R \circ (\mathbb{L}_{G,\mathrm{cusp}})^R.$$

4.6.6. Hence, it suffices to establish an isomorphism between the comonads

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ \mathrm{FLE}_{G,\mathrm{crit}} \circ \mathrm{FLE}^R_{G,\mathrm{crit}} \circ (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R \simeq$$
$$\simeq \mathbb{L}_{G,\mathrm{irred}} \circ (\mathrm{Loc}_{G,\mathrm{cusp}} \otimes \mathfrak{l}) \circ ((\mathrm{Loc}_{G,\mathrm{cusp}} \otimes \mathfrak{l})^R \circ (\mathbb{L}_{G,\mathrm{cusp}})^R,$$

which is the same as

$$(\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ \mathrm{FLE}_{G,\mathrm{crit}}) \circ (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ \mathrm{FLE}_{G,\mathrm{crit}})^R \simeq$$
$$\simeq (\mathbb{L}_{G,\mathrm{irred}} \circ (\mathrm{Loc}_{G,\mathrm{cusp}} \otimes \mathfrak{l})) \circ (\mathbb{L}_{G,\mathrm{irred}} \circ (\mathrm{Loc}_{G,\mathrm{cusp}} \otimes \mathfrak{l}))^R.$$

However, this follows formally from the commutativity of (3.15).

□[Theorem 4.1.5]

Remark 4.6.7. Note that Theorem 4.1.5 only says that $\mathcal{B}_{G,\mathrm{irred}}$ and $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ are isomorphic as objects of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$, but not as co-associative co-algebras. One can upgrade the proof given above to an isomorphism of coalgebras along the following lines:

It follows from the construction that both comonads in (4.4) are linear with respect to the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ on $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$ via the functor $\jmath^* \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$, and the isomorphism between them constructed above respects this structure.

It also follows from the construction that the $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$-linear structure on $\mathbb{L}_{G,\mathrm{cusp}} \circ (\mathbb{L}_{G,\mathrm{cusp}})^R$ agrees with the above $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$-linear structure.

The comonad given by tensor product with $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ has a tautological linear structure with respect to $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$, and hence also with respect to $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$. It follows from the proof of Theorem 4.6.3 given in the next section that the above $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$-linear structure on $- \otimes \mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ agrees with the $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$-linear structure on $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R$.

Thus, we obtain that the two comonads

$$- \otimes \mathcal{B}_{G,\mathrm{irred}} \text{ and } - \otimes \mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$$

are identified as $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$-linear comonads. Since the functor $\jmath^* \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$ is a Verdier quotient, this implies that the above identification is automatically $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$-linear. The latter is equivalent to the identification of $\mathcal{B}_{G,\mathrm{irred}}$ and $\mathcal{B}^{\mathrm{Op}}_{G,\mathrm{irred}}$ as co-associative coalgebras in $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}})$.

5. Proof of Theorem 4.6.3

The rest of the paper is devoted to the proof of Theorem 4.6.3. In particular, it takes place purely on the spectral side.

We will break up Theorem 4.6.3 into two assertions: Propositions 5.1.2 and 5.1.5. The former can be informally phrased as "the Ran integral equates the quasi-coherent and de Rham direct images". The latter can be informally phrased as "the Ran integral erases the difference between the local and the global".

It turns out that both these assertions are quite general, i.e., have nothing to do with the specifics of opers or local systems.

5.1. Strategy of the proof.

5.1.1. Consider the tautological natural transformation

$$(\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}} \circ \mathbf{oblv}^r_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X^{\mathrm{gen}})_{\mathrm{Ran}}} \to \mathbf{oblv}^r_{\mathrm{LS}_{\check{G}}} \circ (\pi_{\mathrm{Ran}})_{*,\mathrm{dR}}, \tag{5.1}$$

as functors

$$\text{D-mod}(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X^{\mathrm{gen}})_{\mathrm{Ran}}) \rightrightarrows \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}).$$

We will prove

Proposition 5.1.2. *The natural transformation* (5.1) *is an isomorphism when evaluated on objects in the essential image of the functor*

$$\pi^!_{\mathrm{Ran}} : \text{D-mod}(\mathrm{LS}_{\check{G}}) \to \text{D-mod}(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X^{\mathrm{gen}})_{\mathrm{Ran}}).$$

Remark 5.1.3. For the validity of Proposition 5.1.2, it is essential that we work with the entire Ran and not a fixed $\underline{x} \in \mathrm{Ran}$.

5.1.4. Let $(s_{\mathrm{Ran}})_*^{\mathrm{IndCoh}}$ denote the functor

$$\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X^{\mathrm{gen}})_{\mathrm{Ran}}) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathrm{Ran}},$$

and let $(s_{\mathrm{Ran}})^{\mathrm{IndCoh},*}$ denote its left adjoint.

The counit of the $((s_{\mathrm{Ran}})^{\mathrm{IndCoh},*}, (s_{\mathrm{Ran}})_*^{\mathrm{IndCoh}})$-adjunction defines a natural transformation

$$(\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}} \circ (s_{\mathrm{Ran}})^{\mathrm{IndCoh},*} \circ (s_{\mathrm{Ran}})_*^{\mathrm{IndCoh}} \to (\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}}. \tag{5.2}$$

We will prove:

Proposition 5.1.5. *The natural transformation* (5.2), *composed with the coarsening functor*

$$\Psi_{\mathrm{LS}_{\check{G}}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}),$$

is an isomorphism, when evaluated on objects in the essential image of the functor

$$\pi^!_{\mathrm{Ran}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X^{\mathrm{gen}})_{\mathrm{Ran}}).$$

5.1.6. We claim that the combination of Propositions 5.1.2 and 5.1.5 implies the assertion of Theorem 4.6.3.

Recall that the morphism $\pi^{\mathrm{irred}}_{\mathrm{Ran}}$ is pseudo-proper, so we can identify $(\pi^{\mathrm{irred}}_{\mathrm{Ran}})^! \simeq ((\pi^{\mathrm{irred}}_{\mathrm{Ran}})_*)^R$. Hence, the comonad

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}} \circ (\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{irred}})^R$$

identifies with

$$\jmath^* \circ (\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}} \circ (s_{\mathrm{Ran}})^{\mathrm{IndCoh},*} \circ (s_{\mathrm{Ran}})_*^{\mathrm{IndCoh}} \circ (\pi_{\mathrm{Ran}})^! \circ \jmath_*.$$

According to Proposition 5.1.5, this comonad maps isomorphically to the comonad

$$\jmath^* \circ (\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}} \circ (\pi_{\mathrm{Ran}})^! \circ \jmath_* \simeq (\pi^{\mathrm{irred}}_{\mathrm{Ran}})_*^{\mathrm{IndCoh}} \circ (\pi^{\mathrm{irred}}_{\mathrm{Ran}})^!.$$

In particular, we obtain that this comonad is obtained by the !-tensor product with the coalgebra object

$$(\pi^{\mathrm{irred}}_{\mathrm{Ran}})_*^{\mathrm{IndCoh}} \circ (\pi^{\mathrm{irred}}_{\mathrm{Ran}})^!(\omega_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}}) \simeq (\pi^{\mathrm{irred}}_{\mathrm{Ran}})_*^{\mathrm{IndCoh}}(\omega_{\mathrm{Op}_{\check{G}}^{\text{mon-free,irred}}(X^{\mathrm{gen}})_{\mathrm{Ran}}}). \tag{5.3}$$

5.1.7. Restricting along the horizontal arrows in the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Op}^{\text{mon-free,irred}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}} & \xrightarrow{\jmath} & \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}} \\ \pi^{\text{irred}}_{\text{Ran}} \downarrow & & \downarrow \pi_{\text{Ran}} \\ \mathrm{LS}^{\text{irred}}_{\check{G}} & \xrightarrow{\jmath} & \mathrm{LS}_{\check{G}}, \end{array}$$

from Proposition 5.1.2 we obtain that the natural transformation

$$(\pi^{\text{irred}}_{\text{Ran}})^{\text{IndCoh}}_{*} \circ \mathbf{oblv}^{r}_{\mathrm{Op}^{\text{mon-free,irred}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}}} \to \mathbf{oblv}^{r}_{\mathrm{LS}^{\text{irred}}_{\check{G}}} \circ (\pi^{\text{irred}}_{\text{Ran}})_{*,\text{dR}} \tag{5.4}$$

is an isomorphism, when evaluated on objects lying in the essential image of

$$(\pi^{\text{irred}}_{\text{Ran}})^{!} : \text{D-mod}(\mathrm{LS}^{\text{irred}}_{\check{G}}) \to \text{D-mod}(\mathrm{Op}^{\text{mon-free,irred}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}}).$$

Hence, we obtain that the coalgebra (5.3) maps isomorphically to

$$\mathbf{oblv}^{r}_{\mathrm{LS}^{\text{irred}}_{\check{G}}}\left((\pi^{\text{irred}}_{\text{Ran}})_{*,\text{dR}} \circ (\pi^{\text{irred}}_{\text{Ran}})^{!}(\omega_{\mathrm{LS}^{\text{irred}}_{\check{G}}})\right) \simeq \mathbf{oblv}^{r}_{\mathrm{LS}^{\text{irred}}_{\check{G}}} \circ (\pi^{\text{irred}}_{\text{Ran}})_{*,\text{dR}}(\omega_{\mathrm{Op}^{\text{mon-free,irred}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}}}).$$

Finally, we note that

$$\mathbf{oblv}^{r}_{\mathrm{LS}^{\text{irred}}_{\check{G}}}(-) \simeq \mathbf{oblv}^{l}_{\mathrm{LS}^{\text{irred}}_{\check{G}}}(-) \otimes \omega_{\mathrm{LS}^{\text{irred}}_{\check{G}}}.$$

☐[Theorem 4.6.3]

5.2. **Framework for the proof of Proposition 5.1.2.** In this subsection we will explain a general framework for the proof of Proposition 5.1.2: it has to do with a morphism between D-prestacks over X.

5.2.1. Consider the prestack

$$(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}})_{\text{dR}^{\text{rel}}} := (\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}})_{\text{dR}} \underset{(\mathrm{LS}_{\check{G}})_{\text{dR}}}{\times} \mathrm{LS}_{\check{G}},$$

so that

$$\mathrm{IndCoh}((\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}})_{\text{dR}^{\text{rel}}}),$$

is the category of *relative* D-modules on $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}}$ with respect to the projection π_{Ran}.

Denote by

$$\mathbf{ind}^{\text{rel}} : \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}}) \rightleftarrows \mathrm{IndCoh}((\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}})_{\text{dR}^{\text{rel}}}) : \mathbf{oblv}^{\text{rel}}$$

the resulting pair of adjoint functors.

Consider also the functors

$$(\pi_{\text{Ran}})^{\text{IndCoh}}_{*,\text{dR}^{\text{rel}}} : \mathrm{IndCoh}((\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}})_{\text{dR}^{\text{rel}}}) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \tag{5.5}$$

and

$$(\pi_{\text{Ran}})^{!}_{\text{dR}^{\text{rel}}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{IndCoh}((\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}})_{\text{dR}^{\text{rel}}}). \tag{5.6}$$

5.2.2. As in Sect. 5.1.1 we have a natural transformation

$$(\pi_{\text{Ran}})^{\text{IndCoh}}_{*} \circ \mathbf{oblv}^{\text{rel}} \to (\pi_{\text{Ran}})_{*,\text{dR}^{\text{rel}}}, \tag{5.7}$$

as functors

$$\mathrm{IndCoh}((\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\text{gen}})_{\text{Ran}})_{\text{dR}^{\text{rel}}}) \rightrightarrows \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}).$$

The assertion of Proposition 5.1.2 follows immediately from the one:

Proposition 5.2.3. *The natural transformation* (5.7) *is an isomorphism when evaluated on objects in the essential image of the functor* (5.6)*.*

In its turn, Proposition 5.2.3 follows from the next assertion:

Proposition 5.2.4. *The counit of the adjunction*

$$\mathbf{ind}^{\mathrm{rel}} \circ \mathbf{oblv}^{\mathrm{rel}} \to \mathrm{Id}$$

is an isomorphism, when evaluated on objects in the essential image of the functor (5.6).

5.3. **An abstract version of Proposition 5.2.4.** In this subsection we will show that Proposition 5.2.4 is a particular case of a general assertion that has to do with a morphism $f : \mathcal{Z} \to \mathcal{Y}$ of D-prestacks over X.

5.3.1. Consider the prestack $\mathrm{Sect}_\nabla(X, \mathcal{Y})$ of horizontal sections of $\mathcal{Y}$, as well as

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Y})_{\mathrm{Ran}} \text{ and } \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}}$$

that associate to a point $\underline{x} \in \mathrm{Ran}$ the spaces of horizontal sections of $\mathcal{Y}$ and $\mathcal{Z}$ over $X - \underline{x}$, respectively.

Note that we have a tautological map

$$\mathrm{Sect}_\nabla(X, \mathcal{Y}) \times \mathrm{Ran} \to \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Y})_{\mathrm{Ran}}$$

(and similarly for $\mathcal{Z}$).

Denote

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} := \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Y})_{\mathrm{Ran}}}{\times} (\mathrm{Sect}_\nabla(X, \mathcal{Y}) \times \mathrm{Ran}).$$

Denote by π_{Ran} the natural projection

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} \to \mathrm{Sect}_\nabla(X, \mathcal{Y})$$

and by $\pi_{\mathrm{Ran},\mathrm{dR}^{\mathrm{rel}}}$ the map

$$(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}})_{\mathrm{dR}^{\mathrm{rel}}} \to \mathrm{Sect}_\nabla(X, \mathcal{Y})$$

5.3.2. We will impose the following finiteness conditions on $\mathcal{Y}$ and $\mathcal{Z}$:

- $\mathcal{Y}$ is *sectionally laft* in the sense of [Ro, Sect. 3.1.3(ii)], i.e.,
  - The prestack $\mathrm{Sect}_\nabla(X, \mathcal{Y})$ is locally almost of finite type;
  - The condition of [Ro, Sect. 3.1.3(ii)] is satisfied for points of $\mathrm{Sect}_\nabla(X, \mathcal{Y})$;
- $\mathcal{Z}$ is *meromorphically sectionally laft* relative to $\mathcal{Y}$, i.e.,
  - The prestack $\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}$ is locally almost of finite type;
  - The condition of [Ro, Sect. 3.1.3(ii)] is satisfied for points of $\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}$.

