PROOF OF THE GEOMETRIC LANGLANDS CONJECTURE II: KAC-MOODY LOCALIZATION AND THE FLE

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Contents

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Introduction

0.1. **Overview.** This paper is the second in a series of five that together prove the geometric Langlands conjecture. In this paper, we study the interaction between Kac-Moody localization and the global geometric Langlands functor of [GLC1]. We do so following the methodology of Beilinson-Drinfeld, using chiral (a.k.a, factorization) homology.

0.1.1. The main result of this paper, which appears in the main body as Theorem 18.5.2, says:

Theorem 0.1.2. *There is a commutative diagram*

$$\begin{tikzcd} \text{D-mod}_{\text{crit}}(\text{Bun}_G) \arrow[r, "\mathbb{L}_G"] & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}) \\ \text{KL}(G)_{\text{crit},\text{Ran}} \arrow[u, "\text{Loc}_{G,\text{crit}} \otimes \mathfrak{l}"] \arrow[r, "\text{FLE}_{G,\text{crit}}"] & \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}. \arrow[u, "\text{Poinc}^{\text{spec}}_{\check{G},*}"'] \end{tikzcd}$$

The terms appearing in the above diagram warrant further discussion. We will do so at more length later in this introduction, but here is a brief synopsis:

- $\text{D-mod}_{\text{crit}}(\text{Bun}_G)$ is the category of *critically twisted* D-modules on Bun_G, as considered originally in [BD1];
- $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ was defined in [AG] as the spectral category in geometric Langlands;
- The functor $\mathbb{L}_G$ is the *Langlands functor* as constructed in [GLC1];
- $\text{KL}(G)_{\text{crit},\text{Ran}}$ is the Ran space[1] version of the Kazhdan-Lusztig category at the critical level (i.e., Kac-Moody modules at the critical level integrable with respect to the arc group $\mathfrak{L}^+(G)$);
- $\text{Op}^{\text{mon-free}}_{\check{G}}$ is the factorization space parametrizing local systems on the formal disc equipped with an oper structure on the punctured disc;
- $\text{FLE}_{G,\text{crit}}$ is the *fundamental local equivalence* at critical level. This equivalence of factorization categories appears in Theorem 6.1.4 and extends the pointwise equivalence of [FG2]. It is the main theorem of Part I of this paper;
- $\text{Loc}_{G,\text{crit}}$ is the functor of critical level Kac-Moody localization;
- The Poincaré series functor, denoted functor $\text{Poinc}^{\text{spec}}_{\check{G},*}$, is given at each finite set $\underline{x} \in \text{Ran}$ of points in X by pull-push along the correspondence
$$\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \leftarrow \text{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}} \to \text{LS}_{\check{G}},$$
where the middle term parametrizes local systems on the global curve X equipped with an oper structure on $X - \underline{x}$;
- $\mathfrak{l}$ is a cohomologically shifted 1-dimensional vector space that can be ignored at first approximation. Using notation defined in the paper, it is $\mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}]$.

0.1.3. The above theorem has had folklore status in the subject. Its main ingredients were discussed at the 2014 conference "Towards the proof of the geometric Langlands conjecture."

However, some of the key technical aspects have not been addressed in the existing literature. This is most notably true for the category $\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}$, i.e., the category of ind-coherent sheaves on the *Ran space* version of the space of monodromy-free opers (see Sect. 0.2.1 below).

0.1.4. The role of Theorem 0.1.2 in the geometric Langlands program is as follows:

The functor $\text{Loc}_{G,\text{crit}}$ is not surjective, but neither is it so far from being surjective (see [Ga1, Prop. 10.1.6]). Therefore, understanding the interaction between $\mathbb{L}_G$ and Kac-Moody localization plays a crucial role in understanding $\text{D-mod}_{\text{crit}}(\text{Bun}_G)$ in spectral terms. See Remark 0.2.10 for an example where this idea is applied.

0.1.5. *What goes into the proof?* As indicated above, we first need to define the various categories.

Second, we need to construct the functors appearing in the commutative diagram. Perhaps the most interesting is $\text{FLE}_{G,\text{crit}}$, which is the subject of Part I of this paper. We discuss it further below.

Finally, we need to prove the diagram commutes. Ultimately, we do this by expressing both circuits in terms of chiral homology for the critical level $\mathcal{W}$-algebra and appealing to the Feigin-Frenkel isomorphism.

[1] The Ran space of X parameterizes finite collection of points of X.

0.1.6. As was mentioned already, this paper builds on the ideas of Beilinson and Drinfeld.

In their seminal works [BD1] and [BD2], Beilinson and Drinfeld introduced the theory of *chiral algebras* – which are equivalent to *factorization algebras* and, suitably understood, vertex algebras – and of *chiral homology* as a tool for studying interactions between categories of *local* nature, such as sheaves on the affine Grassmannian, and categories of *global* nature, such as sheaves on Bun_G.

The functors appearing in Theorem 0.1.2 are of local-to-global nature, and may be viewed as generalizations of the functor of chiral homology. It is in this sense that one can view the present work as a continuation of [BD1, BD2].

0.1.7. In writing this text, we found that we needed to refine many foundational parts of the original work of Beilinson-Drinfeld. This ultimately accounts for the length of the present work.

A significant part of these refinements has to do with the fact that we (have to) work with ∞-categories (whereas in [BD1, BD2] one mostly works with abelian categories).

0.2. **What is done in this paper?** We now highlight what we think are the most important contributions of this paper.

0.2.1. *Monodromy-free opers.* First, as an algebro-geometric object, $\mathrm{Op}_{\check{G}}^{\text{mon-free}}$ parametrizes a point $\underline{x} \in \mathrm{Ran}$, a local system σ on the formal disc $\mathcal{D}_{\underline{x}}$ at x, and an oper structure on the restriction $\sigma|_{\mathcal{D}_{\underline{x}}^\times}$ of σ to the punctured disc.

When we work over a fixed point $x \in X$, the corresponding space $\mathrm{Op}_{\check{G},x}^{\text{mon-free}}$ was introduced and studied in [FG2]. However, the Ran space version presents a host of new challenges.

This space has infinite type, so it is not immediately obvious how to define (ind-)coherent sheaves on it. We explain the relevant geometry needed to make sense of $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})$ in Sect. 3.

In Sect. 4.4, we show that $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})$ can *almost* be realized as a category of factorization modules. More precisely, in Sect. 4.4.4 we define a factorization algebra $R_{\check{G},\mathrm{Op}} \in \mathrm{Rep}(\check{G})$ and prove in Proposition 4.4.7 that its category of factorization modules is equivalent to $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})$ modulo homological convergence issues (more precisely: the corresponding bounded below categories are equivalent).

0.2.2. *The critical FLE.* This result appears as Theorem 6.1.4. It asserts that we have a t-exact equivalence of factorization categories

$$\mathrm{FLE}_{G,\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}).$$

One can view this equivalence as a (substantially amplified) categorical incarnation of the Feigin-Frenkel isomorphism.

The idea of the proof is as follows:

First, we construct the functor $\mathrm{FLE}_{G,\mathrm{crit}}$. The ingredients are Feigin's Drinfeld-Sokolov functor and Beilinson-Drinfeld's birth of opers construction.

Second, we prove that $\mathrm{FLE}_{G,\mathrm{crit}}$ preserves compact objects (in the sense suitable for factorization categories). This expresses a finiteness property of Drinfeld-Sokolov reductions that is not immediate using classical VOA methods; we show that it is immediate from the categorical construction of $\mathcal{W}$-algebras from [Ra2] (i.e., the *affine Skryabin theorem*).

Thanks to the preservation of compactness mentioned above, we are reduced to proving that $\mathrm{FLE}_{G,\mathrm{crit}}$ is a *pointwise* equivalence. This is a theorem of [FG2]. We actually reprove this theorem here to illustrate a more modern point of view on studying Kac-Moody representations using categorical tools.

Namely, we show that in general, for a category $\mathbf{C}$ with an $\mathfrak{L}(G)$-action, the *tempered quotient*[2] $\mathrm{Sph}(\mathbf{C})_{\mathrm{temp}}$ of $\mathrm{Sph}(\mathbf{C}) := \mathbf{C}^{\mathfrak{L}^+(G)}$ can be algorithmically recovered from $\mathrm{Whit}(\mathbf{C})$, the Whittaker

[2]See Sect. 7.1, where this notion is defined.

model (i.e., $\mathfrak{L}(N)$-invariants against a non-degenerate character) of $\mathbf{C}$. Heuristically, $\mathrm{Whit}(\mathbf{C})$ should live as a sheaf over $\mathrm{LS}_{\check{G}}(\mathcal{D}^{\times})$ and its sections over $\mathrm{LS}_{\check{G}}(\mathcal{D})$ should recover $\mathrm{Sph}(\mathbf{C})_{\mathrm{temp}}$; we give a precise assertion of this type in Proposition 7.5.5. The key input for this result is the pointwise version of derived Satake.

We then show that $\mathrm{KL}(G)_{\mathrm{crit}} = \mathrm{Sph}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},x})$ equals its tempered quotient. However, by [Ra2],

$$\mathrm{Whit}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},x}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}),$$

and hence we obtain the FLE from the previous paragraph.

To summarize: we deduce the FLE at critical level as an essentially formal consequence of derived Satake and affine Skryabin.

Remark 0.2.3. We should add that a particular case of the pointwise abelian category version of the FLE was established already in [BD1]:

Namely, in *loc. cit.* it was shown that the subcategory of $(\mathrm{KL}(G)_{\mathrm{crit},x})^{\heartsuit}$ consisting of modules with *regular central characters* is freely generated over $(\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},x}))^{\heartsuit}$ by the vacuum module.

Note, however, that a parallel statement would be *false* at the derived level; this observation is what led the authors of [FG2] to considering the ind-scheme of monodromy-free opers,

0.2.4. *The FLE and duality.* The category $\mathrm{KL}(G)_{\mathrm{crit}}$ is canonically self-dual by a construction with semi-infinite cohomology, see Sect. 2.2.4.

In Sect. 3, we show that $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$ is canonically self-dual, which we express as an equivalence

$$\Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}).$$

This equivalence comes from a similar equivalence $\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}$ using all opers in place of monodromy-free opers; the latter should be thought of as a critical limit of the semi-infinite cohomology for $\mathcal{W}$-algebras considered by Dhillon in [Dh].

In Sect. 8, we prove that these two self-duality constructions match under the FLE. As indicated above, this result should be considered as a compatibility between the FLE and two flavors of semi-infinite cohomology.

0.2.5. *The formalism of local-to-global functors.* We develop axiomatics in Sect. 11. In some part, the constructions here abstract Beilinson-Drinfeld's construction of chiral homology.

There is a separate introduction to this material in Sect. 12.0, so we describe the material briefly:

One often finds the following situation: there is a local (factorization) category $\underline{\mathbf{C}}^{\mathrm{loc}}$, a global category $\mathbf{C}^{\mathrm{glob}}$, and a *local-to-global* functor

$$\mathsf{F} : \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \to \mathbf{C}^{\mathrm{glob}}.$$

Here are examples we have in mind:

- For a chiral algebra $\mathcal{A}$, take[3] $\underline{\mathbf{C}}^{\mathrm{loc}} = \mathcal{A}\text{-mod}^{\mathrm{fact}}$, $\mathbf{C}^{\mathrm{glob}} = \mathrm{Vect}$, and $\mathsf{F} = \underline{C}^{\mathrm{fact}}_{\cdot}(X, \mathcal{A}, -)$, i.e., the functor of chiral homology;
- Take $\underline{\mathbf{C}}^{\mathrm{loc}} = \mathrm{KL}(G)_{\kappa}$, $\mathbf{C}^{\mathrm{glob}} = \mathrm{D\text{-}mod}_{\kappa}(\mathrm{Bun}_G)$, and $\mathsf{F} = \mathrm{Loc}_{G,\kappa}$;
- Take $\underline{\mathbf{C}}^{\mathrm{loc}} = \mathrm{Whit}_{\kappa}(G)$ the category of κ-twisted Whittaker D-modules on the affine Grassmannian, $\mathbf{C}^{\mathrm{glob}} = \mathrm{D\text{-}mod}_{\kappa}(\mathrm{Bun}_G)$, and $\mathsf{F} = \mathrm{Poinc}_{G,!}$ (or $\mathrm{Poinc}_{G,*}$);
- Take $\underline{\mathbf{C}}^{\mathrm{loc}} = \mathrm{Rep}(\check{G})$, $\mathbf{C}^{\mathrm{glob}} = \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$, and $\mathsf{F} = \mathrm{Loc}^{\mathrm{spec}}$ the spectral localization functor;
- Take $\underline{\mathbf{C}}^{\mathrm{loc}} = \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$, $\mathbf{C}^{\mathrm{glob}} = \mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$, and $\mathsf{F} = \underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},*}$ (or $\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},!}$).

[3] Technically, $\mathcal{A}\text{-mod}^{\mathrm{fact}}$ only forms a *lax* factorization category. In fact, the material of Sect. 11 does not assume any sort of factorization, just the existence of suitable categories over the (unital) Ran space.

The last two examples can be considered as $\kappa \to \infty$ limits of the second and third examples.

One key feature of each of the above constructions is that they are *unital*, in the sense that our factorization categories are themselves unital and F (canonically) commutes with vacuum insertion.

The following construction plays a key role:

One starts with a non-unital (but *lax* unital) functor F_0 and it turns out that there exists a procedure that canonically produces from it a *strictly unital* local-to-global functor F.

For example, for a chiral algebra $\mathcal{A}$, we could take F_0 as

$$\mathcal{A}\text{-mod}^{\text{fact}}_{\text{Ran}} \xrightarrow{\mathbf{oblv}} \text{D-mod}(\text{Ran}) \xrightarrow{\mathrm{C}^{\cdot}_c(\text{Ran},-)} \text{Vect}$$

with second functor that of compactly supported de Rham cochains. The resulting functor F is that of chiral homology. I.e., chiral homology can be thought of as a universal procedure that forces F_0 to commute with vacuum insertion.

We discuss this passage from lax to strict unital globlization functors at length in Sect. 11.

Remark 0.2.6. In favorable cases, Betti analogues of local-to-global functors have TQFT interpretations. Namely, given a 4d TQFT Z and a boundary condition $\mathcal{B}$ for it, we obtain $\mathbf{C}^{\text{glob}}$ as $Z(X)$; $\underline{\mathbf{C}}^{\text{loc}}$ as the evaluation of Z on a closed disc $\mathcal{D}^{\text{Betti}}$, putting $\mathcal{B}$ on the boundary $\partial\mathcal{D}^{\text{Betti}}$; and for $\underline{x} \in \text{Ran}$, we suture the disc $\mathcal{D}^{\text{Betti}}_{\underline{x}}$ around $\underline{x}$ into $X \setminus (\mathcal{D}^{\text{Betti}}_{\underline{x}} \setminus \partial\mathcal{D}^{\text{Betti}}_{\underline{x}})$ to obtain F. See Remark 11.3.9 for a related discussion.

0.2.7. *Localization.* In Sect. 10, we construct and study the Kac-Moody localization functor, which appears in Theorem 0.1.2. We do this in a loop group equivariant way, which has the effect of making Loc_G respect the Hecke actions.

We reproduce some results from [CF]. For the purposes of Theorem 0.1.2, the most important outcome is Theorem 12.8.8, which says we have a commutative diagram

$$\begin{CD}
\text{D-mod}_\kappa(\text{Bun}_G) @>{\text{coeff}^{\text{Vac,glob}}_G}>> \text{Vect} \\
@A{\text{Loc}_{G,\kappa} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}[\delta_{N_{\rho(\omega_X)}}]}AA @AA{\underline{\mathrm{C}}^{\text{fact}}_{\cdot}(X,\mathcal{W},-)}A \\
\text{KL}(G)_{\kappa,\text{Ran}} @>{\text{DS}^{\text{enh}}}>> \mathcal{W}_\kappa\text{-mod}^{\text{fact}}.
\end{CD}$$

Here the bottom horizontal arrow is Drinfeld-Sokolov reduction, the right vertical arrow is chiral homology, and the top horizontal arrow is the functor of *vacuum Whittaker coefficient.*

The similarity between this result and Theorem 0.1.2 is plain. In fact, the Langlands functor $\mathbb{L}_G$ is *characterized* using $\text{coeff}^{\text{Vac,glob}}_G$, so this result is quite close to Theorem 0.1.2.

0.2.8. *Localization at the critical level and Hecke eigensheaves.* In Sect. 15, we prove the *Hecke eigenproperty* of localization at the critical level. This result extends one of the main theorems of [BD1]; there Beilinson-Drinfeld considered the vacuum representation, but our result allows consideration of arbitrary objects of $\text{KL}(G)_{\text{crit}}$.

More precisely, Feigin-Frenkel duality (or the FLE) allows us to consider $\text{KL}(G)_{\text{crit,Ran}}$ as a module category for $\text{QCoh}(\text{Op}^{\text{mon-free}}_{\check{G}})_{\text{Ran}}$.

Now let $\text{Op}^{\text{mon-free,glob}}_{\check{G}}$ be the space over Ran parametrizing $\underline{x} \in \text{Ran}$, a $\check{G}$-local system σ on X, and an oper structure on $\sigma|_{X \setminus \underline{x}}$. There is an evident map $\text{Op}^{\text{mon-free,glob}}_{\check{G}} \to \text{Op}^{\text{mon-free}}_{\check{G}}$.

We prove in Corollary 15.5.9 that $\text{Loc}_{G,\text{crit}}$ factors through a functor

$$\text{Loc}^{\text{Op}}_{G,\text{crit}} : \text{KL}(G)_{\text{crit,Ran}} \underset{\text{QCoh}(\text{Op}^{\text{mon-free}}_{\check{G}})}{\otimes} \text{QCoh}(\text{Op}^{\text{mon-free,glob}}_{\check{G}}) \to \text{D-mod}_{\text{crit}}(\text{Bun}_G)$$

that is $\text{Rep}(\check{G})_{\text{Ran}}$-linear with respect to the following actions:

- $\text{Rep}(\check{G})_{\text{Ran}}$ acts on $\text{D-mod}_{\text{crit}}(\text{Bun}_G)$ through the Hecke action.

- $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ acts on

$$\mathrm{KL}(G)_{\mathrm{crit,Ran}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathrm{Ran}}$$

by

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \overset{\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

and pullback along the tautological map $\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}} \to \mathrm{LS}_{\check{G}}$.

Applying the FLE, we can rewrite $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit}}$ as a functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}}) \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G).$$

We have canonical functors

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \longrightarrow \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \xrightarrow{\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}$$

that we can compose with the above to obtain a functor

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}}) \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G).$$

Our Hecke property implies that this functor sends the skyscraper sheaf at an oper $\chi \in \mathrm{Op}^{\text{mon-free,glob}}_{\check{G}}$ to an eigensheaf for the local system underlying χ. When χ is a *regular* oper on X, this is the main construction of [BD1].[4]

Remark 0.2.9. Beilinson-Drinfeld show that their eigensheaves are non-zero by computing their characteristic cycles. This does not directly apply in the monodromy-free setting, but one can use Theorem 0.1.2 to verify that they are non-zero by calculating the Whittaker coefficients of these localized D-modules.

Remark 0.2.10. The above eigen-property for Kac-Moody localization was used in Drinfeld-Gaitsgory's proof of the *spectral action* of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ on $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$, cf. [Ga1] Theorem 4.5.2. The proof presented there is based on Kac-Moody localization, with [Ga1, Theorem 10.3.4] essentially asserting the Hecke property discussed above. In this sense, the present work fills an important gap in the literature.

0.3. **Structure of this paper.**

0.3.1. Overall, the paper proceeds as follows. In Part I, we formulate and prove the critical level FLE, which is the main local result of this paper. We also consider the interactions of the FLE with various duality functors.

Part II considers the vertical local-to-global functors from Theorem 0.1.2 as well as the Hecke actions. Some of our main results here reproduce results and arguments from [CF]. We put particular emphasis on the role of *unital* structures, building on ideas from [BD2], [Ra6], [Ro2] and [Ga4].

The proof of Theorem 0.1.2 relies on chiral homology and some mild variants thereof. Because [BD2] largely considered abelian categories of chiral modules, we need some extensions of their ideas to the derived setting; these appear in the appendices to this paper.

[4] Actually, even in this case, our notion of eigensheaf is somewhat more homotopically robust than the one from [BD1].

0.3.2. In more granular detail, the paper is structured as follows.

Part I deals with the local theory:

Sect. 1 reviews various factorization categories on the geometric side associated with the affine Grassmannian of G, as well as their spectral counterparts. The key points here are the geometric Casselman-Shalika formula (Theorem 1.4.2) and (derived) geometric Satake equivalence (Theorem 1.7.2). This material is largely taken from [CR].

In Sect. 2 we discuss the Kazhdan-Lusztig category and quantum Drinfeld-Sokolov reduction for its modules.

In Sect. 3, we consider ind-coherent sheaves on various spaces of local opers. We reinterpret these objects using factorization module categories in Sect. 4.4.

In Sect. 4 we explain how various factorization categories of interest can be expressed as factorization module categories over factorization algebras. We use this to define a *categorical* action of the Feigin-Frenkel center on Kac-Moody modules at the critical level.

Sect. 5 reviews the Feigin-Frenkel isomorphism at the critical level and Beilinson-Drinfeld's *birth of opers*, i.e., the local interplay of Hecke symmetries and Feigin-Frenkel duality.

In Sect. 6, we formulate the FLE at critical level. We reduce the statement to the pointwise assertion, which was proved in [FG2]. We also prove some important compatibilities for the FLE here.

As discussed above, we also provide another proof of the pointwise assertion in the present paper, by combining general considerations about derived Satake with Feigin and Frenkel's duality for $\mathcal{W}$-algebras.

In Sect. 8, we study the interaction between the FLE and natural duality functors between the DG categories appearing in it.

0.3.3. We now turn to Part II, which deals with local-to-global constructions.

Sect. 9 reviews the definition of Whittaker coefficient functors and establishes some conventions related to them.

In Sect. 10, we introduce the Kac-Moody localization functor Loc_G.

In Sect. 11, we discuss axiomatics for local-to-global functors for factorization categories (or even just categories over Ran). The key construction produces *unital* local-to-global functors from *lax unital* such functors, abstracting the construction of chiral homology.

Sect. 12 considers the interaction between Kac-Moody localization and restriction/inflation along group homomorphisms $H \to G$. This material appeared previously in [CF].

In Sect. 13, we discuss an alternative construction of Kac-Moody localization from the eighth author's thesis. The idea is to realize D-modules on Bun_G as quasi-coherent sheaves equipped with infinitesimal Hecke equivariance structures at every point of Ran.

In Sect. 14, we apply the material from Sect. 12 to calculate Whittaker coefficients of Kac-Moody representations in terms of chiral homology for the critical level $\mathcal{W}$-algebra.

Sect. 15 constructs a Hecke equivariance structure for Loc_G. This can be considered as an extension of the main construction of [BD1, Sect. 7], allowing more choices of characters for the Feigin-Frenkel center while adding homotopy coherence to *loc. cit.*

Sect. 16 contains the proof of Theorem 15.2.8, a technical point from Sect. 15.

Sect. 17 constructs the functor $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}$ and its relative $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}$.

Finally, Sect. 18 considers the interaction between the Langlands functor $\mathbb{L}_G$ and the constructions of earlier sections. Most importantly, we conclude the proof of Theorem 0.1.2 here. We also prove that $\mathbb{L}_G$ is compatible with the factorizable derived Satake equivalence here, to be used in the sequel to this paper.

0.3.4. This paper relies on a lot of foundational material, a big part (but not all) of which has not been previously written down. This material is developed in the Appendix[5] to this paper.

In Appendix A we develop the IndCoh theory for algebro-geometric objects that are *not* of finite type. As it turns out, there are two versions, denoted $\mathrm{IndCoh}^!(-)$ and $\mathrm{IndCoh}^*(-)$, respectively, which in good situations are mutually dual. We introduce the property of schemes, called *placidity*, which guarantees that these categories behave particularly well. In addition, we introduce another category, denoted $\mathrm{QCoh}_{\mathrm{co}}(-)$, useful in many situations, and which is a *pre-dual* of $\mathrm{QCoh}_{\mathrm{co}}$.

In Appendix B we discuss the pattern of factorization. We introduce factorization spaces, and construct examples of such (e.g., loops or arcs into a given target, or various spaces attached to the formal disc). We introduce factorization algebras and modules, and various operations between them. One of the central notions in this paper is that of *factorization category*. We show how various categories of algebro-geometric or representation-theoretic nature acquire this structure (notably, $\mathrm{IndCoh}^*(-)$ of monodromy-free opers and the category of Kac-Moody representations).

In Appendix C we discuss the phenomenon of *unitality*. We introduce categorical prestacks, D-modules and sheaves of categories on them. Our main example is the unital Ran space. We introduce unital and counital factorization spaces, and their common generalization, called "unital-in-correspondences" factorization spaces; it is this latter notion that plays the most important role. We introduce unital factorization algebras and categories. We emphasize that some phenomena (such as restriction of module categories) work differently in unital and non-unital settings, and it is the former that are responsible for some of the fundamental constructions in this paper.

In Appendix D we prove one of the general fundamental theorems that describe a category of algebro-geometric nature as modules over a factorization algebra. Namely, we show that (at the level of bounded below categories), the category $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))$ (here $\mathcal{Y}$ is an affine D-scheme, and $\mathfrak{L}_\nabla(\mathcal{Y})$) is the space of its horizontal sections on the punctured disc) identifies with factorization modules over the (commutative) factorization algebra of regular functions on $\mathcal{Y}$. The equivalence at the level of abelian categories is nearly evident. However, at the derived level, it is quite non-trivial, and requires that $\mathcal{Y}$ be *of finite presentation in the D-sense*.

In Appendix E we explain how to make sense of the *spectral spherical category*, i.e., the category of ind-coherent sheaves on the local spectral Hecke stack $\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}$, as a factorization category. The problem is that $\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}$ does not quite fit into the paradigm of Sect. A, in which we can make sense of $\mathrm{IndCoh}(-)$ by an algorithmic procedure. Yet, we give an algebro-geometric definition of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$, and then compare it with a representation-theoretic one of [CR].

In Appendix F we recap (essentially, following [BD2]) the relation between the scheme of horizontal sections of an affine D-scheme $\mathcal{Y}$ and the factorization homology of the factorization algebra of regular functions on $\mathcal{Y}$.

In Appendix G we describe a procedure that attaches factorization module categories over $\mathrm{Rep}(\check{G})$ to module categories over $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}}_{\check{G}})$, and show that this functor is fully faithful on a certain subcategory.

In Appendix H we recast some of the material from Sect. 11 using the notion of the "independent" category, attached to a crystal of categories on the unital Ran space. We then discuss various notions of action of a factorization monoidal category on a DG category.

Appendix I contains some complementary material to Sect. 11: we give an interpretation to the functor of the integrated insertion of the unit in terms of left-lax functors between crystals of categories over the unital Ran space.

Appendix J is homotopy-theoretic. Here we introduce a device that allows us to construct us monoidal actions from Sects. 4.6 and 5.3 up to *coherent* homotopy. These monoidal actions play a key role in the definition of the FLE functor.

[5]The Appendix is coauthored by J. Campbell, L. Chen, D. Gaitsgory, K. Lin, S. Raskin and N. Rozenblyum.

0.4. **Conventions and notation: generalities.**

0.4.1. *The players.* Throughout the paper we work over a fixed algebraically closed field k of characteristic 0. Thus, all algebro-geometric objects are defined over k.

In particular, X is a smooth projective curve over k, G is a reductive group over k, and $\check{G}$ is the Langlands dual of G.

0.4.2. *Categories.* When we say "category", we mean an ∞-category. Conventions pertaining to the ∞-categorical language are borrowed from [GaRo3, Chapter 1, Sect. 1].

0.4.3. Conventions pertaining to DG categories follow those in [GaRo3, Chapter 1, Sect. 10]. Unless explicitly stated otherwise, a DG category $\mathbf{C}$ is assumed cocomplete (i.e., to contain arbitrary direct sums). (An exception would be, e.g., the category of compact objects in a given $\mathbf{C}$, denoted $\mathbf{C}^c$.)

Unless explicitly stated otherwise, given a pair of DG categories $\mathbf{C}_1$ and $\mathbf{C}_2$, by a functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ we will always understand a *continuous* functor, i.e., one that commutes with arbitrary direct sums (equivalently, colimits).

0.4.4. Given a DG category $\mathbf{C}$ with a t-structure, we will use *cohomological* conventions. I.e., $\mathbf{C}^{\leq 0}$ will denote the subcategory of *connective* objects. We will denote by $\mathbf{C}^{\heartsuit}$ the heart of the t-structure.

0.4.5. Conventions adopted in this paper regarding higher algebra and derived algebraic geometry follow closely those of [AGKRRV].

0.4.6. *Factorization.* Conventions and notation pertaining to the Ran space and *factorization* are explained in Sect. B.

There are several pieces of notation associated with factorization categories:

Given a factorization category $\mathbf{C}$, we will denote by $\underline{\mathbf{C}}$ the corresponding sheaf of categories over Ran, by $\mathbf{C}_{\mathrm{Ran}}$ its category of global sections, and for $\mathcal{Z} \to \mathrm{Ran}$ by $\mathbf{C}_{\mathcal{Z}}$ the category of sections of the pullback of $\underline{\mathbf{C}}$ to $\mathcal{Z}$. In particular, for a k-point $\underline{x} \in \mathrm{Ran}$, we will denote by $\mathbf{C}_{\underline{x}}$ the fiber of $\underline{\mathbf{C}}$ at $\underline{x}$.

Given a pair of factorization categories $\mathbf{C}_1$ and $\mathbf{C}_2$ and a functor Φ between, we will distinguish between a property of this functor (such as admitting an adjoint or being an equivalence) taking place at the *pointwise* or *factorization* level.

The former means that the given property holds for the corresponding functor

$$\Phi : \mathbf{C}_{1,\underline{x}} \to \mathbf{C}_{2,\underline{x}}$$

for any k-point $\underline{x}$ of the Ran space. The latter means that the given property holds for

$$\Phi : \mathbf{C}_{1,\mathcal{Z}} \to \mathbf{C}_{2,\mathcal{Z}}$$

for any prestack $\mathcal{Z} \to \mathrm{Ran}$ (equivalently, one can take $\mathcal{Z}$ to be Ran itself).

0.5. **Acknowledgements.** As should be clear from what we said above, the majority of the second part of this paper can be traced back to the ideas of A. Beilinson and V. Drinfeld recorded in [BD1] and [BD2].

The FLE as presented in Part I relies crucially on the Feigin-Frenkel isomorphism, as a passage between G and $\check{G}$, see Sect. 5.

A crucial role in local and local-to-global constructions is played by the concept of *factorization.* Its appearance in representation theory was pioneered by M. Finkelberg, I. Mirković and V. Schechtman, and it was further subsequently elucidated by A. Beilinson and J. Lurie.

Separate thanks are due to J. Lurie for enabling representation theorists to work within Higher Algebra. The mathematics developed in this paper would not be possible if one worked "up to homotopy".

The fifth and seventh authors wish to thank IHÉS, where a significant part of this paper was written, for creating an excellent working environment.

Part I: Local Theory

This Part is mainly dedicated to the proof of a key local result: the critical FLE. It says that the Kazhdan-Lusztig category at the critical level (for G) is equivalent to the category of ind-coherent sheaves on the space of monodromy-free opers on the punctured disc (for $\check{G}$).

The FLE involves crossing the Langlands bridge. I.e., at some point, we will need to know something about the relationship between G and $\check{G}$. In fact, there are exactly two sources of such results (as long as we stay at the critical level for G and level ∞ for $\check{G}$): one is the geometric Casselman-Shalika formula (Theorem 1.4.2), and the other is the Feigin-Frenkel isomorphism (Theorem 5.1.2). The compatibility between the two is encapsulated by Theorem 5.2.5. The other results of local Langlands nature, including the FLE, are ultimately deduced from one (or a combination) of these two.

Once the FLE is proved, we will use it in Part II to establish a certain global compatibility of the Langlands functor, which will play a key role in subsequent papers in this series. This property will essentially say that the Langlands functor is compatible with the Beilinson-Drinfeld construction of eigensheaves via Kac-Moody localization and opers.

1. Geometric Satake and Casselman-Shalika formula: recollections

In this section we will review the constructions of categories of *geometric nature* associated, on the geometric side, to spaces of maps

$$\mathcal{D} \to G \text{ and } \mathcal{D}^\times \to G,$$

and (twisted) D-modules on these spaces, and on the spectral side to spaces of maps

$$\mathcal{D}_{\mathrm{dR}} \to \check{G} \text{ and } \mathcal{D}^\times_{\mathrm{dR}} \to \check{G}$$

and ind-coherent sheaves on these spaces.

Thus, the main players are:

- The category $\mathrm{Whit}(G)$ of Whittaker D-modules on the affine Grassmannian;
- Its spectral counterpart $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}}) \simeq \mathrm{Rep}(\check{G})$;
- The equivalence $\mathrm{Whit}(G) \simeq \mathrm{Rep}(\check{G})$, which we call the geometric Casselman-Shalika formula (Theorem 1.4.2);
- The local spherical category Sph_G;
- Its spectral counterpart $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$;
- The (derived) geometric Satake equivalence $\mathrm{Sat}_G : \mathrm{Sph}_G \simeq \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ (Theorem 1.7.2).

When dealing with these objects there is one major trouble and three "annoyances", all of which will be introduced in this section, and that will plague us throughout the paper:

1. The trouble is that the local algebro-geometric objects on the spectral side are *not* of finite type (once we consider their factorization versions), so the $\mathrm{IndCoh}(-)$ categories associated to them need extra work to define;
2. This paper is concerned with the *classical* geometric Langlands. However, "classical" for G means the *critical* level. This means that the categories on the geometric side[6] will consist not of D-modules, but of critically or half-twisted D-modules. As a result, throughout the paper, we will have to watch carefully what happens with these twistings as we move between different spaces.
3. Ultimately, on the geometric side, the object we need to consider is not the constant group-scheme on X with fiber G, but rather its twist by the T-torsor $\rho(\omega_X)$. This twist is analogous to the usual ρ-shift in the representation theory of the finite-dimensional G. Thus, all spaces associated with G will undergo the corresponding twist.

[6]We avoid using the word "automorphic" in the local context, as automorphy refers to the global situation.

(4) Both categories Sph_G and $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ are endowed with anti-involutions, denoted σ and σ^{spec}. A source of constant headache throughout this paper is that these anti-involutions are compatible under Sat_G, *up to the Chevalley involution* on G, denoted τ_G. This can be seen as a vestige (in a rather precise sense) of the fact that the square of the usual Fourier transform is not the identity, but rather is given by the action of -1.

1.1. **The critical twist.**

1.1.1. We choose once and for all a square root $\omega_X^{\otimes \frac{1}{2}}$ of the canonical line bundle ω_X on X.

Warning: In this series of papers, ω_X denotes the canonical line bundle on X, and *not* the dualizing sheaf on X, which is the [1] shift of that. (So, properly, we should have used Ω^1_X, rather than ω_X.) This deviates from the convention, according to which, for a prestack $\mathcal{Y}$ we denote by $\omega_{\mathcal{Y}}$ its dualizing sheaf. So the only exception for this rule is when $\mathcal{Y}$ is the curve X itself.

1.1.2. Consider the affine Grassmannian Gr_G as a factorization space over X, equipped with an action of the (factorization) group indscheme $\mathfrak{L}(G)$.

We refer the reader to Sects. B.1.8 and B.5.5, respectively, where the definition of these objects is recalled, and to Sect. B.1.6, where the general theory of factorization spaces is set up.

1.1.3. Let $\det_{\mathrm{Gr}_G}$ denote the determinant (factorization) line bundle on Gr_G.

Remark 1.1.4. According to [BD1, Sect. 4], the choice of $\omega_X^{\otimes \frac{1}{2}}$ gives rise to a square root of $\det_{\mathrm{Gr}_G}$, as a line bundle over $\mathrm{Gr}_{G,\mathrm{Ran}}$. However, this square root is *incompatible* with factorization.[7]

1.1.5. For a line bundle $\mathcal{L}$ on a space $\mathcal{Y}$ and an integer n, let $\mathcal{L}^{\frac{1}{n}}$ denote the étale μ_n-gerbe of nth roots of $\mathcal{L}$.

Recall now that given a μ_n-gerbe $\mathcal{G}$ on a space $\mathcal{Y}$, we can consider the $\mathcal{G}$-twisted category of D-modules on $\mathcal{Y}$, to be denoted

$$\text{D-mod}_{\mathcal{G}}(\mathcal{Y}).$$

Thus, for $(\mathcal{Y}, \mathcal{L}, n)$ as above we can consider the corresponding category

$$\text{D-mod}_{\mathcal{L}^{\frac{1}{n}}}(\mathcal{Y}).$$

1.1.6. Consider the μ_2-gerbe $\det^{\frac{1}{2}}_{\mathrm{Gr}_G}$.

We will use the short-hand notation

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)$$

to denote the (factorization) category

$$\text{D-mod}_{\det^{\frac{1}{2}}_{\mathrm{Gr}_G}}(\mathrm{Gr}_G)$$

of $\det^{\frac{1}{2}}_{\mathrm{Gr}_G}$-twisted D-modules on Gr_G.

Remark 1.1.7. According to Remark 1.1.4, a choice of $\omega_X^{\otimes \frac{1}{2}}$ gives rise to a trivialization of the gerbe $\det^{\frac{1}{2}}_{\mathrm{Gr}_G}$. However, this trivialization is incompatible with factorization.

For that reason, henceforth, we will avoid using it.

[7] More precisely, this square root exists as a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle, where the grading over the connected component Gr^λ_G of Gr_G (here $\lambda \in \Lambda_{G,G} = \pi_0(\mathrm{Gr}_G)$) equals $\langle \lambda, 2\check{\rho}\rangle \bmod 2$.

1.1.8. Recall that for a space $\mathcal{Y}$, we can consider *de Rham* twistings on $\mathcal{Y}$ (see, [GaRo2, Sect. 6]). These are by definition $\mathcal{O}^\times$-gerbes on $\mathcal{Y}_{\mathrm{dR}}$, equipped with a trivialization of their pullback to $\mathcal{Y}$.

Given a de Rham twisting $\mathcal{T}$, we can consider the corresponding twisted category of D-modules

$$\text{D-mod}_{\mathcal{T}}(\mathcal{Y}),$$

see [GaRo2, Sect. 7].

Recall also that to a line bundle $\mathcal{L}$ on $\mathcal{Y}$, we can associate a de Rham twisting, which in this paper we denote by $\mathrm{dlog}(\mathcal{L})$ (the corresponding $\mathcal{O}^\times$-gerbe on $\mathcal{Y}_{\mathrm{dR}}$ is trivial, but the trivialization of its pullback to $\mathcal{Y}$ differs from the tautological one by tensoring with $\mathcal{L}$).

Note that tensoring by $\mathcal{L}$ defines an equivalence

$$\text{D-mod}(\mathcal{Y}) \to \text{D-mod}_{\mathrm{dlog}(\mathcal{L})}(\mathcal{Y}). \tag{1.1}$$

Finally, recall (see [GaRo2, Corollary 6.4.5]) that the space of de Rham twistings on a given space $\mathcal{Y}$ carries a natural a k-linear structure. Thus, for $c \in k$, we have a well-defined twisting $c \cdot \mathrm{dlog}(\mathcal{L})$, and the corresponding category

$$\text{D-mod}_{c \cdot \mathrm{dlog}(\mathcal{L})}(\mathcal{Y}).$$

1.1.9. Let $(\mathcal{Y}, \mathcal{L}, n)$ be as above. Note that for $c = n \in \mathbb{Z} \subset k$, we have

$$n \cdot \mathrm{dlog}(\mathcal{L}) = \mathrm{dlog}(\mathcal{L}^{\otimes n}).$$

In particular, we have a canonical identification of the corresponding twisted categories of D-modules:

$$\text{D-mod}_{\mathcal{L}^{\frac{1}{n}}}(\mathcal{Y}) \overset{\sim}{\to} \text{D-mod}_{\frac{1}{n} \cdot \mathrm{dlog}(\mathcal{L})}(\mathcal{Y}). \tag{1.2}$$

For example, when $n = 1$, the identification (1.2) is the identification of (1.1).

1.1.10. We will use the short-hand notation

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G)$$

for the (factorization) category

$$\text{D-mod}_{\frac{1}{2} \cdot \mathrm{dlog}(\det_{\mathrm{Gr}_G})}(\mathrm{Gr}_G).$$

1.1.11. Applying (1.2) to $\mathcal{Y} = \mathrm{Gr}_G$ and $\mathcal{L} = \det_{\mathrm{Gr}_G}$, we obtain a canonical equivalence of (factorization) categories

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \simeq \text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G).$$

Remark 1.1.12. According to Remark 1.1.4, we can also identify

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \simeq \text{D-mod}(\mathrm{Gr}_{G,\mathrm{Ran}}),$$

or equivalently

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \simeq \text{D-mod}(\mathrm{Gr}_{G,\mathrm{Ran}}),$$

as plain categories, but these identifications are incompatible with the factorization structures.

Remark 1.1.13. We distinguish $\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G)$ and $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)$ notationally for the following two reasons:

(1) The étale gerbe-twisted version makes sense not just in the context of D-modules, but also in other sheaf-theoretic contexts (e.g., Betti, ℓ-adic).

(2) The category $\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G)$ comes equipped with a natural forgetful functor to $\mathrm{IndCoh}(\mathrm{Gr}_G)$, while for a general étale gerbe, the gerbe-twisted category of D-modules does not carry such a functor.

Thus, the distinction between gerbes and twistings becomes relevant when discussing connections between D-modules and modules over Lie algebras, as we often do in this paper. We use the $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)$ (or $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$) to evoke the sheaf-theoretic geometry of these spaces, while $\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G)$ (or $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$) evokes the connection to Kac-Moody representation theory at the critical level.

1.1.14. We can also consider the corresponding *multiplicative* factorization μ_2-gerbe on $\mathfrak{L}(G)$, equipped with a multiplicative trivialization of its restriction to $\mathfrak{L}^+(G)$.

Since the group indscheme $\mathfrak{L}(N)$ is contractible, the restriction of the above gerbe to it also admits a canonical multiplicative trivialization.

In particular, if H is a factorization subgroup of either $\mathfrak{L}^+(G)$ or $\mathfrak{L}(N)$, it makes sense to consider the (factorization) category

$$\text{D-mod}_{\frac{1}{2}}(\text{Gr}_G)^H$$

of H-equivariant D-modules.

1.2. **A geometric twisting construction.**

1.2.1. Let H be a group mapping to G, and let $\mathcal{P}_H$ be an H-torsor over X. Taking sections over the formal disc, $\mathcal{P}_H$ gives rise to a factorization torsor over $\mathfrak{L}^+(H)$; by a slight abuse of notation, we will denote this $\mathfrak{L}^+(H)$-factorization torsor by the same symbol $\mathcal{P}_H$.

Given a space $\mathcal{Y}$ over X, equipped with an action of $\mathfrak{L}^+(H)$, we can form a twist, to be denoted $\mathcal{Y}_{\mathcal{P}_H}$, i.e.,

$$\mathcal{Y}_{\mathcal{P}_H} := (\mathcal{P}_H \times \mathcal{Y})/\mathfrak{L}^+(H).$$

If $\mathcal{Y}$ was endowed with a factorization structure compatible with the $\mathfrak{L}^+(H)$-action, then so is $\mathcal{Y}_{\mathcal{P}_H}$.

1.2.2. The space $\mathcal{Y}_{\mathcal{P}_H}$ is acted on by the adjoint twist $\mathfrak{L}^+(H)_{\mathcal{P}_H}$ of $\mathfrak{L}^+(H)$.

Note that we have a canonical isomorphism

$$\mathcal{Y}/\mathfrak{L}^+(H) \simeq \mathcal{Y}_{\mathcal{P}_H}/\mathfrak{L}^+(H)_{\mathcal{P}_H}. \tag{1.3}$$

1.2.3. We will denote by the subscript $\mathcal{P}_H$ the various categories of D-modules associated with the above geometric objects, such as

$$\text{D-mod}(\mathcal{Y}) \rightsquigarrow \text{D-mod}(\mathcal{Y})_{\mathcal{P}_H} \text{ and } \text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)} \rightsquigarrow (\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)})_{\mathcal{P}_H}.$$

Note, however, that thanks to the identification (1.3), the category $(\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)})_{\mathcal{P}_H}$ is canonically equivalent to the original category $\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)}$. We will denote this equivalence by

$$\alpha_{\mathcal{P}_H,\text{taut}} : \text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)} \overset{\sim}{\to} (\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)})_{\mathcal{P}_H}.$$

1.2.4. A typical example of the above situation that we will consider is when $H = T$, and the T-bundle is $\rho(\omega_X)$, i.e., the bundle induced from $\omega_X^{\otimes \frac{1}{2}}$ by means of

$$2\rho : \mathbb{G}_m \to T.$$

1.3. **The Whittaker category on the affine Grassmannian.**

1.3.1. We apply the construction of Sect. 1.2.4 to $\mathcal{Y} := \text{Gr}_G$, viewed as a scheme acted on by $\mathfrak{L}^+(T) \subset \mathfrak{L}^+(G)$, and the group indscheme $\mathfrak{L}(N)$.

Thus, we can form the (factorization) space $\text{Gr}_{G,\rho(\omega_X)}$, which is acted on by $\mathfrak{L}(G)_{\rho(\omega_X)}$, and in particular $\mathfrak{L}(N)_{\rho(\omega_X)}$.

1.3.2. The group indscheme $\mathfrak{L}(N)_{\rho(\omega_X)}$ is equipped with a homomorphism

$$\chi : \mathfrak{L}(N)_{\rho(\omega_X)} \to \mathbb{G}_a, \tag{1.4}$$

equal to the composition

$$\mathfrak{L}(N)_{\rho(\omega_X)} \to \mathfrak{L}(N/[N,N])_{\rho(\omega_X)} \simeq \prod_I \mathfrak{L}(\mathbb{G}_a)_{\omega_X} \overset{\mathrm{Res}}{\to} \prod_I \mathbb{G}_a \overset{\chi_0}{\to} \mathbb{G}_a,$$

where:

- I is the set of vertices of the Dynkin diagram of G;
- $\mathfrak{L}(\mathbb{G}_a)_{\omega_X}$ is the twist formed with respect to the $\mathfrak{L}^+(\mathbb{G}_m)$-action on $\mathfrak{L}(\mathbb{G}_a)$;
- $\mathrm{Res} : \mathfrak{L}(\mathbb{G}_a)_{\omega_X} \to \mathbb{G}_a$ is the canonical residue map;
- χ_0 is a non-degenerate character (i.e., a character non-trivial along each factor).

1.3.3. Let $\mathbf{C}$ be a category acted on by $\mathfrak{L}(G)_{\rho(\omega_X)}$ at the critical level.[8] Denote:

$$\mathrm{Whit}^!(\mathbf{C}) := \mathbf{C}^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi} \text{ and } \mathrm{Whit}_*(\mathbf{C}) := \mathbf{C}_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi},$$

where we impose equivariance against the pullback of

$$\exp \in \text{D-mod}(\mathbb{G}_a)$$

by means of χ (see [Ra2] for more details). Our normalization for exp is that it is a character sheaf in the *-sense, i.e.,

$$\mathrm{add}^*(\exp) \simeq \exp \boxtimes \exp .$$

Note that

$$(\mathrm{Whit}_*(\mathbf{C}))^\vee \simeq \mathrm{Whit}^!(\mathbf{C}^\vee), \tag{1.5}$$

up to replacing χ_0 by its inverse, where[9]

$$(-)^\vee := \mathrm{Funct}((-), \mathrm{Vect}).$$

1.3.4. Although the assignments

$$\mathbf{C} \rightsquigarrow \mathrm{Whit}^!(\mathbf{C}) \text{ and } \mathbf{C} \rightsquigarrow \mathrm{Whit}_*(\mathbf{C})$$

involve the group *ind*-scheme $\mathfrak{L}(N)_{\rho(\omega_X)}$, they behave nicely on the 2-category of $\mathfrak{L}(G)_{\rho(\omega_X)}$-module categories (see [Ra2]).

Namely, they both commute with limits and colimits. Combined with (1.5), this implies that if $\mathbf{C}$ is dualizable, then so are $\mathrm{Whit}^!(\mathbf{C})$ and $\mathrm{Whit}_*(\mathbf{C})$.

However, more is true.

1.3.5. Let $\omega^{\mathrm{ren}}_{\mathfrak{L}(N)_{\rho(\omega_X)}} \in \text{D-mod}(\mathfrak{L}(N)_{\rho(\omega_X)})$ be *the renormalized* dualizing sheaf on $\mathfrak{L}(N)_{\rho(\omega_X)}$, defined to be the *-pullback of the dualizing sheaf along the projection

$$\mathfrak{L}(N)_{\rho(\omega_X)} \to \mathfrak{L}(N)_{\rho(\omega_X)}/\mathfrak{L}^+(N)_{\rho(\omega_X)}.$$

Consider the object

$$\omega^{\mathrm{ren}}_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi} := \omega^{\mathrm{ren}}_{\mathfrak{L}(N)_{\rho(\omega_X)}} \overset{*}{\otimes} \chi^*(\exp) \in \text{D-mod}(\mathfrak{L}(N)_{\rho(\omega_X)}).$$

[8] The discussion here is applicable both when we work over a fixed point $\underline{x} \in \mathrm{Ran}$ and in the factorization setting.

[9] In the next formula $\mathrm{Funct}(-,-)$ stands for colimit-preserving functors. We will always use this convention when talking about functors between cocomplete categories, unless explicitly specified otherwise.

1.3.6. Let $\mathbf{C}$ be as above. The operation of *-convolution with $\omega^{\mathrm{ren}}_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}$ is an endofunctor of $\mathbf{C}$ (as a plain DG category), and this endofunctor factors as

$$\mathbf{C} \twoheadrightarrow \mathrm{Whit}_*(\mathbf{C}) \to \mathrm{Whit}^!(\mathbf{C}) \hookrightarrow \mathbf{C}.$$

Denote the resulting functor $\mathrm{Whit}_*(\mathbf{C}) \to \mathrm{Whit}^!(\mathbf{C})$ by

$$\Theta_{\mathrm{Whit}(\mathbf{C})} : \mathrm{Whit}_*(\mathbf{C}) \to \mathrm{Whit}^!(\mathbf{C}).$$

The following fundamental result was established in [Ra2]:

Theorem 1.3.7. *The functor $\Theta_{\mathrm{Whit}(\mathbf{C})}$ is an equivalence.*

Remark 1.3.8. The proof of Theorem 1.3.13, as recorded in [Ra2], is given for a fixed formal disc, but the same argument applies to prove a version of this theorem over Ran.

1.3.9. We apply the above discussion to

$$\mathbf{C} := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}).$$

Thus we obtain the (factorization) categories

$$\mathrm{Whit}^!(\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})) \text{ and } \mathrm{Whit}_*(\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})).$$

We will use for them short-hand notations

$$\mathrm{Whit}^!(G) \text{ and } \mathrm{Whit}_*(G),$$

respectively.

Remark 1.3.10. The categories $\mathrm{Whit}^!(G)$ and $\mathrm{Whit}_*(G)$ are canonically independent of the choice of χ_0:

Indeed, given two non-degenerate characters χ_0^1 and χ_0^2, there exists an element $t \in T$ that conjugates χ_0^1 to χ_0^2. Translation by t on $\mathrm{Gr}_{G,\rho(\omega_X)}$ defines then an equivalence between the corresponding Whittaker categories.

The choice of t is unique up to an element $z \in Z_G$. However, the translation action of z on $\mathrm{Gr}_{G,\rho(\omega_X)}$ is trivial.

1.3.11. By (1.5), the categories $\mathrm{Whit}^!(G)$ and $\mathrm{Whit}_*(G)$ are naturally mutually dual, up to replacing χ_0 by its inverse. Note, however, that due to Remark 1.3.10, they are actually mutually dual.

Furthermore, as is shown in [Ga6], both $\mathrm{Whit}^!(G)$ and $\mathrm{Whit}_*(G)$ are compactly generated (see Sect. B.11.9 for what compact generation means in the factorization setting).

1.3.12. Let

$$\Theta_{\mathrm{Whit}(G)} : \mathrm{Whit}_*(G) \to \mathrm{Whit}^!(G)$$

denote the functor from Sect. 1.3.6.

As a particular case of Theorem 1.3.7, we obtain:

Theorem 1.3.13. *The functor $\Theta_{\mathrm{Whit}(G)}$ is an equivalence (of factorization categories).*

1.3.14. The factorization categories $\mathrm{Whit}^!(G)$ and $\mathrm{Whit}_*(G)$ are unital (see Sect. C.11.1) for what this means. Here is the explicit description of their factorization units:

The factorization unit $\mathbf{1}_{\mathrm{Whit}_*(G)} \in \mathrm{Whit}_*(G)$ is the object, denoted $\mathrm{Vac}_{\mathrm{Whit}_*(G)}$, equal to the projection along

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}) \to \mathrm{Whit}_*(G)$$

of $\delta_{1_{\mathrm{Gr}_{G,\rho(\omega_X)}}} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})$, the latter being the factorization unit $\mathbf{1}_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})}$ for $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})$ itself.

1.3.15. The factorization unit $\mathbf{1}_{\mathrm{Whit}^!(G)} \in \mathrm{Whit}^!(G)$ is the object, denoted $\mathrm{Vac}_{\mathrm{Whit}^!(G)}$, equal to the *-direct image along the locally-closed embedding

$$\mathfrak{L}(N)_{\rho(\omega_X)}/\mathfrak{L}^+(N)_{\rho(\omega_X)} \hookrightarrow \mathrm{Gr}_{G,\rho(\omega_X)}$$

of

$$\omega_{\mathfrak{L}(N)_{\rho(\omega_X)}/\mathfrak{L}^+(N)_{\rho(\omega_X)}} \otimes \chi^*(\exp) \in \text{D-mod}(\mathfrak{L}(N)_{\rho(\omega_X)}/\mathfrak{L}^+(N)_{\rho(\omega_X)})^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}.$$

Note that the above *-extension is *clean*, i.e., receives an isomorphism from the !-extension.

This implies that the functor co-represented by $\mathrm{Vac}_{\mathrm{Whit}^!(G)}$ identifies with the functor of !-fiber at the unit point $1_{\mathrm{Gr}_{G,\rho(\omega_X)}} \in \mathrm{Gr}_{G,\rho(\omega_X)}$, restricted to

$$\mathrm{Whit}^!(G) \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}).$$

1.4. **The geometric Casselman-Shalika formula.**

1.4.1. The following is the statement of the geometric Casselman-Shalika formula (see [Ra3, Theorem 6.36.1][10]):

Theorem 1.4.2. *There exists a canonically defined equivalence of factorization categories:*

$$\mathrm{CS}_G : \mathrm{Whit}^!(G) \to \mathrm{Rep}(\check{G}).$$

Remark 1.4.3. In the course of the proof of Theorem 1.4.2 one uses the *naive* (i.e., non-derived) geometric Satake to construct a functor

$$\mathrm{Rep}(\check{G}) \to \mathrm{Whit}^!(G),$$

and one shows that it is an equivalence, see Remark 1.7.7.

1.4.4. The functor CS_G is normalized so that it sends the standard object

$$\Delta^\lambda \in \mathrm{Whit}^!(G), \quad \lambda \in \Lambda^+_G,$$

corresponding to the $\mathfrak{L}(N)_{\rho(\omega_X)}$-orbit

$$S^\lambda := \mathfrak{L}(N)_{\rho(\omega_X)} \cdot t^\lambda$$

to the highest weight module

$$V^{-w_0(\lambda)} \in \mathrm{Rep}(\check{G}).$$

(In the above formula, t denotes the uniformizer on $\mathcal{D}$.)

Remark 1.4.5. By fixing the above normalization for CS_G we made a choice. We could have made a different choice by applying the Chevalley involution on G, or equivalently, on $\check{G}$.

The reason for our particular choice is that it is compatible with the standard normalization for the Feigin-Frenkel isomorphism, see Theorem 5.2.5.

Remark 1.4.6. For the validity of Theorem 1.4.2 at the factorization level, it is crucial that in the definition of $\mathrm{Whit}^!(G)$ we use the twisted category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)$, rather than the untwisted one, i.e., $\text{D-mod}(\mathrm{Gr}_G)$.

1.4.7. The following is a basic pattern of how the equivalence CS_G interacts with duality.

Let us denote by

$$\mathrm{FLE}_{\check{G},\infty} : \mathrm{Rep}(\check{G}) \to \mathrm{Whit}_*(G)$$

the functor equal to $\mathrm{CS}_G^\vee$, with respect to the canonical dualities:

$$\mathrm{Whit}_*(G) = (\mathrm{Whit}^!(G))^\vee \text{ and } \mathrm{Rep}(\check{G})^\vee \simeq \mathrm{Rep}(\check{G}).$$

Remark 1.4.8. The notation $\mathrm{FLE}_{\check{G},\infty}$ stems from the fact that the above functor is indeed the limiting value of the (positive level) FLE equivalence.

[10]The original result in this direction is the main theorem of [FGV].

1.4.9. *Example.* Note, in particular, that the functor $\mathrm{FLE}_{\check{G},\infty}$ sends

$$V^\lambda \in \mathrm{Rep}(\check{G}) \mapsto \nabla^\lambda \in \mathrm{Whit}_*(G),$$

where for $\mu \in \Lambda^+$ we denote by

$$\nabla^\mu \in \mathrm{Whit}_*(G)$$

the object dual to $\Delta^\mu \in \mathrm{Whit}^!(G)$, i.e.,

$$\langle \mathcal{F}, \nabla^\mu \rangle = \mathcal{H}om_{\mathrm{Whit}^!(G)}(\Delta^\mu, \mathcal{F}), \quad \mathcal{F} \in \mathrm{Whit}^!(G),$$

where

$$\langle -, - \rangle : \mathrm{Whit}^!(G) \otimes \mathrm{Whit}_*(G) \to \mathrm{Vect}$$

is the canonical pairing.

1.4.10. Note that the Whittaker category is canonically attached to the pair (G, B). Hence, the group of *outer automorphisms* of G (i.e., the group of automorphisms of the polarized[11] root datum of G) acts on both versions of the Whittaker category.

Let τ_G be the Chevalley involution, viewed as an outer automorphism of G. The corresponding automorphism of the polarized root datum acts as $\lambda \mapsto -w_0(\lambda)$.

1.4.11. We have:

Lemma 1.4.12. *The composition*

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\longrightarrow} \mathrm{Whit}_*(G) \overset{\Theta_{\mathrm{Whit}(G)}}{\longrightarrow} \mathrm{Whit}^!(G)$$

identifies canonically with

$$\tau_G \circ (\mathrm{CS}_G)^{-1}.$$

Remark 1.4.13. As a reality check, note that both functors in (1.4.12) send

$$V^\lambda \in \mathrm{Rep}(\check{G}) \;\mapsto\; \Delta^\lambda \in \mathrm{Whit}^!(G).$$

Indeed, the functor $\Theta_{\mathrm{Whit}(G)}$ is easily seen to send Δ^λ to ∇^λ.

The proof of Lemma 1.4.12 follows easily from the construction of CS_G via naive geometric Satake.

1.5. **The spherical category.**

1.5.1. We denote by $\mathrm{Sph}_G^{\text{non-ren}}$ the (factorization) monoidal category

$$\text{D-mod}_{\frac{1}{2}}(\mathfrak{L}^+(G)\backslash \mathfrak{L}(G)/\mathfrak{L}^+(G)).$$

We have a naturally defined right action of $\mathrm{Sph}_G^{\text{non-ren}}$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)$, compatible with the left action of $\mathfrak{L}(G)$.

1.5.2. We let Sph_G denote its *renormalized* version, which is defined as the ind-completion of the full subcategory in $\mathrm{Sph}_G^{\text{non-ren}}$ consisting of objects whose image under (either of) the forgetful functors

$$\text{D-mod}_{\frac{1}{2}}(\mathfrak{L}^+(G)\backslash \mathfrak{L}(G)) \leftarrow \text{D-mod}_{\frac{1}{2}}(\mathfrak{L}^+(G)\backslash \mathfrak{L}(G)/\mathfrak{L}^+(G)) \to \text{D-mod}_{\frac{1}{2}}(\mathfrak{L}(G)/\mathfrak{L}^+(G))$$

is compact (see [CR, Proposition 6.3.2] for more details).

By construction, the monoidal (and also factorization) unit

$$\mathbf{1}_{\mathrm{Sph}_G} \simeq \delta_{1_{\mathrm{Gr}_G}} \in \mathrm{Sph}_G$$

is compact.

[11] By a polarization of a root datum we mean a choice of the subset of positive roots.

1.5.3. We have an adjoint pair of functors

$$\text{ren} : \text{Sph}_G^{\text{non-ren}} \rightleftarrows \text{Sph}_G : \text{non-ren},$$

with ren being fully faithful and non-ren monoidal. This makes $\text{Sph}_G^{\text{non-ren}}$ into a monoidal colocalization of Sph_G.

In particular, we have a right action of Sph_G on $\text{D-mod}_{\frac{1}{2}}(\text{Gr}_G)$, compatible with the left action of $\mathfrak{L}(G)$ and factorization.

1.5.4. Inversion on the group $\mathfrak{L}(G)$ defines an anti-involution, denoted σ, of Sph_G. We will refer to it as the "flip" anti-involution.

Henceforth, we will use σ to pass between left and right module categories over Sph_G. In light of this, we will not necessarily distinguish between left and right actions of Sph_G.

1.5.5. The fact that Gr_G is ind-proper implies that the composition of the involution σ with Verdier duality (on compact objects) defines an equivalence

$$\text{Sph}_G^\vee \simeq \text{Sph}_G, \tag{1.6}$$

which identifies both with right and left monoidal dualization.

Combined with the fact that the unit in Sph_G is compact, we obtain that Sph_G is *rigid* as a monoidal category.[12]

1.5.6. Recall the setting of Sect. 1.2. For any G-bundle $\mathcal{P}_G$ on X, we can form the twisted version

$$\text{Sph}_{G,\mathcal{P}_G}$$

of $\mathcal{P}_G$.

We have a naturally defined action of $\text{Sph}_{G,\mathcal{P}_G}$ on $\text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\mathcal{P}_G})$, compatible with the left action of $\mathfrak{L}(G)_{\mathcal{P}_G}$ and factorization.

1.5.7. In particular, we have a natural action of $\text{Sph}_{G,\rho(\omega_X)}$ on $\text{Whit}^!(G)$ and $\text{Whit}_*(G)$.

These actions are compatible both with the duality

$$(\text{Whit}^!(G))^\vee \simeq \text{Whit}_*(G) \tag{1.7}$$

(see Sect. 1.5.4) and the functor $\Theta_{\text{Whit}(G)}$.

1.5.8. Note, however, that according to Sect. 1.2.3, we can identify[13]

$$\text{Sph}_G \overset{\alpha_{\rho(\omega_X),\text{taut}}}{\simeq} \text{Sph}_{G,\rho(\omega_X)},$$

and thus we can regard $\text{Whit}^!(G)$ and $\text{Whit}_*(G)$ as acted on by Sph_G itself.

1.6. **The spectral spherical category.**

1.6.1. Consider the *local spectral Hecke stack*

$$\text{Hecke}_{\check{G}}^{\text{spec,loc}} := \text{LS}_{\check{G}}^{\text{reg}} \underset{\text{LS}_{\check{G}}^{\text{mer}}}{\times} \text{LS}_{\check{G}}^{\text{reg}}.$$

as a factorization space.

In the above formula $\text{LS}_{\check{G}}^{\text{reg}}$ (resp., $\text{LS}_{\check{G}}^{\text{mer}}$) is the factorization space that attaches to $\underline{x} \in \text{Ran}$ the stack $\text{LS}_{\check{G}}(\mathcal{D}_{\underline{x}})$ (resp., $\text{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}^\times)$) of $\check{G}$-local systems on the formal multi-disc $\mathcal{D}_{\underline{x}}$ (resp., the punctured multi-disc $\mathcal{D}_{\underline{x}}^\times := \mathcal{D}_{\underline{x}} - \underline{x}$), see Sect. B.7.1.

[12] Being a monoidal colocalization of a rigid category, $\text{Sph}_G^{\text{non-ren}}$ is semi-rigid (see [AGKRRV, Appendix C]).

[13] In the formula below we consider $\mathfrak{L}(G)$ as acted on by $\mathfrak{L}^+(G) \times \mathfrak{L}^+(G)$.

1.6.2. The fiber $\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\underline{x}}$ of $\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}$ over a given point $\underline{x} \in \mathrm{Ran}$ is the stack

$$\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\underline{x}} := \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} \underset{\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}}}{\times} \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}. \tag{1.8}$$

The stack (1.8) is locally of finite type. In fact, its is isomorphic of the product of copies of

$$\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}^{\wedge}_0/\mathrm{Ad}(\check{G})}{\times} \mathrm{pt}/\check{G} \tag{1.9}$$

for each distinct point that comprises $\underline{x}$.

1.6.3. Hence, it makes sense to consider the category

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}} := \mathrm{IndCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\underline{x}}).$$

We endow $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ with a monoidal structure via *-pull and *-push along the standard convolution diagram.

1.6.4. As we let $\underline{x}$ move along Ran (or X^n for a fixed integer n), the resulting prestack is no longer locally almost of finite type, so the category of ind-coherent sheaves on it is not a priori-defined.

In fact, $\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}$ violates the condition of being (locally almost) of finite type so badly, that we do not really know how to define the corresponding category $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ algorithmically.

We refer the reader to Sect. E, where the definition is given (and is compared to another working definition, adopted in [CR]).

Accordingly, the proofs of all the statements that involve $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ are also delegated to Sect. E. In the main body of the text, we will supply prototypes of the corresponding proofs for the pointwise version $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$.

1.6.5. The pointwise version of the spectral Hecke category $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$, i.e., $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$, is equipped with a tautological action on $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}})$.

This construction persists in the factorization setting, i.e., we have an action of the monoidal factorization category $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on

$$\mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}}), \tag{1.10}$$

viewed as a plain[14] factorization category (see Sect. B.13.5 where the equivalence (1.10) is established).

1.6.6. By construction, the category $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ receives a monoidal functor, denoted

$$\mathrm{nv} : \mathrm{Rep}(\check{G}) \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}},$$

to be thought of[15] as the direct image functor along

$$\mathrm{LS}^{\mathrm{reg}}_{\check{G}} \to \mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}},$$

where we now view the categories appearing in (1.10) as (symmetric) *monoidal* factorization categories.

1.6.7. The flip of two factors defines an anti-involution on $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ to be denoted σ^{spec}.

We will use σ^{spec} to pass between left and right $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$-module categories.

Note that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Rep}(\check{G}) \arrow[r, "\mathrm{nv}"] \arrow[d, "\mathrm{Id}"'] & \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \arrow[d, "\sigma^{\mathrm{spec}}"] \\ \mathrm{Rep}(\check{G}) \arrow[r, "\mathrm{nv}"] & \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}, \end{tikzcd} \tag{1.11}$$

where Id makes sense as an anti-involution of $\mathrm{Rep}(\check{G})$, since this category is symmetric monoidal.

[14] As opposed to a (symmetric) monoidal factorization category.
[15] And literally so over a fixed point of Ran.

1.7. **Geometric Satake equivalence.**

1.7.1. The following is the statement of the factorization version of the *derived* geometric Satake equivalence (see [CR]):

Theorem 1.7.2. *There exists a unique equivalence of monoidal factorization categories*

$$\mathrm{Sat}_G : \mathrm{Sph}_G \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}},$$

compatible with the actions of Sph_G *on* $\mathrm{Whit}(G)$ *and* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ *on* $\mathrm{Rep}(\check{G})$ *via the equivalence*

$$\mathrm{CS}_G : \mathrm{Whit}^!(G) \simeq \mathrm{Rep}(\check{G}).$$

The construction of the functor Sat_G will be recalled in Sect. E.10.

Remark 1.7.3. In this series of papers we will refer to Sat_G just as "geometric Satake equivalence", omitting the word "derived". What is more commonly referred to as "geometric Satake" is not an equivalence, but a functor in one direction, which we will refer to as "naive Satake" and denote by $\mathrm{Sat}_G^{-1,\mathrm{nv}}$, see Sect. 1.7.6.

1.7.4. *Example.* Unwinding the construction, we obtain that Sat_G sends the object in Sph_G corresponding to the double coset of the point t^λ (for $\lambda \in \Lambda^+$) to the object

$$\mathrm{nv}(V^\lambda) \in \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}.$$

The above object object in Sph_G is what is usually denoted by

$$\mathrm{IC}_{\overline{\mathrm{Gr}}^\lambda_G},$$

the intersection cohomology sheaf on the closure of the $\mathfrak{L}^+(G)$-orbit Gr^λ of t^λ.

Remark 1.7.5. As in the case of Theorem 1.4.2, for the validity of Theorem 1.7.2 at the factorization level, it is crucial that we work with the twisted category

$$\text{D-mod}_{\frac{1}{2}}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G))$$

rather than with $\text{D-mod}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G))$.

1.7.6. In what follows, we will denote by $\mathrm{Sat}_G^{-1,\mathrm{nv}}$ the functor

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{nv}}{\to} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \overset{\mathrm{Sat}^{-1}}{\to} \mathrm{Sph}_G.$$

Remark 1.7.7. Note, for example that the functor

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\longrightarrow} \mathrm{Sph}_G \overset{-\star\mathrm{Vac}_{\mathrm{Whit}^!(G)}}{\longrightarrow} \mathrm{Whit}^!(G)$$

is CS_G^{-1}.

The functor

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\longrightarrow} \mathrm{Sph}_G \overset{\sigma}{\to} \mathrm{Sph}_G \overset{-\star\mathrm{Vac}_{\mathrm{Whit}_*(G)}}{\longrightarrow} \mathrm{Whit}_*(G)$$

is $\mathrm{FLE}_{\check{G},\infty}$.

1.8. **The curse of σ and τ.**

1.8.1. The following statement results from the uniqueness assertion in Theorem 1.7.2 combined with Lemma 1.4.12:

Corollary 1.8.2. *The following diagram of anti-equivalences commutes:*

$$
\begin{tikzcd}
\mathrm{Sph}_G \arrow[r, "\mathrm{Sat}_G"] \arrow[d, "\sigma"'] & \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \arrow[dd, "\sigma^{\mathrm{spec}}"] \\
\mathrm{Sph}_G \arrow[d, "\tau_G"'] & \\
\mathrm{Sph}_G \arrow[r, "\mathrm{Sat}_G"] & \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}.
\end{tikzcd}
$$

1.8.3. Denote by $\mathrm{Sat}_{G,\tau}$ the (factorization) equivalence

$$\mathrm{Sph}_G \xrightarrow{\tau} \mathrm{Sph}_G \xrightarrow{\mathrm{Sat}_G} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}.$$

Denote by $\mathrm{Sat}^{-1,\mathrm{nv}}_{G,\tau}$ the functor

$$\tau_G \circ \mathrm{Sat}^{-1,\mathrm{nv}}_G, \quad \mathrm{Rep}(\check{G}) \to \mathrm{Sph}_G.$$

1.8.4. As another corollary of Lemma 1.4.12 we obtain:

Corollary 1.8.5. *The equivalence*

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G)$$

is compatible with the actions of Sph_G *and* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ *via* $\mathrm{Sat}_{G,\tau}$*.*

1.8.6. *Warning.* As has been mentioned above, we will use σ (resp., σ^{spec}) to pass between left and right module categories for Sph_G (resp., $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$).

Note, however, that due to Corollary 1.8.2, this procedure is compatible with the geometric Satake equivalence *up to* the Chevalley involution.

In practice, this will manifest itself as follows. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ (resp., $\mathbf{C}_1^{\mathrm{spec}}$ and $\mathbf{C}_2^{\mathrm{spec}}$) be left module categories over Sph_G (resp., $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$). Thanks to the above left-right passage, we can form the tensor products

$$\mathbf{C}_1 \underset{\mathrm{Sph}_G}{\otimes} \mathbf{C}_2 \text{ and } \mathbf{C}_1^{\mathrm{spec}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathbf{C}_2^{\mathrm{spec}}.$$

Suppose that we have a given a functor

$$F_1 : \mathbf{C}_1 \to \mathbf{C}_1^{\mathrm{spec}},$$

which is compatible with the actions via

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_G}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \tag{1.12}$$

and a functor

$$F_2 : \mathbf{C}_2 \to \mathbf{C}_2^{\mathrm{spec}},$$

which is compatible with the actions via

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_{G,\tau}}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}. \tag{1.13}$$

In this case, we obtain a functor

$$F_1 \otimes F_2 : \mathbf{C}_1 \underset{\mathrm{Sph}_G}{\otimes} \mathbf{C}_2 \simeq \mathbf{C}_1^{\mathrm{spec}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathbf{C}_2^{\mathrm{spec}}. \tag{1.14}$$

1.8.7. *Warning.* Similarly, let $\mathbf{C}$ and $\mathbf{C}'$ be left module categories over Sph_G and $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$, respectively. Let us view $\mathbf{C}^\vee$ (resp., $\mathbf{C}'^\vee$) again as a left module, using σ (resp., σ^{spec}).

Let $\mathbf{C} \simeq \mathbf{C}'$ be an equivalence compatible with the actions via (1.12). Then the induced equivalence

$$\mathbf{C}^\vee \simeq \mathbf{C}'^\vee$$

is compatible with the actions via (1.13).

2. Kac-Moody modules and the Kazhdan-Lusztig category

In this section we study the local representation-theoretic category on the geometric side, which we will later connect to the global category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ by a local-to-global procedure.

The category in question is the Kazhdan-Lusztig category at the critical level,

$$\mathrm{KL}(G)_{\mathrm{crit}} := \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(G)}_{\mathrm{crit}}.$$

We will need the following aspects of the theory associated with $\mathrm{KL}(G)_{\mathrm{crit}}$:

- Self-duality;
- The functor of Drinfeld-Sokolov reduction.

2.1. **Definition and basic properties.**

2.1.1. Let κ be a level for $\mathfrak{g}$. We consider

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa,$$

the category of Kac-Moody modules at level κ. This category carries a natural action of $\mathfrak{L}(G)$ at level κ.

The definition of this category at a fixed point $\underline{x} \in \mathrm{Ran}$ is given in [Ra5]. The factorization version is defined in Sect. B.14.

2.1.2. Let

$$\mathrm{KL}(G)_\kappa := \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(G)}_\kappa,$$

denote the corresponding category of spherical objects.

We have an adjunction

$$\mathbf{oblv}_{\mathfrak{L}^+(G)} : \mathrm{KL}(G)_\kappa \rightleftarrows \widehat{\mathfrak{g}}\text{-mod}_\kappa : \mathrm{Av}^{\mathfrak{L}^+(G)}_* .$$

2.1.3. We have a monadic adjunction

$$\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} : \mathrm{Rep}(\mathfrak{L}^+(G)) \rightleftarrows \mathrm{KL}(G)_\kappa : \mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}. \tag{2.1}$$

In particular, $\mathrm{KL}(G)_\kappa$ is compactly generated by the image of compact generators of $\mathrm{Rep}(\mathfrak{L}^+(G))$, where the latter is *by definition* the ind-completion of the small category consisting of finite-dimensional representations.

2.1.4. Let $\mathrm{Vac}(G)_\kappa$ denote the factorization unit in $\mathrm{KL}(G)_\kappa$. By by a slight abuse of notation, we will denote by the same symbol $\mathrm{Vac}(G)_\kappa$ its image under the (strictly unital) factorization functor $\mathbf{oblv}_{\mathfrak{L}^+(G)}$.

We let $\mathbb{V}_{\mathfrak{g},\kappa}$ denote the image of $\mathrm{Vac}(G)_\kappa$ under the tautological forgetful functor

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa \to \mathrm{Vect}\,.$$

The latter is the usual factorization (a.k.a. chiral) algebra attached to the pair $(\mathfrak{g}, \kappa)$.

2.1.5. Our primary interest in this paper is the case when $\kappa = -\frac{1}{2} \cdot \mathrm{Kil}$, where Kil is the Killing form. We will denote the corresponding level by the symbol crit.

By construction, the category $\mathrm{KL}(G)_{\mathrm{crit}}$ carries a monoidal action of Sph_G.

2.2. **Duality.**

2.2.1. For a given level κ, denote
$$\kappa' := -\kappa + 2 \cdot \mathrm{crit}\,.$$

(In particular, $\mathrm{crit}' = \mathrm{crit}$.)

2.2.2. It is known that the categories
$$\widehat{\mathfrak{g}}\text{-mod}_\kappa \text{ and } \widehat{\mathfrak{g}}\text{-mod}_{\kappa'}$$
are canonically dual to one another, in a way compatible with the (unital) factorization structure and the $\mathfrak{L}(G)$-action, see [Ra5, Sect. 9.16.1].

The counit of the duality is the functor
$$\widehat{\mathfrak{g}}\text{-mod}_\kappa \otimes \widehat{\mathfrak{g}}\text{-mod}_{\kappa'} \overset{\otimes}{\to} \widehat{\mathfrak{g}}\text{-mod}_{-\mathrm{Kil}} \to \mathrm{Vect},$$
where the second arrow is the functor of semi-infinite cohomology.

By Sect. C.11.5, the above functor pairing has a structure of (lax unital[16]) factorization functor.

2.2.3. The above duality induces a duality between
$$(\mathrm{KL}(G)_\kappa)^\vee \simeq \mathrm{KL}(G)_{\kappa'}, \tag{2.2}$$
so that
$$\left(\mathbf{oblv}_{\mathfrak{L}^+(G)}\right)^\vee \simeq \mathrm{Av}_*^{\mathfrak{L}^+(G)} \text{ and } \left(\mathrm{Av}_*^{\mathfrak{L}^+(G)}\right)^\vee \simeq \mathbf{oblv}_{\mathfrak{L}^+(G)}.$$

The unit of the duality (2.2) is the object
$$\mathfrak{CDO}(G)_{\kappa,\kappa'} \in \mathrm{KL}(G)_\kappa \otimes \mathrm{KL}(G)_{\kappa'}.$$

Under this duality and the canonical self-duality of $\mathrm{Rep}(\mathfrak{L}^+(G))$, we have
$$\left(\mathbf{ind}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}\right)^\vee \simeq \mathbf{oblv}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa} \text{ and } \left(\mathbf{oblv}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}\right)^\vee \simeq \mathbf{ind}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}.$$

2.2.4. Specializing to the critical level, we obtain a canonical self-duality
$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^\vee \simeq \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}, \tag{2.3}$$
compatible with the $\mathfrak{L}(G)$-actions, and
$$(\mathrm{KL}(G)_{\mathrm{crit}})^\vee \simeq \mathrm{KL}(G)_{\mathrm{crit}}, \tag{2.4}$$
compatible with the Sph_G-actions.

2.3. **The functor of Drinfeld-Sokolov reduction.**

2.3.1. The duality in Sect. 2.2.2 is applicable to *any* finite-dimensional Lie algebra (where the role of the level $2 \cdot \mathrm{crit}$ is played by the Tate extension). In particular, for a unipotent Lie algebra $\mathfrak{n}'$, the corresponding category $\mathfrak{L}(\mathfrak{n}')$ is canonically self-dual, in a way compatible with the $\mathfrak{L}(N')$-action.

This construction is functorial. Hence, if a group H acts on $\mathfrak{n}'$, and $\mathcal{P}_H$ is an H-torsor on X, we obtain a canonical self-duality on $\mathfrak{L}(\mathfrak{n}')_{\mathcal{P}_H}$-mod.

In the particular case $N' = N$, $H = T$ and $\mathcal{P}_H = \rho(\omega_X)$, we obtain a canonical duality on
$$\mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod}.$$

[16]See Sect. C.11.3 for what "lax unital" means.

2.3.2. The character χ (see Sect. 1.3.2) on the group $\mathfrak{L}(N)_{\rho(\omega_X)}$ gives rise to a chracter (denoted by the same symbol)

$$\mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)} \to k.$$

We can regard this character as an object

$$k_\chi \in \mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod}.$$

The factorization property of χ equips k_χ with a structure of *factorization algebra* in $\mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}$-mod, where the latter is regarded as a lax factorization category.

The fact that $\chi|_{\mathfrak{L}(\mathfrak{n})^+_{\rho(\omega_X)}} = 0$ implies that this factorization algebra is naturally unital.

2.3.3. We define the functor

$$\mathrm{BRST}_{\mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)},\chi} : \mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod} \to \mathrm{Vect} \tag{2.5}$$

to be given by

$$\mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod} \overset{\mathrm{Id}\otimes k_\chi}{\longrightarrow} \mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod} \otimes \mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod} \to \mathrm{Vect},$$

where the second arrow is the counit of the self-duality on $\mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}$-mod.

By the above, the functor (2.5) has a natural (lax unital) factorization structure.

2.3.4. Precomposing with

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)} \to \mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod},$$

we obtain a functor of Drinfeld-Sokolov reduction, which we denote by

$$\mathrm{DS} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)} \to \mathrm{Vect}\,. \tag{2.6}$$

The functor (2.6) inherits a (lax unital) factorization structure.

2.3.5. It follows from the construction that the functor DS factors as

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)}) \to \mathrm{Vect}, \tag{2.7}$$

We denote the resulting functor

$$\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)}) \to \mathrm{Vect}$$

by

$$\overline{\mathrm{DS}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)}) \to \mathrm{Vect}\,. \tag{2.8}$$

Remark 2.3.6. The category $\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)})$ and the functor DS are canonically independent of the choice of the character χ_0 of $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$. This happens by the same mechanism as in Remark 1.3.10: the action of the center of the derived group of G on $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ is trivial.

2.3.7. In the sequel, we will use the following assertion (see [FG2, Theorem 3.2.2]):

Lemma 2.3.8. *For a fixed* $\underline{x} \in \mathrm{Ran}$*, the functor*

$$\mathrm{KL}(G)_{\kappa,\rho(\omega_X),\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}} \overset{\mathrm{DS}}{\to} \mathrm{Vect}$$

is t-exact.

Remark 2.3.9. One can show that the assertion of Lemma 2.3.8 holds also in the factorization setting: the proof of [Ra2, Corollary 7.2.2] given at the end of Sect. B.3 of *loc. cit.* adapts to the factorization setting.

3. Ind-coherent sheaves on monodromy-free opers

In this section we study the local counterpart of the Kazhdan-Lusztig category on the spectral side: this is the category

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$$

of ind-coherent sheaves on the space of monodromy-free $\check{G}$-opers on the punctured disc. This category will be related to the global spectral category (in this case $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$) by a local-to-global procedure.

We study $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ along with its cousins, the factorization categories

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{reg}}_{\check{G}}) \text{ and } \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}}).$$

Since the geometric objects involved are *not* locally of finite type, the definition of $\mathrm{IndCoh}(-)$ on them is not automatic. However, thankfully, these objects turn out to be *(ind)-placid* (see Sect. A.9.1 for what this means), so we have well-behaved categories $\mathrm{IndCoh}^*(-)$ and $\mathrm{IndCoh}^!(-)$ attached to them.

3.1. **Monodromy-free opers.**

3.1.1. Let Y be a D-scheme over X. Recall the notion of $\check{G}$-oper on Y. This is a datum of a triple

$$(\mathcal{P}_{\check{B}}, \epsilon, \nabla),$$

where

- $\mathcal{P}_{\check{B}}$ is a $\check{B}$-bundle on $\mathcal{Y}$;
- ϵ is the identification between the induced $\check{T}$-bundle $\mathcal{P}_{\check{T}}$ with $\check{\rho}(\omega_X)|_Y := 2\check{\rho}(\omega^{\otimes \frac{1}{2}})|_Y$;
- ∇ is a connection *along* X on the induced $\check{G}$-bundle $\mathcal{P}_{\check{G}}$.

These data are supposed to satisfy the following compatibility condition:

The *incompatibilty* of $\mathcal{P}_{\check{B}}$ and ∇, which is an element

$$\nabla \bmod \check{\mathfrak{b}} \in (\check{\mathfrak{g}}/\check{\mathfrak{b}})_{\mathcal{P}_{\mathcal{G}}} \otimes \omega_X|_Y$$

belongs to

$$\mathrm{Fil}_{-1}(\check{\mathfrak{g}}/\check{\mathfrak{b}})_{\mathcal{P}_{\mathcal{G}}} \otimes \omega_X|_Y \subset (\check{\mathfrak{g}}/\check{\mathfrak{b}})_{\mathcal{P}_{\mathcal{G}}} \otimes \omega_X|_Y$$

(here $\mathrm{Fil}_{-1}(\check{\mathfrak{g}}/\check{\mathfrak{b}}) \subset \check{\mathfrak{g}}/\check{\mathfrak{b}}$ is the bottom piece of the principal filtration), and its evaluation by means of every negative simple root $-\alpha_i$ of $\check{\mathfrak{g}}$

$$-\alpha_i(\nabla \bmod \check{\mathfrak{b}}) \in -\alpha_i(\mathcal{P}_{\check{T}}) \otimes \omega_X|_Y \overset{\epsilon}{\simeq} -\alpha_i(\check{\rho}(\omega_X))|_Y \otimes \omega_X|_Y \simeq \mathcal{O}_Y$$

is the unit section.

3.1.2. A priori, opers form a D-prestack over X. However, one shows (see, e.g., Sect. 3.1.7) that it is actually an affine D-scheme over X.

3.1.3. We will denote the D-scheme of $\check{G}$-opers by $\mathrm{Op}_{\check{G}}$. Its fiber over a given point $x \in X$ is the scheme $\mathrm{Op}_{\check{G}}(\mathcal{D}_x)$ of $\check{G}$-opers on the formal disc $\mathcal{D}_x$ around x.

We will denote by the symbol

$$\mathrm{Op}^{\text{reg}}_{\check{G}} := \mathfrak{L}^+_\nabla(\mathrm{Op}_{\check{G}})$$

the corresponding factorization (affine) scheme (see Sect. B.4.2), i.e., its fiber $\mathrm{Op}^{\text{reg}}_{\check{G},\underline{x}}$ over a given $\underline{x} \in \mathrm{Ran}$ is the scheme $\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}})$ of $\check{G}$-opers on the formal multi-disc $\mathcal{D}_{\underline{x}}$ around $\underline{x}$.

We let

$$\mathrm{Op}^{\text{mer}}_{\check{G}} := \mathfrak{L}_\nabla(\mathrm{Op}_{\check{G}})$$

denote the factorization ind-scheme of $\check{G}$-opers on the formal punctured disc (see Sect. B.4.6). Its fiber $\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}}$ over a given $\underline{x} \in \mathrm{Ran}$ is the ind-scheme $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})$ of $\check{G}$-opers on the *punctured* multi-disc $\mathcal{D}^\times_{\underline{x}}$.

3.1.4. We recall the following basic fact about opers:

Once the ambient curve X is fixed, we can assume that the G-bundle underlying an oper (on X itself, a multi-disc in X, or a punctured multi-disc in X) is induced from a fixed B-bundle, to be denoted $\mathcal{P}^{\mathrm{Op}}_{\check{B}}$ (see [BD1, Proposition 3.1.10(iii)]).

In what follows we will denote by $\mathcal{P}^{\mathrm{Op}}_{\check{G}}$ the induced $\check{G}$-bundle.

3.1.5. By construction, we have a map

$$\mathrm{Op}^{\mathrm{reg}}_{\check{G}} \to \mathrm{LS}^{\mathrm{reg}}_{\check{G}},$$

to be denoted $\mathfrak{r}^{\mathrm{reg}}$.

Note now that thanks to Sect. 3.1.4 we also have a map[17]

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G}} \to \mathrm{LS}^{\mathrm{mer}}_{\check{G}},$$

to be denoted $\mathfrak{r}$.

We have a commutative *but non-Cartesian* diagram

$$\begin{tikzcd}
\mathrm{Op}^{\mathrm{reg}}_{\check{G}} \arrow[r, "\iota"] \arrow[d, "\mathfrak{r}^{\mathrm{reg}}"'] & \mathrm{Op}^{\mathrm{mer}}_{\check{G}} \arrow[d, "\mathfrak{r}"] \\
\mathrm{LS}^{\mathrm{reg}}_{\check{G}} \arrow[r] & \mathrm{LS}^{\mathrm{mer}}_{\check{G}} .
\end{tikzcd}$$

3.1.6. We define the factorization ind-scheme of *monodromy-free opers* as the fiber product

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}} := \mathrm{LS}^{\mathrm{reg}}_{\check{G}} \underset{\mathrm{LS}^{\mathrm{mer}}_{\check{G}}}{\times} \mathrm{Op}^{\mathrm{mer}}_{\check{G}}.$$

I.e., for a fixed $\underline{x} \in \mathrm{Ran}$, the fiber $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$ is the fiber product

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} := \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} \underset{\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}.$$

Denote by $\iota^{\text{mon-free}}$ and $\iota^{+,\text{mon-free}}$ the resulting maps

$$\mathrm{Op}^{\mathrm{reg}} \overset{\iota^{+,\text{mon-free}}}{\longrightarrow} \mathrm{Op}^{\text{mon-free}}_{\check{G}} \overset{\iota^{\text{mon-free}}}{\to} \mathrm{Op}^{\mathrm{mer}}_{\check{G}}.$$

3.1.7. Recall also that the D-scheme $\mathrm{Op}_{\check{G}}$ is acted on simply transitively by the D-scheme $\mathrm{Jets}(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$ (see Sect. B.5.1) of jets into $\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}$, where $\mathfrak{a}(\check{\mathfrak{g}}) \subset \check{\mathfrak{g}}$ is the centralizer of a regular nilpotent element, and the twist by ω_X is performed with respect to the canonical $\mathbb{G}_m$-action on $\mathfrak{a}(\check{\mathfrak{g}})$ (see, e.g., [BD1, Sect. 3.1.9]).