5.3.3. Denote

$$(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}})_{\mathrm{dR}^{\mathrm{rel}}} := (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}})_{\mathrm{dR}} \underset{\mathrm{Sect}_\nabla(X, \mathcal{Y})_{\mathrm{dR}}}{\times} \mathrm{Sect}_\nabla(X, \mathcal{Y}).$$

Let

$$\mathbf{ind}^{\mathrm{rel}} : \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}) \rightleftarrows \mathrm{IndCoh}\left((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}})_{\mathrm{dR}^{\mathrm{rel}}}\right) : \mathbf{oblv}^{\mathrm{rel}} \tag{5.8}$$

the resulting pair of adjoint functors.

We have:

Proposition 5.3.4. *The counit of the adjunction*

$$\mathbf{ind}^{\mathrm{rel}} \circ \mathbf{oblv}^{\mathrm{rel}} \to \mathrm{Id}$$

is an isomorphism, when evaluated on objects in the essential image of the pullback functor

$$(\pi_{\mathrm{Ran},\mathrm{dR}^{\mathrm{rel}}})^! : \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})) \to \mathrm{IndCoh}\left((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}})_{\mathrm{dR}^{\mathrm{rel}}}\right).$$

The proof will be given in Sect. A.

5.3.5. Note that Proposition 5.2.4 is indeed a particular case of Proposition 5.3.4: we take $\mathcal{Y}$ to be the constant D-stack with fiber $\mathrm{pt}/\check{G}$ and $\mathcal{Z} := \mathrm{Op}_{\check{G}}$.

□[Proposition 5.2.4]

5.4. **A digression: the category** $\mathrm{QCoh}_{\mathrm{co}}$**.** In order to formulate (an abstract version of) Proposition 5.1.5, it will be convenient to introduce a general construction of a certain variant of the category of quasi-coherent sheaves on a prestack, denoted $\mathrm{QCoh}_{\mathrm{co}}(-)$ (see [GLC2, Sect. A.1] for a more detailed discussion).

5.4.1. Let $\mathcal{W}$ be a prestack. We define the category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})$ by

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}) := \underset{S\to\mathcal{W},\, S\in\mathrm{Sch}^{\mathrm{aff}}}{\mathrm{colim}} \mathrm{QCoh}(S),$$

where the colimit is taken with respect to the *direct image functors*[14].

5.4.2. If $\mathcal{W}$ has an affine diagonal, we have a functor

$$\Omega_{\mathcal{W}} : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}) \to \mathrm{QCoh}(\mathcal{W}).$$

Namely, it corresponds to the compatible family of direct image functors

$$\mathrm{QCoh}(S) \to \mathrm{QCoh}(W), \quad S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{W}},$$

which are well-defined since the morphisms $S \to \mathcal{W}$ are affine.

5.4.3. *Example.* Suppose that $\mathcal{W}$ is a *scheme*. Then it is easy to see that in this case the functor $\Omega_{\mathcal{W}}$ is an equivalence.

In fact, according to [Ga4, Proposition 6.2.7 and Theorem 2.2.6], the same is true when $\mathcal{W}$ is an eventually coconnective quasi-compact algebraic stack of finite type with an affine diagonal.

Note that $\mathcal{W} = \mathrm{LS}_{\check{G}}$ is an example of such an algebraic stack.

Remark 5.4.4. We do not know whether $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})$ is dualizable. However, $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})$ is tautologically the pre-dual of $\mathrm{QCoh}(\mathcal{W})$, i.e.,

$$\mathrm{QCoh}(\mathcal{W}) \simeq \mathrm{Funct}(\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}), \mathrm{Vect}),$$

where $\mathrm{Funct}(-,-)$ is the category of colimit-preserving functors.

In particular, if $\mathcal{W}$ is such that $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})$ is dualizable, then so is $\mathrm{QCoh}(\mathcal{W})$.

5.4.5. *Example.* Let $\mathcal{W}$ be an ind-scheme, written as

$$\mathcal{W} = \underset{i}{\mathrm{colim}}\, W^i, \quad W^i \in \mathrm{Sch},$$

where the transition maps $W^i \to W^j$ are closed embeddings.

In this case,

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}) \simeq \underset{i}{\mathrm{colim}}\, \mathrm{QCoh}(W^i)$$

where the colimit is taken with respect to the direct image functors.

Note that if $\mathcal{W}$ is of ind-finite type, we have a naturally defined functor

$$\Psi_{\mathcal{W}} : \mathrm{IndCoh}(\mathcal{W}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}). \tag{5.9}$$

Indeed, we can write

$$\mathrm{IndCoh}(\mathcal{W}) \simeq \underset{i}{\mathrm{colim}}\, \mathrm{IndCoh}(W^i)$$

(under direct image functors) and (5.9) is given by the compatible family of functors

$$\Psi_{W^i} : \mathrm{IndCoh}(W^i) \to \mathrm{QCoh}(W^i).$$

One can show that (5.9) is an equivalence if $\mathcal{W}$ is formally smooth.

[14]The reason for the notation "$\mathrm{QCoh}_{\mathrm{co}}$" is that it is a version of the QCoh category, i.e., we take the colimit with respect to the *-direct image maps, instead of the limit with respect to the *-pullback maps.

5.4.6. The assignment

$$\mathcal{W} \rightsquigarrow \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})$$

has the following functoriality properties for maps $f : \mathcal{W}_1 \to \mathcal{W}_2$:

- We have the direct image functor
$$f_* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_1) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_2).$$
- If f is *schematic*, we also have the pullback functor
$$f^* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_2) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_1),$$
which is a left adjoint of f_*.
- For a pullback square
$$\begin{array}{ccc} \mathcal{W}_1 & \xrightarrow{f} & \mathcal{W}_2 \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow{f} & S_2, \end{array}$$
where S_1 and S_2 are affine schemes, the functor
$$\mathrm{QCoh}(S_1) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_2) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_1),$$
defined by f^*, is an equivalence;
- If f is schematic *and of finite Tor dimension*, we also have the !-pullback functor
$$f^! : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_2) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_1).$$
Note that if f is also proper, then the functors $(f_*, f^!)$ are mutually adjoint.

5.4.7. Let $\mathcal{W}_{\mathrm{Ran}}$ be a prestack over Ran. Set

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}} := \lim_{S\in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}},\, S\to \mathrm{Ran}} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}(S), \tag{5.10}$$

where the limit is formed using the *-pullback functors along the $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S)$-factors and !-pullback functors along the $\mathrm{IndCoh}(S)$-factors.

Thus, an object $\mathcal{F} \in \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}}$ gives rise to an object

$$\mathcal{F}_{S,\underline{x}} \in \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}(S)$$

for every $\underline{x} \in \mathrm{Ran}(S)$.

In the case when $\mathcal{W}_{\mathrm{Ran}} \to \mathrm{Ran}$ is schematic, so that for every $(S, \underline{x})$ as above we have

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S) \simeq \mathrm{QCoh}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S),$$

we will simply write $\mathrm{QCoh}(\mathcal{W})_{\mathrm{Ran}}$ instead of $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}}$.

Remark 5.4.8. The assignment

$$S \rightsquigarrow \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S)$$

naturally forms a sheaf of categories over Ran, to be denoted

$$\underline{\mathrm{QCoh}_{\mathrm{co}}}(\mathcal{W})_{\mathrm{Ran}}.$$

The above definition of $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}}$ is a particular case of the following construction: for any sheaf of categories $\underline{\mathbf{C}}_{\mathrm{Ran}}$ over Ran, we can assign the category

$$\mathbf{C}_{\mathrm{Ran}} := \lim_{S\in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}},\, S\to \mathrm{Ran}} \mathbf{C}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}(S).$$

Note that since Ran is 1-affine, we have

$$\mathbf{C}_{\mathrm{Ran}} \simeq \Gamma(\mathrm{Ran}, \underline{\mathbf{C}}_{\mathrm{Ran}}) \underset{\mathrm{QCoh}(\mathrm{Ran})}{\otimes} \mathrm{IndCoh}(\mathrm{Ran}).$$

In particular, since the functor

$$\mathrm{QCoh}(\mathrm{Ran}) \to \mathrm{IndCoh}(\mathrm{Ran}), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \omega_{\mathrm{Ran}}$$

is an equivalence, we have an equivalence

$$\mathbf{C}_{\mathrm{Ran}} \simeq \Gamma(\mathrm{Ran}, \underline{\mathbf{C}}_{\mathrm{Ran}}).$$

5.4.9. Let $p_{\mathcal{W}_{\mathrm{Ran}}}$ denote the projection

$$\mathcal{W}_{\mathrm{Ran}} \to \mathrm{Ran}.$$

Note that we have a well-defined functor

$$(p_{\mathcal{W}_{\mathrm{Ran}}})_* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}} \to \mathrm{IndCoh}(\mathrm{Ran}) \simeq \text{D-mod}(\mathrm{Ran}).$$

Let us denote by

$$\Gamma^{\mathrm{IndCoh}_{\mathrm{Ran}}}(\mathcal{W}_{\mathrm{Ran}}, -) : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}} \to \mathrm{Vect}$$

the functor equal to the composition

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}} \overset{(p_{\mathcal{W}_{\mathrm{Ran}}})_*}{\longrightarrow} \text{D-mod}(\mathrm{Ran}) \overset{\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-)}{\longrightarrow} \mathrm{Vect},$$

where we can alternatively think of $\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}, -)$ as the functor

$$\mathrm{C}^\cdot_c(\mathrm{Ran}, -) : \text{D-mod}(\mathrm{Ran}) \to \mathrm{Vect},$$

left adjoint to $k \mapsto \omega_{\mathrm{Ran}}$.

5.4.10. Assume now that $\mathcal{W}_{\mathrm{Ran}}$ is locally almost of finite type (so that $\mathrm{IndCoh}(\mathcal{W}_{\mathrm{Ran}})$ is defined) and assume that $\mathcal{W}_{\mathrm{Ran}} \to \mathrm{Ran}$ is a relative ind-scheme.

We claim that in this case, there exists a well-defined functor

$$\Psi_{\mathcal{W}_{\mathrm{Ran}}} : \mathrm{IndCoh}(\mathcal{W}_{\mathrm{Ran}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}}, \tag{5.11}$$

which is a variant of (5.9).

Indeed, we can write

$$\mathrm{IndCoh}(\mathcal{W}_{\mathrm{Ran}}) \simeq \lim_{S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}},\, S \to \mathrm{Ran}} \mathrm{IndCoh}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S),$$

so it is enough to define a compatible family of functors

$$\mathrm{IndCoh}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S). \tag{5.12}$$

Write

$$\mathcal{W}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S \simeq \underset{i}{\mathrm{colim}}\, W^i_S,$$

where W^i_S are schemes, and the transition maps $W^i_S \to W^j_S$ are closed embeddings.

The functors (5.12) are given by the compatible family of functors

$$\mathrm{IndCoh}(W^i_S) \to \mathrm{QCoh}(W^i_S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}(S),$$

Serre-dual to the tautological functors

$$\mathrm{QCoh}(W^i_S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(W^i_S),$$

given by !-pullback along $W^i_S \to S$.

5.5. **Abstract version of Proposition 5.1.5: the absolute case.** As with Proposition 5.1.2, we will prove an abstract statement, of which Proposition 5.1.5 is a particular case. The general set-up involves a morphism

$$\mathcal{Z} \to \mathcal{Y}$$

of D-prestacks as in Sect. 5.3.1. For expository purposes, we will first consider the absolute situation, i.e., one when $\mathcal{Y} = \mathrm{pt}$.

5.5.1. Let $\mathcal{Z}$ be an affine D-scheme over X. For $\underline{x} \in \mathrm{Ran}$ we will denote by $\mathfrak{L}^+_\nabla(\mathcal{Z})_{\underline{x}}$ (resp., $\mathfrak{L}_\nabla(\mathcal{Z})_{\underline{x}}$) the scheme (resp., ind-scheme) $\mathrm{Sect}_\nabla(\mathcal{D}_{\underline{x}}, \mathcal{Z})$ (resp., $\mathrm{Sect}_\nabla(\mathcal{D}_{\underline{x}} - \underline{x}, \mathcal{Z})$).

Consider the corresponding categories

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z})_{\underline{x}}) \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\underline{x}}).$$

In addition, we can consider the ind-scheme $\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Z})$, and the categories

$$\mathrm{IndCoh}(\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Z})) \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Z})).$$

5.5.2. Letting $\underline{x} \in \mathrm{Ran}$ move in a family over Ran, we obtain the spaces $\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}}$ and $\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}$, where

$$\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}} \to \mathrm{Ran}$$

is a relative scheme, and

$$\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}} \to \mathrm{Ran}$$

is a relative ind-scheme. Consider also the relative ind-scheme $\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}}$.

We define the categories $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}}$, $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}}$ and $\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}))_{\mathrm{Ran}}$ by the recipe of Sect. 5.4.7.

5.5.3. Consider the map

$$s_{\mathcal{Z},\mathrm{Ran}} : \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}} \to \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}},$$

obtained by restricting horizontal sections along

$$\mathcal{D}_{\underline{x}} - \underline{x} \to X - \underline{x}.$$

When $\mathcal{Z}$ is unambiguous, we will simply write s_{Ran} instead of $s_{\mathcal{Z},\mathrm{Ran}}$.