From here we obtain that $\mathrm{Op}^{\mathrm{reg}}_{\check{G}}$ (resp., $\mathrm{Op}^{\mathrm{mer}}_{\check{G}}$) is acted on simply-transitively by $\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$ (resp., $\mathfrak{L}(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$).

3.1.8. Recall now the notion of *formal smoothness* of a prestack (see, e.g., [GaRo1, Sect. 8.1]). This notion has an evident relative variant.

We record the following (well-known) assertion:

Lemma 3.1.9. *The morphism $\mathfrak{r} : \mathrm{Op}^{\mathrm{mer}}_{\check{G}} \to \mathrm{LS}^{\mathrm{mer}}_{\check{G}}$ is formally smooth.*

Proof. We will show that for any classical affine scheme S and a map $\sigma : S \to \mathrm{Op}^{\mathrm{mer}}_{\check{G}}$ relative procotangent sheaf

$$T^*_\sigma(\mathrm{Op}^{\mathrm{mer}}_{\check{G}} / \mathrm{LS}^{\mathrm{mer}}_{\check{G}}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0}) \tag{3.1}$$

is a *Tate* vector bundle. This would imply the assertion of the lemma by [GaRo1, Proposition 8.2.2].

[17] We alert the reader to the fact that $\mathrm{LS}^{\mathrm{mer}}_{\check{G}}$ is *not* the space of loops into $\mathrm{pt}/\check{G}$, see Sect. B.7.9.

In order to simplify the notation we will assume that $S = \mathrm{pt}$ (and in particular, we work over a fixed point $\underline{x} \in \mathrm{Ran}$). However, we will perform the analysis in such a way that it will be clear that it works in families.

Let $\mathcal{P}^{\mathrm{Op}}_{\check{B}}$ be as in Sect. 3.1.4. We can represent the tangent space to $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$ at σ is

$$\mathrm{coFib}\left((\check{\mathfrak{n}} \otimes \mathcal{O}_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}} \overset{\nabla_\sigma}{\to} (\check{\mathfrak{b}} \otimes \omega_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}}\right),$$

where:

- $\mathcal{O}_{\mathcal{D}^\times_{\underline{x}}}$ is the ring of functions on the punctured multi-disc $\mathcal{D}^\times_{\underline{x}}$;
- $\omega_{\mathcal{D}^\times_{\underline{x}}}$ is the space of 1-forms on $\mathcal{D}^\times_{\underline{x}}$;
- The notation $(-)_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}}$ indicates the twist by $\mathcal{P}^{\mathrm{Op}}_{\check{B}}$, viewed as a $\check{B}$-bundle on $\mathcal{D}^\times_{\underline{x}}$;
- ∇_σ is the connection defined by σ.

The tangent space to $\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}}$ at the image of σ is

$$\mathrm{coFib}\left((\check{\mathfrak{g}} \otimes \mathcal{O}_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{G}}} \overset{\nabla_\sigma}{\to} (\check{\mathfrak{g}} \otimes \omega_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{G}}}\right). \tag{3.2}$$

Hence, the relative tangent space along $\mathfrak{r}$ is

$$\mathrm{Fib}\left((\check{\mathfrak{g}}/\check{\mathfrak{n}}) \otimes \mathcal{O}_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}} \overset{\nabla_\sigma}{\to} ((\check{\mathfrak{g}}/\check{\mathfrak{b}}) \otimes \omega_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}}\right).$$

Thus, choosing a non-degenerate invariant form on $\check{\mathfrak{g}}$, we can identify the *cotangent* space at σ again with[18]

$$\mathrm{coFib}\left((\check{\mathfrak{n}} \otimes \mathcal{O}_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}} \overset{\nabla_\sigma}{\to} (\check{\mathfrak{b}} \otimes \omega_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}}\right). \tag{3.3}$$

Thus, we need to show that (3.3) is indeed a Tate vector bundle (in degree 0). Consider the principal filtration on $\check{\mathfrak{g}}$, and the induced filtrations on $\check{\mathfrak{n}}$ and $\check{\mathfrak{b}}$. The map in (3.3) sends

$$\mathrm{Fil}_i(\check{\mathfrak{n}} \otimes \mathcal{O}_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}} \overset{\nabla_\sigma}{\to} \mathrm{Fil}_{i-1}(\check{\mathfrak{b}} \otimes \omega_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}}.$$

It suffices to show that for every i,

$$\mathrm{coFib}\left(\mathrm{gr}_i(\check{\mathfrak{n}} \otimes \mathcal{O}_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}} \overset{\nabla_\sigma}{\to} \mathrm{gr}_{i-1}(\check{\mathfrak{b}} \otimes \omega_{\mathcal{D}^\times_{\underline{x}}})_{\mathcal{P}^{\mathrm{Op}}_{\check{B}}}\right) \tag{3.4}$$

is a Tate vector bundle (in degree 0).

However, the latter is evident. In fact, the maps in (3.4) are independent of σ, and the assertion follows fro the fact that the maps

$$\mathrm{gr}_i(\check{\mathfrak{n}}) \overset{\mathrm{ad}_f}{\to} \mathrm{gr}_{i-1}(\check{\mathfrak{b}})$$

are injective (where f is a negative principal nilpotent, fixed as an element in $\check{\mathfrak{g}}/\check{\mathfrak{b}}$).

□

3.1.10. As a corollary of lemmas 3.1.9 and B.7.7, we obtain:

Corollary 3.1.11. *The ind-scheme* $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$ *is formally smooth.*[19]

3.2. **The** IndCoh^* **categories.** In this subsection, for expositional purposes, we will work over a fixed point $\underline{x} \in \mathrm{Ran}$. However, the entire discussion works when $\underline{x}$ forms a family over Ran.

---

[18] The fact that the tangent space and the relative cotangent space to opers are isomorphic is no coincidence: it reflects the interaction of the Poisson structure on $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$ and the symplectic structure on $\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}}$. In fact, the morphism $\mathfrak{r}$ is Lagrangian.

[19] See Sect. B.1.9 for what formal smoothness means in the factorization setting.

3.2.1. First, since $\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}$ (resp., $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$, $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$) is a scheme (resp., ind-scheme), we have a priori defined categories $\mathrm{IndCoh}^*(-)$ and $\mathrm{IndCoh}^!(-)$ attached to them, see Sects. A.4 and A.5 (the Ran space version is discussed in Sects. B.13.16 and B.13.22).

3.2.2. Using Sect. 3.1.7, we can write

$$\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}} \simeq \lim_{\mathbf{L}} \mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}/\mathbf{L},$$

where $\mathbf{L}$ runs over the filtered poset of lattices in $\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}$, viewed as a Tate vector space.

This exhibits $\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}$ as a limit of smooth schemes with smooth transition maps.

3.2.3. In particular, we obtain that $\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}$ is *placid* (see Sect. A.9.1 for what this means), so that the categories

$$\mathrm{IndCoh}_*(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}) \text{ and } \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}})$$

are well-behaved; in particular, they are both compactly generated and are mutually dual.

Note, however, that since $\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}$ is pro-smooth, the coarsening functor

$$\Psi_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}) \to \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}})$$

is an equivalence, as is the functor

$$\Upsilon_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}} : \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}).$$

3.2.4. We can identify

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}} \simeq (\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}} \times \mathfrak{L}(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}})/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}.$$

In particular, we have a pro-smooth projection

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}} \to \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}} \simeq \mathfrak{L}(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}.$$

This exhibits $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$ as an *ind-placid* ind-scheme (see Sect. A.9.8 for what this means).

3.2.5. This ensures that the categories

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \text{ and } \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$$

are well-behaved; in particular, they are both compactly generated and are mutually dual.

3.2.6. Note that the map

$$\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} \to \mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}}$$

is an ind-closed embedding, locally almost of finite presentation (see Lemma B.7.13).

This implies that

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \overset{\iota^{\text{mon-free}}}{\to} \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$$

is also an ind-closed embedding locally almost of finite presentation. In particular, since $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$ is ind-placid, we obtain that $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$ is also ind-placid (see Corollary A.9.10).

Hence, we obtain that the categories

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \text{ and } \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

are also well-behaved; in particular, they are both compactly generated and are mutually dual.

3.2.7. We will now show how one can explicitly exhibit $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$ as a colimit of placid schemes.

To simplify the notation, we will assume that $\underline{x}$ consists of a single point x. Henceforth, we will omit x from the subscript, so we will write $\mathcal{D}$ instead of $\mathcal{D}_{\underline{x}}$.

Recall (see Sect. 3.1.4) that opers can be thought of as connections on a fixed $\check{G}$-bundle $\mathcal{P}^{\mathrm{Op}}_{\check{G}}$. Trivializing this bundle on $\mathcal{D}$, we will think of opers as connection forms, to be denoted α.

Thus, we can write $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)$ as

$$\{(\alpha \in \mathrm{Op}_{\check{G}}(\mathcal{D}^\times), g \in \mathfrak{L}(\check{G})) \,|\, g\cdot\alpha \in \check{\mathfrak{g}} \otimes \omega_{\mathcal{D}^\times_{\underline{x}}}\}/\mathfrak{L}^+(\check{G}) \subset \mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{Gr}_{\check{G}}, \tag{3.5}$$

Therefore, we can write $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)$ as[20]

$$\text{``}\underset{Y}{\mathrm{colim}}\text{''}\, \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \underset{\mathrm{Gr}_{\check{G}}}{\times} Y,$$

where Y runs over the filtered poset of closed subschemes of $\mathrm{Gr}_{\check{G}}$.

Let us show that each $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \underset{\mathrm{Gr}_{\check{G}}}{\times} Y$ is a limit of schemes almost of finite type with smooth transition maps.

Let us consider $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ as acted on by $\mathfrak{L}(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$. Now, it is clear that for a fixed closed subscheme $Y \subset \mathrm{Gr}_{\check{G}}$, there is an action of *any* small enough lattice $\mathbf{L} \subset \mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$ on

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \underset{\mathrm{Gr}_{\check{G}}}{\times} Y$$

via

$$(\alpha, g) \mapsto (\alpha + \alpha_0, g), \quad \alpha_0 \in \mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}).$$

Further, it easy to see that for any such $\mathbf{L}$, the quotient of (3.5) by it is locally almost of finite type.

3.3. Properties of the $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ functor.

3.3.1. The map $\iota^{\text{mon-free}}$ gives rise to the IndCoh-pushforward functor

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}). \tag{3.6}$$

The functor $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ is t-exact with respect to the natural t-structures.

Note that since $\iota^{\text{mon-free}}$ is a closed embedding locally almost of finite presentation, the right adjoint $(\iota^{\text{mon-free}})^!$ of $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ is continuous, in particular, the functor $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ preserves compactness, see Sect. A.10.11.

3.3.2. By the same logic, the functor

$$(\iota^{\text{mon-free}})^! : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

admit a left adjoint, to be denoted also by

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}).$$

(We allow ourselves to use the same symbol $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ in both instances, as the two are unlikely to be confused.)

[20] In the next formula, the symbol "colim" (i.e., with quotes) refers to the fact that we are forming an ind-scheme, rather than taking the colimit in the category of schemes.

3.3.3. The adjoint pairs

$$(\iota^{\text{mon-free}})_*^{\text{IndCoh}} : \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \rightleftarrows \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}}) : (\iota^{\text{mon-free}})^!$$

and

$$(\iota^{\text{mon-free}})_*^{\text{IndCoh}} : \text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \rightleftarrows \text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G},\underline{x}}) : (\iota^{\text{mon-free}})^!$$

are dual to one another with respect to the identifications

$$\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}})^\vee \simeq \text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

and

$$\text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\vee \simeq \text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G},\underline{x}}).$$

3.3.4. We will prove:

Proposition 3.3.5.

(a) *The functor* (3.6) *is conservative.*

(b) *An object of* $\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ *is compact if (and only if) its image under* $(\iota^{\text{mon-free}})_*^{\text{IndCoh}}$ *is compact.*

The rest of this subsection is devoted to the proof of this proposition.

3.3.6. Let

$$(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}$$

denote the formal completion of $\text{Op}^{\text{mer}}_{\check{G},\underline{x}}$ along $\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$.

Since $\iota^{\text{mon-free}}$ is (locally) almost of finite presentation, so is the embedding

$$(\iota^{\text{mon-free}})^\wedge : (\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}} \hookrightarrow \text{Op}^{\text{mer}}_{\check{G},\underline{x}}.$$

In particular, $(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}$ is ind-placid, and we have a well-behaved category

$$\text{IndCoh}^*((\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}).$$

3.3.7. The functor

$$((\iota^{\text{mon-free}})^\wedge)_*^{\text{IndCoh}} : \text{IndCoh}^*((\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}) \to \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})$$

gives rise to an equivalence

$$\text{IndCoh}^*((\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}) \overset{\sim}{\to} \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})_{\text{mon-free}},$$

where

$$\text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})_{\text{mon-free}} \subset \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})$$

is the full subcategory of objects with set-theoretic support on $\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$.

Furthermore, the functor $((\iota^{\text{mon-free}})^\wedge)_*^{\text{IndCoh}}$ admits a continuous right adjoint, to be denoted $((\iota^{\text{mon-free}})^\wedge)^!$, so $((\iota^{\text{mon-free}})^\wedge)_*^{\text{IndCoh}}$ preserves compactness.

3.3.8. The functor $(\iota^{\text{mon-free}})_*^{\text{IndCoh}}$ of (3.6) factors as

$$\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \text{IndCoh}^*((\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}) \overset{((\iota^{\text{mon-free}})^\wedge)_*^{\text{IndCoh}}}{\longrightarrow} \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}}),$$

and in order to prove Proposition 3.3.5, it is enough to establish the corresponding properties for the above functor

$$('\iota^{\text{mon-free}})_*^{\text{IndCoh}} : \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \text{IndCoh}^*((\text{Op}^{\text{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}). \tag{3.7}$$

3.3.9. Recall that according to Corollary 3.1.11, the ind-scheme $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$ is formally smooth. Hence, its embedding into any nilpotent thickening[21] admits a retraction. This implies that the embedding

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \hookrightarrow (\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{mon-free}} \tag{3.8}$$

admits a retraction

$$(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{mon-free}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}. \tag{3.9}$$

3.3.10. The existence of the retraction (3.9) readily implies that $({}'\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ is conservative:

Indeed, the functor of IndCoh-pushforward along (3.9) is a left inverse of $({}'\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$.

This proves point (a) of Proposition 3.3.5.

Remark 3.3.11. Note that the above argument implies that the functor $({}'\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ of (3.7) is *co-monadic*. Indeed, according to what we just proved, it is conservative, and it admits a right adjoint. Hence, it remains to check that it preserves totalizations of $({}'\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$-split cosimplicial objects.

However, since $({}'\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ admits a left inverse, such cosimplicial objects are themselves split, and hence their totalizations are preserved by any functor.

3.3.12. Let $\mathcal{F} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ be an object, such that $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\mathcal{F})$ is compact. Let us show that $\mathcal{F}$ is itself compact.

Since $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\mathcal{F})$ is compact, it is cohomologically bounded, i.e., there exists an integer n such that the map

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\mathcal{F}) \to \tau^{\geq -n}((\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\mathcal{F}))$$

is an isomorphism.

Since the functor $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ is t-exact, we obtain that

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\mathcal{F}) \to (\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\tau^{\geq -n}(\mathcal{F}))$$

is an isomorphism.

However, since we already know that $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ is conservative, this implies that

$$\mathcal{F} \to \tau^{\geq -n}(\mathcal{F})$$

is an isomorphism, i.e., $\mathcal{F}$ is itself cohomologically bounded.

Hence, it remains to check that the individual cohomologies $H^i(\mathcal{F})$ of $\mathcal{F}$ are coherent. However,

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(H^i(\mathcal{F})) \simeq H^i((\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\mathcal{F})),$$

and it is easy to see that an object $\mathcal{F}^0 \in \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})^{\heartsuit}$ is coherent if and only if

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\mathcal{F}^0) \in \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\heartsuit}$$

is coherent (this is true for any closed embedding almost of finite presentation between ind-placid ind-schemes).

$\square$[Proposition 3.3.5]

Remark 3.3.13. The implication "$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*(\mathcal{F})$ is compact" $\Rightarrow$ "$\mathcal{F}$ is compact" can also be proved using the retraction (3.9): since this map is ind-finite, the functor IndCoh-pushforward along it preserves compactness.

Remark 3.3.14. Note that the existence of a retraction implies that the ind-scheme $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$ is *classical*. Indeed, the ind-scheme $\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}}$ is classical, and hence so is its formal completion $(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{mon-free}}$ (the latter follows from [GaRo1, Proposition 6.8.2]: we reduce to the Noetherian situation using placidity).

[21] Assuming all ind-schemes involved are $\aleph_0$.

Remark 3.3.15. The contents of this subsection apply "as-is" when $\underline{x}$ forms family over Ran. In particular, the functor $(\iota^{\text{mon-free}})_*^{\text{IndCoh}}$ has a natural factorization structure.

Moreover, when viewed as a functor between unital factorization categories, $(\iota^{\text{mon-free}})_*^{\text{IndCoh}}$ has a natural lax unital factorization structure.

We claim, however, that this unital structure is actually strict. Indeed, this follows from the fact that $(\iota^{\text{mon-free}})_*^{\text{IndCoh}}$ sends

$$\mathcal{O}_{\check{G}}^{\text{reg}} = \mathbf{1}_{\text{IndCoh}^*(\text{Op}_{\check{G}}^{\text{mon-free}})} \to \mathbf{1}_{\text{IndCoh}^*(\text{Op}_{\check{G}}^{\text{mer}})} = \mathcal{O}_{\check{G}}^{\text{reg}},$$

see Lemma C.11.23.

3.4. **A direct product decomposition.**

3.4.1. We will show that we actually have a (non-canonical) isomorphism

$$(\text{Op}_{\check{G},\underline{x}}^{\text{mer}})^\wedge_{\text{mon-free}} \simeq \text{Op}_{\check{G},\underline{x}}^{\text{mon-free}} \times (\check{\mathfrak{g}}_0^\wedge)^n, \quad |\underline{x}| = n \tag{3.10}$$

(here $\check{\mathfrak{g}}_0^\wedge$ is the formal completion of $\check{\mathfrak{g}}$ at 0), so that (3.8) identifies with the base change of

$$0 \to \check{\mathfrak{g}}_0^\wedge.$$

Remark 3.4.2. The material in this subsection is specific to the situation over a given $\underline{x} \in \text{Ran}$. I.e., we do not know what how to even formulate the corresponding statement over Ran (or even X^n for $n \geq 2$).

3.4.3. With no restriction of generality, we can assume that $\underline{x}$ consists of a single point x. Henceforth in this proof, we will drop the subscript "x" and simply write $\mathcal{D}$ instead of $\mathcal{D}_x$.

We have

$$\text{Op}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{mon-free}} \simeq \text{Op}_{\check{G}}(\mathcal{D}^\times) \underset{\text{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \text{LS}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{reg}},$$

where $\text{LS}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{reg}}$ is the formal completion of $\text{LS}_{\check{G}}(\mathcal{D}^\times)$ along $\text{LS}_{\check{G}}(\mathcal{D})$.

Note that we can identify

$$\text{LS}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{reg}} \simeq \check{\mathfrak{g}}_0^\wedge / \text{Ad}(\check{G}),$$

so that

$$\text{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}) \simeq \text{Op}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{mon-free}} \underset{\check{\mathfrak{g}}_0^\wedge / \text{Ad}(\check{G})}{\times} \text{pt} / \check{G}.$$

3.4.4. Note that the $\check{G}$-bundle on $\text{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)$ corresponding to the map

$$\text{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times) \xrightarrow{\mathfrak{r}} \text{LS}_{\check{G}}(\mathcal{D}) \simeq \text{pt} / \check{G}$$

can be (non-canonically) trivialized, see Sect. 3.1.4. I.e., the map

$$\text{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times) \to \text{Op}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{mon-free}} \xrightarrow{\mathfrak{r}} \text{LS}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{reg}} \simeq \check{\mathfrak{g}}_0^\wedge / \text{Ad}(\check{G}) \to \text{pt} / \check{G}$$

factors though a map

$$\text{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times) \to \text{pt}.$$

Hence, so does the map

$$\text{Op}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{mon-free}} \xrightarrow{\mathfrak{r}} \text{LS}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{reg}} \simeq \check{\mathfrak{g}}_0^\wedge / \text{Ad}(\check{G}) \to \text{pt} / \check{G}.$$

Hence, the map

$$\text{Op}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{mon-free}} \xrightarrow{\mathfrak{r}} \text{LS}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{reg}} \simeq \check{\mathfrak{g}}_0^\wedge / \text{Ad}(\check{G})$$

can be (non-canonically) lifted to a map

$$\text{Op}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{mon-free}} \to \check{\mathfrak{g}}_0^\wedge. \tag{3.11}$$

3.4.5. Combining the maps (3.9) and (3.11), we obtain a map

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)^\wedge_{\text{mon-free}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \times \check{\mathfrak{g}}_0^\wedge, \tag{3.12}$$

such that if we base change both sides with respect to $0 \to \check{\mathfrak{g}}_0^\wedge$, we obtain the identity map on $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)$.

Thus, the map (3.12) becomes an isomorphism after a base change by a nil-isomorphism. This implies that the map (3.12) is itself an isomorphism.

3.5. **The action of** $\mathrm{IndCoh}^!$**.** As in Sect. 3.2, for expositional purposes, we will work over a fixed point $\underline{x} \in \mathrm{Ran}$. However, the entire discussion works when $\underline{x}$ forms a family over Ran.

3.5.1. Recall that the category $\mathrm{IndCoh}^*(\mathcal{Y})$ of an ind-scheme $\mathcal{Y}$ is naturally acted on by $\mathrm{IndCoh}^!(\mathcal{Y})$. For a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, the corresponding functor $f_* : \mathrm{IndCoh}^*(\mathcal{Y}_1) \to \mathrm{IndCoh}^*(\mathcal{Y}_2)$ is $\mathrm{IndCoh}^!(\mathcal{Y}_2)$-linear, where $\mathrm{IndCoh}^!(\mathcal{Y}_2)$ acts on $\mathrm{IndCoh}^*(\mathcal{Y}_1)$ via $f^! : \mathrm{IndCoh}^!(\mathcal{Y}_2) \to \mathrm{IndCoh}^!(\mathcal{Y}_1)$, see Sect. A.6.6.

In particular, we obtain that the category $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$ (resp., $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$) is acted on by $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$ (resp., $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$), and the functor

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$$

is linear with respect to $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$.

3.5.2. Being the right adjoint of a $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linear functor, the functor

$$(\iota^{\text{mon-free}})^! : \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

is right-lax $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linear .

However, it is easy to see that this right-lax $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linearity structure is actually strict, i.e., the adjunction

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \rightleftarrows \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) : (\iota^{\text{mon-free}})^! \tag{3.13}$$

takes place in the 2-category of $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-module categories.

3.5.3. Since $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$ (resp., $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$) is *symmetric* monoidal, we can view the dual categories

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\vee \text{ and } \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})^\vee$$

as modules over $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$ and $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$, respectively.

Note that the identifications

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$$

and

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

are compatible with the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$- and $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$-actions, respectively.

Recall (see Sect. 3.3.2) that the dual of the adjunction (3.13) identifies with

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \rightleftarrows \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) : (\iota^{\text{mon-free}})^! \tag{3.14}$$

It is easy to see that the resulting $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linear structure on (3.14) arising from the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linear structure on (3.13) is the natural $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linear structure on

$$(\iota^{\text{mon-free}})^! : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}).$$

3.5.4. Being a $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linear, the functor

$$(\iota^{\text{mon-free}})^! : \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

induces a functor

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}). \tag{3.15}$$

We will prove:

Proposition 3.5.5. *The functor* (3.15) *is an equivalence.*

The proof will be given in Sect. 3.7.14.

3.5.6. Note that in addition to the functor (3.15), we have a tautologically defined functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}), \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})). \tag{3.16}$$

We claim:

Lemma 3.5.7. *The functor* (3.16) *is an equivalence.*

Proof. Recall that the category $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$ identifies with the dual of $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$, and this identification is compatible with the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-module structures. Therefore, for any $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-module category $\mathbf{C}$, we have

$$\mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})}(\mathbf{C}, \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})) \simeq \mathrm{Funct}(\mathbf{C}, \mathrm{Vect}),$$

where $\mathrm{Funct}(-,-)$ refers to colimit-preserving functors.

Applying this to $\mathbf{C} = \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$, we obtain that the right-hand side on (3.16) identifies with

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})^\vee \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}).$$

It is easy to see, however, that the endomorphism of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$, induced by (3.16) and the above identfication, is the identity functor.

□

3.6. **Action of the spherical category.** The discussion in this subsection will be specific to the situation when $\underline{x} \in \mathrm{Ran}$ is fixed. The generalization in the factorization setting will be discussed in Sect. E.8.

3.6.1. Let us write $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$ as

$$(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}} \underset{(\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\mathrm{reg}}}{\times} \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} \simeq (\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}} \underset{(\check{\mathfrak{g}}^\wedge_0 / \mathrm{Ad}(\check{G}))^{|\underline{x}|}}{\times} (\mathrm{pt} / \check{G})^{|\underline{x}|}.$$

From this presentation it is clear that

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}} \simeq \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} \underset{\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}}}{\times} \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}) \simeq \mathrm{IndCoh}(\mathrm{pt} / \check{G} \underset{\check{\mathfrak{g}}^\wedge_0 / \mathrm{Ad}(\check{G})}{\times} \mathrm{pt} / \check{G})^{\otimes |\underline{x}|}$$

acts on both

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \text{ and } \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}).$$

Moreover, these actions are $\mathrm{IndCoh}^!((\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}})$-linear, and hence $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linear.

3.6.2. The identification

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

is compatible with the structure of $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}, \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}))$-bimodule on the two sides.

3.6.3. It follows from the constructions that the functor (3.15) (resp., (3.16)) respects the $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$-actions on the two sides, where the action on the left-hand side of (3.15) (resp., right-hand side of (3.16)) is via the $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$-factor.

3.6.4. Furthermore, it again follows from the construction that the functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \overset{\otimes}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \overset{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})_{\text{mon-free}}$$

canonically factors via a functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})_{\text{mon-free}}. \tag{3.17}$$

We claim:

Proposition 3.6.5. *The functor* (3.17) *is an equivalence.*

Proof. Since the monoidal categories $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ and $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}})$ are rigid, the projection functor

$$\begin{aligned}\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \\ \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})\end{aligned} \tag{3.18}$$

admits a continuous right adjoint. Moreover, the corresponding adjunction is monadic.

The functor

$$((\iota^{\text{mon-free}})^\wedge)^!|_{\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})_{\text{mon-free}}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})_{\text{mon-free}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

is also monadic.

Hence, we need to show that the functor (3.17) induces an isomorphism between the two monads acting on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ as plain endofunctors.

It is easy to see that the composition of (3.18) with (3.17) is the functor

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})_{\text{mon-free}}.$$

This gives rise to a map between the two monads. Let us show that this map is indeed an isomorphism of the underlying endofunctors.

The monad corresponding to (3.18) is given by the action on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ by the (algebra) object in $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$, equal to the !-pullback of

$$\mathcal{O}_{\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}} \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}) \overset{\Psi_{\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}}}{\simeq} \mathrm{IndCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}})$$

along the first projection

$$p_1 : \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} \underset{\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}}}{\times} \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} \to \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}.$$

The monad corresponding to $(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*$ is given by !-pull followed by *-push along

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \leftarrow \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \underset{(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}}{\times} \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}.$$

However, it is easy to see that this functor is given by the action of the same object in $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$. □

3.7. **Self-duality for** IndCoh **on opers.**

3.7.1. First, we claim that there is a canonically defined equivalence

$$\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}), \tag{3.19}$$

compatible with the monoidal action of $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$ on both sides.

By $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$-linearity, the datum of a functor (3.19) is equivalent to a choice of an object in $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$.

The corresponding object, to be denoted

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}),$$

is constructed as follows.

3.7.2. Consider $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$ as equipped with an action of $\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}$, and note that the quotient

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}$$

is an ind-scheme of ind-finite type. In particular, we have a well-defined category

$$\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}),$$

and an object

$$\omega_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}} \in \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}).$$

The operation of *-pullback along

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}} \to \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}$$

is a well-defined functor

$$\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}). \tag{3.20}$$

We let $\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}$ be the image of $\omega_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}}$ under (3.20).

3.7.3. We claim:

Lemma 3.7.4. *The functor $\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}$ of* (3.19)*, defined by $\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}$, is an equivalence.*

Proof. We can write

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \simeq \underset{\mathbf{L}}{\mathrm{colim}}\, \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathbf{L})$$

(where the transition functors are given by !-pullback) and

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \simeq \underset{\mathbf{L}}{\mathrm{colim}}\, \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathbf{L})$$

(where the transition functors are given by *-pullback), and where the $\mathbf{L}$'s run over the poset of lattices in $\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}$.

The functor $\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}$ is given by the compatible family of (endo)functors

$$\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathbf{L}) \to \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}/\mathbf{L}),$$

each given by tensoring by the graded line

$$\det(\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}/\mathbf{L})[-\dim(\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})_{\underline{x}}/\mathbf{L})].$$

Since all these functors are equivalences, so is their colimit. □

Remark 3.7.5. We can combine the functor $\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}$ of (3.19) with the identification

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$$

and thus view it as the datum of self-duality on $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}})$.

Remark 3.7.6. By the same token, we can define the functor

$$\Theta_{\mathrm{Op}^{\mathrm{reg}}(\check{G})} : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}) \simeq \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}})$$

and show that it is an equivalence.

Note that the corresponding object

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}) \simeq \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}})$$

is $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}}$.

3.7.7. Let

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}})$$

be defined by

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}} := (\iota^{\mathrm{mon\text{-}free}})^!(\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}).$$

Let

$$\Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}), \tag{3.21}$$

be the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}})$-linear functor, corresponding to $\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}}$.

3.7.8. Note that the functor $\Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}}$ is rigged so that it makes the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \arrow[r, "(\iota^{\mathrm{mon\text{-}free}})^!"] \arrow[d, "\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}"'] & \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}) \arrow[d, "\Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}}"] \\
\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \arrow[r, "(\iota^{\mathrm{mon\text{-}free}})^!"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}})
\end{tikzcd} \tag{3.22}$$

commute.

In addition, it follows from Sect. 3.6.3 that the functor $\Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}}$ of (3.21) intertwines the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on the two sides.

3.7.9. We claim:

Proposition 3.7.10. *The functor* $\Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}}$ *of* (3.21) *is an equivalence.*

Proof. Write

$$\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}} = \text{“}\underset{i}{\mathrm{colim}}\text{”}\, Y_i^0 \text{ and } \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}} = \text{“}\underset{i}{\mathrm{colim}}\text{”}\, Y_i,$$

where Y_i^0 and Y_i are schemes, and the map $\iota^{\mathrm{mon\text{-}free}}$ is given by a compatible family of maps

$$Y_i^0 \overset{\iota_i^{\mathrm{mon\text{-}free}}}{\to} Y_i$$

almost of finite presentation. Moreover, we can choose Y_i so that its map to $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$ is almost of finite presentation. In this case both Y_i and Y_i^0 are placid.

Set

$$\omega^{*,\mathrm{fake}}_{Y_i^0} := (\iota_i^{\mathrm{mon\text{-}free}})^!(\omega^{*,\mathrm{fake}}_{Y_i}),$$

where $\omega^{\mathrm{fake},*}_{Y_i}$ is the !-restriction of $\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}$ along $Y_i \to \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$. Let

$$\Theta_{Y_i^0} : \mathrm{IndCoh}^!(Y_i^0) \to \mathrm{IndCoh}^*(Y_i^0)$$

be the $\mathrm{IndCoh}^!(Y_i^0)$-linear functor, defined by $\omega^{*,\mathrm{fake}}_{Y_i^0}$. We will show that each $\Theta_{Y_i^0}$ is an equivalence.

Write

$$Y_i = \lim_{\alpha} Y_{i,\alpha},$$

where Y_α are schemes almost of finite type, with smooth transition maps.

Since $\iota_i^{\text{mon-free}}$ is almost of finite presentation (up to truncation)[22], we can find an index α such that Y_i^0 fits into a Cartesian diagram

$$\begin{tikzcd} Y_i^0 \arrow[r, "\iota_i^{\text{mon-free}}"] \arrow[d, "\pi_\alpha^0"'] & Y_i \arrow[d, "\pi_\alpha"] \\ Y_{i,\alpha}^0 \arrow[r, "(\iota^{\text{mon-free}})_i^0"] & Y_{i,\alpha}. \end{tikzcd}$$

Furthermore, up to enlarging α, by the construction of $\omega^{*,\text{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}$, we can assume that

$$\omega^{\text{fake},*}_{Y_i} \simeq \pi_\alpha^*(\omega_{Y_{i,\alpha}} \otimes \mathcal{L}_{i,\alpha}),$$

where $\mathcal{L}_{i,\alpha}$ is a (comologically graded) line bundle on $Y_{i,\alpha}$. Hence,

$$\omega^{*,\text{fake}}_{Y_i^0} \simeq (\pi_\alpha^0)^*(\omega_{Y_{i,\alpha}^0} \otimes \mathcal{L}_{i,\alpha}^0),$$

where

$$\mathcal{L}_{i,\alpha}^0 := ((\iota^{\text{mon-free}})_i^0)^*(\mathcal{L}_{i,\alpha}).$$

Since the category of indices α is filtered, we can write

$$Y_i^0 \simeq \lim_{\beta\geq\alpha} Y_{i,\beta}^0, \quad Y_{i,\beta}^0 := Y_{i,\alpha}^0 \underset{Y_{i,\alpha}}{\times} Y_{i,\beta},$$

so that

$$\mathrm{IndCoh}^*(Y_i^0) \simeq \underset{\beta\geq\alpha}{\mathrm{colim}}\, \mathrm{IndCoh}(Y_{i,\beta}^0)$$

under *-pullbacks, and

$$\mathrm{IndCoh}^!(Y_i^0) \simeq \underset{\beta\geq\alpha}{\mathrm{colim}}\, \mathrm{IndCoh}(Y_{i,\beta}^0)$$

under !-pullbacks.

For each β, let $\mathcal{L}_{i,\beta}^0$ be the (canonically defined) line bundle on $Y_{i,\beta}^0$ so that

$$\omega_{Y_{i,\beta}^0} \otimes \mathcal{L}_{i,\beta}^0 \simeq (\pi_{\beta,\alpha}^0)^*(\omega_{Y_{i,\alpha}^0} \otimes \mathcal{L}_{i,\alpha}^0), \quad \pi_{\beta,\alpha}^0 : Y_{i,\beta}^0 \to Y_{i,\alpha}^0.$$

We obtain that the functor $\Theta_{Y_i^0}$ is given by the compatible system of (endo)functors

$$\mathrm{IndCoh}(Y_{i,\beta}^0) \overset{\mathcal{L}_{i,\beta}^0\otimes -}{\to} \mathrm{IndCoh}(Y_{i,\beta}^0),$$

which are all equivalences.

□

Remark 3.7.11. One could make the above proof more explicit by using the presentation of $\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ as in Sect. 3.2.7.

Remark 3.7.12. Note that combined with the identification

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}),$$

functor $\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}$ of (3.21) can be viewed as the datum of self-duality on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$.

3.7.13. As a first consequence, passing to the right adjoint functors along the horizontal arrows in (3.22), and knowing that the vertical arrows are equivalences, we obtain another commutative diagram

$$\begin{tikzcd} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \arrow[r, "(\iota^{\text{mon-free}})_*^{\mathrm{IndCoh}}"] \arrow[d, "\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}"'] & \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \arrow[d, "\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}"] \\ \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \arrow[r, "(\iota^{\text{mon-free}})_*^{\mathrm{IndCoh}}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}). \end{tikzcd}$$

[22] The issue of truncation is taken over by passing to the limit.

3.7.14. *Proof of Proposition 3.5.5.* From (3.22) we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}}) \arrow[r, "(\iota^{\text{mon-free}})^!"] \arrow[d, "\mathrm{Id}\otimes\Theta_{\mathrm{Op}^{\text{mer}}_{\check{G}}}"'] & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \arrow[d, "\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}"] \\
\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}}) \arrow[r, "(\iota^{\text{mon-free}})^!"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}).
\end{tikzcd}$$

The vertical arrows in this diagram are equivalences by Lemma 3.7.4 and Proposition 3.7.10, respectively. Since the top horizontal arrow is an equivalence, we obtain that so is the bottom horizontal arrow.

□[Proposition 3.5.5]

3.7.15. Let us observe now that once we know Proposition 3.5.5, we could view the construction of $\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}$ differently:

We start with the equivalence

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

and pass to dual categories. We obtain

$$\mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}), \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})) \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}). \tag{3.23}$$

Applying $\Theta_{\mathrm{Op}^{\text{mer}}_{\check{G}}}$, we replace the left-hand side in (3.23) by

$$\mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}), \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})),$$

and applying Lemma 3.5.7, we rewrite it further as $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$.

Thus, we can interpret (3.23) as an equivalence

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}). \tag{3.24}$$

It is easy to see, however, that (3.24) equals the (inverse of the) equivalence $\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}$ constructed above.

3.7.16. The following results from the definition of the $\mathrm{Sph}^{\text{spec}}_{\check{G},x}$-action on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ and $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ in Sect. 3.6.1:

Lemma 3.7.17. *The equivalence* (3.24) *is compatible with the* $\mathrm{Sph}^{\text{spec}}_{\check{G},x}$*-actions.*

3.7.18. All the preceding discussion in this subsection applies also in the factorization setting.

3.8. **Relation to quasi-coherent sheaves.** We now consider the relationship between ind-coherent and quasi-coherent sheaves on $\mathrm{Op}^{\text{mon-free}}_{\check{G}}$. We observe two pleasant categorical properties over a point and ask if they extend to the factorization setting.

3.8.1. For any prestack $\mathcal{Y}$ we have a canonically defined (symmetric monoidal) functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}^!(\mathcal{Y}).$$

It is known that if $\mathcal{Y}$ is a formally smooth ind-scheme locally almost of finite type, then $\Upsilon_{\mathcal{Y}}$ is an equivalence (see [GaRo1, Theorem 10.1.1])[23].

[23]This result was originally proved by J. Lurie.

3.8.2. Consider the functors

$$\Upsilon_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}} : \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \tag{3.25}$$

and

$$\Upsilon_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}} : \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}), \tag{3.26}$$

respectively.

3.8.3. First, we claim:

Lemma 3.8.4. *The functor $\Upsilon_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}$ is an equivalence.*

Proof. Repeats that of Lemma 3.7.4. □

Remark 3.8.5. Both the statement and the proof of Lemma 3.8.4 carry over to the factorization setting.

3.8.6. We now claim:

Proposition 3.8.7. *For a fixed $\underline{x} \in \mathrm{Ran}$, the functor $\Upsilon_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}$ is an equivalence.*

Proof. We will use the direct product decomposition of Sect. 3.4.

First, it is easy to see that the fact that $\Upsilon_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}$ is an equivalence implies that the functor

$$\Upsilon_{(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}} : \mathrm{QCoh}((\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}) \to \mathrm{IndCoh}^!((\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}})$$

is also an equivalence.

Since the category $\mathrm{QCoh}(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G}))$ is dualizable, the $\boxtimes$ functor

$$\mathrm{QCoh}(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G})) \otimes \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{QCoh}(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G}) \times \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

is an equivalence.

The $\boxtimes$ functor

$$\mathrm{IndCoh}(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G})) \otimes \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G}) \times \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

is an equivalence tautologically.

Since the functor

$$\Upsilon_{\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G})} : \mathrm{QCoh}(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G})) \to \mathrm{IndCoh}(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G}))$$

is an equivalence, in order to prove that $\Upsilon_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}$ is an equivalence, it suffices to show that

$$\Upsilon_{\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G}) \times \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}} : \mathrm{QCoh}(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G}) \times \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G}) \times \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

is an equivalence.

However, this follows from the fact that $\Upsilon_{(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}}$ is an equivalence, combined with the existence of an isomorphism

$$\check{\mathfrak{g}}_0^\wedge / \mathrm{Ad}(\check{G}) \times \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \simeq (\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}.$$

□

3.8.8. The proof of Proposition 3.8.7 given above is specific to the situation when $\underline{x} \in \mathrm{Ran}$ is fixed. Yet, we propose:

Question 3.8.9. *Is the functor*

$$\Upsilon_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}} : \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$$

a factorization equivalence?

3.8.10. Note that we can write

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \simeq (\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{mon-free}} \underset{(\mathrm{LS}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{reg}}}{\times} \mathrm{LS}^{\text{reg}}_{\check{G},\underline{x}}.$$

Hence, the functor of !-pullback along

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \to (\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{mon-free}}$$

gives rise to a functor

$$\mathrm{QCoh}(\mathrm{LS}^{\text{reg}}_{\check{G},\underline{x}}) \underset{\mathrm{QCoh}((\mathrm{LS}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{reg}})}{\otimes} \mathrm{IndCoh}^*((\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{mon-free}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}). \tag{3.27}$$

We claim:

Proposition 3.8.11. *The functor* (3.27) *is an equivalence.*

Proof. Given that the functors

$$\Upsilon_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}} \text{ and } \Upsilon_{(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{mon-free}}},$$

as well as

$$\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}} \text{ and } \Theta_{\mathrm{Op}^{\text{mer}}_{\check{G}}}$$

are equivalences, in order to prove Proposition 3.8.7, it suffices to show that the functor

$$\mathrm{QCoh}(\mathrm{LS}^{\text{reg}}_{\check{G},\underline{x}}) \underset{\mathrm{QCoh}((\mathrm{LS}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{reg}})}{\otimes} \mathrm{QCoh}((\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{mon-free}}) \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

is an equivalence.

However, this follows from the fact that the prestack

$$(\mathrm{LS}^{\text{mer}}_{\check{G},\underline{x}})^{\wedge}_{\text{reg}} \simeq \check{\mathfrak{g}}^{\wedge}_0 / \mathrm{Ad}(\check{G})$$

is *passable* (see [GaRo3, Chapter 3, Proposition 3.5.3]).

□

3.8.12. As in the case of Proposition 3.8.7, the assertion of Proposition 3.8.11 is specific to the situation when $\underline{x} \in \mathrm{Ran}$ is fixed. Parallel to Question 3.8.9, we propose:

Question 3.8.13. *Is the functor*

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \underset{\mathrm{QCoh}((\mathrm{LS}^{\text{mer}}_{\check{G}})^{\wedge}_{\text{reg}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})^{\wedge}_{\text{mon-free}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$$

a factorization equivalence?

4. Digression: IndCoh* via factorization algebras

In this section we discuss the approach to factorization categories arising in the local Langlands theory, on both the geometric and spectral sides, as *factorization modules* over *factorization algebras*[24].

This approach is most efficient when we want to cross the Langlands bridge, i.e., map a category on the geometric side and a category on the spectral side to one another. Indeed, it is often possible to compare the corresponding factorization algebras directly (a prominent example of this is the Feigin-Frenkel isomorphism, see Theorem 5.1.2).

However, this approach comes with a caveat: typically, the given representation-theoretic or algebro-geometric category will *not* be exactly equivalent to the corresponding category of factorization modules. Rather, the two will be related by a *renormalization procedure*. Most often, this will be manifested by the fact that both sides will be endowed with t-structures, and the corresponding *eventually coconnective* subcategories will be equivalent on the nose.

[24] The factorization algebras in question may be either plain ones (i.e., in Vect) or in some simpler or better understood factorization categories.

We apply these ideas to construct a *categorical* action of the Feigin-Frenkel center on Kac-Moody modules at the critical level; see Sect. 4.6.

4.1. **Factorization algebras and modules.**

4.1.1. Let $\mathcal{A}$ be a unital factorization algebra. To it we can attach a *lax* factorization category

$$\mathcal{A}\text{-mod}^{\text{fact}}$$

of unital $\mathcal{A}$-factorization modules (see Sect. B.11.12).

It comes equipped with a conservative forgetful functor

$$\mathbf{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\text{fact}} \to \text{Vect}\,.$$

4.1.2. By definition, the value of $\mathcal{A}\text{-mod}^{\text{fact}}$ over a given $\underline{x} \in \text{Ran}$ is the category

$$\mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}}$$

of factorization $\mathcal{A}$-modules at $\underline{x}$.

In general, we cannot say much about homological properties of the category $\mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}}$. In particular, we do not know whether it is compactly generated.

4.1.3. This is also reflected by the following phenomenon:

For a pair of disjoint points $\underline{x}_1$ and $\underline{x}_2$, the lax factorization structure on $\mathcal{A}\text{-mod}^{\text{fact}}$ recovers the naturally defined functor

$$\mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}_1} \otimes \mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}_2} \to \mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}_1 \sqcup \underline{x}_2}.$$

However, it is not clear whether this functor is an equivalence. (If it were, and if this were true in families over $\underline{x}_1, \underline{x}_2$ moving over Ran, this would mean that the lax factorization structure on $\mathcal{A}\text{-mod}^{\text{fact}}$ is strict.)

4.1.4. Assume for a moment that $\mathcal{A}$ is connective, i.e., $\mathbf{oblv}^l(\mathcal{A}_X) \in \text{QCoh}(X)$ is connective. Then the category $\mathcal{A}\text{-mod}^{\text{fact}}$ carries a (uniquely defined) t-structure (see Sect. B.11.11 for what this means in the factorization setting), for which the functor $\mathbf{oblv}_{\mathcal{A}}$ is t-exact, see Sect. B.11.15.

In addition, it follows from the definition that $\mathcal{A}\text{-mod}^{\text{fact}}$ is *left-complete* in its t-structure.

Remark 4.1.5. The left-completeness of $\mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}}$ is an indication of its failure of compact generation:

Let $\mathbf{C}$ be a category, equipped with a t-structure, in which it is left-complete. Then every object $\mathbf{c} \in \mathbf{C}^c$ is of bounded projective dimension, i.e., the functor $\mathcal{H}om_{\mathbf{C}}(\mathbf{c}, -)$ is of bounded cohomological amplitude. However, typically, the category $\mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}}$ does not contain any objects of bounded projective dimension.

4.1.6. Here is how factorization algebras and modules will typically arise in this paper. Let $\mathbf{A}$ be a (unital) factorization category equipped with a (lax unital) factorization functor

$$F : \mathbf{A} \to \text{Vect}\,.$$

Then $F(\mathbf{1}_{\mathbf{A}})$ is a factorization algebra (in Vect). Moreover, the functor F naturally upgrades to a (factorization) functor, denoted

$$F^{\text{enh}} : \mathbf{A} \to F(\mathbf{1}_{\mathbf{A}})\text{-mod}^{\text{fact}},$$

see Lemma C.15.3.

Remark 4.1.7. The factorization structure on F^{enh} means for example that for disjoint points $\underline{x}_1, \underline{x}_2 \in$ Ran, the diagram

$$\begin{array}{ccc}
\mathbf{A}_{\underline{x}_1} \otimes \mathbf{A}_{\underline{x}_2} & \xrightarrow{\sim} & \mathbf{A}_{\underline{x}_1 \sqcup \underline{x}_2} \\
{\scriptstyle F_{\underline{x}_1} \otimes F_{\underline{x}_2}} \downarrow & & \downarrow {\scriptstyle F_{\underline{x}_1 \sqcup \underline{x}_2}} \\
F(\mathbf{1}_{\mathbf{A}})\text{-mod}^{\text{fact}}_{\underline{x}_1} \otimes F(\mathbf{1}_{\mathbf{A}})\text{-mod}^{\text{fact}}_{\underline{x}_2} & \longrightarrow & F(\mathbf{1}_{\mathbf{A}})\text{-mod}^{\text{fact}}_{\underline{x}_1 \sqcup \underline{x}_2}
\end{array}$$

commutes (even though the bottom horizontal arrow is not in general an equivalence).

4.1.8. If in the situation of Sect. 4.1.6, the category $\mathbf{C}$ is equipped with a t-structure so that $\mathbf{1_C}$ lies in the heart, and the functor F is t-exact, we obtain that $F(\mathbf{1_C})$ is a *classical* factorization algebra, so that the category $F(\mathbf{1_C})\text{-mod}^{\text{fact}}$ carries a t-structure.

In this case, the functor F^{enh} is obviously t-exact.

4.1.9. More generally, if $F : \mathbf{C}_1 \to \mathbf{C}_2$ is a lax unital factorization functor between unital factorization categories, the object

$$F(\mathbf{1}_{\mathbf{C}_1}) \in \mathbf{C}_2$$

has a natural structure of factorization algebra, and the functor F upgrades to a functor

$$F^{\text{enh}} : \mathbf{C}_1 \to F(\mathbf{1}_{\mathbf{C}_1})\text{-mod}^{\text{fact}}(\mathbf{C}_2).$$

4.2. **Kac-Moody modules as factorization modules.** Here is a typical example of the paradigm described in Sects. 4.1.6-4.1.8.

4.2.1. Consider the tautological forgetful functor

$$\mathbf{oblv}_{\widehat{\mathfrak{g}}} : \widehat{\mathfrak{g}}\text{-mod}_\kappa \to \text{Vect}\,.$$

Recall that $\mathbb{V}_{\mathfrak{g},\kappa}$ denotes the factorization algebra $\mathbf{oblv}_{\widehat{\mathfrak{g}}}(\text{Vac}(G)_\kappa)$, so that $\mathbf{oblv}_{\widehat{\mathfrak{g}}}$ upgrades to a (t-exact) functor:

$$\mathbf{oblv}^{\text{enh}}_{\widehat{\mathfrak{g}}} : \widehat{\mathfrak{g}}\text{-mod}_\kappa \to \mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}}. \tag{4.1}$$

4.2.2. We have the following basic observation:

Lemma 4.2.3.

(a) *The functor* $\mathbf{oblv}^{\text{enh}}_{\widehat{\mathfrak{g}}}$ *of* (4.1) *induces an equivalence between the eventually coconnective subcategories of the two sides.*

(b) *The essential image of the subcategory of compact objects of* $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ *under is* $\mathbf{oblv}^{\text{enh}}_{\widehat{\mathfrak{g}}}$ *is contained in* $(\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}})^{>-\infty}$.

4.2.4. The rest of this subsection is devoted to the proof of Lemma 4.2.3. We will prove a pointwise version for $\underline{x} = x \in \text{Ran}$. The factorization version is just a variant of this in families.

Let $\mathbb{V}^{\text{ch}}_{\mathfrak{g},\kappa}$ be the *chiral algebra* corresponding to $\mathbb{V}_{\mathfrak{g},\kappa}$. I.e., as a D-module on X,

$$\mathbb{V}^{\text{ch}}_{\mathfrak{g},\kappa} = \mathbb{V}_{\mathfrak{g},\kappa,X}[-1].$$

First, by [BD1, Proposition 3.4.19] (see [FraG] for the derived version), we have an equivalence

$$\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\text{fact}}_x \simeq \mathbb{V}^{\text{ch}}_{\mathfrak{g},\kappa}\text{-mod}^{\text{ch}}_x,$$

which commutes with the forgetful functors of both sides into Vect, and hence preserves the t-structures on the two sides.

4.2.5. Let $L_{\mathfrak{g},\kappa}$ be the Lie-* algebra

$$\omega_X \oplus (\mathfrak{g} \otimes \mathrm{D}_X)$$

of [BD1, Sect. 2.5.9]. Then by [BD1, Proposition 3.7.17] (which applies as-is in the derived setting), we have

$$\mathbb{V}^{\text{ch}}_{\mathfrak{g},\kappa}\text{-mod}^{\text{ch}}_x \simeq L_{\mathfrak{g},\kappa}\text{-mod}^{\text{ch}}_x,$$

which commutes with the forgetful functors of both sides into Vect, and hence preserves the t-structures on the two sides.

4.2.6. By the construction of $\widehat{\mathfrak{g}}\text{-mod}_{\kappa,x}$ in [Ra5] (or, equivalently, in [FG6, Sect. 23.1]), we have an embedding

$$(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,x})^c \hookrightarrow (L_{\mathfrak{g},\kappa}\text{-mod}_x^{\mathrm{ch}})^{>-\infty}, \tag{4.2}$$

which satisfies the conditions of [FG6, Sect. 22.1.4] (see Sect. D.3.6 for an explanation of why this happens).

Hence,

$$(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,x})^{>-\infty} \to (L_{\mathfrak{g},\kappa}\text{-mod}_x^{\mathrm{ch}})^{>-\infty}$$

is an equivalence by [FG6, Proposition 22.1.5 and 22.2.1].

□[Lemma 4.2.3]

4.3. **The case of commutative factorization algebras.**

4.3.1. Let $\mathcal{Y}$ be an affine D-scheme over X, i.e., $\mathcal{Y} = \mathrm{Spec}_X(A)$, where $A \in \mathrm{ComAlg}(\mathrm{D}\text{-mod}(X))$ with $\mathbf{oblv}^l(A) \in \mathrm{QCoh}(X)^{\leq 0}$.

Let $\mathcal{A} \in \mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X))$ denote the corresponding commutative factorization algebra, i.e., $\mathcal{A} := \mathrm{Fact}(A)$, see Sect. B.10.2.

4.3.2. Let $\mathfrak{L}^+_\nabla(\mathcal{Y})$ denote the affine factorization scheme corresponding to $\mathcal{Y}$ (see Sect. B.4.2), i.e., the fiber $\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}$ of $\mathfrak{L}^+_\nabla(\mathcal{Y})$ at $\underline{x} \in \mathrm{Ran}$ is the space

$$\mathrm{Sect}_\nabla(\mathcal{D}_{\underline{x}}, \mathcal{Y}). \tag{4.3}$$

According to Sect. C.8.11, for $\mathcal{Z} \to \mathrm{Ran}$,

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathcal{Z}} = \mathrm{Spec}_{\mathcal{Z}}(\mathcal{A}_{\mathcal{Z}}).$$

In particular, for $\underline{x} = \{x_1, ..., x_n\}$, we have

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}} \simeq \prod_{i=1,\dots,n} \mathcal{Y}_{x_i}, \quad \underline{x} = \{x_1, ..., x_n\}.$$

(Note that this is compatible with (4.3), since for a singleton $\underline{x} = \{x\}$, we have $\mathrm{Sect}_\nabla(\mathcal{D}_x, \mathcal{Y}) \simeq \mathcal{Y}_x$, in agreement with (4.3)).

4.3.3. Consider the corresponding factorization category $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}))$ (see Sect. B.13.2), so that for $\underline{x} \in \mathrm{Ran}$, we have

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}))_{\underline{x}} := \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}).$$

The factorization category $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}))$ is unital, with the structure sheaf $\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})}$ being the factorization unit.

4.3.4. The functor of global sections $\Gamma(\mathfrak{L}^+_\nabla(\mathcal{Y}), -)$ sends

$$\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})} \mapsto \mathcal{A}$$

and induces an equivalence

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})) \simeq \mathcal{A}\text{-mod}^{\mathrm{com}}.$$

4.3.5. Let $\mathfrak{L}_\nabla(\mathcal{Y})$ denote the factorization D-ind-scheme that attaches to a point $\underline{x} \in \mathrm{Ran}$ the space

$$\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}} := \mathrm{Sect}_\nabla(\mathcal{D}^\times_{\underline{x}}, \mathcal{Y}),$$

see Sect. B.4.6.

We consider the corresponding lax factorization category $\mathrm{QCoh}(\mathfrak{L}_\nabla(\mathcal{Y}))$ (see Sect. B.13.2), so that for $\underline{x} \in \mathrm{Ran}$, we have

$$\mathrm{QCoh}(\mathfrak{L}_\nabla(\mathcal{Y}))_{\underline{x}} := \mathrm{QCoh}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}).$$

For a general $\mathcal{Y}$, this category may be quite ill-behaved (basically, because the category of quasi-coherent sheaves on an ind-scheme may be quite unwieldy); in particular, it is not clear whether $\mathrm{QCoh}(\mathfrak{L}_\nabla(\mathcal{Y}))$ is unital.

4.3.6. Recall now that we can also consider the factorization category

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})),$$

see Sect. B.13.8.

Its factorization unit is the direct image of the structure sheaf on $\mathfrak{L}^+_\nabla(\mathcal{Y})$ along the tautological closed embedding $\mathfrak{L}^+_\nabla(\mathcal{Y}) \xrightarrow{\iota} \mathfrak{L}_\nabla(\mathcal{Y})$. By a slight abuse of notation, we will denote it by the same symbol $\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})}$.

4.3.7. The operation of taking global sections is a (t-exact[25]) factorization functor

$$\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -) : \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \to \mathrm{Vect}\,.$$

Hence, by Sects. 4.1.6-4.1.8, the functor $\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)$ upgrades to a (t-exact) lax unital factorization functor

$$\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}} : \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \to \mathcal{A}\text{-mod}^{\mathrm{fact}}. \tag{4.4}$$

4.3.8. We have the following basic assertion:

Theorem 4.3.9. *Assume that $\mathcal{Y}$ is almost finitely presented in the D-sense[26]. Then the functor* (4.4) *induces an equivalence between the eventually coconnective subcategories of the two sides.*

The proof of this theorem will be given in Sect. D.5. We note that the assertion of the theorem would be *false* without the finite presentation hypothesis, see Sect. D.6.

4.3.10. Recall that if $\mathcal{Z}$ is an ind-scheme, we have a well-defined (t-exact) functor

$$\Psi_{\mathcal{Z}} : \mathrm{IndCoh}^*(\mathcal{Z}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Z}),$$

which induces an equivalence

$$\mathrm{IndCoh}^*(\mathcal{Z})^{>-\infty} \xrightarrow{\sim} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Z})^{>-\infty},$$

see Lemma A.8.10.

Furthermore, if $\mathcal{Z}$ is ind-placid, $\Psi_{\mathcal{Z}}$ gives rise to an equivalence between $\mathrm{IndCoh}^*(\mathcal{Z})^c$ and the subcategory of almost compact objects in $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Z})^{>-\infty}$.

Note also that the composition

$$\Gamma(\mathcal{Z}, -) \circ \Psi_{\mathcal{Z}} : \mathrm{IndCoh}^*(\mathcal{Z}) \to \mathrm{Vect}$$

is the functor $\Gamma^{\mathrm{IndCoh}}(\mathcal{Z}, -)$ of IndCoh-global sections.

4.3.11. Assume for a moment that $\mathfrak{L}_\nabla(\mathcal{Y})$ is ind-placid. Applying Sect. 4.3.10, we obtain a factorization functor

$$\Gamma^{\mathrm{IndCoh}}(\mathfrak{L}_\nabla(\mathcal{Y}), -) \simeq \Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -) \circ \Psi_{\mathfrak{L}_\nabla(\mathcal{Y})}, \quad \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y})) \to \mathrm{Vect}$$

and its enhancement

$$\Gamma^{\mathrm{IndCoh}}(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y})) \to \mathcal{O}_{\mathcal{Y}}\text{-mod}^{\mathrm{fact}}. \tag{4.5}$$

Combining with Theorem 4.3.9 we obtain:

Corollary 4.3.12.

(a) *The functor $\Gamma^{\mathrm{IndCoh}}(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}}$ of* (4.5) *is t-exact and induces an equivalence between the eventually coconnective subcategories of the two sides.*

(b) *The essential image of the subcategory of compact objects in* $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y}))$ *under the functor $\Gamma^{\mathrm{IndCoh}}(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}}$ is contained in* $(\mathcal{O}_{\mathcal{Y}}\text{-mod}^{\mathrm{fact}})^{>-\infty}$.

---

[25]See Sect. A.2.8 for the definition of the t-structure on $\mathrm{QCoh}_{\mathrm{co}}$ of an ind-scheme.

[26]See Sect. B.6.3 for what this means.

Remark 4.3.13. Point (b) in Corollary 4.3.12 can be strengthened as follows: the essential image of $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y}))^c$ under $\Gamma^{\mathrm{IndCoh}}(\mathfrak{L}_\nabla(\mathcal{Y}),-)^{\mathrm{enh}}$ equals the category of *almost compact* objects[27] in $(\mathcal{O}_\mathcal{Y}\text{-mod}^{\mathrm{fact}})^{>-\infty}$.

Note that from Corollary 4.3.12 allows us to recover the (factorization) category $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y}))$ from $\mathcal{O}_\mathcal{Y}\text{-mod}^{\mathrm{fact}}$ equipped with its t-structure:

Namely, we can identify $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y}))$ with the ind-completion of the category of almost compact objects in $(\mathcal{O}_\mathcal{Y}\text{-mod}^{\mathrm{fact}})^{\geq-\infty}$.

4.4. **Recovering** IndCoh* **of opers as factorization modules.**

4.4.1. The setup of Sect. 4.3.11 is directly applicable to the case when $\mathcal{Y}=\mathrm{Op}_{\check{G}}$ (the D-afp assumption is satisfied by Sect. 3.1.7), so that

$$\mathfrak{L}^+_\nabla(\mathcal{Y})=\mathrm{Op}^{\mathrm{reg}}_{\check{G}} \text{ and } \mathfrak{L}_\nabla(\mathcal{Y})=\mathrm{Op}^{\mathrm{mer}}_{\check{G}}.$$

By a slight abuse of notation, we will denote by $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}$ (rather than $\Gamma(\mathrm{Op}^{\mathrm{reg}}_{\check{G}},\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}})$) the corresponding factorization algebra in Vect.

In particular, we obtain that the functor

$$\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}},-):\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})\to\mathrm{Vect}$$

upgrades to a (t-exact) functor

$$\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}},-)^{\mathrm{enh}}:\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})\to\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}, \tag{4.6}$$

and we have:

Corollary 4.4.2.
(a) *The functor* (4.6) *induces an equivalence between the corresponding eventually coconnective (a.k.a. bounded below) subcategories.*
(b) *The essential image of the subcategory of compact objects in* $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$ *under the functor* (4.6) *is contained in* $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}_{\underline{x}})^{>-\infty}$.

We now consider the case of *monodromy-free* opers.

4.4.3. Direct image along the projection

$$\mathfrak{r}^{\mathrm{reg}}:\mathrm{Op}^{\mathrm{reg}}_{\check{G}}\to\mathrm{LS}^{\mathrm{reg}}_{\check{G}}$$

defines a t-exact (lax unital factorization) functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})\simeq\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})\overset{\mathfrak{r}^{\mathrm{reg}}_*}{\to}\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}})\simeq\mathrm{Rep}(\check{G}). \tag{4.7}$$

Denote

$$R_{\check{G},\mathrm{Op}}:=\mathfrak{r}^{\mathrm{reg}}_*(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}). \tag{4.8}$$

This is naturally a commutative factorization algebra in $\mathrm{Rep}(\check{G})$.

4.4.4. Explicitly,

$$R_{\check{G},\mathrm{Op}}:=\left(\Gamma(\mathrm{Op}^{\mathrm{reg}}_{\check{G}},-)\otimes\mathrm{Id}\right)\circ\left((\mathfrak{r}^{\mathrm{reg}})^*\otimes\mathrm{Id}\right)(R_{\check{G}}),$$

where

$$R_{\check{G}}\in\mathrm{Rep}(\check{G})\otimes\mathrm{Rep}(\check{G})$$

is the regular representation.

[27]Recall that an object $\mathbf{c}$ in a DG category $\mathbf{C}$ equipped with a t-structure (assumed compatible with filtered colimits) is said to be *almost compact* if the functor $\mathcal{H}om_{\mathbf{C}}(\mathbf{c},-)$ commutes with filtered colimits on $\mathbf{C}^{\geq-n}$ for all n.

4.4.5. Consider now the factorization functor

$$\mathfrak{r}_*^{\mathrm{IndCoh}}:\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})\to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{reg}})\simeq \mathrm{Rep}(\check{G}). \tag{4.9}$$

Note that we can interpret $R_{\check{G},\mathrm{Op}}$ also as

$$\mathfrak{r}_*^{\mathrm{IndCoh}}(\mathcal{O}_{\mathrm{Op}_{\check{G}}^{\mathrm{reg}}}),$$

where by a slight abuse of notation we view $\mathcal{O}_{\mathrm{Op}_{\check{G}}^{\mathrm{reg}}}$ as an object of $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})$ using

$$\mathrm{QCoh}(\mathrm{Op}_{\check{G}}^{\mathrm{reg}})\overset{\Psi_{\mathrm{Op}_{\check{G}}^{\mathrm{reg}}}}{\simeq}\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\mathrm{reg}})\overset{(\iota^{+,\text{mon-free}})_*^{\mathrm{IndCoh}}}{\longrightarrow}\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}).$$

4.4.6. The functor (4.9) naturally upgrades to a t-exact factorization functor

$$(\mathfrak{r}_*^{\mathrm{IndCoh}})^{\mathrm{enh}}:\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})\to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})). \tag{4.10}$$

We will prove:

Proposition 4.4.7.

(a) *The functor* (4.10) *induces an equivalence between the eventually coconnective subcategories of the two sides.*

(b) *The essential image of the subcategory* $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})^c$ *under* (4.10) *is contained in* $(R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})))^{>-\infty}$.

4.5. **Proof of Proposition 4.4.7.** We will provide a general framework, of which Proposition 4.4.7 is a particular case. The assertion is local, so we can assume that X is affine.

4.5.1. Let $\mathcal{Y}$ be an affine D-scheme, and consider $\mathfrak{L}_\nabla^+(\mathcal{Y}):=\mathcal{T}^+$ as a factorization space. Let $\mathcal{T}$ be a factorization space, equipped with a map

$$\iota:\mathcal{T}^+\to\mathcal{T}$$

that extends to a *unital-in-correspondences structure* on $\mathcal{T}$ relative to $\mathcal{T}^+$ (see Sect. C.10.6).

Consider the factorization category

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}),$$

see Sect. B.13.8.

Note its factorization unit is given by

$$\iota_*(\mathcal{O}_{\mathcal{T}^+}).$$

We will assume:

- (i) $\mathcal{T}$ is an ind-placid ind-scheme.[28]

4.5.2. Let us be given a D-prestack $\mathcal{Y}_0$, equipped with a map

$$f:\mathcal{Y}\to\mathcal{Y}_0,$$

and an extension of the map

$$\mathcal{T}^+=\mathfrak{L}_\nabla^+(\mathcal{Y})\overset{\mathfrak{L}^+(f)}{\to}\mathfrak{L}_\nabla^+(\mathcal{Y}_0)=:\mathcal{T}_0^+$$

to a map

$$\mathcal{T}\overset{\mathfrak{L}(f)}{\to}\mathcal{T}_0^+.$$

We will assume:

- (ii) $\mathcal{T}_0^+$ has an affine diagonal.

Note that assumptions (i) and (ii) imply in particular that the map $\mathfrak{L}(f)$ is ind-schematic.

[28]See Sect. B.1.9 for what this means.

4.5.3. Consider the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}) \overset{\mathfrak{L}(f)_*}{\to} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_0^+) \overset{\Omega_{\mathfrak{L}^+_\nabla(\mathcal{Y}_0)}}{\to} \mathrm{QCoh}(\mathcal{T}_0^+), \tag{4.11}$$

where $\Omega_{\mathfrak{L}^+_\nabla(\mathcal{Y}_0)}$ is as in (A.8). By a slight abuse of notation, we will denote the composite functor in (4.11) by the same symbol $\mathfrak{L}(f)_*$.

The functor $\mathfrak{L}(f)_*$ of (4.11) upgrades to a (factorization) functor

$$(\mathfrak{L}(f)_*)^{\mathrm{enh}} : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}) \to \\ \to \mathfrak{L}(f)_* \circ \iota_*(\mathcal{O}_{\mathcal{T}^+})\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\mathcal{T}_0^+)) = \mathfrak{L}^+(f)_*(\mathcal{O}_{\mathcal{T}^+})\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\mathcal{T}_0^+)).. \tag{4.12}$$

4.5.4. We now make an additional assumptions:

- (iii) The prestack $\mathcal{Y}_0$ admits a map
$$g : \widetilde{\mathcal{Y}}_0 \to \mathcal{Y}_0,$$
where $\widetilde{\mathcal{Y}}_0$ is an affine D-scheme, such that the map
$$\widetilde{\mathcal{T}}_0^+ := \mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}_0) \overset{\mathfrak{L}^+(g)}{\to} \mathfrak{L}^+_\nabla(\mathcal{Y}_0) =: \mathcal{T}_0^+$$
is an fpqc cover.[29]
- (iv) For $\widetilde{\mathcal{Y}} := \mathcal{Y} \underset{\mathcal{Y}_0}{\times} \widetilde{\mathcal{Y}}^0$, the resulting map
$$\widetilde{\mathcal{T}}^+ := \mathcal{T}^+ \underset{\mathcal{T}_0^+}{\times} \widetilde{\mathcal{T}}_0^+ \overset{\iota\times\mathrm{id}}{\to} \mathcal{T} \underset{\mathcal{T}_0^+}{\times} \widetilde{\mathcal{T}}_0^+ =: \widetilde{\mathcal{T}}$$
identifies with
$$\mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}) \to \mathfrak{L}_\nabla(\widetilde{\mathcal{Y}}) \underset{\mathfrak{L}_\nabla(\widetilde{\mathcal{Y}}_0)}{\times} \mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}_0).$$

Note that assumption (iii) implies, in particular, that the category $\mathrm{QCoh}(\mathcal{T}_0^+)$ has a well-behaved t-structure: it is characterized uniquely by the property that the functor

$$(\mathfrak{L}^+(g))^* : \mathrm{QCoh}(\mathcal{T}_0^+) \to \mathrm{QCoh}(\widetilde{\mathcal{T}}_0^+)$$

is t-exact.

It follows from assumption (iv) and base change that the functor $\mathfrak{L}(f)_*$ of (4.11) is t-exact (see Sect. B.13.8, where the t-structure on the left-hand side is defined). Hence, $(\mathfrak{L}(f)_*)^{\mathrm{enh}}$ is also t-exact.

4.5.5. Finally, we make the following assumption:

- (v) The map
$$\widetilde{\mathcal{Y}} := \mathcal{Y} \underset{\mathcal{Y}_0}{\times} \widetilde{\mathcal{Y}}_0 \overset{\tilde{f}}{\to} \widetilde{\mathcal{Y}}_0$$
is D-afp (see Sect. B.6 for what this means).

[29]See Sect. B.1.9 for what this means.

4.5.6. We claim:

Corollary 4.5.7. *Under the above assumptions, the functor* (4.12) *induces an equivalence between the eventually coconnective subcategories of the two sides.*

Proof. Let $\widetilde{\mathcal{Y}}^\bullet_0$ be the Čech nerve of the map g. Denote

$$\widetilde{\mathcal{Y}}^\bullet := \mathcal{Y} \underset{\mathcal{Y}_0}{\times} \widetilde{\mathcal{Y}}^\bullet_0$$

and

$$\widetilde{\mathcal{T}}^{+,\bullet}_0 := \mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}^\bullet_0), \quad \widetilde{\mathcal{T}}^{+,\bullet} := \mathcal{T}^+ \underset{\mathcal{T}^+_0}{\times} \widetilde{\mathcal{T}}^{+,\bullet}_0 \simeq \mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}^\bullet)$$

and

$$\widetilde{\mathcal{T}}^\bullet := \mathcal{T}^+ \underset{\mathcal{T}^+_0}{\times} \widetilde{\mathcal{T}}^{+,\bullet}_0 \simeq \mathfrak{L}_\nabla(\widetilde{\mathcal{Y}}^\bullet) \underset{\mathfrak{L}_\nabla(\widetilde{\mathcal{Y}}^\bullet_0)}{\times} \mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}^\bullet_0).$$

Consider the resulting maps:

$$\widetilde{\mathcal{T}}^\bullet \overset{\mathfrak{L}(\widetilde{f})^\bullet}{\to} \widetilde{\mathcal{T}}^{+,\bullet}_0 \text{ and } \widetilde{\mathcal{T}}^{+,\bullet} \overset{\mathfrak{L}^+(\widetilde{f})^\bullet}{\to} \widetilde{\mathcal{T}}^{+,\bullet}_0.$$

First, by fpqc descent we have a t-exact equivalence

$$\mathrm{QCoh}(\mathcal{T}^+_0) \simeq \mathrm{Tot}(\mathrm{QCoh}(\widetilde{\mathcal{T}}^{+,\bullet}_0)),$$

from which we obtain a t-exact equivalence

$$\mathfrak{L}^+(f)_*(\mathcal{O}_{\mathcal{T}^+})\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\mathcal{T}^+_0)) \simeq \mathrm{Tot}\left(\mathfrak{L}^+(\widetilde{f}^\bullet)_*(\mathcal{O}_{\widetilde{\mathcal{T}}^{+,\bullet}})\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\widetilde{\mathcal{T}}^{+,\bullet}_0))\right)$$

and hence

$$\mathfrak{L}^+(f)_*(\mathcal{O}_{\mathcal{T}^+})\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\mathcal{T}^+_0))^{>-\infty} \simeq \mathrm{Tot}\left(\mathfrak{L}^+(\widetilde{f}^\bullet)_*(\mathcal{O}_{\widetilde{\mathcal{T}}^{+,\bullet}})\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\widetilde{\mathcal{T}}^{+,\bullet}_0))^{>-\infty}\right).$$

Next, by assumption (i) in Sect. 4.5.1 and Proposition A.3.3, the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})^{>-\infty} \to \mathrm{Tot}\left(\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{T}}^\bullet)^{>-\infty}\right)$$

is also an equivalence.

Finally, by assumption (v) and a relative version of Theorem 4.3.9, the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{T}}^\bullet)^{>-\infty} \simeq \mathrm{QCoh}_{\mathrm{co}}\left(\mathfrak{L}_\nabla(\widetilde{\mathcal{Y}}^\bullet) \underset{\mathfrak{L}_\nabla(\widetilde{\mathcal{Y}}^\bullet_0)}{\times} \mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}^\bullet_0)\right)^{>-\infty} \to$$
$$\to \mathfrak{L}^+(\widetilde{f}^\bullet)_*(\mathcal{O}_{\widetilde{\mathcal{T}}^{+,\bullet}})\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\widetilde{\mathcal{T}}^{+,\bullet}_0))^{>-\infty} \simeq \mathfrak{L}^+(\widetilde{f}^\bullet)_*(\mathcal{O}_{\mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}^\bullet)})\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\widetilde{\mathcal{Y}}^\bullet_0)))^{>-\infty}$$

is a term-wise equivalence.

Combining, we obtain that (4.12) is also an equivalence, as required.

□

4.5.8. Precomposing the equivalence (4.12) with the equivalence

$$\mathrm{IndCoh}^*(\mathcal{T})^{>-\infty} \xrightarrow{\Psi_{\mathcal{T}}} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})^{>-\infty}$$

of Lemma A.8.10, we obtain that under assumptions (i)-(iv) above, the functor

$$((\mathfrak{L}(f))^{\mathrm{IndCoh}}_*)^{\mathrm{enh}} := (\mathfrak{L}(f)_*)^{\mathrm{enh}} \circ \Psi_{\mathcal{T}}$$

induces an equivalence

$$\mathrm{IndCoh}^*(\mathcal{T})^{>-\infty} \to ((\mathfrak{L}(f))^{\mathrm{IndCoh}}_*)^{\mathrm{enh}}\text{-mod}^{\mathrm{fact}}(\mathrm{QCoh}(\mathcal{T}^+_0))^{>-\infty}. \tag{4.13}$$

4.5.9. We apply the above to

$$\mathcal{Y} = \mathrm{Op}_{\check{G}}, \quad \mathcal{Y}_0 = \mathrm{pt}/\check{G} \text{ and } \mathcal{T} := \mathrm{Op}_{\check{G}}^{\text{mon-free}} := \mathrm{Op}_{\check{G}}^{\mathrm{mer}} \underset{\mathrm{LS}_{\check{G}}^{\mathrm{mer}}}{\times} \mathrm{LS}_{\check{G}}^{\mathrm{reg}}.$$

Hence, in order to deduce the assertion of Proposition 4.4.7, we have to show that conditions (i)-v) above hold.

4.5.10. Condition (i) says that $\mathrm{Op}_{\check{G}}^{\text{mon-free}}$ is placid; this has been established in Sect. 3.2.6.

We take $\widetilde{\mathcal{Y}}_0 = \mathrm{pt}$ with the tautological map $\mathrm{pt} \to \mathrm{LS}_{\check{G}}^{\mathrm{reg}}$.

Condition (iii) is the content of Lemma B.7.4. Condition (ii) also follows from Lemma B.7.4, since the property of a map being affine can be checked fpqc-locally, and

$$\mathrm{LS}_{\check{G}}^{\mathrm{reg}} \underset{\mathrm{LS}_{\check{G}}^{\mathrm{reg}} \times \mathrm{LS}_{\check{G}}^{\mathrm{reg}}}{\times} (\mathrm{pt} \times \mathrm{pt}) \simeq \mathrm{pt} \underset{\mathrm{LS}_{\check{G}}^{\mathrm{reg}}}{\times} \mathrm{pt} \simeq \mathfrak{L}_{\nabla}^{+}(\check{G}).$$

Condition (iv) is automatic from the construction. Finally, condition (v) is the content of the next lemma:

Lemma 4.5.11. *The affine D-scheme* $\mathrm{Op}_{\check{G}} \underset{\mathrm{pt}/\check{G}}{\times} \mathrm{pt}$ *is D-afp.*

Proof. We have:

$$\mathrm{Op}_{\check{G}} \underset{\mathrm{pt}/\check{G}}{\times} \mathrm{pt} \simeq \mathrm{Op}_{\check{G}} \underset{\mathrm{Jets}(\check{\mathfrak{g}} \otimes \omega_X)}{\times} \mathrm{Jets}(\check{G}),$$

where:

- The map $\mathrm{Op}_{\check{G}} \to \mathrm{Jets}(\check{\mathfrak{g}} \otimes \omega_X)$ is well-defined (Zariski-locally on X) thanks to Sect. 3.1.4;
- The map $\mathrm{Jets}(\check{G}) \to \mathrm{Jets}(\check{\mathfrak{g}} \otimes \omega_X)$ is given by the gauge action on the trivial connection.

This makes the assertion of the lemma manifest, as

$$\mathrm{Op}_{\check{G}},\ \mathrm{Jets}(\check{\mathfrak{g}} \otimes \omega_X) \text{ and } \mathrm{Jets}(\check{G})$$

are all D-afp.

$\square$

$\square$[Proposition 4.4.7]

4.6. **Action of the center on Kac-Moody modules.**

4.6.1. Let $\mathfrak{z}_{\mathfrak{g}}$ be the (classical) center of $\mathbb{V}_{\mathfrak{g},\mathrm{crit}}$, viewed as a plain factorization (chiral) algebra (i.e., a factorization algebra in Vect).

By construction, $\mathfrak{z}_{\mathfrak{g}}$ is a commutative factorization algebra. It acts as such on $\mathrm{Vac}(G)_{\mathrm{crit}} \in \mathrm{KL}(G)_{\mathrm{crit}}$. In particular, we obtain a map of factorization algebras in $\mathrm{KL}(G)_{\kappa}$:

$$\mathfrak{z}_{\mathfrak{g}} \otimes \mathrm{Vac}(G)_{\mathrm{crit}} \to \mathrm{Vac}(G)_{\mathrm{crit}}. \tag{4.14}$$

4.6.2. We will denote by the symbols $\mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}})$ and "Spec"$(\mathfrak{Z}_{\mathfrak{g}})$ the corresponding factorization scheme and ind-scheme, respectively, see Sect. 4.3.

In this subsection we will construct the an action of $\mathrm{IndCoh}^!(\text{"Spec"}(\mathfrak{Z}_{\mathfrak{g}}))$ on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$, compatible with factorization.

Remark 4.6.3. The existence of such an action at the level of abelian categories is essentially evident: the topological algebra of global functions on "Spec"$(\mathfrak{Z}_{\mathfrak{g}})$ maps to the center of the completed universal enveloping algebra of $U(\widehat{\mathfrak{g}}_{\mathrm{crit}})$.

At the derived level (for a fixed point $\underline{x} \in \mathrm{Ran}$) the construction of such an action was carried out in [FG6, Sect. 23.2-23.4] and [Ra5, Sect. 11] in the language of topological associative algebras.

The methods of *loc. cit.* could be adapted to the factorization setting. However, below we present a different construction. Even though it looks more complicated (at least more abstract), its advantage is that it is compatible with the construction of Sect. 5.3, where we do not know how to make other methods work.

The construction of the action presented below has another advantage in that it is manifestly compatible with the action of $\mathfrak{L}(G)$, see Sect. 4.7 below.

4.6.4. Let $\mathcal{Z}$ be an ind-placid ind-scheme. The categories $\mathrm{IndCoh}^!(\mathcal{Z})$ and $\mathrm{IndCoh}^*(\mathcal{Z})$ are each mutually dual, with the pairing given by

$$\mathrm{IndCoh}^!(\mathcal{Z}) \otimes \mathrm{IndCoh}^*(\mathcal{Z}) \overset{\overset{!}{\otimes}}{\to} \mathrm{IndCoh}^*(\mathcal{Z}) \overset{\Gamma^{\mathrm{IndCoh}}(\mathcal{Z},-)}{\longrightarrow} \mathrm{Vect}\,.$$

Hence, we can view $\mathrm{IndCoh}^*(\mathcal{Z})$ as a comonoidal category. Moreover, the datum of an action of $\mathrm{IndCoh}^!(\mathcal{Z})$ on a (factorization) category $\mathbf{C}$ is is equivalent to the datum of a coaction of $\mathrm{IndCoh}^*(\mathcal{Z})$ on $\mathbf{C}$.

4.6.5. As we shall see shortly (see Theorem 5.1.2 and Sect. 3.2.4), the factorization ind-scheme "Spec"$(\mathfrak{Z}_{\mathfrak{g}})$ is ind-placid.

In particular, the category $\mathrm{IndCoh}^*(\text{"Spec"}(\mathfrak{Z}_{\mathfrak{g}}))$ is well-defined (see Sect. B.13.22). Moreover, it is compactly generated and identifies with the dual of $\mathrm{IndCoh}^!(\text{"Spec"}(\mathfrak{Z}_{\mathfrak{g}}))$.

The contents of Sect. 4.6.4 apply also in the factorization context. Hence, our task will be to define a coaction of $\mathrm{IndCoh}^*(\text{"Spec"}(\mathfrak{Z}_{\mathfrak{g}}))$, viewed as a comonoidal category, on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$.

4.6.6. We will first explain how to construct the coaction functor

$$\text{coact} : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\text{"Spec"}(\mathfrak{Z}_{\mathfrak{g}})) \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}. \tag{4.15}$$

Since $\mathrm{Vac}(G)_{\mathrm{crit}}$ is the factorization unit in $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$, we can write

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \simeq \mathrm{Vac}(G)_{\mathrm{crit}}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})$$

and

$$(\mathfrak{z}_{\mathfrak{g}} \otimes \mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}) \simeq \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}).$$

Now, restriction along the map (4.14) gives rise to a t-exact functor

$$\mathrm{Vac}(G)_{\mathrm{crit}}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}) \to (\mathfrak{z}_{\mathfrak{g}} \otimes \mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}),$$

i.e., a functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}).$$

In particular, since the compact generators of $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$ are eventually coconnective, we obtain a functor

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^c \to \left(\mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})\right)^{>-\infty}. \tag{4.16}$$

4.6.7. Note now that by combining Corollary C.16.12 and Corollary 4.4.2(a), we obtain:

Corollary 4.6.8. *The functor*

$$\mathrm{IndCoh}^*(\text{"Spec"}(\mathfrak{Z}_{\mathfrak{g}})) \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}) \tag{4.17}$$

is t-exact and induces an equivalence between the eventually coconnective subcategories of the two sides.

Hence, (4.16) can be thought of as a functor

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^c \to (\mathrm{IndCoh}^*(\text{"Spec"}(\mathfrak{Z}_{\mathfrak{g}})) \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^{>-\infty} \hookrightarrow \mathrm{IndCoh}^*(\text{"Spec"}(\mathfrak{Z}_{\mathfrak{g}})) \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}. \tag{4.18}$$

Ind-extending, from (4.18), we obtain the desired functor (4.15).

4.6.9. Our next task is to extend the functor (4.15) to a coaction of $\mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$. In doing so we will have to overcome two hurdles:

(i) Homological-algebraic, which has to do with inverting the functor (4.15) on the eventually coconnective subcategories.

(ii) Homotopic-algebraic, which has to do with equipping the functor (4.15) with a homotopy-coherent associativity datum.

We will deal with (i) in the rest of this subsection, and with (ii) in Sect. J.

4.6.10. First, proceeding as in Sect. 4.6.6, for an integer n, we define an n-ry operation

$$\mathrm{coact}_n : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}, \tag{4.19}$$

so that the composition with

$$\mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to \mathfrak{z}_{\mathfrak{g}}^{\otimes n}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}) \tag{4.20}$$

is the restriction functor along the action map

$$\mathfrak{z}_{\mathfrak{g}}^{\otimes n} \otimes \mathrm{Vac}(G)_{\mathrm{crit}} \to \mathrm{Vac}(G)_{\mathrm{crit}}. \tag{4.21}$$

4.6.11. We claim:

Lemma 4.6.12. *The functor* (4.19) *is t-exact.*

Proof. By construction, the composition of (4.19) with (4.20) is t-exact. Since the functor (4.20) induces an equivalence on eventually eventually coconnective subcategories (see Corollary 4.6.8), it suffices to show that (4.19) has a bounded cohomological amplitude (over each X^I). By factorization, this reduces to the case when I is a singleton, and by evaluating at field-valued points of X, we reduce to the pointwise case. The latter was established in [Ra5, Sect. 11.13].

□

4.6.13. From Lemma 4.6.12 we obtain:

Corollary 4.6.14. *The functor* (4.15) *satisfies associativity* at the homotopy level.

Proof. We need to show:

- For every $n = n_1 + n_2$, the diagram

$$\begin{tikzcd}
\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \arrow[r, "\mathrm{coact}_{n_2}"] \arrow[d, "\mathrm{coact}_n"'] & \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n_2} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \arrow[d, "\mathrm{Id} \otimes \mathrm{coact}_{n_1}"] \\
\mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \arrow[r, "\sim"] & \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n_2} \otimes \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n_1} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}
\end{tikzcd}$$

  commutes;

- For any n, the diagram

$$\begin{tikzcd}
\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \arrow[r, "\mathrm{coact}"] \arrow[d, "\mathrm{coact}_n"'] & \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})) \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \arrow[d, "\otimes\, \mathrm{comult}_n \otimes \mathrm{Id}"] \\
\mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \arrow[r, "\sim"] & \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}
\end{tikzcd}$$

  commutes.

In both cases, it suffices to show that the natural transformation in question is an isomorphism when evaluated on compact objects. In particular, it suffices to show that it is an isomorphism when evaluated on eventually coconnected subcategories.

We know that the natural transformation becomes an isomorphism after composing with the functor (4.20). Since the functor (4.20) is an equivalence on eventually coconnected subcategories (see Corollary 4.6.8), it suffices to show that all the functors involved have cohomological amplitude bounded on the left. However, this follows from Lemma 4.6.12 (for coact_n), while the functor

$$\mathrm{comult}_n : \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})) \to \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \simeq \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})^n)$$

is t-exact (see Corollary A.8.8).

□

4.7. **Action of the center and the loop group action.**

4.7.1. Our current goal is to show that the $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$-module structure on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$ is compatible with the action of $\mathfrak{L}(G)$ on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$.

By the construction of the module structure, we need to show that the each of the categories

$$(\mathfrak{z}_{\mathfrak{g}}^{\otimes n} \otimes \mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})$$

carries an action of $\mathfrak{L}(G)$, such that:

- It is compatible with the functor

$$\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \overset{(4.20)}{\longrightarrow}$$
$$\to \mathfrak{z}_{\mathfrak{g}}^{\otimes n}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}) \simeq (\mathfrak{z}_{\mathfrak{g}}^{\otimes n} \otimes \mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}});$$

  where the $\mathfrak{L}(G)$ on the left-hand side is via the $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$-factor;
- The restriction functors

$$(\mathfrak{z}_{\mathfrak{g}}^{\otimes n_1} \otimes \mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}) \to (\mathfrak{z}_{\mathfrak{g}}^{\otimes n_2} \otimes \mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})$$

  along the maps

$$\mathfrak{z}_{\mathfrak{g}}^{\otimes n_2} \otimes \mathrm{Vac}(G)_{\mathrm{crit}} \to \mathfrak{z}_{\mathfrak{g}}^{\otimes n_1} \otimes \mathrm{Vac}(G)_{\mathrm{crit}}$$

  that encode the $\mathfrak{z}_{\mathfrak{g}}$-action on $\mathrm{Vac}(G)_{\mathrm{crit}}$ carry a natural $\mathfrak{L}(G)$-equivariant structure.

4.7.2. In order to do so, it suffices to show that for *any* factorization algebra $\mathcal{A} \in \mathrm{KL}(G)_{\mathrm{crit}}$, the lax factorization category

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})$$

carries an action of $\mathfrak{L}(G)$, compatible with the forgetful functor

$$\mathbf{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}) \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}},$$

and this construction is functorial with respect to the functors

$$\mathrm{Res}_{\phi} : \mathcal{A}_2\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}) \to \mathcal{A}_1\text{-mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}),$$

corresponding to homomorphisms

$$\phi : \mathcal{A}_1 \to \mathcal{A}_2$$

of factorization algebras in $\mathrm{KL}(G)_{\mathrm{crit}}$.

4.7.3. Let $\mathbf{oblv}_{\mathfrak{L}^+(G)}$ denote the forgetful functor. For $\mathcal{Z} \to \mathrm{Ran}$, consider

$$\mathrm{Res}_{\mathbf{oblv}_{\mathfrak{L}^+(G)}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}^{\mathrm{fact}_{\mathcal{Z}}}) \in \mathrm{KL}(G)_{\mathrm{crit}}\text{-}\mathbf{mod}_{\mathcal{Z}}^{\mathrm{fact}}. \tag{4.22}$$

By Lemma C.14.18, for $\mathcal{A} \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathrm{KL}(G)_{\mathrm{crit}})$, we have

$$\mathbf{oblv}_{\mathfrak{L}^+(G)}(\mathcal{A})\text{-}\mathbf{mod}^{\mathrm{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})_{\mathcal{Z}} \simeq \mathcal{A}\text{-}\mathbf{mod}^{\mathrm{fact}}\left(\mathrm{Res}_{\mathbf{oblv}_{\mathfrak{L}^+(G)}}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}^{\mathrm{fact}_{\mathcal{Z}}})\right)_{\mathcal{Z}}.$$

Hence, it suffices to show that (4.22) carries an action of $\mathfrak{L}(G)_{\mathcal{Z}}$. We will show this in the following general framework.

4.7.4. Let $\mathbf{A}$ be a factorization category, and let $\mathbf{C}$ be a factorization module category over $\mathbf{A}$ at $\mathcal{Z} \to \mathrm{Ran}$. We assume that $\mathbf{A}$ and $\mathbf{C}$ carry compatible actions of $\mathfrak{L}(G)$ at some level κ.

Set $\mathbf{A}_0 := \mathbf{A}^{\mathfrak{L}^+(G)}$; denote by $\mathbf{oblv}_{\mathfrak{L}^+(G)}$ the forgetful functor

$$\mathbf{A}_0 \to \mathbf{A}.$$

Consider

$$\mathrm{Res}_{\mathbf{oblv}_{\mathfrak{L}^+(G)}}(\mathbf{C}) \in \mathbf{A}_0\text{-}\mathbf{mod}_{\mathcal{Z}}^{\mathrm{fact}}.$$

We claim that $\mathrm{Res}_{\mathbf{oblv}_{\mathfrak{L}^+(G)}}(\mathbf{C})$, viewed as a factorization module category over $\mathbf{A}_0$, carries an action of $\mathfrak{L}(G)_{\mathcal{Z}}$, such that the induced action on

$$\mathrm{Res}_{\mathbf{oblv}_{\mathfrak{L}^+(G)}}(\mathbf{C})_{\mathcal{Z}} \simeq \mathbf{C}_{\mathcal{Z}}$$

is the original $\mathfrak{L}(G)_{\mathcal{Z}}$-action on $\mathbf{C}_{\mathcal{Z}}$.

We sketch the construction of this action below; a more detailed exposition will be given in [CFGY].

4.7.5. Let $\mathrm{Gr}_G^{\mathrm{level}_{\mathcal{Z}}}$ be the factorization Gr_G-module space from Sect. B.2.7. It is equipped with a compatible action of $\mathfrak{L}(G)$ in the left and a commuting $\mathfrak{L}(G)_{\mathcal{Z}}$-action on the right.

Let $\mathfrak{L}(G)^{\mathrm{fact}_{\mathcal{Z}}}$ be the vacuum factorization module space over $\mathfrak{L}(G)$ at $\mathcal{Z}$ (see Sect. B.2.6); it carries a compatible action of $\mathfrak{L}(G) \times \mathfrak{L}(G)$.

We have a naturally defined projection

$$\pi_{\mathcal{Z}} : \mathfrak{L}(G)^{\mathrm{fact}_{\mathcal{Z}}} \to \mathrm{Gr}_G^{\mathrm{level}_{\mathcal{Z}}},$$

(see Sect. B.4.10) which gives rise to the pullback functor

$$\pi_{\mathcal{Z}}^* : \text{D-mod}(\mathrm{Gr}_G^{\mathrm{level}_{\mathcal{Z}}}) \to \text{D-mod}(\mathfrak{L}(G)^{\mathrm{fact}_{\mathcal{Z}}}),$$

compatible[30] with the factorization functor

$$\pi^* : \text{D-mod}(\mathrm{Gr}_G) \to \text{D-mod}(\mathfrak{L}(G)),$$

given by pullback along the projection

$$\pi : \mathfrak{L}(G) \to \mathrm{Gr}_G.$$

In particular, $\pi_{\mathcal{Z}}^*$ gives rise to a functor

$$\text{D-mod}(\mathrm{Gr}_G^{\mathrm{level}_{\mathcal{Z}}}) \to \mathrm{Res}_{\pi^*}(\text{D-mod}(\mathfrak{L}(G)^{\mathrm{fact}_{\mathcal{Z}}}), \tag{4.23}$$

as factorization module categories over $\text{D-mod}(\mathrm{Gr}_G)$, see Sect. B.12.11.

4.7.6. We claim:

Lemma 4.7.7. *The functor* (4.23) *is an equivalence.*

Proof. Follows from Lemma B.15.9. □

[30] See Sect. B.12.10 for what this means.

4.7.8. We now return to the setting of Sect. 4.7.4. Note that

$$\mathbf{C} \simeq \text{D-mod}(\mathfrak{L}(G)^{\text{fact}_{\mathcal{Z}}}) \underset{\mathfrak{L}(G)}{\otimes} \mathbf{C}$$

as factorization module categories over

$$\mathbf{A} \simeq \text{D-mod}(\mathfrak{L}(G)) \underset{\mathfrak{L}(G)}{\otimes} \mathbf{A}$$

at $\mathcal{Z}$, where we use the action of $\mathfrak{L}(G)$ on itself on the *left* to form the tensor product.

Note that the functor $\pi_{\mathcal{Z}}^*$ and hence the equivalence (4.23) are compatible with the actions of $\mathfrak{L}(G)$ on the left. Hence, from Lemma 4.7.7 we obtain:

Corollary 4.7.9. *There is a canonical equivalence*

$$\text{Res}_{\mathbf{oblv}_{\mathfrak{L}^+(G)}}(\mathbf{C}) \simeq \text{D-mod}(\text{Gr}_G^{\text{level}_{\mathcal{Z}}}) \underset{\mathfrak{L}(G)}{\otimes} \mathbf{C},$$

as factorization categories over

$$\mathbf{A}_0 \simeq \text{D-mod}(\text{Gr}_G) \underset{\mathfrak{L}(G)}{\otimes} \mathbf{A}.$$

4.7.10. Now, the action of $\mathfrak{L}(G)_{\mathcal{Z}}$-action on $\text{Gr}_G^{\text{level}_{\mathcal{Z}}}$ on the right gives rise to an action of $\mathfrak{L}(G)_{\mathcal{Z}}$ on $\text{D-mod}(\text{Gr}_G^{\text{level}_{\mathcal{Z}}}) \underset{\mathfrak{L}(G)}{\otimes} \mathbf{C}$, commuting with the factorization module structure over $\text{D-mod}(\text{Gr}_G) \underset{\mathfrak{L}(G)}{\otimes} \mathbf{A}$.

Applying Corollary 4.7.9, we produce the sought-for $\mathfrak{L}(G)_{\mathcal{Z}}$-action on $\text{Res}_{\mathbf{oblv}_{\mathfrak{L}^+(G)}}(\mathbf{C})$.

4.7.11. As a consequence of the compatibility of the $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ and $\mathfrak{L}(G)$-actions, we obtain:

Corollary 4.7.12.

(a) *The category* $\text{KL}(G)_{\text{crit}}$ *carries an action of* $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ *compatible with the action of* $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ *on* $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}$ *and the forgetful functor*

$$\text{KL}(G)_{\text{crit}} \to \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}.$$

(b) *The action of* $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ *on* $\text{KL}(G)_{\text{crit}}$ *is compatible with the action of* Sph_G.

Remark 4.7.13. Note that we could have equivalently defined the action of $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ on $\text{KL}(G)_{\text{crit}}$ directly, by repeating the procedure in Sect. 4.6, replacing $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}$ by $\text{KL}(G)_{\text{crit}}$. An analog of Lemma 4.6.12 follows from the original variant of this lemma, since the corresponding functors

$$\text{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \otimes \text{KL}(G)_{\text{crit}} \to \text{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}$$

are conservative.

4.7.14. Note that since $\mathfrak{z}_{\mathfrak{g}} \subset \mathbb{V}_{\mathfrak{g},\text{crit}}$ is invariant under the adjoint action, we can view it also as the (classical) center of the twisted version $\mathbb{V}_{\mathfrak{g},\text{crit},\rho(\omega_X)}$.

In particular, we can regard $\mathfrak{z}_{\mathfrak{g}}$ as acting on $\text{Vac}(G)_{\text{crit},\rho(\omega_X)}$ as an object of $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X)}$ (or $\text{KL}(G)_{\text{crit},\rho(\omega_X)}$).

In particular, we obtain an action of $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ on $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)}$, compatible with the action of $\mathfrak{L}(G)_{\rho(\omega_X)}$. The conclusion of Lemma 4.7.12 renders automatically to the present twisted context.

In addition, we have:

Corollary 4.7.15.

(a) *The category* $\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)})$ *carries a unique action of* $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$, *compatible with the action of* $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ *on* $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)}$ *and the projection*

$$\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)} \to \text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)}).$$

(b) *The category* $\mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)})$ *carries a unique action of* $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$, *compatible with the action of* $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ *on* $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}$ *and the embedding*

$$\mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}) \hookrightarrow \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}.$$

(c) *The functor*

$$\Theta_{\mathrm{Whit}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)})} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}) \to \mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)})$$

carries a natural $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$*-linear structure.*

4.8. **The enhanced functor of Drinfeld-Sokolov reduction at the critical level.** In this subsection we will study the functor

$$\mathrm{DS} : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Vect} \tag{4.24}$$

of (2.6).

4.8.1. Consider the factorization unit

$$\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)} \in \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}.$$

The functor (4.24) has a natural lax unital factorization structure (see Sect. 2.3.4). In particular, the object

$$\mathrm{DS}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)})$$

is naturally a factorization algebra (in Vect).

4.8.2. The action map

$$\mathfrak{z}_{\mathfrak{g}} \otimes \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}$$

gives rise to a map

$$\mathfrak{z}_{\mathfrak{g}} \otimes \mathrm{DS}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{DS}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)})$$

as factorization algebras.

Pre-composing with the unit for $\mathrm{DS}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)})$, we obtain a map of factorization algebras

$$\mathfrak{z}_{\mathfrak{g}} \to \mathrm{DS}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}). \tag{4.25}$$

We have the following fundamental result, see [FF]:

Theorem 4.8.3. *The map* (4.25) *is an isomorphism.*

4.8.4. By Sect. 4.1.6, the functor DS of (4.24) naturally lifts to a functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{DS}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)})\text{-mod}^{\mathrm{fact}}.$$

Restricting along (4.25), we can view it as a functor, to be denoted

$$\mathrm{DS}^{\mathrm{enh}} : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \to \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}.$$

4.8.5. Consider the functor

$$\Gamma^{\mathrm{IndCoh}}(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}), -) : \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})) \to \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}.$$

We claim:

Proposition 4.8.6. *There exists a uniquely defined (continuous) functor*

$$\mathrm{DS}^{\mathrm{enh,rfnd}} : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})),$$

satisfying

- *There exists an isomorphism*

$$\mathrm{DS}^{\mathrm{enh}} \simeq \Gamma^{\mathrm{IndCoh}}(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}), -)^{\mathrm{enh}} \circ \mathrm{DS}^{\mathrm{enh,rfnd}}; \tag{4.26}$$

- $\mathrm{DS}^{\mathrm{enh,rfnd}}$ *sends compact objects in* $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$ *to eventually coconnective (i.e., bounded below) objects in* $\mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$.

Furthermore, $\mathrm{DS}^{\mathrm{enh,rfnd}}$ *carries a uniquely defined factorization structure, so that* (4.26) *is an isomorphism of factorization functors.*

Proof. It is enough to show that the restriction of $\mathrm{DS}^{\mathrm{enh}}$ to the subcategory

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^c \subset \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$$

can be uniquely lifted to a functor

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^c \to \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))^{>-\infty}.$$

However, this follows from Corollary 4.3.12(a), using the fact that the initial functor DS sends

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^c \to \mathrm{Vect}^{>-\infty}.$$

□

4.8.7. Recall now that the functor DS factors via a functor

$$\overline{\mathrm{DS}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{Vect}.$$

It follows formally that the functor $\mathrm{DS}^{\mathrm{enh}}$ also factors via a functor, denoted

$$\overline{\mathrm{DS}}^{\mathrm{enh}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}.$$

We now quote the following fundamental result of [Ra2]:

Theorem 4.8.8. *The functor* $\mathrm{DS}^{\mathrm{enh,rfnd}}$ *factors via a functor*

$$\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})), \tag{4.27}$$

and the resulting functor $\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}$ *is an equivalence of factorization categories.*

Note that by construction

$$\Gamma^{\mathrm{IndCoh}}(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}), -) \circ \overline{\mathrm{DS}}^{\mathrm{enh,rfnd}} \simeq \overline{\mathrm{DS}}^{\mathrm{enh}}.$$

4.8.9. According to Sect. 4.6 (applied to the twist $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$ instead of the original $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$), the category $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$ carries an action of the monoidal category $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$.

By construction, the functor $\mathrm{DS}^{\mathrm{enh,rfnd}}$ intertwines the above $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$-action on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$ and the natural $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$-action on $\mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$.

According to Corollary 4.7.15(a), the $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$-action on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$ descends to an (a priori, uniquely defined) action on $\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$.

It follows formally that the functor $\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}$ intertwines this action and the $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$-action on $\mathrm{IndCoh}^*(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$.

5. The Feigin-Frenkel isomorphism and its applications

In this section we review the Feigin-Frenkel isomorphism, which provides a bridge between Kac-Moody representations and opers.

Using the Feigin-Frenkel isomorphism, we construct an action of $\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}})$ on $\mathrm{KL}(G)_{\mathrm{crit}}$, which is a key ingredient of the critical FLE functor, studied in the next section.

5.1. The Feigin-Frenkel isomorphism.

5.1.1. We quote the following fundamental result of Feigin and Frenkel ([FF]):

Theorem 5.1.2. *There exists a canonically defined isomorphism of factorization algebras*

$$\mathfrak{z}_{\mathfrak{g}} \overset{\mathrm{FF}_G}{\simeq} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}.$$

Below we will complement Theorem 5.1.2 by an assertion that describes how it interacts with geometric Satake, see Theorem 5.2.5.

5.1.3. *Example.* Let $G = T$ be a torus. Then $\mathfrak{z}_\mathfrak{g}$ is the commutative factorization algebra associated with the commutative algebra object

$$\mathrm{Sym}^!(\mathfrak{t} \otimes \mathrm{D}_X[1]) \in \mathrm{ComAlg}(\mathrm{D\text{-}mod}(X)),$$

i.e.,

$$\mathfrak{z}_\mathfrak{g} = \mathrm{Fact}(\mathrm{Sym}^!(\mathfrak{t} \otimes \mathrm{D}_X[1])),$$

see Sect. B.10.2 for the notation.

When we think of $\mathrm{D\text{-}mod}(X)$ as "left D-modules", the above object is

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathfrak{t} \otimes \mathrm{D}_X \otimes \omega_X^{\otimes -1}) \in \mathrm{ComAlg}(\mathrm{D\text{-}mod}^l(X)).$$

The affine D-scheme $\mathrm{Op}_{\check{T}}$ identifies with the scheme of jets $\mathrm{Jets}(\check{\mathfrak{t}} \otimes \omega_X)$ (see Sect. B.5.1). The isomorphism FF_T amounts to the tautological identification

$$\mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathfrak{t} \otimes \mathrm{D}_X \otimes \omega_X^{\otimes -1})) \simeq J(\check{\mathfrak{t}} \otimes \omega_X).$$

At the level of fibers at a given point $x \in X$, this is the isomorphism

$$\mathrm{Spec}(\mathrm{Sym}(\mathfrak{t} \otimes \mathcal{K}_x/\mathcal{O}_x)) \simeq \check{\mathfrak{t}} \otimes \omega_{\mathcal{D}_x},$$

corresponding to the canonical identification of pro-finite dimensional vector spaces

$$(\mathfrak{t} \otimes \mathcal{K}_x/\mathcal{O}_x)^\vee \simeq \check{\mathfrak{t}} \otimes \omega_{\mathcal{D}_x}.$$

5.2. **The "birth" of opers.** In this subsection we will formulate Theorem 5.2.5, which in [BD1, Sect. 5.3] was called "the birth of opers", that explains how the isomorphism FF_G interacts with geometric Satake.

5.2.1. Consider the (symmetric monoidal) functor

$$\mathrm{Rep}(\check{G}) \xrightarrow{\mathfrak{r}^*} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \simeq \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{com}} \overset{\mathrm{FF}_G}{\simeq} \mathfrak{z}_\mathfrak{g}\text{-mod}^{\mathrm{com}}. \tag{5.1}$$

In particular, the functor (5.1) allows us to view $\mathfrak{z}_\mathfrak{g}\text{-mod}^{\mathrm{com}}$ as a $\mathrm{Rep}(\check{G})$-module category, in a way compatible with factorization.

5.2.2. Let us view $\mathrm{KL}(G)_{\mathrm{crit}}$ as a module category over $\mathrm{Rep}(\check{G})$ via

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\longrightarrow} \mathrm{Sph}_G$$

and the Sph_G-action on $\mathrm{KL}(G)_{\mathrm{crit}}$. This structure is also compatible with factorization.

5.2.3. Finally, note that the action of $\mathfrak{z}_\mathfrak{g}$ on $\mathrm{Vac}(G)_{\mathrm{crit}}$ gives rise to a factorization functor

$$\mathfrak{z}_\mathfrak{g}\text{-mod}^{\mathrm{com}} \to \mathrm{KL}(G)_{\mathrm{crit}}, \quad - \underset{\mathfrak{z}_\mathfrak{g}}{\otimes} \mathrm{Vac}(G)_{\mathrm{crit}}. \tag{5.2}$$

5.2.4. We claim (see [BD1, Theorem 5.5.3]):

Theorem 5.2.5. *The functor* (5.2) *admits a lift to a functor between* $\mathrm{Rep}(\check{G})$*-module categories. This structure is compatible with factorization.*

Remark 5.2.6. Concretely, Theorem 5.2.5 says that the object $\mathrm{Vac}(G)_{\mathrm{crit}}$ satisfies the *Hecke property* with respect to the action of $\mathrm{Rep}(\check{G})$ on $\mathrm{KL}(G)_{\mathrm{crit}}$: i.e., we have

$$\mathrm{Sat}_G^{-1,\mathrm{nv}}(V) \star \mathrm{Vac}(G)_{\mathrm{crit}} \simeq \mathrm{Vac}(G)_{\mathrm{crit}} \underset{\mathfrak{z}_\mathfrak{g}}{\otimes} (\mathrm{FF}_G \circ (\mathfrak{r}^{\mathrm{reg}})^*(V)), \quad V \in \mathrm{Rep}(\check{G}),$$

where in the right-hand side we denoted by FF_G the equivalence

$$\mathfrak{z}_\mathfrak{g}\text{-mod}^{\mathrm{com}} \simeq \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{com}} = \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}),$$

induced by the isomorphism of algebras FF_G.

The above isomorphisms are compatible with tensor products of the V's in the natural sense.

5.2.7. From now on we will identify

$$\mathrm{Op}^{\mathrm{reg}}_{\check{G}} \simeq \mathrm{Spec}(\mathfrak{z}_{\mathfrak{g}}) \text{ and } \mathrm{Op}^{\mathrm{mer}}_{\check{G}} \simeq \text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})$$

using FF_G.

5.2.8. In particular, we will view the category $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$ as acted on by $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$. Similarly, we will view the functors $\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}$ (resp., $\mathrm{DS}^{\mathrm{enh,rfnd}}$) as $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-linear functors from $\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$ (resp., $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$) to $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$.

5.2.9. Here is one particular application of Theorem 5.2.5 that will be used in the sequel. Recall the commutative algebra (factorization) object

$$R_{\check{G},\mathrm{Op}} \in \mathrm{Rep}(\check{G}),$$

Sect. 4.8.

We claim that it acts on $\mathrm{Vac}(G)_{\mathrm{crit}} \in \mathrm{KL}(G)_{\mathrm{crit}}$, when we consider $\mathrm{KL}(G)_{\mathrm{crit}}$ as a $\mathrm{Rep}(\check{G})$-module category as in Sect. 5.2.2.

Indeed, by Theorem 5.2.5, in order to construct this structure, it suffices to construct an action of $R_{\check{G},\mathrm{Op}}$ on

$$\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}} \in \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}),$$

when we consider $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})$ as a $\mathrm{Rep}(\check{G})$-module category via $(\mathfrak{r}^{\mathrm{reg}})^*$. However, the latter structure comes from the map of commutative algebras in $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})$

$$(\mathfrak{r}^{\mathrm{reg}})^*(R_{\check{G},\mathrm{Op}}) \to \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}},$$

given by the counit of the $((\mathfrak{r}^{\mathrm{reg}})^*, \mathfrak{r}^{\mathrm{reg}}_*)$-adjunction.

5.3. **The Kazhdan-Lusztig category at the critical level and monodromy-free opers.** In this subsection we will show that the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-action on $\mathrm{KL}(G)_{\mathrm{crit}}$ factors through an action of $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$.

The construction will emulate the construction of the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-action on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$ in Sect. 4.6, with a “decoration” by $\mathrm{Rep}(\check{G})$.

Remark 5.3.1. The construction of such an action (along with its properties discussed in Sect. 5.4) at fixed $\underline{x} \in \mathrm{Ran}$ was the subject of the paper [FG5]. However, we do not know how to adapt the methods of *loc. cit.* to the factorization setting.

5.3.2. Using the duality between $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ and $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$, it suffices to construct a coaction of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$, viewed as a comonoidal factorization category, on $\mathrm{KL}(G)_{\mathrm{crit}}$.

As in Sect. 4.6.6, we first explain how to construct the coaction functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}. \tag{5.3}$$

We will then upgrade this to the datum of coaction.

5.3.3. We start with the monoidal action of $\mathrm{Rep}(\check{G})$ on $\mathrm{KL}(G)_{\mathrm{crit}}$ as in Sect. 5.2.2. Since $\mathrm{Rep}(\check{G})$ is rigid, the right adjoint to the action is a (lax unital) factorization functor

$$\mathrm{coact}_{\mathrm{Rep}(\check{G}),\mathrm{KL}(G)_{\mathrm{crit}}} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}. \tag{5.4}$$

This functor upgrades to a factorization functor

$$\begin{aligned} \mathrm{coact}^{\mathrm{enh}}_{\mathrm{Rep}(\check{G}),\mathrm{KL}(G)_{\mathrm{crit}}} &: \mathrm{KL}(G)_{\mathrm{crit}} \to \\ &\to \mathrm{coact}_{\mathrm{Rep}(\check{G}),\mathrm{KL}(G)_{\mathrm{crit}}}(\mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}). \end{aligned} \tag{5.5}$$

5.3.4. Recall now (see Sect. 5.2.9) that $R_{\check{G},\mathrm{Op}}$, viewed as an associative algebra object in $\mathrm{Rep}(\check{G})$ acts on $\mathrm{Vac}(G)_{\mathrm{crit}}$, compatibly with factorization. By adjunction, we obtain a map of factorization algebras

$$R_{\check{G},\mathrm{Op}} \otimes \mathrm{Vac}(G)_{\mathrm{crit}} \to \mathrm{coact}_{\mathrm{Rep}(\check{G}),\mathrm{KL}(G)_{\mathrm{crit}}}(\mathrm{Vac}(G)_{\mathrm{crit}}).$$

Restriction along the above map defines a functor

$$\begin{aligned}(5.6)\quad &\mathrm{coact}_{\mathrm{Rep}(\check{G}),\mathrm{KL}(G)_{\mathrm{crit}}}(\mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}) \to \\ &\to (R_{\check{G},\mathrm{Op}} \otimes \mathrm{Vac}(G)_{\mathrm{crit}})\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}) = R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}})\end{aligned}$$

Composing (5.5) and (5.6) we obtain a functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}). \tag{5.7}$$

The functor (5.4) is left t-exact. Hence, since the compact generators of $\mathrm{KL}(G)_{\mathrm{crit}}$ are eventually coconnective, the functor (5.7) gives rise to a functor

$$(\mathrm{KL}(G)_{\mathrm{crit}})^c \to (R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}))^{>-\infty}. \tag{5.8}$$

5.3.5. Consider the functor

$$\begin{aligned}(5.9)\quad &\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \to \\ &\to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}})\end{aligned}$$

As in Corollary 4.6.8, by combining Corollary C.16.12 and Proposition 4.4.7(a), we obtain:

Lemma 5.3.6. *The functor* (5.9) *induces an equivalence between the eventually coconnective subcategories of the two sides.*

Hence, we can view (5.8) as a functor

$$(\mathrm{KL}(G)_{\mathrm{crit}})^c \to \left(\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}\right)^{>-\infty} \hookrightarrow \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}) \otimes \mathrm{KL}(G)_{\mathrm{crit}}. \tag{5.10}$$

Ind-extending (5.10) we obtain the sought-for functor (5.3).

5.3.7. Our next goal is upgrade (5.3) to a datum of coaction of $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$ on $\mathrm{KL}(G)_{\mathrm{crit}}$. We do so by mimicking the strategy in Sects. 4.6.10-4.6.13.

First, we generalize the construction in Sects. 5.3.3-5.3.5 above and define the n-ry operation

$$\mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})^{\otimes n} \otimes \mathrm{KL}(G)_{\mathrm{crit}}. \tag{5.11}$$

5.3.8. We have the following analog of Lemma 4.6.12:

Lemma 5.3.9. *The functor* (5.11) *is t-exact.*

Proof. Note that the functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})^{\otimes n} \otimes \mathrm{KL}(G)_{\mathrm{crit}} \xrightarrow{(\iota^{\mathrm{mon\text{-}free}})^{\mathrm{IndCoh}}_* \otimes \mathbf{oblv}_{\mathfrak{L}^+(G)}} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \tag{5.12}$$

is t-exact and conservative.

Hence, it is enough to show that the composition of (5.11) with (5.12) is t-exact.

Note, however, that by construction, we have a commutative diagram

$$\begin{tikzcd} \mathrm{KL}(G)_{\mathrm{crit}} \arrow[r, "(5.11)"] \arrow[d, "\mathbf{oblv}_{\mathfrak{L}^+(G)}"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})^{\otimes n} \otimes \mathrm{KL}(G)_{\mathrm{crit}} \arrow[d, "(\iota^{\mathrm{mon\text{-}free}})^{\mathrm{IndCoh}}_* \otimes \mathbf{oblv}_{\mathfrak{L}^+(G)}"] \\ \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \arrow[r, "(4.20)"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}. \end{tikzcd}$$

Since $\mathbf{oblv}_{\mathfrak{L}^+(G)}$ is t-exact, the assertion follows from that of Lemma 4.6.12. □

5.3.10. As in Corollary 4.6.14, from Lemma 5.3.9, we obtain that the coaction functor (5.3) is associative *at the homotopy level*. We will equip it with a structure of *coherent homotopy* in Sect. J.5.

5.4. **Properties of the** $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$**-action on** $\mathrm{KL}(G)_{\mathrm{crit}}$**.** In this subsection we will discuss those properties of the $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$-action on $\mathrm{KL}(G)_{\mathrm{crit}}$ that will be used in the sequel.

5.4.1. First, unwinding the construction, we obtain:

Lemma 5.4.2. *The action of* $\mathrm{Rep}(\check{G})$ *on* $\mathrm{KL}(G)_{\mathrm{crit}}$ *given by*

$$\mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}}) \xrightarrow{\mathfrak{r}^*} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \xrightarrow{-\otimes\omega_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \tag{5.13}$$

and the above action of $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ *on* $\mathrm{KL}(G)_{\mathrm{crit}}$ *identifies canonically with the action given by*

$$\mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}^{-1,\mathrm{nv}}_{G}} \mathrm{Sph}_G$$

and the Sph_G*-action on* $\mathrm{KL}(G)_{\mathrm{crit}}$.

5.4.3. Further, comparing with the construction of the action of $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ on $\mathrm{KL}(G)_{\mathrm{crit}}$ given by Corollary 4.7.12, we obtain that this action coincides with the precomposition of the above action of $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ on $\mathrm{KL}(G)_{\mathrm{crit}}$ with

$$\iota^! : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}).$$

5.4.4. We now claim:

Corollary 5.4.5. *For a fixed* $\underline{x} \in \mathrm{Ran}$*, the action functor of* $\mathrm{Rep}(\check{G})_{\underline{x}}$ *on* $\mathrm{KL}(G)_{\mathrm{crit},\underline{x}}$ *via* $\mathrm{Sat}^{-1,\mathrm{nv}}_G$ *and the* $\mathrm{Sph}_{G,\underline{x}}$*-action on* $\mathrm{KL}(G)_{\mathrm{crit},\underline{x}}$ *is t-exact.*

Proof. By Lemma 5.4.2, we need to show that for $V \in \mathrm{Rep}(\check{G})^{\heartsuit}_{\underline{x}}$, the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \overset{V\otimes\mathrm{Id}}{\to} \mathrm{Rep}(\check{G})_{\underline{x}} \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})) \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}}$$

is t-exact, where $\mathrm{Rep}(\check{G}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ is the functor (5.13).

Hence, it suffices to show that if $\mathcal{E}^{\text{mon-free}} \in \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ is a *vector bundle*, then its action on $\mathrm{KL}(G)_{\mathrm{crit},\underline{x}}$ via

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \xrightarrow{-\otimes\omega_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

and the $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$-action on $\mathrm{KL}(G)_{\mathrm{crit},\underline{x}}$ is t-exact.

With no restriction of generality, we can assume that $\mathcal{E}^{\text{mon-free}}$ is the restriction of a vector bundle $\mathcal{E}^{\mathrm{mer}}$ over $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$. Hence, by Sect. 5.4.3, it suffices to show that for a vector bundle $\mathcal{E}^{\mathrm{mer}}$ on $\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}$, its action on $\mathrm{KL}(G)_{\mathrm{crit},\underline{x}}$ via

$$\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \xrightarrow{-\otimes\omega_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}} \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$$

and the action of $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$ given by Corollary 4.7.12 is t-exact.

Since the forgetful functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\underline{x}}$$

is t-exact and conservative, it suffices to show that the action of $\mathcal{E}^{\mathrm{mer}}$ on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\underline{x}}$ is t-exact.

Unwinding the construction, it suffices to show that the composition

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\underline{x}} \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \xrightarrow{\mathrm{Id}\otimes(\mathcal{E}^{\mathrm{mer}}\otimes-)}$$

$$\to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\underline{x}} \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \xrightarrow{\mathrm{Id}\otimes\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}},-)} \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\underline{x}}$$

is t-exact.

In the above composition, the first and the third arrows are t-exact. Hence, it suffices to show that the functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\underline{x}} \otimes \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}}) \overset{\text{Id}\otimes(\mathcal{E}^{\text{mer}}\otimes -)}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\underline{x}} \otimes \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})$$

is t-exact.

However, this easily follows from the fact that the functor

$$\text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}}) \overset{(\mathcal{E}^{\text{mer}}\otimes -)}{\longrightarrow} \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},\underline{x}})$$

is t-exact.

□

6. The critical FLE

In this section we prove the main result of Part I, namely, the *critical FLE*, Theorem 6.1.4, which says that there exists a canonical equivalence of factorization categories

$$\text{FLE}_{G,\text{crit}} : \text{KL}(G)_{\text{crit}} \overset{\sim}{\to} \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}}), \tag{6.1}$$

The functor in one direction in (6.1) is a variation on the theme of the functor $\overline{\text{DS}}^{\text{enh,rfnd}}$ from Sect. 4.8. Essentially $\text{FLE}_{G,\text{crit}}$ is obtained by base changing $\overline{\text{DS}}^{\text{enh,rfnd}}$ along the map from $\text{Op}^{\text{mon-free}}_{\check{G}}$ to $\text{Op}^{\text{mer}}_{\check{G}}$.

6.1. **Construction of the critical FLE functor.**

6.1.1. Let $\mathbf{C}$ be a category equipped with a $\mathfrak{L}(G)_{\rho(\omega_X)}$-action at the critical level, in a way compatible with factorization. Consider the functor

$$\text{Sph}(\mathbf{C}) := \mathbf{C}^{\mathfrak{L}(G)^+_{\rho(\omega_X)}} \to \mathbf{C} \to \mathbf{C}_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi} =: \text{Whit}_*(\mathbf{C}). \tag{6.2}$$

We apply this to $\mathbf{C} = \widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X)}$. Consider the resulting (factorization) functor

$$\text{KL}(G)_{\text{crit},\rho(\omega_X)} \to \text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X)}) \tag{6.3}$$

Composing, we obtain a functor

$$\text{KL}(G)_{\text{crit},\rho(\omega_X)} \to \text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X)}) \overset{\overline{\text{DS}}^{\text{enh,rfnd}}}{\longrightarrow} \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G}}). \tag{6.4}$$

6.1.2. We regard $\text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G}})$ as equipped with a natural action of $\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G}})$. We regard $\text{KL}(G)_{\text{crit},\rho(\omega_X)}$ as acted on by $\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}})$.

By Sects. 5.4.3 and 4.8.9 and Corollary 4.7.12(a), the functor (6.4) is compatible with the $\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G}})$-actions on the two sides. Furthermore, by Sect. 5.4.3, the $\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G}})$-action on $\text{KL}(G)_{\text{crit},\rho(\omega_X)}$ factors through an action of $\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}})$.

Hence, the functor (6.4) gives rise to a (factorization) functor

$$\text{KL}(G)_{\text{crit},\rho(\omega_X)} \to \text{Funct}_{\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G}})}(\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}}), \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G}})). \tag{6.5}$$

Finally, recall that by Lemma 3.5.7, we have a canonical identification

$$\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}}) \simeq \text{Funct}_{\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G}})}(\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}}), \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G}})).$$

Thus, we can interpret (6.5) as a $\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}})$-linear functor

$$\text{KL}(G)_{\text{crit},\rho(\omega_X)} \to \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}}) \tag{6.6}$$

6.1.3. Precomposing (6.6) with $\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\simeq} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}$, we obtain a functor

$$\mathrm{FLE}_{G,\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}) \tag{6.7}$$

The functor (6.7) is the critical FLE functor. The main result of Part I of this paper reads:

Theorem 6.1.4. *The functor* $\mathrm{FLE}_{G,\mathrm{crit}}$ *is an equivalence of factorization categories.*

This theorem will be proved in the course of this and the next two sections.

6.1.5. Unwinding the definitions, we observe that the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ carries a natural lax unital structure (as a functor between unital factorization categories). In particular, we obtain a canonical homomorphism

$$\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}} = \mathbf{1}_{\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})} \to \mathrm{FLE}_{G,\mathrm{crit}}(\mathbf{1}_{\mathrm{KL}(G)_{\mathrm{crit}}}) = \mathrm{Vac}(G)_{\mathrm{crit}} \tag{6.8}$$

as factorization algebras in $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$.

However, we claim:

Lemma 6.1.6. *The map* (6.8) *is an isomorphism.*

Proof. By Proposition 3.3.5(a), it suffices to show that the map (6.8) becomes an isomorphism after applying the functor $(\iota^{\mathrm{mon\text{-}free}})^{\mathrm{IndCoh}}_*$. Since the latter is also a strict unital factorization functor, the resulting homomorphism identifies with a homomorphism of factorization algebras

$$\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}} = \mathbf{1}_{\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})} \to \mathrm{DS}^{\mathrm{enh,rfnd}}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}),$$

corresponding to the lax unital functor

$$\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \hat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh,rfnd}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).$$

However, the latter isomorphism is the content of Theorem 4.8.3.

□

6.1.7. Combining Lemmas 6.1.6 and C.11.23, we obtain:

Corollary 6.1.8. *The functor* $\mathrm{FLE}_{G,\mathrm{crit}}$ *is strictly unital.*

6.2. **Reduction to the pointwise version.**

6.2.1. Fix a point $x \in X$. The pointwise version of Theorem 6.1.4 reads:

Theorem 6.2.2. *The functor* $\mathrm{FLE}_{G,\mathrm{crit}}$ *induces an equivalence*

$$\mathrm{KL}(G)_{\mathrm{crit},x} \overset{\sim}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},x}).$$

Obviously, Theorem 6.1.4 implies Theorem 6.2.2. However, in this subsection we will show that the coverse implication also takes place.

In its turn, Theorem 6.2.2 is known: it is the main result of the paper [FG2]. We will, however, supply a different proof, in which we deduce it from Theorem 4.8.8, see Sect. 7.2.

6.2.3. A key step in proving the implication Theorem 6.2.2 ⇒ Theorem 6.1.4 is the following:

Proposition 6.2.4. *The functor* $\mathrm{FLE}_{G,\mathrm{crit}}$ *preserves compactness.*[31]

[31] See Sect. B.11.10 for what it means for a factorization functor to preserve compactness.

Proof. By Proposition 3.3.5(b), it suffices to show that the composite functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}) \overset{(\iota^{\mathrm{mon\text{-}free}})^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$$

preserves compactness.

I.e., it suffices to show that (6.4) preserves compactness. Since $\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}$ is an equivalence, it suffices to show that the functor (6.3) preserves compactness. However, we claim that this is true more generally.

Namely, we claim that the functor (6.2) admits a continuous right adjoint (and hence, preserves compactness). Indeed, the right adjoint in question is given by[32] convolution with the vacuum object (i.e., the factorization unit)

$$\mathrm{Vac}_{\mathrm{Whit}^!(G)} \in \mathrm{Whit}^!(G) \simeq \text{D-mod}_{\frac{1}{2}}(\mathfrak{L}(G)_{\rho(\omega_X)})^{(\mathfrak{L}(N)_{\rho(\omega_X)},\chi),\mathfrak{L}^+(G)_{\rho(\omega_X)}}.$$

□

6.2.5. Given Proposition 6.2.4 and Theorem 6.2.2, we will deduce Theorem 6.1.4 using the following principle:

Let $F : \mathbf{C}^1 \to \mathbf{C}^2$ be a factorization functor between factorization categories. Assume that $\mathbf{C}^1$ is compactly generated, and assume that F preserves compactness.

Proposition 6.2.6. *If the induced functor $F_x : \mathbf{C}^1_x \to \mathbf{C}^2_x$ is an equivalence for any field-valued point x, then the original functor F is also an equivalence.*

Proof. The assumption that F preserves compactness implies that its right adjoint F^R is also equipped with a factorization structure. We need to show that the unit and the counit of the (F, F^R)-adjunction are isomorphisms.

The latter assertion can be checked strata-wise on Ran. I.e., we have to show that for every n, the corresponding functor

$$F_{\overset{\circ}{X}{}^{(n)}} : \mathbf{C}^1_{\overset{\circ}{X}{}^{(n)}} \to \mathbf{C}^2_{\overset{\circ}{X}{}^{(n)}}$$

is an equivalence.

By factorization, the latter statement reduces to the case $n = 1$, i.e., we have to show that

$$F_X : \mathbf{C}^1_X \to \mathbf{C}^2_X$$

is an equivalence.

The latter fact can be also checked after base-changing to field-valued points.

□

□[Theorem 6.1.4]

6.3. **The inverse of the critical FLE functor.** In this subsection we will assume the statement of Theorem 6.1.4, which was proved modulo Theorem 6.2.2.

6.3.1. Let $\mathbf{C}$ be a category, acted on by $\mathfrak{L}(G)_{\rho(\omega_X)}$. Note that in addition to the functor

$$\mathrm{Sph}(\mathbf{C}) \to \mathbf{C} \to \mathrm{Whit}_*(\mathbf{C}), \tag{6.9}$$

one can consider the functor

$$\mathrm{Whit}^!(\mathbf{C}) \to \mathbf{C} \overset{\mathrm{Av}_*^{\mathfrak{L}^+(G)_{\rho(\omega_X)}}}{\to} \mathrm{Sph}(\mathbf{C}). \tag{6.10}$$

We have the following elementary assertion:

[32] In Lemma 6.3.2 we will give another description of this right adjoint.

Lemma 6.3.2. *The composite*

$$\mathrm{Whit}_*(\mathbf{C}) \overset{\Theta_{\mathrm{Whit}(\mathbf{C})}}{\to} \mathrm{Whit}^!(\mathbf{C}) \overset{(6.10)}{\longrightarrow} \mathrm{Sph}(\mathbf{C}) \tag{6.11}$$

identifies canonically with the right adjoint of (6.9).

Remark 6.3.3. This lemma is embedded into the machinery developed in [Ra2]. We supply a proof for completeness.

Proof. We need to check that for $\mathcal{F}_{\mathrm{Sph}} \in \mathrm{Sph}(\mathbf{C})$ and $\mathcal{F} \in \mathbf{C}$, we have

$$\mathcal{H}om_{\mathrm{Whit}_*(\mathbf{C})}(\overline{\mathcal{F}}_{\mathrm{Sph}}, \overline{\mathcal{F}}) \simeq \mathcal{H}om_{\mathbf{C}}(\mathcal{F}_{\mathrm{Sph}}, \Theta_{\mathrm{Whit}(\mathbf{C})}(\overline{\mathcal{F}})),$$

where $\overline{\mathcal{F}}_{\mathrm{Sph}}, \overline{\mathcal{F}}$ denotes the image of $\mathcal{F}_{\mathrm{Sph}}$ and $\mathcal{F}$ along $\mathbf{C} \to \mathrm{Whit}_*(\mathbf{C})$.

Unwinding the definitions, we reduce the assertion to the case when $\mathbf{C} := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})$ and $\mathcal{F}_{\mathrm{Sph}} = \delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}}$.

Applying the definition of $\mathrm{Whit}_*(G)$, we calculate

$$\mathcal{H}om_{\mathrm{Whit}_*(G)}(\overline{\delta}_{1,\mathrm{Gr}_{G,\rho(\omega_X)}}, \overline{\mathcal{F}}) = \underset{\alpha}{\mathrm{colim}}\, \mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})_{N^\alpha,\chi}}((\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}})^\alpha, \mathcal{F}^\alpha), \tag{6.12}$$

where:

- N^α is a filtered family of group subschemes that comprise $\mathfrak{L}(N)_{\rho(\omega_X)}$;
- $(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}})^\alpha$ and $\mathcal{F}^\alpha$ denote the projections of the corresponding objects along

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})_{N^\alpha,\chi}.$$

We have

$$\mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})_{N^\alpha}}((\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}})^\alpha, \mathcal{F}^\alpha) \simeq \mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})}(\mathrm{Av}_*^{N^\alpha,\chi}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}}), \mathcal{F}),$$

which we further rewrite as

$$\mathrm{C}^\cdot\left(\mathrm{Gr}_{G,\rho(\omega_X)}, \mathbb{D}\left(\mathrm{Av}_*^{N^\alpha,\chi}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}})\right) \overset{!}{\otimes} \mathcal{F}\right) \simeq \mathrm{C}^\cdot(\mathrm{Gr}_{G,\rho(\omega_X)}, \mathrm{Av}_!^{N^\alpha,\chi}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}}) \overset{!}{\otimes} \mathcal{F}).$$

Hence, we can rewrite (6.12) as

$$\mathrm{C}^\cdot\left(\mathrm{Gr}_{G,\rho(\omega_X)}, \underset{\alpha}{\mathrm{colim}}\,(\mathrm{Av}_!^{N^\alpha,\chi}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}})) \overset{!}{\otimes} \mathcal{F}\right) \simeq \\ \simeq \mathrm{C}^\cdot\left(\mathrm{Gr}_{G,\rho(\omega_X)}, \mathrm{Av}_!^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}}) \overset{!}{\otimes} \mathcal{F}\right). \tag{6.13}$$

Now, the cleanness property from Sect. 1.3.15 implies that the natural map

$$\mathrm{Av}_!^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}}) \to \mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}})$$

is an isomorphism, where $\mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}$ is the functor of $*$-convolution with $\omega^{\mathrm{ren}}_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}$.

Hence, we further rewrite (6.13) as

$$\mathrm{C}^\cdot\left(\mathrm{Gr}_{G,\rho(\omega_X)}, \mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}}) \overset{!}{\otimes} \mathcal{F}\right) \simeq \\ \simeq \mathrm{C}^\cdot\left(\mathrm{Gr}_{G,\rho(\omega_X)}, \delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}} \overset{!}{\otimes} \mathrm{Av}_{*,\mathrm{ren}}^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}(\mathcal{F})\right) \simeq \\ \simeq \mathcal{H}om_{\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})}(\delta_{1,\mathrm{Gr}_{G,\rho(\omega_X)}}, \Theta_{\mathrm{Whit}(G)}(\overline{\mathcal{F}})),$$

as desired.

□

6.3.4. We will now use Lemma 6.3.2 to give an explicit description of the inverse of the functor $\mathrm{FLE}_{G,\mathrm{crit}}$.

Consider the functor (6.11) for $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$

$$\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}. \tag{6.14}$$

By the same logic as in Sect. 6.1.2, the functor (6.14) gives rise to a functor

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}. \tag{6.15}$$

Combining with the equivalence $\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}$, we obtain a functor

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}. \tag{6.16}$$

Combining with the equivalence (3.15), from (6.16) we obtain a functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}. \tag{6.17}$$

We will prove:

Proposition 6.3.5. *The functor* (6.17) *is the inverse of* (6.6).

The rest of this subsection is devoted to the proof of Proposition 6.3.5.

6.3.6. We need to show that the composition

$$\begin{split}\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\mathrm{Id}\otimes\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}}{\longrightarrow} \\ \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})\end{split} \tag{6.18}$$

is isomorphic to

$$\begin{split}\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{(6.16)}{\longrightarrow} \\ \to \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{(6.6)}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}).\end{split} \tag{6.19}$$

Both functors are $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$-linear. Hence, by the

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})\text{-}\mathbf{mod} \rightleftarrows \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})\text{-}\mathbf{mod}$$

adjunction, it suffices to show that the functors

$$\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \overset{(\iota^{\text{mon-free}})^!}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \tag{6.20}$$

and

$$\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{(6.15)}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{(6.6)}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$$

are isomorphic as $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-linear functors, i.e., that the diagram

$$\begin{CD} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) @>{(6.15)}>> \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \\ @V{\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}}VV @VV{(6.6)}V \\ \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) @>{(\iota^{\text{mon-free}})^!}>> \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \end{CD}$$

commutes, in a way compatible with the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-actions.

6.3.7. Since $\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}$ and (6.6) are both equivalences, it suffices to show that the diagram obtained by passing to left adjoints along the horizontal arrows, i.e.,

$$\begin{tikzcd}
\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \arrow[d, "\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}"'] & & \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \arrow[ll] \arrow[d, "(6.6)"] \\
\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) & & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \arrow[ll, "(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*"']
\end{tikzcd}$$

commutes, in a way compatible with the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-actions.

6.3.8. However, according to Lemma 6.3.2, the top vertical arrow in the latter diagram is the functor (6.3), and the corresponding diagram commutes by construction.

□[Proposition 6.3.5]

6.4. Compatibility of $\mathrm{FLE}_{G,\mathrm{crit}}$ and $\mathrm{FLE}_{\check{G},\infty}$.

6.4.1. Note that by construction, the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ makes the following diagram commute

$$\begin{tikzcd}
\mathrm{KL}(G)_{\mathrm{crit}} \arrow[r, "\alpha_{\rho(\omega_X),\mathrm{taut}}", "\sim"'] \arrow[d, "\mathrm{FLE}_{G,\mathrm{crit}}"'] & \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \arrow[r] & \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \arrow[d, "\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}"] \\
\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \arrow[rr, "(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*"] & & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).
\end{tikzcd}$$

6.4.2. Note that the functor (6.2) can be expanded to a functor

$$\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{C}) \to \mathrm{Whit}_*(\mathbf{C}). \tag{6.21}$$

Applying this to $\mathbf{C} = \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$, we obtain a functor

$$\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}). \tag{6.22}$$

Composing with $\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\simeq} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}$ and $\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}$, we obtain a (factorization) functor

$$\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}). \tag{6.23}$$

6.4.3. Similarly, the functor

$$(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$$

can be expanded to a (factorization) functor

$$\mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}), \tag{6.24}$$

6.4.4. We claim:

Theorem 6.4.5.

(a) *The functor* $\mathrm{FLE}_{G,\mathrm{crit}}$ *can be canonically endowed with the datum of compatibility with the* Sph_G*-action on* $\mathrm{KL}(G)_{\mathrm{crit}}$ *and the* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$*-action on* $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$*, where we identify*

$$\mathrm{Sph}_G \simeq \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$$

via Sat_G.

(b) *Under the identification of point (a), the diagram*

$$
\begin{tikzcd}
\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \arrow[r, "(6.23)"] \arrow[d, "\mathrm{FLE}^{-1}_{\check{G},\infty} \otimes \mathrm{FLE}_{G,\mathrm{crit}}"', "\sim"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \arrow[d, "\sim"', "\mathrm{Id}"] \\
\mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \arrow[r, "(6.24)"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})
\end{tikzcd} \tag{6.25}
$$

canonically commutes.

This theorem will be proved in the factorization setting in Sect. E.10.

6.4.6. Let us denote by $\mathsf{P}^{\mathrm{loc,enh,rfnd}}_G$ the precomposition of (6.23) with the projection

$$\mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}}.$$

Explicitly, it is given by

$$
\begin{aligned}
\mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit}} &\overset{\mathrm{Id}\otimes\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \\
&\to \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{(6.22)}{\longrightarrow} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).
\end{aligned} \tag{6.26}
$$

Let $\mathsf{P}^{\mathrm{loc}}_G$ and $\mathsf{P}^{\mathrm{loc,enh}}_G$ denote the compositions of $\mathsf{P}^{\mathrm{loc,enh,rfnd}}_{\check{G}}$ with the forgetful functors

$$\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}, -) : \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathrm{Vect} \tag{6.27}$$

and

$$\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}, -)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}, \tag{6.28}$$

respectively. (These two functors are obtained by replacing the last arrow in (6.26) by $\overline{\mathrm{DS}}$ and $\overline{\mathrm{DS}}^{\mathrm{enh}}$, respectively.)

6.4.7. Let us denote by $\mathsf{P}^{\mathrm{loc,enh,rfnd}}_{\check{G}}$ the precomposition of (6.24) with the projection

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to \mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}).$$

Explicitly, it is given by

$$
\begin{aligned}
\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) &\overset{\mathfrak{r}^*\otimes\mathrm{Id}}{\to} \\
&\to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \overset{\otimes}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \overset{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*}{\to} \\
&\to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).
\end{aligned} \tag{6.29}
$$

Let $\mathsf{P}^{\mathrm{loc}}_{\check{G}}$ and $\mathsf{P}^{\mathrm{loc,enh}}_{\check{G}}$ denote the compositions of $\mathsf{P}^{\mathrm{loc,enh,rfnd}}_{\check{G}}$ with the forgetful functors (6.27) and (6.28), respectively.

6.4.8. From Theorem 6.4.5 we immediately obtain:

Corollary 6.4.9. *The functors* $\mathsf{P}^{\mathrm{loc,enh,rfnd}}_G$ *and* $\mathsf{P}^{\mathrm{loc,enh,rfnd}}_{\check{G}}$ *match under the equivalences*

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \text{ and } \mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G).$$

And hence:

Corollary 6.4.10. *The functors* $\mathsf{P}^{\mathrm{loc}}_G$ *and* $\mathsf{P}^{\mathrm{loc}}_{\check{G}}$ *(resp.,* $\mathsf{P}^{\mathrm{loc,enh}}_G$ *and* $\mathsf{P}^{\mathrm{loc,enh}}_{\check{G}}$*) match under the equivalences*

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \text{ and } \mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G).$$

6.5. **The functor** pre-$\mathrm{FLE}_{G,\mathrm{crit}}$.

6.5.1. By the construction of the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ we have the following explicit descriptions of its compositions with various forgetful functors out of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$:

- The composition with the functor
$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \overset{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$$
is the functor
$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh,rfnd}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}});$$
- The composition with the functor
$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \overset{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \overset{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}},-)^{\mathrm{enh}}}{\longrightarrow} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}$$
is the functor
$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}}{\to} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}};$$
- The composition with the functor
$$\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\text{mon-free}}_{\check{G}},-) : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to \mathrm{Vect}$$
is the functor
$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}}{\to} \mathrm{Vect}; \tag{6.30}$$

In this subsection we will describe explicitly the composition of $\mathrm{FLE}_{G,\mathrm{crit}}$ with the functor
$$\mathfrak{r}^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to \mathrm{Rep}(\check{G}).$$

6.5.2. Define the (factorization) functor
$$\text{pre-FLE}_{G,\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{Rep}(\check{G}) \tag{6.31}$$
as the composition
$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}^!(G) \otimes \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\mathrm{CS}_G \otimes \mathrm{Id}}{\longrightarrow} \\ \to \mathrm{Rep}(\check{G}) \otimes \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\mathrm{Id} \otimes \overline{\mathrm{DS}}}{\longrightarrow} \mathrm{Rep}(\check{G}), \tag{6.32}$$
where the second arrow is obtained by duality from the pairing
$$\mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}).$$

6.5.3. We claim:

Proposition 6.5.4. *The functor* $\text{pre-FLE}_{G,\mathrm{crit}}$ *identifies canonically with* $\mathfrak{r}^{\mathrm{IndCoh}}_* \circ \mathrm{FLE}_{G,\mathrm{crit}}$.

The rest of this subsection is devoted to the proof of this proposition.

6.5.5. The next assertion results from the construction of the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ and Lemma 5.4.2:

Lemma 6.5.6. *The functor* $\mathfrak{r}^{\mathrm{IndCoh}}_* \circ \mathrm{FLE}_{G,\mathrm{crit}}$ *identifies with the composition*
$$\mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{Id} \otimes \alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \\ \to \mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Rep}(\check{G}) \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{Id} \otimes \mathrm{DS}}{\to} \mathrm{Rep}(\check{G}), \tag{6.33}$$
where the first arrow is the functor, right adjoint to the action of $\mathrm{Rep}(\check{G})$ *on* $\mathrm{KL}(G)_{\mathrm{crit}}$*, given by*
$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}^{-1,\mathrm{nv}}_G}{\longrightarrow} \mathrm{Sph}_G$$
and the Sph_G*-action on* $\mathrm{KL}(G)_{\mathrm{crit}}$.

Hence, in order to prove Proposition 6.5.4, it suffices to establish an isomorphism between (6.32) and (6.33).

6.5.7. We rewrite the functor in (6.33) as

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to$$
$$\to \mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Rep}(\check{G}) \otimes \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\mathrm{Id}\otimes\overline{\mathrm{DS}}}{\longrightarrow} \mathrm{Rep}(\check{G}).$$

Hence, in order to prove an isomorphism between (6.32) and (6.33), it suffices to show that the functors

$$\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Rep}(\check{G}) \otimes \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$$

and

$$\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}^!(G) \otimes \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\mathrm{CS}_G\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Rep}(\check{G}) \otimes \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$$

are canonically isomorphic.

6.5.8. By duality, this amounts to showing that the functors

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{Sat}_G^{-1,\mathrm{nv}}\otimes\mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{Sph}_G \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$$

and

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{FLE}_{\check{G},\infty}\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$$

are canonically identified.

6.5.9. This follows by combining the following observations:

- The functor $\mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$ factors as
$$\mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)});$$
- The functor $\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$ identifies with
$$\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{Vac}_{\mathrm{Whit}_*(G)}\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}),$$
where $\mathbf{1}_{\mathrm{Whit}_*(G)} \in \mathrm{Whit}_*(G)$ is the vacuum object;
- The functor $\mathrm{FLE}_{\check{G},\infty}$ identifies with
$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\longrightarrow} \mathrm{Sph}_G \overset{\mathrm{Vac}_{\mathrm{Whit}_*(G)}\star-}{\longrightarrow} \mathrm{Whit}_*(G)$$
(see Remark 1.7.7).

$\square$[Proposition 6.5.4]

6.6. **An alternative construction of the critical FLE functor.**

6.6.1. We start by observing:

Lemma 6.6.2. *There exists a canonical identification of factorization algebras in* $\mathrm{Rep}(\check{G})$

$$\text{pre-FLE}_{G,\mathrm{crit}}(\mathrm{Vac}(G)_{\mathrm{crit}}) \simeq R_{\check{G},\mathrm{Op}}. \tag{6.34}$$

Proof. Follows from Proposition 6.5.4 and the fact that the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ is unital (see Corollary 6.1.8).

$\square$

Remark 6.6.3. One can establish the isomorphism (6.34) directly, i.e., without appealing to the functor $\mathrm{FLE}_{G,\mathrm{crit}}$.

6.6.4. By Sect. 4.1.9 and Lemma 6.6.2, the functor pre-$\mathrm{FLE}_{G,\mathrm{crit}}$ upgrades to a functor

$$\text{pre-FLE}^{\mathrm{enh}}_{G,\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit}} \to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})).$$

Note that by construction we have a canonical isomorphism

$$\text{pre-FLE}^{\mathrm{enh}}_{G,\mathrm{crit}} \simeq \Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}, -)^{\mathrm{enh}} \circ \mathrm{FLE}_{G,\mathrm{crit}}\,.$$

Because of this isomorphism we will also use the notation

$$\mathrm{FLE}^{\mathrm{coarse}}_{G,\mathrm{crit}} := \text{pre-FLE}^{\mathrm{enh}}_{G,\mathrm{crit}}\,.$$

6.6.5. We claim now that the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ can be uniquely recovered from $\mathrm{FLE}^{\mathrm{coarse}}_{G,\mathrm{crit}}$.

Namely, by Proposition 4.4.7(a), it suffices to show that the functor $\mathrm{FLE}^{\mathrm{coarse}}_{G,\mathrm{crit}}$ sends compact objects in $\mathrm{KL}(G)_{\mathrm{crit}}$ to eventually coconnective objects in $R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))$.

6.6.6. Since the forgetful functor

$$R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})) \to \mathrm{Rep}(\check{G})$$

is t-exact and conservative, it suffices to show that the functor pre-$\mathrm{FLE}_{G,\mathrm{crit}}$ sends compact objects in $\mathrm{KL}(G)_{\mathrm{crit}}$ to eventually coconnective objects in $\mathrm{Rep}(\check{G})$.

Since the compact generators of $\mathrm{KL}(G)_{\mathrm{crit}}$ are eventually coconnective, it suffices to prove the following:

Lemma 6.6.7. *For a fixed* $\underline{x} \in \mathrm{Ran}$*, the functor*

$$\text{pre-FLE}_{G,\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{Rep}(\check{G})_{\underline{x}}$$

is t-exact.

Proof. We rewrite the functor pre-$\mathrm{FLE}_{G,\mathrm{crit}}$ as (6.33), or equivalently

(6.35)

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{R_{\check{G}}\otimes \mathrm{Id}}{\to} \mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{Id}\otimes \mathrm{Sat}_G^{-1,\mathrm{nv}}\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Rep}(\check{G}) \otimes \mathrm{Sph}_G \otimes \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{Id}\otimes(-\star-)}{\longrightarrow}$$
$$\to \mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{Id}\otimes \alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{Id}\otimes \mathrm{DS}}{\to} \mathrm{Rep}(\check{G})$$

In this composition, the first arrow is tautologically exact, and the second arrow is t-exact by Corollary 5.4.5. Hence, the assertion follows from Lemma 2.3.8.

□

6.6.8. Note that as a corollary of Lemma 6.6.7 and Proposition 6.5.4, we obtain:

Corollary 6.6.9. *For a fixed* $\underline{x} \in \mathrm{Ran}$*, the functor*

$$\mathrm{FLE}_{G,\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$$

is t-exact.

7. Proof of the pointwise version of the critical FLE

In this section we will give a proof of the pointwise version of the critical FLE by deducing it from Theorem 4.8.8.

The idea of the proof is the that the critical FLE is essentially the *base change* of the equivalence of Theorem 4.8.8 along $\mathrm{LS}^{\mathrm{reg}}_{\check{G},x} \to \mathrm{LS}^{\mathrm{mer}}_{\check{G},x}$. In fact, such an equivalence is a general phenomenon for categories acted on by $\mathfrak{L}(G)_x$, given a *temperedness* condition (see Proposition 7.5.5 for a precise statement).

The reason this proof only works for the pointwise version is that it is only in this case that we have a good grip on the base change operation alluded to above.

We note that a completely different proof of the pointwise FLE was given in the paper [FG2].

7.1. **Temperedness.**

7.1.1. Let

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x} \hookrightarrow \mathrm{Sph}^{\mathrm{spec}}_{\check{G},x} \tag{7.1}$$

be the *tempered subcategory*.

By definition, this is the essential image of

$$\Xi_{\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}} : \mathrm{QCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}) \to \mathrm{IndCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}).$$

The embedding (7.1) admits a right adjoint, namely,

$$\Psi_{\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}} : \mathrm{IndCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}) \to \mathrm{QCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}),$$

whose kernel is a monoidal ideal.

This allows us to view $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$ as a monoidal colocalization of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$.

Remark 7.1.2. The definition of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$ is specific to the pointwise version. We do not know how to define it in the factorization setting. The reason for this is the following:

Although we can define $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp}} := \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ in the factorization setting, we do not have the (Ξ,Ψ)-adjunction. The latter is a feature of a locally of finite type situation, which we *are* in at a fixed $\underline{x} \in \mathrm{Ran}$, but *not* when we are allowed to vary over Ran in families.

7.1.3. Let us regard

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}) \simeq \mathrm{Rep}(\check{G})$$

as a bimodule with respect to $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})$ and $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$.

Note, however, that the $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$-action on $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x})$ factors via $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$: indeed, the action is given by t-exact functors and the t-structure on $\mathrm{Rep}(\check{G})$ is separared.

7.1.4. Consider the corresponding functor

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x} \to \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x})). \tag{7.2}$$

The following results from the definitions:

Lemma 7.1.5. *The functor* (7.2) *is an equivalence.*

7.1.6. Let $\mathbf{C}$ be a module category over $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$. Denote

$$\mathbf{C}_{\mathrm{temp}} := \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}{\otimes} \mathbf{C}.$$

The adjunction

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x} \rightleftarrows \mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$$

gives rise to an adjunction

$$\mathbf{C}_{\mathrm{temp}} \rightleftarrows \mathbf{C}, \tag{7.3}$$

making $\mathbf{C}_{\mathrm{temp}}$ into a colocalization of $\mathbf{C}$.

We let $\mathrm{temp}_{\mathbf{C}}$ denote the comonad on $\mathbf{C}$ corresponding to the adjunction (7.3)

7.1.7. Let us regard the two sides of (7.2) as right modules with respect to $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$, where:

- $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$ acts on $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$ by right multiplication;
- $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$ acts on $\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}))$ via the target.

Tensoring (7.2) over $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$ with $\mathbf{C}$ we obtain a functor

$$\mathbf{C}_{\mathrm{temp}} \to \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})}\left(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}{\otimes} \mathbf{C}\right) \tag{7.4}$$

From Lemma 7.1.5 and the fact that $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$ is rigid we obtain:

Corollary 7.1.8. *The functor* (7.4) *is an equivalence.*

7.1.9. We will say that $\mathbf{C}$ is *tempered* if the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$ on $\mathbf{C}$ factors via $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$.

This is equivalent to the condition that the functors (7.3) are mutually inverse equivalences.

Remark 7.1.10. Note that Corollary 7.1.8 says that if $\mathbf{C}$ is tempered, then it can be recovered from the $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})$-module category

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}{\otimes} \mathbf{C}$$

by applying the functor

$$\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})}\left(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), -\right): \\ \mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})\text{-}\mathbf{mod} \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}\text{-}\mathbf{mod} \hookrightarrow \mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}\text{-}\mathbf{mod}.$$

7.1.11. From Corollary 7.1.8 we obtain:

Corollary 7.1.12. *The functor*

$$\mathbf{C} \mapsto \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}{\otimes} \mathbf{C}, \quad \mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}\text{-}\mathbf{mod} \to \mathrm{DGCat}$$

is conservative, when restricted to the subcategory

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}\text{-}\mathbf{mod} \subset \mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}\text{-}\mathbf{mod}.$$

7.1.13. Let

$$\mathrm{Sph}_{G,x} \to \mathrm{Sph}_{G,\mathrm{temp},x} \tag{7.5}$$

be the colocalization corresponding to the colocalization

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x} \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$$

(we can use either Sat_G or $\mathrm{Sat}_{G,\tau}$ to identify Sph_G with $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$; the resulting colocalizations are the same).

The definitions and results from this subsection render automatically to the setting of $\mathrm{Sph}_{G,x}$-module categories.

7.2. **Proof of Theorem 6.2.2.** We are now ready to prove the pointwise version of the FLE.

7.2.1. Recall (see Theorem 6.4.5) that the functor

$$\mathrm{FLE}_{G,\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit},x} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},x}) \tag{7.6}$$

intertwines the actions of $\mathrm{Sph}_{G,x}$ on the left-hand side with the $\mathrm{Sph}^{\mathrm{spec}}_{G,x}$-action on the right-hand side (via Sat_G), and makes the diagram

$$\begin{tikzcd} \mathrm{Whit}_*(G)_x \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},x} \arrow[r, "\mathrm{FLE}^{-1}_{\check{G},\infty} \otimes \mathrm{FLE}_{G,\mathrm{crit}}"] \arrow[d, "(6.23)"'] & \mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},x}) \arrow[d, "(6.24)"] \\ \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x}) \arrow[r, "\mathrm{Id}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x}) \end{tikzcd} \tag{7.7}$$

commute.

Note that Proposition 3.6.5 says that the right vertical arrow in (7.7) is fully faithful with essential image equal to

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x})_{\text{mon-free}} \subset \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x}).$$

We will prove:

Proposition 7.2.2. *The left vertical arrow in* (7.7) *is fully faithful with essential image equal to*

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x})_{\text{mon-free}} \subset \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x}).$$

7.2.3. We now claim:

Proposition 7.2.4. *The* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$*-module category* $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},x})$ *is tempered.*

Proof. Recall the functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}) \underset{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\mathrm{reg}})}{\otimes} \mathrm{IndCoh}^*((\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})^\wedge_{\text{mon-free}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}). \tag{7.8}$$

It intertwines the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$, where the action on the left-hand side is via the first factor.

Now, according to Proposition 3.8.11, the functor (7.8) is an equivalence. Hence, it is enough to show that the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$ on the left-hand side of (7.8) factors through $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$.

However, this follows from the fact that the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$ on

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}) \simeq \mathrm{Rep}(\check{G})$$

factors through $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x}$.

□

7.2.5. Finally, we claim:

Proposition 7.2.6. *The* $\mathrm{Sph}_{G,x}$*-module category* $\mathrm{KL}(G)_{\kappa,x}$ *is tempered.*

We now observe that the combination of Propositions 7.2.2, 7.2.4 and 7.2.6, together with Corollary 7.1.12, implies that (7.6) is an equivalence.

□[Pointwise FLE]

The rest of this section is devoted to the proof of Propositions 7.2.2 and 7.2.6.

7.3. **Proof of Proposition 7.2.2.**

7.3.1. Let $\mathbf{C}$ be a category equipped with a $\mathfrak{L}(G)_x$-action at the critical level. Consider now the category

$$\mathrm{Sph}(\mathbf{C}) := \mathbf{C}^{\mathfrak{L}^+(G)_x}.$$

Interpreting $\mathbf{C}^{\mathfrak{L}^+(G)_x}$ as

$$\mathrm{Funct}_{\mathfrak{L}(G)_x}(\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,x}), \mathbf{C}),$$

we obtain that $\mathrm{Sph}(\mathbf{C})$ carries a natural action of

$$\mathrm{Sph}_{G,x} \simeq \mathrm{End}_{\mathfrak{L}(G)_x}\left(\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,x})\right).$$

7.3.2. Denote

$$\mathbf{C}^{\text{Sph-gen}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,x}) \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{Sph}(\mathbf{C}).$$

We have a tautological functor

$$\mathbf{C}^{\text{Sph-gen}} \to \mathbf{C} \tag{7.9}$$

commuting with the $\mathfrak{L}(G)_x$-action.

The following is a standard result that results from the ind-properness of the affine Grassmannian:

Lemma 7.3.3. *The functor* (7.9) *is fully faithful and admits a continuous right adjoint.*[33]

7.3.4. We shall say that $\mathbf{C}$ is *spherically generated* if the functor (7.9) is an equivalence.

This is equivalent to requiring that $\mathbf{C}$ is generated, as a category acted on by $\mathfrak{L}(G)_x$, by the essential image of the forgetful functor

$$\mathrm{Sph}(\mathbf{C}) \to \mathbf{C}.$$

7.3.5. The above definitions apply when we replace $\mathfrak{L}(G)_x$ by $\mathfrak{L}(G)_{\rho(\omega_X),x}$. Applying to both sides of (7.9) the functor $\mathrm{Whit}_*(-)$, we obtain a functor

$$\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{Sph}(\mathbf{C}) \to \mathrm{Whit}_*(\mathbf{C}), \tag{7.10}$$

i.e., the functor (6.21).

From Lemma 7.3.3 we obtain:

Corollary 7.3.6. *The functor* (7.10) *is fully faithful.*

7.3.7. We apply Corollary 7.3.6 to

$$\mathbf{C} := \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X),x}.$$

Hence, we obtain that the left vertical arrow in (7.7) is fully faithful. Thus, to complete the proof of Proposition 7.2.2, it suffices to show that the essential image of (6.23) is contained in and generates

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x})_{\text{mon-free}} \subset \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x}).$$

[33]This right adjoint automatically commutes with the $\mathfrak{L}(G)_x$-action, essentially because $\mathfrak{L}(G)_x$ is a group.

7.3.8. Note that the essential image of

$$\mathrm{KL}(G)_{\mathrm{crit},x} \overset{\mathbf{1}_{\mathrm{Whit}_*(G)}\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Whit}_*(G)_x \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},x}$$

generates the target.

Indeed, this follows by interpreting the above functor as

$$\mathrm{KL}(G)_{\mathrm{crit},x} \simeq \mathrm{Rep}(\check{G}) \underset{\mathrm{Rep}(\check{G}),\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},x} \overset{\mathrm{FLE}_{\check{G},\infty}\otimes \mathrm{Id}}{\simeq}$$
$$\simeq \mathrm{Whit}_*(G)_x \underset{\mathrm{Sat}_{G,\tau}^{-1,\mathrm{nv}},\mathrm{Rep}(\check{G}),\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},x} \to \mathrm{Whit}_*(G)_x \underset{\mathrm{Sat}_{G,\tau}^{-1},\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x},\mathrm{Sat}_G^{-1}}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},x} \simeq$$
$$\simeq \mathrm{Whit}_*(G)_x \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},x}.$$

Since the essential image of $\mathrm{FLE}_{G,x}$ is contained in $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x})_{\mathrm{mon\text{-}free}}$, we obtain that so is the essential image of (6.23).

7.3.9. Hence, it remains to prove the following:

Lemma 7.3.10. *The essential image* $\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),x}$ *under*

$$\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),x} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X),x} \overset{\mathrm{DS}^{\mathrm{enh,rfnd}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},x})$$

generates $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x})_{\mathrm{mon\text{-}free}}$.

Proof. We prove the lemma by matching the generators.

The compact generators of $\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),x}$ are the Weyl modules

$$\mathbb{V}^{\lambda}_{\mathrm{crit}} := \mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_{\mathrm{crit}}}_{\mathfrak{L}^+(G)}(V^{\check{\lambda}}), \tag{7.11}$$

where $V^{\check{\lambda}} \in \mathrm{Rep}(G)$ is the irreducible representation of highest weight λ.

According to [FG3], the image of $\mathbb{V}^{\lambda}_{\mathrm{crit}}$ under $\mathrm{DS}^{\mathrm{enh,rfnd}}$ is the structure sheaf $\mathcal{O}_{\mathrm{Op}^{\lambda-\mathrm{reg}}_{\check{G},x}}$ of the subscheme

$$\mathrm{Op}^{\lambda-\mathrm{reg}}_{\check{G},x} \subset \mathrm{Op}^{\mathrm{mer}}_{\check{G},x}$$

of $\check{\lambda}$-opers.

We have

$$^{\mathrm{red}}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},x}) = \underset{\lambda}{\sqcup}\, \mathrm{Op}^{\lambda-\mathrm{reg}}_{\check{G},x},$$

from which it is clear that the objects

$$\mathcal{O}_{\mathrm{Op}^{\lambda-\mathrm{reg}}_{\check{G},x}} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x})$$

are the (compact) generators of $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},x})_{\mathrm{mon\text{-}free}}$.

□

7.4. **Proof of Proposition 7.2.6.**

7.4.1. We need to show that the functor

$$\mathrm{KL}(G)_{\mathrm{crit},x} := \mathrm{Sph}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},,x}) \to \mathrm{Sph}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},x})_{\mathrm{temp}} =: (\mathrm{KL}(G)_{\mathrm{crit},x})_{\mathrm{temp}} \tag{7.12}$$

is an equivalence.

First, note that by combining Corollary 7.1.8 with Propositions 7.2.2 and 7.2.4 we obtain that the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ factors as

$$\mathrm{KL}(G)_{\mathrm{crit},x} \overset{(7.12)}{\to} (\mathrm{KL}(G)_{\mathrm{crit},x})_{\mathrm{temp}} \overset{\mathrm{FLE}_{G,\mathrm{crit},\mathrm{temp}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},x}),$$

where $\mathrm{FLE}_{G,\mathrm{crit},\mathrm{temp}}$ is an equivalence.

Combined with Proposition 6.2.4, we obtain that the functor (7.12) preserves compactness. Since (7.12) is a colocalization, we obtain that (7.12) restricts to a colocalization on compacts. Hence, it is sufficient to prove that (7.12) is conservative on $(\mathrm{KL}(G)_{\mathrm{crit},x})^c$.

To prove the latter, it is sufficient to prove that the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ is conservative on $(\mathrm{KL}(G)_{\mathrm{crit},x})^c$.

7.4.2. Since $(\mathrm{KL}(G)_{\mathrm{crit},x})^c \subset (\mathrm{KL}(G)_{\mathrm{crit},x})^b$ and since $\mathrm{FLE}_{G,\mathrm{crit}}$ is t-exact (by Corollary 6.6.9), its suffices to show that $\mathrm{FLE}_{G,\mathrm{crit}}$ is conservative on $(\mathrm{KL}(G)_{\mathrm{crit},x})^\heartsuit$.

Using the fact that $\mathrm{FLE}_{G,\mathrm{crit},\mathrm{temp}}$ is an equivalence, we obtain that it is enough to show that the functor (7.12) is conservative on $(\mathrm{KL}(G)_{\mathrm{crit},x})^\heartsuit$.

7.4.3. Let $\mathrm{temp}_{\mathrm{KL}(G)_{\mathrm{crit},,x}}$ denote the temperization functor (see Sect. 7.1.6). We will prove:

Lemma 7.4.4. *The functor* $\mathrm{temp}_{\mathrm{KL}(G)_{\mathrm{crit},x}}$ *is right t-exact, and the counit map*

$$\mathrm{temp}_{\mathrm{KL}(G)_{\mathrm{crit},x}} \to \mathrm{Id}$$

induces an isomorphism on H^0 when applied to objects in $(\mathrm{KL}(G)_{\mathrm{crit},x})^\heartsuit$.

The lemma immediately implies the conservativity of (7.12) on $(\mathrm{KL}(G)_{\mathrm{crit},x})^\heartsuit$.

□[Proposition 7.2.6]

7.4.5. *Proof of Lemma 7.4.4.* The assertion of the lemma holds more generally for a $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$-module category, equipped with a t-structure, such that the action functor is t-exact. The corresponding property for $\mathrm{KL}(G)_{\mathrm{crit},x} \simeq \mathrm{KL}(G)_{\mathrm{crit},x}$ is guaranteed by Corollary 5.4.5.

Consider the temperization functor $\mathrm{temp}_{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}$ on $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$ itself, i.e., the composition of

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x} = \mathrm{IndCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}) \overset{\Psi_{\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}}}{\longrightarrow}$$

$$\to \mathrm{QCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}) \overset{\Xi_{\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x}) = \mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}.$$

It suffices to show that the object

$$\mathrm{temp}_{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}(\mathbf{1}_{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}})$$

lives in cohomological degrees ≤ 0 and that its 0th cohomology maps isomorphically to $\mathbf{1}_{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}$.

However, this is a general property of the composition $\Xi_Y \circ \Psi_Y$ on an eventually coconnective stack locally almost of finite type. Namely, for every $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})$, the counit of the adjuction

$$\Xi_Y \circ \Psi_Y(\mathcal{F}) \to \mathcal{F}$$

induces an isomorphism on the truncation $\tau^{\geq -n}$ for *any* n.

□[Lemma 7.4.4]

7.5. **Spherical vs Whittaker.** This subsection is not logically necessary for the sequel, but it carries an ideological significance. Here we explain how to realize the pointwise $\mathrm{FLE}_{G,\mathrm{crit}}$ functor as the base change of the functor $\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}$ along $\mathrm{LS}^{\mathrm{reg}}_{\check{G},x} \to \mathrm{LS}^{\mathrm{mer}}_{\check{G},x}$.

7.5.1. Let $\mathbf{C}$ be a category equipped with a $\mathfrak{L}(G)_{\rho(\omega_X),x}$-action, and assume that $\mathbf{C}$ is spherically generated.

Consider the corresponding category

$$\mathrm{Whit}_*(\mathbf{C}) := \mathbf{C}_{\mathfrak{L}(N)_{\rho(\omega_X),x},\chi}.$$

We claim that $\mathrm{Whit}_*(\mathbf{C})$ has a natural structure of module category over $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})$.

7.5.2. Indeed, we have

$$\mathrm{Whit}_*(\mathbf{C}) \simeq \mathrm{Whit}_*(G)_x \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{Sph}(\mathbf{C}), \tag{7.13}$$

so it is enough to show that $\mathrm{Whit}_*(G)_x$ carries a structure of $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})$-module category in a way that commutes with the $\mathrm{Sph}_{G,x}$-action.

We identify

$$\mathrm{Whit}_*(G)_x \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}),$$

where the $\mathrm{Sph}_{G,x}$-action on $\mathrm{Whit}_*(G)_x$ corresponds to the natural $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$-action on $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x})$ via $\mathrm{Sat}_{G,\tau}$ (see Corollary 1.8.5).

The desired $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})$-module structure on $\mathrm{Whit}(G)_x$ comes from the natural action of $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})$-action on $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x})$, which naturally commutes with the $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$-action.

7.5.3. For $\mathbf{C}$ as above, consider the category

$$\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{Whit}_*(\mathbf{C})). \tag{7.14}$$

Recall the functor (6.2)

$$\mathrm{Sph}(\mathbf{C}) \to \mathrm{Whit}_*(\mathbf{C}). \tag{7.15}$$

7.5.4. We claim:

Proposition 7.5.5. *The functor* (7.15) *lifts to a functor*

$$\mathrm{Sph}(\mathbf{C}) \to \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{Whit}_*(\mathbf{C})). \tag{7.16}$$

Moreover, the functor (7.16) *factors as*

$$\mathrm{Sph}(\mathbf{C}) \to \mathrm{Sph}(\mathbf{C})_{\mathrm{temp}} \overset{\sim}{\to} \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{Whit}_*(\mathbf{C})), \tag{7.17}$$

where the second arrow is an equivalence.

Proof. By (7.13), we have

$$\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{Whit}_*(\mathbf{C})) \simeq \tag{7.18}$$

$$\simeq \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{Whit}_*(G)_x) \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{Sph}(\mathbf{C}) \overset{\mathrm{FLE}_{G,\infty}}{\simeq}$$

$$\simeq \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^\wedge_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x})) \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{Sph}(\mathbf{C}) \simeq \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp}\,.x} \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{Sph}(\mathbf{C}).$$

$\square$

7.5.6. Take $\mathbf{C} := \widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X),x}$. So we can regard

$$\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X),x})$$

as acted on by $\text{QCoh}((\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})$ via the recipe of Sect. 7.5.1.

Let us regard

$$\text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}} \simeq \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},x})^\wedge_{\text{mon-free}}$$

as acted on by $\text{QCoh}((\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})$ via pullback along

$$\mathfrak{r} : (\text{Op}^{\text{mer}}_{\check{G},x})^\wedge_{\text{mon-free}} \to (\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}}.$$

Recall now that according to Proposition 7.2.2, the functor $\overline{\text{DS}}^{\text{enh,rfnd}}$ gives rise to an equivalence

$$\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X),x}) \overset{\overline{\text{DS}}^{\text{enh,rfnd}}}{\longrightarrow} \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}}. \tag{7.19}$$

We claim:

Proposition 7.5.7. *The equivalence* (7.19) *is compatible with the above actions of* $\text{QCoh}((\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})$.

Proof. Follows from the fact that the functor (7.19) can be recovered from $\text{FLE}_{G,\text{crit}}$ by applying to both sides the functor

$$\text{QCoh}(\text{LS}^{\text{reg}}_{G,x}) \underset{\text{Sph}^{\text{spec}}_{\check{G},x}}{\otimes} -.$$

□

Remark 7.5.8. The above proof of Proposition 7.5.7 relies relies on Theorem 6.4.5 as an essential ingredient.

In Sect. G we will give a different proof of Proposition 7.5.7, by showing that the functor (7.19) is compatible with the structure of *factorization module category* over $\text{Rep}(\check{G})$.

7.5.9. From Proposition 7.5.7 we obtain that we can regard the FLE functor

$$\text{FLE}_{G,\text{crit}} : \text{KL}(G)_{\text{crit},x} \to \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},x})$$

as obtained from the functor (7.16) for

$$\mathbf{C} := \widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X),x}$$

by precomposing with

$$\text{KL}(G)_{\text{crit},x} \overset{\alpha_{\rho(\omega_X),\text{taut}}}{\longrightarrow} \text{KL}(G)_{\text{crit},\rho(\omega_X),x} = \text{Sph}(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X),x})$$

and post-composing with

$$\begin{gathered}\text{Funct}_{\text{QCoh}((\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})}\left(\text{QCoh}(\text{LS}^{\text{reg}}_{\check{G},x}), \text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X),x})\right) \overset{\text{Proposition 7.5.7}}{\simeq} \\ \simeq \text{Funct}_{\text{QCoh}((\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})}\left(\text{QCoh}(\text{LS}^{\text{reg}}_{\check{G},x}), \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}}\right) \simeq \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},x}),\end{gathered} \tag{7.20}$$

where the last equivalence is obtained as follows:

$$\begin{gathered}\text{Funct}_{\text{QCoh}((\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})}\left(\text{QCoh}(\text{LS}^{\text{reg}}_{\check{G},x}), \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}}\right) \simeq \\ \text{Funct}_{\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}}}\left(\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}} \underset{\text{QCoh}((\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})}{\otimes} \text{QCoh}(\text{LS}^{\text{reg}}_{\check{G},x}), \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}}\right) \\ \overset{\text{Proposition 3.8.11}}{\simeq} \text{Funct}_{\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}}}\left(\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G},x}), \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},x})_{\text{mon-free}}\right) \simeq \\ \simeq \text{Funct}_{\text{IndCoh}^!(\text{Op}^{\text{mer}}_{\check{G},x})}\left(\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G},x}), \text{IndCoh}^*(\text{Op}^{\text{mer}}_{\check{G},x})\right) \overset{\text{Lemma 3.5.7}}{\simeq} \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},x}).\end{gathered}$$

Remark 7.5.10. The proof of the pointwise FLE given in Sect. 7.2 relied on Proposition 7.5.7, and hence on Theorem 6.4.5 ingredient.

As was mentioned in Remark 7.5.8, we will supply a different proof of Proposition 7.5.7.

This allows us to give a proof of the pointwise FLE, avoiding Theorem 6.4.5:

Then one interprets the pointwise $\mathrm{FLE}_{G,\mathrm{crit}}$ functor as in Sect. 7.5.9 above. The assertion of the pointwise FLE follows now by combining Propositions 7.2.6 and 7.5.5.

7.5.11. We remark also that one can deduce the pointwise version of Theorem 6.4.5 from Proposition 7.5.7 by the following argument:

First, we note that in the context of Sect. 7.5.3, the functor

$$\mathrm{Sph}(\mathbf{C}) \to \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{Whit}_*(\mathbf{C}))$$

intertwines the action of $\mathrm{Sph}_{G,x}$ on $\mathrm{Sph}(\mathbf{C})$ and the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$ on

$$\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{Whit}_*(\mathbf{C}))$$

via the source.

Unwinding the construction, we obtain that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_{G,x}}{\otimes} \mathrm{Sph}(\mathbf{C}) \arrow[r, "(7.13)", "\sim"'] \arrow[d, "\mathrm{FLE}^{-1}_{\check{G},\infty} \otimes (7.16)"'] & \mathrm{Whit}_*(\mathbf{C}) \arrow[d, "\mathrm{Id}"] \\
\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}}{\otimes} \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{\check{G},x})^{\wedge}_{\mathrm{reg}})}(\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},x}), \mathrm{Whit}_*(\mathbf{C})) \arrow[r] & \mathrm{Whit}_*(\mathbf{C}),
\end{tikzcd}$$

where the bottom horizontal arrow is the natural evaluation functor.

This proves the desired compatibility, since the functor (7.20) is compatible with the $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x}$-actions.

8. Compatibility of the critical FLE with duality

In this section we show that the FLE equivalence is compatible with the natural self-dualities of the two sides.

The proof proceeds along the following steps:

(1) We show that the self-duality on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$ is compatible with the $\mathrm{IndCoh}^!(\text{“}\mathrm{Spec}\text{”}(\mathfrak{Z}_{\mathfrak{g}}))$-action. This boils down to a particular property of the factorization algebra $\mathrm{CDO}(G)_{\mathrm{crit},\mathrm{crit}}$, given by Lemma 8.3.3;

(2) We show that the equivalence $\overline{\mathrm{DS}}^{\mathrm{enh},\mathrm{rfnd}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$ is compatible with the self-dualities of the two sides. We deduce this from the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-linearity (guaranteed by the previous point), combined with a general uniqueness statement;

(3) Finally, we establish the compatibility of $\mathrm{FLE}_{G,\mathrm{crit}}$ with the self-dualities by essentially base-changing it from the previous point along $\mathrm{Op}^{\text{mon-free}}_{\check{G}} \to \mathrm{Op}^{\mathrm{mer}}_{\check{G}}$.

As we highlight below, our methods are robust enough to show that established compatibility is automatically compatible with the actions of

$$\mathrm{Sph}_G \simeq \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$$

on the two sides.

8.1. **Statement of the compatibility theorem.**

8.1.1. Recall that according to (2.4), we have a canonical identification

$$(\mathrm{KL}(G)_{\mathrm{crit}})^{\vee} \simeq \mathrm{KL}(G)_{\mathrm{crit}}. \tag{8.1}$$

By construction, the equivalence (8.1) respects the actions of Sph_G.

8.1.2. In addition, we have an equivalence

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}) \overset{\Theta_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}). \tag{8.2}$$

This equivalence respects the actions of $\mathrm{Sph}_{\check{G}}^{\text{spec}}$ (see Sects. 3.6.2, 3.7.15 and Lemma 3.7.17).

8.1.3. The goal of this section is to prove the following:

Theorem 8.1.4. *With respect to the identifications* (8.1) *and* (8.2)*, the (factorization) functor*

$$(\mathrm{FLE}_{G,\mathrm{crit}})^\vee : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}) \to \mathrm{KL}(G)_{\mathrm{crit}}$$

identifies with

$$\tau_G \circ (\mathrm{FLE}_{G,\mathrm{crit}})^{-1}.$$

Moreover, this identification of functors respects the compatibility with the actions of

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_{G,\tau}}{\simeq} \mathrm{Sph}_{\check{G}}^{\text{spec}}.$$

One can rephrase Theorem 8.1.4 as a commutative diagram of factorization categories

$$\begin{tikzcd} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})^\vee \arrow[r, "(8.2)", "\sim"'] \arrow[d, "(\mathrm{FLE}_{G,\mathrm{crit}})^\vee"', "\sim"] & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}) \\ (\mathrm{KL}(G)_{\mathrm{crit}})^\vee \arrow[r, "\sim", "(8.1)"'] & \mathrm{KL}(G)_{\mathrm{crit}}. \arrow[u, "\sim", "\tau_G \circ \mathrm{FLE}_{G,\mathrm{crit}}"'] \end{tikzcd} \tag{8.3}$$

Remark 8.1.5. Note the similarity between the statement of Theorem 8.1.4 and Lemma 1.4.12: in both cases a non-tautological self-equivalence of the Whittaker side makes the FLE inverse to its dual, up to the Chevalley involution.

Remark 8.1.6. Note again that the appearance of the Chevalley involution in Theorem 8.1.4 is in line with the curse in Sect. 1.8.7.

8.1.7. The following assertion will not be needed in the sequel, but it provides a concrete perspective on what Theorem 8.1.4 really says.

Recall that the unit of the self-duality (8.1) is the object

$$\mathfrak{CDO}_{\mathrm{crit,crit}} \in \mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{KL}(G)_{\mathrm{crit}}.$$

The unit of the self-duality of $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})$ is

$$(\Delta_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}})_*^{\mathrm{IndCoh}}(\omega^{*,\text{fake}}_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}}),$$

where

$$\omega^{*,\text{fake}}_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}} \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})$$

is as in Sect. 3.7.7, and $\Delta_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}}$ is the diagonal map of $\mathrm{Op}_{\check{G}}^{\text{mon-free}}$.

Thus, from Theorem 8.1.4, we obtain:

Corollary 8.1.8. *We have a canonical isomorphism (of factorization algebra objects)*

$$(\mathrm{FLE}_{G,\mathrm{crit}} \otimes \mathrm{FLE}_{G,\mathrm{crit}})(\mathfrak{CDO}_{\mathrm{crit,crit}}) \simeq (\mathrm{Id} \otimes \tau_G) \circ (\Delta_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}})_*^{\mathrm{IndCoh}}(\omega^{*,\text{fake}}_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}}).$$

Remark 8.1.9. Note that, on the one hand, the statement of Corollary 8.1.8 is actually equivalent to Theorem 8.1.4 (without the $\mathrm{Sph}_G \simeq \mathrm{Sph}_{\check{G}}^{\text{spec}}$ compatibility).

One the other hand, the two factorization algebras appearing in Corollary 8.1.8 are *classical* (i.e., the corresponding chiral algebras lie in $\mathrm{D\text{-}mod}(X)^\heartsuit$), and one can actually prove the existence of an isomorphism between them directly.