We have an adjoint pair of functors

$$(s_{\mathcal{Z},\mathrm{Ran}})^* : \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \rightleftarrows \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}))_{\mathrm{Ran}} : (s_{\mathcal{Z},\mathrm{Ran}})_*.$$

5.5.4. An abstract version of Proposition 5.1.5 (in the absolute case) case reads:

Proposition 5.5.5. *The natural transformation*

$$\Gamma^{\mathrm{IndCoh}_{\mathrm{Ran}}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}}, -) \circ (s_{\mathcal{Z},\mathrm{Ran}})^* \circ (s_{\mathcal{Z},\mathrm{Ran}})_* \to \Gamma^{\mathrm{IndCoh}_{\mathrm{Ran}}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}}, -)$$

is an isomorphism, when evaluated on the image of $\omega_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}))_{\mathrm{Ran}}}$ *along*

$$\mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}}) \overset{\Psi_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}}}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}))_{\mathrm{Ran}}.$$

The proof will be given in Sect. B.

5.6. **Abstract version of Proposition 5.1.5: the relative case.** In this section we will introduce a relative version of the set-up of Sect. 5.5.

5.6.1. Let $\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}$ have the same meaning as in Sect. 5.3.1. We will view it as a relative ind-scheme over

$$\mathrm{Sect}_\nabla(X, \mathcal{Y}) \times \mathrm{Ran}.$$

Consider the corresponding category

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}}.$$

Let π_{Ran} denote the projection

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} \to \mathrm{Sect}_\nabla(X, \mathcal{Y}).$$

Combined with the projection

$$p_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}} : \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} \to \mathrm{Ran},$$

we obtain a map

$$\pi_{\mathrm{Ran}} \times p_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}} : \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} \to \mathrm{Sect}_\nabla(X, \mathcal{Y}) \times \mathrm{Ran}.$$

5.6.2. Let us denote by

$$(\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}_{\mathrm{Ran}}} : \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y}))$$

the composite functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}} \overset{(\pi_{\mathrm{Ran}} \times p_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}})_*}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X, \mathcal{Y})) \otimes \mathrm{IndCoh}(\mathrm{Ran}) \to$$
$$\overset{\Omega_{\mathrm{Sect}_\nabla(X,\mathcal{Y})} \otimes \mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})) \otimes \mathrm{IndCoh}(\mathrm{Ran}) \overset{\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-)}{\longrightarrow} \mathrm{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})).$$

Note that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}) & \xrightarrow{\Psi_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}}} & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}} \\ {\scriptstyle (\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}}} \downarrow & & \downarrow {\scriptstyle (\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}_{\mathrm{Ran}}}} \\ \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})) & \xrightarrow{\Psi_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}} & \mathrm{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})), \end{array} \tag{5.13}$$

where the top horizontal arrow is the functor of (5.11).

5.6.3. Let us be in the situation described in [GLC2, Sects. 4.5.1-4.5.2.], with the following change of notations: what was denoted by $\mathcal{Y}$ (resp., $\mathcal{Y}_0$) in *loc. cit.* we denote by $\mathcal{Z}$ (resp., $\mathcal{Y}$).

Let us be given an affine map

$$s_{\mathcal{T},\mathrm{Ran}} : \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} \to \mathcal{T}_{\mathrm{Ran}}$$

that fits into a commutative (but not necessarily Cartesian) diagram

$$\begin{array}{ccc} \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} & \xrightarrow{s_{\mathcal{T},\mathrm{Ran}}} & \mathcal{T}_{\mathrm{Ran}} \\ {\scriptstyle \pi_{\mathrm{Ran}} \times p_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}}} \downarrow & & \downarrow {\scriptstyle \mathfrak{L}(f)} \\ \mathrm{Sect}_\nabla(X, \mathcal{Y}) \times \mathrm{Ran} & \xrightarrow{s_{\mathcal{Y},\mathrm{Ran}}} & \mathfrak{L}^+(\mathcal{Y})_{\mathrm{Ran}}. \end{array} \tag{5.14}$$

5.6.4. We will now make the following additional structural assumption:

Consider the fiber product

$$\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y}),\mathrm{Ran}} := (\mathrm{Sect}_\nabla(X, \mathcal{Y}) \times \mathrm{Ran}) \underset{\mathfrak{L}^+(\mathcal{Y})_{\mathrm{Ran}}}{\times} \mathcal{T}_{\mathrm{Ran}}.$$

We require that there exist a $\mathrm{Sect}_\nabla(X, \mathcal{Y})$-family of affine D-schemes

$$\mathcal{Z}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})} \to \mathrm{Sect}_\nabla(X, \mathcal{Y}) \times X_{\mathrm{dR}},$$

such that:

- $\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y}),\mathrm{Ran}}$ identifies with the (relative over $\mathrm{Sect}_\nabla(X, \mathcal{Y})$) factorization space of horizontal loops $\mathfrak{L}_\nabla(\mathcal{Z}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}}/\,\mathrm{Sect}_\nabla(X, \mathcal{Y})$;
- The map
$$s'_{\mathcal{T},\mathrm{Ran}} : \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} \to \mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y}),\mathrm{Ran}},$$
arising from (5.14), identifies with the evaluation map
$$s_{\mathcal{Z}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}} : \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}/\,\mathrm{Sect}_\nabla(X, \mathcal{Y}))_{\mathrm{Ran}} \to \mathfrak{L}_\nabla(\mathcal{Z}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}}/\,\mathrm{Sect}_\nabla(X, \mathcal{Y}).$$

Finally, we require that the above data be compatible with the unital structures in the natural sense.

Remark 5.6.5. For our applications, we will take $\mathcal{Z} = \mathrm{Op}_{\check{G}}$ and $\mathcal{Y} = \mathrm{pt}/\check{G}$ and $\mathcal{T} := \mathrm{Op}_{\check{G}}^{\mathrm{mon\text{-}free}}$. In this case $\mathrm{Sect}_\nabla(X, \mathcal{Y}) = \mathrm{LS}_{\check{G}}$, and $\mathcal{Z}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}$ is the D-scheme parameterized by $\mathrm{LS}_{\check{G}}$ that classifies oper structures on a given local system.

5.6.6. We have an adjoint pair of functors

$$(s_{\mathcal{T},\mathrm{Ran}})^* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})_{\mathrm{Ran}} \rightleftarrows \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}} : (s_{\mathcal{T},\mathrm{Ran}})_*.$$

5.6.7. We claim:

Proposition 5.6.8. *The natural transformation*

$$(\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}_{\mathrm{Ran}}} \circ (s_{\mathcal{T},\mathrm{Ran}})^* \circ (s_{\mathcal{T},\mathrm{Ran}})_* \to (\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}_{\mathrm{Ran}}}$$

is an isomorphism, when evaluated on objects that lie in the essential image of the functor

$$\mathrm{IndCoh}(\mathrm{Sect}_\nabla(X,\mathcal{Y})) \overset{\pi^!_{\mathrm{Ran}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}) \to \overset{\Psi_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}}}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}}. \tag{5.15}$$

The proof will be given in Sect. C.

5.6.9. Let us show how Proposition 5.6.8 implies Proposition 5.1.5.

We apply Proposition 5.6.8 to the spaces specified in Remark 5.6.5, so that

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} = \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}.$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}) & \xrightarrow{\Psi_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}}} & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}}))_{\mathrm{Ran}} \\ \Big\downarrow{\scriptstyle (s_{\mathrm{Ran}})_*^{\mathrm{IndCoh}}} & & \Big\downarrow{\scriptstyle (s_{\mathrm{Ran}})_*} \\ \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} & \xrightarrow{\Psi_{(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}}} & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}. \end{array} \tag{5.16}$$

Moreover, the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}) & \xrightarrow{\Psi_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}})_{\mathrm{Ran}}}} & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X^{\mathrm{gen}}))_{\mathrm{Ran}} \\ \Big\uparrow{\scriptstyle (s_{\mathrm{Ran}})^{*,\mathrm{IndCoh}}} & & \Big\uparrow{\scriptstyle (s_{\mathrm{Ran}})^*} \\ \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} & \xrightarrow{\Psi_{(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}}} & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}. \end{array} \tag{5.17}$$

obtained from (5.16) by passing to left adjoints along the vertical arrows, commutes as well.

The conclusion of Proposition 5.1.5 follows now from Proposition 5.6.8, by juxtaposing the commutative diagrams (5.13), (5.16) and (5.17).

☐[Proposition 5.1.5]

Appendix A. Proof of Proposition 5.3.4

The idea of the proof of Proposition 5.3.4 can be summarized by the following slogan: the unital version of the space of *rational* horizontal sections maps isomorphically to its own de Rham prestack.

We will deduce it from the main theorem of [Ro] by a rather formal manipulation.

A.1. **The unital Ran space.** In order to prove Proposition 5.3.4 we will need to work with the *unital Ran space*, which is no longer a prestack (i.e., a functor from affine schemes to ∞-groupoids) but rather a *categorical prestack*, i.e., a functor from affine schemes to ∞-categories (see [GLC2, Sect. C.5] for a more detailed discussion).

A.1.1. Recall the notion of *categorical prestack*, see [Ro, Appendix C]. By definition, this is a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to 1\text{-}\mathrm{Cat},$$

where 1-Cat denotes the $(\infty,1)$-category of $(\infty,1)$-categories.

Thus, a categorical prestack $\mathcal{X}$ assigns to an affine scheme S a category, to be denoted $\mathcal{X}(S)$, and to a map $f : S_1 \to S_2$ a functor

$$\mathcal{X}(f) : \mathcal{X}(S_2) \to \mathcal{X}(S_1),$$

equipped with a datum of compatibility for compositions.

A.1.2. Let $\mathrm{Ran}^{\mathrm{untl}}$ be the unital version of the Ran space, see [Ga4, Sect. 4.2] or [Ro, Sect. 2.1]. I.e., $\mathrm{Ran}^{\mathrm{untl}}$ associates to an affine scheme S the *category* of finite subsets of $\mathrm{Hom}(S, X_{\mathrm{dR}})$, where the morphisms are given by inclusion.

Let

$$\mathsf{t} : \mathrm{Ran} \to \mathrm{Ran}^{\mathrm{untl}}$$

denote the tautological map.

A.1.3. Along with the prestacks

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}},\ \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}},\ (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}})_{\mathrm{dR}^{\mathrm{rel}}},\ \text{etc}$$

one can consider their unital versions, which are now *categorical prestacks*, denoted

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}^{\mathrm{untl}}},\ \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}},\ (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}, \tag{A.1}$$

respectively, see [Ro, Sect. 3.3.1].

Explicitly, for an affine scheme S, the category $\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}^{\mathrm{untl}}}(S)$ consists of pairs $(\underline{x}, z)$, where $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}(S)$ and z is a horizontal section of $\mathcal{Z}$ on $X \times S - \mathrm{Graph}_{\underline{x}}$.

A morphism $(\underline{x}_1, z_1) \to (\underline{x}_2, z_2)$ is an inclusion $\underline{x}_1 \subseteq \underline{x}_2$ and an identification

$$z_1|_{X\times S - \mathrm{Graph}_{\underline{x}_2}} \simeq z_2.$$

And similarly for the other two categorical prestacks in (A.1).

A.1.4. By definition, the projections from the categorical prestacks in (A.1) to $\mathrm{Ran}^{\mathrm{untl}}$ are *value-wise co-Cartesian fibrations in groupoids.*

Denote by $\pi_{\mathrm{Ran}^{\mathrm{untl}}}$ the projection from

$$(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}} \to \mathrm{Sect}_\nabla(X, \mathcal{Y}).$$

A.1.5. We will denote by t the maps from the non-unital to the unital versions. We have

$$\pi_{\mathrm{Ran}^{\mathrm{untl}}} \circ \mathsf{t} = \pi_{\mathrm{Ran}}.$$

A.2. IndCoh on categorical prestacks.

A.2.1. Let $\mathcal{X}$ be a categorical prestack *locally almost of finite type*, see [Ro, Sect. C.1.3] for what this means. In this case, it makes sense to talk about the category $\mathrm{IndCoh}(\mathcal{X})$ (see [Ga4, Sect. 2.2] or [Ro, Sect. C.3]).

Namely, an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ associates to an affine scheme S (assumed almost of finite type) a functor

$$\mathcal{X}(S) \to \mathrm{IndCoh}(S),$$

in a way compatible with !-pullback for morphisms between affine schemes.

We will denote this data as follows:

- For an object $x \in \mathcal{X}(S)$, we have an object

$$x^!(\mathcal{F}) \in \mathrm{IndCoh}(S);$$

- For a morphism $x_1 \overset{\alpha}{\to} x_2$ in $\mathcal{X}(S)$ a morphism
$$x_1^!(\mathcal{F}) \to x_2^!(\mathcal{F})$$
in $\text{IndCoh}(S)$.

A.2.2. We let
$$\text{IndCoh}(\mathcal{X})_{\text{str}} \subset \text{IndCoh}(\mathcal{X})$$
be the full subcategory, consisting of objects $\mathcal{F} \in \text{IndCoh}(\mathcal{X})$ such that for every affine test-scheme S and an arrow
$$x_1 \overset{\alpha}{\to} x_2, \quad x_1, x_2 \in \mathcal{X}(S),$$
the resulting map
$$x_1^!(\mathcal{F}) \to x_2^!(\mathcal{F})$$
is an isomorphism.

In other words, if we denote by
$$\mathcal{X} \overset{\text{str}}{\to} \mathcal{X}_{\text{str}}$$
the prestack, obtained from $\mathcal{X}$ by inverting all arrows, the pullback functor
$$\text{IndCoh}(\mathcal{X}_{\text{str}}) \overset{\text{str}^!}{\longrightarrow} \text{IndCoh}(\mathcal{X})$$
defines an equivalence
$$\text{IndCoh}(\mathcal{X}_{\text{str}}) \overset{\sim}{\to} \text{IndCoh}(\mathcal{X})_{\text{str}}.$$

A.2.3. We claim:

Lemma A.2.4. *The natural diagram of categories*
$$\begin{array}{ccc} \text{IndCoh}(\mathcal{X}_{\text{dR}})_{\text{str}} & \longrightarrow & \text{IndCoh}(\mathcal{X})_{\text{str}} \\ \downarrow & & \downarrow \\ \text{IndCoh}(\mathcal{X}_{\text{dR}}) & \longrightarrow & \text{IndCoh}(\mathcal{X}) \end{array}$$
is a pullback square.