8.1.10. Applying the forgetful functor to Vect, from Corollary 8.1.8 we obtain:

Corollary 8.1.11. *We have a canonical isomorphism of factorization algebras*

$$(\mathrm{DS}\otimes\mathrm{DS})\circ(\alpha_{\rho(\omega_X),\mathrm{taut}}\otimes\alpha_{\rho(\omega_X),\mathrm{taut}})(\mathfrak{CDO}_{\mathrm{crit},\mathrm{crit}})\simeq\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\text{mon-free}}_{\check{G}},\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}). \tag{8.4}$$

Remark 8.1.12. We note that the factorization algebra

$$\mathcal{B}_G:=(\mathrm{DS}\otimes\mathrm{DS})\circ(\alpha_{\rho(\omega_X),\mathrm{taut}}\otimes\alpha_{\rho(\omega_X),\mathrm{taut}})(\mathfrak{CDO}_{\mathrm{crit},\mathrm{crit}})$$

was studied in [FG2].

One can view (8.4) as an extension of the Feigin-Frenkel isomorphism FF_G: indeed according to Lemma 8.3.3 below, the factorization algebra $\mathcal{B}_{\mathfrak{g}}$ receives a homomorphism from

$$(\mathrm{DS}\otimes\mathrm{DS})(\mathbb{V}_{\mathfrak{g},\mathrm{crit},\rho(\omega_X)}\underset{\mathfrak{z}_{\mathfrak{g}}}{\otimes}\mathbb{V}_{\mathfrak{g},\mathrm{crit},\rho(\omega_X)})\simeq\mathfrak{z}_{\mathfrak{g}},$$

while

$$\mathcal{B}^{\mathrm{spec}}_{\check{G}}:=\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\text{mon-free}}_{\check{G}},\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}})$$

receives a homomorphism from

$$\Gamma(\mathrm{Op}^{\mathrm{reg}}_{\check{G}},\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}})=:\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}.$$

8.2. Feigin-Frenkel center and self-duality.

8.2.1. Recall the duality identification

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^\vee\simeq\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}. \tag{8.5}$$

of (2.3).

Recall also that $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$ carries a canonical action of $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$, see Sect. 4.6. Since the category $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ is *symmetric* monoidal, this action induces an action of $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ on $(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^\vee$.

8.2.2. The goal of this subsection is to prove the following:

Theorem-Construction 8.2.3. *The identification* (8.5) *carries a canonical structure of compatibility with the* $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$*-actions, up to the automorphism of* “Spec”$(\mathfrak{Z}_{\mathfrak{g}})$ *induced by the Chevalley involution* τ_G.

The rest of this subsection is devoted to the proof of this theorem.

8.2.4. Let

$$\mathrm{u}_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}},\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}}\in\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}\otimes\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$$

be the unit of the self-duality. The statement of Theorem 8.2.3 is equivalent to the assertion that $\mathrm{u}_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}},\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}}$ can be lifted to an object of the category

$$\mathrm{Funct}_{\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))\otimes\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))}(\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})),\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}\otimes\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})$$

where $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))\otimes\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ acts on $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$ via τ_G on one of the factors.

8.2.5. Consider the (factorization) category

$$\text{D-mod}(\mathfrak{L}(G))_{\text{crit},\text{crit}}$$

of critically twisted D-modules on the loop group. We have a naturally defined (factorization) functor

$$\Gamma^{\text{IndCoh}}(\mathfrak{L}(G),-):\text{D-mod}(\mathfrak{L}(G))_{\text{crit},\text{crit}}\to\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}\otimes\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}. \tag{8.6}$$

We have:

$$\mathrm{u}_{\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}},\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}}\simeq\Gamma^{\text{IndCoh}}(\mathfrak{L}(G),\delta_{1,\mathfrak{L}(G)}),$$

where $\delta_{1,\mathfrak{L}(G)}\in\text{D-mod}(\mathfrak{L}(G))_{\text{crit},\text{crit}}$ is the δ-function at the origin.

The required property of $\mathrm{u}_{\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}},\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}}$ follows from the next general assertion:

Proposition 8.2.6. *The functor* (8.6) *factors as*

$$\text{D-mod}(\mathfrak{L}(G))_{\text{crit},\text{crit}}\to$$
$$\to\text{Funct}_{\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))\otimes\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))}(\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})),\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}\otimes\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}})\to$$
$$\to\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}\otimes\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}.$$

□[Theorem 8.2.3]

8.3. **Proof of Proposition 8.2.6.**

8.3.1. First, we record the initial input, from which we will deduce Proposition 8.2.6. Recall the (factorization algebra) object

$$\mathfrak{CDO}(G)_{\text{crit},\text{crit}}\in\text{KL}(G)_{\text{crit}}\otimes\text{KL}(G)_{\text{crit}}.$$

By a slight abuse of notation we will denote by the same symbol the image of $\mathfrak{CDO}(G)_{\text{crit},\text{crit}}$ under the forgetful functor

$$\text{KL}(G)_{\text{crit}}\otimes\text{KL}(G)_{\text{crit}}\to\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}\otimes\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}.$$

Let us denote by $\text{CDO}(G)_{\text{crit},\text{crit}}$ the image of $\mathfrak{CDO}(G)_{\text{crit},\text{crit}}$ along the further forgetful functor

$$\mathbf{oblv}_{\widehat{\mathfrak{g}}\times\widehat{\mathfrak{g}}}:\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}\otimes\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}\to\text{Vect}.$$

8.3.2. The unit of $\mathfrak{CDO}(G)_{\text{crit},\text{crit}}$ as a factorization algebra in $\text{KL}(G)_{\text{crit}}\otimes\text{KL}(G)_{\text{crit}}$ is a map (of factorization algebras)

$$\text{Vac}(G)_{\text{crit}}\otimes\text{Vac}(G)_{\text{crit}}\to\mathfrak{CDO}(G)_{\text{crit},\text{crit}}, \tag{8.7}$$

which gives rise to a map

$$\mathbb{V}_{\mathfrak{g},\text{crit}}\otimes\mathbb{V}_{\mathfrak{g},\text{crit}}\to\text{CDO}(G)_{\text{crit},\text{crit}}. \tag{8.8}$$

The following was established in [FG1, Theorem 5.4]:

Lemma 8.3.3. *The map* (8.8) *factors as*

$$\mathbb{V}_{\mathfrak{g},\text{crit}}\otimes\mathbb{V}_{\mathfrak{g},\text{crit}}\to\mathbb{V}_{\mathfrak{g},\text{crit}}\underset{\mathfrak{z}_{\mathfrak{g}}}{\otimes}\mathbb{V}_{\mathfrak{g},\text{crit}}\to\text{CDO}(G)_{\text{crit},\text{crit}},$$

where the $\mathfrak{z}_{\mathfrak{g}}$-action on one of the tensor factors is twisted by τ_G.

Since the factorization algebras involved lie in the heart of the t-structure, from Lemma 8.3.3 we obtain:

Corollary 8.3.4. *The map* (8.7) *factors as*

$$\text{Vac}(G)_{\text{crit}}\otimes\text{Vac}(G)_{\text{crit}}\to\text{Vac}(G)_{\text{crit}}\underset{\mathfrak{z}_{\mathfrak{g}}}{\otimes}\text{Vac}(G)_{\text{crit}}\to\mathfrak{CDO}(G)_{\text{crit},\text{crit}},$$

where the $\mathfrak{z}_{\mathfrak{g}}$-action on the right tensor factor is twisted by τ_G.

We will now show how to use Corollary 8.3.4 to prove Proposition 8.2.6.

8.3.5. Note that $\text{D-mod}(\mathfrak{L}(G))_{\text{crit,crit}}$ is equipped with a t-structure (see Sect. B.11.11 for what this means in the factorization setting), so that the object $\delta_{\mathfrak{L}^+(G)\subset\mathfrak{L}(G)} \in \text{D-mod}(\mathfrak{L}(G))_{\text{crit,crit}}$ lies in the heart.

The category $\text{D-mod}(\mathfrak{L}(G))_{\text{crit,crit}}$ is compactly generated by objects that lie in $\text{D-mod}(\mathfrak{L}(G))^{>-\infty}_{\text{crit,crit}}$. Hence, in order to prove Proposition 8.2.6, it suffices to show that the restriction of the functor $\Gamma^{\text{IndCoh}}(\mathfrak{L}(G),-)$ to $\text{D-mod}(\mathfrak{L}(G))^{>-\infty}_{\text{crit,crit}}$ factors as

$$\text{D-mod}(\mathfrak{L}(G))^{>-\infty}_{\text{crit,crit}} \to$$
$$\to \text{Funct}_{\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_\mathfrak{g}))\otimes\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_\mathfrak{g}))}(\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_\mathfrak{g})), \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}) \to$$
$$\to \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}.$$

8.3.6. Consider the (factorization) functor $\Gamma^{\text{IndCoh}}(\mathfrak{L}(G),-)$. It sends the factorization unit

$$\mathbf{1}_{\text{D-mod}(\mathfrak{L}(G))_{\text{crit,crit}}} \simeq \delta_{\mathfrak{L}^+(G)\subset\mathfrak{L}(G)} \in \Gamma^{\text{IndCoh}}(\mathfrak{L}(G),-)$$

to

$$\mathfrak{CDO}(G)_{\text{crit,crit}} \in \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}.$$

In particular, by Sect. 4.1.9, it upgrades to a functor

$$\Gamma^{\text{IndCoh}}(\mathfrak{L}(G),-)^{\text{enh}} : \text{D-mod}(\mathfrak{L}(G))_{\text{crit,crit}} \to \mathfrak{CDO}\text{-mod}^{\text{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}).$$

8.3.7. Applying Corollary 8.3.4, we obtain that the functor $\Gamma^{\text{IndCoh}}(\mathfrak{L}(G),-)$ factors as

$$\text{D-mod}(\mathfrak{L}(G))_{\text{crit,crit}} \to$$
$$\to (\text{Vac}(G)_{\text{crit}} \underset{\mathfrak{Z}_\mathfrak{g}}{\otimes} \text{Vac}(G)_{\text{crit}})\text{-mod}^{\text{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}) \to \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}.$$

In particular, the restriction of the functor $\Gamma^{\text{IndCoh}}(\mathfrak{L}(G),-)$ to $\text{D-mod}(\mathfrak{L}(G))^{>-\infty}_{\text{crit,crit}}$ factors via the forgetful functor

$$\left((\text{Vac}(G)_{\text{crit}} \underset{\mathfrak{Z}_\mathfrak{g}}{\otimes} \text{Vac}(G)_{\text{crit}})\text{-mod}^{\text{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}})\right)^{>-\infty} \to \left(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}\right)^{>-\infty}. \tag{8.9}$$

8.3.8. However, unwinding the construction of the $\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_\mathfrak{g}))$-action on $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}$, we obtain that the functor (8.9) factors as

$$\left((\text{Vac}(G)_{\text{crit}} \underset{\mathfrak{Z}_\mathfrak{g}}{\otimes} \text{Vac}(G)_{\text{crit}})\text{-mod}^{\text{fact}}(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}})\right)^{>-\infty} \to$$
$$\to \text{Funct}_{\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_\mathfrak{g}))\otimes\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_\mathfrak{g}))}(\text{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_\mathfrak{g})), \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}) \to$$
$$\to \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\text{crit}}.$$

$\square$[Proposition 8.2.6]

8.4. **Self-duality on opers via Kac-Moody.**

8.4.1. Recall the identification (1.5). Applying this to the category $\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)}$ and using the identification (8.5), we obtain an identification

$$\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)})^\vee \simeq \text{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)}), \tag{8.10}$$

which fits into the commutative diagram

$$\begin{tikzcd}
\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)})^\vee \arrow[r, "(8.10)"] \arrow[d] & \text{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)}) \arrow[d] \\
(\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)})^\vee \arrow[r, "(8.5)"] & \widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega)},
\end{tikzcd}$$

where the left vertical arrow is the dual of the projection

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}).$$

Since the vertical arrows in the above diagram are fully faithful, from Theorem 8.2.3[34] (combined with Corollary 4.7.15(a,b)), we obtain:

Corollary 8.4.2. *The functor* (8.10) *is equipped with a natural* $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$*-linear structure, up to the automorphism of* “Spec”$(\mathfrak{Z}_{\mathfrak{g}})$*, induced by the Chevalley involution* τ_G*.*

8.4.3. Recall now identification of Theorem 1.3.7. Applying this to the category $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}$, we obtain an identification

$$\Theta_{\mathrm{Whit}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)})} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}) \simeq \mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}). \tag{8.11}$$

Concatenating (8.10) with (8.11) we obtain an identification

$$\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)})^\vee \simeq \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}). \tag{8.12}$$

Combining with Corollaries 8.4.2 and 4.7.15(c), we obtain that the functor (8.12) is also endowed with an $\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}}))$-linear structure, up to the automorphism of “Spec”$(\mathfrak{Z}_{\mathfrak{g}})$ induced by the Chevalley involution τ_G.

8.4.4. Recall now that we have an identification

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \overset{\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}). \tag{8.13}$$

We will prove:

Theorem 8.4.5. *With respect to the identifications* (8.12) *and* (8.13)*, the functor dual to*

$$\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}) \xrightarrow{\sim} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$$

identifies with $\tau_G \circ (\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}})^{-1}$*, compatibly with the actions of*

$$\mathrm{IndCoh}^!(\text{“Spec”}(\mathfrak{Z}_{\mathfrak{g}})) \overset{\mathrm{FF}_G}{\simeq} \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).$$

One can rephrase Theorem 8.1.4 as a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})^\vee \arrow[r, "(8.13)", "\sim"'] \arrow[d, "(\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}})^\vee"', "\sim"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \\
\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)})^\vee \arrow[r, "\sim", "(8.12)"'] & \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}). \arrow[u, "\sim"', "\tau_G\circ\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}"]
\end{tikzcd} \tag{8.14}$$

Remark 8.4.6. As we shall see below, Theorem 8.4.5 is actually easy. However, it can be seen as a particular case of a conjecture, proposed by G. Dhillon, which says that at any level κ, the self-dualities of the (renormalized) categories of factorization modules

$$\mathcal{W}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}} \simeq \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}\text{-mod}^{\mathrm{fact}}$$

that come from the identifications

$$\mathcal{W}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}} = \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_\kappa) \text{ and } \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}\text{-mod}^{\mathrm{fact}} = \mathrm{Whit}_*(\widehat{\check{\mathfrak{g}}}\text{-mod}_{\check{\kappa}})$$

and Theorem 1.3.7, respectively, agree.

For non-critical κ, this conjecture is completely open. What makes it tractable at the critical level is precisely the interpretation of $\mathcal{W}_{\mathfrak{g},\mathrm{crit}}$ as the Feigin-Frenkel center.

8.5. **Proof of Theorem 8.4.5.**

[34]Here we apply Theorem 8.2.3 in the twisted setting, i.e., to $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega)}$ instead of $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$.

8.5.1. Consider the $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-linear self-equivalence of $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$ obtained by going clockwise along the edges of (8.14). We need to show that this functor is isomorphic to the identity.

Using the equivalence $\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}$, and further, the equivalence

$$\Upsilon_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}} : \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$$

of Proposition 3.8.7, we can transform the above $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-linear self-equivalence of the category $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$ into a $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$-linear self-equivalence of $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$.

Such a self-equivalence is given by a (graded) line bundle, to be denoted $\mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}$, on $\mathrm{Op}^{\mathrm{mer}}_{\check{G}}$. Since all the equivalences in sight are compatible with factorization, this line bundle has a natural factorization structure.

We will now show that any such (graded) line bundle is automatically trivial.

8.5.2. The question is local, so let us choose a $\check{G}$-oper σ on X.

The datum of σ gives rise to a section

$$\sigma_{\mathrm{Ran}} : \mathrm{Ran} \to \mathrm{Op}^{\mathrm{reg}}_{\check{G},\mathrm{Ran}} \to \mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathrm{Ran}},$$

compatible with factorization.

Set $\mathcal{L}_{\mathrm{Ran},\sigma} := (\sigma_{\mathrm{Ran}})^*(\mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}})$. This is a factorization line bundle on Ran. We claim that it is automatically trivial.

8.5.3. Indeed, write

$$\mathrm{Ran} \simeq \underset{I}{\mathrm{colim}}\, X^I_{\mathrm{dR}},$$

where the index I runs over the category of non-empty finite sets and surjective maps.

Set $\mathcal{L}_{X^I_{\mathrm{dR}},\sigma} := \mathcal{L}_{\mathrm{Ran},\sigma}|_{X^I_{\mathrm{dR}}}$. The collection of local systems

$$I \rightsquigarrow \mathcal{L}_{X^I_{\mathrm{dR}},\sigma} \tag{8.15}$$

is compatible with the factorization structure.

In particular,

$$\mathcal{L}_{X^I_{\mathrm{dR}},\sigma}|_{\overset{o}{X}{}^I} \simeq (\mathcal{L}_{X_{\mathrm{dR}},\sigma})^{\boxtimes I}|_{\overset{o}{X}{}^I},$$

where $\overset{o}{X}{}^I \subset X^I$ is the complement of the diagonal divisor.

Hence,

$$\mathcal{L}_{X^I_{\mathrm{dR}},\sigma} \simeq (\mathcal{L}_{X_{\mathrm{dR}},\sigma})^{\boxtimes I}, \tag{8.16}$$

compatibly with the factorization structure.

Consider (8.16) for $I = \{1, 2\}$. Restricting to the diagonal $X \to X \times X$, we obtain

$$\mathcal{L}_{X_{\mathrm{dR}},\sigma} \simeq (\mathcal{L}_{X_{\mathrm{dR}},\sigma})^{\otimes 2}.$$

Hence, $\mathcal{L}_{X_{\mathrm{dR}},\sigma}$ is trivial. By (8.16), this trivializes the system (8.15).

8.5.4. For a fixed I, consider the fiber products

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^I_{\mathrm{dR}}} := X^I_{\mathrm{dR}} \underset{\mathrm{Ran}}{\times} \mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathrm{Ran}} \text{ and } \mathrm{Op}^{\mathrm{mer}}_{\check{G},X^I} := X^I \underset{\mathrm{Ran}}{\times} \mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathrm{Ran}}$$

and the line bundles

$$\mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^I_{\mathrm{dR}}}} := \mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathrm{Ran}}}|_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},X^I_{\mathrm{dR}}}} \text{ and } \mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^I}} := \mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathrm{Ran}}}|_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},X^I}}.$$

The map $\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^I} \to X^I$ is a Zariski-locally trivial fibration with ind-pro-affine spaces as fibers. Since X^I is smooth, we obtain that $\mathcal{L}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},X^I}}$ descends to a canonically defined line bundle $\mathcal{L}_{X^I}$ on X^I. Moreover, the collection

$$I \rightsquigarrow \mathcal{L}_{X^I}$$

has a natural factorization structure.

Furthermore, by construction, we have

$$\mathcal{L}_{X^I} \simeq \mathcal{L}_{X^I_{\mathrm{dR}},\sigma}|_{X^I},$$

compatibly with factorization.

Hence, by Sect. 8.5.2, we obtain that the system

$$I \mapsto \mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^I}} \tag{8.17}$$

is canonically trivial, compatibly with factorization.

It remains to show that this trivialization descends to a trivialization of the system

$$I \mapsto \mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^I_{\mathrm{dR}}}}. \tag{8.18}$$

8.5.5. Note a priori, the obstruction to triviality is given by a function on $\mathcal{L}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^I}}$ with values in the pullback of the sheaf of 1-forms on X^I; denote this function by α_{X^I}.

Note that by factorization,

$$\alpha_{X^I}|_{\overset{o}{X}{}^I} \simeq \alpha_X^{\boxtimes I}.$$

Hence, it is enough to show that $\alpha_X = 0$.

8.5.6. We will first show that $\alpha_X|_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},X}}$ is trivial.

Recall that the factorization scheme $\mathrm{Op}^{\mathrm{reg}}_{\check{G}}$ is counital (see Sect. C.6.15 for what this means). In particular, there exist canonical projections

$$p_i : \mathrm{Op}^{\mathrm{reg}}_{\check{G},X^2} \to \mathrm{Op}^{\mathrm{reg}}_{\check{G},X}, \quad i = 1, 2$$

covering the two projections $p_i : X^2 \to X$.

We claim that

$$\alpha_{X^2} = p_1^*(\alpha_X) + p_2^*(\alpha_X).$$

Indeed, the equality takes place over $\overset{o}{X}{}^2$, by factorization, and hence over the entire X^2 by density.

8.5.7. Recall now that $\mathrm{Op}^{\mathrm{mer}}_{\check{G}}$ has a unital-in-correspondences structure relative to $\mathrm{Op}^{\mathrm{reg}}_{\check{G}}$ (see Sect. C.10.6 for what this means). We claim that the connection forms α_I are compatible with this structure in the following sense:

For an injection of finite sets $\phi : I_1 \to I_2$, let

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^{I_1}} \overset{\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}}}{\longleftarrow} \mathrm{Op}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{\check{G},X^{\phi}} \overset{\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big}}}{\longrightarrow} \mathrm{Op}^{\mathrm{mer}}_{\check{G},X^{I_2}}$$

be the correspondence covering

$$X^{I_1} \overset{\Delta_\phi}{\longleftarrow} X^{I_2} \overset{\mathrm{id}}{\longrightarrow} X^{I_2}.$$

We claim that

$$(\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}})^*(\alpha_{X^{I_1}}) = (\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big}})^*(\alpha_{X^{I_2}}). \tag{8.19}$$

Indeed, write $I_2 = I_1 \sqcup J$, and let

$$(X^{I_1} \times X^J)_{\mathrm{disj}} \subset X^{I_1} \times X^J$$

be the corresponding open subset.

It suffices that the equality (8.19) takes place over

$$\mathrm{Op}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{\check{G},X^{\phi}} \underset{X^{I_2}}{\times} (X^{I_1} \times X^J)_{\mathrm{disj}}.$$

We have

$$\mathrm{Op}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{\check{G},X^{\phi}} \underset{X^{I_2}}{\times} (X^{I_1} \times X^J)_{\mathrm{disj}} \simeq (\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^{I_1}} \times \mathrm{Op}^{\mathrm{reg}}_{\check{G},X^J}) \underset{X^{I_1}\times X^J}{\times} (X^{I_1} \times X^J)_{\mathrm{disj}} \tag{8.20}$$

and

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^{I_2}} \underset{X^{I_2}}{\times} (X^{I_1} \times X^J)_{\mathrm{disj}}) \simeq (\mathrm{Op}^{\mathrm{mer}}_{\check{G},X^{I_1}} \times \mathrm{Op}^{\mathrm{mer}}_{\check{G},X^J}) \underset{X^{I_1}\times X^J}{\times} (X^{I_1} \times X^J)_{\mathrm{disj}},$$

where the map $\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}}$ identifies with projection on the first factor in (8.20), and the map $\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big}}$ is the inclusion

$$\mathrm{Op}^{\mathrm{reg}}_{\check{G},X^J} \to \mathrm{Op}^{\mathrm{mer}}_{\check{G},X^J}$$

along the second factor.

By factorization, we obtain that

$$(\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big}})^*(\alpha_{X^{I_2}})|_{\mathrm{Op}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{\check{G},X^{\phi}} \underset{X^{I_2}}{\times} (X^{I_1}\times X^J)_{\mathrm{disj}}}$$

equals the sum of $(\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}})^*(\alpha_{X^{I_1}})$ and the pullback of $\alpha_{X^J}|_{\mathrm{Op}^{\mathrm{reg}}_{\check{G},X^J}}$ along the projection of (8.20) on the second factor.

However, the latter is zero by Sect. 8.5.6.

8.5.8. We are now ready to show that $\alpha_X = 0$. In doing so we will mimic the argument in [BD2, Proposition 3.4.7].

Write $\Omega^1_{X^2}$ as

$$\omega_X \boxtimes \mathcal{O}_X \oplus \mathcal{O}_X \boxtimes \omega_X.$$

We will show that the restriction of α_{X^2} to

$$\mathrm{Op}^{\mathrm{mer}}_{\check{G},X} \simeq X \underset{X^2}{\times} \mathrm{Op}^{\mathrm{mer}}_{\check{G},X^2},$$

viewed as a function on $\mathrm{Op}^{\mathrm{mer}}_{\check{G},X}$ with values in the pullback of $\Omega^1_{X^2}$, takes values both in the pullback of $\omega_X \boxtimes \mathcal{O}_X$ *and* in the pullback of $\mathcal{O}_X \boxtimes \omega_X$. This would implies that this restriction is 0, and hence also that $\alpha_X = 0$.

We will in fact show that the restriction of α_{X^2} to

$$(X^2)^\wedge \underset{X^2}{\times} \mathrm{Op}^{\mathrm{mer}}_{\check{G},X^2} \tag{8.21}$$

is 0, where $(X^2)^\wedge$ is the formal completion of the diagonal in X^2.

8.5.9. By symmetry, it suffices to show that the restriction of α_{X^2} to takes values in the pullback of $\omega_X \boxtimes \mathcal{O}_X$.

Consider the inclusion $I_1 := \{1\} \overset{\phi}{\hookrightarrow} \{1,2\} := I_2$, and the corresponding map

$$\mathrm{Op}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{\check{G},X^\phi} \overset{\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big}}}{\longrightarrow} \mathrm{Op}^{\mathrm{mer}}_{\check{G},X^2}.$$

This map is an isomorphism over $X \subset X^2$, and hence, induces an isomorphism

$$(X^2)^\wedge \underset{X^2}{\times} \mathrm{Op}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{\check{G},X^\phi} \overset{\sim}{\to} (X^2)^\wedge \underset{X^2}{\times} \mathrm{Op}^{\mathrm{mer}}_{\check{G},X^2}.$$

Hence, it is enough to prove that the pullback of α_{X^2} along target takes values in the pullback of $\omega_X \boxtimes \mathcal{O}_X$. However, this has been established in Sect. 8.5.7.

$\square$[Theorem 8.4.5]

8.6. **Proof of Theorem 8.1.4.** By a slight abuse of notation we will use the symbol $\mathrm{FLE}_{G,\mathrm{crit}}$ for the functor (6.6).

8.6.1. Consider the following diagram

$$\begin{tikzcd}[column sep=small]
(\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)})^\vee \arrow[d, "\sim"] & & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^\vee \arrow[ll, "\tau_G\circ(\mathrm{FLE}_{G,\mathrm{crit}})^\vee"'] \arrow[dd, "\sim"] \\
(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}))^\vee \arrow[d, "\sim"] & & \\
(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}))^\vee \arrow[d, "\sim"'] & & (\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}))^\vee \arrow[ll, "(\mathrm{Id}\otimes\tau_G\circ\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}})^\vee"'] \arrow[d, "\sim"] \\
\mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^\vee) \arrow[d, "\sim"'] & & \mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})^\vee) \arrow[ll, "\tau_G\circ(\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}})^\vee"'] \arrow[d, "\sim"] \\
\mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})) & & \mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})) \arrow[d, "\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G}}}"', "\sim"] \\
\mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})) \arrow[u, "\sim"', "\Theta_{\mathrm{Whit}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})}"] \arrow[rr, "\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}"] & & \mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \mathrm{IndCoh}_*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})) \\
\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \arrow[u, "\sim"'] \arrow[rr, "\mathrm{FLE}_{G,\mathrm{crit}}"] & & \mathrm{IndCoh}_*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \arrow[u, "\sim"]
\end{tikzcd}$$

in which the upper vertical arrows are the duals of

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{(6.15)}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}$$

and

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}),$$

respectively.

We will show:

- The left vertical composition is the identification (8.1);
- The right vertical composition is the identification (8.2);
- All inner squares commute.

This will establish the commutativity of (8.3). The compatibility of this isomorphism with the actions of

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_{G,\tau}}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}.$$

is automatic from the construction.

8.6.2. *The left vertical composition.* We need to establish the commutativity of the following diagram

$$\begin{tikzcd}
(\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)})^\vee \arrow[r] \arrow[dd, "(8.1)"'] & (\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}))^\vee \arrow[d, "\sim"] \\
& \mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^\vee) \arrow[d, "\sim"] \\
\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \arrow[r] & \mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}), \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})).
\end{tikzcd}$$

However, this is just the fact that in the context of Sect. 6.3.1, the dual of the functor (6.9) for $\mathbf{C}$ is the functor (6.10) for $\mathbf{C}^\vee$.

8.6.3. *The right vertical composition.* The identification of the right vertical composition follows from Sect. 3.7.15.

8.6.4. *The top square.* Passing to the dual functors, we need to establish the commutativity of

$$\begin{tikzcd}
\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}) \arrow[r, "\mathrm{FLE}_{G,\mathrm{crit}}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \\
\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}^!(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \arrow[u, "\sim"] & \\
\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \arrow[u, "\sim"] \arrow[r, "\mathrm{Id}\otimes\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}", "\sim"'] & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})). \arrow[uu, "\sim"]
\end{tikzcd}$$

However, this is the content of Proposition 6.3.5.

8.6.5. *The 2nd square from the top.* This square commutes tautologically.

8.6.6. *The bottom square.* The commutation follows from the definition of the functor $\mathrm{FLE}_{G,\mathrm{crit}}$.

8.6.7. Finally, it remains to show that the 3rd square from the top commutes.[35]

However, the required commutation is given by Theorem 8.4.5.

□[Theorem 8.1.4]

[35] This is the only non-tautological point in the proof.

Part II: Local-to-global constructions

9. The coefficient and Poincaré functor(s)

This section begins by introducing our main object of study: the critically twisted category of D-modules on Bun_G. In this section we will mostly think of its incarnation as $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$, see Remark 1.1.13, as the main characters in this section are sheaf-theoretic in nature.

The focus of this section is Poincaré and Whattaker coefficient functors. In fact, there are two Poincaré functors

$$\mathrm{Poinc}_{G,!} : \mathrm{Whit}^!(G)_{\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \text{ and } \mathrm{Poinc}_{G,*} : \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G),$$

where $\text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$ is the dual category of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$. These two functors are Verdier-conjugate: the dual functor of $\mathrm{Poinc}_{G,*}$ is isomorphic to the right adjoint of $\mathrm{Poinc}_{G,!}$; this is the functor

$$\mathrm{coeff}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\mathrm{Ran}}.$$

One can also give a global interpretation of the above functors, where instead of the affine Grassmannian, one uses the *twisted Drinfeld compactification*

$$\overline{\mathrm{Bun}}_{N,\rho(\omega_X)} \to \mathrm{Bun}_G\,.$$

This is how the global geometric Whittaker model had been mostly approached so far (see, e.g., [Ga1]). The two approaches are, however, equivalent (see [Ga6]).

For the purposes of this paper, we will only explicitly need the global interpretation of the *vacuum* cases of the above functors, see Sect. 9.6.

9.1. Twisted D-modules on Bun_G.

9.1.1. Let $\det_{\mathrm{Bun}_G}$ be the determinant line bundle on Bun_G, normalized so that it sends a G-bundle $\mathcal{P}_G$ to

$$\det\left(\Gamma(X,\mathfrak{g}_{\mathcal{P}_G})\right) \otimes \det\left(\Gamma(X,\mathfrak{g}_{\mathcal{P}^0_G})\right)^{\otimes -1},$$

where $\mathcal{P}^0_G$ is the trivial bundle.

9.1.2. Note that we have

$$\pi^*(\det_{\mathrm{Bun}_G}) \simeq \det_{\mathrm{Gr}_{G,\mathrm{Ran}}},$$

where π denotes the projection

$$\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Bun}_G\,. \tag{9.1}$$

9.1.3. Note also that up to the (constant) line $\det\left(\Gamma(X,\mathfrak{g}_{\mathcal{P}^0_G})\right)$, the line bundle $\det_{\mathrm{Bun}_G}$ identifies with the canonical line bundle on Bun_G.

9.1.4. Let crit be half of the de Rham twisting defined by $\det_{\mathrm{Bun}_G}$, i.e.,

$$\mathrm{crit} = \frac{1}{2}\cdot \mathrm{dlog}(\det_{\mathrm{Bun}_G}).$$

We will denote by

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

the corresponding category of twisted D-modules.

Note that by Sect. 9.1.3, the critical twisting on Bun_G is canonically isomorphic to the half-canonical twisting.

9.1.5. As in Sect. 1.1.11, we obtain a *canonical* identification

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \overset{\sim}{\to} \text{D-mod}_{\text{crit}}(\text{Bun}_G), \tag{9.2}$$

where $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ is the short-hand for

$$\text{D-mod}_{\det^{\frac{1}{2}}_{\text{Bun}_G}}(\text{Gr}_G),$$

cf. Sect. 1.1.6.

Remark 9.1.6. According to [BD1, Sect. 4], the choice of $\omega_X^{\otimes \frac{1}{2}}$ gives rise to a choice of the square root of $\det_{\text{Bun}_G}$ as a line bundle. This allows us to identify $\text{D-mod}_{\text{crit}}(\text{Bun}_G)$ (or equivalently $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$) with the untwisted category $\text{D-mod}(\text{Bun}_G)$.

However, we will avoid using this identification.

9.1.7. Pullback along π defines functors

$$\pi^! : \text{D-mod}_{\text{crit}}(\text{Bun}_G) \to \text{D-mod}_{\text{crit}}(\text{Gr}_{G,\text{Ran}})$$

and

$$\pi^! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\text{Ran}}),$$

so that the diagram

$$\begin{CD}
\text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\text{Ran}}) @>>> \text{D-mod}_{\text{crit}}(\text{Gr}_{G,\text{Ran}}) \\
@A{\pi^!}AA @AA{\pi^!}A \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) @>>> \text{D-mod}_{\text{crit}}(\text{Bun}_G)
\end{CD}$$

commutes.

9.2. Restricting to (twists of) Bun_N.

9.2.1. Let $\mathcal{P}_T$ be any T-bundle. Consider the stack

$$\text{Bun}_{N,\mathcal{P}_T} \simeq \text{Bun}_B \underset{\text{Bun}_T}{\times} \text{pt}, \tag{9.3}$$

where $\text{pt} \to \text{Bun}_T$ is the point $\mathcal{P}_T$.

Denote by $\mathfrak{p}$ the map

$$\text{Bun}_{N,\mathcal{P}_T} \to \text{Bun}_G\,.$$

Note that the pullback of $\det_{\text{Bun}_G}$ along this map is canonically constant. Denote the resulting line by

$$\mathfrak{l}_{G,N_{\mathcal{P}_T}}.$$

9.2.2. We obtain that $\mathfrak{p}$ gives rise to well-defined functors

$$\mathfrak{p}^!_{\text{crit}} : \text{D-mod}_{\text{crit}}(\text{Bun}_G) \to \text{D-mod}_{\frac{1}{2}\cdot \text{dlog}(\mathfrak{l}_{G,N_{\mathcal{P}_T}})}(\text{Bun}_{N,\mathcal{P}_T}) \overset{\sim}{\to} \text{D-mod}(\text{Bun}_{N,\mathcal{P}_T})$$

(the second identification is due to the fact that the dlog map over pt is trivial), and

$$\mathfrak{p}^! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{D-mod}_{\mathfrak{l}^{\frac{1}{2}}_{G,N_{\mathcal{P}_T}}}(\text{Bun}_{N,\mathcal{P}_T}),$$

where the subscript $\mathfrak{l}^{\frac{1}{2}}_{G,N_{\mathcal{P}_T}}$ means the twist by the constant[36] μ_2-gerbe of square roots of the line $\mathfrak{l}_{G,N_{\mathcal{P}_T}}$.

[36] I.e., pulled back from pt.

We have a commutative diagram

$$
\begin{tikzcd}
\mathrm{D}\text{-}\mathrm{mod}_{\mathfrak{l}^{\frac{1}{2}}_{G,N_{\mathcal{P}_T}}}(\mathrm{Bun}_{N,\mathcal{P}_T}) \arrow[r, "\sim"] & \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \\
\mathrm{D}\text{-}\mathrm{mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "\mathfrak{p}^!"] \arrow[r, "(9.2)"] & \mathrm{D}\text{-}\mathrm{mod}_{\mathrm{crit}}(\mathrm{Bun}_G), \arrow[u, "\mathfrak{p}^!_{\mathrm{crit}}"']
\end{tikzcd}
\tag{9.4}
$$

where the top horizontal row comes from the identification

$$\mathrm{D}\text{-}\mathrm{mod}_{\mathfrak{l}^{\frac{1}{2}}_{G,N_{\mathcal{P}_T}}}(\mathrm{pt}) \overset{(1.2)}{\simeq} \mathrm{D}\text{-}\mathrm{mod}_{\frac{1}{2}\cdot \mathrm{dlog}(\mathfrak{l}_{G,N_{\mathcal{P}_T}})}(\mathrm{pt}) \simeq \mathrm{D}\text{-}\mathrm{mod}(\mathrm{pt}) \simeq \mathrm{Vect}\,.$$

9.2.3. We take $\mathcal{P}_T = \rho(\omega_X)$. The following was proved in [GLC1, Proposition 1.3.3]:

Proposition 9.2.4. *The line $\mathfrak{l}_{G,N_{\rho(\omega_X)}}$ admits a canonical square root.*

9.2.5. Let

$$\mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \tag{9.5}$$

denote the square root of the line $\mathfrak{l}_{G,N_{\rho(\omega_X)}}$ from Proposition 9.2.4.

From Proposition 9.2.4 we obtain that there exists an a priori identification

$$\mathrm{D}\text{-}\mathrm{mod}_{\mathfrak{l}^{\frac{1}{2}}_{G,N_{\rho(\omega_X)}}}(\mathrm{Bun}_{N,\rho(\omega)}) \simeq \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Bun}_{N,\rho(\omega)}). \tag{9.6}$$

Denote by

$$\mathfrak{p}^!_{\frac{1}{2}} : \mathrm{D}\text{-}\mathrm{mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Bun}_{N,\rho(\omega)})$$

the functor equal to the composition

$$\mathrm{D}\text{-}\mathrm{mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathfrak{p}^!}{\to} \mathrm{D}\text{-}\mathrm{mod}_{\mathfrak{l}^{\frac{1}{2}}_{G,N_{\rho(\omega_X)}}}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{(9.6)}{\longrightarrow} \mathrm{D}\text{-}\mathrm{mod}_{\frac{1}{2}}(\mathrm{Bun}_{N,\rho(\omega_X)}).$$

Note that we have a commutative diagram

$$
\begin{tikzcd}
\mathrm{D}\text{-}\mathrm{mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \arrow[r, "\otimes \mathfrak{l}^{\otimes -\frac{1}{2}}_{G,N_{\rho(\omega_X)}}"] & \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \\
\mathrm{D}\text{-}\mathrm{mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "\mathfrak{p}^!_{\frac{1}{2}}"] \arrow[r, "(9.2)"] & \mathrm{D}\text{-}\mathrm{mod}_{\mathrm{crit}}(\mathrm{Bun}_G). \arrow[u, "\mathfrak{p}^!_{\mathrm{crit}}"']
\end{tikzcd}
\tag{9.7}
$$

9.3. **The coefficient functor.** In this subsection we will recall the definition of the functor of Whittaker coefficient(s).

9.3.1. The functor of Whittaker coefficient(s), denoted coeff_G, maps

$$\mathrm{D}\text{-}\mathrm{mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\mathrm{Ran}},$$

and is defined as follows.

To simplify the notation, we will work over a particular point $\underline{x} \in \mathrm{Ran}$. So we need to define the functor

$$\mathrm{coeff}_{G,\underline{x}} : \mathrm{D}\text{-}\mathrm{mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\underline{x}}.$$

9.3.2. Consider the $\rho(\omega_X)$-twisted version of the map (9.1)

$$\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}} \to \mathrm{Bun}_G\,.$$

By a slight abuse of notation, we will denote it by the same symbol π. By further abuse of notation, we will keep the same notation for the restriction of this map to

$$\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}} \to \mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}. \tag{9.8}$$

9.3.3. Due to the trivialization of the μ_2-gerbe $\mathfrak{l}^{\frac{1}{2}}_{G,N_{\rho(\omega_X)}}$ given by Proposition 9.2.4, the map π gives rise to a well-defined functor

$$\pi^!_{\frac{1}{2}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}}).$$

9.3.4. Write $\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}$ as a filtered union of subschemes N^α. For every α, consider the functor

$$\text{Av}_*^{(N^\alpha,\chi)} : \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}})^{N^\alpha,\chi} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}}).$$

For $N^\alpha \subset N^{\alpha'}$, we have a canonically defined natural transformation

$$\text{Av}_*^{(N^{\alpha'},\chi)} \to \text{Av}_*^{(N^\alpha,\chi)}. \tag{9.9}$$

We have the following (elementary) observation:

Lemma 9.3.5. *The natural transformation*

$$\text{Av}_*^{(N^{\alpha'},\chi)} \circ \pi^!_{\frac{1}{2}} \to \text{Av}_*^{(N^\alpha,\chi)} \circ \pi^!_{\frac{1}{2}},$$

induced by (9.9)*, is an isomorphism when N^α is large enough.*

Proof. Let

$$\text{Sect}(X-\underline{x}, N_{\rho(\omega_X)}) \subset \text{Sect}(X-\underline{x}, G_{\rho(\omega_X)})$$

be the group ind-schemes of sections of

$$N_{\rho(\omega_X)} \subset G_{\rho(\omega_X)}$$

over $X-\underline{x}$. Laurent expansion defines maps

$$\text{Sect}(X-\underline{x}, N_{\rho(\omega_X)}) \to \mathfrak{L}(N)_{\rho(\omega_X),\underline{x}} \text{ and } \text{Sect}(X-\underline{x}, G_{\rho(\omega_X)}) \to \mathfrak{L}(G)_{\rho(\omega_X),\underline{x}}.$$

Note that the restriction of χ to $\text{Sect}(X-\underline{x}, N_{\rho(\omega_X)})$ is trivial.

For $\mathcal{F} \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$, the pullback

$$\pi^!_{\frac{1}{2}}(\mathcal{F}) \in \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}})$$

is $\text{Sect}(X-\underline{x}, G_{\rho(\omega_X)})$-equivariant, and hence $\text{Sect}(X-\underline{x}, N_{\rho(\omega_X)})$-equivariant.

Hence, the map in the lemma is an isomorphism any time

$$N^\alpha \cdot \text{Sect}(X-\underline{x}, N_{\rho(\omega_X)}) = \mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}. \tag{9.10}$$

□

9.3.6. By Lemma 9.9, for N^α large enough, the functor

$$\text{Av}_*^{(N^\alpha,\chi)} \circ \pi^!_{\frac{1}{2}} \tag{9.11}$$

does not depend on the choice of N^α. In particular, its essential image is contained in

$$\bigcap_{N^\alpha} \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}})^{N^\alpha,\chi} = \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}})^{\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}},\chi} = \text{Whit}^!(G)_{\underline{x}}.$$

Thus, we let $\text{coeff}_{G,\underline{x}}$ be the functor (9.11) for N^α large enough.

9.3.7. By construction, the functor $\text{coeff}_{G,\underline{x}}$ is compatible with the action of $\text{Sph}_{G,\underline{x}}$.

When working over the Ran space, we consider the functor, to be denoted $\text{coeff}_{G,\text{Ran}}$,

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G \times \text{Ran}) \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \otimes \text{D-mod}(\text{Ran}) \overset{\text{coeff}_G \otimes \text{Id}}{\longrightarrow}$$

$$\to \text{Whit}^!(G)_{\text{Ran}} \otimes \text{D-mod}(\text{Ran}) \overset{\overset{!}{\otimes}}{\to} \text{Whit}^!(G)_{\text{Ran}}.$$

This functor is compatible with the natural action of $\text{Sph}_{G,\text{Ran}}$ on the two sides.

9.3.8. We let

$$\mathrm{coeff}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\mathrm{Ran}}$$

denote the composition

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{Id}\otimes\omega_{\mathrm{Ran}}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G \times \mathrm{Ran}) \overset{\mathrm{coeff}_{G,\mathrm{Ran}}}{\longrightarrow} \mathrm{Whit}^!(G)_{\mathrm{Ran}}.$$

9.3.9. The functors $\mathrm{coeff}_{G,\underline{x}}$ have the following property:

For $\underline{x} \subseteq \underline{x}'$ consider the natural embedding

$$\mathrm{incl}_{\underline{x}\subseteq\underline{x}'} : \mathrm{Gr}_{G,\rho(\omega_X),\underline{x}} \hookrightarrow \mathrm{Gr}_{G,\rho(\omega_X),\underline{x}'}.$$

The functor $(\mathrm{incl}_{\underline{x}\subseteq\underline{x}'})^!$ maps $\mathrm{Whit}(G)_{\underline{x}'} \to \mathrm{Whit}(G)_{\underline{x}}$, and we have

$$\mathrm{coeff}_{G,\underline{x}} \simeq \mathrm{incl}^!_{\underline{x}\subseteq\underline{x}'} \circ \mathrm{coeff}_{G,\underline{x}'}\,. \tag{9.12}$$

The isomorphism (9.12) expresses the *unital* structure on the functor coeff_G, to be discussed in Sect. 11.

9.3.10. Let

$$\mathrm{coeff}_G^{\mathrm{Vac}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Vect}$$

denote the composition of $\mathrm{coeff}_{G,\underline{x}}$ with the functor

$$\mathrm{Whit}^!(G)_{\underline{x}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \mathrm{Vect},$$

where the second arrow is the functor of !-fiber at the unit point.

By (9.12), the above definition of $\mathrm{coeff}_G^{\mathrm{Vac}}$ is canonically independent of the choice of $\underline{x}$.

Equivalently, $\mathrm{coeff}_G^{\mathrm{Vac}}$ is the unique functor $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Vect}$ so that the diagram commutes

$$\begin{tikzcd}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[r] & \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}) \arrow[r, "(\mathbf{1}_{\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}})^!"] & \text{D-mod}(\mathrm{Ran}) \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G"] \arrow[rr, "\mathrm{coeff}_G^{\mathrm{Vac}}"] & & \mathrm{Vect}\,. \arrow[u]
\end{tikzcd}$$

(In the above diagram the right vertical arrow is the !-pullback along $\mathrm{Ran} \to \mathrm{pt}$, which is fully faithful by the contractibility of the Ran space.)

9.3.11. As in Remark 1.3.10, both the category $\mathrm{Whit}^!(G)_{\mathrm{Ran}}$ and the functor coeff_G are canonically independent of the choice of a non-degenerate character $\chi_0 : N \to \mathbb{G}_a$.

9.4. **The !-Poincaré functor.**

9.4.1. We start again by working with a fixed $\underline{x} \in \mathrm{Ran}$. We claim:

Proposition 9.4.2. *The functor* $\mathrm{coeff}_{G,\underline{x}}$ *admits a left adjoint, to be denoted* $\mathrm{Poinc}_{G,!,\underline{x}}$.

Remark 9.4.3. In fact, as we work over a fixed point $\underline{x} \in \mathrm{Ran}$, all objects in $\mathrm{Whit}^!(G)_{\underline{x}}$ are ind-holonomic, which implies the assertion of the proposition. Below we give a different argument, which works also when $\underline{x}$ is allowed to move in families over Ran, see Sect. 9.4.6.

Proof of Proposition 9.4.2. Consider the *partially defined*[37] functor

$$\pi_{!,\frac{1}{2}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

left adjoint to $\pi^!_{\frac{1}{2}}$.

The assertion of the proposition is equivalent to the fact that $\pi_{!,\frac{1}{2}}$ *is* defined on $\mathrm{Whit}^!(G)_{\mathrm{Ran}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}})$.

[37]The issue here is that the "lower-!" functors are not necessarily defined on non-holonomic objects.

First, it is easy to see that if $\pi_{!,\frac{1}{2}}$ is defined on some object $\mathcal{F} \in \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}})$ and $\mathcal{S}$ is an object of $\text{Sph}_{G,\underline{x}}$, then $\pi_{!,\frac{1}{2}}$ is defined on $\mathcal{S} \star \mathcal{F}$, and in fact

$$\pi_{!,\frac{1}{2}}(\mathcal{S} \star \mathcal{F}) \simeq \mathcal{S} \star \pi_{!,\frac{1}{2}}(\mathcal{F}).$$

This follows from the properness of the convolution diagram that defines the $\text{Sph}_{G,\underline{x}}$-action on $\text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}})$ and $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$.

Next, we observe that $\pi_{!,\frac{1}{2}}$ is defined on the vacuum object

$$\text{Vac}_{\text{Whit}^!(G),\underline{x}} \in \text{Whit}^!(G)_{\underline{x}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}}).$$

Indeed, this follows from the fact that $\text{Vac}_{\text{Whit}^!(G),\underline{x}}$ is ind-holonomic, and the !-direct image functor is defined on holonomic D-modules.

Finally, we claim that any object of $\text{Whit}^!(G)_{\underline{x}}$ can be obtained as a convolution of an object of $\text{Sph}_{G,\underline{x}}$ with $\text{Vac}_{\text{Whit}^!(G),\underline{x}}$. In fact, by Remark 1.7.7, the functor

$$\text{Rep}(\check{G})_{\underline{x}} \overset{\text{Sat}^{-1,\text{nv}}}{\longrightarrow} \text{Sph}_{\underline{x}} \overset{-\star\text{Vac}_{\text{Whit}^!(G),\underline{x}}}{\longrightarrow} \text{Whit}^!(G)_{\underline{x}}$$

is an equivalence.

□

9.4.4. The above proof shows that the functor $\text{Poinc}_{G,!,\underline{x}}$ is also compatible with the action of $\text{Sph}_{G,\underline{x}}$.

Note, however, that this also follows a priori from the compatibility of $\text{coeff}_{G,x}$ with $\text{Sph}_{G,x}$-actions and the observation that $\text{Sph}_{G,x}$ is rigid as a monoidal category.[38]

9.4.5. For a pair of points $\underline{x}, \underline{x}'$ of Ran with $\underline{x} \subseteq \underline{x}'$, let

$$\text{ins. vac}_{\underline{x}\subseteq\underline{x}'} : \text{Whit}(G)_{\underline{x}} \to \text{Whit}(G)_{\underline{x}'}$$

be the functor left adjoint to

$$(\text{incl}_{\underline{x}\subseteq\underline{x}'})^! : \text{Whit}(G)_{\underline{x}'} \to \text{Whit}(G)_{\underline{x}}.$$

Explicitly, if $\underline{x}' = \underline{x} \sqcup \underline{x}''$, so that

$$\text{Whit}(G)_{\underline{x}'} \simeq \text{Whit}(G)_{\underline{x}} \otimes \text{Whit}(G)_{\underline{x}''},$$

then

$$\text{ins. vac}_{\underline{x}\subseteq\underline{x}'} \simeq \text{Id} \otimes \text{Vac}_{\text{Whit}^!(G),\underline{x}''}.$$

By adjunction, from (9.12), we obtain:

$$\text{Poinc}_{G,!,\underline{x}'} \circ \text{ins. vac}_{\underline{x}\subseteq\underline{x}'} \simeq \text{Poinc}_{G,!,\underline{x}}. \tag{9.13}$$

In Sect. 11.3.7 we will formulate a version of this isomorphism when the points $\underline{x}$ and $\underline{x}'$ move in families over the Ran space.

9.4.6. By the same token, the functor

$$\text{coeff}_{G,\text{Ran}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G \times \text{Ran}) \to \text{Whit}^!(G)_{\text{Ran}}$$

admits a left adjoint, to be denoted

$$\text{Poinc}_{G,!,\text{Ran}} : \text{Whit}^!(G)_{\text{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G \times \text{Ran}).$$

The functor $\text{Poinc}_{G,!,\text{Ran}}$ is compatible with the actions of $\text{Sph}_{G,\text{Ran}}$ on the two sides.

[38] In fact, this was implicitly used in the proof of Proposition 9.4.2: the properness of the convolution diagram is the reason for the rigidity of $\text{Sph}_{G,x}$.

9.4.7. The functor

$$\mathrm{coeff}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\mathrm{Ran}}$$

admits a left adjoint, to be denoted

$$\mathrm{Poinc}_{G,!} : \mathrm{Whit}^!(G)_{\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

and given by restricting the partially defined functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}) \overset{\pi_{!,\frac{1}{2}}}{\to} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

to

$$\mathrm{Whit}^!(G)_{\mathrm{Ran}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}).$$

Explicitly, the $\mathrm{Poinc}_{G,!}$ identifies with the composition

$$\mathrm{Whit}^!(G)_{\mathrm{Ran}} \overset{\mathrm{Poinc}_{G,!,\mathrm{Ran}}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G \times \mathrm{Ran}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

where the second arrow is the functor of !-direct image.

The functor $\mathrm{Poinc}_{G,!,\underline{x}}$ is obtained from $\mathrm{Poinc}_{G,!}$ by restriction along (9.8).

9.4.8. It follows formally from Sect. 9.3.10 that the object

$$\mathrm{Poinc}_{G,!,\underline{x}}(\mathbf{1}_{\mathrm{Whit}^!(G),\underline{x}}) \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is canonically independent of the choice of $\underline{x}$.

We will denote it by

$$\mathrm{Poinc}^{\mathrm{Vac}}_{G,!} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

We also have

$$\mathrm{Poinc}^{\mathrm{Vac}}_{G,!} \simeq \mathrm{Poinc}_{G,!}(\mathrm{Vac}_{\mathrm{Whit}^!(G),\mathrm{Ran}}),$$

where

$$\mathrm{Vac}_{\mathrm{Whit}^!(G),\mathrm{Ran}} \in \mathrm{Whit}^!(G)_{\mathrm{Ran}}$$

is the factorization unit spread over the Ran space.

9.4.9. As we saw in the proof of Proposition 9.4.2, we can recover the functor $\mathrm{Poinc}_{G,!,\underline{x}}$ from the object $\mathrm{Poinc}^{\mathrm{Vac}}_{G,!}$ using the Hecke action.

By adjunction, we obtain that the functor $\mathrm{coeff}_{G,\underline{x}}$ can be uniquely recovered from the knowledge of $\mathrm{coeff}^{\mathrm{Vac}}_G$ and the action of $\mathrm{Rep}(G)_{\underline{x}}$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ via

$$\mathrm{Rep}(G)_{\underline{x}} \overset{\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\longrightarrow} \mathrm{Sph}_{G,\underline{x}}.$$

The same applies to the functors $(\mathrm{Poinc}^{\mathrm{Vac}}_{G,!,\mathrm{Ran}}, \mathrm{coeff}^{\mathrm{Vac}}_{G,\mathrm{Ran}})$.

9.5. **The *-Poincaré functor.**

9.5.1. Recall that along with the category $\text{D-mod}(\mathrm{Bun}_G)$, one can consider its version $\text{D-mod}_{\mathrm{co}}(\mathrm{Bun}_G)$, and similarly for gerbe-twisted versions $\text{D-mod}_{\mathcal{G}}(\mathrm{Bun}_G)$.

In the untwisted case, we have the identification

$$(\text{D-mod}(\mathrm{Bun}_G))^\vee \simeq \text{D-mod}_{\mathrm{co}}(\mathrm{Bun}_G).$$

In the twisted case, this becomes

$$(\text{D-mod}_{\mathcal{G}}(\mathrm{Bun}_G))^\vee \simeq \text{D-mod}_{\mathcal{G}^{\otimes -1},\mathrm{co}}(\mathrm{Bun}_G). \tag{9.14}$$

For $\mathcal{G} = \det^{\frac{1}{2}}_{\mathrm{Bun}_G}$, the identification (9.14) becomes a self-duality

$$(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G))^\vee \simeq \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G). \tag{9.15}$$

9.5.2. Let

$$\mathrm{Poinc}_{G,*} : \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$$

be the functor *dual* to coeff_G.

Let

$$\mathrm{Poinc}_{G,*,\underline{x}} : \mathrm{Whit}_*(G)_{\underline{x}} \to \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$$

be the functor dual to $\mathrm{coeff}_{G,\underline{x}}$. It is easy to see that the functor $\mathrm{Poinc}_{G,*,\underline{x}}$ is obtained from $\mathrm{Poinc}_{G,*}$ by restriction along (9.8).

The functor $\mathrm{Poinc}_{G,*,\underline{x}}$ is also compatible with the action of $\mathrm{Sph}_{G,\underline{x}}$.

9.5.3. Let

$$\mathrm{Poinc}^{\mathrm{Vac}}_{G,*} \in \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$$

be the image under $\mathrm{Poinc}_{G,*,\underline{x}}$ of the factorization unit

$$\mathbf{1}_{\mathrm{Whit}_*(G),\underline{x}} \in \mathrm{Whit}_*(G)_{\underline{x}}$$

at some/any $\underline{x} \in \mathrm{Ran}$ or, equivalently, of

$$\mathbf{1}_{\mathrm{Whit}_*(G),\mathrm{Ran}} \in \mathrm{Whit}_*(G)_{\mathrm{Ran}}$$

under $\mathrm{Poinc}_{G,*}$.

By definition, the pairing with $\mathrm{Poinc}^{\mathrm{Vac}}_{G,*}$, viewed as a functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Vect},$$

is the functor $\mathrm{coeff}^{\mathrm{Vac}}_G$.

9.5.4. The functor $\mathrm{Poinc}_{G,*,\underline{x}}$ can be explicitly described as follows. For N^α as in Sect. 9.3.4, consider the composition

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \overset{\mathrm{Av}_*^{(N^\alpha,\chi)}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \overset{\pi_{*,\frac{1}{2}}}{\to} \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G).$$

For $N^\alpha \subset N^{\alpha'}$, we have a canonically defined natural transformation

$$\pi_{*,\frac{1}{2}} \circ \mathrm{Av}_*^{(N^{\alpha'},\chi)} \to \pi_{*,\frac{1}{2}} \circ \mathrm{Av}_*^{(N^\alpha,\chi)}, \tag{9.16}$$

and it follows from Lemma 9.3.5 that the maps (9.16) are isomorphisms for N^α large enough.

It follows formally that the resulting functor $\pi_{*,\frac{1}{2}} \circ \mathrm{Av}_*^{(N^\alpha,\chi)}$

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G),$$

for some/any α that is large enough, factors via the projection

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \Big(\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}})\Big)_{\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}},\chi} \to \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G).$$

The resulting functor

$$\mathrm{Whit}_*(G)_{\underline{x}} := \Big(\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}})\Big)_{\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}},\chi} \to \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$$

is the functor $\mathrm{Poinc}_{G,*,\underline{x}}$.

9.6. **Coefficient and Poincaré functors: global interpretation.**

9.6.1. Consider the stack $\mathrm{Bun}_{N,\rho(\omega_X)}$ and the map

$$\mathfrak{p} : \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathrm{Bun}_G .$$

Recall that, by Sect. 9.2.5, we have a well-defined functor

$$\mathfrak{p}^!_{\frac{1}{2}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}). \tag{9.17}$$

9.6.2. The character χ has a global counterpart, which is a map

$$\chi^{\mathrm{glob}} : \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathbb{G}_a.$$

Namely, it is the composition

$$\mathrm{Bun}_{N,\rho(\omega_X)} \to \mathrm{Bun}_{N/[N,N],\rho(\omega_X)} \simeq \prod_i \mathrm{Bun}_{(\mathbb{G}_a)_{\omega_X}} \to \prod_i H^1(X,\omega_X) \simeq \prod_i \mathbb{G}_a \overset{\chi_0}{\to} \mathbb{G}_a,$$

where:

- $(\mathbb{G}_a)_{\omega_X}$ is the twist of the constant group-scheme with fiber $\mathbb{G}_a$ using the $\mathbb{G}_m$-action on $\mathbb{G}_a$ and the line bundle ω_X, viewed as a $\mathbb{G}_m$-torsor[39];
- $\mathrm{Bun}_{(\mathbb{G}_a)_{\omega_X}} \to H^1(X,\omega_X)$ is the map that records the class of a torsor.

9.6.3. We let

$$\mathrm{coeff}_G^{\mathrm{Vac,glob}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Vect}$$

denote the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_{\frac{1}{2}}}{\to} \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{-\overset{*}{\otimes}(\chi^{\mathrm{glob}})^*(\exp)}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Bun}_{N,\rho(\omega_X)},-)}{\longrightarrow} \mathrm{Vect}\,.$$

9.6.4. Let

$$\mathrm{Poinc}_{G,!}^{\mathrm{Vac,glob}} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

be the object

$$\mathfrak{p}_{!,\frac{1}{2}} \circ (\chi^{\mathrm{glob}})^*(\exp).$$

I.e., the functor

$$\mathrm{Vect} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G), \quad k \mapsto \mathrm{Poinc}_{G,!}^{\mathrm{Vac,glob}}$$

is the left adjoint of $\mathrm{coeff}_G^{\mathrm{Vac,glob}}$.

Let

$$\mathrm{Poinc}_{G,*}^{\mathrm{Vac,glob}} \in \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$$

be the object

$$\mathfrak{p}_{*,\frac{1}{2}} \circ (\chi^{\mathrm{glob}})^*(\exp).$$

9.6.5. Denote

$$\delta_{N_{\rho(\omega_X)}} := \dim(\mathrm{Bun}_{N,\rho(\omega_X)}).$$

We have

$$\mathbb{D}^{\mathrm{Verdier}}(\mathrm{Poinc}_{G,!}^{\mathrm{Vac,glob}}) = \mathrm{Poinc}_{G,*}^{\mathrm{Vac,glob}}[2\delta_{N_{\rho(\omega_X)}}],$$

where $\mathbb{D}^{\mathrm{Verdier}}$ is the usual Verdier dualization functor

$$(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c)^{\mathrm{op}} \to (\text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G))^c.$$

In other words, the object $\mathrm{Poinc}_{G,*}^{\mathrm{Vac,glob}}[2\delta_{N_{\rho(\omega_X)}}]$, viewed as a functor

$$\mathrm{Vect} \to \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G),$$

is the dual of $\mathrm{coeff}_G^{\mathrm{Vac,glob}}$.

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[39] I.e., it is the total space of ω_X as a line bundle.

9.6.6. We claim:

Lemma 9.6.7.
(i) $\mathrm{coeff}_G^{\mathrm{Vac,glob}} \simeq \mathrm{coeff}_G^{\mathrm{Vac}}[2\delta_{N_{\rho(\omega_X)}}]$;
(ii) $\mathrm{Poinc}_{G,!}^{\mathrm{Vac,glob}} \simeq \mathrm{Poinc}_{G,!}^{\mathrm{Vac}}[-2\delta_{N_{\rho(\omega_X)}}]$;
(iii) $\mathrm{Poinc}_{G,*}^{\mathrm{Vac,glob}} \simeq \mathrm{Poinc}_{G,*}^{\mathrm{Vac}}$.

Proof. The three statements are logically equivalent. We will prove point (iii).

Pick $\underline{x} \in \mathrm{Ran}$, and consider the map

$$N^\alpha/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}} \hookrightarrow \mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}} \overset{\text{act on the unit}}{\longrightarrow} \mathrm{Gr}_{G,\rho(\omega_X),\underline{x}} \to \mathrm{Bun}_G$$

for $N^\alpha \supset \mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}}$ as in Sect. 9.3.4.

By definition, the object $\mathrm{Poinc}_{G,*}^{\mathrm{Vac,glob}}$ is the direct image along this map of $(\chi|^*_{N^\alpha/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}}}(\exp)$, where:

- By a slight abuse of notation, we regard χ as a map $\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}} \to \mathbb{G}_a$;
- The index α is taken to be large enough so that (9.10) holds.

Note that, however, the map

$$\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}} \overset{\text{act on the unit}}{\longrightarrow} \mathrm{Gr}_{G,\rho(\omega_X),\underline{x}} \overset{\pi}{\to} \mathrm{Bun}_G$$

factors as

$$\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}} \to \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathrm{Bun}_G\,.$$

Hence, it suffices to show that if (9.10) holds, then

$$(\pi_{N,\rho(\omega_X)}|_{N^\alpha/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}}})_* \circ (\chi|^*_{N^\alpha/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}}}(\exp) \simeq (\chi^{\mathrm{glob}})^*(\exp). \tag{9.18}$$

Note that the map

$$\chi : \mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}} \to \mathbb{G}_a$$

identifies with

$$\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}} \overset{\pi_{N,\rho(\omega_X)}}{\longrightarrow} \mathrm{Bun}_{N,\rho(\omega_X)} \overset{\chi^{\mathrm{glob}}}{\longrightarrow} \mathbb{G}_a.$$

This implies (9.18) by the projection formula, since if (9.10) holds, the map

$$\pi_{N,\rho(\omega_X)}|_{N^\alpha/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}}} : N^\alpha/\mathfrak{L}^+(N)_{\rho(\omega_X),\underline{x}} \to \mathrm{Bun}_{N,\rho(\omega_X)}$$

is smooth with homologically contractible fibers; in fact, fibers are isomorphic to

$$N^\alpha \cap \mathrm{Sect}(X - \underline{x}, N_{\rho(\omega_X)}).$$

□

10. The localization functor

A fundamental insight of Beilinson-Drinfeld in [BD1] is that the localization functor

$$\mathrm{Loc}_{G,\kappa} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G). \tag{10.1}$$

may be used as a key local-to-global tool in geometric Langlands theory.

There are multiple (equivalent) ways to set this up, and in this section we will describe one of them.[40]

For the sake of completeness we will define $\mathrm{Loc}_{G,\kappa}$ for any level κ. We will specialize to the critical value of κ starting from Sect. 14.

10.1. **The de Rham twisting on** Bun_G **corresponding to a level.** In this subsection we will show how a level κ gives rise to a de Rham twisting $\mathcal{T}_\kappa^{\mathrm{glob}}$ on Bun_G.

[40]The reader who is willing to take the existence of $\mathrm{Loc}_{G,\kappa}$ and its basic properties of faith may choose to skip directly to Sect. 12.

10.1.1. Let $\mathfrak{L}(G)^\wedge$ denote the formal completion of $\mathfrak{L}(G)$ along $\mathfrak{L}^+(G)$, viewed as a factorization group ind-scheme.

Note that a level κ may be thought of as a central extension of $\mathfrak{L}(G)^\wedge$ equipped with a splitting over $\mathfrak{L}^+(G)$. Equivalently, we can think of it as a factorization line bundle $\mathcal{L}^{\mathrm{loc}}_\kappa$ on the groupoid

$$\mathrm{pt}/\mathfrak{L}^+(G) \overset{\overleftarrow{h}^{\mathrm{loc},\wedge}}{\longleftarrow} \mathrm{Hecke}^{\mathrm{loc},\wedge}_G \overset{\overrightarrow{h}^{\mathrm{loc},\wedge}}{\longrightarrow} \mathrm{pt}/\mathfrak{L}^+(G),$$

compatible with the groupoid structure, where:

- $\mathrm{Hecke}^{\mathrm{loc}}_G$ is the local Hecke stack, i.e., $\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G)$, viewed as a groupoid on $\mathrm{pt}/\mathfrak{L}^+(G)$;
- $\mathrm{Hecke}^{\mathrm{loc},\wedge}_G$ is the formal completion of $\mathrm{Hecke}^{\mathrm{loc}}_G$ along the unit section $\mathrm{pt}/\mathfrak{L}^+(G) \to \mathrm{Hecke}^{\mathrm{loc}}_G$.

10.1.2. Let

$$\mathrm{Bun}_G \times \mathrm{Ran} \overset{\overleftarrow{h}^{\mathrm{glob}}}{\longleftarrow} \mathrm{Hecke}^{\mathrm{glob}}_{G,\mathrm{Ran}} \overset{\overrightarrow{h}^{\mathrm{glob}}}{\longrightarrow} \mathrm{Bun}_G \times \mathrm{Ran}$$

be the global Hecke groupoid, and let $\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathrm{Ran}}$ denote its formal completion along the unit section

$$\mathrm{Bun}_G \times \mathrm{Ran} \to \mathrm{Hecke}^{\mathrm{glob}}_{G,\mathrm{Ran}}.$$

Note that we have a map of groupoids

$$\begin{tikzcd}
\mathrm{Bun}_G \times \mathrm{Ran} \arrow[d, "\mathrm{ev}_{\mathrm{Ran}}"'] & \mathrm{Hecke}^{\mathrm{glob}}_{G,\mathrm{Ran}} \arrow[l, "\overleftarrow{h}^{\mathrm{glob}}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{glob}}"] \arrow[d, "\mathrm{ev}_{\mathrm{Ran}}"'] & \mathrm{Bun}_G \times \mathrm{Ran} \arrow[d, "\mathrm{ev}_{\mathrm{Ran}}"] \\
(\mathrm{pt}/\mathfrak{L}^+(G))_{\mathrm{Ran}} & \mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}} \arrow[l, "\overleftarrow{h}^{\mathrm{loc}}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{loc}}"] & (\mathrm{pt}/\mathfrak{L}^+(G))_{\mathrm{Ran}},
\end{tikzcd} \tag{10.2}$$

in which both squares are Cartesian, where we denote by $\mathrm{ev}_{\mathrm{Ran}}$ the "global-to-local" map given by retsriction to the parameterized multi-disc.

Taking the formal completion along the unit sections yields the diagram

$$\begin{tikzcd}
\mathrm{Bun}_G \times \mathrm{Ran} \arrow[d, "\mathrm{ev}_{\mathrm{Ran}}"'] & \mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathrm{Ran}} \arrow[l, "\overleftarrow{h}^{\mathrm{glob},\wedge}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{glob},\wedge}"] \arrow[d, "\mathrm{ev}_{\mathrm{Ran}}"'] & \mathrm{Bun}_G \times \mathrm{Ran} \arrow[d, "\mathrm{ev}_{\mathrm{Ran}}"] \\
(\mathrm{pt}/\mathfrak{L}^+(G))_{\mathrm{Ran}} & \mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}} \arrow[l, "\overleftarrow{h}^{\mathrm{loc},\wedge}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{loc},\wedge}"] & (\mathrm{pt}/\mathfrak{L}^+(G))_{\mathrm{Ran}}.
\end{tikzcd} \tag{10.3}$$

10.1.3. The pullback of the line bundle $\mathcal{L}^{\mathrm{loc}}_{\kappa,\mathrm{Ran}}$ on $\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}}$ along $\mathrm{ev}_{\mathrm{Ran}}$ gives rise to a line bundle, to be denoted $\mathcal{L}^{\mathrm{glob}}_{\kappa,\mathrm{Ran}}$, on $\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathrm{Ran}}$ that is multiplicative with respect to the groupoid structure.

Consider the prestack quotient

$$(\mathrm{Bun}_G \times \mathrm{Ran})/\,\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}}.$$

The datum of $\mathcal{L}^{\mathrm{glob}}_{\kappa,\mathrm{Ran}}$ is equivalent to that of a $\mathcal{O}^\times$-gerbe on $(\mathrm{Bun}_G \times \mathrm{Ran})/\,\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}}$, to be denoted $\mathcal{G}^{/\,\mathrm{Hecke}}_{\kappa,\mathrm{Ran}}$, equipped with a trivialization of its pullback to $\mathrm{Bun}_G \times \mathrm{Ran}$.

10.1.4. Note now that we have a tautological map

$$(\mathrm{Bun}_G \times \mathrm{Ran})/\,\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}} \to (\mathrm{Bun}_G)_{\mathrm{dR}} \times \mathrm{Ran}.$$

Consider the composite map

$$(\mathrm{Bun}_G \times \mathrm{Ran})/\,\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}} \to (\mathrm{Bun}_G)_{\mathrm{dR}} \tag{10.4}$$

We have the following fundamental assertion (see [Ro2, 4.5.3]):

Theorem 10.1.5. *The functor of pullback along* (10.4) *on* $\mathrm{IndCoh}(-)$ *is fully faithful.*

Corollary 10.1.6. *The functor of pullback along* (10.4) *is fully faithful on*[41] $\mathrm{Perf}(-)$.

[41]For a prestack $\mathcal{Y}$, we denote by $\mathrm{Perf}(\mathcal{Y})$ the category of dualizable objects in $\mathrm{QCoh}(\mathcal{Y})$, i.e., the objects whose pullback to any affine scheme is perfect.

10.1.7. The construction of the twising $\mathcal{T}_\kappa$ on Bun_G corresponding to κ is provided by the following assertion:

Corollary 10.1.8. *There exists a uniquely defined de Rham twisting $\mathcal{T}^{\mathrm{glob}}_\kappa$ on Bun_G, such that:*

- *The pullback of the underlying $\mathcal{O}^\times$-gerbe $\mathcal{G}^{\mathrm{glob}}_\kappa$ on $(\mathrm{Bun}_G)_{\mathrm{dR}}$ along (10.4) identifies with $\mathcal{G}^{/\,\mathrm{Hecke}}_{\kappa,\mathrm{Ran}}$;*
- *The trivialization of $\mathcal{G}^{\mathrm{glob}}_\kappa|_{\mathrm{Bun}_G}$ is compatible with the trivialization of $\mathcal{G}^{/\,\mathrm{Hecke}}_{\kappa,\mathrm{Ran}}|_{\mathrm{Bun}_G \times \mathrm{Ran}}$.*

Proof. According to [GaRo2, Sect. 6.3], using the exponential isomorphism

$$(\mathbb{G}_a)^\wedge \overset{\exp}{\to} (\mathbb{G}_m)^\wedge,$$

we can think of a twisting on a prestack $\mathcal{Y}$ as a point in $\mathrm{Maps}(\mathcal{O}_{\mathcal{Y}_{\mathrm{dR}}}, \mathcal{O}_{\mathcal{Y}_{\mathrm{dR}}}[2])$ equipped with a trivialization of its pullback to $\mathcal{Y}$.

By the same logic, we can think of $\mathcal{G}^{/\,\mathrm{Hecke}}_{\kappa,\mathrm{Ran}}$ as a point of

$$\mathrm{Maps}(\mathcal{O}_{(\mathrm{Bun}_G \times \mathrm{Ran})/\,\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}}}, \mathcal{O}_{(\mathrm{Bun}_G \times \mathrm{Ran})/\,\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}}}[2])$$

equipped with a trivialization of its pullback to $\mathrm{Bun}_G \times \mathrm{Ran}$.

The assertion of the corollary follows now from Corollary 10.1.6, combined with the fact that the functor

$$\mathrm{Vect} \to \mathrm{QCoh}(\mathrm{Ran}), \quad k \mapsto \mathcal{O}_{\mathrm{Ran}}$$

is fully faithful.

□

10.1.9. In what follows we will denote the category of $\mathcal{T}^{\mathrm{glob}}_\kappa$-twisted D-modules on Bun_G by

$$\text{D-mod}_\kappa(\mathrm{Bun}_G).$$

10.1.10. Take $\kappa = 2 \cdot \mathrm{crit}$. We claim:

Proposition 10.1.11. *The resulting twisting $\mathcal{T}_{2\cdot\mathrm{crit}}$ on Bun_G identifies canonically with* $\mathrm{dlog}(\det_{\mathrm{Bun}_G})$.

Remark 10.1.12. This proposition implies that our notations for $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$ (see Sect. 9.1.4) are consistent.

Proof. Unwinding the construction, we need to show that the multiplicative line bundle $\mathcal{L}^{\mathrm{glob}}_{2\cdot\mathrm{crit},\mathrm{Ran}}$ on $\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathrm{Ran}}$ identifies with the restriction of the multiplicative line bundle on $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$ given by

$$\overrightarrow{h}^*(\det_{\mathrm{Bun}_G}) \otimes \overleftarrow{h}^*(\det_{\mathrm{Bun}_G})^{\otimes -1}.$$

Recall that that the multiplicative line bundle $\mathcal{L}^{\mathrm{loc}}_{2\cdot\mathrm{crit}}$ on $\mathrm{Hecke}^{\mathrm{loc},\wedge}_G$ is itself obtained as the restriction of the *inverse* of the Tate line bundle $\mathcal{L}^{\mathrm{loc}}_{\mathrm{Tate}}$ on $\mathrm{Hecke}^{\mathrm{loc}}_G$, constructed as follows:

The line bundle $\mathcal{L}^{\mathrm{loc}}_{\mathrm{Tate}}$ associates to a pair of G-bundles $\mathcal{P}^1_G$ and $\mathcal{P}^2_G$ on $\mathcal{D}_{\underline{x}}$ equipped with an identification

$$\mathcal{P}^1_G|_{\mathcal{D}^\times_{\underline{x}}} \simeq \mathcal{P}^2_G|_{\mathcal{D}^\times_{\underline{x}}}$$

the relative determinant of the two lattices

$$\Gamma(\mathcal{D}_{\underline{x}}, \mathfrak{g}_{\mathcal{P}^1_G}) \subset \Gamma(\mathcal{D}^\times_{\underline{x}}, \mathfrak{g}_{\mathcal{P}^1_G}) = \Gamma(\mathcal{D}^\times_{\underline{x}}, \mathfrak{g}_{\mathcal{P}^2_G}) \supset \Gamma(\mathcal{D}_{\underline{x}}, \mathfrak{g}_{\mathcal{P}^2_G}),$$

i.e.,

$$\det(\Gamma(\mathcal{D}_{\underline{x}}, \mathfrak{g}_{\mathcal{P}^1_G})/\mathbf{L}) \otimes \det(\Gamma(\mathcal{D}_{\underline{x}}, \mathfrak{g}_{\mathcal{P}^2_G})/\mathbf{L})^{-1} \tag{10.5}$$

for some/any lattice $\mathbf{L}$ contained in both.

Given a pair of G-bundles $\mathcal{P}^1_G$ and $\mathcal{P}^2_G$ on X equipped with an identification

$$\mathcal{P}^1_G|_{X-\underline{x}} \simeq \mathcal{P}^2_G|_{X-\underline{x}},$$

the fiber of $\overrightarrow{h}^*(\det_{\mathrm{Bun}_G}) \otimes \overleftarrow{h}^*(\det_{\mathrm{Bun}_G})^{\otimes -1}$ at the corresponding point of $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$ is given by

$$\det(\Gamma(X, \mathfrak{g}_{\mathcal{P}^2_G})) \otimes \det(\Gamma(X, \mathfrak{g}_{\mathcal{P}^1_G}))^{\otimes -1}. \tag{10.6}$$

We claim that the lines (10.5) and (10.6) are indeed canonically inverse to one another.

Indeed, we can rewrite (10.5) as

$$\det\Big(\mathrm{Fib}\Big(\Gamma(\mathcal{D}_{\underline{x}},\mathfrak{g}_{\mathcal{P}^1_G})\oplus\mathfrak{g}^{\mathrm{out}}\to\mathfrak{g}^{\mathrm{mer}}\Big)\Big)\otimes\det\Big(\mathrm{Fib}\Big(\Gamma(\mathcal{D}_{\underline{x}},\mathfrak{g}_{\mathcal{P}^2_G})\oplus\mathfrak{g}^{\mathrm{out}}\to\mathfrak{g}^{\mathrm{mer}}\Big)\Big)^{\otimes -1},$$

where:

- $\Gamma(\mathcal{D}^\times_{\underline{x}},\mathfrak{g}_{\mathcal{P}^1_G}) =: \mathfrak{g}^{\mathrm{mer}} := \Gamma(\mathcal{D}^\times_{\underline{x}},\mathfrak{g}_{\mathcal{P}^2_G})$;
- $\Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}^1_G}) =: \mathfrak{g}^{\mathrm{out}} := \Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}^2_G})$.

However,

$$\mathrm{Fib}\Big(\Gamma(\mathcal{D}_{\underline{x}},\mathfrak{g}_{\mathcal{P}^i_G})\oplus\mathfrak{g}^{\mathrm{out}}\to\mathfrak{g}^{\mathrm{mer}}\Big)\simeq\Gamma(X,\mathfrak{g}_{\mathcal{P}^i_G}).$$

□

10.2. **The functor Γ_G.** Our approach to the construction of the localization functor is by defining it as the left adjoint of the functor Γ_G of *global sections* (not quite literally, though, see Sect. 10.3).

In this subsection we introduce the functor Γ_G.

To simplify the notation, for most of this subsection we fix a point $\underline{x}\in\mathrm{Ran}$.

10.2.1. For an integer n, consider the stack $\mathrm{Bun}_G^{\mathrm{level}_{n\cdot\underline{x}}}$ of G-bundles with structure of level n at $\underline{x}$.

Consider the corresponding category

$$\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot\underline{x}}}),$$

see [Ra5, Sect. 1.4.25].

We endow it with the forgetful functor

$$\mathbf{oblv}^{l,\mathrm{ren}}_\kappa:\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot\underline{x}}})\to\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot\underline{x}}}),\tag{10.7}$$

which is the composition with the usual *left* forgetful functor

$$\mathbf{oblv}^{l}_\kappa:\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot\underline{x}}})\to\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot\underline{x}}})$$

(see Corollary A.1.16), followed by the cohomological shift $[2n\dim(\mathfrak{g})]$.

Note that for $n_1\le n_2$, we have a commutative diagram

$$\begin{array}{ccc}\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n_2\cdot\underline{x}}}) & \xrightarrow{\mathbf{oblv}^{l,\mathrm{ren}}_\kappa} & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n_2\cdot\underline{x}}})\\ \uparrow & & \uparrow\\ \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n_1\cdot\underline{x}}}) & \xrightarrow{\mathbf{oblv}^{l,\mathrm{ren}}_\kappa} & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n_1\cdot\underline{x}}}),\end{array}$$

where:

- The left vertical arrow is the functor of *-pullback on twisted D-modules;
- The right vertical arrow is the functor of *-pullback on $\mathrm{QCoh}_{\mathrm{co}}$.

Note that by definition, for $n=0$, we have $\mathbf{oblv}^{l,\mathrm{ren}}_\kappa=\mathbf{oblv}^{l}_\kappa$.

10.2.2. Consider the stack (in fact, a scheme)

$$\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}=\lim_n\mathrm{Bun}_G^{\mathrm{level}_{n\cdot\underline{x}}}$$

of G-bundles with full level structure at $\underline{x}$.

Define

$$\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}):=\operatorname*{colim}_n\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{n\cdot\underline{x}}}),$$

where the colimit is formed using the *-pullback functors

$$\text{D-mod}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{n_1\cdot\underline{x}}})\to\text{D-mod}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{n_2\cdot\underline{x}}}),\quad n_2\ge n_1.$$

The functors (10.7) combine to a functor

$$\mathbf{oblv}^{l,\mathrm{ren}}_\kappa : \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}).$$

Consider the composite functor, to be denoted $\Gamma^{\mathrm{ren}}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}, -)$,

$$\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}) \overset{\mathbf{oblv}^{l,\mathrm{ren}}_\kappa}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}) \overset{\Gamma(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}},-)}{\longrightarrow} \mathrm{Vect}; \tag{10.8}$$

10.2.3. According to [Ra5, Sect. 1.4.25], the category $\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}})$ carries a *strong* action of $\mathfrak{L}(G)_{\underline{x}}$ at level κ. Furthermore, the functor $\Gamma^{\mathrm{ren}}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}, -)$ of (10.8) is *weakly* $\mathfrak{L}(G)_{\underline{x}}$-equivariant.

Hence, by the *universal property of* $\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}}$ (see [Ra5, Sect. 1.4.25]), the functor $\Gamma^{\mathrm{ren}}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}, -)$ upgrades to a functor

$$\Gamma^{\mathrm{ren}}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}, -)^{\mathrm{enh}} : \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}) \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}},$$

strongly compatible with the $\mathfrak{L}(G)_{\underline{x}}$-actions.

10.2.4. In particular, the functor $\Gamma^{\mathrm{ren}}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}, -)^{\mathrm{enh}}$ gives rise to a functor, to be denoted $\Gamma_{G,\kappa,\underline{x}}$:

$$\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G) \simeq \left(\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}})\right)^{\mathfrak{L}^+(G)_{\underline{x}}} \to (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}})^{\mathfrak{L}^+(G)_{\underline{x}}} = \mathrm{KL}(G)_{\kappa,\underline{x}}.$$

10.2.5. By a similar token, letting $\underline{x}$ vary over the Ran space, we obtain a functor

$$\Gamma_{G,\kappa} : \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G) \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}.$$

Furthermore, we can consider a D-mod(Ran)-linear functor

$$\Gamma_{G,\kappa,\mathrm{Ran}} : \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}},$$

so that $\Gamma_{G,\kappa}$ is the composition

$$\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G) \overset{\mathrm{Id}\otimes\omega_{\mathrm{Ran}}}{\longrightarrow} \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \overset{\Gamma_{G,\kappa,\mathrm{Ran}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}.$$

10.2.6. Let us specialize for a moment to the case when $\kappa = \mathrm{crit}$. In this case, both sides of

$$\Gamma_{G,\mathrm{crit},\underline{x}} : \text{D-mod}_{\mathrm{crit},\mathrm{co}}(\mathrm{Bun}_G) \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}}$$

are acted on by $\mathrm{Sph}_{G,\underline{x}}$, and it follows from the construction that the functor $\Gamma_{G,\mathrm{crit},\underline{x}}$ is compatible with these actions.

Similarly, the functor $\Gamma_{G,\kappa,\mathrm{Ran}}$ is compatible with the action of $\mathrm{Sph}_{G,\mathrm{Ran}}$.

10.2.7. Note that by construction, we have a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G) & \xrightarrow{\Gamma_{G,\kappa,\underline{x}}} & \mathrm{KL}(G)_{\kappa,\underline{x}} \\ {\scriptstyle \mathbf{oblv}^l_\kappa}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}} \\ \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G) & \xrightarrow[\Gamma^{\mathrm{QCoh}}_{G,\kappa,\underline{x}}]{} & \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}}), \end{array} \tag{10.9}$$

where $\Gamma^{\mathrm{QCoh}}_{G,\kappa,\underline{x}}$ is the functor of pushforward along

$$\mathrm{ev}_{\underline{x}} : \mathrm{Bun}_G \to \mathrm{pt}\,/\mathfrak{L}^+(G)_{\underline{x}}.$$

Remark 10.2.8. Note that $\mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}})$ is the renormalized version of $\mathrm{QCoh}(\mathrm{pt}\,/\mathfrak{L}^+(G)_{\underline{x}})$ (see Sect. B.14.1); however, this difference is immaterial for the definition of the functor$\Gamma^{\mathrm{QCoh}}_{G,\kappa,\underline{x}}$:

We have:

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G) \simeq \underset{U}{\mathrm{colim}}\, \mathrm{QCoh}(U),$$

where:

- The index U runs over the filtered posets of quasi-compact open substacks of Bun_G;
- For $U_1 \overset{j_{1,2}}{\hookrightarrow} U_2$, the transition functor $\mathrm{QCoh}(U_1) \to \mathrm{QCoh}(U_2)$ is given by $(j_{1,2})_*$.

A functor out of $\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G)$ amounts to a compatible collection of functors out of $\mathrm{QCoh}(U)$. Thus, in order to define

$$\Gamma^{\mathrm{QCoh}}_{G,\kappa,\underline{x}} : \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G) \to \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}}),$$

we need to define the functors

$$\Gamma^{\mathrm{QCoh}}_{G,\kappa,\underline{x},U} : \mathrm{QCoh}(U) \to \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}}).$$

The sought-for functor $\Gamma^{\mathrm{QCoh}}_{G,\kappa,\underline{x},U}$ are defined as the ind-extension of the functor

$$\mathrm{QCoh}(U)^c \hookrightarrow \mathrm{QCoh}(U)^{>-\infty} \overset{(\mathrm{ev}_{\underline{x},U})_*}{\longrightarrow} \mathrm{QCoh}(\mathrm{pt}\,/\mathfrak{L}^+(G)_{\underline{x}})^{>-\infty} \simeq$$
$$\simeq \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}})^{>-\infty} \hookrightarrow \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}}),$$

where $\mathrm{ev}_{\underline{x},U}$ denotes the restriction of $\mathrm{ev}_{\underline{x}}$ to U.

10.2.9. Similarly, we have a commutative diagram

$$\begin{tikzcd}
\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \arrow[r, "\Gamma_{G,\kappa,\mathrm{Ran}}"] \arrow[d, "\mathbf{oblv}^l_\kappa"'] & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \arrow[d, "\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"] \\
\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \arrow[r, "\Gamma^{\mathrm{QCoh}}_{G,\kappa,\mathrm{Ran}}"'] & \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}}.
\end{tikzcd} \tag{10.10}$$

10.2.10. Let now $\underline{x}$ and $\underline{x}'$ be two points of Ran with $\underline{x} \subseteq \underline{x}'$. Consider the functor

$$\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'} : \mathrm{KL}(G)_{\kappa,\underline{x}} \to \mathrm{KL}(G)_{\kappa,\underline{x}'}$$

obtained by inserting the vacuum objects at the points $\underline{x}' - \underline{x}$.

Consider its right adjoint, $(\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'})^R$. Explicitly, the functor $(\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'})^R$ is given by

$$\mathrm{KL}(G)_{\kappa,\underline{x}'} \simeq \mathrm{KL}(G)_{\kappa,\underline{x}} \otimes \mathrm{KL}(G)_{\kappa,\underline{x}'-\underline{x}} \overset{\mathrm{Id}\otimes\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}}{\longrightarrow}$$
$$\to \mathrm{KL}(G)_{\kappa,\underline{x}} \otimes \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}'-\underline{x}}) \overset{\mathrm{Id}\otimes\mathrm{inv}^{\mathfrak{L}^+(G)_{\underline{x}'-\underline{x}}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\underline{x}}.$$

It follows from the commutation of (10.9) that we have a canonical isomorphism

$$\Gamma_{G,\kappa,\underline{x}} \simeq (\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'})^R \circ \Gamma_{G,\kappa,\underline{x}'}. \tag{10.11}$$

This isomorphism expresses the *unital* structure on the assignment

$$\underline{x} \rightsquigarrow \Gamma_{G,\mathrm{crit},\underline{x}},$$

to be discussed in Sect. 11.

10.2.11. In particular, for any $\underline{x}$, the functor

$$\text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G) \overset{\Gamma_{G,\kappa,\underline{x}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\underline{x}} \overset{\mathrm{inv}^{\mathfrak{L}^+(G)_{\underline{x}}}}{\longrightarrow} \mathrm{Vect}$$

identifies with $\Gamma(\mathrm{Bun}_G, \mathbf{oblv}^l_\kappa(-))$.

10.3. **Localization functor as a left adjoint.** As was mentioned previously, we construct the localization functor $\mathrm{Loc}_{G,\kappa}$ to be *essentially* the left adjoint of Γ_G. However, there is a caveat: this adjunction takes place over quasi-compact open substacks $U \subset \mathrm{Bun}_G$, and we obtain the corresponding functors $\mathrm{Loc}_{G,\kappa,U}$. We then obtain the sought-for functor $\mathrm{Loc}_{G,\kappa}$ by passing to the limit.