Proof. Follows from the fact that for an affine scheme S almost of finite type, the !-pullback functor with respect to $S_{\text{red}} \to S$ is conservative.

□

A.3. A reformulation.

A.3.1. Note that the projection
$$(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}})_{\text{dR}^{\text{rel}}} \overset{\pi_{\text{Ran},\text{dR}^{\text{rel}}}}{\to} \text{Sect}_\nabla(X, \mathcal{Y})$$
factors as
$$(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}})_{\text{dR}^{\text{rel}}} \overset{\text{t}}{\to} (\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{dR}^{\text{rel}}} \overset{\text{str}}{\to}$$
$$\to \left(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}_{\text{str}}}\right)_{\text{dR}^{\text{rel}}} \overset{(\pi_{\text{Ran}^{\text{untl}},\text{dR}^{\text{rel}}})_{\text{str}}}{\to} \text{Sect}_\nabla(X, \mathcal{Y}),$$
where
$$\left(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}_{\text{str}}}\right)_{\text{dR}^{\text{rel}}} := \left((\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{dR}^{\text{rel}}}\right)_{\text{str}}.$$

Hence, the pullback functor
$$\pi^!_{\text{Ran}^{\text{untl}},\text{dR}^{\text{rel}}} : \text{IndCoh}(\text{Sect}_\nabla(X, \mathcal{Y})) \to \text{IndCoh}\left((\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{dR}^{\text{rel}}}\right)$$
maps to
$$\text{IndCoh}\left((\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{dR}^{\text{rel}}}\right)_{\text{str}} \subset \text{IndCoh}\left((\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{dR}^{\text{rel}}}\right).$$

A.3.2. We obtain that Proposition 5.3.4 follows from the next more precise statement:

Proposition A.3.3. *The counit of the adjunction*

$$\mathbf{ind}^{\mathrm{rel}} \circ \mathbf{oblv}^{\mathrm{rel}} \to \mathrm{Id}$$

is an isomorphism, when evaluated on objects in the essential image along $\mathsf{t}^!$ *of*

$$\mathrm{IndCoh}\left((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}\right)_{\mathrm{str}} \subset \mathrm{IndCoh}\left((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}\right).$$

A.3.4. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} & \xrightarrow{\mathsf{t}} & \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}} \\ \downarrow & & \downarrow \\ (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}})_{\mathrm{dR}^{\mathrm{rel}}} & \xrightarrow{\mathsf{t}} & (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}} \end{array} \tag{A.2}$$

This diagram is value-wise Cartesian. Hence, we have a well-defined pair of adjoint functors

$$\mathbf{ind}^{\mathrm{rel}}_{\mathrm{untl}} : \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}}) \rightleftarrows \mathrm{IndCoh}\left((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}\right) : \mathbf{oblv}^{\mathrm{rel}}_{\mathrm{untl}}, \tag{A.3}$$

and both functors are compatible with their non-unital counterparts (5.8) via $\mathsf{t}^!$.

A.3.5. We also have a commutative diagram

$$\begin{array}{ccc} \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}} & \xrightarrow{\mathrm{str}} & \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}_{\mathrm{str}}} \\ \downarrow & & \downarrow \\ (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}} & \xrightarrow{\mathrm{str}} & \left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}_{\mathrm{str}}}\right)_{\mathrm{dR}^{\mathrm{rel}}}, \end{array}$$

where

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}_{\mathrm{str}}} := (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{str}},$$

which is value-wise Cartesian. Hence, we have another pair of adjoint functors

$$\mathbf{ind}^{\mathrm{rel}}_{\mathrm{untl,str}} : \mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}_{\mathrm{str}}}\right) \rightleftarrows \rightleftarrows \mathrm{IndCoh}\left(\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}_{\mathrm{str}}}\right)_{\mathrm{dR}^{\mathrm{rel}}}\right) : \mathbf{oblv}^{\mathrm{rel}}_{\mathrm{untl,str}}, \tag{A.4}$$

where both functors are compatible with their non-strict counterparts (A.3) via $\mathrm{str}^!$.

We can equivalently think of (A.4) as an adjunction

$$\mathbf{ind}^{\mathrm{rel}}_{\mathrm{untl,str}} : (\mathrm{IndCoh}\,(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}}))_{\mathrm{str}} \rightleftarrows \rightleftarrows (\mathrm{IndCoh}\,((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}))_{\mathrm{str}} : \mathbf{oblv}^{\mathrm{rel}}_{\mathrm{untl,str}}. \tag{A.5}$$

A.3.6. The assertion of Proposition A.3.3 follows from the following even more precise statement:

Proposition A.3.7. *The functor*

$$\mathbf{oblv}^{\mathrm{rel}}_{\mathrm{untl,str}} : (\mathrm{IndCoh}\,((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}))_{\mathrm{str}} \to (\mathrm{IndCoh}\,(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}}))_{\mathrm{str}}$$

is an equivalence.

A.4. **A description of relative D-modules.** In order to prove Proposition A.3.7, we will describe the category

$$\mathrm{IndCoh}\,((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}})$$

à la [Ro, Corollary 4.6.10].

A.4.1. As a warm-up, let us fix a point $\underline{x}$, and consider the prestack

$$\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y}) := \mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}) \underset{\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y})}{\times} \mathrm{Sect}_\nabla(X,\mathcal{Y})$$

along with its variant

$$\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y})_{\mathrm{dR}^{\mathrm{rel}}} := \mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z})_{\mathrm{dR}} \underset{\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y})_{\mathrm{dR}}}{\times} \mathrm{Sect}_\nabla(X,\mathcal{Y}) \simeq$$
$$\simeq (\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y}))_{\mathrm{dR}} \underset{\mathrm{Sect}_\nabla(X,\mathcal{Y})_{\mathrm{dR}}}{\times} \mathrm{Sect}_\nabla(X,\mathcal{Y}).$$

We will describe the category

$$\mathrm{IndCoh}(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y})_{\mathrm{dR}^{\mathrm{rel}}})$$

along with its forgetful (i.e., pullback) functor to $\mathrm{IndCoh}(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y}))$.

A.4.2. Consider the map

$$\mathrm{add}_{\underline{x}} : \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}},$$

given by

$$\underline{y} \mapsto \underline{y} \cup \underline{x}.$$

Set

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}},\underline{x}} := \mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}} \underset{\mathrm{Ran}^{\mathrm{untl}},\mathrm{add}_{\underline{x}}}{\times} \mathrm{Ran}^{\mathrm{untl}}.$$

Restriction along $X-(\underline{y}\cup\underline{x}) \subset X-\underline{x}$ gives rise to a map

$$\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y}) \times \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}},\underline{x}}. \tag{A.6}$$

A.4.3. Denote by

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})^\wedge_{\mathrm{Ran}^{\mathrm{untl}},\underline{x}}$$

the formal completion of $\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}},\underline{x}}$ along (A.6).

The projection

$$\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y}) \times \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y})_{\mathrm{dR}^{\mathrm{rel}}}$$

tautologically extends to map

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})^\wedge_{\mathrm{Ran}^{\mathrm{untl}},\underline{x}} \to \mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y})_{\mathrm{dR}^{\mathrm{rel}}}. \tag{A.7}$$

A.4.4. The following is a version of [Ro, Corollary 4.6.10], where we allow poles at $\underline{x}$:

Theorem A.4.5. *The functor*

$$\mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y})_{\mathrm{dR}^{\mathrm{rel}}}\right) \to$$
$$\to \mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y})\right) \underset{\mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y})\times\mathrm{Ran}^{\mathrm{untl}}\right)}{\times} \mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})^\wedge_{\mathrm{Ran}^{\mathrm{untl}},\underline{x}}\right),$$

given by pullback along the maps $\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y}) \to \mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Z}/\mathcal{Y})_{\mathrm{dR}^{\mathrm{rel}}}$ *and* (A.7)*, is an equivalence.*

Remark A.4.6. In fact, this theorem is a particular case of [Ro, Corollary 4.6.10]: replace the original $\mathcal{Z}$ by its restriction of scalars along $X-\underline{x}\to X$.

A.4.7. We will now state a version of Theorem A.4.5, where we let $\underline{x}$ vary along $\mathrm{Ran}^{\mathrm{untl}}$. Consider the map

$$\mathrm{add}: \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}}, \quad \underline{x}_1, \underline{x}_2 \mapsto \underline{x}_1 \cup \underline{x}_2.$$

Set

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{add}} := \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}} \underset{\mathrm{Ran}^{\mathrm{untl}},\mathrm{add}}{\times} (\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}),$$

and

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{pr}_1} := \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}} \underset{\mathrm{Ran}^{\mathrm{untl}},\mathrm{pr}_1}{\times} (\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}),$$

where

$$\mathrm{pr}_1 : \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}}$$

is the projection on the first factor. In other words,

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{pr}_1} \simeq \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}} \times \mathrm{Ran}^{\mathrm{untl}}.$$

Restriction along $X - (\underline{x}_1 \cup \underline{x}_2) \subset X - \underline{x}_1$ gives rise to a map

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{pr}_1} \to \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{add}}. \tag{A.8}$$

A.4.8. Denote by

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})^\wedge_{\mathrm{add}}$$

the formal completion of $\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{add}}$ along (A.8).

The projection

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{pr}_1} \to (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}$$

tautologically extends to a map

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})^\wedge_{\mathrm{add}} \to (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}. \tag{A.9}$$

The following is a version of Theorem A.4.5 in families:

Theorem A.4.9. *The functor*

$$\begin{aligned} &\mathrm{IndCoh}\left((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}\right) \to \\ &\to \mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}}\right) \underset{\mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{pr}_1}\right)}{\times} \mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})^\wedge_{\mathrm{add}}\right), \end{aligned} \tag{A.10}$$

given by pullback along the maps $\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}} \to (\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}$ *and* (A.9), *is an equivalence.*

A.5. **Proof of Proposition A.3.7.**

A.5.1. The functor (A.10) induces a functor

$$\begin{aligned} &\mathrm{IndCoh}\left((\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{dR}^{\mathrm{rel}}}\right)_{\mathrm{str}} \to \\ &\to \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}})_{\mathrm{str}} \underset{\mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{pr}_1}\right)_{\mathrm{str}}}{\times} \mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})^\wedge_{\mathrm{add}}\right)_{\mathrm{str}}, \end{aligned} \tag{A.11}$$

where the two sides in (A.11) are full subcategories in the corresponding sides in (A.10).

Since the functor (A.10) is an equivalence, we obtain that (A.11) is fully faithful.

A.5.2. We will prove:

Lemma A.5.3. *The functor*

$$\mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})^\wedge_{\mathrm{add}}\right)_{\mathrm{str}} \to \mathrm{IndCoh}\left(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{pr}_1}\right)_{\mathrm{str}}$$

is an equivalence.

Let us assume this lemma for a moment and finish the proof of Proposition A.3.7.

A.5.4. By Lemma A.5.3, we obtain that the right-hand side in (A.11) projects isomorphically onto the first factor. Hence, we obtain that the pullback functor

$$\text{IndCoh}\left((\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{dR}^{\text{rel}}}\right)_{\text{str}} \to \text{IndCoh}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{str}}, \tag{A.12}$$

which is the functor $\mathbf{oblv}^{\text{rel}}_{\text{untl,str}}$ of Proposition A.3.7, is fully faithful.

It remains to show that the functor (A.12) is essentially surjective.

A.5.5. Let $\mathcal{F}$ be an object in $\text{IndCoh}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{str}}$, which, by Lemma A.5.3, we interpret as an object in the right-hand side of (A.11).

By Theorem A.4.9 it corresponds to an object $\mathcal{F}_{\text{dR}} \in \text{IndCoh}\left((\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{Ran}^{\text{untl}}})_{\text{dR}^{\text{rel}}}\right)$, and we only need to show that $\mathcal{F}_{\text{dR}}$ is strict. However, this follows from (a relative version of) Lemma A.2.4.
□[Proposition A.3.7]

A.6. **Proof of Lemma A.5.3.**

A.6.1. We will prove that the map

$$(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{pr}_1})_{\text{str}} \to \left(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})^\wedge_{\text{add}}\right)_{\text{str}}$$

is an isomorphism of prestacks.

A.6.2. We claim:

Lemma A.6.3. *Let $\mathcal{W} \to \text{Ran}^{\text{untl}} \times \text{Ran}^{\text{untl}}$ be a map of categorical prestacks, which is a value-wise co-Cartesian fibration. Then then the induced map*

$$\mathcal{W} \underset{\text{Ran}^{\text{untl}} \times \text{Ran}^{\text{untl}}, \Delta}{\times} \text{Ran}^{\text{untl}} \to \mathcal{W}$$

induces an isomorphism

$$\left(\mathcal{W} \underset{\text{Ran}^{\text{untl}} \times \text{Ran}^{\text{untl}}, \Delta}{\times} \text{Ran}^{\text{untl}}\right)_{\text{str}} \to \mathcal{W}_{\text{str}}.$$

Proof. This follows from the fact that the diagonal map $\Delta : \text{Ran}^{\text{untl}} \to \text{Ran}^{\text{untl}} \times \text{Ran}^{\text{untl}}$ is value-wise cofinal.
□

A.6.4. Applying Lemma A.6.3, it suffices to show that the map

$$\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{pr}_1} \underset{\text{Ran}^{\text{untl}} \times \text{Ran}^{\text{untl}}, \Delta}{\times} \text{Ran}^{\text{untl}} \to \text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})^\wedge_{\text{add}} \underset{\text{Ran}^{\text{untl}} \times \text{Ran}^{\text{untl}}, \Delta}{\times} \text{Ran}^{\text{untl}}$$

induces an isomorphism on strictifications.

We claim that the above map is actually an isomorphism as-is.

A.6.5. Since the operation of formal completion commutes with fiber products, it suffices to show that the map

$$\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{pr}_1} \underset{\text{Ran}^{\text{untl}} \times \text{Ran}^{\text{untl}}, \Delta}{\times} \text{Ran}^{\text{untl}} \to \text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}/\mathcal{Y})_{\text{add}} \underset{\text{Ran}^{\text{untl}} \times \text{Ran}^{\text{untl}}, \Delta}{\times} \text{Ran}^{\text{untl}}$$

is an isomorphism.

However, the latter is evident on the nose.
□[Lemma A.5.3]

Appendix B. Proof of Proposition 5.5.5

In this section we let $\mathcal{Z}$ be an arbitrary affine D-scheme over X.

We will show that the assertion of Proposition 5.5.5 essentially amounts to [BD1, Proposition 4.6.5], combined with some *unitality* considerations.

B.1. **A reformulation in terms of unital structure.**

B.1.1. Let us return to the setting of Sect. 5.4.7. Suppose that $\mathcal{W}_{\text{Ran}}$ extends to a categorical prestack $\mathcal{W}_{\text{Ran}^{\text{untl}}}$ over Ran^{untl}, so that

$$\mathcal{W}_{\text{Ran}} \simeq \mathcal{W}_{\text{Ran}^{\text{untl}}} \underset{\text{Ran}^{\text{untl}}}{\times} \text{Ran},$$

and

$$\mathcal{W}_{\text{Ran}^{\text{untl}}} \to \text{Ran}^{\text{untl}}$$

is a value-wise co-Cartesian fibration in groupoids.

Effectively, this means that for $S \in \text{Sch}^{\text{aff}}$ and a map $\alpha : \underline{x}_1 \to \underline{x}_2$ in $\text{Ran}^{\text{untl}}(S)$ we have a map of prestacks

$$\alpha_{\mathcal{W}} : \mathcal{W}_{\text{Ran}} \underset{\text{Ran},\underline{x}_1}{\times} S \to \mathcal{W}_{\text{Ran}} \underset{\text{Ran},\underline{x}_2}{\times} S.$$

We will refer to $\mathcal{W}_{\text{Ran}^{\text{untl}}}$ as the unital structure on $\mathcal{W}_{\text{Ran}}$.

Let t denote the tautological map

$$\mathcal{W}_{\text{Ran}} \to \mathcal{W}_{\text{Ran}^{\text{untl}}}.$$

B.1.2. To the data as above we can attach a category $\text{QCoh}_{\text{co}}(\mathcal{W})_{\text{Ran}^{\text{untl}}}$. Namely, the data of an object of $\text{QCoh}_{\text{co}}(\mathcal{W})_{\text{Ran}^{\text{untl}}}$ consists of an object $\mathcal{F} \in \text{QCoh}_{\text{co}}(\mathcal{W})_{\text{Ran}}$, and for every α as above of a map

$$\left((\alpha_{\mathcal{W}})_* \otimes \text{Id}_{\text{IndCoh}(S)}\right)(\mathcal{F}_{S,\underline{x}_1}) \to \mathcal{F}_{S,\underline{x}_2}$$

in

$$\text{QCoh}_{\text{co}}(\mathcal{W}_{\text{Ran}} \underset{\text{Ran},\underline{x}_2}{\times} S) \underset{\text{QCoh}(S)}{\otimes} \text{IndCoh}(S).$$

Let $\mathsf{t}^!$ denote the tautological forgetful functor

$$\text{QCoh}_{\text{co}}(\mathcal{W})_{\text{Ran}^{\text{untl}}} \to \text{QCoh}_{\text{co}}(\mathcal{W})_{\text{Ran}}.$$

B.1.3. Note that the space $\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z})_{\text{Ran}^{\text{untl}}}$ from Sect. A.1.3 provides a unital structure on $\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z})_{\text{Ran}}$.

We will deduce Proposition 5.5.5 from the following more precise statement:

Proposition B.1.4. *The natural transformation*

$$\Gamma^{\text{IndCoh}_{\text{Ran}}}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z})_{\text{Ran}}, -) \circ (s_{\mathcal{Z},\text{Ran}})^* \circ (s_{\mathcal{Z},\text{Ran}})_* \to \Gamma^{\text{IndCoh}_{\text{Ran}}}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z})_{\text{Ran}}, -)$$

is an isomorphism, when evaluated on the essential image of the functor

$$\mathsf{t}^! : \text{QCoh}_{\text{co}}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}))_{\text{Ran}^{\text{untl}}} \to \text{QCoh}_{\text{co}}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}))_{\text{Ran}}.$$

In the rest of this subsection we will show how Proposition B.1.4 implies Proposition 5.5.5.

B.1.5. Assume that $\mathcal{W}_{\text{Ran}^{\text{untl}}}$ is locally almost of finite type (in particular, $\mathcal{W}_{\text{Ran}}$ is locally almost of finite type, so that the category $\text{IndCoh}(\mathcal{W}_{\text{Ran}})$ is well-ddefined). We claim that we have a well-defined category $\text{IndCoh}(\mathcal{W}_{\text{Ran}^{\text{untl}}})$.

By definition, the data of an object of $\text{IndCoh}(\mathcal{W}_{\text{Ran}^{\text{untl}}})$ consists of an object $\mathcal{F} \in \text{QCoh}_{\text{co}}(\mathcal{W}_{\text{Ran}})$, i.e., for every $\underline{x} \in \text{Ran}(S)$ we have an object $\mathcal{F}_{S,\underline{x}} \in \text{IndCoh}(\mathcal{W}_{\text{Ran}} \underset{\text{Ran},\underline{x}}{\times} S)$, and for every α as above of a map

$$(\alpha_{\mathcal{W}})_*(\mathcal{F}_{S,\underline{x}_1}) \to \mathcal{F}_{S,\underline{x}_2}$$

in $\text{IndCoh}(S)$.

Let $\mathsf{t}^!$ denote the tautological forgetful functor

$$\text{IndCoh}(\mathcal{W}_{\text{Ran}^{\text{untl}}}) \to \text{IndCoh}(\mathcal{W}_{\text{Ran}}).$$

B.1.6. Assume now that $\mathcal{W}_{\mathrm{Ran}} \to \mathrm{Ran}$ is a relative ind-scheme. Then as in Sect. 5.4.10 we have a functor

$$\Psi_{\mathcal{W}_{\mathrm{Ran}^{\mathrm{untl}}}} : \mathrm{IndCoh}(\mathcal{W}_{\mathrm{Ran}^{\mathrm{untl}}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}^{\mathrm{untl}}}, \tag{B.1}$$

which makes the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{W}_{\mathrm{Ran}^{\mathrm{untl}}}) & \xrightarrow{\Psi_{\mathcal{W}_{\mathrm{Ran}^{\mathrm{untl}}}}} & \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}^{\mathrm{untl}}} \\ {\scriptstyle \mathsf{t}^!}\downarrow & & \downarrow{\scriptstyle \mathsf{t}^!} \\ \mathrm{IndCoh}(\mathcal{W}_{\mathrm{Ran}}) & \xrightarrow{\Psi_{\mathcal{W}_{\mathrm{Ran}}}} & \mathrm{QCoh}_{\mathrm{co}}(\mathcal{W})_{\mathrm{Ran}} \end{array}$$

commutes.

B.1.7. Thus, we obtain that in order to prove Proposition 5.5.5, it suffices to show that the object

$$\omega_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z})_{\mathrm{Ran}}} \in \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z})_{\mathrm{Ran}})$$

lies in the essential image of the functor

$$\mathsf{t}^! : \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z})_{\mathrm{Ran}^{\mathrm{untl}}}) \to \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z})_{\mathrm{Ran}}).$$

However, this is true for any $\mathcal{W}_{\mathrm{Ran}^{\mathrm{untl}}}$ for which the maps $\alpha_{\mathcal{W}}$ are proper, which is the case for $\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z})_{\mathrm{Ran}^{\mathrm{untl}}}$.

□[Proposition 5.5.5]

B.2. **The local unital structure.**

B.2.1. Recall the categories $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}}$ and $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}}$, see Sect. 5.4.7. We will now introduce their variants, to be denoted

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}} \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}},$$

respectively.

In order to do so, as in Sect. B.1.2, we must attach to a map $\alpha : \underline{x}_1 \to \underline{x}_2$ in $\mathrm{Ran}^{\mathrm{untl}}(S)$ functors

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Ran},\underline{x}_1}{\times} S) \to \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Ran},\underline{x}_2}{\times} S) \tag{B.2}$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Ran},\underline{x}_1}{\times} S) \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Ran},\underline{x}_2}{\times} S), \tag{B.3}$$

respectively.

B.2.2. Recall that for a point $\underline{x}$ of Ran we have

$$\mathfrak{L}^+_\nabla(\mathcal{Z})_{\underline{x}} \simeq \mathrm{Sect}_\nabla(\mathcal{D}_{\underline{x}},\mathcal{Z}) \text{ and } \mathfrak{L}_\nabla(\mathcal{Z})_{\underline{x}} \simeq \mathrm{Sect}_\nabla(\mathcal{D}_{\underline{x}} - \underline{x},\mathcal{Z}).$$

For $\underline{x}_1 \subseteq \underline{x}_2$, set

$$\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Z})_{\underline{x}_1\subseteq\underline{x}_2} := \mathrm{Sect}_\nabla(\mathcal{D}_{\underline{x_2}} - \underline{x}_1,\mathcal{Z}).$$

Restriction along

$$\mathcal{D}_{\underline{x_1}} \subseteq \mathcal{D}_{\underline{x_2}}$$

gives rise to a map

$$\mathfrak{L}^+_\nabla(\mathcal{Z})_{\underline{x}_2} \to \mathfrak{L}^+_\nabla(\mathcal{Z})_{\underline{x}_1}. \tag{B.4}$$

We define the functor

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z})_{\underline{x}_1}) \to \mathrm{QCoh}(\mathfrak{L}^+_\nabla((\mathcal{Z})_{\underline{x}_2}) \tag{B.5}$$

to be given by pullback along (B.4).

B.2.3. Restriction along

$$\mathcal{D}_{\underline{x_1}} - \underline{x}_1 \subseteq \mathcal{D}_{\underline{x_2}} - \underline{x}_1 \supseteq \mathcal{D}_{\underline{x_2}} - \underline{x}_2$$

defines maps

$$\mathfrak{L}_\nabla(\mathcal{Z})_{\underline{x}_1} \leftarrow \mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Z})_{\underline{x}_1 \subseteq \underline{x}_2} \to \mathfrak{L}_\nabla(\mathcal{Z})_{\underline{x}_2}. \tag{B.6}$$

The operation of *-pull and *-push along (B.6) gives rise to a functor

$$\text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\underline{x}_1}) \to \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\underline{x}_2}). \tag{B.7}$$

B.2.4. The operations in (B.5) and (B.7) make sense when $\underline{x}_1$ and $\underline{x}_2$ are S-points of Ran, and give rise to the sought-for functors (B.2) and (B.3), respectively.

We will denote by $\mathsf{t}^!$ the corresponding forgetful functors

$$\text{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\text{Ran}^{\text{untl}}} \to \text{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\text{Ran}} \text{ and } \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\text{Ran}^{\text{untl}}} \to \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\text{Ran}},$$

respectively.

B.2.5. We will deduce Proposition B.1.4 from the following even more precise assertion:

Proposition B.2.6. *The natural transformation*

$$\Gamma^{\text{IndCoh}_{\text{Ran}}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\text{Ran}}, -) \to \Gamma^{\text{IndCoh}_{\text{Ran}}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\text{Ran}}, -) \circ (s_{\mathcal{Z},\text{Ran}})_* \circ (s_{\mathcal{Z},\text{Ran}})^* \simeq$$
$$\simeq \Gamma^{\text{IndCoh}_{\text{Ran}}}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z})_{\text{Ran}}, -) \circ (s_{\mathcal{Z},\text{Ran}})^*,$$

arising from the unit of the $((s_{\mathcal{Z},\text{Ran}})^*, (s_{\mathcal{Z},\text{Ran}})_*)$-adjunction, is an isomorphism, when evaluated on objects lying in the essential image of the functor

$$\mathsf{t}^! : \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\text{Ran}^{\text{untl}}} \to \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\text{Ran}}.$$

In the rest of this subsection we will show how Proposition B.2.6 implies Proposition B.1.4.

B.2.7. Let $\mathcal{F}_{\text{glob}}$ be an object of $\text{QCoh}_{\text{co}}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}))_{\text{Ran}}$, and assume that it lies in the essential image of

$$\mathsf{t}^! : \text{QCoh}_{\text{co}}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}))_{\text{Ran}^{\text{untl}}} \to \text{QCoh}_{\text{co}}(\text{Sect}_\nabla(X^{\text{gen}}, \mathcal{Z}))_{\text{Ran}}.$$

We wish to show that the map

$$\Gamma^{\text{IndCoh}_{\text{Ran}}}\left(\mathfrak{L}_\nabla(\mathcal{Z})_{\text{Ran}}, (s_{\mathcal{Z},\text{Ran}})_* \circ (s_{\mathcal{Z},\text{Ran}})^* \circ (s_{\mathcal{Z},\text{Ran}})_*(\mathcal{F}_{\text{glob}})\right) \to$$
$$\to \Gamma^{\text{IndCoh}_{\text{Ran}}}\left(\mathfrak{L}_\nabla(\mathcal{Z})_{\text{Ran}}, (s_{\mathcal{Z},\text{Ran}})_*(\mathcal{F}_{\text{glob}})\right),$$

induced by the *counit* of the $((s_{\mathcal{Z},\text{Ran}})^*, (s_{\mathcal{Z},\text{Ran}})_*)$-adjunction, is an isomorphism.

It is sufficient to show that the map

$$\Gamma^{\text{IndCoh}_{\text{Ran}}}\left(\mathfrak{L}_\nabla(\mathcal{Z})_{\text{Ran}}, (s_{\mathcal{Z},\text{Ran}})_*(\mathcal{F}_{\text{glob}})\right) \to$$
$$\to \Gamma^{\text{IndCoh}_{\text{Ran}}}\left(\mathfrak{L}_\nabla(\mathcal{Z})_{\text{Ran}}, (s_{\mathcal{Z},\text{Ran}})_* \circ (s_{\mathcal{Z},\text{Ran}})^* \circ (s_{\mathcal{Z},\text{Ran}})_*(\mathcal{F}_{\text{glob}})\right),$$

induced by the *unit* of the adjunction, is an isomorphism.