10.3.1. Let

$$U \overset{j_U}{\hookrightarrow} \mathrm{Bun}_G \tag{10.12}$$

be a quasi-compact open substack. Consider the corresponding functor

$$j_{*,\mathrm{co}} : \text{D-mod}_\kappa(U) \to \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G).$$

Denote

$$\Gamma_{G,\kappa,\underline{x},U} := \Gamma_{G,\kappa,\underline{x}} \circ j_{*,\mathrm{co}}, \quad \text{D-mod}_\kappa(U) \to \mathrm{KL}(G)_{\kappa,\underline{x}}.$$

10.3.2. We claim:

Lemma 10.3.3. *The functor $\Gamma_{G,\kappa,\underline{x},U}$ admits a left adjoint.*

Proof. Since the essential image of

$$\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} : \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}}) \to \mathrm{KL}(G)_{\kappa,\underline{x}}$$

generates the target category, it suffices to show that the composite functor

$$\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} \circ \Gamma_{G,\kappa,\underline{x},U}, \quad \text{D-mod}_\kappa(U) \to \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}})$$

admits a left adjoint.

The above functor identifies with

$$(\mathrm{ev}_{\underline{x}} \circ j)_* \circ \mathbf{oblv}^l_\kappa.$$

In this composition, both arrows admit left adjoints: the left adjoint of $\mathbf{oblv}^l_\kappa$ is $\mathbf{ind}^l_\kappa$, and the left adjoint of $(\mathrm{ev}_{\underline{x}} \circ j)_*$ is $(\mathrm{ev}_{\underline{x}} \circ j)^*$.

$\square$

10.3.4. Let us denote the left adjoint in Lemma 10.3.3 by $\mathrm{Loc}_{G,\kappa,\underline{x},U}$.

For an inclusion between quasi-compact open substacks

$$U_1 \overset{j_{1,2}}{\hookrightarrow} U_2,$$

we have

$$(j_2)_{*,\mathrm{co}} \circ (j_{1,2})_* \simeq (j_1)_{*,\mathrm{co}}.$$

Hence, we obtain a canonical identification

$$\mathrm{Loc}_{G,\kappa,\underline{x},U_1} \simeq j^*_{1,2} \circ \mathrm{Loc}_{G,\kappa,\underline{x},U_2}.$$

Therefore, the system of functors

$$U \rightsquigarrow \{\mathrm{Loc}_{G,\kappa,\underline{x},U}\}$$

gives rise to a functor

$$\mathrm{Loc}_{G,\kappa,\underline{x}} : \mathrm{KL}(G)_{\kappa,\underline{x}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G), \tag{10.13}$$

so that for every (10.12), we have

$$j^* \circ \mathrm{Loc}_{G,\kappa,\underline{x}} \simeq \mathrm{Loc}_{G,\kappa,\underline{x},U}.$$

The functor (10.13) is the sought-for localization functor.

10.3.5. The entire preceding discussion generalizes to the case when $\underline{x}$ is allowed to move in families over Ran. In particular, we obtain a functor

$$\mathrm{Loc}_{G,\kappa,\mathrm{Ran}} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G \times \mathrm{Ran}).$$

Let $\mathrm{Loc}_{G,\kappa}$ denote the composition

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \xrightarrow{\mathrm{Loc}_{G,\kappa,\mathrm{Ran}}} \text{D-mod}_\kappa(\mathrm{Bun}_G \times \mathrm{Ran}) \to \text{D-mod}_\kappa(\mathrm{Bun}_G),$$

where the second arrow is the functor of !-pushforward.

Remark 10.3.6. The above construction of the localization functor is essentially equivalent to the one from [CF, Sect. 4.1].

10.3.7. Properties of the functor $\Gamma_{G,\kappa',\mathrm{Ran}}$ induce corresponding properties of the functor $\mathrm{Loc}_{G,\kappa,\mathrm{Ran}}$. We will now list some of them.

10.3.8. By adjunction, for every quasi-compact open as in (10.12), from diagram (10.9) we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{QCoh}(U) \arrow[r, "\mathbf{ind}^l_\kappa"] & \text{D-mod}_\kappa(U) \\
\mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}}) \arrow[u, "j^*\circ \mathrm{Loc}^{\mathrm{QCoh}}_{G,\underline{x}}"] \arrow[r, "\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"'] & \mathrm{KL}(G)_{\kappa,\underline{x}}, \arrow[u, "\mathrm{Loc}_{G,\kappa,\underline{x},U}"']
\end{tikzcd} \tag{10.14}$$

where:

$$\mathrm{Loc}^{\mathrm{QCoh}}_{G,\underline{x}} : \mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}}) \to \mathrm{QCoh}(\mathrm{Bun}_G)$$

is the functor of pullback along $\mathrm{ev}_{\underline{x}} : \mathrm{Bun}_G \to \mathrm{pt}/\mathfrak{L}^+(G)_{\underline{x}}$.

Remark 10.3.9. As in Remark 10.2.8, the difference between $\mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}})$ and $\mathrm{QCoh}(\mathrm{pt}/\mathfrak{L}^+(G)_{\underline{x}})$ does not play a role in the definition of the functor $\mathrm{Loc}^{\mathrm{QCoh}}_{G,\underline{x}}$.

Namely, $\mathrm{Loc}^{\mathrm{QCoh}}_{G,\underline{x}}$ is defined as the ind-extension of

$$\mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}})^c \hookrightarrow \mathrm{QCoh}(\mathrm{pt}/\mathfrak{L}^+(G)_{\underline{x}}) \overset{(\mathrm{ev}_{\underline{x}})^*}{\hookrightarrow} \mathrm{QCoh}(\mathrm{Bun}_G).$$

Note also that for a quasi-compact U, the functor

$$\mathrm{Loc}^{\mathrm{QCoh}}_{G,\underline{x},U} := j^* \circ \mathrm{Loc}^{\mathrm{QCoh}}_{G,\underline{x}}$$

is the left adjoint of the functor $\Gamma^{\mathrm{QCoh}}_{G,\kappa,\underline{x},U}$.

10.3.10. Passing to the limit over (10.12), we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Bun}_G) \arrow[r, "\mathbf{ind}^l_\kappa"] & \text{D-mod}_\kappa(\mathrm{Bun}_G) \\
\mathrm{Rep}(\mathfrak{L}^+(G)_{\underline{x}}) \arrow[u, "\mathrm{Loc}^{\mathrm{QCoh}}_{G,\underline{x}}"] \arrow[r, "\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"'] & \mathrm{KL}(G)_{\kappa,\underline{x}}. \arrow[u, "\mathrm{Loc}_{G,\kappa,\underline{x}}"']
\end{tikzcd} \tag{10.15}$$

Similarly, we have

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \arrow[r, "\mathbf{ind}^l_\kappa"] & \text{D-mod}_\kappa(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \arrow[u, "\mathrm{Loc}^{\mathrm{QCoh}}_{G,\mathrm{Ran}}"] \arrow[r, "\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"'] & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \arrow[u, "\mathrm{Loc}_{G,\kappa,\mathrm{Ran}}"']
\end{tikzcd} \tag{10.16}$$

and

$$
\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Bun}_G) \arrow[r, "\mathbf{ind}^l_\kappa"] & \text{D-mod}_\kappa(\mathrm{Bun}_G) \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \arrow[u, "\mathrm{Loc}^{\mathrm{QCoh}}_G"] \arrow[r, "\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"] & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}, \arrow[u, "\mathrm{Loc}_{G,\kappa}"']
\end{tikzcd}
\tag{10.17}
$$

where $\mathrm{Loc}^{\mathrm{QCoh}}_G$ is the functor of pull-push along

$$\mathrm{Bun}_G \leftarrow \mathrm{Bun}_G \times \mathrm{Ran} \to (\mathrm{pt}\,/\mathfrak{L}^+(G))_{\mathrm{Ran}}.$$

10.3.11. Let now $\underline{x}$ and $\underline{x}'$ be two points of Ran with $\underline{x} \subseteq \underline{x}'$. Recall the functor

$$\mathrm{ins.\,vac}_{\underline{x}\subseteq\underline{x}'} : \mathrm{KL}(G)_{\kappa,\underline{x}} \to \mathrm{KL}(G)_{\kappa,\underline{x}'}. \tag{10.18}$$

For every (10.12), from (10.11) we obtain a canonical isomorphism

$$\mathrm{Loc}_{G,\kappa,\underline{x}',U} \circ \mathrm{ins.\,vac}_{\underline{x}\subseteq\underline{x}'} \simeq \mathrm{Loc}_{G,\kappa,\underline{x},U}\,. \tag{10.19}$$

Passing to the limit over (10.12), we obtain a canonical isomorphism

$$\mathrm{Loc}_{G,\kappa,\underline{x}'} \circ \mathrm{ins.\,vac}_{\underline{x}\subseteq\underline{x}'} \simeq \mathrm{Loc}_{G,\kappa,\underline{x}}\,. \tag{10.20}$$

In Sect. 11.3.7 we will formulate a version of (10.20) when the points $\underline{x}$ and $\underline{x}'$ move in families over the Ran space.

10.3.12. It follows from (10.15) that for any $\underline{x}$, we have

$$\mathrm{Loc}_{G,\kappa,\underline{x}}(\mathrm{Vac}(G)_{\kappa,\underline{x}}) \simeq \mathbf{ind}^l_\kappa(\mathcal{O}_{\mathrm{Bun}_G}).$$

Equivalently,

$$\mathrm{Loc}_{G,\kappa,\mathrm{Ran}}(\mathrm{Vac}(G)_{\kappa,\mathrm{Ran}}) \simeq \mathbf{ind}^l_\kappa(\mathcal{O}_{\mathrm{Bun}_G}) \boxtimes \omega_{\mathrm{Ran}}. \tag{10.21}$$

Note that

$$\mathbf{ind}^l_\kappa(\mathcal{O}_{\mathrm{Bun}_G}) \simeq \mathrm{D}_{\mathrm{Bun}_G,\kappa},$$

where

$$\mathrm{D}_{\mathrm{Bun}_G,\kappa} \in \text{D-mod}_\kappa(\mathrm{Bun}_G)$$

is the D-module of differential operators, viewed a twisted *left* D-module.

10.3.13. Let us specialize for a moment to the case when $\kappa = \mathrm{crit}$. Then from Sect. 10.2.6 we obtain that for every U, the functor $\mathrm{Loc}_{G,\kappa,\underline{x},U}$ is compatible with the action of $\mathrm{Sph}_{G,\underline{x}}$. Hence, so is the functor $\mathrm{Loc}_{G,\kappa,\underline{x}}$.

Similarly, the functor $\mathrm{Loc}_{G,\kappa,\mathrm{Ran}}$ is compatible with the action of $\mathrm{Sph}_{G,\mathrm{Ran}}$.

10.3.14. For future reference we note that the entire discussion in this subsection applies to to the case of an infinite level structure at a given $\underline{x}_0 \in \mathrm{Ran}$. I.e., for a given quasi-compact $U \subset \mathrm{Bun}_G$ and

$$U^{\mathrm{level}_{\underline{x}_0}} := U \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\mathrm{level}_{\underline{x}_0}},$$

we can consider the left adjoint

$$\mathrm{Loc}_{G,\kappa,\underline{x}_0,U} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}_0} \to \text{D-mod}_{\kappa,\mathrm{co}}(U_G^{\mathrm{level}_{\underline{x}_0}})$$

of

$$\Gamma(U^{\mathrm{level}_{\underline{x}_0}}, -)^{\mathrm{enh}} : \text{D-mod}_{\kappa,\mathrm{co}}(U_G^{\mathrm{level}_{\underline{x}_0}}) \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}_0}.$$

The functors $\mathrm{Loc}_{G,\kappa,\underline{x}_0,U}$ glue to a functor

$$\mathrm{Loc}_{G,\kappa,\underline{x}_0,U} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}_0} \to \text{D-mod}_{\kappa,\mathrm{co}}(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}_0}}).$$

For the Ran space version, one should consider

$$\mathrm{Ran}_{\underline{x}_0} := \{\underline{x}_0\} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq},$$

where:

- $\mathrm{Ran}^{\subseteq}$ is as in Sect. 11.2.2;
- The fiber product is formed using the map $\mathrm{pr}_{\mathrm{small}} : \mathrm{Ran}^{\subseteq} \to \mathrm{Ran}$.

10.4. **The fiber of the localization functor.**

10.4.1. Fix a k-point $\mathcal{P}_G \in \mathrm{Bun}_G$. The goal of this subsection is to describe the functor

$$\mathrm{Loc}_{G,\kappa,\underline{x}} \overset{\mathrm{Loc}_{G,\kappa,\underline{x}}}{\longrightarrow} \text{D-mod}_\kappa(\mathrm{Bun}_G) \overset{!\text{-fiber at } \mathcal{P}_G}{\longrightarrow} \mathrm{Vect}\,. \tag{10.22}$$

10.4.2. Consider the Lie algebra $\Gamma(X - \underline{x}, \mathfrak{g}_{\mathcal{P}_G})$. Laurent expansion defines a map

$$\Gamma(X - \underline{x}, \mathfrak{g}_{\mathcal{P}_G}) \to \mathfrak{L}(\mathfrak{g}_{\mathcal{P}_G})_{\underline{x}},$$

and recall that the Kac-Moody extension

$$0 \to k \to \widehat{\mathfrak{g}}_{\kappa,\mathcal{P}_G,\underline{x}} \to \mathfrak{L}(\mathfrak{g}_{\mathcal{P}_G})_{\underline{x}} \to 0$$

admits a canonical splitting over $\Gamma(X - \underline{x}, \mathfrak{g}_{\mathcal{P}_G})$. Hence, we have a well-defined restriction functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}} \to \Gamma(X - \underline{x}, \mathfrak{g}_{\mathcal{P}_G})\text{-mod}. \tag{10.23}$$

10.4.3. Composing with

$$\mathrm{KL}(G)_{\kappa,\underline{x}} \overset{\alpha_{\mathcal{P}_G,\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_G,\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}}$$

we obtain a functor

$$\mathrm{KL}(G)_{\kappa,\underline{x}} \to \Gamma(X - \underline{x}, \mathfrak{g}_{\mathcal{P}_G})\text{-mod}. \tag{10.24}$$

10.4.4. We claim:

Proposition 10.4.5. *The functor* (10.22) *identifies canonically with the composition of* (10.24) *and the functor of* $\Gamma(X - \underline{x}, \mathfrak{g}_{\mathcal{P}_G})$*-coinvariants*

$$\Gamma(X - \underline{x}, \mathfrak{g}_{\mathcal{P}_G})\text{-mod} \to \mathrm{Vect}\,.$$

Remark 10.4.6. An analog of Proposition 10.4.5 for the localization functor in the finite-dimensional situation is obvious:

Let $\mathfrak{h}$ be a (discrete[42]) Lie algebra and let $\mathcal{Y}$ be a smooth variety equipped with an action of $\mathfrak{h}$ by vector fields. Then the corresponding localization functor

$$\mathrm{Loc}_{\mathfrak{h},\mathcal{Y}} : \mathfrak{h}\text{-mod} \to \text{D-mod}(\mathcal{Y}),$$

left adjoint to

$$\Gamma(\mathcal{Y}, \mathbf{oblv}^l(-))^{\mathrm{enh}} : \text{D-mod}(\mathcal{Y}) \to \mathfrak{h}\text{-mod},$$

is given by

$$\mathrm{D}_{\mathcal{Y}} \underset{U(\mathfrak{h})}{\otimes} - \,. \tag{10.25}$$

If $y \in \mathcal{Y}$ is a point for which the action map $\mathfrak{h} \to T_y(\mathcal{Y})$ is surjective, the composition

$$\mathfrak{h}\text{-mod} \overset{\mathrm{Loc}_{\mathfrak{h},\mathcal{Y}}}{\to} \text{D-mod}(\mathcal{Y}) \overset{!\text{-fiber at } y}{\longrightarrow} \mathrm{Vect}$$

identifies with the functor of coinvariants with respect to

$$\mathrm{Stab}_y(\mathfrak{h}) \subset \mathfrak{h}.$$

This follows from the fact that the *-fiber at y of $\mathrm{D}_{\mathcal{Y}}$ (as an object of $\mathrm{QCoh}(\mathcal{Y})$ via left multiplication) identifies, as a $\mathfrak{h}$-module, with

$$\mathbf{ind}^{\mathfrak{h}}_{\mathrm{Stab}_y(\mathfrak{h})}(k).$$

[42]As opposed to Tate.

10.4.7. One can prove Proposition 10.4.5 directly by emulating the argument in Remark 10.4.6.

In fact, such an assertion is valid for $\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}$ replaced by a pro-scheme $\mathcal{Y}$ equipped with an action of $\mathfrak{L}(G)^\wedge_{\underline{x}}$ (the formal completion of $\mathfrak{L}(G)_{\underline{x}}$ along $\mathfrak{L}^+(G)_{\underline{x}}$), such that $\mathcal{Y}/\mathfrak{L}^+(G)_{\underline{x}}$ is locally of finite type, and a point $y \in \mathcal{Y}$ at which the action is infinitesimally transitive, i.e.,

$$\mathfrak{L}(\mathfrak{g})_{\underline{x}} \to T_y(\mathcal{Y})$$

is surjective.

We will, however, supply a different argument, specific to the case of Bun_G, see Sect. 12.2.

10.4.8. As an immediate corollary of Proposition 10.4.5 we obtain:

Corollary 10.4.9. *The functor*

$$\mathrm{Loc}_{G,\kappa,\underline{x}} : \mathrm{KL}(G)_{\kappa,\underline{x}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G)$$

is right t-exact, when D-mod$_\kappa(\mathrm{Bun}_G)$ *is equipped with the* left *t-structure, i.e., one for which the functor* $\mathbf{oblv}^l_\kappa$ *is t-exact.*

Proof. We need to show that the composite functor

$$\mathrm{KL}(G)_{\kappa,\underline{x}} \overset{\mathrm{Loc}_{G,\kappa,\underline{x}}}{\longrightarrow} \text{D-mod}_\kappa(\mathrm{Bun}_G) \overset{\mathbf{oblv}^l_\kappa}{\to} \mathrm{QCoh}(\mathrm{Bun}_G)$$

is right t-exact.

In order to prove that, it suffices to show that the composition of the above functor with the functor of *-fiber at any field-valued point of Bun_G is right t-exact.

By base change, we can assume that the point in question is rational. In this case, the corresponding functor identifies with the functor (10.22).

□

Corollary 10.4.10. *The functor* $\mathrm{Loc}_{G,\kappa,\underline{x}}$ *annihilates infinitely connective objects (i.e., objects that belong to* $(\mathrm{KL}(G)_{\kappa,\underline{x}})^{<-n}$ *for any* n*).*

Proof. Follows from the fact that the t-structure on D-mod$_\kappa(\mathrm{Bun}_G)$ is separated.

□

10.5. **Localization functor as the dual.**

10.5.1. Let κ' be the reflected level, i.e.,

$$\kappa' := -\kappa + 2 \cdot \mathrm{crit}.$$

We claim that that we have a canonical duality

$$(\text{D-mod}_{\kappa',\mathrm{co}}(\mathrm{Bun}_G))^\vee \simeq \text{D-mod}_\kappa(\mathrm{Bun}_G) \tag{10.26}$$

for which the dual of the functor

$$\mathbf{oblv}^l_{\kappa'} : \text{D-mod}_{\kappa',\mathrm{co}}(\mathrm{Bun}_G) \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G)$$

is the functor

$$\mathbf{ind}^l_\kappa : \mathrm{QCoh}(\mathrm{Bun}_G) \to \text{D-mod}_\kappa(\mathrm{Bun}_G),$$

with respect to the identification[43]

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Bun}_G), \mathrm{Vect}) \simeq \mathrm{QCoh}(\mathrm{Bun}_G).$$

[43] We warn the reader that the category $\mathrm{QCoh}(\mathrm{Bun}_G)$ is not dualizable.

10.5.2. Indeed, we start with the identification

$$(\text{D-mod}_{\kappa',\text{co}}(\text{Bun}_G))^\vee \simeq \text{D-mod}_{-\kappa'}(\text{Bun}_G),$$

given by Verdier duality, and compose it with the functor

$$\text{D-mod}_{-\kappa'}(\text{Bun}_G) \overset{\otimes K_{\text{Bun}_G}}{\longrightarrow} \text{D-mod}_{-\kappa'+\text{dlog}(K_{\text{Bun}_G})}(\text{Bun}_G) \simeq \text{D-mod}_{-\kappa'+2\cdot\text{crit}}(\text{Bun}_G) =$$
$$= \text{D-mod}_\kappa(\text{Bun}_G) \overset{[\dim(\text{Bun}_G)]}{\longrightarrow} \text{D-mod}_\kappa(\text{Bun}_G),$$

where:

- K_{Bun_G} is the canonical line bundle on Bun_G, so that $K_{\text{Bun}_G}[\dim(\text{Bun}_G)] \simeq \omega_{\text{Bun}_G}$;
- We have used the identification $\text{dlog}(K_{\text{Bun}_G}) = \text{dlog}(\det_{\text{Bun}_G}) = 2 \cdot \text{crit}$ from Sect. 9.1.3.

10.5.3. We have the following assertion:

Proposition 10.5.4. *With respect to the identifications* (10.26) *and*

(10.27) $$(\text{KL}(G)_{\kappa,\underline{x}})^\vee \simeq \text{KL}(G)_{\kappa',\underline{x}}$$

of (2.2)*, the functor*

$$\text{Loc}_{G,\kappa,\underline{x}} : \text{KL}(G)_{\kappa,\underline{x}} \to \text{D-mod}_\kappa(\text{Bun}_G)$$

identifies canonically with the dual of

$$\Gamma_{G,\kappa',\underline{x}} : \text{D-mod}_{\kappa',\text{co}}(\text{Bun}_G) \to \text{KL}(G)_{\kappa',\underline{x}}.$$

The induced identification

$$\mathbf{ind}^l_\kappa \circ (\text{ev}_{\underline{x}})^* \simeq \text{Loc}_{G,\kappa,\underline{x}} \circ \mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} \simeq (\Gamma_{G,\kappa',\underline{x}})^\vee \circ (\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_{\kappa'}}_{\mathfrak{L}^+(G)})^\vee \simeq$$
$$\simeq \left(\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_{\kappa'}}_{\mathfrak{L}^+(G)} \circ \Gamma_{G,\kappa',\underline{x}}\right)^\vee \simeq ((\text{ev}_{\underline{x}})_* \circ \mathbf{oblv}^l_{\kappa'})^\vee \simeq (\mathbf{oblv}^l_{\kappa'})^\vee \circ ((\text{ev}_{\underline{x}})_*)^\vee \simeq$$
$$\simeq \mathbf{ind}^l_\kappa \circ ((\text{ev}_{\underline{x}})_*)^\vee \simeq \mathbf{ind}^l_\kappa \circ (\text{ev}_{\underline{x}})^*$$

is the identity map.

This assertion is proved in [CF, Theorem 4.0.5(2)].[44]

Remark 10.5.5. The proof of Proposition 10.5.4 in [CF] essentially emulates the following finite-dimensional phenomenon.

Let $\mathcal{Y}$ and $\mathfrak{h}$ be as in Remark 10.4.6. On the one hand, we can consider the adjoint pair

$$\text{Loc}_{\mathfrak{h},\mathcal{Y}} : \mathfrak{h}\text{-mod} \leftrightarrows \text{D-mod}(\mathcal{Y}) : \Gamma(\mathcal{Y}, \mathbf{oblv}^l(-))^{\text{enh}}.$$

On the other hand, consider the canonical line bundle $K_\mathcal{Y}$ as a line bundle acted on by $\mathfrak{h}$, and consider the corresponding functor

$$\Gamma(\mathcal{Y}, K_\mathcal{Y} \otimes \mathbf{oblv}^l(-))^{\text{enh}} : \text{D-mod}(\mathcal{Y}) \to \mathfrak{h}\text{-mod}.$$

Let

$$\text{Loc}_{\mathfrak{h},\mathcal{Y},K_\mathcal{Y}} : \mathfrak{h}\text{-mod} \to \text{D-mod}(\mathcal{Y})$$

denote the left adjoint of $\Gamma(\mathcal{Y}, K_\mathcal{Y} \otimes \mathbf{oblv}^l(-))^{\text{enh}}$.

Then the functors $\text{Loc}_{\mathfrak{h},\mathcal{Y},K_\mathcal{Y}}[\dim(\mathcal{Y})]$ and $\Gamma(\mathcal{Y}, \mathbf{oblv}^l(-))^{\text{enh}}$ are mutually dual in terms of the Verdier duality identification

$$\text{D-mod}(\mathcal{Y})^\vee \simeq \text{D-mod}(\mathcal{Y}).$$

This follows from the expression for $\text{Loc}_{\mathfrak{h},\mathcal{Y}}$ given by formula (10.25), and a similar formula for $\text{Loc}_{\mathfrak{h},\mathcal{Y},K_\mathcal{Y}}$.

[44] In *loc. cit.* the dual functor to $\text{Loc}_{G,\kappa}$ is denoted Loc_{co}.

10.5.6. The assertion of Proposition 10.5.4 admits an immediate generalization when $\underline{x}$ moves in families over the Ran space:

Proposition 10.5.7. *With respect to the identifications* (10.26) *and*

$$(\mathrm{KL}(G)_{\kappa,\mathrm{Ran}})^\vee \simeq \mathrm{KL}(G)_{\kappa',\mathrm{Ran}}$$

(a) *The functor*

$$\mathrm{Loc}_{G,\kappa,\mathrm{Ran}} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran})$$

identifies canonically with the dual of

$$\Gamma_{G,\kappa',\mathrm{Ran}} : \text{D-mod}_{\kappa',\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \to \mathrm{KL}(G)_{\kappa',\mathrm{Ran}}.$$

The induced identification

$$\begin{aligned}\mathbf{ind}^l_\kappa \circ (\mathrm{ev}_{\mathrm{Ran}})^* \simeq \mathrm{Loc}_{G,\kappa,\mathrm{Ran}} \circ \mathbf{ind}^{(\hat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} \simeq (\Gamma_{G,\kappa',\mathrm{Ran}})^\vee \circ (\mathbf{oblv}^{(\hat{\mathfrak{g}},\mathfrak{L}^+(G))_{\kappa'}}_{\mathfrak{L}^+(G)})^\vee \simeq \\ \simeq \left(\mathbf{oblv}^{(\hat{\mathfrak{g}},\mathfrak{L}^+(G))_{\kappa'}}_{\mathfrak{L}^+(G)} \circ \Gamma_{G,\kappa',\mathrm{Ran}}\right)^\vee \simeq ((\mathrm{ev}_{\mathrm{Ran}})_* \circ \mathbf{oblv}^l_{\kappa'})^\vee \simeq (\mathbf{oblv}^l_{\kappa'})^\vee \circ ((\mathrm{ev}_{\mathrm{Ran}})_*)^\vee \simeq \\ \simeq \mathbf{ind}^l_\kappa \circ ((\mathrm{ev}_{\mathrm{Ran}})_*)^\vee \simeq \mathbf{ind}^l_\kappa \circ (\mathrm{ev}_{\mathrm{Ran}})^*\end{aligned}$$

is the identity map.

(b) *The functor*

$$\mathrm{Loc}_{G,\kappa} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran})$$

identifies canonically with the dual of

$$\Gamma_{G,\kappa'} : \text{D-mod}_{\kappa',\mathrm{co}}(\mathrm{Bun}_G) \to \mathrm{KL}(G)_{\kappa',\mathrm{Ran}}.$$

For the proof, see [CF, Theorem 4.0.5(2)].

10.5.8. Note that by combining Sect. 10.2.11 and Proposition 10.5.7 with the fact that the functors

$$\mathrm{KL}(G)_{\kappa'} \overset{\mathrm{inv}_{\mathfrak{L}^+(G)}}{\longrightarrow} \mathrm{Vect} \text{ and } \mathrm{Vect} \overset{k \mapsto \mathrm{Vac}(G)_\kappa}{\longrightarrow} \mathrm{KL}(G)_\kappa$$

and

$$\text{D-mod}_{\kappa',\mathrm{co}} \overset{\Gamma(\mathrm{Bun}_G,-)\circ \mathbf{oblv}^l_\kappa}{\longrightarrow} \mathrm{Vect} \text{ and } \mathrm{Vect} \overset{k\mapsto \mathbf{ind}^l_\kappa(\mathcal{O}_{\mathrm{Bun}_G})}{\longrightarrow} \text{D-mod}_\kappa(\mathrm{Bun}_G)$$

are mutually dual, we obtain an identification

$$\mathrm{Loc}_{G,\kappa,\mathrm{Ran}}(\mathrm{Vac}_{G,\mathrm{Ran}}) \simeq \mathbf{ind}^l_\kappa(\mathcal{O}_{\mathrm{Bun}_G}) \boxtimes \omega_{\mathrm{Ran}}. \tag{10.28}$$

However, it follows formally that the identification (10.28) is the same as that in (10.21).

11. Digression: local-to-global functors and unitality

In this section we will introduce a general framework that formalizes the unital property of the functors

$$\mathrm{Poinc}_{G,!,\mathrm{Ran}},\ \mathrm{Poinc}_{G,*,\mathrm{Ran}} \text{ and } \mathrm{Loc}_{G,\kappa,\mathrm{Ran}}.$$

The unital property says, roughly speaking, that the insertion of the vacuum[45] does not change the value of the functor (see Sect. 11.3.3).

A key phenomenon that we will observe is the following: insertion of the vacuum along the entire Ran space improves the unital property of the functor, see Sect. 11.4. The functor of factorization homology and its generalizations are particular cases of this construction, see Sect. 11.9.

11.0. What is this section about? As this section deals with some abstract material, a general introduction is in order.

[45] In the main body of this section, we use the word "unit" instead of "vacuum".

11.0.1. In this section, we study the general formalism of local-to-global functors. The (*local*) source of such a functor is a crystal of categories $\underline{\mathbf{C}}^{\mathrm{loc}}$ over the Ran space, while its (*global*) target is a single category $\mathbf{C}^{\mathrm{glob}}$. Roughly, for a space $\mathcal{Z}$ equipped with a map $\underline{x} : \mathcal{Z} \to \mathrm{Ran}$, the value of $\underline{\mathbf{C}}^{\mathrm{loc}}$ on $\mathcal{Z}$ is a category $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} = \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}}$, and a local-to-global functor $\underline{\mathsf{F}}$ is a compatible collection of functors

$$\mathsf{F}_{\mathcal{Z}} = \mathsf{F}_{\mathcal{Z},\underline{x}} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}} \to \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z})$$

for every $\mathcal{Z}$ and $\underline{x}$. (Here and below, the words "compatible collection" mean "collection equipped with higher coherence data".)

11.0.2. Next, we introduce the notion of a unital crystal of categories over the Ran space (see Sect. 11.2). Informally, a unital structure on a sheaf $\underline{\mathbf{C}}^{\mathrm{loc}}$ is a compatible collection of functors

$$\mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}_1} \to \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}_2}$$

for every space $\mathcal{Z}$ and two maps $\underline{x}_1, \underline{x}_2 : \mathcal{Z} \to \mathrm{Ran}$ such that $\underline{x}_1 \subseteq \underline{x}_2$. Here we view $\mathcal{Z}$-points of the Ran space as $\mathcal{Z}$-families of finite subsets of X.

11.0.3. Suppose now that $\underline{\mathbf{C}}^{\mathrm{loc}}$ is a unital crystal of categories, and $\underline{\mathsf{F}}$ is a local-to-global functor from $\underline{\mathbf{C}}^{\mathrm{loc}}$ to a category $\mathbf{C}^{\mathrm{glob}}$. We then introduce the notion of a unital structure on $\underline{\mathsf{F}}$; informally, it is a compatible collection of natural transformations

$$\mathsf{F}_{\mathcal{Z},\underline{x}_1} \to \mathsf{F}_{\mathcal{Z},\underline{x}_2} \circ \mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2}\,. \tag{11.1}$$

for every $\mathcal{Z}$, $\underline{x}_1$, and $\underline{x}_2$ as above.

In fact, we have two notions: a (*strict*) unital structure, where the transformations (11.1) are required to be isomorphisms, and a *lax* unital structure, where (11.1) can be arbitrary transformations.

Accordingly, we obtain two categories of unital local-to-global functors: the category of (strictly) unital local-to-global functors, and the category of lax unital local-to-global functors; the former is a full subcategory of the latter. We denote the categories by

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \subset \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}). \tag{11.2}$$

11.0.4. The main subject of this section is a construction on local-to-global functors, which we call *the integrated insertion of the unit.* It can be defined as the left adjoint of the embedding (11.2):

$$\int \mathrm{ins.\,unit} : \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}).$$

However, the functor admits a geometric description. Remarkably, the description makes sense for all (i.e., not necessarily lax unital) local-to-global functors.

11.0.5. We will use this formalism in Sect. 12 in the context of compatibility between certain natural constructions and local-to-global functors. We will see that, in three different situations, the compatibility is only lax at the start, but composition with the functor $\int \mathrm{ins.\,unit}$ makes it strict.

11.1. Setup for local-to-global functors.

11.1.1. Let $\underline{\mathbf{C}}^{\mathrm{loc}}$ be a crystal of categories over Ran (see Sect. B.8). Let $\mathbf{C}^{\mathrm{glob}}$ be a DG category, and let us be given a functor

$$\underline{\mathsf{F}} : \underline{\mathbf{C}}^{\mathrm{loc}} \to \mathbf{C}^{\mathrm{glob}} \otimes \underline{\mathrm{D}\text{-}\mathrm{mod}}(\mathrm{Ran}),$$

where $\underline{\mathrm{D}\text{-}\mathrm{mod}}(\mathrm{Ran})$ is the unit crystal of categories over Ran.

Thus, for every space $\mathcal{Z}$ mapping to Ran, we have a category $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}}$, tensored over $\mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z})$ and a functor

$$\mathsf{F}_{\mathcal{Z}} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}). \tag{11.3}$$

Remark 11.1.2. In the above procedure, we associate to $\mathcal{Z} \to \mathrm{Ran}$ the category of *cristalline* sections of $\underline{\mathbf{C}}^{\mathrm{loc}}$ over $\mathcal{Z}$, i.e., the category of sections of $\underline{\mathbf{C}}^{\mathrm{loc}}$ over $\mathcal{Z}_{\mathrm{dR}}$, cf. Sect. C.2.10.

11.1.3. Assume for a moment that $\mathcal{Z}$ is pseudo-proper, so that the functor

$$\mathrm{C}^{\cdot}_c(\mathcal{Z}, -) : \text{D-mod}(\mathcal{Z}) \to \mathrm{Vect}$$

left adjoint to $k \mapsto \omega_{\mathcal{Z}}$ is defined (see Sect. C.4.12). In this case we will denote by $\mathsf{F}_{\int_{\mathcal{Z}}}$ the composition

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \overset{\mathsf{F}_{\mathcal{Z}}}{\to} \text{D-mod}(\mathcal{Z}) \otimes \mathbf{C}^{\mathrm{glob}} \overset{\mathrm{C}^{\cdot}_c(\mathcal{Z},-)\otimes \mathrm{Id}}{\longrightarrow} \mathbf{C}^{\mathrm{glob}}.$$

11.1.4. In particular, for $\mathcal{Z} = \mathrm{Ran}$ and the identity map, we obtain the category $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}$ and a functor

$$\mathsf{F}_{\mathrm{Ran}} : \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \to \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(\mathrm{Ran}).$$

We will also use the symbol $\mathsf{F} : \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \to \mathbf{C}^{\mathrm{glob}}$ for the functor $\mathsf{F}_{\int_{\mathrm{Ran}}}$.

11.1.5. Note that the datum of F recovers that of $\underline{\mathsf{F}}$. Namely, the functor $\mathsf{F}_{\mathcal{Z}}$ identifies with

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}\times\mathcal{Z}} \simeq \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \otimes \text{D-mod}(\mathcal{Z}) \overset{\mathsf{F}\otimes\mathrm{Id}}{\longrightarrow} \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(\mathcal{Z}),$$

where:

- $\mathrm{Ran} \times \mathcal{Z}$ is viewed as a space over Ran via the projection on the first factor;
- The arrow $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}\times\mathcal{Z}}$ is the !-pushforward along the graph $\mathcal{Z} \to \mathrm{Ran} \times \mathcal{Z}$ of the original map $\mathcal{Z} \to \mathrm{Ran}$.

11.1.6. For a general pseudo-proper $\mathcal{Z}$, the functor $\mathsf{F}_{\int_{\mathcal{Z}}}$ factors as

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \overset{\mathsf{F}}{\to} \mathbf{C}^{\mathrm{glob}},$$

where the first arrow is the functor of !-puhsforward (see Corollary C.4.10).

For $\mathcal{Z} = \mathrm{pt}$ and the map $\mathcal{Z} \to \mathrm{Ran}$ given by $\underline{x} \in \mathrm{Ran}$, we obtain the category denoted $\mathbf{C}^{\mathrm{loc}}_{\underline{x}}$ and a functor

$$\mathsf{F}_{\underline{x}} : \mathbf{C}^{\mathrm{loc}}_{\underline{x}} \to \mathbf{C}^{\mathrm{glob}}.$$

11.1.7. The main examples of the above are when $\underline{\mathbf{C}}^{\mathrm{loc}}$ is one of the factorization categories

$$\mathrm{Whit}^!(G),\ \mathrm{Whit}_*(G),\ \mathrm{KL}(G)_\kappa.$$

In each of these cases, the corresponding global category is

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),\ \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G),\ \text{and}\ \text{D-mod}_\kappa(\mathrm{Bun}_G),$$

and the functor $\mathsf{F}_{\mathrm{Ran}}$ is

$$\mathrm{Poinc}_{G,!,\mathrm{Ran}},\ \mathrm{Poinc}_{G,*,\mathrm{Ran}}\ \text{and}\ \mathrm{Loc}_{G,\kappa,\mathrm{Ran}},$$

respectively.

11.1.8. For given $\underline{\mathbf{C}}^{\mathrm{loc}}$ and $\mathbf{C}^{\mathrm{glob}}$, we can consider the totality of functors $\underline{\mathsf{F}}$ as above as a category, denoted

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}).$$

By Sect. 11.1.5, this is the same as just the category

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}, \mathbf{C}^{\mathrm{glob}}).$$

11.2. **The local unital structure.**

11.2.1. Let $\underline{\mathbf{C}}^{\mathrm{loc}}$ be a crystal of categories over Ran. By a local unital structure on $\underline{\mathbf{C}}^{\mathrm{loc}}$: we mean an extension $\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}$ of $\underline{\mathbf{C}}^{\mathrm{loc}}$ to a crystal of categories over $\mathrm{Ran}^{\mathrm{untl}}$ (see Sect. C.2 for what this means).

An example of such a structure is provided by a unital lax factorization category.

Let us explain what the unital structure means in concrete terms.

11.2.2. Let $\mathrm{Ran}^{\subseteq}$ be the moduli space of pairs

$$(\underline{x}, \underline{x}' \,|\, \underline{x} \subseteq \underline{x}'),$$

see Sect. B.2.1.

We have the maps

$$\mathrm{pr}_{\mathrm{small}}, \mathrm{pr}_{\mathrm{big}} : \mathrm{Ran}^{\subseteq} \rightrightarrows \mathrm{Ran}$$

that remember $\underline{x}$ and $\underline{x}'$, respectively.

Let diag denote the diagonal map

$$\mathrm{Ran} \to \mathrm{Ran}^{\subseteq}.$$

Note that

$$\mathrm{pr}_{\mathrm{small}} \circ \mathrm{diag} \simeq \mathrm{Id} \simeq \mathrm{pr}_{\mathrm{diag}} \circ \mathrm{diag}\,.$$

11.2.3. Denote

$$\mathrm{Ran}^{\subseteq^2} := \mathrm{Ran}^{\subseteq} \underset{\mathrm{pr}_{\mathrm{small}}, \mathrm{Ran}, \mathrm{pr}_{\mathrm{big}}}{\times} \mathrm{Ran}^{\subseteq}.$$

In addition to the two projections

$$\mathrm{pr}_{\mathrm{small}^2}, \mathrm{pr}_{\mathrm{big}^2} : \mathrm{Ran}^{\subseteq^2} \rightrightarrows \mathrm{Ran}^{\subseteq},$$

we have a map

$$\mathrm{pr}_{\mathrm{comp}} : \mathrm{Ran}^{\subseteq^2} \to \mathrm{Ran}^{\subseteq}$$

that sends

$$(\underline{x}, \underline{x}', \underline{x}'' \,|\, \underline{x} \subseteq \underline{x}' \subseteq \underline{x}'') \mapsto (\underline{x}, \underline{x}'').$$

11.2.4. Note that $\mathrm{Ran}^{\subseteq}$ is the prestack of morphisms of $\mathrm{Ran}^{\mathrm{untl}}$. Hence, at the level of 1-morphisms, an extension of $\underline{\mathbf{C}}^{\mathrm{loc}}$ to $\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}$ amounts to a functor

$$(\mathrm{pr}_{\mathrm{small}})^*(\underline{\mathbf{C}}^{\mathrm{loc}}) \to (\mathrm{pr}_{\mathrm{big}})^*(\underline{\mathbf{C}}^{\mathrm{loc}})$$

as crystals of categories over $\mathrm{Ran}^{\subseteq}$, or equivalently, to a functor

$$\underline{\mathbf{C}}^{\mathrm{loc}} \to (\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\underline{\mathbf{C}}^{\mathrm{loc}}) \tag{11.4}$$

as crystals of categories over Ran.

In the above formula:

- $(\mathrm{pr}_{\mathrm{big}})^*$ (resp., $(\mathrm{pr}_{\mathrm{small}})^*$) is the functor of pullback along $\mathrm{pr}_{\mathrm{big}}$ (resp., $\mathrm{pr}_{\mathrm{small}}$) from crystals of categories over Ran to crystals of categories over $\mathrm{Ran}^{\subseteq}$;
- $(\mathrm{pr}_{\mathrm{small}})_*$ is the functor of pushforward along $\mathrm{pr}_{\mathrm{small}}$ from sheaves of categories over $\mathrm{Ran}^{\subseteq}$ to crystals of categories over Ran.

We refer the reader to Sect. C.3, where the operation of pushforward for crystals of categories is reviewed.

11.2.5. Denote the functor (11.4) by[46]

$$\mathrm{ins.\,unit} : \underline{\mathbf{C}}^{\mathrm{loc}} \to (\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\underline{\mathbf{C}}^{\mathrm{loc}}).$$

The functor ins. unit has an associativity structure explained in Sect. 11.2.9. The full datum of the upgrade

$$\underline{\mathbf{C}}^{\mathrm{loc}} \rightsquigarrow \underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}$$

is encoded by ins. unit, together with the associativity structure satisfying a homotopy-coherent system of compatibilities.

We will now explain the concrete meaning of the functor ins. unit.

[46] In the formula below ins. unit is the abbreviation of "insert unit".

11.2.6. For $\mathcal{Z} \to \mathrm{Ran}$, denote

$$\mathcal{Z}^{\subseteq} := \mathcal{Z} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq},$$

where in the formation of the fiber product the map $\mathrm{Ran}^{\subseteq} \to \mathrm{Ran}$ is $\mathrm{pr}_{\mathrm{small}}$, see Sect. B.2.2.

Denote by $\mathrm{pr}_{\mathrm{small},\mathcal{Z}}$ the map

$$\mathcal{Z}^{\subseteq} \to \mathcal{Z},$$

and by $\mathrm{pr}_{\mathrm{big}}$ the projection

$$\mathcal{Z}^{\subseteq} \to \mathrm{Ran}^{\subseteq} \overset{\mathrm{pr}_{\mathrm{big}}}{\longrightarrow} \mathrm{Ran}.$$

We view $\mathcal{Z}^{\subseteq}$ as mapping to Ran via $\mathrm{pr}_{\mathrm{big}}$. The map diag induces a map

$$\mathrm{diag}_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}^{\subseteq}.$$

11.2.7. The map $\mathrm{diag}_{\mathcal{Z}}$ gives rise to a functor

$$(\mathrm{diag}_{\mathcal{Z}})^! : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}} \to \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}}.$$

Since $\mathrm{diag}_{\mathcal{Z}}$ is pseudo-proper, the functor $(\mathrm{diag}_{\mathcal{Z}})^!$ admits a left adjoint, to be denoted $(\mathrm{diag}_{\mathcal{Z}})_!$ (see Corollary C.4.10). Thus, we have an adjoint pair:

$$(\mathrm{diag}_{\mathcal{Z}})_! : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \rightleftarrows \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}} : (\mathrm{diag}_{\mathcal{Z}})^!. \tag{11.5}$$

11.2.8. The functor ins. unit assigns to $\mathcal{Z}$ a D-mod($\mathcal{Z}$)-linear functor

$$\mathrm{ins.\,unit}_{\mathcal{Z}} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}}. \tag{11.6}$$

Note also that

$$(\mathrm{diag}_{\mathcal{Z}})^! \circ \mathrm{ins.\,unit}_{\mathcal{Z}} \simeq \mathrm{Id} \tag{11.7}$$

as endofunctors of $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}}$.

11.2.9. Denote

$$\mathcal{Z}^{\subseteq^2} := (\mathcal{Z}^{\subseteq})^{\subseteq} \simeq \mathcal{Z} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}^2}}{\times} \mathrm{Ran}^{\subseteq^2} \simeq \mathcal{Z}^{\subseteq} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}}}{\times} \mathrm{Ran}^{\subseteq}.$$

We view $\mathcal{Z}^{\subseteq^2}$ as mapping to Ran via $\mathrm{pr}_{\mathrm{big}^2}$. The map $\mathrm{pr}_{\mathrm{comp}} : \mathrm{Ran}^{\subseteq^2} \to \mathrm{Ran}^{\subseteq}$ gives rise to a map

$$\mathrm{pr}_{\mathrm{comp},\mathcal{Z}} : \mathcal{Z}^{\subseteq^2} \to \mathcal{Z}^{\subseteq}$$

as spaces over Ran.

The associativity property of the functor ins. unit is encoded by the following diagram

$$\begin{tikzcd}
\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \arrow[r, "\mathrm{ins.unit}_{\mathcal{Z}}"] \arrow[d, "\mathrm{ins.unit}_{\mathcal{Z}}"'] & \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}} \arrow[d, "\mathrm{pr}^!_{\mathrm{comp},\mathcal{Z}}"] \\
\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}} \arrow[r, "\mathrm{ins.unit}_{\mathcal{Z}^{\subseteq}}"'] & \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq^2}}.
\end{tikzcd}$$

11.2.10. *Example.* At the pointwise level, the datum of (11.6) is a system of functors

$$\mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2} : \mathbf{C}^{\mathrm{loc}}_{\underline{x}_1} \to \mathbf{C}^{\mathrm{loc}}_{\underline{x}_2} \text{ for } \underline{x}_1 \subseteq \underline{x}_2.$$

When

$$\underline{x}_2 = \underline{x}_1 \sqcup \underline{x}',$$

and $\underline{\mathbf{C}}^{\mathrm{loc}}$ is a unital lax factorization category $\mathbf{C}$, the above functor is

$$\mathbf{C}_{\underline{x}_1} \xrightarrow{\mathrm{Id} \otimes \mathbf{1}_{\mathbf{C},\underline{x}'}} \mathbf{C}_{\underline{x}_1} \otimes \mathbf{C}_{\underline{x}'} \to \mathbf{C}_{\underline{x}_2},$$

where the last arrow is given by the lax factorization structure.

11.2.11. Assume for a moment that $\mathcal{Z}$ is pseudo-proper. In this case the map $\mathrm{pr}_{\mathrm{big}} : \mathcal{Z}^{\subseteq} \to \mathrm{Ran}$ is pseudo-proper, and hence the functor

$$(\mathrm{pr}_{\mathrm{big}})_! : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}$$

left adjoint to $(\mathrm{pr}_{\mathrm{big}})^!$ is defined (see Corollary C.4.10).

We will consider the functor

$$\int_{\mathcal{Z}} \mathrm{ins.\,unit} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}$$

equal to the composition

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \overset{\mathrm{ins.unit}_{\mathcal{Z}}}{\longrightarrow} \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}} \overset{(\mathrm{pr}_{\mathrm{big}})_!}{\longrightarrow} \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}.$$

11.2.12. In particular, we obtain an endofunctor

$$\int_{\mathrm{Ran}} \mathrm{ins.\,unit} : \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}. \tag{11.8}$$

Note that the adjunction (11.6) and the identification (11.7) give rise to a natural transformation

$$\mathrm{Id} \to \int_{\mathrm{Ran}} \mathrm{ins.\,unit} \tag{11.9}$$

as endofunctors of $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}$. Indeed, (11.9) is given by

$$\mathrm{Id} \simeq (\mathrm{pr}_{\mathrm{big}})_! \circ (\mathrm{diag}_{\mathrm{Ran}})_! \simeq (\mathrm{pr}_{\mathrm{big}})_! \circ (\mathrm{diag}_{\mathrm{Ran}})_! \circ (\mathrm{diag}_{\mathrm{Ran}})^! \circ \mathrm{ins.\,unit}_{\mathrm{Ran}} \to$$
$$\to (\mathrm{pr}_{\mathrm{big}})_! \circ \mathrm{ins.\,unit}_{\mathrm{Ran}} = \int_{\mathrm{Ran}} \mathrm{ins.\,unit}\,.$$

11.2.13. *Inventory of notation.* We briefly summarize the notation related to insertion of the unit.

We denote by $\mathrm{ins.\,unit}_{\mathcal{Z}}$ the functor

$$\mathrm{ins.\,unit}_{\mathcal{Z}} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}}$$

For $\mathcal{Z}$ pseudo-proper, we denote by

$$\int_{\mathcal{Z}} \mathrm{ins.\,unit} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}$$

the composition of $\mathrm{ins.\,unit}_{\mathcal{Z}}$ with $(\mathrm{pr}_{\mathrm{big}})_!$.

In particular, for $\mathcal{Z} = \mathrm{Ran}$, we have $\int_{\mathrm{Ran}} \mathrm{ins.\,unit}$, which is an endofunctor of $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}$.

11.2.14. Yet, in (11.15) we will introduce yet another symbol: just $\int \mathrm{ins.\,unit}$. It will be an endofunctor of the category

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}), \quad \underline{\mathsf{F}} \mapsto \underline{\mathsf{F}}^{\int \mathrm{ins.unit}}.$$

(see Sect. 11.1.8), defined when $\underline{\mathbf{C}}^{\mathrm{loc}}$ is equipped with a local unital structure.

We will have

$$\mathsf{F}_{\int_{\mathcal{Z}^{\subseteq}}} \circ \mathrm{ins.\,unit}_{\mathcal{Z}} \simeq \mathsf{F}'_{\int_{\mathcal{Z}}}$$

as functors $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{glob}}$, where:

- $\underline{\mathsf{F}}' := \mathsf{F}^{\int \mathrm{ins.unit}} \in \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$;
- The notation $\mathsf{F}'_{\int_{\mathcal{Z}}}$ is as in Sect. 11.1.3.

11.3. A (lax) unital structure on a local-to-global functor.

11.3.1. Let $(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}, \underline{\mathsf{F}})$ be as in Sect. 11.1.1. Assume now that $\underline{\mathbf{C}}^{\mathrm{loc}}$ is equipped with a local unital structure.

A lax unital structure on $\underline{\mathsf{F}}$ is its upgrade to a *right-lax* functor

$$\underline{\mathsf{F}}^{\mathrm{untl}} : \underline{\mathbf{C}}^{\mathrm{loc,untl}} \to \mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}) \tag{11.10}$$

between crystals of categories over $\mathrm{Ran}^{\mathrm{untl}}$, see Sect. C.2.8 for what this means.

11.3.2. Concretely, a lax unital structure on $\underline{\mathsf{F}}$ means the following. Let $\mathcal{Z}$ be a space, and let

$$\underline{x}_1 \overset{\alpha}{\to} \underline{x}_2$$

be a morphism in the category $\mathrm{Maps}(\mathcal{Z}, \mathrm{Ran}^{\mathrm{untl}})$.

The maps $\underline{x}_i$ give rise to categories $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}_i}$ tensored over D-mod$(\mathcal{Z})$, $i = 1, 2$. The datum of $\underline{\mathsf{F}}$ gives rise to D-mod$(\mathcal{Z})$-linear functors

$$\mathsf{F}_{\mathcal{Z},\underline{x}_i} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}_i} \to \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(\mathcal{Z}).$$

The local unital structure on $\underline{\mathbf{C}}^{\mathrm{loc}}$ gives rise to a D-mod$(\mathcal{Z})$-linear functor

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}_1} \overset{\mathbf{C}^{\mathrm{loc}}_\alpha}{\to} \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}_2}.$$

Then the datum of $\underline{\mathsf{F}}^{\mathrm{untl}}$ gives rise to a natural transformation

$$\mathsf{F}_{\mathcal{Z},\underline{x}_1} \to \mathsf{F}_{\mathcal{Z},\underline{x}_2} \circ \mathbf{C}^{\mathrm{loc}}_\alpha \tag{11.11}$$

as functors

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z},\underline{x}_1} \to \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(\mathcal{Z}).$$

11.3.3. *Example.* Set $\mathcal{Z} = \mathrm{pt}$, so that $\underline{x}_1 \overset{\alpha}{\to} \underline{x}_2$ corresponds to an inclusion

$$\underline{x}_1 \subseteq \underline{x}_2.$$

Then $\underline{\mathsf{F}}^{\mathrm{untl}}$ gives rise to a natural transformation

$$\mathsf{F}_{\underline{x}_1} \to \mathsf{F}_{\underline{x}_2} \circ \mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2}.$$

11.3.4. We can rewrite the datum of natural transformations (11.11) as follows:

Let $\underline{\mathsf{F}}$ be as in Sect. 11.1. Evaluating $\underline{\mathsf{F}}$ on $\mathcal{Z}^{\subseteq}$, we obtain a D-mod$(\mathcal{Z})$-linear functor

$$\mathsf{F}_{\mathcal{Z}^{\subseteq}} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}} \to \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(\mathcal{Z}^{\subseteq}).$$

The datum of $\underline{\mathsf{F}}^{\mathrm{untl}}$ gives rise to a natural transformation

$$(\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^!) \circ \mathsf{F}_{\mathcal{Z}} \to \mathsf{F}_{\mathcal{Z}^{\subseteq}} \circ \mathrm{ins.\,unit}_{\mathcal{Z}} \tag{11.12}$$

as functors

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(\mathcal{Z}^{\subseteq}).$$

The natural transformation (11.12) encodes the datum of $\underline{\mathsf{F}}^{\mathrm{untl}}$ at the level of 1-morphisms. One can recover the full datum of $\underline{\mathsf{F}}^{\mathrm{untl}}$ by imposing a datum of associativity that (11.12) is supposed to satisfy.

11.3.5. We shall say that a lax unital structure on $\underline{\mathsf{F}}$ is *strict* if $\underline{\mathsf{F}}^{\mathrm{untl}}$ is a strict functor between crystals of categories over $\mathrm{Ran}^{\mathrm{untl}}$, see Sect. C.2.8 for what this means.

By definition, this means that the natural transformations (11.11) are isomorphisms.

In this case we will call $\underline{\mathsf{F}}^{\mathrm{untl}}$ a *unital structure* on $\underline{\mathsf{F}}$.

11.3.6. Equivalently, a lax unital structure on $\underline{\mathsf{F}}$ is strict if the natural transformation (11.12) is an isomorphism for any $\mathcal{Z}$.

11.3.7. Each of the examples from Sect. 11.1.7 has a natural unital structure.

11.3.8. We can consider the categories

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\subset \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \tag{11.13}$$

of local-to-global functors equipped with a unital or lax unital structures, respectively, with the former being a full subcategory of the latter.

Note that we have a forgetful functor

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}), \tag{11.14}$$

where $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$ is as in Sect. 11.1.8.

Remark 11.3.9. In [HR, Sect. 1.2.6], axioms for an algebro-geometric avatar of [1,2]-extended 3d quantum field theories on X were considered, although a detailed definition was not provided. In the present setting, we can easily spell out the complete axioms:

We should have the data of a unital factorization category $\mathbf{C}$ (viewed as a crystal of categories over $\mathrm{Ran}^{\mathrm{untl}}$), a local-to-global functor $\mathsf{F}:\mathbf{C}_{\mathrm{Ran}}\to\mathrm{Vect}$, and a unital structure on $\underline{\mathsf{F}}$.

We refer to *loc. cit.* for a discussion of why these axiomatics can be geometrically interpreted in terms of 3d QFTs.

Moreover, the discussion from [HR, Sect. 1.2.13-15] suggests that local-to-global functors valued in more general global categories $\mathbf{C}^{\mathrm{glob}}$ should generally be interpreted in terms of boundary conditions for 4d QFTs; this applies for all the examples we consider here.

11.4. **Integrated insertion of the unit.** The main construction in this subsection (i.e., the operation $\int\mathrm{ins.\,unit}$) may be viewed as an abstraction of the definition of chiral (a.k.a. factorization) homology in [BD2, Sect. 4.2].

As we shall see, the framework introduced above allows us to reproduce this construction automatically: it amounts to the left of adjoint to the embedding

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}).$$

See also Sect. 11.9, where the specific example of the functor of factorization homology is considered.

11.4.1. Let $\underline{\mathbf{C}}^{\mathrm{loc}}$ and $\mathbf{C}^{\mathrm{glob}}$ be as in Sect. 11.1. Let

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$$

be the corresponding category of local-to-global functors.

Assume now that $\underline{\mathbf{C}}^{\mathrm{loc}}$ is equipped with a local unital structure.

11.4.2. We define an endofunctor

$$\int\mathrm{ins.\,unit}:\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \tag{11.15}$$

$$\underline{\mathsf{F}}\mapsto \underline{\mathsf{F}}^{\int\mathrm{ins.unit}}$$

by

$$\mathsf{F}_{\mathcal{Z}}^{\int\mathrm{ins.unit}}:=(\mathrm{Id}\otimes(\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!)\circ\mathsf{F}_{\mathcal{Z}^{\subseteq}}\circ\mathrm{ins.\,unit}_{\mathcal{Z}}\,.$$

In other words, $\mathsf{F}_{\mathcal{Z}}^{\int\mathrm{ins.unit}}$ is the composition

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}}\xrightarrow{\mathrm{ins.unit}_{\mathcal{Z}}}\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}^{\subseteq}}\xrightarrow{\mathsf{F}_{\mathcal{Z}^{\subseteq}}}\mathbf{C}^{\mathrm{glob}}\otimes\mathrm{D\text{-}mod}(\mathcal{Z}^{\subseteq})\xrightarrow{\mathrm{Id}\otimes(\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!}\mathbf{C}^{\mathrm{glob}}\otimes\mathrm{D\text{-}mod}(\mathcal{Z}).$$

11.4.3. Note that we have a natural transformation

$$\mathrm{Id} \to \int \mathrm{ins.\,unit} \tag{11.16}$$

so that for a given $\mathcal{Z}$ the corresponding map

$$\mathsf{F}_{\mathcal{Z}} \to \mathsf{F}_{\mathcal{Z}}^{\int \mathrm{ins.unit}} \tag{11.17}$$

is given by

$$\begin{aligned} \mathsf{F}_{\mathcal{Z}} \simeq \big(\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!\big) \circ (\mathrm{Id} \otimes (\mathrm{diag}_{\mathcal{Z}})_!) \circ \mathsf{F}_{\mathcal{Z}} \simeq \big(\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!\big) \circ \mathsf{F}_{\mathcal{Z}^{\subseteq}} \circ (\mathrm{diag}_{\mathcal{Z}})_! \overset{(11.7)}{\simeq} \\ \simeq \big(\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!\big) \circ \mathsf{F}_{\mathcal{Z}^{\subseteq}} \circ (\mathrm{diag}_{\mathcal{Z}})_! \circ (\mathrm{diag}_{\mathcal{Z}})^! \circ \mathrm{ins.\,unit}_{\mathcal{Z}} \to \\ \to \big(\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!\big) \circ \mathsf{F}_{\mathcal{Z}^{\subseteq}} \circ \mathrm{ins.\,unit}_{\mathcal{Z}} = \underline{\mathsf{F}}_{\mathcal{Z}}^{\int \mathrm{ins.unit}}. \end{aligned} \tag{11.18}$$

11.4.4. Assume for a moment that $\mathcal{Z}$ is pseudo-proper. Applying $\mathrm{C}^\cdot_c(\mathcal{Z}, -) \otimes \mathrm{Id}$ to both sides of (11.17), we obtain a natural transformation

$$\mathsf{F}_{\int_{\mathcal{Z}}} \to \mathsf{F} \circ \int_{\mathcal{Z}} \mathrm{ins.\,unit}, \tag{11.19}$$

where $\int_{\mathcal{Z}} \mathrm{ins.\,unit}$ is as in Sect. 11.2.11.

Take $\mathcal{Z} = \mathrm{Ran}$. In this case, the resulting natural transformation (11.19) is

$$\mathsf{F} \to \mathsf{F} \circ \int_{\mathrm{Ran}} \mathrm{ins.\,unit}, \tag{11.20}$$

where $\int_{\mathrm{Ran}} \mathrm{ins.\,unit}$ is as in Sect. 11.2.12.

It is easy to see, however, that (11.20) equals the natural transformation obtained by applying F to the natural transformation (11.9).

11.4.5. Suppose for a moment that $\underline{\mathsf{F}}$ is equipped with a unital structure, i.e., it is the image under the forgetful functor (11.14) of an object $\underline{\mathsf{F}}^{\mathrm{untl}} \in \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$.

We claim that in this case the map (11.17) is an isomorphism. Indeed, in this case, the *isomorphism* (11.12) identifies

$$\mathsf{F}_{\mathcal{Z}}^{\int \mathrm{ins.unit}} \simeq (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^! \circ \mathsf{F}_{\mathcal{Z}}, \tag{11.21}$$

and the map (11.16) is the map

$$\mathsf{F}_{\mathcal{Z}} \simeq (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{diag}_{\mathcal{Z}})_! \circ (\mathrm{diag}_{\mathcal{Z}})^! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^! \circ \mathsf{F}_{\mathcal{Z}} \to (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^! \circ \mathsf{F}_{\mathcal{Z}}. \tag{11.22}$$

Now, the contractibility of the Ran space implies that the counit of the $((\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!, (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^!)$-adjunction is an isomorphism. Hence, the right-hand side of (11.22) maps isomorphically to $\mathsf{F}_{\mathcal{Z}}$, and the composition

$$\mathsf{F}_{\mathcal{Z}} \overset{(11.22)}{\longrightarrow} (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^! \circ \mathsf{F}_{\mathcal{Z}} \to \mathsf{F}_{\mathcal{Z}}$$

is the identity map.

In particular, in this case the natural transformation (11.20) is an isomorphism.

11.4.6. Our next goal, carried out in Sects. 11.5-11.6, is to perform similar constructions with the same input, but in the unital context, i.e., working over $\mathrm{Ran}^{\mathrm{untl}}$ rather than over Ran. Namely, we will show that, parallel to (11.15), there exists an endofunctor

$$\int \mathrm{ins.\,unit} : \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \tag{11.23}$$

$$\underline{\mathsf{F}}^{\mathrm{untl}} \mapsto \underline{\mathsf{F}}^{\mathrm{untl},\int \mathrm{ins.unit}}$$

that makes the diagram

$$\begin{CD} \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) @>{\int \mathrm{ins.unit}}>> \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \\ @VVV @VVV \\ \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) @>{\int \mathrm{ins.unit}}>> \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \end{CD} \tag{11.24}$$

commute.

11.4.7. In addition, the functor (11.23) will be equipped with a natural transformation

$$\mathrm{Id} \to \int \mathrm{ins.\,unit}, \tag{11.25}$$

which is compatible with (11.16) via (11.24).

We will also show:

- The essential image of (11.23) belongs to $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$.
- The natural transformation (11.25) evaluates to an isomorphism on objects that belong to $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$;
- The two natural transformations

$$\int \mathrm{ins.\,unit} \rightrightarrows \int \mathrm{ins.\,unit} \circ \int \mathrm{ins.\,unit}, \tag{11.26}$$

  arising from (11.25) coincide (it follows that they are isomorphisms).

11.4.8. The above properties combined imply that the functor (11.23) is the left adjoint of the embedding

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \hookrightarrow \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}).$$

Remark 11.4.9. The reason for discussing both versions of $\int \mathrm{ins.\,unit}$, i.e., (11.23) and (11.15), is that the former has a clear categorical meaning (i.e., it is the left adjoint of the forgetful functor), while the latter is easily computable (a priori, the functor (11.23) involves taking cohomology over categorical prestacks).

However, the commutation of (11.24) implies that (11.23) is computable as well.

11.5. **Construction of the integrated functor.** This and the next subsection are devoted to the construction of the functor (11.23) and the verification of its properties. The reader who is willing to take this on faith may choose to skip these two subsections.

We are going to present the construction of the functor (11.23) in a hands-on manner. See, however, Sect. C.3.10 for its abstract interpretation.

11.5.1. Let $\mathcal{Y}$ be a categorical prestack and let $\mathcal{Y}^{\to}$ be the categorical prestack of 1-morphisms in $\mathcal{Y}$. I.e., for an affine scheme S, objects of $\mathrm{Maps}(S, \mathcal{Y}^{\to})$ are

$$y_1, y_2 \in \mathrm{Maps}(S, \mathcal{Y}),\ \ y_1 \to y_2,$$

and morphisms are commutative diagrams

$$\begin{tikzcd} y_1 \arrow[r] \arrow[d] & y_2 \arrow[d] \\ y_1' \arrow[r] & y_2'. \end{tikzcd}$$

We have the projections

$$\mathrm{pr}_{\mathrm{source}}, \mathrm{pr}_{\mathrm{target}} : \mathcal{Y}^{\to} \to \mathcal{Y},$$

with $\mathrm{pr}_{\mathrm{source}}$ being a Cartesian fibration.

Let $\underline{\mathbf{C}}$ be a crystal of categories on $\mathcal{Y}$. Tautologically, we have a (strict) functor

$$(\mathrm{pr}_{\mathrm{source}})^*(\underline{\mathbf{C}}) \to (\mathrm{pr}_{\mathrm{target}})^*(\underline{\mathbf{C}}), \tag{11.27}$$

as crystals of categories on $\mathcal{Y}^{\to}$.

Recall the construction of the direct image of a crystal of categories , reviewed in Sect. C.3. According to Sect. C.3.8, we have a (strict) functor

$$\underline{\mathbf{C}} \to (\mathrm{pr}_{\mathrm{source}})_{*,\mathrm{strict}} \circ (\mathrm{pr}_{\mathrm{source}})^*(\underline{\mathbf{C}}).$$

Composing with (11.27) we obtain a (strict) functor

$$\underline{\mathbf{C}} \to (\mathrm{pr}_{\mathrm{source}})_{*,\mathrm{strict}} \circ (\mathrm{pr}_{\mathrm{target}})^*(\underline{\mathbf{C}}). \tag{11.28}$$

11.5.2. We apply the construction in Sect. 11.5.1 to $\mathcal{Y} = \mathrm{Ran}^{\mathrm{untl}}$. Denote

$$\mathrm{Ran}^{\mathrm{untl},\to} =: \mathrm{Ran}^{\subseteq,\mathrm{untl}},$$

viewed as a categorical prestacks.

Note that the prestack in groupoids underlying $\mathrm{Ran}^{\subseteq,\mathrm{untl}}$ is the prestack $\mathrm{Ran}^{\subseteq}$ introduced in Sect. 11.2.2. We will use the symbols $\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}}$ and $\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}$ for the corresponding maps $\mathrm{pr}_{\mathrm{source}}$ and $\mathrm{pr}_{\mathrm{target}}$.

Thus, for $\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}$ as in Sect. 11.2, the functor (11.28) is a functor

$$\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}} \to (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_{*,\mathrm{strict}} \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}). \tag{11.29}$$

11.5.3. Let now $\underline{\mathsf{F}}^{\mathrm{untl}}$ be an object of $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$. Applying pullback along $\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}$, we obtain a *lax* functor

$$(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}) \overset{(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\underline{\mathsf{F}}^{\mathrm{untl}})}{\longrightarrow} (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}))$$

as crystals of categories on $\mathrm{Ran}^{\subseteq,\mathrm{untl}}$.

Using Sect. C.3.7 we obtain a lax functor of crystal of categories on $\mathrm{Ran}^{\mathrm{untl}}$

$$(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_{*,\mathrm{lax}} \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}) \overset{(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_* \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\underline{\mathsf{F}}^{\mathrm{untl}})}{\longrightarrow} (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_{*,\mathrm{lax}} \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})).$$

Combining, we obtain a functor

$$\begin{aligned} \underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}} &\to (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_{*,\mathrm{strict}} \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}) \to (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_{*,\mathrm{lax}} \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}) \to \\ &\to (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_{*,\mathrm{lax}} \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})) \simeq \mathbf{C}^{\mathrm{glob}} \otimes (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_{*,\mathrm{lax}}(\underline{\text{D-mod}}(\mathrm{Ran}^{\subseteq,\mathrm{untl}})). \end{aligned} \tag{11.30}$$

11.5.4. Consider the (strict) functor

$$\underline{\text{D-mod}}(\text{Ran}^{\text{untl}}) \overset{\text{(C.9)}}{\longrightarrow} (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{strict}}(\underline{\text{D-mod}}(\text{Ran}^{\subseteq,\text{untl}})) \to (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{lax}}(\underline{\text{D-mod}}(\text{Ran}^{\subseteq,\text{untl}})). \tag{11.31}$$

Lemma 11.5.5. *The functor* (11.31) *admits a value-wise left adjoint, to be denoted* $(\text{pr}^{\text{untl}}_{\text{small}})_!$. *Moreover, this value-wise left adjoint, which is a priori a* left-lax *functor, is strict.*

The proof will be given in Sect. 11.7.2.

11.5.6. Thus, composing the lax functor (11.30) with $(\text{pr}^{\text{untl}}_{\text{small}})_!$, we obtain a lax functor

$$\underline{\mathbf{C}}^{\text{loc,untl}} \to \mathbf{C}^{\text{glob}} \otimes (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{lax}}(\underline{\text{D-mod}}(\text{Ran}^{\subseteq,\text{untl}})) \overset{\text{Id}\otimes(\text{pr}^{\text{untl}}_{\text{small}})_!}{\longrightarrow} \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}). \tag{11.32}$$

11.5.7. The functor (11.32) is the sought-for object

$$\underline{\mathsf{F}}^{\text{untl},\int \text{ins.unit}}.$$

11.6. **Properties of the integrated functor.** We now proceed to establish the properties of the functor (11.23).

11.6.1. Note that the construction in Sect. 11.5.1 is functorial in the following sense:

If $\underline{\Phi} : \underline{\mathbf{C}}^{\text{loc,untl}}_1 \to \underline{\mathbf{C}}^{\text{loc,untl}}_2$ is a *strict* functor between sheaves of categories on Ran^{untl}, and

$$\underline{\mathsf{F}}^{\text{untl}} : \underline{\mathbf{C}}^{\text{loc,untl}}_2 \to \underline{\text{D-mod}}(\text{Ran}^{\text{untl}})$$

is a lax unital functor, then we have a (tautological) isomorphism

$$(\underline{\mathsf{F}}^{\text{untl}} \circ \Phi)^{\int \text{ins.unit}} \simeq \underline{\mathsf{F}}^{\text{untl},\int \text{ins.unit}} \circ \Phi.$$

11.6.2. We first show that $\int$ ins. unit acts as identity on objects $\underline{\mathsf{F}}^{\text{untl}} \in \text{Funct}^{\text{loc}\to\text{glob,untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}})$.

Indeed, if the functor

$$\underline{\mathsf{F}}^{\text{untl}} : \underline{\mathbf{C}}^{\text{loc,untl}} \to \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}^{\text{untl}})$$

is strict, then by Sect. 11.6.1 above, the functor (11.30) identifies with the composition of $\underline{\mathsf{F}}^{\text{untl}}$ with the tensor product of the identity endofunctor of $\mathbf{C}^{\text{glob}}$ with

$$\underline{\text{D-mod}}(\text{Ran}^{\text{untl}}) \overset{(11.28)}{\longrightarrow} (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{strict}} \circ (\text{pr}^{\text{untl}}_{\text{big}})^*(\underline{\text{D-mod}}(\text{Ran}^{\text{untl}})) = \\ = (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{strict}}(\underline{\text{D-mod}}(\text{Ran}^{\subseteq,\text{untl}})) \overset{(\text{pr}^{\text{untl}}_{\text{small}})_!}{\longrightarrow} \underline{\text{D-mod}}(\text{Ran}^{\text{untl}}). \tag{11.33}$$

We claim that (11.33) is the identity endofunctor of $\underline{\text{D-mod}}(\text{Ran}^{\text{untl}})$. Indeed, observe that for $\underline{\mathbf{C}} = \underline{\text{D-mod}}(\text{Ran}^{\text{untl}})$, the functor (11.27) is the identity endofunctor of $\underline{\text{D-mod}}(\text{Ran}^{\subseteq,\text{untl}})$, so the composition

$$\underline{\text{D-mod}}(\text{Ran}^{\text{untl}}) \overset{(11.28)}{\longrightarrow} (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{strict}} \circ (\text{pr}^{\text{untl}}_{\text{big}})^*(\underline{\text{D-mod}}(\text{Ran}^{\text{untl}})) = \\ = (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{strict}}(\underline{\text{D-mod}}(\text{Ran}^{\subseteq,\text{untl}}))$$

is the functor

$$\underline{\text{D-mod}}(\text{Ran}^{\text{untl}}) \overset{(11.28)}{\longrightarrow} (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{strict}} \circ (\text{pr}^{\text{untl}}_{\text{small}})^*(\underline{\text{D-mod}}(\text{Ran}^{\text{untl}})) = \\ = (\text{pr}^{\text{untl}}_{\text{small}})_{*,\text{strict}}(\underline{\text{D-mod}}(\text{Ran}^{\subseteq,\text{untl}})).$$

The assertion follows now from the next lemma:

Lemma 11.6.3. *The functor* (11.31) *is fully faithful.*[47]

The proof will be given in Sect. 11.7.1.

[47] See Sect. C.2.7 for what this means.

11.6.4. Next we show that the essential image of the functor (11.23) lies in the subcategory $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$, i.e., $\underline{\mathsf{F}}^{\mathrm{untl},\int \mathrm{ins.unit}}$ is strict.

Let S be an affine scheme and let $\underline{x}_1 \xrightarrow{\alpha} \underline{x}_2$ be a 1-morphism in $\mathrm{Maps}(S, \mathrm{Ran}^{\mathrm{untl}})$. Consider the fiber products

$$S^{\subseteq,\mathrm{untl}}_{\underline{x}_i} := S \underset{\underline{x}_i, \mathrm{Ran}^{\mathrm{untl}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{intl}}, \quad i = 1, 2,$$

and the resulting map

$$S^{\subseteq,\mathrm{untl}}_{\underline{x}_2} \xrightarrow{\alpha^*} S^{\subseteq,\mathrm{untl}}_{\underline{x}_1}. \tag{11.34}$$

Consider the diagram

$$\begin{tikzcd}
\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}_{S,\underline{x}_1} \arrow[r] \arrow[d] & \Gamma^{\mathrm{lax}}(S^{\subseteq,\mathrm{untl}}_{\underline{x}_1}, \underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}) \arrow[r, "\underline{\mathsf{F}}^{\mathrm{untl}}"] \arrow[d] & \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(S^{\subseteq,\mathrm{untl}}_{\underline{x}_1}) \arrow[d, "\mathrm{Id} \otimes (\alpha^*)^!"] \arrow[dl, Rightarrow] \\
\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}_{S,\underline{x}_2} \arrow[r] & \Gamma^{\mathrm{lax}}(S^{\subseteq,\mathrm{untl}}_{\underline{x}_2}, \underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}) \arrow[r, "\underline{\mathsf{F}}^{\mathrm{untl}}"] & \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(S^{\subseteq,\mathrm{untl}}_{\underline{x}_2}),
\end{tikzcd}$$

where:

- The left vertical arrow is given by the structure of crystal of categories on $\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}$;
- The middle vertical arrow is (C.8);
- Both left horizontal arrows are (11.28);
- The left square commutes because the functor (C.9) is strict;
- The natural transformation in the right square is given by the lax functor structure on $\underline{\mathsf{F}}^{\mathrm{untl}}$.

Given Lemma 11.5.5, it suffices to show that the above natural transformation is an isomorphism.

Note, however, that the middle vertical arrow in the above diagram is the functor (C.7) corresponding to the morphism α^* of (11.34). This makes the assertion manifest.

11.6.5. We now construct the natural transformation (11.25). This is done in the same way as in (11.18) using the map

$$\mathrm{diag}^{\mathrm{untl}} : \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\subseteq,\mathrm{untl}},$$

corresponding to the identity morphisms in $\mathrm{Ran}^{\mathrm{untl}}$ as a categorical prestack.

11.6.6. We now show that the two maps in (11.26) coincide. This amounts to the following assertion.

Let S be an affine scheme equipped with a map to $\mathrm{Ran}^{\mathrm{untl}}$. Consider

$$S^{\subseteq,\mathrm{untl}} := S \underset{\mathrm{Ran}^{\mathrm{untl}}, \mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{intl}} \text{ and } S^{\subseteq^2,\mathrm{untl}} := S^{\subseteq,\mathrm{untl}} \underset{\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}, \mathrm{Ran}^{\mathrm{untl}}, \mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{intl}}.$$

We have the naturally defined maps

$$\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2} : S^{\subseteq^2,\mathrm{untl}} \to \mathrm{Ran}^{\subseteq,\mathrm{intl}} \text{ and } \mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S} : S^{\subseteq^2,\mathrm{untl}} \to S^{\subseteq,\mathrm{untl}}.$$

We also have the maps

$$\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}} \text{ and } \mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{small}}, \quad S^{\subseteq,\mathrm{untl}} \rightrightarrows S^{\subseteq^2,\mathrm{untl}}$$

so that

$$\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S} \circ \mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}} = \mathrm{Id}, \quad \mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S} \circ \mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{small}} = \mathrm{diag}^{\mathrm{untl}}_{S} \circ \mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S}$$

and

$$\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2} \circ \mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{small}} = \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}, \quad \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2} \circ \mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}} = \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}.$$

11.6.7. Let us explain explicitly what these maps are when $S = \mathrm{pt}$, and the map $S \to \mathrm{Ran}^{\mathrm{untl}}$ corresponds to a point $\underline{x} \in \mathrm{Ran}$.

The categorical prestacks classify $S^{\subseteq,\mathrm{untl}}$ and $S^{\subseteq^2,\mathrm{untl}}$

$$(\underline{x} \subseteq \underline{x}_1) \text{ and } (\underline{x} \subseteq \underline{x}_1 \subseteq \underline{x}_2)$$

respectively, with the morphisms given by inclusions of the $\underline{x}_1$'s and $\underline{x}_2$.

The map $\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S}$ sends

$$(\underline{x} \subseteq \underline{x}_1 \subseteq \underline{x}_2) \mapsto (\underline{x} \subseteq \underline{x}_1);$$

the map $\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2}$ sends

$$(\underline{x} \subseteq \underline{x}_1 \subseteq \underline{x}_2) \mapsto \underline{x}_2;$$

the map $\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{small}}$ sends

$$(\underline{x} \subseteq \underline{x}_1) \mapsto (\underline{x} \subseteq \underline{x} \subseteq \underline{x}_1);$$

the map $\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}}$ sends

$$(\underline{x} \subseteq \underline{x}_1) \mapsto (\underline{x} \subseteq \underline{x}_1 \subseteq \underline{x}_1).$$

11.6.8. We obtain the natural transformations

$$\begin{multline}(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^! \simeq (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^! \simeq \\ \simeq (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}})^! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2})^! \to (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2})^! \tag{11.35}\end{multline}$$

and

$$\begin{multline}(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^! \simeq (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})_! \simeq \\ \simeq (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{small}})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{small}})^! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2})^! \to (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}^2,S})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2})^! \tag{11.36}\end{multline}$$

as functors

$$\text{D-mod}(\mathrm{Ran}^{\mathrm{untl}}) \to \text{D-mod}(S).$$

We need to show that the natural transformations (11.35) and (11.36) coincide.

This follows from the following observation: there exists a 1-morphism between the maps

$$\mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{small}} \xrightarrow{\alpha} \mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}},$$

so that the induced map

$$\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}} \simeq \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2} \circ \mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{small}} \xrightarrow{\alpha} \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}^2} \circ \mathrm{diag}^{\mathrm{untl}}_{S,\mathrm{big}} \simeq \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}$$

is the identity map.

11.6.9. Finally, we prove the commutativity of (11.24).

Let $\underline{\mathsf{F}}^{\mathrm{untl}}$ be an object of $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax}\text{-}\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$, and let $\underline{\mathsf{F}}$ be the corresponding object of $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$. We need to establish an isomorphism between $\underline{\mathsf{F}}^{\int \mathrm{ins.unit}}$ and

$$\underline{\mathsf{F}}^{\mathrm{untl},\int \mathrm{ins.unit}}|_{\underline{\mathbf{C}}^{\mathrm{loc}}} : \underline{\mathbf{C}}^{\mathrm{loc}} \to \mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}).$$

Let S be an affine scheme and let us be given an S-point of Ran. Denote

$$S^{\subseteq,\mathrm{untl}} := S \underset{\mathrm{Ran}^{\mathrm{untl}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{intl}} \text{ and } S^{\subseteq} := S \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq},$$

so that $S^{\subseteq}$ is the prestack in groupoids underlying $S^{\subseteq,\mathrm{untl}}$.

The value of $\underline{\mathsf{F}}^{\mathrm{untl},\int \mathrm{ins.unit}}|_{\underline{\mathbf{C}}^{\mathrm{loc}}}$ at the above S-point of Ran is given by the composition

$$\begin{multline}\mathbf{C}^{\mathrm{loc}}_S = \underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}_S \overset{(11.28)}{\longrightarrow} \Gamma^{\mathrm{lax}}(S^{\subseteq,\mathrm{untl}}, \underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}}) \overset{\underline{\mathsf{F}}^{\mathrm{untl}}}{\longrightarrow} \\ \to \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(S^{\subseteq,\mathrm{untl}}) \overset{(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})_!}{\longrightarrow} \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(S) \tag{11.37}\end{multline}$$

The value of $\underline{\mathsf{F}}^{\int \text{ins.unit}}$ at the above S-point of Ran is given by

$$\mathbf{C}^{\text{loc}}_S = \underline{\mathbf{C}}^{\text{loc},\text{untl}}_S \overset{(11.28)}{\longrightarrow} \Gamma^{\text{lax}}(S^{\subseteq,\text{untl}}, \underline{\mathbf{C}}^{\text{loc},\text{untl}}) \overset{\mathsf{F}^{\text{untl}}}{\longrightarrow} \tag{11.38}$$

$$\to \mathbf{C}^{\text{glob}} \otimes \text{D-mod}(S^{\subseteq,\text{untl}}) \to \mathbf{C}^{\text{glob}} \otimes \text{D-mod}(S^{\subseteq}) \overset{(\text{pr}_{\text{small},S})_!}{\longrightarrow} \mathbf{C}^{\text{glob}} \otimes \text{D-mod}(S).$$

Let t denote the tautological map $S^{\subseteq} \to S^{\subseteq,\text{untl}}$. We have a natural transformation

$$(\text{pr}_{\text{small},S})_! \circ \mathsf{t}^! \simeq (\text{pr}^{\text{untl}}_{\text{small},S})_! \circ \mathsf{t}_! \circ \mathsf{t}^! \to (\text{pr}^{\text{untl}}_{\text{small},S})_! \tag{11.39}$$

as functors

$$\text{D-mod}(S^{\subseteq,\text{untl}}) \to \text{D-mod}(S).$$

Thus, to establish an isomorphism between (11.37) and (11.38), it suffices to prove that the natural transformation (11.39) is an isomorphism. However, this is a variant of Lemma C.5.12 (with the same proof).

11.6.10. The compatibility of (11.16) and (11.25) follows from the commutativity of the diagram

$$\begin{tikzcd}
(\text{diag}_S)^! \circ \mathsf{t}^! \arrow[r, "\sim"] \arrow[d] & (\text{pr}_{\text{small},S})_! \circ (\text{diag}_S)_! \circ (\text{diag}_S)^! \circ \mathsf{t}^! \arrow[r] & (\text{pr}_{\text{small},S})_! \circ \mathsf{t}^! \arrow[d] \\
(\text{diag}^{\text{untl}}_S)^! \arrow[r, "\sim"] & (\text{pr}^{\text{untl}}_{\text{small},S})_! \circ (\text{diag}^{\text{untl}}_S)_! \circ (\text{diag}^{\text{untl}}_S)^! \arrow[r] & (\text{pr}^{\text{untl}}_{\text{small},S})_!,
\end{tikzcd}$$

as functors $\text{D-mod}(S^{\subseteq,\text{untl}}) \to \text{D-mod}(S)$, where

$$\text{diag}_S : S \to S^{\subseteq} \text{ and } \text{diag}^{\text{untl}}_S : S \to S^{\subseteq,\text{untl}}$$

are the corresponding maps.

11.7. Proofs of Lemmas 11.5.5, Lemma 11.6.3 and C.5.13.

11.7.1. *Proof of Lemma 11.6.3.* We need to show that, for any affine scheme S equipped with a map $S \to \text{Ran}^{\text{untl}}$, the functor

$$\text{D-mod}(S) \overset{(\text{pr}_{\text{small},S})^!}{\longrightarrow} \text{D-mod}(S^{\subseteq,\text{untl}})^{\text{strict}} \hookrightarrow \text{D-mod}(S^{\subseteq,\text{untl}})^{\text{lax}},$$

is fully faithful.

It suffices to show that the first arrow, i.e., $(\text{pr}_{\text{small},S})^!$ is fully faithful. However, this follows from the fact that morphism $\text{pr}_{\text{small},S}$ admits a left adjoint. Namely, it is given by diag_S.

□[Lemma 11.6.3]

11.7.2. *Proof of Lemma 11.5.5.* Let S be an affine scheme and let $\underline{x}$ be an S-point of Ran^{untl}. Consider the map

$$S^{\subseteq,\text{untl}}_{\underline{x}} := S \underset{\underline{x},\text{Ran}^{\text{untl}}}{\times} \text{Ran}^{\subseteq,\text{untl}} \overset{\text{pr}^{\text{untl}}_{\text{small},S,\underline{x}}}{\longrightarrow} S.$$

The first assertion of the lemma is that the functor

$$\text{D-mod}(S) \overset{(\text{pr}^{\text{untl}}_{\text{small},S,\underline{x}})^!}{\to} \text{D-mod}(S^{\subseteq,\text{untl}}_{\underline{x}})^{\text{strict}} \to \text{D-mod}(S^{\subseteq,\text{untl}}_{\underline{x}})$$

admits a left adjoint, to be denoted $(\text{pr}^{\text{untl}}_{\text{small},S,\underline{x}})_!$. This is a particular case of Corollary C.4.12.

The second assertion of the lemma is that for a 1-morphism $\underline{x}_1 \overset{\alpha}{\to} \underline{x}_2$ in $\text{Maps}(S, \text{Ran}^{\text{untl}})$ and the corresponding map

$$S^{\subseteq,\text{untl}}_{\underline{x}_2} \overset{\alpha^*}{\to} S^{\subseteq,\text{untl}}_{\underline{x}_1},$$

the natural transformation

$$(\text{pr}^{\text{untl}}_{\text{small},S,\underline{x}_2})_! \circ (\alpha^*)^! \simeq (\text{pr}^{\text{untl}}_{\text{small},S,\underline{x}_1})_! \circ (\alpha^*)_! \circ (\alpha^*)^! \to (\text{pr}^{\text{untl}}_{\text{small},S,\underline{x}_1})_!$$

is an isomorphism.

This follows from the fact that the map α^* admits a value-wise left adjoint. Namely, the left adjoint in question attaches to an affine scheme S' with a map $g_1 : S' \to S^{\subseteq,\mathrm{untl}}_{\underline{x}_1}$ the map $g_2 : S' \to S^{\subseteq,\mathrm{untl}}_{\underline{x}_2}$ defined as follows: the corresponding map

$$S' \overset{g_2}{\to} S^{\subseteq,\mathrm{untl}}_{\underline{x}_2} \overset{\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}}{\longrightarrow} \mathrm{Ran}^{\mathrm{untl}}$$

is obtained by applying the map

$$\mathrm{union} : \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}}$$

to

$$S' \overset{g_1}{\to} S^{\subseteq,\mathrm{untl}}_{\underline{x}_1} \overset{\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}}{\longrightarrow} \mathrm{Ran}^{\mathrm{untl}}$$

and

$$S' \overset{g_1}{\to} S^{\subseteq,\mathrm{untl}}_{\underline{x}_1} \overset{\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S,\underline{x}_1}}{\longrightarrow} S \overset{\underline{x}_2}{\to} \mathrm{Ran}^{\mathrm{untl}}.$$

$\square$[Lemma 11.5.5]

11.8. **Unitality as a property.** A somewhat surprising fact is that, given a local unital structure on $\underline{\mathbf{C}}^{\mathrm{loc}}$, a *unital*[48] structure on (a non-unital local-to-global functor) $\underline{\mathsf{F}}$ is actually a *property*, and not an additional piece of structure, as we shall presently explain.

The contents of this subsection are not necessary for the sequel.

11.8.1. Let $\underline{\mathsf{F}}$ be as in Sect. 11.1. Recall the morphism (11.18)

$$\mathsf{F}_{\mathcal{Z}} \to \big(\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!\big) \circ \mathsf{F}_{\mathcal{Z}^{\subseteq}} \circ \mathrm{ins.\,unit}_{\mathcal{Z}} \tag{11.40}$$

as functors $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathcal{Z})$.