B.2.8. Denote

$$\mathcal{F}_{\text{loc}} := (s_{\mathcal{Z},\text{Ran}})_*(\mathcal{F}_{\text{glob}}) \in \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\text{Ran}}.$$

Thus, we have to show that the map

$$\Gamma^{\text{IndCoh}_{\text{Ran}}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\text{Ran}}, \mathcal{F}_{\text{loc}}) \to \Gamma^{\text{IndCoh}_{\text{Ran}}}(\mathfrak{L}_\nabla(\mathcal{Z})_{\text{Ran}}, (s_{\mathcal{Z},\text{Ran}})_* \circ (s_{\mathcal{Z},\text{Ran}})^*(\mathcal{F}_{\text{loc}})), \tag{B.8}$$

induced by the unit of the adjunction, is an isomorphism.

B.2.9. Note that the functor

$$(s_{\mathcal{Z},\mathrm{Ran}})_* : \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}))_{\mathrm{Ran}} \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}}$$

gives rise to a functor

$$(s_{\mathcal{Z},\mathrm{Ran}^{\mathrm{untl}}})_* : \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}},$$

so that the diagram

$$\begin{array}{ccc} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}} & \xrightarrow{(s_{\mathcal{Z},\mathrm{Ran}^u})_*} & \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}} \\ {\scriptstyle \mathrm{t}^!}\downarrow & & \downarrow{\scriptstyle \mathrm{t}^!} \\ \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}))_{\mathrm{Ran}} & \xrightarrow{(s_{\mathcal{Z},\mathrm{Ran}})_*} & \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \end{array}$$

commutes.

B.2.10. Hence, the assumption on $\mathcal{F}_{\mathrm{glob}}$ implies that $\mathcal{F}_{\mathrm{loc}}$ lies in the essential image of

$$\mathrm{t}^! : \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}}.$$

Hence, the isomorphism (B.8) follows from Proposition B.2.6.

□[Proposition B.1.4]

B.3. **An expression for the global sections functor.** In this subsection we will recall the expression for the functor

$$\Gamma^{\mathrm{IndCoh}_{\mathrm{Ran}}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}}, -) \circ (s_{\mathcal{Z},\mathrm{Ran}})^*$$

in terms of *factorization homology* à la [BD1, Sect. 4.6].

B.3.1. To any categorical prestack $\mathcal{W}$ we can attach the prestack $\mathcal{W}^{\to}$ that classifies arrows in $\mathcal{W}$. I.e., for a test affine scheme S, the groupoid $\mathcal{W}^{\to}(S)$ classifies triples

$$(w_1 \in \mathcal{W}(S), w_2 \in \mathcal{W}(S), \alpha : w_1 \to w_2).$$

B.3.2. Denote

$$\mathrm{Ran}^{\subseteq} := (\mathrm{Ran}^{\mathrm{untl}})^{\to}.$$

Denote by

$$\mathrm{pr}_{\mathrm{small}}, \mathrm{pr}_{\mathrm{big}} : \mathrm{Ran}^{\subseteq} \to \mathrm{Ran}$$

the maps that correspond to the source and the target of the arrow, respectively.

Explicitly, the groupoid $\mathrm{Ran}^{\subseteq}(S)$ consists of

$$\{\underline{x}_1, \underline{x}_2 \in \mathrm{Ran}(S) \,|\, \underline{x}_1 \subseteq \underline{x}_2\},$$

and the maps $\mathrm{pr}_{\mathrm{small}}$ and $\mathrm{pr}_{\mathrm{big}}$ send a point as above to $\underline{x}_1$ and $\underline{x}_2$, respectively.

B.3.3. Denote

$$\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{small}} := \mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}}}{\times} \mathrm{Ran}^{\subseteq},\ \mathcal{Z}_{\mathrm{Ran}^{\subseteq},\mathrm{big}} := \mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{big}}}{\times} \mathrm{Ran}^{\subseteq}$$

and

$$\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{small}} := \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}}}{\times} \mathrm{Ran}^{\subseteq},\ \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}} := \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{big}}}{\times} \mathrm{Ran}^{\subseteq}.$$

Proceeding as in Sect. 5.4.7, one can define the corresponding categories

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{small}},\ \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{big}},$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{small}} \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{big}}$$

respectively.

B.3.4. Denote by

$$p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}}} : \mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}} \to \mathrm{Ran}$$

and

$$p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{small}}} : \mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{small}} \to \mathrm{Ran}^\subseteq \text{ and } p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}}} : \mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}} \to \mathrm{Ran}^\subseteq,$$

as well as

$$p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}} : \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}} \to \mathrm{Ran}$$

and

$$p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{small}}} : \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{small}} \to \mathrm{Ran}^\subseteq \text{ and } p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}}} : \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}} \to \mathrm{Ran}^\subseteq,$$

the resulting maps.

We will consider the corresponding functors

$$(p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_* : \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \to \mathrm{IndCoh}(\mathrm{Ran})$$

and

$$(p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}}})_* : \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{big}} \to \mathrm{IndCoh}(\mathrm{Ran}^\subseteq)$$

as well

$$(p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_* : \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \to \mathrm{IndCoh}(\mathrm{Ran})$$

and

$$(p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}}})_* : \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{big}} \to \mathrm{IndCoh}(\mathrm{Ran}^\subseteq).$$

B.3.5. Note now that the unital structures on the categories $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}}$ and $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}}$, determined by $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}}$, and $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}}$, respectively, give rise to functors

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{small}} \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{big}} \tag{B.9}$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{small}} \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{big}}. \tag{B.10}$$

Denote the compositions

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \overset{\mathrm{pr}^!_{\mathrm{small}}}{\longrightarrow} \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{small}} \overset{(B.9)}{\longrightarrow} \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{big}}$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \overset{\mathrm{pr}^!_{\mathrm{small}}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{small}} \overset{(B.10)}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^\subseteq,\mathrm{big}}$$

in both instances by ins. unit; we will refer to this functor as the "insertion of the unit".

B.3.6. Let

$$\mathrm{diag} : \mathrm{Ran} \to \mathrm{Ran}^\subseteq$$

denote the diagonal map,

$$\underline{x} \mapsto (\underline{x} \subseteq \underline{x}).$$

In terms of Sect. B.3.1, it corresponds to the identity morphisms on objects of $\mathcal{W}_{\mathrm{Ran}^{\mathrm{untl}}}(S)$.

Note that we have pullback squares

$$\begin{array}{ccc} \mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}} & \xrightarrow{\mathrm{diag}} & \mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}} \\ {\scriptstyle p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}}}}\downarrow & & \downarrow{\scriptstyle p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}}}} \\ \mathrm{Ran} & \xrightarrow{\mathrm{diag}} & \mathrm{Ran}^\subseteq. \end{array} \tag{B.11}$$

and

$$\begin{array}{ccc} \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}} & \xrightarrow{\mathrm{diag}} & \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}} \\ {\scriptstyle p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}}}\downarrow & & \downarrow{\scriptstyle p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^\subseteq,\mathrm{big}}}} \\ \mathrm{Ran} & \xrightarrow{\mathrm{diag}} & \mathrm{Ran}^\subseteq. \end{array} \tag{B.12}$$

Note also that we have a canonical identification

$$\mathrm{diag}^! \circ \mathrm{ins.\,unit} \simeq \mathrm{Id},$$

in both instances.

B.3.7. Recall that $s_{\mathcal{Z},\mathrm{Ran}}$ (or simply $s_{\mathcal{Z},\mathrm{Ran}}$) denotes the map

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z})_{\mathrm{Ran}} \to \mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}.$$

We will use the same symbol $s_{\mathcal{Z},\mathrm{Ran}}$ (or simply $s_{\mathcal{Z},\mathrm{Ran}}$) to denote the map

$$\mathrm{Sect}_\nabla(X, \mathcal{Z}) \times \mathrm{Ran} \to \mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}}.$$

The following assertion is a variant with parameters of [BD2, Proposition 4.6.5]:

Proposition B.3.8.
(a) *The functor*

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \overset{(s_{\mathcal{Z},\mathrm{Ran}})^*}{\longrightarrow} \mathrm{QCoh}(\mathrm{Sect}_\nabla(X,\mathcal{Z})) \otimes \mathrm{IndCoh}(\mathrm{Ran}) \overset{\Gamma(\mathrm{Sect}_\nabla(X,\mathcal{Z}),-)\otimes \mathrm{Id}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Ran})$$

identifies canonically with

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{big}} \overset{(p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Ran}^{\subseteq}) \to$$
$$\overset{(\mathrm{pr}_{\mathrm{small}})^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Ran}).$$

Under the above identification, the map

$$(p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_* \to (p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_* \circ (s_{\mathcal{Z},\mathrm{Ran}})_* \circ (s_{\mathcal{Z},\mathrm{Ran}})^*,$$

given by the unit of the $((s_{\mathcal{Z},\mathrm{Ran}})^*, (s_{\mathcal{Z},\mathrm{Ran}})_*)$*-adjunction, corresponds to the map*

$$(p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_* \simeq (\mathrm{pr}_{\mathrm{small}})_* \circ (p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_* \circ \mathrm{diag}_* \simeq$$
$$\simeq (\mathrm{pr}_{\mathrm{small}})_* \circ (p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_* \circ \mathrm{diag}_* \circ \mathrm{diag}^! \circ \mathrm{ins.\,unit} \to (\mathrm{pr}_{\mathrm{small}})_* \circ (p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_* \circ \mathrm{ins.\,unit}\,.$$

(b) *The functor*

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \overset{(s_{\mathcal{Z},\mathrm{Ran}})^*}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}))_{\mathrm{Ran}} \overset{(p_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}))_{\mathrm{Ran}}})_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Ran})$$

identifies canonically with

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{big}} \overset{(p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Ran}^{\subseteq})$$
$$\to \overset{(\mathrm{pr}_{\mathrm{small}})_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Ran}).$$

Under the above identification, the map

$$(p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_* \to (p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_* \circ (s_{\mathcal{Z},\mathrm{Ran}})_* \circ (s_{\mathcal{Z},\mathrm{Ran}})^*,$$

given by the unit of the $((s_{\mathcal{Z},\mathrm{Ran}})^*, (s_{\mathcal{Z},\mathrm{Ran}})_*)$*-adjunction, corresponds to the map*

$$(p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_* \simeq (\mathrm{pr}_{\mathrm{small}})_* \circ (p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_* \circ \mathrm{diag}_* \simeq$$
$$\simeq (\mathrm{pr}_{\mathrm{small}})_* \circ (p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_* \circ \mathrm{diag}_* \circ \mathrm{diag}^! \circ \mathrm{ins.\,unit} \to (\mathrm{pr}_{\mathrm{small}})_* \circ (p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_* \circ \mathrm{ins.\,unit}\,.$$

B.4. **Inputing the unitality structure.** In this subsection we will prove Proposition B.2.6 by combining Proposition B.3.8(b) with a cofinality argument.

B.4.1. Note that in the situation of Sect. B.3.1, the prestack $\mathcal{W}^{\to}$ itself can also be extended to a categorical prestack.

Applying this to $\mathrm{Ran}^{\mathrm{untl}}$, we obtain a categorical prestack, denoted $\mathrm{Ran}^{\subseteq,\mathrm{untl}}$. Explicitly, the space of morphisms

$$(\underline{x}_1 \subseteq \underline{x}_2) \to (\underline{x}_1' \subseteq \underline{x}_2')$$

is

$$\begin{cases} \{*\} \text{ if } \underline{x}_1 \subseteq \underline{x}_1' \text{ and } \underline{x}_2 \subseteq \underline{x}_2', \\ \emptyset, \text{ otherwise.} \end{cases}$$

Denote by

$$\mathsf{t}^{\subseteq} : \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\subseteq,\mathrm{untl}}$$

the corresponding map.

B.4.2. The following assertion is obtained by unwinding the constructions:

Lemma B.4.3. *The composite functor*

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}} \overset{\mathrm{t}^!}{\to} \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow}$$
$$\to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{big}} \overset{(p_{\mathfrak{L}^+_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Ran}^{\subseteq})$$

factors via a functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{IndCoh}(\mathrm{Ran}^{\subseteq,\mathrm{untl}}),$$

followed by $(\mathsf{t}^{\subseteq})^!$.

B.4.4. Note that from (B.12), we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{big}} & \xrightarrow{\mathrm{diag}^!} & \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} \\ {\scriptstyle (p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_*}\downarrow & & \downarrow{\scriptstyle (p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_*} \\ \mathrm{IndCoh}(\mathrm{Ran}^{\subseteq}) & \xrightarrow{\mathrm{diag}^!} & \mathrm{IndCoh}(\mathrm{Ran}) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}} & \xrightarrow{\mathrm{diag}_*} & \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Z}))_{\mathrm{Ran}^{\subseteq},\mathrm{big}} \\ {\scriptstyle (p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}}})_*}\downarrow & & \downarrow{\scriptstyle (p_{\mathfrak{L}_\nabla(\mathcal{Z})_{\mathrm{Ran}^{\subseteq},\mathrm{big}}})_*} \\ \mathrm{IndCoh}(\mathrm{Ran}) & \xrightarrow{\mathrm{diag}_*} & \mathrm{IndCoh}(\mathrm{Ran}^{\subseteq}). \end{array}$$

Hence, combining Proposition B.3.8(b) and Lemma B.4.3, we obtain that in order deduce Proposition B.2.6, it suffices to prove the following assertion:

Proposition B.4.5. *The natural transformation*

$$\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-) \circ \mathrm{diag}^! \simeq \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-) \circ (\mathrm{pr}_{\mathrm{small}})_* \circ \mathrm{diag}_* \circ \mathrm{diag}^! \to$$
$$\to \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-) \circ (\mathrm{pr}_{\mathrm{small}})_* \simeq \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\subseteq},-)$$

of functors $\mathrm{IndCoh}(\mathrm{Ran}^{\subseteq}) \rightrightarrows \mathrm{Vect}$, *is an isomorphism, when evaluated on the essential image of the functor*

$$(\mathsf{t}^{\subseteq})^! : \mathrm{IndCoh}(\mathrm{Ran}^{\subseteq,\mathrm{untl}}) \to \mathrm{IndCoh}(\mathrm{Ran}^{\subseteq}).$$

□[Proposition B.2.6]

B.4.6. *Proof of Proposition B.4.5.* We need to show that the natural transformation

$$\mathrm{C}^\cdot_c(\mathrm{Ran}, -) \circ \mathrm{diag}^! \circ (\mathsf{t}^{\subseteq})^! \to \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq}, -) \circ (\mathsf{t}^{\subseteq})^!, \tag{B.13}$$

as functors

$$\mathrm{IndCoh}(\mathrm{Ran}^{\subseteq,\mathrm{untl}}) \rightrightarrows \mathrm{Vect},$$

is an isomorphism.

First, as in [Ga4, Theorem 4.6.2], one shows that the map $\mathsf{t}^{\subseteq}$ is *universally homologically cofinal*. Hence, the natural transformation

$$\mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq}, -) \circ (\mathsf{t}^{\subseteq})^! \to \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq,\mathrm{untl}}, -),$$

as functors

$$\mathrm{IndCoh}(\mathrm{Ran}^{\subseteq,\mathrm{untl}}) \rightrightarrows \mathrm{Vect},$$

is an isomorphism.

Consider now the composition

$$\mathsf{t}^{\subseteq} \circ \mathrm{diag} : \mathrm{Ran} \to \mathrm{Ran}^{\subseteq,\mathrm{untl}}.$$

It is easily seen to be value-wise cofinal. Hence, the natural transformation

$$\mathrm{C}^\cdot_c(\mathrm{Ran}, -) \circ \mathrm{diag}^! \circ (\mathsf{t}^{\subseteq})^! \to \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq,\mathrm{untl}}, -),$$

as functors

$$\mathrm{IndCoh}(\mathrm{Ran}^{\subseteq,\mathrm{untl}}) \rightrightarrows \mathrm{Vect},$$

is an isomorphism.

Combining, we obtain that (B.13) is also an isomorphism, as desired.

□[Proposition B.4.5]

Appendix C. Proof of Proposition 5.6.8

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We will show that Proposition 5.6.8 amounts to a parameterized version of Proposition 5.5.5, combined with a fully-faithfulness assertion regarding the *localization functor* $\mathrm{Loc}_{\mathcal{Y}}$.

C.1. **Localization functor in the abstract setting.**

C.1.1. Let $\mathcal{Y}_{\mathrm{Ran}}$ satisfy the following conditions:

- The diagonal map $\mathcal{Y}_{\mathrm{Ran}} \to \mathcal{Y}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathcal{Y}_{\mathrm{Ran}}$ is affine. Note that this formally implies that the diagonal map of $\mathrm{Sect}(X, \mathcal{Y})$ is affine;
- For every $S \to \mathrm{Ran}$, the prestack $\mathcal{Y}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S$ is *passable* (see [GaRo2, Chapter 3, Sect. 3.5.1] for what this means);
- The prestack $\mathrm{Sect}(X, \mathcal{Y})$ is passable.

C.1.2. Note that the above conditions imply that the morphism

$$s_{\mathcal{Y},\mathrm{Ran}} : \mathrm{Sect}(X, \mathcal{Y}) \times \mathrm{Ran} \to \mathcal{Y}_{\mathrm{Ran}}$$

behaves nicely with respect to push-forwards:

For any prestack $\mathcal{W}$ mapping to $\mathcal{Y}_{\mathrm{Ran}}$, and the base-changed map

$$s'_{\mathcal{Y},\mathrm{Ran}} : (\mathrm{Sect}(X, \mathcal{Y}) \times \mathrm{Ran}) \underset{\mathcal{Y}_{\mathrm{Ran}}}{\times} \mathcal{W} \to \mathcal{W},$$

the functor

$$(s'_{\mathcal{Y},\mathrm{Ran}})_* : \mathrm{QCoh}((\mathrm{Sect}(X, \mathcal{Y}) \times \mathrm{Ran}) \underset{\mathcal{Y}_{\mathrm{Ran}}}{\times} \mathcal{W}) \to \mathrm{QCoh}(\mathcal{W}),$$

right adjoint to $(s'_{\mathcal{Y},\mathrm{Ran}})^*$, commutes with colimits, and satisfies the base change formula. This follows from [GaRo2, Chapter 3, Proposition 3.5.3].

C.1.3. Consider the resulting pair of adjoint functors

$$(s_{\mathcal{Y},\mathrm{Ran}})^* : \mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}} \rightleftarrows \mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y})) \otimes \mathrm{IndCoh}(\mathrm{Ran}) : (s_{\mathcal{Y},\mathrm{Ran}})_*.$$

Denote

$$\mathrm{Loc}_{\mathcal{Y}} := (\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-)) \circ (s_{\mathcal{Y},\mathrm{Ran}})^*, \quad \mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y})).$$

The right adjoint $\mathrm{Loc}^R_{\mathcal{Y}}$ of $\mathrm{Loc}_{\mathcal{Y}}$ is thus given by

$$(s_{\mathcal{Y},\mathrm{Ran}})_* \circ (\mathrm{Id} \otimes (\omega_{\mathrm{Ran}} \otimes -)), \quad \mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y})) \to \mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}}.$$

C.1.4. We will prove:

Proposition C.1.5. *Let $\mathcal{Y}$ be a D-prestack with an affine diagonal, satisfying the finiteness assumptions of Sect. C.1.1 above. Then the natural transformation*

$$\mathrm{Loc}_{\mathcal{Y}} \circ (s_{\mathcal{Y},\mathrm{Ran}})_* \simeq (\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-)) \circ (s_{\mathcal{Y},\mathrm{Ran}})^* \circ (s_{\mathcal{Y},\mathrm{Ran}})_* \to (\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-)),$$

arising from the counit of the $((s_{\mathcal{Y},\mathrm{Ran}})^*, (s_{\mathcal{Y},\mathrm{Ran}})_*)$-adjunction, is an isomorphism, when evaluated on the essential image of the functor

$$\mathsf{t}^! \otimes \mathrm{Id} : \mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y})) \otimes \mathrm{IndCoh}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y})) \otimes \mathrm{IndCoh}(\mathrm{Ran}).$$

This proposition will be proved in Sect. C.3.

C.1.6. Combined with the contractibility of the Ran space, i.e., the fact that the map

$$\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}, \omega_{\mathrm{Ran}}) \to k,$$

given by the counit of the $(\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-), \omega_{\mathrm{Ran}} \otimes -)$-adjunction, is an isomorphism, from Proposition C.1.5 we obtain:

Proposition C.1.7. *Let $\mathcal{Y}$ be a D-prestack with an affine diagonal, satisfying the finiteness assumptions of Sect. C.1.1 above. Then the counit of the adjunction*

$$\mathrm{Loc}_{\mathcal{Y}} \circ \mathrm{Loc}^R_{\mathcal{Y}} \to \mathrm{Id},$$

is an isomorphism.

Note that Proposition C.1.7 can be restated as:

Corollary C.1.8. *Under the above assumptions on $\mathcal{Y}$, the functor*

$$\mathrm{Loc}^R_{\mathcal{Y}} : \mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y})) \to \mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}}$$

is fully faithful.

C.1.9. Note that for $\mathcal{Y}$ being the constant D-stack with fiber $\mathrm{pt}/\check{G}$, we have

$$\mathrm{Sect}(X,\mathcal{Y}) \simeq \mathrm{LS}_{\check{G}}.$$

Furthermore, we have a tautological identification

$$\mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}} \simeq \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

Under this identification, we have

$$\mathrm{Loc}_{\mathcal{Y}} \simeq \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$$

and

$$\mathrm{Loc}^R_{\mathcal{Y}} \simeq \Gamma^{\mathrm{spec}}_{\check{G}}.$$

Hence, Corollary C.1.8 contains Proposition 1.1.4 as a particular case.

Remark C.1.10. As far as the actual proof of Proposition C.1.5 is concerned, we will first establish Proposition C.1.7, and then deduce the general case stated in Proposition C.1.5.

C.2. **Proof of Proposition 5.6.8.** In this subsection we will assume Proposition C.1.5 and deduce Proposition 5.6.8.

C.2.1. Along with the category $\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}}$, we will consider its unital version

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}^{\mathrm{untl}}},$$

equipped with a forgetful functor

$$\mathsf{t}^! : \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}}. \tag{C.1}$$

We will show that the natural transformation

$$(\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}_{\mathrm{Ran}}} \circ (s_{\mathcal{T},\mathrm{Ran}})^* \circ (s_{\mathcal{T},\mathrm{Ran}})_* \to (\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}_{\mathrm{Ran}}}, \tag{C.2}$$

induced by the counit of the $((s_{\mathcal{T},\mathrm{Ran}})^*, (s_{\mathcal{T},\mathrm{Ran}})_*)$-adjunction, is an isomorphism, when evaluated on objects lying in the essential image of the forgetful functor (C.1).

This will imply the assertion of Proposition 5.6.8, since the functor (5.15) factors as

$$\mathrm{IndCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})) \overset{\pi^!_{\mathrm{Ran}^{\mathrm{untl}}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}}) \to$$
$$\overset{\Psi_{\mathrm{Sect}_\nabla(X^{\mathrm{gen}},\mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl}}}}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}^{\mathrm{untl}}} \overset{\mathsf{t}^!}{\to} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}}.$$

C.2.2. Recall the space

$$\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y}),\mathrm{Ran}} := (\mathrm{Sect}_\nabla(X, \mathcal{Y}) \times \mathrm{Ran}) \underset{\mathfrak{L}^+(\mathcal{Y})_{\mathrm{Ran}}}{\times} \mathcal{T}_{\mathrm{Ran}}.$$

Consider the corresponding category

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}}.$$

Let us denote by π'_{Ran} the projection

$$\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y}),\mathrm{Ran}} \to \mathrm{Sect}(X, \mathcal{Y}).$$

A procedure similar to that defining the functor $(\pi_{\mathrm{Ran}})_*^{\mathrm{IndCoh}_{\mathrm{Ran}}}$ gives rise to a functor

$$(\pi'_{\mathrm{Ran}})_*^{\mathrm{IndCoh}_{\mathrm{Ran}}} : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{Sect}(X, \mathcal{Y})).$$

C.2.3. Note that the morphism

$$s_{\mathcal{T},\mathrm{Ran}} : \mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} \to \mathcal{T}_{\mathrm{Ran}}$$

factors as

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y})_{\mathrm{Ran}} \overset{s'_{\mathcal{T},\mathrm{Ran}}}{\longrightarrow} \mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y}),\mathrm{Ran}} \overset{s'_{\mathcal{Y},\mathrm{Ran}}}{\longrightarrow} \mathcal{T}_{\mathrm{Ran}},$$

where $s'_{\mathcal{Y},\mathrm{Ran}}$ is a base change of the map

$$s_{\mathcal{Y},\mathrm{Ran}} : \mathrm{Sect}(X, \mathcal{Y}) \times \mathrm{Ran} \to \mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}},$$

which appears in Proposition C.1.5.

Thus, we can factor $(s_{\mathcal{T},\mathrm{Ran}})_*$ as

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}} \overset{(s'_{\mathcal{T},\mathrm{Ran}})_*}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}} \overset{(s'_{\mathcal{Y},\mathrm{Ran}})_*}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Ran}})$$

and $(s_{\mathcal{T},\mathrm{Ran}})^*$ as

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Ran}}) \overset{(s'_{\mathcal{Y},\mathrm{Ran}})^*}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}} \overset{(s'_{\mathcal{T},\mathrm{Ran}})^*}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Z}/\mathcal{Y}))_{\mathrm{Ran}}.$$

C.2.4. Consider the natural transformation

$$(\pi'_{\mathrm{Ran}})^{\mathrm{IndCoh}_{\mathrm{Ran}}}_* \circ (s'_{\mathcal{Y},\mathrm{Ran}})^* \circ (s'_{\mathcal{Y},\mathrm{Ran}})_* \circ (s'_{\mathcal{T},\mathrm{Ran}})_* \to \\ \to (\pi_{\mathrm{Ran}})^{\mathrm{IndCoh}_{\mathrm{Ran}}}_* \circ (s'_{\mathcal{T},\mathrm{Ran}})^* \circ (s'_{\mathcal{Y},\mathrm{Ran}})^* \circ (s'_{\mathcal{Y},\mathrm{Ran}})_* \circ (s'_{\mathcal{T},\mathrm{Ran}})_*, \tag{C.3}$$

arising from the *unit* of the $((s'_{\mathcal{T},\mathrm{Ran}})^*, (s'_{\mathcal{T},\mathrm{Ran}})_*)$-adjunction.

Its composition with (C.2) is the natural transformation

$$(\pi'_{\mathrm{Ran}})_* \circ (s'_{\mathcal{Y},\mathrm{Ran}})^* \circ (s'_{\mathcal{Y},\mathrm{Ran}})_* \circ (s'_{\mathcal{T},\mathrm{Ran}})_* \to (\pi'_{\mathrm{Ran}})_* \circ (s'_{\mathcal{T},\mathrm{Ran}})_* \simeq (\pi_{\mathrm{Ran}})_*, \tag{C.4}$$

arising from the *counit* of the $((s'_{\mathcal{Y},\mathrm{Ran}})^*, (s'_{\mathcal{Y},\mathrm{Ran}})_*)$-adjunction.

We will show that both (C.3) and (C.4) are isomorphisms when evaluated on objects lying in the essential image of the functor (C.1). This will imply that (C.2) is also an isomorphism on such objects.