Definition 11.8.2. *We shall say that $\underline{\mathsf{F}}$ satisfies Global Unitality Axiom 1 if the natural transformation* (11.18) *is an isomorphism (for any $\mathcal{Z} \to \mathrm{Ran}$).*

11.8.3. Assume that F satisfies Global Unitality Axiom 1. Then inverting (11.40) and applying the $((\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_!, (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^!)$-adjunction, we obtain a map

$$\mathsf{F}_{\mathcal{Z}^{\subseteq}} \circ \mathrm{ins.\,unit}_{\mathcal{Z}} \to \Big(\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^!\Big)(\mathsf{F}_{\mathcal{Z}}) \tag{11.41}$$

as functors $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathcal{Z}^{\subseteq})$.

Definition 11.8.4. *We shall say that $\underline{\mathsf{F}}$ satisfies Global Unitality Axiom 2 if the natural transformation* (11.41) *is an isomorphism (for any $\mathcal{Z} \to \mathrm{Ran}$).*

Definition 11.8.5. *We shall say that $\underline{\mathsf{F}}$ has a global unital property if it satisfies Axioms 1 and 2.*

11.8.6. It is clear that if $\underline{\mathsf{F}}$ is the image under

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$$

of

$$\underline{\mathsf{F}}^{\mathrm{untl}} \in \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}),$$

then $\underline{\mathsf{F}}$ has a global unitality property, see Sect. 11.4.5.

[48]As opposed to lax unital.

11.8.7. Vice versa, suppose that

$$\underline{\mathsf{F}} \in \operatorname{Funct}^{\text{loc}\to\text{glob}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}})$$

has a global unitality property.

Then the inverse of the isomorphism (11.41) provides an isomorphism as in (11.12).

More generally, one can show that in this case $\underline{\mathsf{F}}$ comes from a uniquely defined object $\underline{\mathsf{F}}^{\text{untl}} \in \operatorname{Funct}^{\text{loc}\to\text{glob,untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}})$.

In other words, we claim:

Proposition 11.8.8. *The composite functor*

$$\operatorname{Funct}^{\text{loc}\to\text{glob,untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}}) \to \operatorname{Funct}^{\text{loc}\to\text{glob,lax-untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}}) \to \operatorname{Funct}^{\text{loc}\to\text{glob}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}})$$

is fully faithful, and its essential image consists of objects that have a global unitality property.

We will give two, rather different in spirit, proofs of Proposition 11.8.8: one in Sect. H.3.3, and another in Sect. I.

11.9. **Factorization homology.**

11.9.1. In this section we will assume that $\underline{\mathbf{C}}^{\text{loc,untl}}$ comes from a unital lax factorization category $\mathbf{A}$, i.e., $\underline{\mathbf{C}}^{\text{loc,untl}} = \underline{\mathbf{A}}$ in the notations of Sect. B.11.1. Let $\underline{\mathsf{F}}$ be a functor

$$\underline{\mathbf{A}} \to \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}),$$

equipped with a lax unital structure.

Let now $\mathcal{A}$ be a unital factorization algebra in $\mathbf{A}$. We will regard $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$ as a lax factorization category (see Sect. C.11.9), and consider the corresponding crystal of categories $\mathcal{A}\text{-}\underline{\text{mod}}^{\text{fact}}(\mathbf{A})$ on Ran, which naturally extends to a crystal of categories over Ran^{untl} (see Sect. C.11.13).

In the particular case of factorization algebras, we will denote the functor $\text{ins. unit}_{\mathcal{Z}}$ of Sect. 11.6 by

$$\text{ins. vac}_{\mathcal{Z}} : \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})_{\mathcal{Z}} \to \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})_{\mathcal{Z}^{\subseteq}}, \quad \mathcal{Z} \to \text{Ran}.$$

This notation is meant to emphasize that the unit in $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$ is the "vacuum module", i.e., $\mathcal{A}$, viewed as a factorization module over itself.

11.9.2. Let

$$\underline{\mathbf{oblv}}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}) \to \mathbf{A}$$

be the tautological forgetful functor, viewed as a functor between crystals of categories over Ran.

Note that the unital structure on $\mathcal{A}$ defines on $\underline{\mathbf{oblv}}_{\mathcal{A}}$ a structure of *lax functor* between crystals of categories over Ran^{untl}, see Sect. C.11.14.

In particular

$$\underline{\mathsf{F}} \circ \underline{\mathbf{oblv}}_{\mathcal{A}} : \mathcal{A}\text{-}\underline{\text{mod}}^{\text{fact}}(\mathbf{A}) \to \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran})$$

also acquires a lax unital structure.

11.9.3. The functor of *factorization homology*

$$\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}} : \mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathbf{A}) \to \mathbf{C}^{\mathrm{glob}} \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran})$$

is by definition

$$(\underline{\mathsf{F}} \circ \underline{\mathbf{oblv}}_{\mathcal{A}})^{\int \mathrm{ins.unit}},$$

see (11.15).

For $\mathcal{Z} \to \mathrm{Ran}$ we will denote the corresponding functor

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}} \to \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathcal{Z})$$

by $\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}_{\mathcal{Z}}$.

For $\mathcal{Z}$ pseudo-proper, we will denote the composition of $\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}_{\mathcal{Z}}$ with $\mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathcal{Z},-)$ by $\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}_{\int_{\mathcal{Z}}}$.

For $\mathcal{Z} = \mathrm{Ran}$ and the identity map we will denote the resulting functor

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathrm{Ran}} \to \mathbf{C}^{\mathrm{glob}}$$

by $\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}$.

11.9.4. Recall the natural transformation (11.16). In our case, this is a map

$$\underline{\mathsf{F}} \circ \underline{\mathbf{oblv}}_{\mathcal{A}} \to \underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}, \tag{11.42}$$

which we will denote by $\underline{\mathrm{Cltr}}_{\mathcal{A}}$ and refer to as the "correlator" map.

For a given $\mathcal{Z} \to \mathrm{Ran}$, this is a map

$$\mathrm{Cltr}_{\mathcal{A},\mathcal{Z}} : \mathsf{F} \circ \underline{\mathbf{oblv}}_{\mathcal{A},\mathcal{Z}} \to \mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}_{\mathcal{Z}}.$$

11.9.5. *Example.* Let $\mathbf{A} = \mathrm{Vect}$, $\mathbf{C}^{\mathrm{glob}} = \mathrm{Vect}$ and $\underline{\mathsf{F}} = \mathrm{Id}$. In this case,

$$\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-) := \underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}, \quad \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A}) \to \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran})$$

is the usual functor of factorization homology.

11.9.6. A key fact for us is that according to Sect. 11.4.6, the functor $\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}$ acquires a natural lax unital structure. Moreover, by Sect. 11.4.7, this lax unital structure is actually *strict*.

11.9.7. Let us apply the functor $\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)^{\mathsf{F}}_{\mathrm{Ran}}$ to the object

$$(\mathbf{1}_{\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})})_{\mathrm{Ran}} = \mathcal{A}^{\mathrm{fact}}_{\mathrm{Ran}} \in \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathrm{Ran}}$$

(see Sect. C.11.15) i.e., to $\mathcal{A}_{\mathrm{Ran}}$, viewed as a factorization module over itself at Ran.

By unitality, the above object is of the form

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}) \otimes \omega_{\mathrm{Ran}},$$

for a canonically defined object

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}) \in \mathbf{C}^{\mathrm{glob}}. \tag{11.43}$$

The object (11.43) is called the *vacuum factorization homology* of $\mathcal{A}$.

Explicitly,

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}) \simeq (\mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran},-)) \circ \mathsf{F}(\mathcal{A}_{\mathrm{Ran}}).$$

11.9.8. Note that again by the unitality (Sect. 11.9.6) of the functor $\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)$, for any $\mathcal{Z} \to \mathrm{Ran}$ and

$$\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}} \in \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}},$$

(see Sects. B.9.7 and B.11.16 for the notation), we have

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}})_{\mathcal{Z}} \simeq \mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}) \otimes \omega_{\mathcal{Z}}. \tag{11.44}$$

In particular, for any $\underline{x} \in \mathrm{Ran}$,

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},\mathcal{A}^{\mathrm{fact}_{\underline{x}}})_{\underline{x}} \simeq \mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}). \tag{11.45}$$

11.9.9. Let us write out explicitly the proof of the fact that the functor $\underset{\cdot}{\mathrm{C}}^{\mathrm{fact}}(X, \mathcal{A}, -)^{\mathsf{F}}$ is unital (we will essentially repeat the argument from Sect. 11.6.4).

Fix $\mathcal{Z} \to \mathrm{Ran}$ and consider an object $\mathcal{M} \in \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}}$. From it we produce an object

$$\mathrm{ins.\,unit}_{\mathcal{Z}}(\mathcal{M}) \in \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}^{\subseteq}},$$

and further

$$\mathrm{ins.\,unit}_{\mathcal{Z}^{\subseteq}}(\mathrm{ins.\,unit}_{\mathcal{Z}}(\mathcal{M})) \in \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}^{\subseteq^2}},$$

where

$$\mathcal{Z}^{\subseteq^2} := (\mathcal{Z}^{\subseteq})^{\subseteq}.$$

Consider the object

$$\mathcal{M}' := \mathsf{F}_{\mathcal{Z}^{\subseteq^2}} \circ \mathbf{oblv}_{\mathcal{A}, \mathcal{Z}^{\subseteq^2}} (\mathrm{ins.\,unit}_{\mathcal{Z}^{\subseteq}}(\mathrm{ins.\,unit}_{\mathcal{Z}}(\mathcal{M}))) \in \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathcal{Z}^{\subseteq^2}).$$

Consider now the map

$$\mathrm{diag}_{\mathcal{Z},\mathrm{big}} : \mathcal{Z}^{\subseteq} \to \mathcal{Z}^{\subseteq^2}$$

(see Sect. 11.6.6).

It gives rise to a map

$$(\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{diag}_{\mathcal{Z},\mathrm{big}})^!(\mathcal{M}') \simeq (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}^{\subseteq}})_! \circ (\mathrm{diag}_{\mathcal{Z},\mathrm{big}})_! \circ (\mathrm{diag}_{\mathcal{Z},\mathrm{big}})^!(\mathcal{M}') \to \\ \to (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}^{\subseteq}})_!(\mathcal{M}'), \tag{11.46}$$

and we wish to show that this map is an isomorphism.

Now, the lax unital structure on $\underline{\mathsf{F}}$ implies that $\mathcal{M}'$ is the pullback of an object in

$$\mathcal{M}'' \in \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathcal{Z}^{\subseteq^2,\mathrm{untl}}).$$

As in Lemma C.5.13, one shows that one can replace both sides in (11.46) by their unital versions, i.e., it is sufficient to show that the corresponding map

$$(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{\mathcal{Z},\mathrm{big}})^!(\mathcal{M}'') \simeq (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},\mathcal{Z}^{\subseteq}})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{\mathcal{Z},\mathrm{big}})_! \circ (\mathrm{diag}^{\mathrm{untl}}_{\mathcal{Z},\mathrm{big}})^!(\mathcal{M}'') \to \\ \to (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},\mathcal{Z}^{\subseteq}})_!(\mathcal{M}'')$$

is an isomorphism.

However, the latter follows from the fact that the map

$$\mathrm{diag}^{\mathrm{untl}}_{\mathcal{Z},\mathrm{big}} : \mathcal{Z}^{\subseteq,\mathrm{untl}} \to \mathcal{Z}^{\subseteq^2,\mathrm{untl}}$$

is value-wise cofinal.

□

11.9.10. In what follows we will need the following variant of the isomorphism we just proved:

The natural transformation $\underline{\mathsf{F}} \to \underline{\mathsf{F}}^{\int \mathrm{ins.unit}}$ induces a natural transformation

$$\underset{\cdot}{\mathrm{C}}^{\mathrm{fact}}(X, \mathcal{A}, -)^{\mathsf{F}} \to \underset{\cdot}{\mathrm{C}}^{\mathrm{fact}}(X, \mathcal{A}, -)^{\mathsf{F}^{\int \mathrm{ins.unit}}}. \tag{11.47}$$

We claim:

Lemma 11.9.11. *The natural transformation* (11.47) *is an isomorphism.*

Proof. By the construction of (11.15), the unit in $\mathcal{A}$ gives rise to a natural transformation

$$\mathsf{F}^{\int \mathrm{ins.unit}} \circ \underline{\mathbf{oblv}}_{\mathcal{A}} \to (\mathsf{F} \circ \underline{\mathbf{oblv}}_{\mathcal{A}})^{\int \mathrm{ins.unit}} = \underset{\cdot}{\mathrm{C}}^{\mathrm{fact}}(X, \mathcal{A}, -)^{\mathsf{F}}. \tag{11.48}$$

Applying $\int \mathrm{ins.\,unit}$ we obtain a natural transformation

$$\underset{\cdot}{\mathrm{C}}^{\mathrm{fact}}(X, \mathcal{A}, -)^{\mathsf{F}^{\int \mathrm{ins.unit}}} \to \int \mathrm{ins.\,unit}(\underset{\cdot}{\mathrm{C}}^{\mathrm{fact}}(X, \mathcal{A}, -)^{\mathsf{F}}),$$

so that the diagram

$$\begin{array}{ccc}
\underline{\mathrm{C}}^{\mathrm{fact}}_{\cdot}(X,\mathcal{A},-)^{\mathsf{F}^{\int \mathrm{ins.unit}}} & \xrightarrow{(11.48)} & \int \mathrm{ins.\,unit} \circ \int \mathrm{ins.\,unit}(\mathsf{F} \circ \underline{\mathbf{oblv}}_{\mathcal{A}}) \\
\uparrow & & \sim \Big\uparrow \int \mathrm{ins.unit}(\underline{\mathrm{Cltr}}) \\
\underline{\mathrm{C}}^{\mathrm{fact}}_{\cdot}(X,\mathcal{A},-)^{\mathsf{F}} & \xrightarrow{=} & \int \mathrm{ins.\,unit}(\mathsf{F} \circ \underline{\mathbf{oblv}}_{\mathcal{A}})
\end{array}$$

commutes.

Hence, it suffices to check that the map (11.48) is an isomorphism. However, this is done by the same argument as in Sect. 11.9.9.

□

11.9.12. Here is a particular case of Lemma 11.9.11 that we will need:

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be a pair of unital factorization algebras in a unital lax factorization category $\mathbf{A}_0$, and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a unital homomorphism. Denote by res^{ϕ} the resulting functor

$$\mathcal{A}_2\text{-mod}(\mathbf{A}_0) \to \mathcal{A}_1\text{-mod}(\mathbf{A}_0).$$

Let us be given a lax unital functor

$$\underline{\mathsf{F}}_0 : \mathbf{A}_0 \to \mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}).$$

For a given $\mathcal{Z} \to \mathrm{Ran}$ we have a natural transformation

$$\begin{aligned}
(11.49)\quad \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A}_2,-)^{\mathsf{F}_0}_{\mathcal{Z}} &= (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ \mathbf{oblv}_{\mathcal{A}_2,\mathcal{Z}^{\subseteq}} \circ \mathrm{ins.\,vac}_{\mathcal{Z},\mathcal{A}_2} \simeq \\
&\simeq (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ \mathbf{oblv}_{\mathcal{A}_1,\mathcal{Z}^{\subseteq}} \circ \mathrm{res}^{\phi} \circ \mathrm{ins.\,vac}_{\mathcal{Z},\mathcal{A}_2} \simeq \\
&\simeq (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}^{\subseteq}})_! \circ (\mathrm{diag}_{\mathcal{Z}^{\subseteq}})_! \circ \mathbf{oblv}_{\mathcal{A}_1,\mathcal{Z}^{\subseteq}} \circ \mathrm{res}^{\phi} \circ \mathrm{ins.\,vac}_{\mathcal{Z},\mathcal{A}_2} \simeq \\
&\simeq (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}^{\subseteq}})_! \circ \mathbf{oblv}_{\mathcal{A}_1,(\mathcal{Z}^{\subseteq})^{\subseteq}} \circ (\mathrm{diag}_{\mathcal{Z}^{\subseteq}})_! \circ \mathrm{res}^{\phi} \circ \mathrm{ins.\,vac}_{\mathcal{Z},\mathcal{A}_2} \simeq \\
&\simeq (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}^{\subseteq}})_! \circ \mathbf{oblv}_{\mathcal{A}_1,(\mathcal{Z}^{\subseteq})^{\subseteq}} \circ (\mathrm{diag}_{\mathcal{Z}^{\subseteq}})_! \circ (\mathrm{diag}_{\mathcal{Z}^{\subseteq}})^! \circ \mathrm{ins.\,vac}_{\mathcal{Z}^{\subseteq},\mathcal{A}_1} \circ \mathrm{res}^{\phi} \circ \mathrm{ins.\,vac}_{\mathcal{Z},\mathcal{A}_2} \to \\
&\to (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ (\mathrm{pr}_{\mathrm{small},\mathcal{Z}^{\subseteq}})_! \circ \mathbf{oblv}_{\mathcal{A}_1,(\mathcal{Z}^{\subseteq})^{\subseteq}} \circ \mathrm{ins.\,vac}_{\mathcal{Z}^{\subseteq},\mathcal{A}_1} \circ \mathrm{res}^{\phi} \circ \mathrm{ins.\,vac}_{\mathcal{Z},\mathcal{A}_2} \simeq \\
&\simeq (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A}_1,-))^{\mathsf{F}_0}_{\mathcal{Z}^{\subseteq}} \circ \mathrm{res}^{\phi} \circ \mathrm{ins.\,vac}_{\mathcal{Z},\mathcal{A}_2},
\end{aligned}$$

We claim:

Corollary 11.9.13. *The natural transformation* (11.49) *is an isomorphism.*

Proof. Take $\mathbf{A} := \mathcal{A}_1\text{-mod}(\mathbf{A}_0)$ and $\mathcal{A} = \mathcal{A}_2$, viewed as a unital factorization algebra in $\mathcal{A}_1\text{-mod}(\mathbf{A}_0)$. Take $\underline{\mathsf{F}} = \mathsf{F}_0 \circ \underline{\mathbf{oblv}}_{\mathcal{A}_1}$.

Then the assertion follows from Lemma 11.9.11, where we note that the natural transformation (11.49) is (11.47).

□

11.9.14. Here is another application of Lemma 11.9.11. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}$ be a lax unital factorization functor between lax factorization categories. Denote by $\underline{\Phi}$ the corresponding right-lax functor

$$\underline{\mathbf{A}}_1 \to \underline{\mathbf{A}}$$

as crystals of categories on $\mathrm{Ran}^{\mathrm{untl}}$.

Let us be given a lax unital local-to-global functor

$$\underline{\mathsf{F}} : \mathbf{A} \to \mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}).$$

Note that $\underline{\mathsf{F}}_1 := \underline{\mathsf{F}} \circ \underline{\Phi}$ also acquires a lax unital structure. By functoriality, the natural transformation

$$\underline{\mathsf{F}}_1 = \underline{\mathsf{F}} \circ \underline{\Phi} \to \underline{\mathsf{F}}^{\int \mathrm{ins.unit}} \circ \underline{\Phi}$$

gives rise to a natural transformation

$$\underline{\mathsf{F}}_1^{\int \mathrm{ins.unit}} \to (\underline{\mathsf{F}}^{\int \mathrm{ins.unit}} \circ \underline{\Phi})^{\int \mathrm{ins.unit}}. \tag{11.50}$$

Corollary 11.9.15. *The natural transformation* (11.50) *is an isomorphism.*

Proof. By Sect. 4.1.6, the functor Φ factors as

$$\mathbf{A}_1 \overset{\Phi^{\mathrm{enh}}}{\longrightarrow} \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A}) \overset{\mathbf{oblv}_{\mathcal{A}}}{\longrightarrow} \mathbf{A},$$

where $\mathcal{A} = \Phi(\mathbf{1}_{\mathbf{A}_1})$.

Since the functor Φ^{enh} is strictly unital, the assertion of the corollary reduces to the case when $\mathbf{A}_1 = \mathcal{A}\text{-mod}(\mathbf{A})$ and $\Phi = \mathbf{oblv}_{\mathcal{A}}$. However, in the latter case, the map (11.50) is the map (11.47). $\square$

12. Properties of the localization functor

In this section we study the composition of the localization functors with three constructions of global nature:

- The forgetful functor $\text{D-mod}_\kappa(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{Bun}_G)$;
- The pullback functor $\text{D-mod}_\kappa(\mathrm{Bun}_G) \to \text{D-mod}_\kappa(\mathrm{Bun}_{G'})$ corresponding to a group homomorphism $G' \to G$;
- For a unipotent group-scheme N', the functor of de Rham cohomology $\text{D-mod}(\mathrm{Bun}_{N'}) \to \mathrm{Vect}$.

The pattern in the three composite functors mentioned above is that they can all be expressed via a local operation, followed by another localization functor:

- Restriction $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \mathrm{Rep}(\mathfrak{L}^+(G))$, followed by $\mathcal{O}$-module localization $\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{Bun}_G)$;
- Restriction $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \mathrm{KL}(G')_{\kappa,\mathrm{Ran}}$, followed by
$$\mathrm{Loc}_{G',\kappa} : \mathrm{KL}(G')_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_{G'});$$
- The functor of BRST reduction $\mathrm{KL}(N')_{\mathrm{Ran}} \to \mathrm{Vect}$.

However, there is a caveat, common to all three of these situations: in order for the local operation to reproduce the global one, we need to precompose the former with an endofunctor of the source given by $\mathrm{ins.vac}_{\int_{\mathrm{Ran}}}$, see (11.8).

12.1. Localization and the forgetful functor.

12.1.1. Note that by adjunction, the commutative diagram (10.17) gives rise to a natural transformation

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Bun}_G) & & \text{D-mod}_\kappa(\mathrm{Bun}_G) \arrow[ll, "\mathbf{oblv}^l_\kappa"'] \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \arrow[u, "\mathrm{Loc}_G^{\mathrm{QCoh}}"] \arrow[urr, Rightarrow] & & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}. \arrow[u, "\mathrm{Loc}_{G,\kappa}"'] \arrow[ll, "\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"]
\end{tikzcd} \tag{12.1}$$

The natural transformation in (12.1) is *not* an isomorphism (unless $G = 1$): namely, evaluate both circuits on

$$\mathrm{Vac}(G)_{\kappa,x} \in \mathrm{KL}(G)_{\kappa,x} \hookrightarrow \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}$$

for some $x \in X$.

We will now draw another diagram, in which a natural transformation will be an isomorphism, which encodes another basic property of the localization functor.

12.1.2. Being unital factorization categories, both $\mathrm{KL}(G)_\kappa$ and $\mathrm{Rep}(\mathfrak{L}^+(G))$, viewed as crystals of categories over Ran, carry local unital structures (see Sect. 11.2.1 for what this means). Furthermore, the local-to-global functors[49]

$$\underline{\mathrm{Loc}}_{G,\kappa} : \underline{\mathrm{KL}}(G)_\kappa \to \mathrm{D\text{-}mod}_\kappa(\mathrm{Bun}_G) \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}) \tag{12.2}$$

and

$$\underline{\mathrm{Loc}}_G^{\mathrm{QCoh}} : \underline{\mathrm{Rep}}(\mathfrak{L}^+(G)) \to \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}) \tag{12.3}$$

both carry naturally defined (strict) unital structures (see Sect. 11.3.1 for what this means).

Note, however, that the restriction functor

$$\underline{\mathbf{oblv}}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa} : \underline{\mathrm{KL}}(G)_\kappa \to \underline{\mathrm{Rep}}(\mathfrak{L}^+(G))$$

is merely right-lax, (as is the case for any factorization functor that is lax unital as opposed to strictly unital).

12.1.3. The natural transformation

$$\mathrm{Loc}_G^{\mathrm{QCoh}} \circ \mathbf{oblv}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa} \to \mathbf{oblv}^l_\kappa \circ \mathrm{Loc}_{G,\kappa}$$

from diagram (12.1) can be viewed as a natural transformation

$$\underline{\mathrm{Loc}}_G^{\mathrm{QCoh}} \circ \underline{\mathbf{oblv}}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa} \to \mathbf{oblv}^l_\kappa \circ \underline{\mathrm{Loc}}_{G,\kappa} \tag{12.4}$$

between functors

$$\underline{\mathrm{KL}}(G)_\kappa \rightrightarrows \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran})$$

between crystals of categories over Ran (see Sect. 11.1.5).

The two sides in (12.4) are lax unital local-to-global functors, and the map (12.4) is compatible with the lax unital structures.

12.1.4. Note now that the right-hand side in (12.4) is strictly unital (because $\underline{\mathrm{Loc}}_{G,\kappa}$ is unital). Hence by Sect. 11.4.8, the map (12.4) gives rise to a map[50]

$$\left(\underline{\mathrm{Loc}}_G^{\mathrm{QCoh}} \circ \underline{\mathbf{oblv}}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}\right)^{\int \mathrm{ins.vac}} \to \mathbf{oblv}^l_\kappa \circ \underline{\mathrm{Loc}}_{G,\kappa}. \tag{12.5}$$

We claim:

Theorem 12.1.5. *The natural transformation* (12.5) *is an isomorphism.*

Remark 12.1.6. One can interpret Theorem 12.1.5 as follows: the natural transformation (12.4) fails to be an isomorphism because the right-hand side is unital (i.e., insertion of vacuum does not change the value of the functor), but the left-hand side is only lax unital. But once we correct this by applying $\int \mathrm{ins.\,vac}$, the corresponding map becomes an isomorphism.

12.1.7. We will now reformulate Theorem 12.1.5 is concrete terms, which do not explicitly involve categorical prestacks:

Let $\int \mathrm{ins.\,vac}$ be the endofunctor of $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}}$ from Sect. 11.2.12. We have:

Theorem 12.1.8. *The natural transformation in* (12.1) *becomes an isomorphism after precomposing with* $\int \mathrm{ins.\,vac}$.

Note that Theorems 12.1.5 and 12.1.8 are logically equivalent. This follows from Sect. 11.1.5 and the commutativity of (11.24).

Theorem 12.1.8 will be proved in Sect. 13.3[51].

[49] In the formulas below, the underline has the meaning from Sect. 11.1.1.

[50] In the formula below, for the factorization category $\mathrm{KL}(G)_\kappa$ we use the notation ins. vac instead of ins. unit.

[51] We supply a proof for completness. An equivalent statement appears in [CF, Lemma 4.4.16 and Variant 4.4.17]

12.1.9. Note that since $\underline{\mathrm{Loc}}_{G,\kappa}$ is strictly unital, the map

$$\mathrm{Loc}_{G,\kappa} \to \mathrm{Loc}_{G,\kappa} \circ \int_{\mathrm{Ran}} \mathrm{ins.\,vac}$$

is an isomorphism (see Sect. 11.4.5). Hence, Theorem 12.1.8 implies:

Corollary 12.1.10. *We have a commutative diagram*

$$\begin{tikzcd} \mathrm{D\text{-}mod}_\kappa(\mathrm{Bun}_G) \arrow[rr, "\mathbf{oblv}^l_\kappa"] & & \mathrm{QCoh}(\mathrm{Bun}_G) \\ \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \arrow[u, "\mathrm{Loc}_{G,\kappa}"] \arrow[r, "\int \mathrm{ins.vac}"] & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \arrow[r, "\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"] & \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}}. \arrow[u, "\mathrm{Loc}^{\mathrm{QCoh}}_G"'] \end{tikzcd}$$

12.1.11. Consider the functor

$$\underline{\mathrm{Rep}}(\mathfrak{L}^+(G)) \xrightarrow{\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_G} \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}).$$

By Sect. 11.9.3, for a factorization algebra $\mathcal{A} \in \mathrm{Rep}(\mathfrak{L}^+(G))$, we can consider the local-to-global functor

$$\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, -)^{\mathrm{Loc}^{\mathrm{QCoh}}_G} : \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G))) \to \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}).$$

Moreover, by Sect. 11.9.6, the functor $\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, -)^{\mathrm{Loc}^{\mathrm{QCoh}}_G}$ is *strictly* unital.

12.1.12. Denote

$$\mathbb{V}_{G,\kappa} := \mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}(\mathrm{Vac}(G)_\kappa),$$

viewed as a factorization algebra in $\mathrm{Rep}(\mathfrak{L}^+(G))$.

The functor $\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}$ upgrades to a strictly unital factorization functor

$$(\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)})^{\mathrm{enh}} : \mathrm{KL}(G)_\kappa \to \mathbb{V}_{G,\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G))).$$

Note that whereas $\underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}$ was right-lax, when viewed as a functor between sheaves of categories over $\mathrm{Ran}^{\mathrm{untl}}$, the functor $(\underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)})^{\mathrm{enh}}$ is strict.

12.1.13. With these notations, from Theorem 12.1.5, we obtain:

Corollary 12.1.14. *We have a commutative diagram*

$$\begin{tikzcd} \mathrm{D\text{-}mod}_\kappa(\mathrm{Bun}_G) \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}) \arrow[r, "\mathbf{oblv}^l_\kappa \otimes \mathrm{Id}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}) \\ \underline{\mathrm{KL}}(G)_\kappa \arrow[u, "\underline{\mathrm{Loc}}_{G,\kappa}"] \arrow[r, "(\underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)})^{\mathrm{enh}}"'] & \mathbb{V}_{G,\kappa}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G))). \arrow[u, "\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathbb{V}_{G,\kappa},-)^{\mathrm{Loc}^{\mathrm{QCoh}}_G}"'] \end{tikzcd} \tag{12.6}$$

Integrating over Ran, we obtain:

Corollary 12.1.15. *We have a commutative diagram*

$$\begin{tikzcd} \mathrm{D\text{-}mod}_\kappa(\mathrm{Bun}_G) \arrow[r, "\mathbf{oblv}^l_\kappa"] & \mathrm{QCoh}(\mathrm{Bun}_G) \\ \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \arrow[u, "\mathrm{Loc}_{G,\kappa}"] \arrow[r, "(\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)})^{\mathrm{enh}}"'] & \mathbb{V}_{G,\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G)))_{\mathrm{Ran}}. \arrow[u, "\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathbb{V}_{G,\kappa},-)^{\mathrm{Loc}^{\mathrm{QCoh}}_G}"'] \end{tikzcd} \tag{12.7}$$

12.2. **Proof of Proposition 10.4.5.** In this subsection we will use Theorem 12.1.8 (or rather Corollary 12.1.15) to deduce Proposition 10.4.5.

Let $\mathcal{P}_G$ be a k-point of Bun_G. Applying Corollary 12.1.15, we need to construct an isomorphism between

$$\mathrm{KL}(G)_{\kappa,\underline{x}} \overset{\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}}{\longrightarrow} \mathrm{Rep}(\mathfrak{L}^+(G))_{\underline{x}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathbb{V}_{G,\kappa},-)_{\underline{x}}^{\mathrm{Loc}_G^{\mathrm{QCoh}}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{Bun}_G) \overset{*\text{-fiber at }\mathcal{P}_G}{\longrightarrow} \mathrm{Vect} \tag{12.8}$$

and

$$\mathrm{KL}(G)_{\kappa,\underline{x}} \overset{\alpha_{\mathcal{P}_G,\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_G,\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}} \overset{(10.23)}{\longrightarrow} \Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})\text{-mod} \overset{\mathrm{coinv}_{\Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})}}{\longrightarrow} \mathrm{Vect}\,. \tag{12.9}$$

12.2.1. First, we note that the functor

$$\underline{\mathrm{Rep}}(\mathfrak{L}^+(G)) \overset{\mathrm{Loc}_G^{\mathrm{QCoh}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \underline{\text{D-mod}}(\mathrm{Ran}) \overset{(*\text{-fiber at }\mathcal{P}_G)\otimes \mathrm{Id}}{\longrightarrow} \underline{\text{D-mod}}(\mathrm{Ran})$$

is a $\mathcal{P}_G$-twisted version of the forgetful functor. Denote it by $\mathbf{oblv}_{\mathfrak{L}^+(G),\mathcal{P}_G}$. We can view it as a factorization functor

$$\mathbf{oblv}_{\mathfrak{L}^+(G),\mathcal{P}_G} : \mathrm{Rep}(\mathfrak{L}^+(G)) \to \mathrm{Vect}\,.$$

Denote

$$\mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G} := \mathbf{oblv}_{\mathfrak{L}^+(G),\mathcal{P}_G} \circ \mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}(\mathrm{Vac}(G)_\kappa).$$

This is a factorization algebra in Vect, which is a $\mathcal{P}_G$-twisted version of the vacuum representation $\mathbb{V}_{\mathfrak{g},\kappa}$. We can rewrite the functor

$$\underline{\mathrm{Rep}}(\mathfrak{L}^+(G)) \overset{\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathbb{V}_{G,\kappa},-)^{\mathrm{Loc}_G^{\mathrm{QCoh}}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \underline{\text{D-mod}}(\mathrm{Ran}) \overset{*\text{-fiber at }\mathcal{P}_G\otimes\mathrm{Id}}{\longrightarrow} \underline{\text{D-mod}}(\mathrm{Ran})$$

as

$$\underline{\mathrm{Rep}}(\mathfrak{L}^+(G)) \overset{\underline{\mathrm{C}}^{\mathrm{fact}}_\cdot(X,\mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G},-)^{\mathbf{oblv}_{\mathfrak{L}^+(G),\mathcal{P}_G}}}{\longrightarrow} \underline{\text{D-mod}}(\mathrm{Ran}).$$

12.2.2. Consider the forgetful (factorization) functor

$$\mathbf{oblv}_{\widehat{\mathfrak{g}},\mathcal{P}_G} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}} \to \mathrm{Vect}\,.$$

Note that

$$\mathbf{oblv}_{\widehat{\mathfrak{g}},\mathcal{P}_G} \circ \mathbf{oblv}_{\mathfrak{L}^+(G)} \circ \alpha_{\mathcal{P}_G,\mathrm{taut}} \simeq \mathbf{oblv}_{\mathfrak{L}^+(G),\mathcal{P}_G} \circ \mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}$$

as factorization functors

$$\mathrm{KL}(G)_\kappa \to \mathrm{Vect}\,.$$

In particular,

$$\mathbf{oblv}_{\widehat{\mathfrak{g}},\mathcal{P}_G}(\mathrm{Vac}(G)_{\kappa,\mathcal{P}_G}) \simeq \mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G},$$

and we obtain that the functor $\mathbf{oblv}_{\widehat{\mathfrak{g}},\mathcal{P}_G}$ enhances to a functor

$$\mathbf{oblv}^{\mathrm{enh}}_{\widehat{\mathfrak{g}},\mathcal{P}_G} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}} \to \mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{fact}}, \tag{12.10}$$

see Sect. 4.1.6.

12.2.3. We obtain that we can rewrite (12.8) as

$$\mathrm{KL}(G)_{\kappa,\underline{x}} \overset{\alpha_{\mathcal{P}_G,\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_G,\underline{x}} \overset{\mathbf{oblv}_{\mathfrak{L}^+(G)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}} \overset{\mathbf{oblv}^{\mathrm{enh}}_{\widehat{\mathfrak{g}},\mathcal{P}_G}}{\longrightarrow}$$
$$\to \mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \overset{\underline{\mathrm{C}}^{\mathrm{fact}}_{\cdot}(X,\mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G},-)}{\longrightarrow} \mathrm{Vect}\,. \tag{12.11}$$

Thus, we obtain that it suffices to show that the composition

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}} \overset{\mathbf{oblv}^{\mathrm{enh}}_{\widehat{\mathfrak{g}},\mathcal{P}_G}}{\longrightarrow} \mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \overset{\underline{\mathrm{C}}^{\mathrm{fact}}_{\cdot}(X,\mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G},-)}{\longrightarrow} \mathrm{Vect} \tag{12.12}$$

identifies with

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}} \overset{(10.23)}{\longrightarrow} \Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})\text{-mod} \overset{\mathrm{coinv}_{\Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})}}{\longrightarrow} \mathrm{Vect}\,. \tag{12.13}$$

We will relate (12.12) to the functor of $\Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})$-coinvariants using the calculation of factorization homology of (twisted) chiral envelopes of Lie-* algebras performed in [BD2].

12.2.4. Observe that the chiral algebra corresponding to the factorization algebra $\mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G}$ identifies with (κ-twisted) the chiral envelope

$$U^{\mathrm{ch}}(L_{\mathfrak{g},\mathcal{P}_G})_\kappa,$$

where $L_{\mathfrak{g},\kappa,\mathcal{P}_G}$ is the $\mathcal{P}_G$-twist of the Lie-algebra

$$\omega_X \oplus (\mathfrak{g}\otimes \mathrm{D}_X).$$

Moreover, we have a canonical equivalence

$$L_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{ch}}_{\underline{x}} \simeq U^{\mathrm{ch}}(L_{\mathfrak{g},\mathcal{P}_G})_\kappa\text{-mod}^{\mathrm{ch}}_{\underline{x}} \simeq \mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{fact}}_{\underline{x}}$$

and a functor

$$L_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{ch}}_{\underline{x}} \to \Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})\text{-mod}. \tag{12.14}$$

Under these identifications, the functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G,\underline{x}} \overset{\mathbf{oblv}^{\mathrm{enh}}_{\widehat{\mathfrak{g}},\mathcal{P}_G}}{\longrightarrow} \mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \simeq L_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{ch}}_{\underline{x}} \overset{(12.14)}{\longrightarrow} \Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})\text{-mod}$$

identifies with the functor (10.23).

12.2.5. Thus, it remains to show that the functor

$$L_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{ch}}_{\underline{x}} \simeq \mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\mathbb{V}_{\mathfrak{g},\kappa,\mathcal{P}_G},-)_{\underline{x}}}{\longrightarrow} \mathrm{Vect}$$

identifies canonically with

$$L_{\mathfrak{g},\kappa,\mathcal{P}_G}\text{-mod}^{\mathrm{ch}}_{\underline{x}} \overset{(12.14)}{\longrightarrow} \Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})\text{-mod} \overset{\mathrm{coinv}_{\Gamma(X-\underline{x},\mathfrak{g}_{\mathcal{P}_G})}}{\longrightarrow} \mathrm{Vect}\,.$$

However, the latter is the assertion of [BD2, Proposition 4.8.2]

Remark 12.2.6. An alternative proof of the latter assertion can be found in [FraG, Corollary 6.4.4] in the special case when the coefficient module is the vacuum representation.

However, the method from [FraG] easily adapts to the present setting and can also be used to reprove [BD2, Proposition 4.8.2] in the generality in which we are using it.

□[Proposition 10.4.5]

12.3. **Localization and restriction.**

12.3.1. Let $\phi : G' \to G$ be a group homomorphism. We restrict the level κ to G' and consider the corresponding Kazhdan-Lusztig category

$$\mathrm{KL}(G')_\kappa := \widehat{\mathfrak{g}}'\text{-mod}_\kappa^{\mathfrak{L}^+(G')}$$

and the localization functor

$$\mathrm{Loc}_{G',\kappa} : \mathrm{KL}(G')_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_{G'}).$$

12.3.2. The map ϕ gives rise to (factorization) restriction functors

$$\mathrm{Rep}(\mathfrak{L}^+(G)) \xrightarrow{\mathrm{res}^\phi} \mathrm{Rep}(\mathfrak{L}^+(G')) \text{ and } \mathrm{KL}(G)_\kappa \xrightarrow{\mathrm{res}^\phi} \mathrm{KL}(G')_\kappa$$

so that the diagram

$$\begin{tikzcd}
\mathrm{KL}(G)_\kappa \arrow[r, "\mathrm{res}^\phi"] \arrow[d, "\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"'] & \mathrm{KL}(G')_\kappa \arrow[d, "\mathbf{oblv}^{(\widehat{\mathfrak{g}}',\mathfrak{L}^+(G'))_\kappa}_{\mathfrak{L}^+(G')}"] \\
\mathrm{Rep}(\mathfrak{L}^+(G)) \arrow[r, "\mathrm{res}^\phi"] & \mathrm{Rep}(\mathfrak{L}^+(G'))
\end{tikzcd}$$

commutes.

In addition, the map ϕ gives rise to a map

$$\phi^{\mathrm{glob}} : \mathrm{Bun}_{G'} \to \mathrm{Bun}_G,$$

which is compatible with the twistings and thus gives rise to a functor

$$(\phi^{\mathrm{glob}})^!_\kappa : \text{D-mod}_\kappa(\mathrm{Bun}_G) \to \text{D-mod}_\kappa(\mathrm{Bun}_{G'}),$$

which makes the diagram

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Bun}_G) \arrow[r, "(\phi^{\mathrm{glob}})^*"] & \mathrm{QCoh}(\mathrm{Bun}_{G'}) \\
\text{D-mod}_\kappa(\mathrm{Bun}_G) \arrow[r, "(\phi^{\mathrm{glob}})^!_\kappa"] \arrow[u, "\mathbf{oblv}^l_\kappa"] & \text{D-mod}_\kappa(\mathrm{Bun}_{G'}) \arrow[u, "\mathbf{oblv}^l_\kappa"']
\end{tikzcd}$$

commute.

12.3.3. Let $U \subset \mathrm{Bun}_G$ and $U' \subset \mathrm{Bun}_{G'}$ be a pair of quasi-compact substacks so that ϕ^{glob} maps $U' \to U$.

Consider the corresponding functors

$$\Gamma_{G,\kappa,U} : \text{D-mod}_\kappa(U) \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \text{ and } \Gamma_{G',\kappa,U} : \text{D-mod}_\kappa(U') \to \mathrm{KL}(G')_{\kappa,\mathrm{Ran}}.$$

By construction, we have a natural transformation

$$\begin{tikzcd}
\text{D-mod}_\kappa(U) \arrow[r, "(\phi^{\mathrm{glob}})^!_\kappa"] \arrow[d, "\Gamma_{G,\kappa,U}"'] & \text{D-mod}_\kappa(U') \arrow[d, "\Gamma_{G',\kappa,U}"] \\
\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \arrow[r, "\mathrm{res}^\phi"'] \arrow[ur, Rightarrow] & \mathrm{KL}(G')_{\kappa,\mathrm{Ran}}.
\end{tikzcd} \tag{12.15}$$

By adjunction, we obtain a natural transformation

$$
\begin{tikzcd}
\text{D-mod}_\kappa(U) \arrow[r, "(\phi^{\text{glob}})^!_\kappa"] & \text{D-mod}_\kappa(U') \\
\text{KL}(G)_{\kappa,\text{Ran}} \arrow[u, "\text{Loc}_{G,\kappa,U}"] \arrow[r, "\text{res}^\phi"'] & \text{KL}(G')_{\kappa,\text{Ran}}. \arrow[u, "\text{Loc}_{G',\kappa,U'}"'] \arrow[ul, Rightarrow]
\end{tikzcd}
\tag{12.16}
$$

Passing to the limit over U, from (12.16), we obtain a natural transformation

$$
\begin{tikzcd}
\text{D-mod}_\kappa(\text{Bun}_G) \arrow[r, "(\phi^{\text{glob}})^!_\kappa"] & \text{D-mod}_\kappa(\text{Bun}_{G'}) \\
\text{KL}(G)_{\kappa,\text{Ran}} \arrow[u, "\text{Loc}_{G,\kappa}"] \arrow[r, "\text{res}^\phi"'] & \text{KL}(G')_{\kappa,\text{Ran}}. \arrow[u, "\text{Loc}_{G',\kappa}"'] \arrow[ul, Rightarrow]
\end{tikzcd}
\tag{12.17}
$$

12.3.4. The natural transformation in (12.17) is not an isomorphism (unless ϕ itself is). We will now draw another diagram, in which the natural transformation is an isomorphism, and which expresses the composition

$$(\phi^{\text{glob}})^!_\kappa \circ \text{Loc}_{G,\kappa}$$

via $\text{Loc}_{G',\kappa}$.

This will be completely parallel to Sects. 12.1.2-12.1.4.

12.3.5. The natural transformation

$$\text{Loc}_{G',\kappa} \circ \text{res}^\phi \to (\phi^{\text{glob}})^!_\kappa \circ \text{Loc}_{G,\kappa}$$

in (12.17) can be viewed as a natural transformation

$$\underline{\text{Loc}}_{G',\kappa} \circ \underline{\text{res}}^\phi \to (\phi^{\text{glob}})^!_\kappa \circ \underline{\text{Loc}}_{G,\kappa} \tag{12.18}$$

between functors

$$\underline{\text{KL}}(G)_\kappa \rightrightarrows \text{D-mod}_\kappa(\text{Bun}_{G'}) \otimes \underline{\text{D-mod}}(\text{Ran})$$

between crystals of categories over Ran (see Sect. 11.1.5).

The two sides in (12.18) are lax unital local-to-global functors, and the map (12.4) is compatible with the lax unital structures.

Note now that the right-hand side in (12.18) is strictly unital. Hence by Sect. 11.4.8, the map (12.18) gives rise to a map

$$\left(\underline{\text{Loc}}_{G',\kappa} \circ \underline{\text{res}}^\phi\right)^{\int \text{ins.vac}} \to (\phi^{\text{glob}})^!_\kappa \circ \underline{\text{Loc}}_{G,\kappa} \tag{12.19}$$

as unital local-to-global functors.

We claim:

Proposition 12.3.6. *The natural transformation* (12.19) *is an isomorphism.*

Parallel to Sect. 12.3.6, we will now reformulate Proposition 12.3.6 is several (equivalent) ways.

Proposition 12.3.7. *The natural transformation in* (12.17) *becomes an isomorphism after precomposing with* $\int$ ins. vac.

Corollary 12.3.8. *We have a commutative diagram*

$$\begin{tikzcd} \text{D-mod}_\kappa(\text{Bun}_G) \arrow[rr, "(\phi^{\text{glob}})^!_\kappa"] & & \text{D-mod}_\kappa(\text{Bun}_{G'}) \\ \text{KL}(G)_{\kappa,\text{Ran}} \arrow[u, "\text{Loc}_{G,\kappa}"] \arrow[r, "\int \text{ins.vac}"] & \text{KL}(G)_{\kappa,\text{Ran}} \arrow[r, "\text{res}^\phi"] & \text{KL}(G')_{\kappa,\text{Ran}} \arrow[u, "\text{Loc}_{G',\kappa}"'] \end{tikzcd}$$

12.3.9. Consider the factorization algebra

$$\text{Vac}(G|_{G'})_\kappa := \text{res}^\phi(\text{Vac}(G)_\kappa) \in \text{KL}(G')_\kappa.$$

The functor res^ϕ upgrades to a factorization functor

$$(\text{res}^\phi)^{\text{enh}} : \text{KL}(G)_\kappa \to \text{Vac}(G|_{G'})_\kappa\text{-mod}^{\text{fact}}(\text{KL}(G')_\kappa).$$

Consider the functor

$$\underline{\text{C}}^{\text{fact}}_\cdot(X, \text{Vac}(G|_{G'})_\kappa, -)^{\text{Loc}_{G',\kappa}} : \text{Vac}(G|_{G'})_\kappa\text{-mod}^{\text{fact}}(\text{KL}(G')_\kappa) \to \text{D-mod}_\kappa(\text{Bun}_{G'}),$$

see Sect. 11.9.3.

Corollary 12.3.10. *We have a commutative diagram*

$$\begin{tikzcd} \text{D-mod}_\kappa(\text{Bun}_G) \otimes \underline{\text{D-mod}}(\text{Ran}) \arrow[r, "(\phi^{\text{glob}})^!_\kappa \otimes \text{Id}"] & \text{D-mod}_\kappa(\text{Bun}_{G'}) \otimes \underline{\text{D-mod}}(\text{Ran}) \\ \text{KL}(G)_\kappa \arrow[u, "\underline{\text{Loc}}_{G,\kappa}"] \arrow[r, "(\text{res}^\phi)^{\text{enh}}"'] & \text{Vac}(G|_{G'})_\kappa\text{-mod}^{\text{fact}}(\text{KL}(G')_\kappa). \arrow[u, "\underline{\text{C}}^{\text{fact}}_\cdot(X, \text{Vac}(G|_{G'})_\kappa, -)^{\text{Loc}_{G',\kappa}}"'] \end{tikzcd} \tag{12.20}$$

Integrating over Ran, we obtain:

Corollary 12.3.11. *We have a commutative diagram*

$$\begin{tikzcd} \text{D-mod}_\kappa(\text{Bun}_G) \arrow[r, "(\phi^{\text{glob}})^!_\kappa"] & \text{D-mod}_\kappa(\text{Bun}_{G'}) \\ \text{KL}(G)_{\kappa,\text{Ran}} \arrow[u, "\text{Loc}_{G,\kappa}"] \arrow[r, "(\text{res}^\phi)^{\text{enh}}"'] & \text{Vac}(G|_{G'})_\kappa\text{-mod}^{\text{fact}}(\text{KL}(G')_\kappa)_{\text{Ran}}. \arrow[u, "\text{C}^{\text{fact}}_\cdot(X, \text{Vac}(G|_{G'})_\kappa, -)"'] \end{tikzcd} \tag{12.21}$$

12.4. **Proof of Proposition 12.3.6.**

12.4.1. Since the functor

$$\mathbf{oblv}^l_{\kappa,G'} : \text{D-mod}_\kappa(\text{Bun}_{G'}) \to \text{QCoh}(\text{Bun}_{G'}) \tag{12.22}$$

is conservative, it is sufficient to prove that the natural transformation in (12.19) becomes an isomorphism after composing with

$$\mathbf{oblv}^l_{\kappa,G'} \otimes \text{Id} : \text{D-mod}_\kappa(\text{Bun}_{G'}) \otimes \underline{\text{D-mod}}(\text{Ran}^{\text{untl}}) \to \text{QCoh}(\text{Bun}_{G'}) \otimes \underline{\text{D-mod}}(\text{Ran}^{\text{untl}}).$$

Remark 12.4.2. The idea of the proof is the following: the diagram

$$\begin{tikzcd} \text{QCoh}(\text{Bun}_G) \arrow[r, "(\phi^{\text{glob}})^*"] & \text{QCoh}(\text{Bun}_{G'}) \\ \text{Rep}(\mathfrak{L}^+(G))_{\text{Ran}} \arrow[u, "\text{Loc}^{\text{QCoh}}_G"] \arrow[r, "\text{res}^\phi"] & \text{Rep}(\mathfrak{L}^+(G'))_{\text{Ran}} \arrow[u, "\text{Loc}^{\text{QCoh}}_{G'}"'] \end{tikzcd}$$

commutes tautologically.

Combining this observation with Corollary 12.1.10, we can express both

$$\mathbf{oblv}^l_\kappa \circ (\phi^{\mathrm{glob}})^!_\kappa \circ \mathrm{Loc}_{G,\kappa} \text{ and } \mathbf{oblv}^l_\kappa \circ \mathrm{Loc}_{G',\kappa} \circ \mathrm{res}^\phi$$

in terms of

$$\mathrm{Loc}^{\mathrm{QCoh}}_{G'} \circ \mathrm{res}^\phi \circ \mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}.$$

The two sides will not match on the nose, but the difference will be accounted for by Corollary 11.9.15.

12.4.3. By construction, we have a commutative diagram of lax unital local-to global functors

$$\begin{tikzcd}
(\mathbf{oblv}^l_{G',\kappa} \otimes \mathrm{Id}) \circ \underline{\mathrm{Loc}}_{G',\kappa} \circ \underline{\mathrm{res}}^\phi \arrow[r, "(\mathbf{oblv}^l_{\kappa,G'} \otimes \mathrm{Id})(12.18)"] & (\mathbf{oblv}^l_{G',\kappa} \otimes \mathrm{Id}) \circ (\phi^{\mathrm{glob}})^!_\kappa \circ \underline{\mathrm{Loc}}_{G,\kappa} \\
\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G'} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}}',\mathfrak{L}^+(G'))_\kappa}_{\mathfrak{L}^+(G')} \circ \underline{\mathrm{res}}^\phi \arrow[u, "(12.4)\circ \underline{\mathrm{res}}^\phi"] & \\
\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G'} \circ \underline{\mathrm{res}}^\phi \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} \arrow[u, "\sim"] & \\
(\phi^{\mathrm{glob}})^* \circ \underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} \arrow[u, "\sim"] \arrow[r, "(\phi^{\mathrm{glob}})^*(12.4)"] & (\phi^{\mathrm{glob}})^* \circ \mathbf{oblv}^l_{G,\kappa} \circ \underline{\mathrm{Loc}}_{G,\kappa}. \arrow[uuu, "\sim"']
\end{tikzcd}$$

By adjunction, we obtain a diagram

$$\begin{tikzcd}
(\mathbf{oblv}^l_{G',\kappa} \otimes \mathrm{Id}) \circ \left(\underline{\mathrm{Loc}}_{G',\kappa} \circ \underline{\mathrm{res}}^\phi\right)^{\int \mathrm{ins.vac}} \arrow[r, "(\mathbf{oblv}^l_{\kappa,G'} \otimes \mathrm{Id})(12.19)"] & (\mathbf{oblv}^l_{G',\kappa} \otimes \mathrm{Id}) \circ (\phi^{\mathrm{glob}})^!_\kappa \circ \underline{\mathrm{Loc}}_{G,\kappa} \\
\left((\mathbf{oblv}^l_{G',\kappa} \otimes \mathrm{Id}) \circ \underline{\mathrm{Loc}}_{G',\kappa} \circ \underline{\mathrm{res}}^\phi\right)^{\int \mathrm{ins.vac}} \arrow[u, "\sim"] \arrow[r] & (\mathbf{oblv}^l_{G',\kappa} \otimes \mathrm{Id}) \circ (\phi^{\mathrm{glob}})^!_\kappa \circ \underline{\mathrm{Loc}}_{G,\kappa} \arrow[u, "="'] \\
\left(\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G'} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}}',\mathfrak{L}^+(G'))_\kappa}_{\mathfrak{L}^+(G')} \circ \underline{\mathrm{res}}^\phi\right)^{\int \mathrm{ins.vac}} \arrow[u, "((12.4)\circ \underline{\mathrm{res}}^\phi)^{\int \mathrm{ins.vac}}"] & \\
\left(\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G'} \circ \underline{\mathrm{res}}^\phi \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}\right)^{\int \mathrm{ins.vac}} \arrow[u, "\sim"] & \\
\left((\phi^{\mathrm{glob}})^* \circ \underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}\right)^{\int \mathrm{ins.vac}} \arrow[u, "\sim"] \arrow[r] & (\phi^{\mathrm{glob}})^* \circ \mathbf{oblv}^l_{G,\kappa} \circ \underline{\mathrm{Loc}}_{G,\kappa} \arrow[uuu, "\sim"'] \\
(\phi^{\mathrm{glob}})^* \circ \left(\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}\right)^{\int \mathrm{ins.vac}} \arrow[u, "\sim"] \arrow[r, "(\phi^{\mathrm{glob}})^*(12.5)", "\sim"'] & (\phi^{\mathrm{glob}})^* \circ \mathbf{oblv}^l_{G,\kappa} \circ \underline{\mathrm{Loc}}_{G,\kappa}, \arrow[u, "="']
\end{tikzcd}$$

where the bottom arrow is an isomorphism by Theorem 12.1.5 (for G).

12.4.4. Hence, to prove that $(\mathbf{oblv}^l_{\kappa,G'} \otimes \mathrm{Id})(12.19)$ is an isomorphism, it suffices to show that the map

$$\left(\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G'} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}}',\mathfrak{L}^+(G'))_\kappa}_{\mathfrak{L}^+(G')} \circ \underline{\mathrm{res}}^\phi\right)^{\int \mathrm{ins.vac}} \overset{((12.4)\circ \underline{\mathrm{res}}^\phi)^{\int \mathrm{ins.vac}}}{\longrightarrow} \\ \to \left((\mathbf{oblv}^l_{G',\kappa} \otimes \mathrm{Id}) \circ \underline{\mathrm{Loc}}_{G',\kappa} \circ \underline{\mathrm{res}}^\phi\right)^{\int \mathrm{ins.vac}} \tag{12.23}$$

induced by

$$\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G'} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}}',\mathfrak{L}^+(G'))_\kappa}_{\mathfrak{L}^+(G')} \circ \underline{\mathrm{res}}^{\phi} \overset{(12.4)\circ\underline{\mathrm{res}}^{\phi}}{\longrightarrow} (\mathbf{oblv}^{l}_{G',\kappa} \otimes \mathrm{Id}) \circ \underline{\mathrm{Loc}}_{G',\kappa} \circ \underline{\mathrm{res}}^{\phi},$$

is an isomorphism.

Note that the map

$$\left(\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G'} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}}',\mathfrak{L}^+(G'))_\kappa}_{\mathfrak{L}^+(G')}\right)^{\int \mathrm{ins.vac}} \overset{(12.4)^{\int \mathrm{ins.vac}}}{\longrightarrow} \left((\mathbf{oblv}^{l}_{G',\kappa} \otimes \mathrm{Id}) \circ \underline{\mathrm{Loc}}_{G',\kappa}\right)^{\int \mathrm{ins.vac}},$$

induced by

$$\underline{\mathrm{Loc}}^{\mathrm{QCoh}}_{G'} \circ \underline{\mathbf{oblv}}^{(\widehat{\mathfrak{g}}',\mathfrak{L}^+(G'))_\kappa}_{\mathfrak{L}^+(G')} \overset{(12.4)}{\longrightarrow} (\mathbf{oblv}^{l}_{G',\kappa} \otimes \mathrm{Id}) \circ \underline{\mathrm{Loc}}_{G',\kappa},$$

is an isomorphism, by Theorem 12.1.5 (for G').

This implies that (12.23) is an isomorphism by Corollary 11.9.15.

$\square$[Proposition 12.3.6]

12.5. **Localization for unipotent group-schemes.** Let N' be a unipotent group-scheme over X. We will make the following technical assumption: N' admits a filtration by normal group-schemes with abelian subquotients.

12.5.1. Consider the factorization category

$$\mathrm{KL}(N') := \mathfrak{L}(\mathfrak{n}')\text{-mod}^{\mathfrak{L}^+(N')}.$$

Note that the critical level for N' is zero. In particular, we have the self-dualites

$$(\mathfrak{L}(\mathfrak{n}')\text{-mod})^\vee \simeq \mathfrak{L}(\mathfrak{n}')\text{-mod}. \tag{12.24}$$

$$\mathrm{KL}(N')^\vee \simeq \mathrm{KL}(N'). \tag{12.25}$$

Both dualities take place in the sense of unital factorization categories, see Sect. C.11.5.

12.5.2. Note that since $\mathfrak{L}^+(N')$ is pro-unipotent, the forgetful functor

$$\mathbf{oblv}_{\mathfrak{L}^+(N')} : \mathrm{KL}(N') \to \mathfrak{L}(\mathfrak{n}')\text{-mod}$$

is fully faithful.

Recall also that the right adjoint

$$\mathrm{Av}_*^{\mathfrak{L}^+(N')} : \mathfrak{L}(\mathfrak{n}')\text{-mod} \to \mathrm{KL}(N')$$

of $\mathbf{oblv}_{\mathfrak{L}^+(N')}$ identifies also with the dual of $\mathbf{oblv}_{\mathfrak{L}^+(N')}$ with respect to the self-dualities (12.24) and (12.25).

12.5.3. Consider the (factorization) functor of semi-infinite cohomology with respect to $\mathfrak{L}(\mathfrak{n}')$:

$$\mathrm{BRST}_{\mathfrak{n}'} : \mathfrak{L}(\mathfrak{n}')\text{-mod} \to \mathrm{Vect}\,.$$

In terms of the duality (12.24), the functor $\mathrm{BRST}_{\mathfrak{n}'}$ is given by

$$\langle k, - \rangle_{\mathfrak{L}(\mathfrak{n}')\text{-mod}},$$

where:

- $k \in \mathrm{KL}_{N'} \subset \mathfrak{L}(\mathfrak{n}')$-mod is the *trivial* representation;
- $\langle -, - \rangle_{\mathfrak{L}(\mathfrak{n}')\text{-mod}}$ denotes the pairing $\mathfrak{L}(\mathfrak{n}')\text{-mod} \otimes \mathfrak{L}(\mathfrak{n}')\text{-mod} \to \mathrm{Vect}$, corresponding to (12.24).

Since k upgrades to an object of $\mathrm{FactAlg}^{\mathrm{untl}}(X, \mathfrak{L}(\mathfrak{n}')\text{-mod})$, the functor $\mathrm{BRST}_{\mathfrak{n}'}$ has a natural lax unital factorization structure.

12.5.4. The value of $\mathrm{BRST}_{\mathfrak{n}'}$ on $\mathrm{Vac}(N')$ is

$$\langle k, \mathrm{Vac}(N')\rangle_{\mathfrak{L}(\mathfrak{n}')\text{-mod}} \simeq \langle \mathrm{Av}_*^{\mathfrak{L}^+(N')}(k), \mathrm{Vac}(N')\rangle_{\mathrm{KL}(N')} \simeq \\ \simeq \langle k, \mathrm{Vac}(N')\rangle_{\mathrm{KL}(N')} \simeq \langle k, \mathbf{1}_{\mathrm{Rep}(\mathfrak{L}^+(N'))}\rangle_{\mathrm{Rep}(\mathfrak{L}^+(N'))}, \tag{12.26}$$

where:

- $\langle -, -\rangle_{\mathrm{KL}(N')}$ is the pairing $\mathrm{KL}(N') \otimes \mathrm{KL}(N') \to \mathrm{Vect}$ corresponding to (12.25);
- $\mathbf{1}_{\mathrm{Rep}(\mathfrak{L}^+(N'))} \in \mathrm{Rep}(\mathfrak{L}^+(N'))$ is the *trivial* representation;
- $\langle -, -\rangle_{\mathrm{Rep}(\mathfrak{L}^+(N'))}$ denotes pairing $\mathrm{Rep}(\mathfrak{L}^+(N')) \otimes \mathrm{Rep}(\mathfrak{L}^+(N')) \to \mathrm{Vect}$, corresponding to the natural self-duality of $\mathrm{Rep}(\mathfrak{L}^+(N'))$, i.e.,
$$\langle V_1, V_2\rangle_{\mathrm{Rep}(\mathfrak{L}^+(N'))} = \mathcal{H}om_{\mathrm{Rep}(\mathfrak{L}^+(N'))}(\mathbf{1}_{\mathrm{Rep}(\mathfrak{L}^+(N'))}, V_1 \otimes V_2).$$

Denote
$$\Omega(\mathfrak{n}') := \mathrm{inv}_{\mathfrak{L}^+(N')}(\mathbf{1}_{\mathrm{Rep}(\mathfrak{L}^+(N'))}) \simeq \mathrm{C}^\cdot(\mathfrak{L}^+(\mathfrak{n}')) \in \mathrm{FactAlg}^{\mathrm{untl}}(X).$$

Remark 12.5.5. Note also that $\Omega(\mathfrak{n}')$ is the commutative factorization algebra canonically isomorphic to
$$\mathrm{Fact}(\mathrm{C}^\cdot_{\mathrm{chev}}(L_{\mathfrak{n}'})),$$
where:

- $L_{\mathfrak{n}'}$ is the Lie-* algebra $\mathfrak{n}' \otimes \mathrm{D}_X$;
- For a Lie-* algebra L, we denote by $\mathrm{C}^\cdot_{\mathrm{chev}}(L) \in \mathrm{ComAlg}(\mathrm{D}\text{-mod}(X))$ its cohomological Chevalley complex, see [BD2, Sect. 1.4.10].

12.5.6. Hence, from (12.26), we obtain
$$\mathrm{BRST}_{\mathfrak{n}'}(\mathrm{Vac}(N')) \simeq \Omega(\mathfrak{n}'), \tag{12.27}$$
as factorization algebras (in Vect).

In particular, the functor $\mathrm{BRST}_{\mathfrak{n}'}$ enhances to a functor
$$\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}'} : \mathfrak{L}(\mathfrak{n}')\text{-mod} \to \Omega(\mathfrak{n}')\text{-mod}^{\mathrm{fact}}.$$

By a slight abuse of notation, we will denote by the same symbols $\mathrm{BRST}_{\mathfrak{n}'}$ and $\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}'}$ the restrictions of the above functors along
$$\mathrm{KL}(N') \to \mathfrak{L}(\mathfrak{n}')\text{-mod}.$$

12.5.7. Let $\delta_{N'}$ denote the integer $\dim(\mathrm{Bun}_{N'})$.

Note that the canonical line bundle $K_{\mathrm{Bun}_{N'}}$ of $\mathrm{Bun}_{N'}$, i.e.,
$$\det(T^*(\mathrm{Bun}_{N'})),$$
is canonically constant. Let $\mathfrak{l}_{N'}$ denote the corresponding (ungraded) line.

The material in Sect. 10 applies as-is to the group scheme N' over X, so we can consider the localization functor
$$\mathrm{Loc}_{N'} : \mathrm{KL}(N')_{\mathrm{Ran}} \to \mathrm{D}\text{-mod}(\mathrm{Bun}_{N'}).$$

Note also that since N' is unipotent, the stack $\mathrm{Bun}_{N'}$ is quasi-compact. In particular, there is no difference between $\mathrm{D}\text{-mod}(\mathrm{Bun}_{N'})$ and $\mathrm{D}\text{-mod}_{\mathrm{co}}(\mathrm{Bun}_{N'})$. Moreover, $\mathrm{Bun}_{N'}$ is *safe* in the sense of [DG1, Sect. 10.2] so the "constant sheaf"
$$\underline{k} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_{N'})$$
is compact, and the functor
$$\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Bun}_{N'}, -) = \mathcal{H}om_{\mathrm{D}\text{-mod}(\mathrm{Bun}_{N'})}(\underline{k}, -)$$
is continuous.

12.5.8. Our next goal is to construct a natural transformation from the composition

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{BRST}_{\mathfrak{n}'}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}) \overset{\mathrm{C}^{\cdot}_c(\mathrm{Ran},-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N'}[\delta_{N'}]}{\longrightarrow} \mathrm{Vect} \tag{12.28}$$

to

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{Loc}_{N'}}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N'}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'},-)}{\longrightarrow} \mathrm{Vect}, \tag{12.29}$$

i.e.,

$$(\mathrm{C}^{\cdot}_c(\mathrm{Ran},-)\circ \mathrm{BRST}_{\mathfrak{n}'})\otimes \mathfrak{l}_{N'}[\delta_{N'}] \to \mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'},-)\circ \mathrm{Loc}_{N'}. \tag{12.30}$$

12.5.9. Let us interpret $\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'},-)\circ \mathrm{Loc}_{N'}$ as

$$\langle \omega_{\mathrm{Bun}_{N'}}, \mathrm{Loc}_{N'}(-)\rangle_{\mathrm{Bun}_{N'}}, \tag{12.31}$$

where

$$\langle -,-\rangle_{\mathrm{Bun}_{N'}} : \text{D-mod}(\mathrm{Bun}_{N'})\otimes \text{D-mod}(\mathrm{Bun}_{N'}) \to \mathrm{Vect}$$

is the Verdier duality pairing.

Using Proposition 10.5.7, we rewrite (12.31) as

$$\langle \Gamma_{N',\mathrm{Ran}}(\omega_{\mathrm{Bun}_{N'}}),-\rangle_{\mathrm{KL}(N')_{\mathrm{Ran}}}\otimes \mathfrak{l}_{N'}[\delta_{N'}], \tag{12.32}$$

where

$$\langle -,-\rangle_{\mathrm{KL}(N')_{\mathrm{Ran}}}$$

is the self-duality on $\mathrm{KL}_{N',\mathrm{Ran}}$ induced by (12.25).

12.5.10. Note now that for any $\underline{x}\in \mathrm{Ran}$

$$\Gamma^{\mathrm{ren}}(\mathrm{Bun}_{N'}^{\mathrm{level}_{\underline{x}}}, \omega_{\mathrm{Bun}_{N'}})$$

(see Equation (10.8) for the definition of $\Gamma^{\mathrm{ren}}(\mathrm{Bun}_{N'}^{\mathrm{level}_{\underline{x}}},-)$) receives a map from

$$\Gamma(\mathrm{Bun}_{N'}, \mathbf{oblv}^l(\omega_{\mathrm{Bun}_{N'}}))\simeq \Gamma(\mathrm{Bun}_{N'}, \mathcal{O}_{\mathrm{Bun}_{N'}}),$$

and hence from

$$k\to \Gamma(\mathrm{Bun}_{N'}, \mathcal{O}_{\mathrm{Bun}_{N'}}).$$

Furthermore, it is easy to see that the resulting map

$$k\to \Gamma^{\mathrm{ren}}(\mathrm{Bun}_{N'}^{\mathrm{level}_{\underline{x}}}, \omega_{\mathrm{Bun}_{N'}})$$

in Vect upgrades to a map

$$k\to \Gamma^{\mathrm{ren}}(\mathrm{Bun}_{N'}^{\mathrm{level}_{\underline{x}}}, \omega_{\mathrm{Bun}_{N'}})^{\mathrm{enh}} = \Gamma_{N',\underline{x}}(\omega_{\mathrm{Bun}_{N'}}).$$

in $\mathrm{KL}(N')_{\underline{x}}$.

Making $\underline{x}$ move in families over Ran, we obtain a map

$$k_{\mathrm{Ran}}\to \Gamma_{N',\mathrm{Ran}}(\omega_{\mathrm{Bun}_{N'}}).$$

12.5.11. Thus, we obtain a map

$$\langle k_{\mathrm{Ran}},-\rangle_{\mathrm{KL}(N'),\mathrm{Ran}}\otimes \mathfrak{l}_{N'}[\delta_{N'}] \to \mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'},-)\circ \mathrm{Loc}_{N'}. \tag{12.33}$$

Finally, we note that the functor

$$\langle k_{\mathrm{Ran}},-\rangle_{\mathrm{KL}(N'),\mathrm{Ran}} : \mathrm{KL}(N')_{\mathrm{Ran}}\to \mathrm{Vect}$$

identifies with

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{BRST}_{\mathfrak{n}'}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}) \overset{\mathrm{C}^{\cdot}_c(\mathrm{Ran},-)}{\longrightarrow} \mathrm{Vect}.$$

Combining with (12.33) we obtain the desired map (12.30).

12.5.12. We will prove:[52]

Theorem 12.5.13. *The natural transformation* (12.30) *becomes an isomorphism after precomposing with the endofunctor*

$$\int \mathrm{ins.\,vac} : \mathrm{KL}(N')_{\mathrm{Ran}} \to \mathrm{KL}(N')_{\mathrm{Ran}}.$$

12.5.14. Let us reformulate Theorem 12.5.13 in more concrete terms. Note that we can rewrite the precomposition of (12.28) with $\int \mathrm{ins.\,vac}$ as

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}'}}{\longrightarrow} \Omega(\mathfrak{n}')\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\Omega(\mathfrak{n}'),-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N'}[\delta_{N'}]}{\longrightarrow} \mathrm{Vect}\,. \tag{12.34}$$

Further, since the functor $\mathrm{Loc}_{N'}$ has a unital structure, the precomposition of (12.29) with $\int \mathrm{ins.\,vac}$ is canonically isomorphic to (12.29) itself.

Hence, Theorem 12.5.13 implies:

Corollary 12.5.15. *There exists a canonical isomorphism between* (12.34) *and the functor* (12.29).

12.5.16. *Variant.* Consider the Lie algebra $\Gamma(X,\mathfrak{n}')$, and let χ^{Lie} be its character. The datum of χ^{Lie} gives rise to a factorization character

$$\chi : \mathfrak{L}(\mathfrak{n}') \to \mathbb{G}_a,$$

trivial in $\mathfrak{L}^+(\mathfrak{n}')$, and a map

$$\chi^{\mathrm{glob}} : \mathrm{Bun}_{N'} \to \mathbb{G}_a.$$

Let $\mathrm{BRST}_{\mathfrak{n}',\chi}$ be the χ-twisted version of the semi-infinite cohomology functor, i.e.

$$\mathrm{BRST}_{\mathfrak{n}',\chi}(-) = \mathrm{BRST}_{\mathfrak{n}'}(-\otimes \chi).$$

Note that

$$\mathrm{Vac}(N')\otimes \chi \simeq \mathrm{Vac}(N').$$

Hence,

$$\mathrm{BRST}_{\mathfrak{n}',\chi}(\mathrm{Vac}(N')) \simeq \mathrm{BRST}_{\mathfrak{n}'}(\mathrm{Vac}(N')) \simeq \Omega(\mathfrak{n}')$$

as factorization algebras.

Let $\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}',\chi}$ be the enhancement of $\mathrm{BRST}_{\mathfrak{n}',\chi}$

$$\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}',\chi} : \mathfrak{L}(\mathfrak{n}')\text{-mod} \to \Omega(\mathfrak{n}')\text{-mod}^{\mathrm{fact}}.$$

12.5.17. As in (12.30) one constructs a natural transformation

$$(\mathrm{C}^{\cdot}_c(\mathrm{Ran},-)\circ \mathrm{BRST}_{\mathfrak{n}',\chi})\otimes \mathfrak{l}_{N'}[\delta_{N'}] \to \mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'}, -\overset{*}{\otimes}\chi^*(\exp))\circ \mathrm{Loc}_{N'}\,. \tag{12.35}$$

And parallel to Theorem 12.5.13, we have:

Theorem 12.5.18. *The map* (12.35) *becomes an isomorphism after precomposing with*

$$\int \mathrm{ins.\,vac} : \mathrm{KL}(N')_{\mathrm{Ran}} \to \mathrm{KL}(N')_{\mathrm{Ran}}.$$

Corollary 12.5.19. *There exists a canonical isomorphism between*

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}',\chi}}{\longrightarrow} \Omega(\mathfrak{n}')\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\Omega(\mathfrak{n}'),-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N'}[\delta_{N'}]}{\longrightarrow} \mathrm{Vect} \tag{12.36}$$

and

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{Loc}_{N'}}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N'}) \overset{-\overset{*}{\otimes}\chi^*(\exp)}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N'}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'},-)}{\longrightarrow} \mathrm{Vect}, \tag{12.37}$$

12.6. **Proof of Theorem 12.5.13.**

[52]This result is established in [CF, Theorem 4.0.5(4)]. We will provide proof for completeness.

12.6.1. Since $\mathfrak{L}^+(N')$ is pro-unipotent, the category $\mathrm{Rep}(\mathfrak{L}^+(G))$ is generated by $\mathbf{1}_{\mathrm{Rep}(\mathfrak{L}^+(N'))}$. And hence the category $\mathrm{KL}(N')$ is generated by $\mathrm{Vac}(N')$. Therefore, $\mathrm{KL}(N')_{\mathrm{Ran}}$ is generated by $\mathrm{Vac}(N')_{\mathrm{Ran}}$ as a D-mod(Ran)-module category.

By unitality, for both sides in Theorem 12.5.13, tensoring the source by an object $\mathcal{F} \in \text{D-mod}(\mathrm{Ran})$ has the effect of tensoring the target by $\mathrm{C}^{\cdot}_c(\mathrm{Ran}, \mathcal{F})$. Hence, it is enough to show that the map in Theorem 12.5.13 evaluates to an isomorphism on $\mathrm{Vac}(N')_{\mathrm{Ran}}$.

12.6.2. By construction, the left-hand side is

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \Omega(\mathfrak{n}'))$$

(see Sect. 11.9.7 for the notation).

By (10.28), we have

$$\mathrm{Loc}_{N'}(\mathrm{Vac}(N)_{\mathrm{Ran}}) \simeq \mathbf{ind}^l(\mathcal{O}_{\mathrm{Bun}_{N'}}).$$

We rewrite

$$\mathbf{ind}^l(\mathcal{O}_{\mathrm{Bun}_{N'}}) \simeq \mathbf{ind}^r(\omega_{\mathrm{Bun}_{N'}}) \simeq \mathbf{ind}^r(\mathcal{O}_{\mathrm{Bun}_{N'}}) \otimes \mathfrak{l}_{N'}[\delta_{N'}].$$

Hence, the map in Theorem 12.5.13 becomes a map

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \Omega(\mathfrak{n}')) \to \Gamma(\mathrm{Bun}_{N'}, \mathcal{O}_{\mathrm{Bun}_{N'}}). \tag{12.38}$$

The fact that (12.38) is an isomorphism is well-known. For completeness, we will supply a proof in the next subsection.

12.6.3. The material in the rest of this subsection is not logically necessary, except for the example considered in Sect. 12.6.7.

According to Remark 12.5.5, the factorization algebra $\Omega(\mathfrak{n}')$ can be thought of as the factorization algebra associated to the cohomological Chevalley complex of a Lie-* algebra. Let us consider $\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \Omega(\mathfrak{n}'))$ in this paradigm.

Let L be a Lie-* algebra, whose underlying D-module is classical, finitely generated and projective (as a D-module). Consider its cohomological Chevalley complex

$$\mathrm{C}^{\cdot}_{\mathrm{chev}}(L) \in \mathrm{ComAlg}(\text{D-mod}(X)).$$

Denote

$$\Omega(L) := \mathrm{Fact}(\mathrm{C}^{\cdot}_{\mathrm{chev}}(L)) \in \mathrm{ComAlg}(\mathrm{FactAlg}(X)),$$

and consider

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \Omega(L)) \in \mathrm{ComAlg}(\mathrm{Vect}).$$

Unfortunately, we do not have a good grip on what $\mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \Omega(L))$ looks like.

12.6.4. Note that $\mathrm{C}^{\cdot}_{\mathrm{chev}}(L)$ is naturally written as

$$\mathrm{C}^{\cdot}_{\mathrm{chev}}(L) \simeq \lim_n \mathrm{C}^{\cdot}_{\mathrm{chev}}(L)_n, \tag{12.39}$$

where $\mathrm{C}^{\cdot}_{\mathrm{chev}}(L)_n \in \mathrm{ComAlg}(\text{D-mod}(X))$ is the n-step cohomological Chevalley complex.

Note, however, that the assumptions on L imply that for every n, the composite map

$$\tau^{\leq n}(\mathrm{C}^{\cdot}(L)) \to \mathrm{C}^{\cdot}(L) \to \mathrm{C}^{\cdot}(L)_n$$

is an isomorphism (here $\tau^{\leq n}$ refers to the left D-module structure, i.e., one for which $\mathbf{oblv}^l$ is t-exact). So, in the formation $\mathrm{C}^{\cdot}_{\mathrm{ch}}(L)$ no "actual completion" is involved.

12.6.5. Denote

$$\Omega(L)_n := \mathrm{Fact}(\mathrm{C}^\cdot_{\mathrm{chev}}(L)_n) \in \mathrm{ComAlg}(\mathrm{FactAlg}(X)).$$

Set:

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))^\wedge := \lim_n \mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L)_n).$$

Unlike $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))$, the algebra $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))^\wedge$ can be described explicitly. Namely, according to [BD2, Proposition 7.4.1],

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))^\wedge \simeq \mathrm{C}^\cdot_{\mathrm{chev}}(\mathrm{C}^\cdot_{\mathrm{dR}}(X, L)) = \mathrm{C}_{\mathrm{chev},\cdot}(\mathrm{C}^\cdot_{\mathrm{dR}}(X, L))^\vee,$$

where:

- $\mathrm{C}^\cdot_{\mathrm{dR}}(X, L)$ is considered as a Lie algebra (in Vect);
- $\mathrm{C}^\cdot_{\mathrm{chev}}(-)$ and $\mathrm{C}_{\mathrm{chev},\cdot}(-)$ are the cohomological and homological Chevalley complexes of a Lie algebra, respectively.

12.6.6. The map $\to$ in (12.39) gives rise to a map

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L)) \to \mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))^\wedge \tag{12.40}$$

but this map is in general *not* an isomorphism.

12.6.7. *Example.* Let L be abelian. Denote $L^\vee := \mathbb{D}(L)[-1]$. Then

$$\mathrm{C}^\cdot_{\mathrm{chev}}(L) \simeq \mathrm{Sym}^!(\mathbb{D}(L)),$$

and hence

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L)) \simeq \mathrm{Sym}(\mathrm{C}^\cdot_{\mathrm{dR}}(X, \mathbb{D}(L))) \simeq \mathrm{Sym}(\mathrm{C}^\cdot_{\mathrm{dR}}(X, L)^\vee[-1]).$$

By contrast,

$$\mathrm{C}^\cdot_{\mathrm{chev}}(\mathrm{C}^\cdot_{\mathrm{dR}}(X, L)) \simeq \mathrm{Sym}(\mathrm{C}^\cdot_{\mathrm{dR}}(X, L)[-1])^\vee.$$

So the difference between the two sides in (12.40) in this case is that between a polynomial algebra and its completion.

12.6.8. The map (12.40) is the best approximation to $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))$ that we have in general. In certain situations, it allows us to recover $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))$ completely.

This happens, for example, if L carries a strictly positive grading. In this case, the map (12.40) defines an isomorphism on each graded component. I.e., we have

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))^d \simeq (\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))^\wedge)^d \simeq \mathrm{C}^\cdot_{\mathrm{chev}}(\mathrm{C}^\cdot_{\mathrm{dR}}(X, L))^d \simeq (\mathrm{C}_{\mathrm{chev},\cdot}(\mathrm{C}^\cdot_{\mathrm{dR}}(X, L))^{-d})^\vee, \quad d \in \mathbb{Z}^{<0}.$$

Note, however, that $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))^\wedge$ is *not* the direct sum of its graded pieces. Rather,

$$(\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))^\wedge)^d = \lim_n \mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L)_n)^d.$$

Remark 12.6.9. Using cohomological truncations on powers of X one can show:

(i) $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))$ is coconnective.

(ii) Suppose that L is such that $H^0(\mathrm{C}^\cdot_{\mathrm{dR}}(X, L)) = 0$. Then $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(L))$ is classical.

12.7. **Global functions on the moduli space of bundles for a unipotent group-scheme.**

12.7.1. Let $\mathcal{Y}$ be a D-prestack over X. Let $A \in \mathrm{ComAlg}(\text{D-mod}(X))$ be the algebra of global functions on $\mathcal{Y}$, and let $\mathcal{A} = \mathrm{Fact}(A) \in \mathrm{ComAlg}(\mathrm{FactAlg}(X))$ be the corresponding factorization algebra (see Sect. B.10.2).

Recall that by (F.3), the evaluation map

$$\mathrm{Sect}_\nabla(X, \mathcal{Y}) \times X \to \mathcal{Y}$$

gives rise to a map

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \to \Gamma(\mathrm{Sect}_\nabla(X, \mathcal{Y}), \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}), \tag{12.41}$$

Unwinding the construction, it is easy to see that the map (12.38) is the map (12.41) for $\mathcal{Y} = \mathrm{Jets}(\mathrm{pt}/N')$, where

$$\Omega(\mathfrak{n}') = \mathcal{O}_{\mathrm{pt}/\mathfrak{L}^+(N')} \overset{(C.44)}{\simeq} \mathrm{Fact}(\mathcal{O}_{\mathrm{Jets}(\mathrm{pt}/N')})$$

and

$$\mathrm{Sect}_\nabla(X, \mathrm{Jets}(\mathrm{pt}/N')) \simeq \mathrm{Sect}(X, \mathrm{pt}/N') \simeq \mathrm{Bun}_{N'}.$$

Thus, we need to show that (12.41) is an isomorphism in this case.

Remark 12.7.2. Recall that (12.41) is an isomorphism for $\mathcal{Y}$ that is affine over X, see Proposition F.1.4. So we want to prove that $\mathcal{Y} = \mathrm{Jets}(\mathrm{pt}/N')$ is not too different from the affine case.

The proof below follows closely that of Proposition F.1.4.

12.7.3. First, we will show that the map (12.41) induces an *isomorphism*

$$\begin{aligned} &\mathrm{Maps}(\mathrm{Spec}(R), \mathrm{Sect}_\nabla(X, \mathrm{Jets}(\mathrm{pt}/N'))) \to \\ &\to \mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(\Gamma(\mathrm{Sect}_\nabla(X, \mathrm{Jets}(\mathrm{pt}/N')), \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathrm{Jets}(\mathrm{pt}/N'))}), R) \to \\ &\to \mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{O}_{\mathrm{pt}/\mathfrak{L}^+(N')}), R) \end{aligned} \tag{12.42}$$

for $R \in \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$.

This essentially follows from the fact that $\mathrm{Jets}(N')$ is pro-unipotent, and hence $\mathrm{Jets}(\mathrm{pt}/N')$ is *as good as affine*[53] (see Sect. 12.7.5 below for what this means), i.e., for $\mathcal{R} \in \mathrm{ComAlg}(\text{D-mod}(X)^{\leq 0})$, the map

$$\mathrm{Maps}_{X,\nabla}(\mathrm{Spec}_X(\mathcal{R}), \mathrm{Jets}(\mathrm{pt}/N')) \to \mathrm{Maps}_{\mathrm{ComAlg}(\text{D-mod}(X))}(\mathcal{O}_{\mathrm{Jets}(\mathrm{pt}/N')}, \mathcal{R}) \tag{12.43}$$

is an isomorphism.

12.7.4. In more detail, for $R \in \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$ we have:

$$\begin{aligned} &\mathrm{Maps}(\mathrm{Spec}(R), \mathrm{Sect}_\nabla(X, \mathrm{Jets}(\mathrm{pt}/N'))) = \mathrm{Maps}_{X,\nabla}(\mathrm{Spec}(R) \times X, \mathrm{Jets}(\mathrm{pt}/N'))) \overset{(12.43)}{\simeq} \\ &\simeq \mathrm{Maps}_{\mathrm{ComAlg}(\text{D-mod}(X))}(\mathcal{O}_{\mathrm{Jets}(\mathrm{pt}/N')}, R \otimes \mathcal{O}_X). \end{aligned} \tag{12.44}$$

Using Corollary C.9.5, we rewrite the expression in the right-hand side in (12.44) as

$$\mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{O}_{\mathrm{pt}/\mathfrak{L}^+(N')}), R). \tag{12.45}$$

Combining (12.44) and (12.45), we obtain an isomorphism

$$\mathrm{Maps}(\mathrm{Spec}(R), \mathrm{Sect}_\nabla(X, \mathrm{Jets}(\mathrm{pt}/N'))) \simeq \mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{O}_{\mathrm{pt}/\mathfrak{L}^+(N')}), R),$$

and unwinding the definitions we obtain that this isomorphism equals the map in (12.42).

12.7.5. Let us call a prestack $\mathcal{Z}$ *as good as affine* if the functor

$$\Gamma(\mathcal{Z}, -) : \mathrm{QCoh}(\mathcal{Z}) \to \mathcal{O}_{\mathcal{Z}}\text{-mod}$$

is an equivalence.

[53] Up to issues of renormalization, which are irrelevant here.

12.7.6. Let R be an object of $\mathrm{ComAlg}(\mathrm{Vect})$. Define the prestack "$\mathrm{Spec}(R)$" by

$$\mathrm{Maps}(\mathrm{Spec}(R'), \text{“}\mathrm{Spec}(R)\text{”}) := \mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(R, R'), \quad R' \in \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}).$$

Note the formation of "$\mathrm{Spec}(R)$" is functorial in R. In particular, we obtain a map

$$R \simeq \mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(k[t], R) \to \mathrm{Maps}(\text{“}\mathrm{Spec}(R)\text{”}, \mathrm{Spec}(k[t])) \simeq \\ \simeq \mathrm{Maps}(\text{“}\mathrm{Spec}(R)\text{”}, \mathbb{A}^1) \simeq \Gamma(\text{“}\mathrm{Spec}(R)\text{”}, \mathcal{O}_{\text{“}\mathrm{Spec}(R)\text{”}}). \tag{12.46}$$

We shall say that R is"as good as connective" if:

- The prestack "$\mathrm{Spec}(R)$" is as good as affine;
- The map (12.46) is an isomorphism.

12.7.7. Thus, given that (12.42) is an isomorphism, we need to show that $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}'))$ is as good as connective.

We will now use the assumption that N' admits a filtration by normal subgroups with abelian quotient. We will argue by induction on the length of such a filtration.

12.7.8. We first consider the base of the induction, i.e., case when N' is a vector group-scheme, i.e., is the total space of a vector bundle $\mathcal{E}$ on X. In this case, the computation of $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}'))$ has been performed in Sect. 12.6.7.

We obtain that $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}'))$ is (non-canonically) isomorphic to the tensor product

$$\mathrm{Sym}(V_1) \otimes \mathrm{Sym}(V_2[-1]),$$

where V_1 and V_2 are classical finite-dimensional vector spaces.

It is clear that the tensor product of two algebras both of which are as good as connective is itself as good as connective. Hence, it remains to see that algebras of the form $\mathrm{Sym}(V[-1])$, where V is a classical finite-dimensional vector space, are as good as connective.

However, this is well-known: in this case

$$\text{“}\mathrm{Spec}(\mathrm{Sym}(V[-1]))\text{”} \simeq \mathrm{pt}/V^\vee$$

and the assertion is manifest.

12.7.9. We now perform the induction step. Thus, we fix a short exact sequence

$$1 \to N'_2 \to N' \to N'_1 \to 1,$$

where N'_2 is a vector group-scheme.

We observe:

Lemma 12.7.10. *Let $R_1 \to R$ be a map of commutative algebras in* Vect. *Assume that:*

- *R_1 is as good as connective;*
- *For any homomorphism $R_1 \to R'$ with R' connective, the base change $R' \underset{R_1}{\otimes} R$ is as good as connective.*

Then R is as good as connective.

We apply this lemma to

$$R := \mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}')) \text{ and } R_1 := \mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}'_1)).$$

By the induction hypothesis, $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}'_1))$ is as good as connective. Hence, it remains to show that for any connective R' and a homomorphism

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}'_1)) \to R',$$

the algebra

$$R' \underset{\mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}'_1))}{\otimes} \mathrm{C}^{\mathrm{fact}}_\cdot(X, \Omega(\mathfrak{n}'))$$

is as good as connective.

12.7.11. We now apply Lemma C.9.14, and hence we can rewrite

$$R' \underset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\Omega(\mathfrak{n}'_1))}{\otimes} \mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\Omega(\mathfrak{n}')) \simeq \mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\Omega(\mathfrak{n}')_{R'}),$$

where

$$\Omega(\mathfrak{n}')_{R'} := \Omega(\mathfrak{n}') \underset{\Omega(\mathfrak{n}'_1)}{\otimes} (R' \otimes \omega_{\mathrm{Ran}}) \in \mathrm{ComAlg}(\mathrm{FactAlg}(X) \otimes R'\text{-mod}).$$

12.7.12. Recall that

$$\Omega(\mathfrak{n}') \simeq \mathrm{Fact}(\mathrm{C}^{\cdot}_{\mathrm{chev}}(L_{\mathfrak{n}'})) \text{ and } \Omega(\mathfrak{n}'_1) \simeq \mathrm{Fact}(\mathrm{Chev}^{\cdot}_{\mathrm{chev}}(L_{\mathfrak{n}'_1})),$$

where

$$L_{\mathfrak{n}'} = \mathfrak{n}' \otimes \mathrm{D}_X, \quad L_{\mathfrak{n}'_1} = \mathfrak{n}'_1 \otimes \mathrm{D}_X\,.$$

Hence, we can rewrite

$$\Omega(\mathfrak{n}')_{R'} \simeq \mathrm{Fact}\left(\mathrm{C}^{\cdot}_{\mathrm{chev}}(L_{\mathfrak{n}'}) \underset{\mathrm{C}^{\cdot}_{\mathrm{chev}}(L_{\mathfrak{n}'_1})}{\otimes} (R' \otimes \mathcal{O}_X)\right).$$

12.7.13. We can interpret the datum of $\mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\Omega(\mathfrak{n}'_1)) \to R'$ as a map

$$\mathrm{Spec}(R') \to \mathrm{Bun}_{N'_1}\,.$$

The adjoint action of N'_1 on N'_2 gives rise to an R'-family of twisted forms of N'_2, denoted $N'_{2,R'}$. Consider the corresponding R'-family of Lie-* algebras

$$L_{\mathfrak{n}'_{2,R'}} := \mathfrak{n}'_{2,R'} \otimes \mathrm{D}_X\,.$$

We have

$$\mathrm{C}^{\cdot}_{\mathrm{chev}}(L_{\mathfrak{n}'}) \underset{\mathrm{C}^{\cdot}_{\mathrm{chev}}(L_{\mathfrak{n}'_1})}{\otimes} (R' \otimes \mathcal{O}_X) \simeq \mathrm{C}^{\cdot}_{\mathrm{chev}}(L_{\mathfrak{n}'_{2,R'}}).$$

Thus, we can consider

$$\Omega(\mathfrak{n}'_{2,R'}) \in \mathrm{ComAlg}(\mathrm{FactAlg}(X) \otimes R'\text{-mod})$$

and we obtain:

$$\Omega(\mathfrak{n}')_{R'} \simeq \mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\Omega(\mathfrak{n}'_{2,R'})).$$

12.7.14. It remains to show that $\mathrm{C}^{\mathrm{fact}}_{\cdot}(X,\Omega(\mathfrak{n}'_{2,R'}))$ is as good as connective.

Recall that $\mathfrak{n}'_2$ is abelian. Hence, $N'_{2,R'}$ is R'-family of vector group-schemes. Hence, the required assertion is a relative (over $\mathrm{Spec}(R')$) version of the case considered in Sect. 12.7.8.

$\square$[Theorem 12.5.13]

12.8. Application: integration over (twists of) Bun_N via BRST.

12.8.1. Let $\mathcal{P}_T$ be a T-bundle on X. Consider the unipotent group-scheme $N_{\mathcal{P}_T}$. Denote the corresponding moduli stack $\mathrm{Bun}_{N_{\mathcal{P}_T}}$; note that it identifies with $\mathrm{Bun}_{N,\mathcal{P}_T}$ (see (9.3)).

The resulting map

$$\mathrm{Bun}_{N_{\mathcal{P}_T}} \simeq \mathrm{Bun}_{N,\mathcal{P}_T} \xrightarrow{\mathfrak{p}} \mathrm{Bun}_G$$

can be thought of as

$$\mathrm{Bun}_{N_{\mathcal{P}_T}} \to \mathrm{Bun}_{G_{\mathcal{P}_T}} \xrightarrow{\alpha^{-1}_{\mathcal{P}_T,\mathrm{taut}}} \mathrm{Bun}_G\,.$$

12.8.2. Since the restriction of κ to $\mathfrak{n}$ is trivial, we obtain that the restriction of the twisting $\mathcal{T}_\kappa$ along $\mathfrak{p}$ is canonically trivial. In particular, we have a well-defined functor

$$\mathfrak{p}^!_\kappa : \text{D-mod}_\kappa(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}). \tag{12.47}$$

12.8.3. Note that the embedding

$$N_{\mathcal{P}_T} \to G_{\mathcal{P}_T}$$

gives rise to a map

$$\mathfrak{L}(\mathfrak{n}_{\mathcal{P}_T}) \to \widehat{\mathfrak{g}}_{\kappa,\mathcal{P}_T},$$

and this map lifts to the Kac-Moody extension.

In particular, we obtain a well-defined restriction functor

$$\mathrm{KL}(G)_\kappa \overset{\alpha_{\mathcal{P}_T,\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T} \to \mathrm{KL}(N_{\mathcal{P}_T}).$$

Denote

$$\Omega(\mathfrak{n}_{\mathcal{P}_T}, \mathfrak{g})_\kappa := \mathrm{BRST}_{\mathfrak{n}_{\mathcal{P}_T}}(\mathrm{Vac}(G)_{\kappa,\mathcal{P}_T}).$$

This is a factorization algebra, which receives a homomorphism from $\Omega(\mathfrak{n}_{\mathcal{P}_T})$.

Thus, the composition

$$\mathrm{KL}(G)_\kappa \overset{\alpha_{\mathcal{P}_T,\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T} \to \mathrm{KL}(N_{\mathcal{P}_T}) \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T})\text{-mod}^{\mathrm{fact}}$$

further enhances to a (factorization) functor

$$\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T}} : \mathrm{KL}(G)_\kappa \to \Omega(\mathfrak{n}_{\mathcal{P}_T}, \mathfrak{g})_\kappa\text{-mod}^{\mathrm{fact}}.$$

12.8.4. We are going to prove:

Theorem 12.8.5. *The composition*

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\kappa}}{\longrightarrow} \text{D-mod}_\kappa(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_\kappa}{\to} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Bun}_{N,\mathcal{P}_T},-)}{\longrightarrow} \mathrm{Vect} \tag{12.48}$$

identifies with the functor

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T}, \mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})_\kappa,-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N_{\mathcal{P}_T}}[\delta_{N_{\mathcal{P}_T}}]}{\longrightarrow} \mathrm{Vect}, \tag{12.49}$$

where the notations $\delta_{N_{\mathcal{P}_T}}$ and $\mathfrak{l}_{N_{\mathcal{P}_T}}$ are as in Sect. 12.5.7.

12.8.6. *Proof of Theorem 12.8.5.* First, we rewrite the functor

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\kappa}}{\longrightarrow} \text{D-mod}_\kappa(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_\kappa}{\to} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T})$$

using (a $\mathcal{P}_T$-twisted version) of Corollary 12.3.11.

We obtain that it identifies with

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\alpha_{\mathcal{P}_T,\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T,\mathrm{Ran}} \overset{\int \mathrm{ins.vac}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T,\mathrm{Ran}} \to$$
$$\to \mathrm{KL}(N_{\mathcal{P}_T})_{\mathrm{Ran}} \overset{\mathrm{Loc}_{N_{\mathcal{P}_T}}}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}).$$

By Theorem 12.5.13, the functor

$$\mathrm{KL}(N_{\mathcal{P}_T})_{\mathrm{Ran}} \overset{\mathrm{Loc}_{N_{\mathcal{P}_T}}}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Bun}_{N,\mathcal{P}_T},-)}{\longrightarrow} \mathrm{Vect}$$

identifies with

$$\mathrm{KL}(N_{\mathcal{P}_T})_{\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}_{\mathcal{F}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\Omega(\mathfrak{n}_{\mathcal{P}_T}),-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N_{\mathcal{P}_T}}[\delta_{N_{\mathcal{P}_T}}]}{\longrightarrow} \mathrm{Vect}.$$

The assertion of the theorem follows now by applying Corollary 11.9.13.

□[Theorem 12.8.5]

12.8.7. Note that the same proof applies in the situation twisted by a character. Namely, χ^{Lie} be a character of $\Gamma(X, \mathfrak{n}_{\mathcal{P}_T})$ as in Sect. 12.5.16.

Denote

$$\Omega(\mathfrak{n}_{\mathcal{P}_T}, \chi, \mathfrak{g})_\kappa := \mathrm{BRST}_{\mathfrak{n}_{\mathcal{P}_T},\chi}(\mathrm{Vac}(G)_{\kappa,\mathcal{P}_T}).$$

Consider the corresponding functor

$$\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T},\chi} : \mathrm{KL}(G)_\kappa \to \Omega(\mathfrak{n}_{\mathcal{P}_T}, \chi, \mathfrak{g})_\kappa\text{-mod}^{\mathrm{fact}}.$$

Then:

Theorem 12.8.8. *The composition*

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \xrightarrow{\mathrm{Loc}_{G,\kappa}} \text{D-mod}_\kappa(\mathrm{Bun}_G) \xrightarrow{\mathfrak{p}^!_\kappa} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \xrightarrow{-\overset{*}{\otimes}\chi^*(\exp)}$$
$$\to \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \xrightarrow{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Bun}_{N,\mathcal{P}_T},-)} \mathrm{Vect}$$

identifies with the functor

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \xrightarrow{\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T},\chi}} \Omega(\mathfrak{n}_{\mathcal{P}_T}, \chi, \mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \xrightarrow{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\Omega(\mathfrak{n}_{\mathcal{P}_T},\chi,\mathfrak{g})_\kappa,-)} \mathrm{Vect} \xrightarrow{-\otimes \mathfrak{l}_{N_{\mathcal{P}_T}}[\delta_{N_{\mathcal{P}_T}}]} \mathrm{Vect}.$$

13. Localization via the infinitesimal Hecke groupoid

This goal of this section is to prove Theorem 12.1.8. This will be based on the approach to the localization functor via the infinitesimal Hecke groupoid, which was developed in the unpublished part of the thesis of the eighth author of this paper.

13.1. Another take on the functor Γ_G.

13.1.1. Recall the local Hecke stack completed along the diagonal, viewed as a groupoid acting on $\mathrm{pt}/\mathfrak{L}^+(G)$:

$$\mathrm{pt}/\mathfrak{L}^+(G) \overset{\overleftarrow{h}^{\mathrm{loc},\wedge}}{\longleftarrow} \mathrm{Hecke}^{\mathrm{loc},\wedge}_G \overset{\overrightarrow{h}^{\mathrm{loc},\wedge}}{\longrightarrow} \mathrm{pt}/\mathfrak{L}^+(G).$$

Recall also that the datum of a level κ gives rise to a *multiplicative* line bundle $\mathcal{L}^{\mathrm{loc}}_\kappa$ on $\mathrm{Hecke}^{\mathrm{loc},\wedge}_G$, see Sect. 10.1.1.

According to [CF, Sect. 3.3], we can identify the category $\mathrm{KL}(G)_\kappa$ with the category

$$\mathrm{Rep}(\mathfrak{L}^+(G))^{\mathrm{Hecke}^{\mathrm{loc},\wedge}_G,\mathcal{L}^{\mathrm{loc}}_\kappa}$$

of $\mathcal{L}^{\mathrm{loc}}_\kappa$-twisted $\mathrm{Hecke}^{\mathrm{loc},\wedge}_G$-equivariant objects in $\mathrm{Rep}(\mathfrak{L}^+(G))$.

I.e., this is the category of $\mathcal{M} \in \mathrm{Rep}(\mathfrak{L}^+(G))$ equipped with an isomorphism

$$(\overleftarrow{h}^{\mathrm{loc},\wedge})^*(\mathcal{M}) \simeq \mathcal{L}^{\mathrm{loc}}_\kappa \otimes (\overrightarrow{h}^{\mathrm{loc},\wedge})^*(\mathcal{M}) \tag{13.1}$$

(the isomorphism taking place in $\mathrm{QCoh}(\mathrm{Hecke}^{\mathrm{loc},\wedge}_G)$), equipped with a homotopy-coherent system of compatibilities.

13.1.2. Let $\inf(\mathrm{Bun}_G)$ denote the infinitesimal groupoid of Bun_G, i.e.,

$$\inf(\mathrm{Bun}_G) := (\mathrm{Bun}_G \times \mathrm{Bun}_G)^\wedge,$$

where $(-)^\wedge$ means formal completion along the diagonal. Consider the corresponding diagram

$$\mathrm{Bun}_G \overset{\overleftarrow{h}^{\inf}}{\longleftarrow} \inf(\mathrm{Bun}_G) \overset{\overrightarrow{h}^{\inf}}{\longrightarrow} \mathrm{Bun}_G.$$

Note that the datum of the de Rham twisting $\mathcal{T}_\kappa$ gives rise to a multiplicative line bundle, to be denoted $\mathcal{L}^{\inf}_\kappa$ on $\inf(\mathrm{Bun}_G)$.

Let $\mathcal{Z}$ be a prestack mapping to Ran. Since Bun_G is eventually coconnective, it follows from [GaRo2, Proposition 3.4.3] that we have a canonical equivalence

$$\text{D-mod}_\kappa(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \simeq (\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}))^{\inf(\mathrm{Bun}_G)\times\mathcal{Z},\mathcal{L}^{\inf}_\kappa}$$

that intertwines the functor

$$(\mathbf{oblv}^l_\kappa \otimes \mathrm{Id}) : \text{D-mod}_\kappa(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \to \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})$$

with the tautological forgetful functor

$$(\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}))^{\inf(\mathrm{Bun}_G)\times\mathcal{Z},\mathcal{L}^{\inf}_\kappa} \to \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}).$$

The same applies when we replace Bun_G by its open substack U.

13.1.3. Note now that we have a tautological map of groupoids

$$\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z}} \to \inf(\mathrm{Bun}_G) \times \mathcal{Z} \tag{13.2}$$

over $\mathcal{Z}$.

By construction, the pullback of $\mathcal{L}^{\inf}_\kappa$ along (13.2) identifies canonically with $\mathcal{L}^{\mathrm{glob}}_\kappa$ as a multiplicative line bundle.

From here we obtain that *-pullback along (13.2) defines a functor

$$\begin{aligned}\text{D-mod}_\kappa(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \simeq (\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}))^{\inf(\mathrm{Bun}_G)\times\mathcal{Z},\mathcal{L}^{\inf}_\kappa} \to \\ \to (\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}))^{\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z}},\mathcal{L}^{\mathrm{glob}}_\kappa},\end{aligned} \tag{13.3}$$

that intertwines the forgetful functor

$$(\mathbf{oblv}^l_\kappa \otimes \mathrm{Id}) : \text{D-mod}_\kappa(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \to \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})$$

with the tautological forgetful functor

$$(\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}))^{\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z}},\mathcal{L}^{\mathrm{glob}}_\kappa} \to \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}).$$

Denote the functor (13.3) by

$$\mathbf{oblv}_{\inf\to\mathrm{Hecke}^\wedge,\mathcal{Z}}.$$

13.1.4. Let $U \subset \mathrm{Bun}_G$ be a quasi-compact open substack. Note that it makes sense to restrict $\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z}}$ to U:

$$(\overleftarrow{h}^{\mathrm{glob},\wedge})^{-1}(U \times \mathcal{Z}) =: \mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z},U} =: (\overrightarrow{h}^{\mathrm{glob},\wedge})^{-1}(U \times \mathcal{Z}).$$

The contents of Sect. 13.1.3 apply over U as well, and we obtain a functor, denoted

$$\mathbf{oblv}_{\inf\to\mathrm{Hecke}^\wedge,\mathcal{Z},U}$$

that maps

$$\begin{aligned}\text{D-mod}_\kappa(U) \otimes \text{D-mod}(\mathcal{Z}) \simeq (\mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z}))^{\inf(\mathrm{Bun}_U)\times\mathcal{Z},\mathcal{L}^{\inf}_\kappa} \to \\ \to (\mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z}))^{\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z},U},\mathcal{L}^{\mathrm{glob}}_\kappa}\end{aligned} \tag{13.4}$$

and that intertwines the forgetful functor

$$(\mathbf{oblv}^l_\kappa \otimes \mathrm{Id}) : \text{D-mod}_\kappa(U) \otimes \text{D-mod}(\mathcal{Z}) \to \mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z})$$

with the tautological forgetful functor

$$(\mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z}))^{\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z},U},\mathcal{L}^{\mathrm{glob}}_\kappa} \to \mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z}).$$

13.1.5. Assume now that U is quasi-compact. Let j denote its embedding into Bun_G. Consider the diagram:

$$\begin{tikzcd} U \times \mathcal{Z} \arrow[d, "\mathrm{ev}_{\mathcal{Z},U}"'] & \mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z},U} \arrow[l, "\overleftarrow{h}^{\mathrm{glob},\wedge}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{glob},\wedge}"] \arrow[d] & U \times \mathcal{Z} \arrow[d, "\mathrm{ev}_{\mathcal{Z},U}"] \\ (\mathrm{pt}/\mathfrak{L}^+(G))_{\mathcal{Z}} & \mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathcal{Z}} \arrow[l, "\overleftarrow{h}^{\mathrm{loc},\wedge}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{loc},\wedge}"] & (\mathrm{pt}/\mathfrak{L}^+(G))_{\mathcal{Z}}, \end{tikzcd} \tag{13.5}$$

where $\mathrm{ev}_{\mathcal{Z},U} := \mathrm{ev}_{\mathcal{Z}} \circ j$.

Since both squares in (13.5) are Cartesian, we obtain that the functor[54]

$$(\mathrm{ev}_{\mathcal{Z},U})_* : \mathrm{QCoh}(U) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}) \to \mathrm{Rep}(\mathrm{pt}/\mathfrak{L}^+(G))_{\mathcal{Z}}$$

gives rise to a functor

$$(\mathrm{QCoh}(U) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}))^{\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z},U}, \mathcal{L}^{\mathrm{glob}}_\kappa} \to \left(\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}}\right)^{\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathcal{Z}}, \mathcal{L}^{\mathrm{loc}}_\kappa} \overset{\text{Sect. 13.1.1}}{\simeq} \mathrm{KL}(G)_{\kappa,\mathcal{Z}}. \tag{13.6}$$

We will denote the functor (13.6) by

$$(\mathrm{ev}_{\mathcal{Z},U})_*^{\mathrm{Hecke}^\wedge\text{-enh}}.$$

13.1.6. Composing, we obtain that the functor

$$(\mathrm{ev}_{\mathcal{Z},U})_* \circ (\mathbf{oblv}^l_\kappa \otimes \mathrm{Id}) : \mathrm{D\text{-}mod}_\kappa(U) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}) \to \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}}$$

lifts to a functor

$$\mathrm{D\text{-}mod}_\kappa(U) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}) \overset{(\mathrm{ev}_{\mathcal{Z},U})_*^{\mathrm{Hecke}^\wedge\text{-enh}} \circ \mathbf{oblv}_{\mathrm{inf}\to\mathrm{Hecke}^\wedge,\mathcal{Z},U}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{Z}}. \tag{13.7}$$

13.1.7. It follows from the construction of the identification

$$\left(\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}}\right)^{\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathcal{Z}}, \mathcal{L}^{\mathrm{loc}}_\kappa} \simeq \mathrm{KL}(G)_{\kappa,\mathcal{Z}} \tag{13.8}$$

that the functor

$$(\mathrm{ev}_{\mathcal{Z},U})_*^{\mathrm{Hecke}^\wedge\text{-enh}} \circ \mathbf{oblv}_{\mathrm{inf}\to\mathrm{Hecke}^\wedge,\mathcal{Z},U}$$

of (13.7) identifies canonically with the functor

$$\Gamma_{G,\kappa,\mathcal{Z},U} := \Gamma_{G,\kappa,\mathcal{Z}} \circ j_{*,\mathrm{co}},$$

so that the diagram

$$\begin{tikzcd} \mathbf{oblv}^{(\hat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} \circ (\mathrm{ev}_{\mathcal{Z},U})_*^{\mathrm{Hecke}^\wedge\text{-enh}} \circ \mathbf{oblv}_{\mathrm{inf}\to\mathrm{Hecke}^\wedge,\mathcal{Z},U} \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathbf{oblv}^{(\hat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} \circ \Gamma_{G,\kappa,\mathcal{Z},U} \arrow[d, "\sim"] \\ (\mathrm{ev}_{\mathcal{Z},U})_* \circ (\mathbf{oblv}^l_\kappa \otimes \mathrm{Id}) \arrow[r, "\mathrm{id}"] & (\mathrm{ev}_{\mathcal{Z},U})_* \circ (\mathbf{oblv}^l_\kappa \otimes \mathrm{Id}) \end{tikzcd}$$

commutes.

13.2. **The functor $\mathrm{Loc}_{G,\kappa}$ and the infinitesimal Hecke groupoid.**

[54]Note that as in Remark 10.2.8, the difference between $\mathrm{Rep}(\mathrm{pt}/\mathfrak{L}^+(G))$ and $\mathrm{QCoh}(\mathrm{pt}/\mathfrak{L}^+(G))$ is immaterial here.

13.2.1. Let $U \subset \mathrm{Bun}_G$ be a quasi-compact open substack, and let $\mathcal{Z}$ be a space mapping to Ran.

By the same logic as in Sect. 13.1.5, the functor

$$(\mathrm{ev}_{\mathcal{Z},U})^* : (\mathrm{pt}\,/\mathfrak{L}^+(G))_{\mathrm{Ran}} \to \mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z})$$

lifts to a functor

$$\mathrm{KL}(G)_{\kappa,\mathcal{Z}} \simeq \left(\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}}\right)^{\mathrm{Hecke}^{\mathrm{loc},\wedge}_{G,\mathcal{Z}},\mathcal{L}^{\mathrm{loc}}_\kappa} \to (\mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z}))^{\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z},U},\mathcal{L}^{\mathrm{glob}}_\kappa}, \tag{13.9}$$

to be denoted

$$(\mathrm{ev}_{\mathcal{Z},U})^{*,\mathrm{Hecke}^\wedge\text{-enh}}.$$

Furthermore, the functors $(\mathrm{ev}_{\mathcal{Z},U})^{*,\mathrm{Hecke}^\wedge\text{-enh}}$ and $(\mathrm{ev}_{\mathcal{Z},U})_*^{\mathrm{Hecke}^\wedge\text{-enh}}$ are adjoint.

13.2.2. Denote by $\mathsf{M}^{\mathrm{loc}}_\kappa$ the (factorization) monad

$$\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} \circ \mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}$$

acting on $\mathrm{Rep}(\mathfrak{L}^+(G))$.

Denote by $\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z},U}$ the monad acting on $\mathrm{QCoh}(U)\otimes\text{D-mod}(\mathcal{Z})$, corresponding to the forgetful functor

$$(\mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z}))^{\mathrm{Hecke}^{\mathrm{glob},\wedge}_{G,\mathcal{Z},U},\mathcal{L}^{\mathrm{glob}}_\kappa} \to \mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z})$$

and its left adjoint.

By adjunction, we obtain that the functor $\mathrm{ev}^*_{\mathcal{Z},U}$ intertwines the monads $\mathsf{M}^{\mathrm{loc}}_\kappa$ and $\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z},U}$, i.e., we have a commutative diagram

$$\begin{tikzcd} \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}} \arrow[r, "\mathrm{ev}^*_{\mathcal{Z},U}"] & \mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z}) \\ \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}} \arrow[r, "\mathrm{ev}^*_{\mathcal{Z},U}"] \arrow[u, "\mathsf{M}^{\mathrm{loc}}_\kappa"] & \mathrm{QCoh}(U) \otimes \text{D-mod}(\mathcal{Z}). \arrow[u, "\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z},U}"'] \end{tikzcd} \tag{13.10}$$

13.2.3. The diagrams (13.10) are compatible under inclusions $U_1 \subset U_2$. Passing to the limit over U, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}} \arrow[r, "\mathrm{ev}^*_{\mathcal{Z}}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \\ \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}} \arrow[r, "\mathrm{ev}^*_{\mathcal{Z}}"] \arrow[u, "\mathsf{M}^{\mathrm{loc}}_\kappa"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}), \arrow[u, "\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z}}"'] \end{tikzcd} \tag{13.11}$$

where $\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z}} := \mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z},U}$ fot $U = \mathrm{Bun}_G$.

13.2.4. Let $\mathsf{M}^{\mathrm{inf}}_{\kappa,\mathcal{Z}}$ denote the monad

$$(\mathbf{oblv}^l_\kappa \circ \mathbf{ind}^k_\kappa) \otimes \mathrm{Id}$$

acting on $\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})$.

The map of groupoids (13.2) gives rise to a map of monads

$$\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z}} \to \mathsf{M}^{\mathrm{inf}}_{\kappa,\mathcal{Z}}.$$

In particular, we obtain a diagram

$$\begin{tikzcd} \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \arrow[r, "\mathrm{id}"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \\ \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \arrow[r, "\mathrm{id}"'] \arrow[u, "\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z}}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}) \arrow[u, "\mathsf{M}^{\mathrm{inf}}_{\kappa,\mathcal{Z}}"'] \end{tikzcd} \tag{13.12}$$

Concatenating diagrams (13.11) and (13.12) we obtain a diagram

$$
\begin{tikzcd}
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}} \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathcal{Z}}"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}) \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}} \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathcal{Z}}"'] \arrow[u, "\mathsf{M}_\kappa^{\mathrm{loc}}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}) \arrow[u, "\mathsf{M}_{\kappa,\mathcal{Z}}^{\mathrm{inf}}"']
\end{tikzcd}
\tag{13.13}
$$

It follows from Sect. 13.1.7 that diagram (13.13) identifies with the outer diagram in

$$
\begin{tikzcd}
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}} \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathcal{Z}}"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}) \\
\mathrm{KL}(G)_{\kappa,\mathcal{Z}} \arrow[r, "(\mathrm{Loc}_{G,\kappa})_{\mathcal{Z}}"'] \arrow[u, "\mathbf{oblv}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}"] & \mathrm{D}\text{-}\mathrm{mod}_\kappa(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}) \arrow[u, "\mathbf{oblv}_\kappa^l"'] \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathcal{Z}} \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathcal{Z}}"'] \arrow[u, "\mathbf{ind}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}), \arrow[u, "\mathbf{ind}_\kappa^l"']
\end{tikzcd}
\tag{13.14}
$$

in which the lower square is (the base change along $\mathcal{Z} \to \mathrm{Ran}$) of (10.16), and the upper square is obtained from the lower square by passing to adjoints along the vertical arrows.

13.3. **Proof of Theorem 12.1.8.** We are now ready to prove Theorem 12.1.8.

13.3.1. We wish to show that the natural transformation in

$$
\begin{tikzcd}
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}^{\subseteq}} \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}^{\subseteq}}"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Ran}^{\subseteq}) \\
\mathrm{KL}(G)_{\kappa,\mathrm{Ran}^{\subseteq}} \arrow[r, "(\mathrm{Loc}_{G,\kappa})_{\mathrm{Ran}^{\subseteq}}"'] \arrow[u, "\mathbf{oblv}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}"] & \mathrm{D}\text{-}\mathrm{mod}_\kappa(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathbf{oblv}_\kappa^l \otimes \mathrm{Id}"']
\end{tikzcd}
$$

becomes an isomorphism, after we:

- We precompose with $\mathrm{ins.vac}_{\mathrm{Ran}} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}^{\subseteq}}$;
- Postcompose with $\mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq}, -)$.

Since the essential image of

$$\mathbf{ind}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa} : \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}$$

generates the target, it is sufficient to show that the natural transformation becomes an isomorphism when we further precompose with this functor.

Thus, we obtain the diagram

$$
\begin{tikzcd}
\operatorname{Rep}(\mathfrak{L}^+(G))_{\operatorname{Ran}^{\subseteq}} \arrow[r, "(\operatorname{Loc}_G^{\operatorname{QCoh}})_{\operatorname{Ran}^{\subseteq}}"] \arrow[dr, Rightarrow, no head] & \operatorname{QCoh}(\operatorname{Bun}_G) \otimes \operatorname{D-mod}(\operatorname{Ran}^{\subseteq}) \\
\operatorname{KL}(G)_{\kappa,\operatorname{Ran}^{\subseteq}} \arrow[u, "\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"] \arrow[r, "(\operatorname{Loc}_{G,\kappa})_{\operatorname{Ran}^{\subseteq}}"'] & \operatorname{D-mod}_\kappa(\operatorname{Bun}_G) \otimes \operatorname{D-mod}(\operatorname{Ran}^{\subseteq}) \arrow[u, "\mathbf{oblv}^l_\kappa \otimes \operatorname{Id}"'] \\
\operatorname{KL}(G)_{\kappa,\operatorname{Ran}} \arrow[u, "\operatorname{ins.vac}_{\operatorname{Ran}}"] \arrow[r, "(\operatorname{Loc}_{G,\kappa})_{\operatorname{Ran}}"'] & \operatorname{D-mod}_\kappa(\operatorname{Bun}_G) \otimes \operatorname{D-mod}(\operatorname{Ran}) \arrow[u, "\operatorname{Id} \otimes (\operatorname{pr}_{\operatorname{small}})^!"'] \\
\operatorname{Rep}(\mathfrak{L}^+(G))_{\operatorname{Ran}} \arrow[u, "\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"] \arrow[r, "(\operatorname{Loc}_G^{\operatorname{QCoh}})_{\operatorname{Ran}}"'] & \operatorname{QCoh}(\operatorname{Bun}_G) \otimes \operatorname{D-mod}(\operatorname{Ran}), \arrow[u, "\mathbf{ind}^l_\kappa \otimes \operatorname{Id}"']
\end{tikzcd}
\tag{13.15}
$$

which we then compose with

$$\operatorname{Id} \otimes \operatorname{C}^\cdot_c(\operatorname{Ran}^{\subseteq}, -) : \operatorname{QCoh}(\operatorname{Bun}_G) \otimes \operatorname{D-mod}(\operatorname{Ran}^{\subseteq}) \to \operatorname{QCoh}(\operatorname{Bun}_G),$$

and we need to show that the resulting natural transformation commutes.

13.3.2. Consider the commutative diagram obtained by concatenating the lower two squares in (13.15). It is easy to see that it identifies with the outer diagram in

$$
\begin{tikzcd}
\operatorname{KL}(G)_{\kappa,\operatorname{Ran}^{\subseteq}} \arrow[r, "(\operatorname{Loc}_{G,\kappa})_{\operatorname{Ran}^{\subseteq}}"] & \operatorname{D-mod}_\kappa(\operatorname{Bun}_G) \otimes \operatorname{D-mod}(\operatorname{Ran}^{\subseteq}) \\
\operatorname{Rep}(\mathfrak{L}^+(G))_{\operatorname{Ran}^{\subseteq}} \arrow[u, "\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"] \arrow[r, "(\operatorname{Loc}_G^{\operatorname{QCoh}})_{\operatorname{Ran}^{\subseteq}}"] & \operatorname{QCoh}(\operatorname{Bun}_G) \otimes \operatorname{D-mod}(\operatorname{Ran}^{\subseteq}) \arrow[u, "\mathbf{ind}^l_\kappa \otimes \operatorname{Id}"'] \\
\operatorname{Rep}(\mathfrak{L}^+(G))_{\operatorname{Ran}} \arrow[u, "\operatorname{ins.vac}_{\operatorname{Ran}}"] \arrow[r, "(\operatorname{Loc}_G^{\operatorname{QCoh}})_{\operatorname{Ran}}"] & \operatorname{QCoh}(\operatorname{Bun}_G) \otimes \operatorname{D-mod}(\operatorname{Ran}). \arrow[u, "\operatorname{Id} \otimes (\operatorname{pr}_{\operatorname{small}})^!"']
\end{tikzcd}
$$

Hence, instead of (13.15), we can consider the diagram

(13.16)
$$\begin{tikzcd}
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}^{\subseteq}} \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}^{\subseteq}}"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \\
\mathrm{KL}(G)_{\kappa,\mathrm{Ran}^{\subseteq}} \arrow[u, "\mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"] \arrow[r, "(\mathrm{Loc}_{G,\kappa})_{\mathrm{Ran}^{\subseteq}}"'] & \text{D-mod}_\kappa(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathbf{oblv}^l_\kappa\otimes \mathrm{Id}"'] \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}^{\subseteq}} \arrow[u, "\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}"] \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}^{\subseteq}}"'] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathbf{ind}^l_\kappa\otimes \mathrm{Id}"'] \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \arrow[u, "\mathrm{ins.vac}_{\mathrm{Ran}}"] \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}}"'] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}). \arrow[u, "\mathrm{Id}\otimes(\mathrm{pr}_{\mathrm{small}})^!"']
\end{tikzcd}$$

13.3.3. Using Sect. 13.2.4, we can replace (13.16) by the diagram

(13.17)
$$\begin{tikzcd}
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}^{\subseteq}} \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}^{\subseteq}}"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}^{\subseteq}} \arrow[u, "\mathsf{M}^{\mathrm{loc}}_\kappa"] \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}^{\subseteq}}"'] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathsf{M}^{\mathrm{inf}}_{\kappa,\mathrm{Ran}^{\subseteq}}"'] \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \arrow[u, "\mathrm{ins.vac}_{\mathrm{Ran}}"] \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}}"'] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}). \arrow[u, "\mathrm{Id}\otimes(\mathrm{pr}_{\mathrm{small}})^!"']
\end{tikzcd}$$

We further rewrite (13.17) as the outer diagram in

(13.18)
$$\begin{tikzcd}
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}^{\subseteq}} \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}^{\subseteq}}"] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \arrow[r, "\mathrm{Id}"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}^{\subseteq}} \arrow[u, "\mathsf{M}^{\mathrm{loc}}_\kappa"] \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}^{\subseteq}}"'] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathrm{Ran}^{\subseteq}}"'] \arrow[r, "\mathrm{Id}"] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathsf{M}^{\mathrm{inf}}_{\kappa,\mathrm{Ran}^{\subseteq}}"'] \\
\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \arrow[u, "\mathrm{ins.vac}_{\mathrm{Ran}}"] \arrow[r, "(\mathrm{Loc}_G^{\mathrm{QCoh}})_{\mathrm{Ran}}"'] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}) \arrow[u, "\mathrm{Id}\otimes(\mathrm{pr}_{\mathrm{small}})^!"'] \arrow[r, "\mathrm{Id}"] & \mathrm{QCoh}(\mathrm{Bun}_G)\otimes \text{D-mod}(\mathrm{Ran}). \arrow[u, "\mathrm{Id}\otimes(\mathrm{pr}_{\mathrm{small}})^!"']
\end{tikzcd}$$

13.3.4. The left portion of (13.18) commutes. Hence, it is enough to show that the outer diagram in

$$
\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Bun}_G) \arrow[r, "\mathrm{Id}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq}, -)"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq}, -)"'] \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathsf{M}^{\mathrm{glob}}_{\kappa, \mathrm{Ran}^{\subseteq}}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathsf{M}^{\mathrm{inf}}_{\kappa, \mathrm{Ran}^{\subseteq}}"'] \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small}})^!"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}). \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small}})^!"']
\end{tikzcd}
\tag{13.19}
$$

commutes.

We will show that already the diagram

$$
\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \arrow[r, "\mathrm{Id}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small}})_!"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small}})_!"'] \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathsf{M}^{\mathrm{glob}}_{\kappa, \mathrm{Ran}^{\subseteq}}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq}) \arrow[u, "\mathsf{M}^{\mathrm{inf}}_{\kappa, \mathrm{Ran}^{\subseteq}}"'] \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small}})^!"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}). \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small}})^!"']
\end{tikzcd}
\tag{13.20}
$$

commutes.

This is a particular case of the next assertion:

Theorem 13.3.5. *For any $\mathcal{Z} \to \mathrm{Ran}$, the natural transformation in the diagram*

$$
\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}) \arrow[r, "\mathrm{Id}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}) \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}^{\subseteq}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small}})_!"] \arrow[dr, Rightarrow] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}^{\subseteq}) \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small}})_!"'] \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}^{\subseteq}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z}^{\subseteq}}"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}^{\subseteq}) \arrow[u, "\mathsf{M}^{\mathrm{inf}}_{\kappa,\mathcal{Z}^{\subseteq}}"'] \\
\mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}) \arrow[r, "\mathrm{Id}"] \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^!"] & \mathrm{QCoh}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}). \arrow[u, "\mathrm{Id} \otimes (\mathrm{pr}_{\mathrm{small},\mathcal{Z}})^!"']
\end{tikzcd}
$$

induced by the map of monads

$$\mathsf{M}^{\mathrm{glob}}_{\kappa,\mathcal{Z}^{\subseteq}} \to \mathsf{M}^{\mathrm{inf}}_{\kappa,\mathcal{Z}^{\subseteq}},$$

is an isomorphism.

This theorem is a particular case of [Ro2, Theorem 4.3.6], combined with Remark 4.5.6 in *loc. cit.*
$\square$[Theorem 12.1.8]

14. The composition of localization and coefficient functors

The goal of this section is to give an expression for the composition

(14.1) $$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\mathrm{crit}}}{\longrightarrow} \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \simeq \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{coeff}}{\longrightarrow} \mathrm{Whit}^!(G)_{\mathrm{Ran}}$$

in terms of factorization homology.

This expression (combined with the compatibility of $\mathrm{FLE}_{G,\mathrm{crit}}$ and $\mathrm{FLE}_{\check{G},\infty}$ expressed by Corollary 6.4.10) will be used in Sect. 18 in order to show that the Langlands functor is compatible with the critical localization and the spectral Poincaré series functors via the critical FLE.

14.1. **The vacuum case.**

14.1.1. Let us specialize the setting of Theorem 12.8.8 to the case $\kappa = \mathrm{crit}$ and $\mathcal{P}_T = \rho(\omega_X)$. In this case, the integer that we denoted $\delta_{N_{\mathcal{P}_T}}$ is $\delta_{N_{\rho(\omega_X)}}$. Denote the corresponding line $\mathfrak{l}_{N_{\mathcal{P}_T}}$ by

(14.2) $$\mathfrak{l}_{N_{\rho(\omega_X)}}.$$

14.1.2. We obtain:

Theorem 14.1.3. *The composition*

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\mathrm{crit}}}{\longrightarrow} \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_{\mathrm{crit}}}{\to} \mathrm{D\text{-}mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{-\overset{*}{\otimes}\chi^*(\exp)}{\longrightarrow}$$
$$\to \mathrm{D\text{-}mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Bun}_{N,\rho(\omega_X)},-)}{\longrightarrow} \mathrm{Vect}$$

identifies with the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),\mathrm{Ran}} \overset{\mathrm{DS}^{\mathrm{enh}}}{\longrightarrow}$$
$$\to \mathfrak{z}_{\mathfrak{g}}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathfrak{z}_{\mathfrak{g}},-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N_{\rho(\omega_X)}}[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}.$$

14.1.4. Denote by Loc_G the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\mathrm{crit}}}{\longrightarrow} \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \overset{(9.2)}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G). \tag{14.3}$$

From (9.7) we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Vect} \arrow[r, "\mathrm{Id}"] & \mathrm{Vect} \\
\text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \arrow[u, "\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N,\rho(\omega_X)},-)\circ(-\overset{*}{\otimes}\chi^*(\exp))"] & \mathrm{Vect} \arrow[u, "-\otimes\mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes\frac{1}{2}}"'] \\
\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \arrow[u, "\mathfrak{p}^!_{\mathrm{crit}}"] \arrow[r, "(9.2)"] & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G^{\mathrm{Vac,glob}}"'] \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[u, "\mathrm{Loc}_{G,\mathrm{crit}}"] \arrow[r, "\mathrm{Id}"] & \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}}. \arrow[u, "\mathrm{Loc}_G"']
\end{tikzcd}$$

Recall also that

$$\mathrm{coeff}_G^{\mathrm{Vac,glob}} \simeq \mathrm{coeff}_G^{\mathrm{Vac}}[2\delta_{N_{\rho(\omega_X)}}].$$

14.1.5. Hence, Theorem 14.1.3 can be restated as:

Theorem 14.1.6. *The composition*

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_G}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{coeff}_G^{\mathrm{Vac}}}{\longrightarrow} \mathrm{Vect} \overset{-\otimes\mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes\frac{1}{2}}\otimes\mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1}[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}$$

identifies with the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),\mathrm{Ran}} \overset{\mathrm{DS}^{\mathrm{enh}}}{\longrightarrow} \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathfrak{z}_{\mathfrak{g}},-)}{\longrightarrow} \mathrm{Vect}\,.$$

14.2. **Composition of coefficient and localization functors: the general case.** We are now ready to state the general theorem, describing the composition of the functors

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_G}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{-\otimes\mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes\frac{1}{2}}\otimes\mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1}[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

and

$$\mathrm{coeff}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\mathrm{Ran}}.$$

14.2.1. Recall that the category $\mathrm{Whit}^!(G)_{\mathrm{Ran}}$ is the dual of $\mathrm{Whit}_*(G)_{\mathrm{Ran}}$. Hence, the description of the above composition is equivalent to describing the pairing

$$\begin{aligned}
&\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{Loc}_G\otimes\mathrm{Id}}{\longrightarrow} \\
&\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{(-\otimes\mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes\frac{1}{2}}\otimes\mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1}[\delta_{N_{\rho(\omega_X)}}])\otimes\mathrm{Id}}{\longrightarrow} \\
&\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{coeff}_G\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Whit}^!(G)_{\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \mathrm{Vect}\,.
\end{aligned} \tag{14.4}$$

14.2.2. Recall the factorization functor $\mathsf{P}_G^{\mathrm{loc,enh}}$, see Sect. 6.4.6. We will think of it as the functor

$$\begin{aligned}
&\mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{Id}\otimes\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \\
&\qquad \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\overline{\mathrm{DS}}^{\mathrm{enh}}}{\to} \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}.
\end{aligned}$$

Let $\mathsf{P}_G^{\mathrm{loc}}$ be the composition of $\mathsf{P}_G^{\mathrm{loc,enh}}$ with

$$\mathbf{oblv}_{\mathfrak{z}_{\mathfrak{g}}} : \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}} \to \mathrm{Vect}\,.$$

14.2.3. We will prove:

Theorem 14.2.4. *The functor* (14.4) *identifies canonically with*

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{ins.vac}_{\mathrm{Ran}} \otimes \mathrm{ins.vac}_{\mathrm{Ran}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}^{\subseteq}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}^{\subseteq}} \to$$
$$\to (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \overset{\mathsf{P}^{\mathrm{loc,enh}}_G}{\longrightarrow}$$
$$\to \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathfrak{z}_{\mathfrak{g}},-)_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}) \to$$
$$\overset{\mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq},-)}{\longrightarrow} \mathrm{Vect}, \tag{14.5}$$

where the fiber product $\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}$ *is formed using the projections* $\mathrm{pr}_{\mathrm{big}} : \mathrm{Ran}^{\subseteq} \to \mathrm{Ran}$.

Remark 14.2.5. Note that the functor (14.5), appearing in Theorem 14.2.4 can also be rewritten as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{ins.vac}_{\mathrm{Ran}} \otimes \mathrm{ins.vac}_{\mathrm{Ran}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}^{\subseteq}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}^{\subseteq}} \to$$
$$\to (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \overset{\mathsf{P}^{\mathrm{loc}}_G}{\longrightarrow} \text{D-mod}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}) \overset{\mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq})}{\longrightarrow} \mathrm{Vect} \tag{14.6}$$

i.e., instead of $\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathfrak{z}_{\mathfrak{g}},-)_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}$ we can use the functor $\mathbf{oblv}_{\mathfrak{z}_{\mathfrak{g}},\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}$.

Indeed, since the functor $\mathsf{P}^{\mathrm{loc,enh}}_G$ is strictly unital, a priori, by Theorem 14.2.4, the pairing (14.5) is

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{ins.unit}_{\mathrm{Ran}} \otimes \mathrm{ins.unit}_{\mathrm{Ran}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}^{\subseteq}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}^{\subseteq}} \to$$
$$\to (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \overset{\mathrm{ins.vac}_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}}{\longrightarrow}$$
$$\to (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq})^{\subseteq}} \overset{\mathsf{P}^{\mathrm{loc}}_G}{\longrightarrow} \text{D-mod}((\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq})^{\subseteq}) \to$$
$$\overset{\mathrm{C}^\cdot_c((\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq})^{\subseteq})}{\longrightarrow} \mathrm{Vect}\,. \tag{14.7}$$

However, the composition of the first three lines in (14.7) lies in the essential image of the functor

$$\mathsf{t}^! : \text{D-mod}((\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq})^{\subseteq,\mathrm{untl}}) \to \text{D-mod}((\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq})^{\subseteq}).$$

Hence, by the same mechanism as in Sect. 11.9.9, the expression in (14.7) is isomorphic to that in (14.6).

14.3. Proof of Theorem 14.2.4.

14.3.1. We rewrite the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{Loc}_G \otimes \mathrm{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{coeff}_G \otimes \mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{Whit}^!(G)_{\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \mathrm{Vect}\,. \tag{14.8}$$

as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*}}{\longrightarrow}$$
$$\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \overset{\langle -,-\rangle_{\mathrm{Bun}_G}}{\longrightarrow} \mathrm{Vect}, \tag{14.9}$$

where the last arrow is the canonical pairing

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \to \mathrm{Vect}\,.$$

14.3.2. By the unital property of Loc_G and $\mathrm{Poinc}_{G,*}$, we can rewrite the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$$

as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{ins.vac}_{\mathrm{Ran}} \otimes \mathrm{ins.vac}_{\mathrm{Ran}}}{\longrightarrow} \\ \to \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}^{\subseteq}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}^{\subseteq}} \overset{(\mathrm{Loc}_G)_{\mathrm{Ran}^{\subseteq}} \otimes (\mathrm{Poinc}_{G,*})_{\mathrm{Ran}^{\subseteq}}}{\longrightarrow} \\ \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \to \\ \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq} \times \mathrm{Ran}^{\subseteq}, -)}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G). \tag{14.10}$$

14.3.3. In this subsection we will use an analog of Sect. C.5.15, which will allow us to replace

$$\mathrm{Ran}^{\subseteq} \times \mathrm{Ran}^{\subseteq} \rightsquigarrow \mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}.$$

We note that the composition in the first three lines in (14.10) actually factors via the tensor product of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$ with

$$\text{D-mod}(\mathrm{Ran} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{untl}}) \otimes \text{D-mod}(\mathrm{Ran} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{untl}}) \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{t}^! \otimes \mathrm{t}^!}{\longrightarrow} \\ \to \text{D-mod}(\mathrm{Ran}^{\subseteq}) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq})$$

Hence, by the same principle as in Sect. C.5.15, its further composition with

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \to \\ \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq} \times \mathrm{Ran}^{\subseteq}, -)}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G)$$

is isomorphic to its composition with

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \Delta^!_{\mathrm{Ran}^{\subseteq}}}{\longrightarrow} \\ \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}) \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}, -)}{\longrightarrow} \\ \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G).$$

Hence, we can rewrite (14.10) as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{ins.vac}_{\mathrm{Ran}} \otimes \mathrm{ins.vac}_{\mathrm{Ran}}}{\longrightarrow} \\ \to \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}^{\subseteq}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}^{\subseteq}} \overset{(\mathrm{Loc}_G)_{\mathrm{Ran}^{\subseteq}} \otimes (\mathrm{Poinc}_{G,*})_{\mathrm{Ran}^{\subseteq}}}{\longrightarrow} \\ \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \to \\ \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes (\Delta_{\mathrm{Ran}})^!}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}^{\subseteq}) \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq}, -)}{\longrightarrow} \\ \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G),$$

which is the same as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{ins.vac}_{\mathrm{Ran}} \otimes \mathrm{ins.vac}_{\mathrm{Ran}}}{\longrightarrow} \\ \to \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}^{\subseteq}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}^{\subseteq}} \to (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \to \\ \overset{(\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}) \to \\ \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}, -)}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G).$$

14.3.4. Thus, we obtain that we can rewrite (14.9) as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{ins.vac} \otimes \mathrm{ins.vac}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}^{\subseteq}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}^{\subseteq}} \to \\ \to (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \to \\ \overset{(\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}) \to \\ \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}, -)}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \overset{\langle -,-\rangle_{\mathrm{Bun}_G}}{\longrightarrow} \mathrm{Vect}. \tag{14.11}$$

Hence, in order to prove the theorem, it is enough to identify the composition

$$(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \to \\ \overset{(\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}}{\longrightarrow} \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}) \to \\ \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}, -)}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \overset{\langle -,-\rangle_{\mathrm{Bun}_G}}{\longrightarrow} \\ \to \mathrm{Vect} \overset{(-\otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect} \tag{14.12}$$

with

$$(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \overset{\mathsf{P}^{\mathrm{loc,enh}}_G}{\longrightarrow} \\ \to \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathfrak{z}_{\mathfrak{g}},-)_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}}{\longrightarrow} \mathrm{D\text{-}mod}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}) \to \\ \overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}, -)}{\longrightarrow} \mathrm{Vect}. \tag{14.13}$$

14.3.5. Applying the functor $(\mathrm{pr}_{\mathrm{big}})_!$, we obtain that it suffices to identify

$$(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}} \overset{(\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}}}{\longrightarrow} \\ \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \overset{\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{C}^\cdot_c(\mathrm{Ran},-)}{\longrightarrow} \\ \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \overset{\langle -,-\rangle_{\mathrm{Bun}_G}}{\longrightarrow} \mathrm{Vect} \overset{(-\otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect} \tag{14.14}$$

with

$$(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}} \overset{\mathsf{P}^{\mathrm{loc,enh}}_G}{\longrightarrow} \mathfrak{z}_{\mathfrak{g}}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathfrak{z}_{\mathfrak{g}},-)}{\longrightarrow} \mathrm{Vect}. \tag{14.15}$$

14.3.6. Note that both functors (14.14) and (14.15) factor naturally via

$$(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}} \to (\mathrm{KL}(G)_{\mathrm{crit}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Whit}_*(G))_{\mathrm{Ran}},$$

and in particular via

$$(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}} \to (\mathrm{KL}(G)_{\mathrm{crit}} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Whit}_*(G))_{\mathrm{Ran}},$$

where $\mathrm{Rep}(\check{G})$ maps to Sph_G via $\mathrm{Sat}_G^{-1,\mathrm{nv}}$.

Hence, using the fact that the action of $\mathrm{Rep}(\check{G})$ on $\mathrm{Vac}_{\mathrm{Whit}_*(G)}$ defines an equivalence

$$\mathrm{Rep}(\check{G}) \to \mathrm{Whit}_*(G),$$

we are reduced to establishing an identification between

$$\begin{aligned} &\mathrm{KL}(G)_{\mathrm{crit,Ran}} \overset{\mathrm{Id}\otimes \mathrm{Vac}_{\mathrm{Whit}_*(G)}}{\longrightarrow} (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}} \to \\ &\overset{(\mathrm{Loc}_G\otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}}}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \overset{\mathrm{Id}\otimes\mathrm{Id}\otimes \mathrm{C}^\cdot_c(\mathrm{Ran},-)}{\longrightarrow} \\ &\to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \overset{\langle -,-\rangle_{\mathrm{Bun}_G}}{\longrightarrow} \mathrm{Vect} \overset{(-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect} \end{aligned} \tag{14.16}$$

and

$$\begin{aligned} &\mathrm{KL}(G)_{\mathrm{crit,Ran}} \overset{\mathrm{Id}\otimes \mathrm{Vac}_{\mathrm{Whit}_*(G)}}{\longrightarrow} (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(G))_{\mathrm{Ran}} \overset{\mathsf{P}_G^{\mathrm{loc,enh}}}{\longrightarrow} \\ &\to \mathfrak{z}_{\mathfrak{g}}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathfrak{z}_{\mathfrak{g}},-)}{\longrightarrow} \mathrm{Vect}. \end{aligned} \tag{14.17}$$

14.3.7. Using the unital property of $\mathrm{Poinc}_{G,*}$ we can identify (14.16) with

$$\begin{aligned} &\mathrm{KL}(G)_{\mathrm{crit,Ran}} \overset{(\mathrm{Loc}_G)_{\mathrm{Ran}}\otimes \mathrm{Poinc}^{\mathrm{Vac}}_{G,*}}{\longrightarrow} \\ &\to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \overset{\mathrm{Id}\otimes \mathrm{C}^\cdot_c(\mathrm{Ran},-)\otimes\mathrm{Id}}{\longrightarrow} \\ &\to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2},\mathrm{co}}(\mathrm{Bun}_G) \overset{\langle -,-\rangle_{\mathrm{Bun}_G}}{\longrightarrow} \mathrm{Vect} \overset{(-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}. \end{aligned} \tag{14.18}$$

By definition, we can identify (14.17) with

$$\mathrm{KL}(G)_{\mathrm{crit,Ran}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\simeq} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),\mathrm{Ran}} \overset{\mathrm{DS}^{\mathrm{enh}}}{\longrightarrow} \mathfrak{z}_{\mathfrak{g}}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathfrak{z}_{\mathfrak{g}},-)}{\longrightarrow} \mathrm{Vect}. \tag{14.19}$$

So, it remains to identify (14.18) and (14.19).

14.3.8. We rewrite (14.18) as

$$\mathrm{KL}(G)_{\mathrm{crit,Ran}} \overset{\mathrm{Loc}_G}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{coeff}^{\mathrm{Vac}}_G}{\longrightarrow} \mathrm{Vect} \overset{(-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}. \tag{14.20}$$

Now the isomorphism between (14.20) and (14.19) is the assertion of Theorem 14.1.6.

□[Theorem 14.2.4]

15. The Hecke eigen-property of critical localization

The goal of this section is to establish the Hecke eigenproprety of the functor $\mathrm{Loc}_{G,\mathrm{crit}}$, which lies in the heart of the manuscript [BD1]. What we do will essentially amount to a souped-up version of the construction in *loc. cit.*

The contents of this (and the next) section are not logically necessary either for this paper, nor for the other papers in the GLC series. However, here we fill the gap in the literature: we supply a proof of a [Ga1, Theorem 10.3.4], which is used in [Ga1, Corollary 4.5.5], while the latter is essential for the GLC series.

15.1. **Statement of the result.**

15.1.1. Consider the prestacks

$$\mathrm{LS}^{\mathrm{mer,glob}}_{\check{G},\mathrm{Ran}} \text{ and } \mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathrm{Ran}}$$

that attach to $\underline{x} \in \mathrm{Ran}$ the spaces

$$\mathrm{LS}_{\check{G}}(X - \underline{x}) \text{ and } \mathrm{Op}_{\check{G}}(X - \underline{x}),$$

respectively (see Sect. B.7.14 for what we mean by local systems on an open curve).

Set

$$\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathrm{Ran}} := (\mathrm{LS}_{\check{G}} \times \mathrm{Ran}) \underset{\mathrm{LS}^{\mathrm{mer,glob}}_{\check{G},\mathrm{Ran}}}{\times} \mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathrm{Ran}}.$$

Thus, we have a Cartesian diagram

$$\begin{tikzcd}
\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathrm{Ran}} \arrow[r, "\iota^{\text{mon-free,glob}}"] \arrow[d, "\mathfrak{r}^{\mathrm{glob}}"'] & \mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathrm{Ran}} \arrow[d, "\mathfrak{r}^{\mathrm{glob}}"'] \\
\mathrm{LS}_{\check{G}} \times \mathrm{Ran} \arrow[r] & \mathrm{LS}^{\mathrm{mer,glob}}_{\check{G},\mathrm{Ran}}.
\end{tikzcd}$$

All of the spaces and maps in the above diagram have natural unital structures (see Sect. C.6.1 for what this means).

As we shall see shortly, $\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathrm{Ran}}$ is a relative ind-affine ind-scheme over Ran.

15.1.2. Restriction to the formal disc gives rise to the maps

$$\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathrm{Ran}} \to \mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathrm{Ran}} \text{ and } \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathrm{Ran}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}};$$

we will denote both by $\mathrm{ev}_{\mathrm{Ran}}$.

Note that the following commutative square is Cartesian

$$\begin{tikzcd}
\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathrm{Ran}} \arrow[r, "\mathrm{ev}_{\mathrm{Ran}}"] \arrow[d, "\iota^{\text{mon-free,glob}}"'] & \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}} \arrow[d, "\iota^{\text{mon-free}}"] \\
\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathrm{Ran}} \arrow[r, "\mathrm{ev}_{\mathrm{Ran}}"] & \mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathrm{Ran}}
\end{tikzcd} \tag{15.1}$$

15.1.3. For a given $\mathcal{Z} \to \mathrm{Ran}$, we will change the subscript

$$\mathrm{Ran} \rightsquigarrow \mathcal{Z}$$

for the corresponding base-changed spaces and maps.[55]

We can consider the assignment

$$\mathcal{Z} \mapsto \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}} := \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathcal{Z}_{\mathrm{dR}}})$$

as a crystal of categories over Ran, see Sect. B.13.8.

[55] We remind that, according to our conventions in Sect. B.8.17, when discussing crystals over Ran, given $\mathcal{Z} \to \mathrm{Ran}$, by default we base change to $\mathcal{Z}_{\mathrm{dR}}$ rather than too $\mathcal{Z}$.

15.1.4. Fix $\mathcal{Z} \to \mathrm{Ran}$, and recall that according to Sect. 5.3, we have a canonically defined action of the (symmetric) monoidal category $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}$ on $\mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}}$. Composing with

$$\Upsilon_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}}} : \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}},$$

we obtain an action of $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}$ on $\mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}}$.

Consider the category

$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} := \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}}. \tag{15.2}$$

Denote by $\mathrm{Id} \otimes (\mathrm{ev}_{\mathcal{Z}})^*$ the resulting functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \to \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}} = \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}}.$$

Note that the assignment

$$\mathcal{Z} \mapsto \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \tag{15.3}$$

is naturally a crystal of categories over Ran.

15.1.5. We define an action of $\mathrm{Rep}(\check{G})_{\mathcal{Z}}$ on the category (15.2) as follows.

Recall that we have a symmetric monoidal functor

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G},\mathcal{Z}} : \mathrm{Rep}(\check{G})_{\mathcal{Z}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathcal{Z}),$$

see Sect. 17.6.1.

Composing with

$$(\mathfrak{r}^{\mathrm{glob}})^* : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathcal{Z}) \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}}$$

we obtain a symmetric monoidal functor

$$(\mathfrak{r}^{\mathrm{glob}})^* \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G},\mathcal{Z}} : \mathrm{Rep}(\check{G})_{\mathcal{Z}} \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}}.$$

We let $\mathrm{Rep}(\check{G})_{\mathcal{Z}}$ act on (15.2) by $(\mathfrak{r}^{\mathrm{glob}})^* \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G},\mathcal{Z}}$ via the second factor.

15.1.6. The main result of this section is the following:

Theorem-Construction 15.1.7. *There exists a canonically defined functor*

$$\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}} : \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathcal{Z}} \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})$$

such that:

(a) *The functor*

$$\mathrm{Loc}_{G,\mathrm{crit},\mathcal{Z}} : \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})$$

factors as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \overset{\mathrm{Id}\otimes(\mathrm{ev}_{\mathcal{Z}})^*}{\longrightarrow} \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathcal{Z}} \overset{\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}}}{\longrightarrow} \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}). \tag{15.4}$$

(b) *The functor* $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}}$ *intertwines the above action of* $\mathrm{Rep}(\check{G})_{\mathcal{Z}}$ *on* (15.2) *and the action of* $\mathrm{Rep}(\check{G})_{\mathcal{Z}}$ *on* $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})$ *obtained from* $\mathrm{Sat}^{\mathrm{nv},-1}_G$ *and the action of* $\mathrm{Sph}_{G,\mathcal{Z}}$ *on* $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})$.

Remark 15.1.8. The Hecke eigen-property of the functor $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}}$ formulated in point (b) of Theorem 15.1.7 is not quite the full Hecke property one wants:

For example, when $\mathcal{Z} = \mathrm{pt}$ and $\mathcal{Z} \to \mathrm{Ran}$ corresponds to $\underline{x} \in \mathrm{Ran}$, the compatibility assertion in point (b) only talks about the Hecke action at $\underline{x}$, whereas one wants the compatibility with Hecke action over the entire Ran space.

See Corollary 15.5.9 for a stronger assertion, which we will deduce from Proposition 15.3.6 below.

15.2. **The key local construction.** In this subsection we will formulate Theorem 15.2.8, which is a local counterpart of Theorem 15.1.7.

15.2.1. Let

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}^{\subseteq}} \to \mathrm{Ran}^{\subseteq}$$

be the relative indscheme that attaches to

$$(\underline{x} \subseteq \underline{x}') \in \mathrm{Ran}^{\subseteq}$$

the space

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}'} := \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}-\underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}).$$

Remark 15.2.2. Note that $\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}^{\subseteq}}$ is exactly the geometric object that encodes the unital-in-correspondences structure on $\mathrm{Op}^{\text{mon-free}}_{\check{G}}$, see Sect. C.10.12.

15.2.3. We have the projections

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}} \overset{\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}}}{\longleftarrow} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}^{\subseteq}} \overset{\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big}}}{\longrightarrow} \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}} \tag{15.5}$$

given by restrictions along

$$\mathcal{D}_{\underline{x}} - \underline{x} \hookrightarrow \mathcal{D}_{\underline{x}'} - \underline{x} \hookleftarrow \mathcal{D}_{\underline{x}'} - \underline{x}',$$

respectively.

15.2.4. *Example.* For $\underline{x}' = \underline{x} \sqcup \underline{x}''$, we have

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) = \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}_{\underline{x}} - \underline{x}) \times \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}''}).$$

15.2.5. For $\mathcal{Z} \to \mathrm{Ran}$, let us denote by $\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathcal{Z}^{\subseteq}}$ the base-change

$$\mathcal{Z} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}^{\subseteq}}.$$

The assignment

$$\mathcal{Z} \mapsto \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}$$

is a crystal of symmetric monoidal categories over Ran, which we will denote by

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}).$$

The map $\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}}$ gives rise to a symmetric monoidal functor

$$(\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}})^* : \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}).$$

15.2.6. Denote

$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{loc}}_{\check{G}}}_{\mathrm{crit}} := \mathrm{KL}(G)_{\mathrm{crit}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}). \tag{15.6}$$

We will regard it as a crystal of categories over Ran. Denote by $\mathrm{Id} \otimes (\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}})^*$ the functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{KL}(G)_{\mathrm{crit}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}) = \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{loc}}_{\check{G}}}_{\mathrm{crit}}.$$

15.2.7. Recall that ins. vac. is a functor between crystals of categories over Ran

$$\mathrm{KL}(G)_{\mathrm{crit}} \to (\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\mathrm{KL}(G)_{\mathrm{crit}}), \tag{15.7}$$

and actually a (strict) functor between crystals of categories over $\mathrm{Ran}^{\mathrm{untl}}$

$$\mathrm{KL}(G)_{\mathrm{crit}} \to (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_{*,\mathrm{strict}} \circ (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^*(\mathrm{KL}(G)_{\mathrm{crit}}).$$

The key construction that we will need says the following:

Theorem-Construction 15.2.8. *There exists a canonically defined functor*

$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{loc}}_{\check{G}}}_{\mathrm{crit}} \overset{\mathrm{ins.vac}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}}{\longrightarrow} (\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\mathrm{KL}(G)_{\mathrm{crit}}), \tag{15.8}$$

such that ins. vac. *factors as*

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{Id}\otimes(\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}})^*}{\longrightarrow} \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{loc}}_{\check{G}}}_{\mathrm{crit}} \overset{\mathrm{ins.vac}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}}{\longrightarrow} (\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\mathrm{KL}(G)_{\mathrm{crit}}).$$

Furthermore, the functor $\mathrm{ins.vac}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}$ *is linear with respect to*

$$(\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})),$$

where:

- $(\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}))$ *acts on the left-hand side via* $(\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big}})^*$*;*
- $(\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}))$ *acts on the right-hand side by applying the functor* $(\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*$ *to the* $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$*-action on* $\mathrm{KL}(G)_{\mathrm{crit}}$.

The proof of Theorem 15.2.8 will be given in Sect. 16.

Remark 15.2.9. Note that Theorem 15.2.8 sounds semantically close to Theorem 15.1.7. And indeed, Theorem 15.1.7 will be deduced from Theorem 15.2.8 by a manipulation that involves factorization homology.

Yet, Theorem 15.2.8 is purely local, while Theorem 15.1.7 is "local-to-global".

That said, we will need Theorem 15.2.8 *not only* for the proof of Theorem 15.1.7. In the next subsection, we will use it to extract a stronger Hecke property for the functor $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}}$.

15.2.10. Let us explain what Theorem 15.2.8 says at the pointwise level, i.e., for a fixed $(\underline{x} \subseteq \underline{x}') \in \mathrm{Ran}^{\subseteq}$.

Write

$$\underline{x}' = \underline{x} \sqcup \underline{x}''.$$

The corresponding functor

$$\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'} : \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}'} \simeq \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}''}$$

acts as

$$\mathcal{M} \mapsto \mathcal{M} \otimes \mathrm{Vac}(G)_{\mathrm{crit},\underline{x}''}.$$

The pointwise statement of Theorem 15.2.8 is that this functor can be factored as

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \otimes \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}) \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}''},$$

where the second arrow is $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}''})$-linear.

In other words, we are saying that the object $\mathrm{Vac}(G)_{\mathrm{crit},\underline{x}''} \in \mathrm{KL}(G)_{\mathrm{crit},\underline{x}''}$ naturally lifts to an object of the category

$$\mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}''})}(\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}), \mathrm{KL}(G)_{\mathrm{crit},\underline{x}''}).$$

This lift is the basic feature of the vacuum object: it says that the structure of factorization $\mathfrak{z}_{\mathfrak{g}}$-module on $\mathrm{Vac}(G)_{\mathrm{crit}}$ as an object of $\mathrm{KL}(G)_{\mathrm{crit}}$ is obtained by restriction from a structure of *commutative* $\mathfrak{z}_{\mathfrak{g}}$-module.

The proof of Theorem 15.2.8 will amount to spelling out the above construction in the factorization setting.

Remark 15.2.11. In the proof of Theorem 15.2.8 that we will give, we will avoid using the FLE. We do this for aesthetical reasons: the construction of the functor Theorem 15.2.8 is more or less tautological if one says the right words.

However, if we use the FLE, there would be almost nothing to prove: the FLE, combined with the equivalence $\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}$, allows us to identity the two sides in (15.8) with

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}) \text{ and } (\mathrm{pr}_{\text{small}})_* \circ (\mathrm{pr}_{\text{big}})^*(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})),$$

respectively, and the functor in question is obtained by taking direct image along

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}^{\subseteq}} \overset{\mathrm{pr}^{\mathrm{Op}}_{\text{big}}}{\longrightarrow} \mathrm{Ran}^{\subseteq} \underset{\mathrm{pr}_{\text{big}},\mathrm{Ran}}{\times} \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}}.$$

That said, the functor $\mathrm{ins.vac}^{\text{mon-free}\rightsquigarrow\text{reg}}$ that we will construct does reproduce the above functor, by the nature of the FLE.

Remark 15.2.12. Note that the space $\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}^{\subseteq}}$, equipped with the maps (15.5), has a structure of groupoid[56] acting on $\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}}$, with the composition given by

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}')}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}(\mathcal{D}_{\underline{x}''} - \underline{x}') \simeq$$

$$\simeq (\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}')}{\times} \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}''} - \underline{x}')) \underset{(\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}')}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}''} - \underline{x}'))}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}''}) \to$$

$$\to \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}''} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}''} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}''}) = \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}}(\mathcal{D}_{\underline{x}''} - \underline{x}),$$

where the middle arrow is given by gluing along

$$\mathcal{D}_{\underline{x}'} - \underline{x} \underset{\mathcal{D}_{\underline{x}'} - \underline{x}'}{\sqcup} \mathcal{D}_{\underline{x}''} - \underline{x}' \simeq \mathcal{D}_{\underline{x}''} - \underline{x}.$$

Note that one can interpret Theorem 15.2.8 as saying that the above action of $\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}^{\subseteq}}$ on $\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}}$ can be lifted to an action on $\mathrm{KL}(G)_{\text{crit}}$ *at the level of 1-morphisms.*

In fact, Theorem 15.2.8 has a natural upgrade to a statement that we have a full datum of action of $\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}^{\subseteq}}$ on $\mathrm{KL}(G)_{\text{crit}}$. This is again an automatic if we allow ourselves to use the FLE.

15.3. **The expanded Hecke eigen-property.** In this subsection we will assume Theorem 15.1.7 and explain that the Hecke property of the functor $\mathrm{Loc}^{\mathrm{Op}}_{G,\text{crit},\mathcal{Z}}$ stated in point (b) of the theorem implies a stronger property (the difference between the two versions of the Hecke property is what was alluded to in Remark 15.1.8).

15.3.1. Let $\mathcal{Z} \to \mathrm{Ran}$ be as above. Consider the space

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg,glob}}_{\check{G},\mathcal{Z}^{\subseteq}} := \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathcal{Z}} \underset{\mathcal{Z},\mathrm{pr}_{\text{small},\mathcal{Z}}}{\times} \mathcal{Z}^{\subseteq}. \tag{15.9}$$

Denote by $\mathrm{pr}^{\mathrm{Op}^{\text{glob}}}_{\text{small},\mathcal{Z}}$ the projection

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg,glob}}_{\check{G},\mathcal{Z}^{\subseteq}} \to \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathcal{Z}}.$$

Note that we have a naturally defined ind-closed embedding

$$\mathrm{pr}^{\mathrm{Op}^{\text{glob}}}_{\text{big},\mathcal{Z}} : \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg,glob}}_{\check{G},\mathcal{Z}^{\subseteq}} \hookrightarrow \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathcal{Z}^{\subseteq}}. \tag{15.10}$$

Denote by $\mathrm{pr}^{\mathrm{Op}^{\text{glob}}}_{\text{big}}$ the composition of (15.10) with the projection

$$\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathcal{Z}^{\subseteq}} \overset{\mathrm{id}\times\mathrm{pr}_{\text{big}}}{\longrightarrow} \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathrm{Ran}}.$$

[56] This groupoid structure is part of the structure of being unital-in-correspondences, see Sect. C.10.

15.3.2. *Example.* Let us explain what the map (15.10) looks like for $\mathcal{Z} = \mathrm{pt}$ so that $\mathcal{Z} \to \mathrm{Ran}$ corresponds to $\underline{x} \in \mathrm{Ran}$ (in which case $\mathcal{Z}^{\subseteq} = \mathrm{Ran}_{\underline{x}}$, so a point on it corresponds to $\underline{x} \subseteq \underline{x}'$).

Then a point of the left-hand (resp., right-hand) side in (15.10) is a local system on X with an oper structure away from $\underline{x}$ (resp., $\underline{x}'$), and the map (15.10) is given by restriction along

$$X - \underline{x}' \subseteq X - \underline{x}.$$

15.3.3. Consider the map

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg},\text{glob}}_{\check{G},\mathcal{Z}^{\subseteq}} = \mathrm{Op}^{\text{mon-free},\text{glob}}_{\check{G},\mathcal{Z}} \underset{\mathcal{Z},\mathrm{pr}_{\mathrm{small},\mathcal{Z}}}{\times} \mathcal{Z}^{\subseteq} \overset{\mathfrak{r}^{\mathrm{glob}}\times\mathrm{id}}{\longrightarrow} (\mathrm{LS}_{\check{G}} \times \mathcal{Z}) \underset{\mathcal{Z}}{\times} \mathcal{Z}^{\subseteq} = \mathrm{LS}_{\check{G}} \times \mathcal{Z}^{\subseteq},$$

to be denoted $'\mathfrak{r}^{\mathrm{glob}}$.

Thus, we can consider the symmetric monoidal functor

$$('\mathfrak{r}^{\mathrm{glob}})^* \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G},\mathcal{Z}^{\subseteq}} : \mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}} \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg},\text{glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}.$$

15.3.4. We can view $\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg},\text{glob}}_{\check{G},\mathcal{Z}^{\subseteq}}$ as mapping to $\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}}$ by

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg},\text{glob}}_{\check{G},\mathcal{Z}^{\subseteq}} = \mathrm{Op}^{\text{mon-free},\text{glob}}_{\check{G},\mathcal{Z}} \underset{\mathcal{Z},\mathrm{pr}_{\mathrm{small},\mathcal{Z}}}{\times} \mathcal{Z}^{\subseteq} \to \mathrm{Op}^{\text{mon-free},\text{glob}}_{\check{G},\mathcal{Z}} \overset{\mathrm{ev}_{\mathcal{Z}}}{\to} \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}}.$$

Hence, we can form the category

$$\mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg},\text{glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}. \tag{15.11}$$

Since

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free},\text{glob}}_{\check{G}})_{\mathcal{Z}} \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq}) \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg},\text{glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}$$

is an equivalence, we can rewrite (15.11) as

$$\begin{aligned} \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free},\text{glob}}_{\check{G}})_{\mathcal{Z}} \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq}) = \\ = \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq}). \end{aligned} \tag{15.12}$$

Thus, we obtain that the category

$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq})$$

carries a monoidal action of $\mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}}$.

15.3.5. Consider the functor

$$\begin{aligned} \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq}) \overset{\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}}\otimes\mathrm{Id}}{\longrightarrow} \\ \to (\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})) \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq}) \simeq \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}^{\subseteq}). \end{aligned} \tag{15.13}$$

We claim:

Proposition 15.3.6. *The functor* (15.13) *intertwines the actions of* $\mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}}$ *on the two sides, where the action of* $\mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}}$ *on the left-hand sides is the one specified in Sect. 15.3.4, and on the right-hand side it is obtained from* $\mathrm{Sat}^{\mathrm{nv},-1}_G$ *and the action of* $\mathrm{Sph}_{G,\mathcal{Z}^{\subseteq}}$ *on* $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z}^{\subseteq})$.

15.4. **Proof of Proposition 15.3.6.**

15.4.1. Note that we have a Cartesian square

$$\begin{tikzcd} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg,glob}}_{\check{G},\mathcal{Z}^{\subseteq}} \arrow[r, "\mathrm{pr}^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{big},\mathcal{Z}}"] \arrow[d, "\mathrm{ev}_{\mathcal{Z}^{\subseteq}}"'] & \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathcal{Z}^{\subseteq}} \arrow[d, "\mathrm{ev}_{\mathcal{Z}^{\subseteq}}"] \\ \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathcal{Z}^{\subseteq}} \arrow[r, "\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big},\mathcal{Z}}"'] & \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}^{\subseteq}}, \end{tikzcd}$$

where $\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big},\mathcal{Z}}$ is the map whose composition with

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}^{\subseteq}} \simeq \mathcal{Z}^{\subseteq} \underset{\mathrm{pr}_{\mathrm{big}},\mathrm{Ran}}{\times} \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}}$$

is the map

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}^{\subseteq}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}^{\subseteq}} \overset{\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big}}}{\longrightarrow} \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}}.$$

We claim:

Lemma 15.4.2. *The functor*

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})_{\mathcal{Z}^{\subseteq}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}} \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}$$

is an equivalence.

The lemma will be proved in Sect. 15.10.

15.4.3. Consider the category

$$(15.14)\quad \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq}) =$$
$$= \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg,glob}}_{\check{G}})_{\mathcal{Z}} \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq}) \simeq$$
$$\simeq \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}.$$

By Lemma 15.4.2, we can rewrite it as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})_{\mathcal{Z}^{\subseteq}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}},$$

i.e.,

$$\mathrm{KL}^{\mathrm{Op}^{\mathrm{loc}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}},$$

where

$$\mathcal{Z} \mapsto \mathrm{KL}^{\mathrm{Op}^{\mathrm{loc}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}}$$

is the factorization category (15.6).

Using the functor $\mathrm{ins.vac}^{\text{mon-free}\rightsquigarrow\text{reg}}$ we obtain a functor

$$(15.15)\quad \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\text{D-mod}(\mathcal{Z})}{\otimes} \text{D-mod}(\mathcal{Z}^{\subseteq}) \simeq$$
$$\simeq \mathrm{KL}^{\mathrm{Op}^{\mathrm{loc}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}} \to$$
$$\overset{\mathrm{ins.vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\mathcal{Z}}\otimes\mathrm{Id}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}^{\subseteq}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}} \simeq$$
$$\simeq \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}^{\subseteq}}.$$

Remark 15.4.4. In the spirit of Remark 15.2.12, one can show that the functors (15.15) upgrade to a *local unital structure* (see Sect. 11.2.1 for what this means) on the crystal of categories over Ran given by (15.3).

15.4.5. The following property will be embedded into the construction of the assignment

$$\mathcal{Z} \mapsto \mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}} : \tag{15.16}$$

The composition

$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \overset{\mathrm{Id}\otimes \mathrm{pr}^!_{\mathrm{small}}}{\longrightarrow} \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z})}{\otimes} \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}^{\subseteq}) \overset{(15.15)}{\longrightarrow}$$
$$\to \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}^{\subseteq}} \overset{\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}^{\subseteq}}}{\longrightarrow} \mathrm{D}\text{-}\mathrm{mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}^{\subseteq}) \tag{15.17}$$

identifies with the functor

$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \overset{\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}}}{\longrightarrow}$$
$$\to \mathrm{D}\text{-}\mathrm{mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}) \overset{\mathrm{Id}\otimes \mathrm{pr}^!_{\mathrm{small}}}{\longrightarrow} \mathrm{D}\text{-}\mathrm{mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}^{\subseteq}). \tag{15.18}$$

Remark 15.4.6. In the spirit of Remark 15.4.4 one can show that the isomorphism between (15.17) and (15.18) upgrades to a *unital structure* (see Sect. 11.3.5 for what this means) on the assignment (15.16).

15.4.7. Since the functors

$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z})}{\otimes} \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}^{\subseteq}) \overset{(15.15)}{\longrightarrow}$$
$$\to \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}^{\subseteq}} \overset{\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}^{\subseteq}}}{\longrightarrow} \mathrm{D}\text{-}\mathrm{mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}^{\subseteq}) \tag{15.19}$$

and (15.13) are $\mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z}^{\subseteq})$-linear, and the isomorphism between (15.17) and (15.18) is $\mathrm{D}\text{-}\mathrm{mod}(\mathcal{Z})$-linear, we obtain that the functor (15.19) identifies with (15.13).

Thus, in order to prove Proposition 15.3.6, it suffices to construct the datum of compatibility with the $\mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}}$-action for the functor (15.19).

15.4.8. We will show that each of the two arrows in (15.19) is compatible with the $\mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}}$-action.

For the second arrow, this follows from Theorem 15.1.7(b).

15.4.9. For the first arrow, unwinding the definition of the functor (15.15), we have to show that the functor

$$\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}_{\check{G}})_{\mathcal{Z}^{\subseteq}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}} \to$$
$$\to \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg,glob}}_{\check{G}})_{\mathcal{Z}^{\subseteq}}$$

is $\mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}}$-linear, where:

- $\mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}}$ acts on the right-hand side via $\mathrm{Loc}^{\mathrm{spec}}_{\check{G},\mathcal{Z}^{\subseteq}}$ followed by pullback along

$$'\mathfrak{r}^{\mathrm{glob}} : \mathrm{Op}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg,glob}}_{\check{G},\mathcal{Z}^{\subseteq}} \to \mathrm{LS}_{\check{G}} \times \mathcal{Z}^{\subseteq}; \tag{15.20}$$

- $\mathrm{Rep}(\check{G})_{\mathcal{Z}^{\subseteq}}$ acts on the left-hand side on the second factor via $\mathrm{Loc}^{\mathrm{spec}}_{\check{G},\mathcal{Z}^{\subseteq}}$ followed by pullback along

$$\mathfrak{r}^{\mathrm{glob}} : \mathrm{Op}^{\mathrm{mon\text{-}free,glob}}_{\check{G},\mathcal{Z}^{\subseteq}} \to \mathrm{LS}_{\check{G}} \times \mathcal{Z}^{\subseteq}.$$

The required compatibility follows from the fact that

$$'\mathfrak{r}^{\mathrm{glob}} = \mathfrak{r}^{\mathrm{glob}} \circ \mathrm{pr}^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{big},\mathcal{Z}}$$

as maps

$$\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg},\mathrm{glob}}_{\check{G},\mathcal{Z}^{\subseteq}} \rightrightarrows \mathrm{LS}_{\check{G}} \times \mathcal{Z}^{\subseteq}.$$

15.5. **The *integrated* Hecke eigen-property.** In this subsection we will apply the paradigm of Sect. H.7, and deduce an ultimate form of compatibility of the functor $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}}$ with the Hecke action.

15.5.1. Consider $\mathrm{Rep}(\check{G})$ as a *unital* monoidal factorization category. Note that we consider the Hecke action as a local Ran-unital action of $\mathrm{Rep}(\check{G})$ on $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$ (see Sect. H.6.1 for what this means), i.e.,

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \in \mathrm{Rep}(\check{G})^{\mathrm{loc}}\textbf{-mod}.$$

In particular, the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, endowed with the *convolution* monoidal structure (i.e., the monoidal category $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\star}$, see Sect. H.5.2), on $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$ factors via

$$(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\star} \twoheadrightarrow \mathrm{Rep}(\check{G})_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}.$$

For any $\mathcal{Z} \to \mathrm{Ran}$, we will consider

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathcal{Z})$$

as a module over

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \otimes \text{D-mod}(\mathcal{Z}).$$

15.5.2. Note also that the spectral localization functor can be viewed as a (strictly) unital (symmetric) monoidal functor

$$\underline{\mathrm{Loc}}^{\mathrm{spec}}_{\check{G}} : \underline{\mathrm{Rep}}(\check{G}) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}),$$

where $\underline{\mathrm{Rep}}(\check{G})$ is the crystal of categories over $\mathrm{Ran}^{\mathrm{untl}}$ corresponding to $\mathrm{Rep}(\check{G})$, viewed as a factorization category.

In particular, the functor

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

factors as

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \twoheadrightarrow \mathrm{Rep}(\check{G})_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}).$$

Given $\mathcal{Z} \to \mathrm{Ran}$, we will consider the category

$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\mathcal{Z}} = \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free},\mathrm{glob}}_{\check{G}})_{\mathcal{Z}}$$

as acted on by

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \otimes \text{D-mod}(\mathcal{Z})$$

via the projection

$$\mathrm{Op}^{\text{mon-free},\mathrm{glob}}_{\check{G},\mathcal{Z}} \to \mathrm{LS}_{\check{G}} \times \mathcal{Z}$$

and the action on the second factor.

15.5.3. We claim:

Corollary 15.5.4. *The functor*

$$\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}} : \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathcal{Z}} \to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z})$$

is compatible with the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \otimes \mathrm{D\text{-}mod}(\mathcal{Z})$.

Proof. First, it is easy to see that we can assume that $\mathcal{Z}$ is an affine scheme S. By Corollary H.7.8, we can consider both sides, i.e.,

$$\mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(S) \text{ and } \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},S},$$

as objects of

$$\mathrm{Rep}(\check{G})^{\mathrm{loc},\mathrm{untl}}_S\textbf{-mod}$$

(see Sect. H.7.3 for the notation), and we have to show that $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},S}$ extends to a map inside this category (see Corollary H.7.8).

Now, Proposition 15.3.6 implies that the functor $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},S}$ gives rise to a functor between the images of these two objects under the forgetful functor

$$\mathrm{Rep}(\check{G})^{\mathrm{loc},\mathrm{untl}}_S\textbf{-mod} \to \mathrm{Rep}(\check{G})^{\mathrm{loc}}_S\textbf{-mod}. \tag{15.21}$$

Now, the assertion follows from the fact that the functor (15.21) is fully faithful, see Sect. H.7.3. □

Corollary 15.5.5. *The functor*

$$\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}} : \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathcal{Z}} \to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z})$$

is compatible with the action of $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\star}$.

15.5.6. Let now $\mathcal{Z}$ be pseudo-proper, and consider the functor

$$(\mathrm{Id} \otimes \mathrm{C}^{\cdot}_c(\mathcal{Z}, -)) : \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathcal{Z}) \to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G).$$

This functor is obviously compatible with the action of $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\star}$.

Denote

$$\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\int_{\mathcal{Z}}} := (\mathrm{Id} \otimes \mathrm{C}^{\cdot}_c(\mathcal{Z}, -)) \circ \mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\mathcal{Z}}, \quad \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathcal{Z}} \to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G).$$

Hence, from Corollary 15.5.5 we obtain:

Corollary 15.5.7. *The functor* $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\int_{\mathcal{Z}}}$ *is compatible with the action of* $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\star}$ *on the two sides.*

15.5.8. Finally, we take $\mathcal{Z} = \mathrm{Ran}$. Denote the corresponding functor

$$\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\int_{\mathrm{Ran}}} : \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathrm{Ran}} \to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

by $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit}}$.

We obtain:

Corollary 15.5.9. *The the functor* $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit}}$ *is compatible with the action of* $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\star}$ *(with the convolution monoidal structure) on the two sides.*

This corollary is the ultimate form of the compatibility between the (globalized) critical localization functor and the (non-derived) Hecke action.

15.5.10. A variant of Corollary 15.5.5 was at the core of the construction of Hecke eigensheaves in [BD1]. In our language, this construction can be reformulated as follows.

Take $\mathcal{Z} = \mathrm{pt}$, so that $\mathcal{Z} \to \mathrm{Ran}$ corresponds to $\underline{x} \in \mathrm{Ran}$.

Recall that according to Theorem 6.1.4, Proposition 3.8.7 and Proposition 3.7.10, we can identify

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \overset{\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}}{\simeq} \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \overset{\Upsilon_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}}{\simeq} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \tag{15.22}$$

as $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$-module categories.

Hence, we can identify the category $\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathcal{Z}}$ with $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$, as a module category over $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$. Since $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})$ is locally almost of finite type and formally smooth, by [GaRo1, Theorem 10.1.1], the functor

$$\Upsilon_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})} : \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})) \to \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$$

is an equivalence. Hence, we can further identify $\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathcal{Z}}$ with $\mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$ as a $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$-module category.

Hence, we can view $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\underline{x}}$ as a functor

$$\mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})) \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

that intertwines the action of $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\star$ on $\mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$, given by

$$(\mathfrak{r}^{\mathrm{glob}})^* \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\star \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$$

and the action of $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$ on $\mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}))$, and the $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\star$-action on $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$.

In particular, for a k-point $\sigma \in \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})$ and the corresponding sky-scraper sheaf

$$k_\sigma \in \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})),$$

the object

$$\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\underline{x}}(k_\sigma) \in \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

is a Hecke eigensheaf with eigenvalue $\mathfrak{r}^{\mathrm{glob}}(\sigma) \in \mathrm{LS}_{\check{G}}$.

Remark 15.5.11. The case that is actually considered in [BD1] is when σ is a regular oper. In our language, this corresponds replacing $\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}}_{\mathrm{crit},\mathcal{Z}}$ with

$$\mathrm{KL}(G)^{\mathrm{reg}}_{\mathrm{crit},\mathcal{Z}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})_{\mathcal{Z}}}{\otimes} (\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(X)) \otimes \text{D-mod}(\mathcal{Z})),$$

where

$$\mathrm{KL}(G)^{\mathrm{reg}}_{\mathrm{crit},\mathcal{Z}} := \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}}(\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})_{\mathcal{Z}}, \mathrm{KL}(G)_{\mathrm{crit},\mathcal{Z}}).$$

15.6. **Proof of Theorem 15.1.7.**

15.6.1. In the proof of Theorem 15.1.7, for expositional purposes we will assume that $\mathcal{Z} = \mathrm{pt}$, so its map to Ran corresponds to $\underline{x} \in \mathrm{Ran}$. We will denote the corresponding space $\mathcal{Z}^{\subseteq}$ by $\mathrm{Ran}_{\underline{x}}$.

15.6.2. Before we launch the proof, let us explain the idea that lies behind it. Let $\mathcal{A}$ be a factorization algebra, and let $\mathcal{A}^{\mathrm{ch}}$ be the corresponding chiral algebra, see Sect. D.1.1. Let $\mathfrak{z}$ be a commutative factorization algebra, such that $\mathfrak{z}^{\mathrm{ch}}$ maps to the center of $\mathcal{A}^{\mathrm{ch}}$.

Let $\mathcal{M}$ be an object of

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}} = \mathcal{A}^{\mathrm{ch}}\text{-mod}^{\mathrm{ch}}_{\underline{x}}$$

equipped with a *commutative* action of $\mathfrak{z}^{\mathrm{ch}}$, which is compatible with the action of $\mathcal{A}^{\mathrm{ch}}$.

We claim that in this case $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{M})_{\underline{x}}$ carries an action of $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathfrak{z})$. Let us construct the action morphism.

We can interpret the given structure on $\mathcal{M}$ as a map of modules

$$\mathfrak{z}_x \otimes \mathcal{M} \to \mathcal{M} \tag{15.23}$$

compatible with a map of the chiral algebras

$$\mathfrak{z}^{\mathrm{ch}} \otimes \mathcal{A}^{\mathrm{ch}} \to \mathcal{A}^{\mathrm{ch}}.$$

By the functoriality of factorization homology, we obtain a map

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathfrak{z}) \otimes \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{M})_{\underline{x}} \simeq \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathfrak{z}, \mathfrak{z}_x) \otimes \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{M})_{\underline{x}} \overset{\sim}{\to}$$
$$\to \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathfrak{z} \otimes \mathcal{A}, \mathfrak{z}_x \otimes \mathcal{M})_{\underline{x}} \to \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{M})_{\underline{x}},$$

which is the required action map.

The resulting action of $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathfrak{z})$ on $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{M})_{\underline{x}}$ has the following property:

The action of $\mathfrak{z}_x$ on $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{M})_{\underline{x}}$ obtained from the homomorphism

$$\mathfrak{z}_x \to \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathfrak{z}, \mathfrak{z}_x) \simeq \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathfrak{z})$$

equals the action obtained from the $\mathfrak{z}_x$-action on $\mathcal{M}$ by endomorphisms of the chiral $\mathcal{A}^{\mathrm{ch}}$-module structure.

It is a souped-up version of this construction that will be used in Sect. 15.6.8 below. See also Remark 15.6.15.

15.6.3. Write

$$\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \simeq \text{“}\underset{R}{\mathrm{colim}}\text{”}\ \mathrm{Spec}(R),$$

where $Y := \mathrm{Spec}(R) \to \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}$ are closed embeddings almost of finite presentation.