C.2.5. *Verification that* (C.3) *is an isomorphism.* Along with $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}}$ we can consider its unital version

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}^{\mathrm{untl}}}.$$

Note that the functors

$$(s'_{\mathcal{Y},\mathrm{Ran}})^*,\ (s'_{\mathcal{Y},\mathrm{Ran}})_*,\ (s'_{\mathcal{T},\mathrm{Ran}})^*,\ (s'_{\mathcal{T},\mathrm{Ran}})_*$$

upgrade to functors

$$(s'_{\mathcal{Y},\mathrm{Ran}^{\mathrm{untl}}})^*,\ (s'_{\mathcal{Y},\mathrm{Ran}^{\mathrm{untl}}})_*,\ (s_{\mathcal{T},\mathrm{Ran}^{\mathrm{untl}}})^*,\ (s_{\mathcal{T},\mathrm{Ran}^{\mathrm{untl}}})_*$$

between the corresponding unital categories.

Hence, the functor

$$(s'_{\mathcal{Y},\mathrm{Ran}})^* \circ (s'_{\mathcal{Y},\mathrm{Ran}})_* \circ (s'_{\mathcal{T},\mathrm{Ran}})_*$$

sends objects that lie in the essential image of the functor (C.1) to objects that lie in the essential image of the corresponding functor

$$\mathsf{t}^! : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})_{\mathrm{Ran}}. \tag{C.5}$$

We obtain that it is enough to show that the natural transformation

$$(\pi'_{\mathrm{Ran}})^{\mathrm{IndCoh}_{\mathrm{Ran}}}_* \circ (s'_{\mathcal{T},\mathrm{Ran}})^* \to (\pi_{\mathrm{Ran}})^{\mathrm{IndCoh}_{\mathrm{Ran}}}_*, \tag{C.6}$$

arising from the *counit* of the $((s'_{\mathcal{T},\mathrm{Ran}})^*, (s'_{\mathcal{T},\mathrm{Ran}})_*)$-adjunction, is an isomorphism when evaluated on objects lying in the essential image of the functor (C.5).

However, thanks to the identification

$$s'_{\mathcal{T},\mathrm{Ran}} \simeq s_{\mathcal{Z}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}}$$

(see Sect. 5.6.4), the latter statement is a parameterized (by $\mathrm{Sect}(X,\mathcal{Y})$) version of Proposition B.2.6 for the relative affine scheme $\mathcal{Z}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}$.

C.2.6. *Verification that* (C.4) *is an isomorphism.* As above, it is enough to show that the natural transformation

$$(\pi'_{\mathrm{Ran}})^{\mathrm{IndCoh}_{\mathrm{Ran}}}_* \circ (s'_{\mathcal{Y},\mathrm{Ran}})^* \circ (s'_{\mathcal{Y},\mathrm{Ran}})_* \to (\pi'_{\mathrm{Ran}})^{\mathrm{IndCoh}_{\mathrm{Ran}}}_* \tag{C.7}$$

is an isomorphism when evaluated on objects lying in the essential image of the functor (C.5).

However, by base change along the Cartesian diagram

$$\begin{array}{ccc} \mathcal{T}_{\mathrm{Sect}_\nabla(X,\mathcal{Y}),\mathrm{Ran}} & \xrightarrow{s'_{\mathcal{Y},\mathrm{Ran}}} & \mathcal{T}_{\mathrm{Ran}} \\ \downarrow & & \downarrow \\ \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Ran} & \xrightarrow{s_{\mathcal{Y},\mathrm{Ran}}} & \mathcal{Y}_{\mathrm{Ran}}, \end{array}$$

this reduces to the assertion of Proposition C.1.5.

□[Proposition 5.6.8]

C.3. **Proof of Proposition C.1.5.**

C.3.1. We will first reduce the assertion of Proposition C.1.5 to that of Proposition C.1.7, and then prove Proposition C.1.7.

We need to show that the natural transformation

$$(\mathrm{Id}\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-))\circ(s_{\mathrm{Ran}^{\mathrm{untl}}})^*\circ(s_{\mathrm{Ran}^{\mathrm{untl}}})_*\to(\mathrm{Id}\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-))$$

arising from the counit of the $((s_{\mathrm{Ran}^{\mathrm{untl}}})^*,(s_{\mathrm{Ran}^{\mathrm{untl}}})_*)$-adjunction, is an isomorphism.

C.3.2. First, note that the left-lax symmetric monoidal structure on the functor

$$\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-):\mathrm{IndCoh}(\mathrm{Ran}^{\mathrm{untl}})\to\mathrm{Vect},$$

arising by adjunction from the monoidal structure on the functor $\omega_{\mathrm{Ran}^{\mathrm{untl}}}\otimes -$, is actually strictly symmetric monoidal structure. Indeed, this follows from the fact that the diagonal morphism

$$\mathrm{Ran}^{\mathrm{untl}}\to\mathrm{Ran}^{\mathrm{untl}}\times\mathrm{Ran}^{\mathrm{untl}}$$

is value-wise cofinal.

C.3.3. Similarly, we obtain that the functor

$$(\mathrm{Id}\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-))\circ(s_{\mathrm{Ran}^{\mathrm{untl}}})^*:\mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}}\to\mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y}))$$

is $\mathrm{IndCoh}(\mathrm{Ran}^{\mathrm{untl}})$-linear, where $\mathrm{IndCoh}(\mathrm{Ran}^{\mathrm{untl}})$ acts on $\mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y}))$ via the symmetric monoidal functor $\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-)$.

C.3.4. This implies that we have a canonical isomorphism between the functor

$$(\mathrm{Id}\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-))\circ(s_{\mathrm{Ran}^{\mathrm{untl}}})^*\circ(s_{\mathrm{Ran}^{\mathrm{untl}}})_*$$

and

$$\Big((\mathrm{Id}\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-))\circ(s_{\mathrm{Ran}^{\mathrm{untl}}})^*\circ(s_{\mathrm{Ran}^{\mathrm{untl}}})_*\circ(\mathrm{Id}\otimes(\omega_{\mathrm{Ran}^{\mathrm{untl}}}\otimes -))\Big)\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-),$$

and this isomorphism is compatible with the map of both to

$$(\mathrm{Id}\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-))\simeq\Big((\mathrm{Id}\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-))\circ(\mathrm{Id}\otimes(\omega_{\mathrm{Ran}^{\mathrm{untl}}}\otimes -))\Big)\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}^{\mathrm{untl}},-).$$

However, the latter map is an isomorphism, by Proposition C.1.7.

□[Proposition C.1.5]

C.4. **Proof of Proposition C.1.7.**

C.4.1. Due to the assumption that $\mathrm{Sect}(X,\mathcal{Y})$ is passable, self-functors on $\mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y}))$ are in bijection with objects of $\mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Y})\times\mathrm{Sect}(X,\mathcal{Y}))$, and the identity endofunctor is given by

$$(\Delta_{\mathrm{Sect}(X,\mathcal{Y})})_*(\mathcal{O}_{\mathrm{Sect}(X,\mathcal{Y})}).$$

Thus, we need to show that the map

$$(\mathrm{Id}\otimes(\mathrm{Loc}_{\mathcal{Y}}\circ\mathrm{Loc}^R_{\mathcal{Y}}))((\Delta_{\mathrm{Sect}(X,\mathcal{Y})})_*(\mathcal{O}_{\mathrm{Sect}(X,\mathcal{Y})}))\to(\Delta_{\mathrm{Sect}(X,\mathcal{Y})})_*(\mathcal{O}_{\mathrm{Sect}(X,\mathcal{Y})}) \tag{C.8}$$

is an isomorphism.

C.4.2. We rewrite the left-hand side in (C.8) as the image of $\mathcal{O}_{\mathrm{Sect}(X,\mathcal{Y})}$ along the push-pull along the diagram

$$
\begin{array}{ccccc}
\mathrm{Sect}(X,\mathcal{Y}) & & & & \\
\downarrow {\scriptstyle \Delta_{\mathrm{Sect}(X,\mathcal{Y})}} & & & & \\
\mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Sect}(X,\mathcal{Y}) & \longleftarrow & \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Ran} & & \\
& & \downarrow {\scriptstyle \mathrm{id} \times s_{\mathcal{Y},\mathrm{Ran}}} & & \\
& & \mathrm{Sect}(X,\mathcal{Y}) \times \mathcal{Y}_{\mathrm{Ran}} & \xleftarrow{\mathrm{id} \times s_{\mathcal{Y},\mathrm{Ran}}} & \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Ran} \\
& & & & \downarrow \\
& & & & \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Sect}(X,\mathcal{Y}).
\end{array}
$$

By base change, we rewrite this as push-pull along

$$
\begin{array}{ccccc}
& & \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Ran} & \longrightarrow & \mathrm{Sect}(X,\mathcal{Y}) \times \mathrm{Sect}(X,\mathcal{Y}) \\
& & \downarrow {\scriptstyle s_{\mathcal{Y},\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} s_{\mathcal{Y},\mathrm{Ran}}} & & \\
\mathcal{Y}_{\mathrm{Ran}} & \xrightarrow{\Delta_{\mathcal{Y}_{\mathrm{Ran}}/\mathrm{Ran}}} & \mathcal{Y}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathcal{Y}_{\mathrm{Ran}} & &
\end{array}
$$

of the object

$$(p_{\mathcal{Y}_{\mathrm{Ran}}})^*(\omega_{\mathrm{Ran}}) \in \mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}}.$$

C.4.3. Consider the following version of the set-up of Sect. B.3.

Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ is an affine morphism of D-prestacks. Consider the following commutative (but non-Cartesian) diagram

$$
\begin{array}{ccc}
\mathrm{Sect}(X,\mathcal{Z}_1) \times \mathrm{Ran} & \xrightarrow{s_{\mathcal{Z}_1,\mathrm{Ran}}} & \mathcal{Z}_{1,\mathrm{Ran}} \\
{\scriptstyle \mathrm{Sect}(f)\times\mathrm{id}} \downarrow & & \downarrow {\scriptstyle f_{\mathrm{Ran}}} \\
\mathrm{Sect}(X,\mathcal{Z}_2) \times \mathrm{Ran} & \xrightarrow[s_{\mathcal{Z}_2,\mathrm{Ran}}]{} & \mathcal{Z}_{2,\mathrm{Ran}}.
\end{array}
$$

The $((s_{\mathcal{Z}_1,\mathrm{Ran}})^*, (s_{\mathcal{Z}_1,\mathrm{Ran}})_*)$- and $((s_{\mathcal{Z}_2,\mathrm{Ran}})^*, (s_{\mathcal{Z}_2,\mathrm{Ran}})_*)$-adjunctions give rise to natural transformation

$$s^*_{\mathcal{Z}_2,\mathrm{Ran}} \circ (f_{\mathrm{Ran}})_* \to (\mathrm{Sect}(f)_* \otimes \mathrm{Id})_* \circ s^*_{\mathcal{Z}_1,\mathrm{Ran}}$$

as functors

$$\mathrm{QCoh}(\mathcal{Z}_1)_{\mathrm{Ran}} \rightrightarrows \mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Z}_2)) \otimes \mathrm{IndCoh}(\mathrm{Ran}).$$

Consider the induced natural transformation

$$
\begin{aligned}
(\mathrm{C.9}) \quad & (\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-)) \circ s^*_{\mathcal{Z}_2,\mathrm{Ran}} \circ (f_{\mathrm{Ran}})_* \to \\
& \to (\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-)) \circ (\mathrm{Sect}(f)_* \otimes \mathrm{Id})_* \circ s^*_{\mathcal{Z}_1,\mathrm{Ran}} \simeq \mathrm{Sect}(f)_* \circ (\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran},-)) \circ s^*_{\mathcal{Z}_1,\mathrm{Ran}}
\end{aligned}
$$

as functors

$$\mathrm{QCoh}(\mathcal{Z}_1)_{\mathrm{Ran}} \rightrightarrows \mathrm{QCoh}(\mathrm{Sect}(X,\mathcal{Z}_2)).$$

The following is a parametrized version of Proposition B.3.8(a):

Proposition C.4.4. *The natural transformation* (C.9) *is an isomorphism, when evaluated on objects lying in the essential image of the forgetful functor*

$$\mathsf{t}^! : \mathrm{QCoh}(\mathcal{Z}_1)_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{QCoh}(\mathcal{Z}_1)_{\mathrm{Ran}}.$$

Corollary C.4.5. *The natural transformation* (C.9) *is an isomorphism, when evaluated on the object*

$$(p_{\mathcal{Z}_{1,\mathrm{Ran}}})^*(\omega_{\mathrm{Ran}}) \in \mathrm{QCoh}(\mathcal{Z}_1)_{\mathrm{Ran}}.$$

C.4.6. We will apply the above to

$$\mathcal{Z}_1 = \mathcal{Y},\ \mathcal{Z}_2 = \mathcal{Y} \times \mathcal{Y}$$

and f being the diagonal map.

Unwinding the definitions, we obtain that the map

$$(\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}, -)) \circ (s_{\mathcal{Y},\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} s_{\mathcal{Y},\mathrm{Ran}})^* \circ (\Delta_{\mathcal{Y}_{\mathrm{Ran}}})_*((p_{\mathcal{Y}_{\mathrm{Ran}}})^*(\omega_{\mathrm{Ran}})) \overset{(C.9)}{\longrightarrow} \tag{C.10}$$

$$\to (\Delta_{\mathrm{Sect}(X,\mathcal{Y})})_* \circ (\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}, -)) \circ (s_{\mathcal{Y},\mathrm{Ran}})^*((p_{\mathcal{Y}_{\mathrm{Ran}}})^*(\omega_{\mathrm{Ran}})) \overset{\sim}{\to} (\Delta_{\mathrm{Sect}(X,\mathcal{Y})})_*(\mathcal{O}_{\mathrm{Sect}(X,\mathcal{Y})})$$

identifies with the map

$$(\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Ran}, -)) \circ (s_{\mathcal{Y},\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} s_{\mathcal{Y},\mathrm{Ran}})^* \circ (\Delta_{\mathcal{Y}_{\mathrm{Ran}}})_*((p_{\mathcal{Y}_{\mathrm{Ran}}})^*(\omega_{\mathrm{Ran}})) \to (\Delta_{\mathrm{Sect}(X,\mathcal{Y})})_*(\mathcal{O}_{\mathrm{Sect}(X,\mathcal{Y})})$$

of (C.8).

Hence, the latter map is an isomorphism by Corollary C.4.5.

□[Proposition C.1.7]

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