We claim:

Lemma 15.6.4. *For any* $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$*-module category* $\mathbf{C}$*, the naturally defined functor*

$$\underset{R}{\mathrm{colim}}\ \mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}(\mathrm{IndCoh}^!(Y), \mathbf{C}) \to \mathbf{C}$$

is an equivalence.

This lemma will be proved in Sect. 15.10.

15.6.5. For every $Y = \mathrm{Spec}(R)$ as above, denote

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} := \mathrm{Funct}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}(\mathrm{IndCoh}^!(Y), \mathrm{KL}(G)_{\mathrm{crit},\underline{x}}).$$

We consider it as a $\mathrm{QCoh}(Y)$-linear category via

$$\Upsilon_Y : \mathrm{QCoh}(Y) \to \mathrm{IndCoh}^!(Y).$$

15.6.6. Denote
$$Y^{\mathrm{glob}} := Y \underset{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(X-\underline{x}).$$

Denote by
$$\mathrm{ev}_{\underline{x}} : Y^{\mathrm{glob}} \to Y \tag{15.24}$$
the evaluation map.

Denote
$$\mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\underline{x},Y} := \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Y^{\mathrm{glob}}).$$

Denote by
$$\mathrm{Id}\otimes \mathrm{ev}^*_{\underline{x}} : \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\underline{x},Y}$$
the corresponding functor.

15.6.7. By Lemma 15.6.4, in order to construct the functor $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\underline{x}}$, it suffices to construct a compatible family of functors
$$\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\underline{x},Y} : \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\underline{x},Y} \to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G),$$
such that:

(a) The functor
$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \xrightarrow{\mathrm{Loc}_{G,\kappa,\underline{x}}} \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \tag{15.25}$$
factors as
$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \xrightarrow{\mathrm{Id}\otimes\mathrm{ev}^*_{\underline{x}}} \mathrm{KL}(G)^{\mathrm{Op}^{\mathrm{glob}}_{\check{G}}}_{\mathrm{crit},\underline{x},Y} \xrightarrow{\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\underline{x},Y}} \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G); \tag{15.26}$$

(b) The functor $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\underline{x},Y}$ is $\mathrm{Rep}(\check{G})_{\underline{x}}$-linear.

15.6.8. Let R^{glob} denote the algebra of functions on the (affine) scheme Y^{glob}. The closed embedding (15.24) gives rise to a homomorphism
$$R \to R^{\mathrm{glob}}. \tag{15.27}$$

The datum of $\mathrm{Loc}^{\mathrm{Op}}_{G,\mathrm{crit},\underline{x},Y}$ together with the factorization in point (a) above is equivalent to the datum of action of R^{glob} on the functor (15.25), such that the action of R obtained by precomposing with (15.27) is identified with the action coming from the action of R on the identity endofunctor of $\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y}$.

15.6.9. By unitality, we can rewrite (15.25) as
$$\begin{aligned}\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} &\xrightarrow{\mathrm{ins.vac}_{\underline{x}}} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}} \xrightarrow{(\mathrm{Loc}_{G,\kappa})_{\mathrm{Ran}_{\underline{x}}}} \\ &\to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G)\otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}) \xrightarrow{\mathrm{Id}\otimes \mathrm{C}^{\cdot}_c(\mathrm{Ran}_{\underline{x}},-)} \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G).\end{aligned} \tag{15.28}$$

Let $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}}$ be the object in $\mathrm{ComAlg}(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\underline{x}})$ from Sect. F.2.6. Let $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ be the underlying object in $\mathrm{ComAlg}(\mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}))$.

Let us view $\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}$ as a category tensored over $\mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})$. In particular, it makes sense to talk about algebras in $\mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})$ acting on objects in $\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}$ or on functors with values in $\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}$.

We will show that $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ acts on the functor
$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \xrightarrow{\mathrm{ins.vac}_{\underline{x}}} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}. \tag{15.29}$$

Moreover, when we evaluate the natural transformation

$$(\mathrm{diag}_{\underline{x}})_! \simeq (\mathrm{diag}_{\underline{x}})_! \circ (\mathrm{diag}_{\underline{x}})^! \circ \mathrm{ins.\,vac}_{\underline{x}} \to \mathrm{ins.\,vac}_{\underline{x}}$$

on $\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y}$, the action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on the right-hand side will be compatible with the action of

$$R \simeq (\mathrm{diag}_{\underline{x}})^!((\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}) \tag{15.30}$$

on the left-hand side.

15.6.10. Assume for a moment the existence of such an action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on (15.29), and let us produce from it an action of R^{glob} on (15.25).

First, by functoriality, we obtain an action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on the functor

$$\begin{aligned} \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \overset{\mathrm{ins.vac}_{\underline{x}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}} &\overset{(\mathrm{Loc}_{G,\kappa})_{\mathrm{Ran}_{\underline{x}}}}{\longrightarrow} \\ &\to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}). \end{aligned} \tag{15.31}$$

15.6.11. Note that $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}} \in \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})$ belongs to the essential image of the restriction functor

$$\mathsf{t}^! : \mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}) \to \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}).$$

In particular, it belongs to the subcategory

$$\mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})^{\text{almost-untl}} \subset \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})$$

(see Sect. C.5.15).

In particular, $\mathrm{C}^\cdot_c(\mathrm{Ran}_{\underline{x}}, (\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}})$ acquires a structure of (commutative) algebra.

15.6.12. The essential image of

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \overset{\mathrm{ins.vac}_{\underline{x}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}$$

belongs to the essential image of

$$\mathsf{t}^! : \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}} \to \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}.$$

Hence, the essential image of (15.31) belongs to the essential image of

$$(\mathrm{Id} \otimes \mathsf{t}^!) : \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}) \to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}).$$

In particular, it belongs to the subcategory

$$\mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})^{\text{almost-untl}} \subset \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}).$$

15.6.13. Hence, by Sect. C.5.15, the action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on (15.31) gives rise to an action of the commutative algebra

$$\mathrm{C}^\cdot_c(\mathrm{Ran}_{\underline{x}}, (\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}})$$

on the functor (15.28).

15.6.14. Finally, by Lemmas F.2.8 and F.3.5, we identify

$$\mathrm{C}^{\cdot}_c(\mathrm{Ran}_{\underline{x}}, (\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}) \simeq \mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}, R) \simeq R^{\mathrm{glob}}$$

as (commutative) algebras.

This gives the sought-for action of R^{glob} on (15.25). The compatbility with the R-action on the identity endofunctor of $\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y}$ follows from the fact that the (iso)morphism

$$\mathrm{Loc}_{G,\kappa,\underline{x}} \simeq \mathrm{Loc}_{G,\kappa,\mathrm{Ran}_{\underline{x}}} \circ (\mathrm{diag}_{\underline{x}})_! \simeq$$
$$\simeq \mathrm{Loc}_{G,\kappa,\mathrm{Ran}_{\underline{x}}} \circ (\mathrm{diag}_{\underline{x}})_! \circ (\mathrm{diag}_{\underline{x}})^! \circ \mathrm{ins.\,vac}_{\underline{x}} \to \mathrm{Loc}_{G,\kappa,\mathrm{Ran}_{\underline{x}}} \circ \mathrm{ins.\,vac}_{\underline{x}}$$

intertwines the R-action on the left-hand side and the $\mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}, R)$-action on the right-hand side.

Remark 15.6.15. A simplified version of the above argument proves the following statement:

Let $\mathcal{A}$ be a commutative factorization algebra. Consider the factorization category $\mathcal{A}\text{-mod}^{\mathrm{com}}$ as a unital sheaf of categories on Ran. Consider the local-to-global functor

$$\mathcal{A}\text{-mod}^{\mathrm{com}} \to \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}^{\mathrm{untl}})$$

given by

$$(\mathcal{Z} \to \mathrm{Ran}) \rightsquigarrow \mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \mathcal{A}, -)_{\mathcal{Z}}.$$

Then:

(a) The above functor is acted on by $\mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \mathcal{A})$; in particular, upgrades to a (strictly unital) local-to-global functor

$$\mathcal{A}\text{-mod}^{\mathrm{com}} \to \mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \mathcal{A})\text{-mod} \otimes \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}^{\mathrm{untl}}). \tag{15.32}$$

(b) The functor (15.32) is *universal* among strictly unital local-to-global functors.

In the language of Sect. H.1, point (b) can reformulated as saying that (15.32) induces an equivalence

$$\mathcal{A}\text{-mod}^{\mathrm{com}}_{\mathrm{Ran,indep}} \to \mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \mathcal{A})\text{-mod}.$$

15.7. **Construction of the algebra action.** In this subsection we will construct the sought-for action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on the functor (15.29).

15.7.1. By Theorem 15.2.8, the functor (15.29) factors as

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \overset{\mathrm{Id}\otimes(\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}})^*}{\longrightarrow} \tag{15.33}$$
$$\to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \overset{\mathrm{ins.vac}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}_{\underline{x}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}},$$

while the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \overset{\mathrm{Id}\otimes(\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small}})^*}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$$

which appears in (15.33), factors naturally as

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \to$$
$$\to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{mon\text{-}free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}).$$

15.7.2. We rewrite $\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$ tautologically as

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \Big(\mathrm{QCoh}(Y) \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}})\Big).$$

We now claim:

Lemma 15.7.3. *The naturally defined functor*

$$\mathrm{QCoh}(Y) \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \to \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big)$$

is an equivalence.

The lemma will be proved in Sect. 15.10.

15.7.4. Applying Lemma 15.7.3, we obtain that the functor (15.29) can be factored as

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big) \to$$
$$\to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \xrightarrow{\mathrm{ins.vac}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\underline{x}}} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}. \tag{15.34}$$

Hence, it is enough to construct an action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on the composition

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big) \to$$
$$\to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \xrightarrow{\mathrm{ins.vac}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\underline{x}}} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}. \tag{15.35}$$

15.7.5. Consider the category

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big) \tag{15.36}$$

as tensored over

$$\mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big) \tag{15.37}$$

and hence over $\mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})$.

We note that, according to Lemma F.3.5,

$$Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}} \simeq \mathrm{Spec}_{\mathrm{Ran}_{\underline{x}}}((\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}).$$

Hence, $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ maps (isomorphically) to endomorphisms of the monoidal unit

$$\mathcal{O}_{Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}} \in \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big).$$

Hence, it acts by endomorphisms of the identity functor on (15.36).

15.7.6. The functors in the composition (15.35) are $\mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})$-linear. This produces the sought-for action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on (15.35).

The compatibility of this action with (15.30) follows from the construction.

15.8. **Verification of the Hecke property: reduction to a local statement.** The goal of this and the next subsections is to verify property (b) from Sect. 15.6.7.

For expositional reasons, we will fix an object $V \in \operatorname{Rep}(\check{G})_{\underline{x}}$ and show that the functor $\operatorname{Loc}^{\operatorname{Op}}_{G,\operatorname{crit},\underline{x},Y}$ intertwines the actions of V on the two sides.

We will reduce the local-to-global assertion we are after to a purely local one, namely, (15.42).

15.8.1. Let $\mathcal{V}$ denote the object of $\operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ equal to $(\mathfrak{r}^{\operatorname{reg}})^*(V)$, where we identify

$$\operatorname{Rep}(\check{G})_{\underline{x}} \simeq \operatorname{QCoh}(\operatorname{LS}^{\operatorname{reg}}_{\check{G},\underline{x}}).$$

Let

$$\mathcal{V}^{\operatorname{glob}} := (\mathfrak{r}^{\operatorname{glob}})^* \circ \operatorname{Loc}^{\operatorname{spec}}_{\check{G},\underline{x}}(V) \in \operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}}_{\check{G}}(X - \underline{x})).$$

Note that we have

$$\mathcal{V}^{\operatorname{glob}} \simeq \operatorname{ev}^*_{\underline{x}}(\mathcal{V}),$$

where

$$\operatorname{ev}_{\underline{x}} : \operatorname{Op}^{\text{mon-free}}_{\check{G}}(X - \underline{x}) \to \operatorname{Op}^{\text{mon-free}}_{\check{G},\underline{x}}.$$

Denote by

$$\mathcal{V}_Y \text{ and } \mathcal{V}^{\operatorname{glob}}_Y$$

the restrictions of $\mathcal{V}$ and $\mathcal{V}^{\operatorname{glob}}$ to

$$Y \hookrightarrow \operatorname{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \text{ and } Y^{\operatorname{glob}} \hookrightarrow \operatorname{Op}^{\text{mon-free}}_{\check{G}}(X - \underline{x}),$$

respectively.

By a slight abuse of notation, we will denote by the same symbols $\mathcal{V}_Y$ and $\mathcal{V}^{\operatorname{glob}}_Y$ global sections of the corresponding vector bundles, viewed as modules over R and R^{glob}, respectively.

15.8.2. Let $\mathcal{M}$ be an object of $\operatorname{KL}(G)_{\kappa,\underline{x},Y}$, and consider the object

$$\operatorname{Loc}_{G,\kappa,\underline{x}}(\mathcal{M}) \in \operatorname{D-mod}_{\operatorname{crit}}(\operatorname{Bun}_G).$$

The construction in Sects. 15.6.10-15.6.14 endows $\operatorname{Loc}_{G,\kappa,\underline{x}}(\mathcal{M})$ with an action of R^{glob}.

Let

$$\operatorname{H}_V : \operatorname{D-mod}_{\operatorname{crit}}(\operatorname{Bun}_G) \to \operatorname{D-mod}_{\operatorname{crit}}(\operatorname{Bun}_G)$$

be the Hecke endofunctor corresponding to V.

On the one hand, by functoriality, the object

$$\operatorname{H}_V(\operatorname{Loc}_{G,\kappa,\underline{x}}(\mathcal{M})) \in \operatorname{D-mod}_{\operatorname{crit}}(\operatorname{Bun}_G)$$

acquires an action of R^{glob}.

On the other hand, we can consider

$$\mathcal{V}^{\operatorname{glob}}_Y \underset{R^{\operatorname{glob}}}{\otimes} \operatorname{Loc}_{G,\kappa,\underline{x}}(\mathcal{M}) \in R^{\operatorname{glob}}\text{-mod}(\operatorname{D-mod}_{\operatorname{crit}}(\operatorname{Bun}_G)).$$

15.8.3. The statement of (b) in Sect. 15.6.7 is that we have a canonical isomorphism

$$\operatorname{H}_V(\operatorname{Loc}_{G,\kappa,\underline{x}}(\mathcal{M})) \simeq \mathcal{V}^{\operatorname{glob}}_Y \underset{R^{\operatorname{glob}}}{\otimes} \operatorname{Loc}_{G,\kappa,\underline{x}}(\mathcal{M}) \tag{15.38}$$

as objects of $R^{\operatorname{glob}}\text{-mod}(\operatorname{D-mod}_{\operatorname{crit}}(\operatorname{Bun}_G))$.

Thus, our goal is to establish (15.38).

15.8.4. First, we rewrite the right-hand side in (15.38). Namely,

$$\mathcal{V}_Y^{\mathrm{glob}} \underset{R^{\mathrm{glob}}}{\otimes} \mathrm{Loc}_{G,\kappa,\underline{x}}(\mathcal{M}) \simeq \mathrm{Loc}_{G,\kappa,\underline{x}}(\mathcal{V}_Y \underset{R}{\otimes} \mathcal{M}),$$

where:

- We regard $\mathcal{V}_Y \underset{R}{\otimes} \mathcal{M}$ as an object $\mathrm{KL}(G)_{\kappa,\underline{x},Y}$;
- $\mathrm{Loc}_{G,\kappa,\underline{x}}(-)$ acquires an action of R^{glob} via the construction in Sects. 15.6.10-15.6.14.

15.8.5. We will now rewrite the left-hand side in (15.38).

Consider the category $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}$. It carries an action of $\mathrm{Sph}_{G,\mathrm{Ran}_{\underline{x}}}$. We will denote the action functor by

$$\mathcal{F}, \mathcal{M}' \mapsto \mathcal{F} \cdot \mathcal{M}'.$$

The unital structure on Sph_G gives rise to a (monoidal) functor

$$\mathrm{ins.unit}_{\underline{x}} : \mathrm{Sph}_{G,\underline{x}} \to \mathrm{Sph}_{G,\mathrm{Ran}_{\underline{x}}}^{\overset{!}{\otimes}}$$

(see Sect. H.5.5 for the notation).

In particular, we obtain a monoidal action of $\mathrm{Sph}_{G,\underline{x}}$ on $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}$.

15.8.6. Recall that the object

$$\mathrm{ins.vac}_{\underline{x}}(\mathcal{M}) \in \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}$$

carries an action of the algebra object $(\mathcal{O}_{\mathrm{Op}_{\check{G}}^{\mathrm{reg},Y}})_{\mathrm{Ran}_{\underline{x}}} \in \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})$. By functoriality, we obtain that

$$\mathrm{ins.unit}_{\underline{x}}(\mathrm{Sat}_G^{-1,\mathrm{nv}}(V)) \cdot \mathrm{ins.vac}_{\underline{x}}(\mathcal{M}) \in \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}} \tag{15.39}$$

also carries an action of $(\mathcal{O}_{\mathrm{Op}_{\check{G}}^{\mathrm{reg},Y}})_{\mathrm{Ran}_{\underline{x}}}$.

Further, the object

$$\mathrm{Loc}_{G,\mathrm{crit},\mathrm{Ran}_{\underline{x}}}\Big(\mathrm{ins.unit}_{\underline{x}}(\mathrm{Sat}_G^{-1,\mathrm{nv}}(V)) \cdot \mathrm{ins.vac}_{\underline{x}}(\mathcal{M})\Big) \in \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}) \tag{15.40}$$

also carries an action of $(\mathcal{O}_{\mathrm{Op}_{\check{G}}^{\mathrm{reg},Y}})_{\mathrm{Ran}_{\underline{x}}}$.

15.8.7. The object (15.39) belongs to the essential image of the restriction functor

$$\mathsf{t}^! : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}} \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}.$$

Hence, the object (15.40) belongs to the essential image of

$$(\mathrm{Id} \otimes \mathsf{t}^!) : \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}) \to \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}).$$

In particular, it belongs to

$$\mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})^{\mathrm{almost\text{-}untl}} \subset \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}).$$

Hence, by Sect. C.5.15, we obtain that the object

$$(\mathrm{Id} \otimes \mathrm{C}^{\cdot}_c(\mathrm{Ran}_{\underline{x}}, -)) \circ \mathrm{Loc}_{G,\mathrm{crit},\mathrm{Ran}_{\underline{x}}}\Big(\mathrm{ins.unit}_{\underline{x}}(\mathrm{Sat}_G^{-1,\mathrm{nv}}(V)) \cdot \mathrm{ins.vac}_{\underline{x}}(\mathcal{M})\Big) \tag{15.41}$$

acquires an action of

$$\mathrm{C}^{\cdot}_c(\mathrm{Ran}_{\underline{x}}, (\mathcal{O}_{\mathrm{Op}_{\check{G}}^{\mathrm{reg},Y}})_{\mathrm{Ran}_{\underline{x}}}) \simeq \mathrm{C}^{\mathrm{fact}}_{\cdot}(X, \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}}, R) \simeq R^{\mathrm{glob}}.$$

15.8.8. Recall now that the functor

$$\mathrm{Loc}_{G,\mathrm{crit},\mathrm{Ran}_{\underline{x}}} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}} \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}_{\underline{x}})$$

is $\mathrm{Sph}^{\overset{!}{\otimes}}_{G,\mathrm{Ran}_{\underline{x}}}$-linear.

Note also that the functor

$$\mathrm{Id} \otimes \mathrm{C}^{\cdot}_c(\mathrm{Ran}_{\underline{x}}, -) : \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}_{\underline{x}}) \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

is $\mathrm{Sph}_{G,\underline{x}}$-linear, where $\mathrm{Sph}_{G,\underline{x}}$ acts on the left-hand side via $\mathrm{ins.\,unit}_{\underline{x}}$.

Combining, we obtain that the functor

$$(\mathrm{Id} \otimes \mathrm{C}^{\cdot}_c(\mathrm{Ran}_{\underline{x}}, -)) \circ \mathrm{Loc}_{G,\mathrm{crit},\mathrm{Ran}_{\underline{x}}}, \quad \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}} \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

is $\mathrm{Sph}_{G,\underline{x}}$-linear.

15.8.9. Combining Sects. 15.8.6-15.8.7 with Sect. 15.8.8, we obtain that

$$\mathrm{H}_V(\mathrm{Loc}_{G,\kappa,\underline{x}}(\mathcal{M})) \in R^{\mathrm{glob}}\text{-mod}(\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G))$$

identifies with the object (15.41), with the R^{glob}-action specified in 15.8.6-15.8.7.

15.8.10. Hence, we obtain that in order to prove (15.38), it suffices to establish an isomorphism

$$\mathrm{ins.\,unit}_{\underline{x}}(\mathrm{Sat}_G^{-1,\mathrm{nv}}(V)) \cdot \mathrm{ins.\,vac}_{\underline{x}}(\mathcal{M}) \simeq \mathrm{ins.\,vac}_{\underline{x}}(\mathcal{V}_Y \underset{R}{\otimes} \mathcal{M}) \tag{15.42}$$

as $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$-modules in $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}$, where:

- We regard $\mathcal{V}_Y \underset{R}{\otimes} \mathcal{M}$ as an object $\mathrm{KL}(G)_{\kappa,\underline{x},Y}$;
- $\mathrm{ins.\,vac}_{\underline{x}}(-)$ on each side acquires an action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ via the construction in Sect. 15.7.

Remark 15.8.11. As was mentioned in the preamble to this subsection, we fixed objects $\mathcal{M} \in \mathrm{KL}(G)_{\kappa,\underline{x},Y}$ and $V \in \mathrm{Rep}(\check{G})_{\underline{x}}$ for expositional reasons. The actual assertion behind (15.42), and one that we actually prove in Sect. 15.9, is that that $\mathrm{ins.\,vac}_{\underline{x}}$, viewed as a functor

$$\mathrm{KL}(G)_{\kappa,\underline{x},Y} \to ((\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}})\text{-mod}(\mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}),$$

is $\mathrm{Rep}(\check{G})_{\underline{x}}$-linear.

15.9. **Verification of the Hecke property at the local level.** In this subsection we will construct the identification (15.42) and thereby complete the verification of point (b) in Sect. 15.6.7.

15.9.1. Let us apply Lemma 5.4.2. It implies that we can rewrite the left-hand side in (15.42) as

$$\mathfrak{r}^*(\mathrm{ins.\,unit}_{\underline{x}}(V)) \otimes \mathrm{ins.\,vac}_{\underline{x}}(\mathcal{M}), \tag{15.43}$$

where:

- $\mathrm{ins.\,unit}_{\underline{x}} : \mathrm{Rep}(\check{G})_{\underline{x}} \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}_{\underline{x}}}$ is the unital structure on $\mathrm{Rep}(\check{G})$;
- We identify $\mathrm{Rep}(\check{G})_{\mathrm{Ran}_{\underline{x}}} \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$;
- $\mathfrak{r}$ denotes the map $\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}} \to \mathrm{LS}^{\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}$;
- $\otimes$ refers to the action of $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$ on $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}$.

The action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on (15.43) is obtained by functoriality from the $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$-action on $\mathrm{ins.\,vac}_{\underline{x}}(\mathcal{M})$.

15.9.2. We now note that the linearity with respect to

$$(\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*(\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}))$$

in Theorem 15.2.8 implies that the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big) \to \tag{15.44}$$

$$\to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \overset{\mathrm{ins.vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\underline{x}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_{\underline{x}}}$$

of (15.35) is linear with respect to $\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$, which acts on

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big)$$

via the pullback along

$$Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}} \overset{p_2}{\to} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}} \overset{\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big},\underline{x}}}{\longrightarrow} \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}},$$

where $\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big},\underline{x}}$ is the map from Sect. 15.4.1.

15.9.3. Hence, we can rewrite (15.43) as the value on $\mathcal{M} \in \mathrm{KL}(G)_{\kappa,\underline{x},Y}$ of the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big) \to \tag{15.45}$$

$$\overset{\mathrm{Id}\otimes(-\otimes(\mathfrak{r}\circ\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big},\underline{x}}\circ p_2)^*(\mathrm{ins.unit}_{\underline{x}}(V)))}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big) \to$$

$$\overset{(15.35)}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}.$$

In terms of this identification, the action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on (15.43) is obtained from the action of $(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg},Y}_{\check{G}}})_{\mathrm{Ran}_{\underline{x}}}$ on the functor (15.44) (which is the same as (15.35)) from Sect. 15.7.6.

15.9.4. Thus, we obtain that in order to prove (15.42), it suffices to establish an isomorphism between the value on $\mathcal{M}$ of the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{IndCoh}^!(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big) \to \tag{15.46}$$

$$\overset{\mathrm{Id}\otimes(-\otimes(\mathfrak{r}\circ\mathrm{pr}^{\mathrm{Op}}_{\mathrm{big},\underline{x}}\circ p_2)^*(\mathrm{ins.unit}_{\underline{x}}(V)))}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big)$$

and the value on $\mathcal{V}_Y \underset{R}{\otimes} \mathcal{M}$ of the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}\Big(Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\Big). \tag{15.47}$$

15.9.5. In order to do that, it suffices to establish an isomorphism between the following vector bundles on

$$Y \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}:$$

- The pullback of $\text{ins.unit}_{\underline{x}}(V) \in \text{Rep}(\check{G})_{\text{Ran}_{\underline{x}}} \simeq \text{QCoh}(\text{LS}^{\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}})$ along

$$Y \underset{\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}} \overset{p_2}{\to} \text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}} \xrightarrow{\text{pr}^{\text{Op}}_{\text{big},\underline{x}}} \text{Op}^{\text{mon-free}}_{\check{G},\text{Ran}_{\underline{x}}} \overset{\mathfrak{r}}{\to} \text{LS}^{\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}};$$

- The pullback of $\mathcal{V}_Y$ along

$$Y \underset{\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}} \to Y.$$

15.9.6. Note that for $(\underline{x} \subseteq \underline{x}') \in \text{Ran}_{\underline{x}}$, restriction along

$$\mathcal{D}_{\underline{x}} \hookrightarrow \mathcal{D}_{\underline{x}'}$$

gives rise to a map

$$\text{LS}^{\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}} \to \text{LS}^{\text{reg}}_{\check{G},\underline{x}}, \tag{15.48}$$

so that

$$\text{ins.unit}_{\underline{x}} \simeq (15.48)^*.$$

The required isomorphism of vector bundles follows from the commutative diagram

$$\begin{tikzcd}
Y \underset{\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} \text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}} \arrow[r, "\text{id}\times p_1"] \arrow[d, "p_2"'] & Y \arrow[r] & \text{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \arrow[dd, "\mathfrak{r}"] \\
\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}} \arrow[d, "\text{pr}^{\text{Op}}_{\text{big},\underline{x}}"'] & & \\
\text{Op}^{\text{mon-free}}_{\check{G},\text{Ran}_{\underline{x}}} \arrow[r, "\mathfrak{r}"] & \text{LS}^{\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}} \arrow[r, "(15.48)"] & \text{LS}^{\text{reg}}_{\check{G},\underline{x}}.
\end{tikzcd}$$

15.10. **Proofs of Lemmas 15.6.4 and 15.7.3.**

15.10.1. We first prove Lemma 15.6.4. In fact, the assertion holds for $\text{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}})$ replaced by an arbitrary ind-placid ind-scheme Z.

Namely, for $Y_1 \overset{\iota_{1,2}}{\hookrightarrow} Y_2$, the functor

$$\text{Funct}_{\text{IndCoh}^!(Z)}(\text{IndCoh}^!(Y_1), \mathbf{C}) \to \text{Funct}_{\text{IndCoh}^!(Z)}(\text{IndCoh}^!(Y_2), \mathbf{C})$$

admits a right adjoint, given by precomposition with $(\iota_{1,2})^{\text{IndCoh}}_*$.

Hence, we can rewrite the colimit

$$\underset{Y}{\text{colim}}\ \text{Funct}_{\text{IndCoh}^!(Z)}(\text{IndCoh}^!(Y), \mathbf{C})$$

as a limit with respect to the above right adjoint functors.

The latter limit is the same as

$$\text{Funct}_{\text{IndCoh}^!(Z)}\left(\underset{Y}{\text{colim}}\ \text{IndCoh}^!(Y), \mathbf{C}\right).$$

We now use the fact that the functor

$$\underset{Y}{\text{colim}}\ \text{IndCoh}^!(Y) \to \text{IndCoh}(Z)$$

is an equivalence.

□[Lemma 15.6.4]

15.10.2. The rest of this subsection is devoted to the proof of Lemma 15.7.3. Consider the Cartesian diagram

$$\begin{CD}
\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}} @>>> \mathrm{Op}^{\text{mer}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}} \\
@V{\mathrm{pr}^{\mathrm{Op}}_{\text{small},\underline{x}}}VV @VV{\mathrm{pr}^{\mathrm{Op}}_{\text{small},\underline{x}}}V \\
\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} @>{\iota^{\text{mon-free}}}>> \mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}},
\end{CD}$$

where $\mathrm{Op}^{\text{mer}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}$ is as in Sect. F.3.3.

15.10.3. We will prove:

Lemma 15.10.4.

(a) *The category* $\mathrm{QCoh}(\mathrm{Op}^{\text{mer}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$ *is dualizable as a* $\mathrm{QCoh}(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})$*-module.*

(b) *For any affine* $Y \to \mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}}$*, the functor*

$$\mathrm{QCoh}(Y) \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}^{\text{mer}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \to \mathrm{QCoh}(Y \underset{\mathrm{Op}^{\text{mer}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\text{mer}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$$

is an equivalence.

It is easy to see that Lemma 15.10.4 implies (15.7.3) by passage to the limit.

Thus, the rest of this subsection is devoted to the proof of Lemma 15.10.4.

15.10.5. First, a standard limit-colimit procedure reduces the assertion to the case when we replace $\mathrm{Ran}_{\underline{x}}$ by

$$\mathcal{Z} := (X^I_{\underline{x}})_{\mathrm{dR}},$$

where:

- We think of $\underline{x}$ as a point of X^J for some finite set J;
- I is a finite set with a map $J \to I$;
- $X^I_{\underline{x}} := X^I \underset{X^J,\underline{x}}{\times} \mathrm{pt}$.

Further, we can assume that X is affine and admits an étale map to $\mathbb{A}^1$.

Second, we can replace the D-scheme $\mathrm{Op}_{\check{G}}$ by $\mathrm{Jets}(\mathcal{E})$, where $\mathcal{E}$ is the total space of a vector bundle on X (see Sect. 3.1.7).

15.10.6. Note that for X as above, we have (non-canonical) isomorphisms

$$\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_{\mathcal{Z}} \simeq \mathfrak{L}^+(\mathcal{E})_{\mathcal{Z}} \times (\mathfrak{L}(\mathcal{E})_{\underline{x}}/\mathfrak{L}^+(\mathcal{E})_{\underline{x}})$$

and

$$\mathfrak{L}(\mathcal{E})_{\underline{x}} \simeq \mathfrak{L}^+(\mathcal{E})_{\underline{x}} \times (\mathfrak{L}(\mathcal{E})_{\underline{x}}/\mathfrak{L}^+(\mathcal{E})_{\underline{x}}),$$

so that the projection

$$\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_{X^I} \to \mathfrak{L}(\mathcal{E})_{\underline{x}}$$

corresponds to the projection

$$\mathfrak{L}^+(\mathcal{E})_{\mathcal{Z}} \to \mathfrak{L}^+(\mathcal{E})_{\underline{x}}.$$

We can therefore identify

$$\mathrm{QCoh}(\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_{\mathcal{Z}}) \simeq \mathrm{QCoh}(\mathfrak{L}^+(\mathcal{E})_{\mathcal{Z}}) \otimes \mathrm{QCoh}(\mathfrak{L}(\mathcal{E})_{\underline{x}}/\mathfrak{L}^+(\mathcal{E})_{\underline{x}})$$

and

$$\mathrm{QCoh}(\mathfrak{L}(\mathcal{E})_{\underline{x}}) \simeq \mathrm{QCoh}(\mathfrak{L}^+(\mathcal{E})_{\underline{x}}) \otimes \mathrm{QCoh}(\mathfrak{L}(\mathcal{E})_{\underline{x}}/\mathfrak{L}^+(\mathcal{E})_{\underline{x}}).$$

15.10.7. To prove point (a), it suffices to show that $\mathrm{QCoh}(\mathfrak{L}^+(\mathcal{E})_{\mathcal{Z}})$ is dualizable as a module over $\mathrm{QCoh}(\mathfrak{L}(\mathcal{E})_{\underline{x}})$. However, this is obvious, since both geometric objects are affine schemes.

15.10.8. To prove point (b), it is sufficient to do so for a cofinal family of Y's. Hence, we can assume that Y is invariant under translations with respect to $\mathfrak{L}^+(\mathcal{E})_{\underline{x}}$. Hence, we can identify

$$Y \underset{\mathfrak{L}(\mathcal{E})_{\underline{x}}}{\times} \mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_{\mathcal{Z}} \simeq \mathfrak{L}^+(\mathcal{E})_{\underline{x}} \times (Y/\mathfrak{L}^+(\mathcal{E})_{\underline{x}}).$$

We have

$$\operatorname{QCoh}(Y \underset{\mathfrak{L}(\mathcal{E})_{\underline{x}}}{\times} \mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_{\mathcal{Z}}) \simeq \operatorname{QCoh}(\mathfrak{L}^+(\mathcal{E})_{\mathcal{Z}}) \otimes \operatorname{QCoh}(Y/\mathfrak{L}^+(\mathcal{E})_{\underline{x}}),$$

which makes the assertion of point (b) manifest.

$\square$[Lemma 15.10.4]

16. Proof of Theorem 15.2.8

16.1. **Reformulation and strategy.**

16.1.1. For expositional purposes we will let $\mathcal{Z} = \text{pt}$, so that $\mathcal{Z} \to \operatorname{Ran}$ corresponds to $\underline{x} \in \operatorname{Ran}$.

Hence, our goal is to construct a functor

$$\operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}) \underset{\operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \operatorname{KL}(G)_{\text{crit},\underline{x}} \overset{\text{ins.vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\underline{x}}}{\longrightarrow} \operatorname{KL}(G)_{\kappa,\operatorname{Ran}_{\underline{x}}}$$

such that

$$\text{ins. vac}_{\underline{x}} : \operatorname{KL}(G)_{\text{crit},\underline{x}} \to \operatorname{KL}(G)_{\kappa,\operatorname{Ran}_{\underline{x}}}$$

factors as

$$\operatorname{KL}(G)_{\text{crit},\underline{x}} \overset{(\text{pr}^{\text{Op}}_{\text{small},\underline{x}})^*\otimes\text{Id}}{\longrightarrow} \operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}) \underset{\operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \operatorname{KL}(G)_{\text{crit},\underline{x}} \to$$
$$\overset{\text{ins.vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\underline{x}}}{\longrightarrow} \operatorname{KL}(G)_{\kappa,\operatorname{Ran}_{\underline{x}}}, \tag{16.1}$$

and such that the functor $\text{ins. vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\underline{x}}$ is $\operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}}_{\check{G},\operatorname{Ran}_{\underline{x}}})$-linear via

$$(\text{pr}^{\text{Op}}_{\text{big},\underline{x}})^* : \operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}}_{\check{G},\operatorname{Ran}_{\underline{x}}}) \to \operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}).$$

16.1.2. Precomposing with the functor

$$\Upsilon_{\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}} : \operatorname{QCoh}(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}) \to \operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}), \tag{16.2}$$

we obtain that it suffices to carry out the construction in Sect. 16.1.1 above for $\operatorname{QCoh}(-)$ replaced[57] by $\operatorname{IndCoh}^!(-)$.

I.e., from now on our goal will be to construct a functor

$$\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}) \underset{\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \operatorname{KL}(G)_{\text{crit},\underline{x}} \overset{\text{ins.vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\underline{x}}}{\longrightarrow} \operatorname{KL}(G)_{\kappa,\operatorname{Ran}_{\underline{x}}} \tag{16.3}$$

such that $\text{ins. vac}_{\underline{x}}$ factors as

$$\operatorname{KL}(G)_{\text{crit},\underline{x}} \overset{(\text{pr}^{\text{Op}}_{\text{small},\underline{x}})^!\otimes\text{Id}}{\longrightarrow} \operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}) \underset{\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \operatorname{KL}(G)_{\text{crit},\underline{x}} \to$$
$$\overset{\text{ins.vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\underline{x}}}{\longrightarrow} \operatorname{KL}(G)_{\kappa,\operatorname{Ran}_{\underline{x}}}, \tag{16.4}$$

and such that the functor $\text{ins. vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\underline{x}}$ is $\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}}_{\check{G},\operatorname{Ran}_{\underline{x}}})$-linear via

$$(\text{pr}^{\text{Op}}_{\text{big},\underline{x}})^! : \operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}}_{\check{G},\operatorname{Ran}_{\underline{x}}}) \to \operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\operatorname{Ran}_{\underline{x}}}).$$

[57]One can show (using Lemma 15.10.4 combined with a parallel statement for $\operatorname{IndCoh}^!$) that the functor (16.2) is actually an equivalence.

16.1.3. Consider the factorization functor

$$\mathcal{V}ac(G)_{\mathrm{crit}} : \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \simeq \mathfrak{z}\text{-mod}^{\mathrm{cl}} \to \mathrm{KL}(G)_\kappa,$$

given by

$$\mathcal{F} \mapsto \mathcal{F} \underset{\mathfrak{z}}{\otimes} \mathrm{Vac}(G)_{\mathrm{crit}}.$$

16.1.4. Recall that according to the conventions in Sect. B.12.4, for a factorization category $\mathbf{A}$, we denote by $\mathbf{A}^{\mathrm{fact}_{\underline{x}}}$ the vacuum object of $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}}$, i.e., the object whose underlying category is $\mathbf{A}_x$.

Thus, we can consider

$$(\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}} \in \mathrm{KL}(G)_\kappa\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}}.$$

Consider the object

$$\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}) \in \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}}.$$

16.1.5. *Example.* For $\underline{x} \sqcup \underline{x}'' = \underline{x}' \in \mathrm{Ran}_{\underline{x}'}$, the fiber of $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})$ at $\underline{x}'$ is

$$\mathrm{KL}(G)_{\kappa,\underline{x}} \otimes \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}).$$

16.1.6. Consider the assignment

$$(\mathcal{Z} \to \mathrm{Ran}_{\underline{x}}) \mapsto \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}} \underset{\mathrm{Ran}_{\underline{x}}}{\times} \mathcal{Z}_{\mathrm{dR}})$$

as a crystal of (symmetric) monoidal categories over $\mathrm{Ran}_{\underline{x}}$, denote it by $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$.

Note that it extends naturally to a crystal of (symmetric) monoidal categories over $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$, in which the monoidal structure is given by *strict* functors between sheaves of categories.

16.1.7. The key step will be to construct an action of

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$$

on

$$\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})$$

(as crystals of categories over $\mathrm{Ran}_{\underline{x}}$). Furthermore, we will show this action extends to a *strict* action as crystals of categories over $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$.

16.1.8. The above strict compatibility means in particular that the functor

$$\mathrm{ins.vac}_{\underline{x}} : \mathrm{KL}(G)_{\kappa,\underline{x}} \to \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\mathrm{Ran}_{\underline{x}}}$$

intertwines the $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$-action on $\mathrm{KL}(G)_{\kappa,\underline{x}}$ and the $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$-action on $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\mathrm{Ran}_{\underline{x}}}$ via the functor

$$\mathrm{ins.unit}_{\underline{x}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}),$$

while the latter is the functor of !-pullback along

$$\mathrm{pr}^{\mathrm{Op}}_{\mathrm{small},\underline{x}} : \mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}.$$

Hence, we obtain a functor

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{KL}(G)_{\kappa,\underline{x}} \to \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\mathrm{Ran}_{\underline{x}}}. \tag{16.5}$$

16.1.9. Composing with the tautological functor

$$\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\mathrm{Ran}_{\underline{x}}} \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}},$$

we obtain the sought-for functor

$$\mathrm{ins.vac}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\underline{x}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \underset{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{KL}(G)_{\kappa,\underline{x}} \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_{\underline{x}}}$$

of (16.3).

16.2. **The acting agents.** In this subsection we will interpret the category $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$, in terms of *factorization restriction.*

16.2.1. Recall the map of factorization spaces

$$\iota^{+,\text{mon-free}}:\mathrm{Op}^{\text{reg}}_{\check{G}}\to\mathrm{Op}^{\text{mon-free}}_{\check{G}}.$$

Consider the corresponding factorization functors

$$\mathrm{QCoh}(\mathrm{Op}^{\text{reg}}_{\check{G}})\simeq\mathrm{IndCoh}^*(\mathrm{Op}^{\text{reg}}_{\check{G}})\overset{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}{\longrightarrow}\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \tag{16.6}$$

and

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{reg}}_{\check{G}})\overset{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}{\longrightarrow}\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}). \tag{16.7}$$

16.2.2. We start with

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}\in\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}}$$

and

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}\in\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}}$$

and consider the resulting objects

$$\mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}})\in\mathrm{QCoh}(\mathrm{Op}^{\text{reg}}_{\check{G}})\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}} \tag{16.8}$$

and

$$\mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}})\in\mathrm{IndCoh}^!(\mathrm{Op}^{\text{reg}}_{\check{G}})\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}}. \tag{16.9}$$

Since the factorization functors (16.6) and (16.7) are unital, the module categories (16.8) and (16.9) have natural unital structures, see Sect. C.14.15.

16.2.3. Let us consider the assignments

$$(\mathcal{Z}\to\mathrm{Ran}_{\underline{x}})\mapsto\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\underset{\mathrm{Ran}_{\underline{x}}}{\times}\mathcal{Z}_{\mathrm{dR}})$$

and

$$(\mathcal{Z}\to\mathrm{Ran}_{\underline{x}})\mapsto\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}}\underset{\mathrm{Ran}_{\underline{x}}}{\times}\mathcal{Z}_{\mathrm{dR}})$$

as crystals of categories over $\mathrm{Ran}_{\underline{x}}$ (the latter is the crystal of categories that we have introduced in Sect. 16.1.6).

They have natural structures of unital module categories (at $\underline{x}$) over

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{reg}}_{\check{G}})\text{ and }\mathrm{IndCoh}^!(\mathrm{Op}^{\text{reg}}_{\check{G}}),$$

respectively. We will denote them by

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}\text{ and }\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}},$$

respectively.

16.2.4. We will regard $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$ (resp., $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$) as equipped with a comonoidal (resp., monoidal) structure given by IndCoh^*-pushforward (resp., !-pullback) along the diagonal morphism.

We note that when we view $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$ as a crystal of categories over $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$, its comonoidal structure is given by right-lax functors.

By contrast, when we view $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$ as a crystal of categories over $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$, its monoidal structure is given by *strict* functors.

16.2.5. The map $\iota^{+,\text{mon-free}}$ gives rise to a map

$$\iota^{+,\text{mon-free}\to\text{reg}} : \mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\mathrm{Ran}_{\underline{x}}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}}.$$

(Note that $\iota^{+,\text{mon-free}}$ is the same as the map $\mathrm{pr}^{\mathrm{Op}}_{\text{big},\underline{x}}$).

We obtain functors of (unital) module categories

$$(\iota^{+,\text{mon-free}\to\text{reg}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\text{fact}_{\underline{x}}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\text{fact}_{\underline{x}}} \tag{16.10}$$

and

$$(\iota^{+,\text{mon-free}\to\text{reg}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\text{fact}_{\underline{x}}} \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\text{fact}_{\underline{x}}}, \tag{16.11}$$

compatible with the functors (16.6) and (16.7), respectively.

16.2.6. By Sect. B.12.11, the functors (16.10) and (16.11) give rise to maps

$$(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\text{fact}_{\underline{x}}} \to \mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\text{fact}_{\underline{x}}}) \tag{16.12}$$

and

$$(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\text{fact}_{\underline{x}}} \to \to \mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\text{fact}_{\underline{x}}}) \tag{16.13}$$

in

$$\mathrm{QCoh}(\mathrm{Op}^{\text{reg}}_{\check{G}})\text{-}\mathbf{mod}^{\text{fact}}_{\underline{x}} \text{ and } \mathrm{IndCoh}^!(\mathrm{Op}^{\text{reg}}_{\check{G}})\text{-}\mathbf{mod}^{\text{fact}}_{\underline{x}},$$

respectively. Moreover, the maps (16.12) and (16.13) are compatible with the unital structures, see Lemma C.14.16.

16.2.7. The following is a variant of Lemma F.5.7:

Lemma 16.2.8. *The functors* (16.12) *and* (16.13) *are equivalences.*

Proof. We prove the assertion for IndCoh^*. The case of $\mathrm{IndCoh}^!$ is analogous.

By Lemma B.15.9, it suffices to check that:

(i) The functor (16.6) admits a right adjoint (as a functor between sheaves of categories);

(ii) The functor (16.10) admits a right adjoint (as a functor between sheaves of categories);

(iii) The functor (16.10) induces an equivalence between the fibers of the two sides at $\underline{x} \in \mathrm{Ran}_{\underline{x}}$.

We note that point (iii) holds tautologically.

Points (i) and (ii) are also automatic: the right adjoints in question are given by the funtors $(\iota^{+,\text{mon-free}})^!$ and $(\iota^{+,\text{mon-free}\to\text{reg}})^!$, respectively.

□

16.3. **Construction of the action.**

16.3.1. Recall the object

$$\mathrm{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\mathrm{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}}) \in \mathrm{QCoh}(\mathrm{Op}^{\text{reg}}_{\check{G}})\text{-}\mathbf{mod}^{\text{fact}}_{\underline{x}}.$$

We claim that $\mathrm{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\mathrm{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})$, viewed as a sheaf of categories over $\mathrm{Ran}_{\underline{x}}$, carries a canonically defined action of $\mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\text{fact}_{\underline{x}}})$.

16.3.2. By duality, a datum of such an action is equivalent to the datum of a coaction of the sheaf of comonoidal categories $\mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}})$.

We will construct the corresponding coaction functor

$$\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}) \to \tag{16.14}$$
$$\to \mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}) \otimes \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}).$$

The full datum of coaction is defined similar to Sect. 5.3, using the device from Sect. J.

16.3.3. We interpret the right-hand side in (16.14) as the restriction of

$$(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \otimes \mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}} \in (\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \otimes \mathrm{KL}(G)_\kappa)\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}}$$

along the factorization functor

$$((\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_* \otimes \mathcal{V}ac(G)_{\mathrm{crit}}) : \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \otimes \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \otimes \mathrm{KL}(G)_\kappa.$$

The functor (16.14) is given by the procedure of restriction from Sect. B.12.15 along the diagram

$$\begin{CD}
\mathrm{KL}(G)_\kappa @>>> \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \otimes \mathrm{KL}(G)_\kappa \\
@A{\mathcal{V}ac(G)_{\mathrm{crit}}}AA @AA{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_* \otimes \mathcal{V}ac(G)_{\mathrm{crit}}}A \\
\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) @>>> \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \otimes \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}),
\end{CD}$$

where:

- The top horizontal arrow is the coaction of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ on $\mathrm{KL}(G)_\kappa$;
- The bottom horizontal arrow is the comonoidal operation, i.e., the functor of direct image along the diagonal map.

16.3.4. *Example.* Here is what the above action (resp., coaction) does at the pointwise level. Write

$$\underline{x}' = \underline{x} \sqcup \underline{x}'',$$

so that

$$\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\underline{x}'} \simeq \mathrm{KL}(G)_{\kappa,\underline{x}} \otimes \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}),$$

$$\mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}})_{\underline{x}'} \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \otimes \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}),$$

$$\mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}})_{\underline{x}'} \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \otimes \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}).$$

The coaction of $\mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}})_{\underline{x}'}$ on $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\underline{x}'}$ acts as the tensor product of

- The coaction of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ on $\mathrm{KL}(G)_{\kappa,\underline{x}}$, and
- The functor of direct image along the diagonal map $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}) \to \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}) \otimes \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''})$.

The action of $\mathrm{Res}_{(\iota^{+,\text{mon-free}})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}})_{\underline{x}'}$ on $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\underline{x}'}$ acts as the tensor product of

- The action of $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})$ on $\mathrm{KL}(G)_{\kappa,\underline{x}}$, and
- The canonical action of $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''})$ on $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''}) \simeq \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}''})$.

16.4. **The unital structure on the action functor.**

16.4.1. Let us regard

$$\mathrm{KL}(G)_\kappa \text{ and } \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$$

as crystals of categories on Ran, equipped with a *unital* structure (see Sect. 11.2.1 for what this means).

The coaction of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ on $\mathrm{KL}(G)_\kappa$ has the following feature: it extends to a coaction in the 2-category of crystals of categories on $\mathrm{Ran}^{\mathrm{untl}}$ with *right-lax* functors.

This follows from the construction of this coaction in Sect. 5.3, using the following observation:

For a map of factorization algebras $\mathcal{A}_1 \to \mathcal{A}_2$ in a given factorization category $\mathbf{A}$, the restriction functor

$$\mathcal{A}_2\text{-mod}(\mathbf{A}) \to \mathcal{A}_1\text{-mod}(\mathbf{A}),$$

viewed as a functor between crystals of categories on Ran, admits a natural extension to a right-lax functor between crystals of categories on $\mathrm{Ran}^{\mathrm{untl}}$.

16.4.2. It follows from the construction in Sect. 16.3.3 and Sect. C.14.20 that the functor (16.14) extends to a right-lax functor between crystals of categories on $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$.

By a similar token, we obtain that the full datum of coaction of the comonoidal category $\mathrm{Res}_{(\iota^+,\text{mon-free})^{\mathrm{IndCoh}}_*}(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}})$ on $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})$ extends to a coaction in the 2-category of crystals of categories on $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$ with right-lax functors.

16.4.3. Combining with Lemma 16.2.8, we obtain that $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})$ carries a coaction of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$.

Furthermore, this coaction extends to a coaction in the 2-category of crystals of categories on $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$ with right-lax functors.

16.4.4. We note now that the unital structures on

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}} \text{ and } \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$$

have the following feature:

The counit of the duality

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}} \otimes \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}} \to \underline{\text{D-mod}}(\mathrm{Ran}_{\underline{x}})$$

extends to a right-lax functor between crystals of categories on $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$.

16.4.5. The coaction of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$ on $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})$ gives rise to an action of $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$ on $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})$.

Moreover, by Sect. 16.4.4, this action extends to an action in the 2-category of sheaves of categories on $\mathrm{Ran}^{\mathrm{untl}}$ with *right-lax* functors.

16.4.6. We now claim:

Lemma 16.4.7. *The right-lax functors that define the action of* $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G}})^{\mathrm{fact}_{\underline{x}}}$ *on* $\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})$ *as crystals of categories on* $\mathrm{Ran}^{\mathrm{untl}}$ *are* strict.

Proof. We need to show the following: for $\underline{x} \subseteq \underline{x}_1 \subseteq \underline{x}_2$, the natural transformation from

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_1}) \otimes \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \overset{\text{action}}{\to} \tag{16.15}$$

$$\to \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \overset{\mathrm{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}}{\longrightarrow} \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2}$$

to

$$\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_1})\otimes \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \overset{\text{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}\otimes\text{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}}{\longrightarrow} \to \operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_2})\otimes \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2} \overset{\text{action}}{\to} \to \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2} \tag{16.16}$$

is an isomorphism.

The question of a functor between crystals of categories (in this case, over $\operatorname{Ran}_{\underline{x}}$) being an isomorphism can be checked strata-wise. So we can assume that

$$\underline{x}_2=\underline{x}_1\sqcup\underline{x}'.$$

We identify

$$\operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2}\simeq \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1}\otimes \operatorname{QCoh}(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''}) \tag{16.17}$$

and

$$\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_2})\otimes \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2}\simeq \simeq \operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_1})\otimes \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \bigotimes \bigotimes \operatorname{IndCoh}^!(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''})\otimes \operatorname{QCoh}(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''}), \tag{16.18}$$

so that the functor

$$\operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \overset{\text{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}}{\longrightarrow} \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2}$$

identifies with

$$\operatorname{Id}\otimes\mathcal{O}_{\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''}}$$

and the functor

$$\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_1})\otimes \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \overset{\text{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}\otimes\text{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}}{\longrightarrow} \to \operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_2})\otimes \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2}$$

identifies with

$$\operatorname{Id}\otimes\omega_{\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''}}\otimes\mathcal{O}_{\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''}}.$$

In terms of (16.17) and (16.18), the functor

$$\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_2})\otimes \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2} \overset{\text{action}}{\to} \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_2}$$

is the tensor product of

$$\operatorname{IndCoh}^!(\operatorname{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_1})\otimes \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \overset{\text{action}}{\to} \operatorname{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\operatorname{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1}$$

along the first factor and

$$\operatorname{IndCoh}^!(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''})\otimes\operatorname{QCoh}(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''})\simeq \operatorname{IndCoh}^!(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''})\otimes\operatorname{IndCoh}^*(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''}) \overset{\overset{!}{\otimes}}{\to} \to \operatorname{IndCoh}^*(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''})\simeq\operatorname{QCoh}(\operatorname{Op}^{\text{reg}}_{\check{G},\underline{x}''})$$

along the second factor.

Hence, we obtain that, in terms of (16.17) and (16.18), both functors (16.15) and (16.16) are identified with

$$\text{IndCoh}^!(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\underline{x}\subseteq\underline{x}_1}) \otimes \text{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\text{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \overset{\text{action}}{\to} \tag{16.19}$$
$$\to \text{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\text{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \xrightarrow{\text{Id}\otimes\omega_{\text{Op}^{\text{reg}}_{\check{G},\underline{x}''}}}$$
$$\to \text{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\text{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\underline{x}\subseteq\underline{x}_1} \otimes \text{IndCoh}^!(\text{Op}^{\text{reg}}_{\check{G},\underline{x}''}).$$

Unwinding the construction, we obtain that the endomorphism of (16.19) defined by the structure of right-lax functor on the action map is the identity map.

□

16.5. **End of the construction.**

16.5.1. Thus, we have carried out the construction announced in Sect. 16.1.7. In particular, we obtain a functor

$$\text{IndCoh}^!(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}}) \underset{\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \text{KL}(G)_{\text{crit},\underline{x}} \to \text{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\text{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\text{Ran}_{\underline{x}}}, \tag{16.20}$$

and its composition of the functor (16.5) with

$$\text{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\text{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\text{Ran}_{\underline{x}}} \to \text{KL}(G)_{\kappa,\text{Ran}_{\underline{x}}} \tag{16.21}$$

produces the sought-for functor

$$\text{ins.}\,\text{vac}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\underline{x}} : \text{IndCoh}^!(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}}) \underset{\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \text{KL}(G)_{\kappa,\underline{x}} \to \text{KL}(G)_{\kappa,\text{Ran}_{\underline{x}}}. \tag{16.22}$$

16.5.2. It remains to show that the functor (16.22) is $\text{IndCoh}^!(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}})$-linear.

The functor (16.20) is $\text{IndCoh}^!(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}})$-linear by construction. So it remains to show that the functor (16.21) is also $\text{IndCoh}^!(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}})$-linear.

16.5.3. Unwinding the construction of the coaction of $\text{Res}_{(\iota^+,\text{mon-free})^{\text{IndCoh}}_*}(\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})^{\text{fact}_{\underline{x}}})$ on $\text{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\text{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})$ in Sect. 16.3.3, we obtain that the functor (16.21) intertwines the coaction of $\text{Res}_{(\iota^+,\text{mon-free})^{\text{IndCoh}}_*}(\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})^{\text{fact}_{\underline{x}}})$ on the left-hand side with the coaction of $\text{IndCoh}^*(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}})$ on the right hand side via the tautological forgetful functor

$$\text{Res}_{(\iota^+,\text{mon-free})^{\text{IndCoh}}_*}(\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})^{\text{fact}_{\underline{x}}}) \to \text{IndCoh}^*(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}}).$$

Hence, the functor (16.21) intertwines the coaction of $\text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G},\text{Ran}_{\underline{x}}})$ on the left-hand side of (16.21) with the action of $\text{IndCoh}^*(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}})$ on the right-hand side of (16.21) via $(\text{pr}^{\text{Op}}_{\text{big},\underline{x}})^{\text{IndCoh}}_*$.

Passing to the dual of the acting agents, we obtain that functor (16.21) intertwines the action of $\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G},\text{Ran}_{\underline{x}}})$ on the left-hand side of (16.21) with the action of $\text{IndCoh}^!(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}})$ on the right-hand side of (16.21) via $(\text{pr}^{\text{Op}}_{\text{big},\underline{x}})^!$, as desired.

□[Theorem 15.2.8]

16.5.4. The next assertion is not needed for the sequel; we mention it for the sake of completeness:

Proposition 16.5.5. *The functor*

$$\text{QCoh}(\text{Op}^{\text{mon-free}\rightsquigarrow\text{reg}}_{\check{G},\text{Ran}_{\underline{x}}}) \underset{\text{QCoh}(\text{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \text{KL}(G)_{\text{crit},\underline{x}} \to \text{Res}_{\mathcal{V}ac(G)_{\text{crit}}}((\text{KL}(G)_\kappa)^{\text{fact}_{\underline{x}}})_{\text{Ran}_{\underline{x}}}, \tag{16.23}$$

induced by (16.20)*, is an equivalence.*

The rest of this subsection is devoted to the proof of this proposition.

16.5.6. The functor (16.23) comes from a morphism in the 2-category $\mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}})\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}}$:

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}})^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}) \tag{16.24}$$

Hence, by Lemma B.15.9, in order to check that (16.23) is an equivalence, it suffices to check that:

(i) The composition of (16.24) with the tautological functor

$$\mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}) \to (\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}$$

admits a right adjoint (as a functor between sheaves of categories);

(ii) The functor $\mathcal{V}ac(G)_{\mathrm{crit}}$ admits a right adjoint (as a functor between sheaves of categories);

(iii) The functor (16.24) induces an equivalence between the fibers of the two sides at $\underline{x} \in \mathrm{Ran}_{\underline{x}}$.

16.5.7. Point (iii) above is immediate. Point (ii) follows from the fact that the functor $\mathcal{V}ac(G)_{\mathrm{crit}}$ preserves compactness.

Hence, it remains to show that the composition in point (i) preserves compactness (and the left-hand side is compactly generated).

16.5.8. Let Y be as in Sect. 15.6.3.

It is easy to see that the corresponding category $\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y}$ is compactly generated. By Lemma 15.6.4, it suffices to show that the composition

$$\begin{aligned}\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}})^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \\ \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}})^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \\ \to \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}) \to (\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}\end{aligned} \tag{16.25}$$

preserves compactness.

We rewrite the left-hand side in (16.25) as

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}})^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y},$$

and further, by Lemma 15.7.3 as

$$\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}} \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} Y)^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y}. \tag{16.26}$$

16.5.9. Now, since $\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}} \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} Y$ is a relative affine *scheme* (as opposed to ind-scheme) over $\mathrm{Ran}_{\underline{x}}$, the category (16.26) is compactly generated by the essential image of compact objects along the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}} \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}}{\times} Y)^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y},$$

given by tensoring with the structure sheaf along the first factor.

Hence, it suffices to show that the functor

$$\begin{aligned}\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}})^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \\ \to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}\rightsquigarrow\mathrm{reg}}_{\check{G}})^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \\ \to \mathrm{Res}_{\mathcal{V}ac(G)_{\mathrm{crit}}}((\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}) \to (\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}\end{aligned} \tag{16.27}$$

preserves compactness.

16.5.10. The forgetful functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x},Y} \to \mathrm{KL}(G)_{\mathrm{crit},\underline{x}}$$

preserves compactness (indeed, it admits a continuous right adjoint). Hence, it suffices to show that the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{QCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free}\rightsquigarrow\text{reg}})^{\mathrm{fact}_x} \underset{\mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}})}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to (\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}}$$

preserves compactness.

However, by construction, the latter functor is

$$\mathrm{ins.vac}_{\underline{x}} : \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to (\mathrm{KL}(G)_\kappa)^{\mathrm{fact}_{\underline{x}}},$$

and the assertion follows.

□[Proposition 16.5.5]

17. Spectral Poincaré functor(s)

In this section we start dealing with the local-to-global constructions on the spectral side, i.e., when the recipient category is $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$.

We introduce two versions of the spectral Poincaré functor:

$$\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathrm{Ran}} \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$$

and

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathrm{Ran}} \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

However, we show (Theorem 17.4.7) that they are intertwined by the “self-duality” functor

$$\Theta_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}} : \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}),$$

up to tensoring by a graded line.

Next we recall the definition of the *spectral localization and global sections functors*

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) : \Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G}}.$$

Finally, we give the expression for the composition

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathrm{Ran}} \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \overset{\Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G}}}{\longrightarrow} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

via factorization homology, which exactly matches the composition (14.1) under $\mathrm{FLE}_{G,\mathrm{crit}}$ and $\mathrm{FLE}_{\check{G},\infty}$.

17.1. Ind-coherent sheaves on local vs. global opers.

17.1.1. For $\mathcal{Z} \to \mathrm{Ran}$ consider the morphism

$$\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathcal{Z}} \overset{\mathrm{ev}_{\mathcal{Z}}}{\to} \mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathcal{Z}}.$$

Note that the prestack $\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathcal{Z}}$ is locally almost of finite type, so we have a well-defined category $\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} := \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathcal{Z}})$.

Consider the pair of *mutually dual* functors

$$(\mathrm{ev}_{\mathcal{Z}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \tag{17.1}$$

and

$$(\mathrm{ev}_{\mathcal{Z}})^! : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}}. \tag{17.2}$$

17.1.2. We claim:

Lemma 17.1.3. *The functor* (17.2) *preserves compactness.*

17.1.4. Before we prove Lemma 17.1.3, we need to introduce some notation. For expositional purposes, we will assume that $\mathcal{Z} = \mathrm{pt}$, so that $\mathcal{Z} \to \mathrm{Ran}$ corresponds to $\underline{x} \in \mathrm{Ran}$.

Let $\mathbf{V}$ denote the Tate vector space

$$\Gamma(\mathcal{D}_{\underline{x}}^{\times}, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}).$$

Let $\mathbf{L}_0 \subset \mathbf{V}$ denote the standard lattice, i.e.,

$$\mathbf{L}_0 := \Gamma(\mathcal{D}_{\underline{x}}, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}).$$

Recall that, according to Sect. 3.1.7, we have a simply-transitive action of $\mathbf{L}_0$ on $\mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}}$, so that

$$\mathbf{V} \overset{\mathbf{L}_0}{\times} \mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}} \simeq \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}.$$

For a lattice $\mathbf{L} \supset \mathbf{L}_0$ denote

$$\mathbf{L} \overset{\mathbf{L}_0}{\times} \mathrm{Op}^{\mathrm{reg}}_{\check{G},\underline{x}} =: \mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}} \subset \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}.$$

Denote

$$\mathbf{V}^{\mathrm{glob}} := \Gamma(X - \underline{x}, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}).$$

We have a simply-transitive action of $\mathbf{V}^{\mathrm{glob}}$ on $\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}$, compatible with the embedding $\mathrm{ev}_{\underline{x}}$ via

$$\mathbf{V}^{\mathrm{glob}} \hookrightarrow \mathbf{V}.$$

For $\mathbf{L} \supset \mathbf{L}_0$ set

$$\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}} := \mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}} \underset{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}.$$

17.1.5. *Proof of Lemma 17.1.3.* We have

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \simeq \underset{\mathbf{L}\subset\mathbf{L}_0}{\mathrm{colim}}\, \mathrm{IndCoh}^!(\mathrm{Op}^{\mathbf{L}}_{\check{G}})_{\mathcal{Z}}, \tag{17.3}$$

where the colimit is taken with respect to the IndCoh-pushforward functors.

Hence, it is enough to show that the composition

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\mathbf{L}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \xrightarrow{\mathrm{ev}^!_{\underline{x}}} \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}}$$

preserves compactness.

The Cartesian diagram

$$\begin{tikzcd}
\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}} \arrow[r, "\mathrm{ev}_{\underline{x}}"] \arrow[d] & \mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}} \arrow[d] \\
\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}} \arrow[r, "\mathrm{ev}_{\underline{x}}"'] & \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}
\end{tikzcd} \tag{17.4}$$

gives rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}^!(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}) \arrow[d, "\mathrm{pushforward}"'] & \mathrm{IndCoh}^!(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}) \arrow[l, "(\mathrm{ev}_{\underline{x}})^!"'] \arrow[d, "\mathrm{pushforward}"] \\
\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}) & \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}), \arrow[l, "(\mathrm{ev}_{\underline{x}})^!"]
\end{tikzcd}$$

see Sect. A.10.12.

Hence, it suffices to show that the functor

$$(\mathrm{ev}_{\underline{x}})^! : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}})$$

preserves compactness.

Write

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}) \simeq \underset{\mathbf{L}'\subset \mathbf{L}_0}{\mathrm{colim}}\, \mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L}}_{\check{G},\mathcal{Z}}/\mathbf{L}'),$$

where the colimit is taken with respect to the !-pullback functors.

Hence, it suffices to show that the !-pullback functors along

$$\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}} \to \mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}} \to \mathrm{Op}^{\mathbf{L}}_{\check{G},\mathcal{Z}}/\mathbf{L}'$$

preserve compactness.

However, the latter is obvious, since the above morphism goes between two smooth schemes.

□[Lemma 17.1.3]

17.1.6. As an immediate corollary of Lemma 17.1.3, we obtain:

Corollary 17.1.7. *The functor* (17.1) *admits a left adjoint, to be denoted* $\mathrm{ev}_{\mathcal{Z}}^{*,\mathrm{IndCoh}}$.

17.1.8. Note that we have a tautological commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} \arrow[r, "(\mathrm{ev}_{\mathcal{Z}})^{\mathrm{IndCoh}}_*"] \arrow[d, "\Psi_{\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathcal{Z}}}"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \arrow[d, "\Psi_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathcal{Z}}}"] \\
\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} \arrow[r, "(\mathrm{ev}_{\mathcal{Z}})_*"'] & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}},
\end{tikzcd} \tag{17.5}$$

see Sect. A.7.3.

Since the morphism $\mathrm{ev}_{\mathcal{Z}}$ is schematic, the functor

$$(\mathrm{ev}_{\mathcal{Z}})_* : \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}}$$

admits a left adjoint, denoted $(\mathrm{ev}_{\mathcal{Z}})^*$, see Sect. A.1.4.

Passing to left adjoints along the horizontal arrows in (17.5), we obtain a diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} \arrow[d, "\Psi_{\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\mathcal{Z}}}"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \arrow[l, "(\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}"'] \arrow[d, "\Psi_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathcal{Z}}}"] \\
\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} & \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}}. \arrow[l, "(\mathrm{ev}_{\mathcal{Z}})^*"] \arrow[ul, Rightarrow]
\end{tikzcd} \tag{17.6}$$

We claim:

Lemma 17.1.9. *The natural transformation in* (17.6) *is an isomorphism.*

Proof. With no restriction of generality, we can assume that $\mathcal{Z} = X^I$; in particular, it is smooth.

We claim that the vertical arrows in (17.5) are in fact equivalences. Indeed, we write

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \simeq \underset{\mathbf{L}\subset \mathbf{L}_0}{\mathrm{colim}}\, \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G}})_{\mathcal{Z}}$$

and

$$\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} \simeq \underset{\mathbf{L}\subset \mathbf{L}_0}{\mathrm{colim}}\, \mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G}})_{\mathcal{Z}}$$

where both colimits are formed with respect to the pushforward functors.

Hence it enough to show that the functors

$$\Psi_{\mathrm{Op}^{\mathbf{L}}_{\check{G},\mathcal{Z}}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{QCoh}(\mathrm{Op}^{\mathbf{L}}_{\check{G}})_{\mathcal{Z}}$$

and

$$\Psi_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\mathcal{Z}}} : \mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{QCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G}})_{\mathcal{Z}}$$

are equivalences.

However, this follows from the fact that $\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\mathcal{Z}}$ (resp., $\mathrm{Op}^{\mathbf{L}}_{\check{G},\mathcal{Z}}$) is smooth (resp., pro-smooth). □

17.2. **Interaction with self-duality.**

17.2.1. Recall now the functor

$$\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathcal{Z}}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}},$$

see Sect. 3.7.1.

17.2.2. Denote by $\mathfrak{l}_{\mathrm{Kost}(\check{G})}$ the (non-graded) line

$$\det(\Gamma(X, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})).$$

Set

$$\delta_G := \dim(\mathrm{Bun}_G) = (g-1)\cdot \dim(G).$$

17.2.3. We claim:

Proposition 17.2.4. *There exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \arrow[r, "\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathcal{Z}}}"] \arrow[d, "(\mathrm{ev}_{\mathcal{Z}})^!"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathcal{Z}} \arrow[d, "(\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}"] \\
\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}} \arrow[r, "-\otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[-\delta_G]"'] & \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G}})_{\mathcal{Z}}.
\end{tikzcd}$$

The rest of this subsection is devoted to proof of Proposition 17.2.4.

17.2.5. For expositional purposes, we will assume that $\mathcal{Z} = \mathrm{pt}$, so that $\mathcal{Z} \to \mathrm{Ran}$ corresponds to $\underline{x} \in \mathrm{Ran}$. We will use the notation from Sect. 17.1.4.

17.2.6. By the definition of the functor $\Theta_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}$, we need to establish an isomorphism of the following two objects in $\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}})$:

$$\omega_{\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[-\delta_G] \simeq (\mathrm{ev}_{\underline{x}})^{*,\mathrm{IndCoh}}(\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}),$$

where

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}})$$

is as in Sect. 3.7.2.

17.2.7. In terms of the presentation

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \simeq \underset{\mathbf{L}\subset\mathbf{L}_0}{\mathrm{colim}}\, \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}) \overset{\Psi_{\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}}}{\simeq} \mathrm{QCoh}(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}),$$

the object $\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}$ is, by construction, the colimit of the images of

$$\mathcal{O}_{\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}} \otimes \det(\mathbf{L}/\mathbf{L}_0)^{\otimes -1}[\dim(\mathbf{L}/\mathbf{L}_0)].$$

In terms of the presentation

$$\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{glob}}_{\check{G},\underline{x}}) \simeq \underset{\mathbf{L}\subset\mathbf{L}_0}{\mathrm{colim}}\, \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}) \overset{\Psi_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}}}{\simeq} \mathrm{QCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}),$$

the object $\omega_{\mathrm{Op}^{\mathrm{glob}}_{\check{G},\underline{x}}}$ is, tautologically, the colimit of the images of

$$\Psi_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}}(\omega_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}}).$$

17.2.8. The Cartesian diagram (17.4) gives rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}) \arrow[r, "(\mathrm{ev}_{\underline{x}})^{\mathrm{IndCoh}}_*"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}) \\
\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{glob}}_{\check{G},\underline{x}}) \arrow[u, "\text{!-pullback}"] \arrow[r, "(\mathrm{ev}_{\underline{x}})^{\mathrm{IndCoh}}_*"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}), \arrow[u, "\text{!-pullback}"']
\end{tikzcd}$$

see Sect. A.10.12.

Passing to the left adjoints, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}) \arrow[d, "*\text{-pushforward}"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}) \arrow[l, "(\mathrm{ev}_{\underline{x}})^{*,\mathrm{IndCoh}}"'] \arrow[d, "*\text{-pushforward}"] \\
\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{glob}}_{\check{G},\underline{x}}) & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}). \arrow[l, "(\mathrm{ev}_{\underline{x}})^{*,\mathrm{IndCoh}}"]
\end{tikzcd} \tag{17.7}$$

As in Lemma 17.1.9, we also have a commutative diagram

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}) & \mathrm{QCoh}(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}) \arrow[l, "(\mathrm{ev}_{\underline{x}})^*"'] \\
\mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}) \arrow[u, "\Psi_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}). \arrow[l, "(\mathrm{ev}_{\underline{x}})^{*,\mathrm{IndCoh}}"] \arrow[u, "\Psi_{\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}}"']
\end{tikzcd}$$

Hence, it is enough to construct a compatible collection of identifications

$$\Psi_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}}(\omega_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}}) \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[-\delta_G] \simeq \mathcal{O}_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}} \otimes \det(\mathbf{L}/\mathbf{L}_0)^{\otimes -1}[\dim(\mathbf{L}/\mathbf{L}_0)],$$

taking place in $\mathrm{QCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}})$.

17.2.9. Note now that $\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}$ is an affine space with respect to

$$\mathbf{L}^{\mathrm{glob}} := \mathbf{V}^{\mathrm{glob}} \cap \mathbf{L}.$$

Hence,

$$\Psi_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}}(\omega_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}}) \simeq \mathcal{O}_{\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}} \otimes \det(\mathbf{L}^{\mathrm{glob}})^{\otimes -1}[\dim(\mathbf{L}^{\mathrm{glob}})].$$

17.2.10. Thus, it remains to establish a compatible collection of isomorphisms between the lines

$$\det(\mathbf{L}^{\mathrm{glob}})^{\otimes -1} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[\dim(\mathbf{L}^{\mathrm{glob}}) - \delta_G] \simeq \det(\mathbf{L}/\mathbf{L}_0)^{\otimes -1}[\dim(\mathbf{L}/\mathbf{L}_0)]. \tag{17.8}$$

However, this follows from the fact that

$$\mathfrak{l}_{\mathrm{Kost}(\check{G})} \simeq \det(\mathbf{V}^{\mathrm{glob}} \cap \mathbf{L}_0) \text{ and } \delta_G = \dim(\Gamma(X, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})) = \dim(\mathbf{V}^{\mathrm{glob}} \cap \mathbf{L}_0).$$

17.3. **Ind-coherent sheaves on local vs. global *monodromy-free* opers.**

17.3.1. For $\mathcal{Z} \to \mathrm{Ran}$ recall the (relative ind-scheme) $\mathrm{Op}^{\mathrm{mon\text{-}free},\mathrm{glob}}_{\check{G},\mathcal{Z}}$, which fits into the Cartesian square

$$\begin{tikzcd}
\mathrm{Op}^{\mathrm{mon\text{-}free},\mathrm{glob}}_{\check{G},\mathcal{Z}} \arrow[r, "\mathrm{ev}_{\mathcal{Z}}"] \arrow[d, "\iota^{\mathrm{mon\text{-}free},\mathrm{glob}}"'] & \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\mathcal{Z}} \arrow[d, "\iota^{\mathrm{mon\text{-}free}}"] \\
\mathrm{Op}^{\mathrm{mer},\mathrm{glob}}_{\check{G},\mathcal{Z}} \arrow[r, "\mathrm{ev}_{\mathcal{Z}}"'] & \mathrm{Op}^{\mathrm{mer}}_{\check{G},\mathcal{Z}}
\end{tikzcd} \tag{17.9}$$

Consider the morphism:

$$\mathrm{Op}^{\mathrm{mon\text{-}free},\mathrm{glob}}_{\check{G},\mathcal{Z}} \overset{\mathrm{ev}_{\mathcal{Z}}}{\to} \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\mathcal{Z}}$$

and the resulting pair of mutually dual functors

$$(\mathrm{ev}_{\mathcal{Z}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free,glob}})_{\mathcal{Z}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathcal{Z}} \tag{17.10}$$

and

$$(\mathrm{ev}_{\mathcal{Z}})^! : \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathcal{Z}} \to \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free,glob}})_{\mathcal{Z}}. \tag{17.11}$$

17.3.2. We claim:

Lemma 17.3.3. *The functor* (17.11) *preserves compactness.*

Proof. From (17.9) we obtain a commutative square

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free,glob}})_{\mathcal{Z}} \arrow[d, "(\iota^{\text{mon-free,glob}})_*^{\mathrm{IndCoh}}"'] & \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathcal{Z}} \arrow[l, "\mathrm{ev}_{\mathcal{Z}}^!"'] \arrow[d, "(\iota^{\text{mon-free}})_*^{\mathrm{IndCoh}}"] \\
\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mer,glob}})_{\mathcal{Z}} & \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mer}})_{\mathcal{Z}}, \arrow[l, "\mathrm{ev}_{\mathcal{Z}}^!"]
\end{tikzcd} \tag{17.12}$$

see Sect. A.10.12.

As in Proposition 3.3.5(b), one shows that an object in $\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free,glob}})_{\mathcal{Z}}$ is compact if and only if its image in $\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mer,glob}})_{\mathcal{Z}}$ under $(\iota)^{\text{glob}})_*^{\mathrm{IndCoh}}$ is compact.

Hence, it suffices to show that the clockwise circuit in (17.12) preserves compactness.

For the functor $(\iota^{\text{mon-free}})_*^{\mathrm{IndCoh}}$ this is evident (since the morphism $\iota^{\text{mon-free}}$ is of finite presentation). For the bottom horizontal arrow in (17.12) this follows from Lemma 17.1.3. □

17.3.4. As a corollary of Lemma 17.3.3 we obtain:

Corollary 17.3.5. *The functor* (17.10) *admits a left adjoint, to be denoted* $\mathrm{ev}_{\mathcal{Z}}^{*,\mathrm{IndCoh}}$.

17.3.6. The fact that (17.9) is Cartesian implies that the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free,glob}})_{\mathcal{Z}} \arrow[r, "(\mathrm{ev}_{\mathcal{Z}})_*^{\mathrm{IndCoh}}"] & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathcal{Z}} \\
\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mer,glob}})_{\mathcal{Z}} \arrow[u, "(\iota^{\text{mon-free,glob}})^!"] \arrow[r, "(\mathrm{ev}_{\mathcal{Z}})_*^{\mathrm{IndCoh}}"'] & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mer}})_{\mathcal{Z}} \arrow[u, "(\iota^{\text{mon-free}})^!"']
\end{tikzcd} \tag{17.13}$$

commutes, see Sect. A.10.12.

By passing to left adjoints along all arrows in (17.13) we obtain that the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free,glob}})_{\mathcal{Z}} \arrow[d, "(\iota^{\text{mon-free,glob}})_*^{\mathrm{IndCoh}}"'] & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathcal{Z}} \arrow[l, "(\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}"'] \arrow[d, "(\iota^{\text{mon-free}})_*^{\mathrm{IndCoh}}"] \\
\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mer,glob}})_{\mathcal{Z}} & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mer}})_{\mathcal{Z}} \arrow[l, "(\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}"]
\end{tikzcd} \tag{17.14}$$

commutes as well.

However, passing left adjoints only along the horizontal arrows in (17.13) we obtain a diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free,glob}})_{\mathcal{Z}} & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}})_{\mathcal{Z}} \arrow[l, "(\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}"'] \arrow[dl, Rightarrow, shorten=2em] \\
\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mer,glob}})_{\mathcal{Z}} \arrow[u, "(\iota^{\text{mon-free,glob}})^!"] & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mer}})_{\mathcal{Z}}. \arrow[l, "(\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}"] \arrow[u, "(\iota^{\text{mon-free}})^!"']
\end{tikzcd} \tag{17.15}$$

We claim:

Lemma 17.3.7. *The natural transformation in* (17.15) *is an isomorphism.*

Proof. For expositional purposes we will assume that $\mathcal{Z} = \mathrm{pt}$, so that $\mathcal{Z} \to \mathrm{Ran}$ corresponds to $\underline{x} \in \mathrm{Ran}$. We will use the notations from Sect. 17.1.4.

For $\mathbf{L} \supset \mathbf{L}_0$ denote

$$\mathrm{Op}^{\mathbf{L}\cap\text{mon-free}}_{\check{G},\underline{x}} := \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} \underset{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}},$$

$$\mathrm{Op}^{\mathbf{L}\cap\text{mon-free,glob}}_{\check{G},\underline{x}} := \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}} \underset{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}{\times} \mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}},$$

Using (17.7) and a similar diagram for "mer" replaced my "mon-free", it suffices to show that the natural transformation in the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L}\cap\text{mon-free,glob}}_{\check{G},\underline{x}}) & & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}\cap\text{mon-free}}_{\check{G},\underline{x}}) \arrow[ll, "(\mathrm{ev}_{\underline{x}})^{*,\mathrm{IndCoh}}"'] \arrow[dll, Rightarrow] \\
\mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}) \arrow[u, "(\iota^{\text{mon-free,glob}})^!"] & & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}). \arrow[ll, "(\mathrm{ev}_{\underline{x}})^{*,\mathrm{IndCoh}}"] \arrow[u, "(\iota^{\text{mon-free}})^!"']
\end{tikzcd}$$

is an isomorphism.

According to Sect. 3.2.7, for a small enough lattice $\mathbf{L}' \subset \mathbf{L}$, we have a well-defined action of $\mathbf{L}'$ on $\mathrm{Op}^{\mathbf{L}\cap\text{mon-free}}_{\check{G},\underline{x}}$ by translations, and the quotient $\mathrm{Op}^{\mathbf{L}\cap\text{mon-free}}_{\check{G},\underline{x}}/\mathbf{L}'$ is a prestack locally almost of finite type.

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}\cap\text{mon-free}}_{\check{G},\underline{x}}) & \mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L}\cap\text{mon-free}}_{\check{G},\underline{x}}/\mathbf{L}') \arrow[l, "*\text{-pullback}"'] \\
\mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}) \arrow[u, "(\iota^{\text{mon-free}})^!"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}/\mathbf{L}'). \arrow[l, "*\text{-pullback}"'] \arrow[u, "(\iota^{\text{mon-free}}/\mathbf{L}')^!"']
\end{tikzcd}$$

Hence, it suffices to show that the natural transformation in the diagram

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L}\cap\text{mon-free,glob}}_{\check{G},\underline{x}}) & & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}\cap\text{mon-free}}_{\check{G},\underline{x}}/\mathbf{L}') \arrow[ll, "*\text{-pullback}"'] \arrow[dll, Rightarrow] \\
\mathrm{IndCoh}(\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}}) \arrow[u, "(\iota^{\text{mon-free,glob}})^!"] & & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}/\mathbf{L}'). \arrow[ll, "*\text{-pullback}"] \arrow[u, "(\iota^{\text{mon-free}}/\mathbf{L}')^!"']
\end{tikzcd}$$

is an isomorphism.

However, this follows from the fact that the diagram

$$\begin{tikzcd}
\mathrm{Op}^{\mathbf{L}\cap\text{mon-free,glob}}_{\check{G},\underline{x}} \arrow[r] \arrow[d] & \mathrm{Op}^{\mathbf{L}\cap\text{mon-free}}_{\check{G},\underline{x}}/\mathbf{L}' \arrow[d] \\
\mathrm{Op}^{\mathbf{L},\mathrm{glob}}_{\check{G},\underline{x}} \arrow[r] & \mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}/\mathbf{L}'
\end{tikzcd}$$

is Cartesian, combined with the fact that $\mathrm{Op}^{\mathbf{L}}_{\check{G},\underline{x}}/\mathbf{L}'$ is a smooth scheme.

□

17.3.8. Recall the functor

$$\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}},$$

see Sect. 3.7.7.

We claim:

Proposition 17.3.9. *There exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}} \arrow[r, "\Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}}}"] \arrow[d, "(\mathrm{ev}_{\mathcal{Z}})^!"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}} \arrow[d, "(\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}"] \\
\mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}} \arrow[r, "-\otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[-\delta_G]"'] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}}.
\end{tikzcd}$$

Proof. Both circuits of the diagram are $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}}$-linear functors. Hence, it suffices to identify the objects that correspond to the image of the unit.

I.e., we wish to identify

$$\omega_{\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\mathcal{Z}}} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[-\delta_G] \simeq (\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}(\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{Z}}}). \tag{17.16}$$

We start with

$$\omega_{\mathrm{Op}^{\text{mer,glob}}_{\check{G},\mathcal{Z}}} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[-\delta_G] \simeq (\mathrm{ev}_{\mathcal{Z}})^{*,\mathrm{IndCoh}}(\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\text{mer}}_{\check{G},\mathcal{Z}}}),$$

given by Proposition 17.2.4 and apply the functor $(\iota^{\text{mon-free,glob}})^!$.

The left-hand side gives the left-hand side of (17.16). The right-hand side gives the right-hand side of (17.16) thanks to Lemma 17.3.7.

□

17.4. **Two versions of the spectral Poincaré functor.**

17.4.1. For $\mathcal{Z} \to \mathrm{Ran}$, we define the spectral !-Poincaré functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\mathcal{Z}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathcal{Z})$$

as

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}} \xrightarrow{\mathrm{ev}^!_{\mathcal{Z}}} \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}} \overset{(\mathfrak{r}^{\mathrm{glob}})^{\mathrm{IndCoh}}_*}{\to} \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathcal{Z}).$$

17.4.2. We define the spectral *-Poincaré functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathcal{Z}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}} \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathcal{Z})$$

as

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathcal{Z}} \xrightarrow{\mathrm{ev}^{*,\mathrm{IndCoh}}_{\mathcal{Z}}} \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G}})_{\mathcal{Z}} \overset{(\mathfrak{r}^{\mathrm{glob}})^{\mathrm{IndCoh}}_*}{\to} \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathcal{Z}).$$

17.4.3. Both

$$\mathcal{Z} \rightsquigarrow \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\mathcal{Z}} \text{ and } \mathcal{Z} \rightsquigarrow \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathcal{Z}} \tag{17.17}$$

are naturally local-to-global functors in the sense of Sect. 11.1.1, to be denoted

$$\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},!} \text{ and } \underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},*},$$

respectively.

17.4.4. Furthermore, the assignments (17.17) have natural *unital* structures, in the sense of Sect. 11.3.5. Let us spell it out explicitly for the *-version (!-version is analogous).

The local unital structure on the source crystal of categories , i.e., $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ assigns to

$$(\underline{x} \subseteq \underline{x}') \in \mathrm{Ran}$$

the functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'})$$

given by *-pull followed by *-push along the diagram

$$\begin{tikzcd}[column sep=tiny]
& \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}) \arrow[dl] \arrow[dr] & \\
\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}) \arrow[d, equal] & & \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}') \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}')}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}) \arrow[d, equal] \\
\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} & & \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'},
\end{tikzcd} \tag{17.18}$$

in which the slanted arrows are given by restriction along the inclusions

$$(\mathcal{D}_{\underline{x}} - \underline{x}) \to (\mathcal{D}_{\underline{x}'} - \underline{x}) \leftarrow (\mathcal{D}_{\underline{x}'} - \underline{x}'),$$

respectively.

17.4.5. Consider the diagram

$$\begin{tikzcd}
\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'} & \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}'} \arrow[l, "\mathrm{ev}_{\underline{x}'}"'] \arrow[r, "\mathfrak{r}^{\mathrm{glob}}_{\underline{x}'}"] & \mathrm{LS}_{\check{G}} \\
\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}) \arrow[u] \arrow[d] & \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}} \arrow[l] \arrow[u] \arrow[r, "\mathfrak{r}^{\mathrm{glob}}_{\underline{x}}"] \arrow[d, "\mathrm{ev}_{\underline{x}}"] & \mathrm{LS}_{\check{G}} \arrow[u, "\mathrm{id}"'] \\
\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}} & \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}, \arrow[l, "\mathrm{id}"'] &
\end{tikzcd}$$

in which the square

$$\begin{tikzcd}
\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'} & \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}'} \arrow[l, "\mathrm{ev}_{\underline{x}'}"'] \\
\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}) \arrow[u] & \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}} \arrow[l] \arrow[u]
\end{tikzcd}$$

is Cartesian.

By Sect. A.10.12, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'}) \arrow[r, "*\text{-pushforward}"] \arrow[d, "!\text{-pullback}"'] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}'}) \arrow[d, "!\text{-pullback}"] \\
\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}})) \arrow[r, "*\text{-pushforward}"] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}}).
\end{tikzcd}$$

Passing to left adjoints, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'}) \arrow[r, "*\text{-pullback}"] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}'}) \\
\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}})) \arrow[r, "*\text{-pullback}"] \arrow[u, "*\text{-pushforward}"] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}}). \arrow[u, "*\text{-pushforward}"']
\end{tikzcd}$$

Now, the unital structure on $\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},*}$ is encoded by the following diagram:

$$\begin{tikzcd}
\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'}) \arrow[r, "\mathrm{ev}^{*,\mathrm{IndCoh}}_{\underline{x}'}"] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}'}) \arrow[r, "(\mathfrak{r}^{\mathrm{glob}}_{\underline{x}'})^{\mathrm{IndCoh}}_*"] & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \\
\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'})) \arrow[u, "*\text{-pushforward}"] \arrow[r, "*\text{-pullback}"] & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}}) \arrow[u, "*\text{-pushforward}"] \arrow[r, "(\mathfrak{r}^{\mathrm{glob}}_{\underline{x}})^{\mathrm{IndCoh}}_*"] & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, "\mathrm{id}"'] \\
\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \arrow[u, "*\text{-pullback}"] \arrow[r, "\mathrm{Id}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}). \arrow[u, "\mathrm{ev}^*_{\underline{x}}"'] &
\end{tikzcd}$$

17.4.6. Note that from Proposition 17.3.9 we obtain:

Theorem 17.4.7. *There is an isomorphism of local-to-global functors*

$$\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},!} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[-\delta_G] \simeq \underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},*} \circ \Theta_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}}.$$

Remark 17.4.8. The reason that we discuss both $\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},!}$ and $\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},*}$ (despite the fact that, thanks to Theorem 17.4.7, they are easily expressible one through another) is that the !-version is naturally compatible with Eisenstein series (which we will exploit in the sequel to this paper), and the *-version is naturally compatible with the functor

$$\Gamma^{\mathrm{IndCoh}}(\mathrm{LS}_{\check{G}}, -) : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{Vect},$$

which we will use in the next section.

17.5. **Action of the spectral spherical category and temperedness.** In this subsection we will work with $\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},*}$, but a parallel discussion is applicable to $\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},!}$.

17.5.1. For a fixed $\underline{x} \in \mathrm{Ran}$ we have a naturally defined action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ on $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$. Namely, it is given by *-pull followed by *-push along the following diagram

$$\begin{tikzcd}
\mathrm{LS}_{\check{G}} \arrow[d, "\mathrm{ev}_{\underline{x}}"'] & \mathrm{Hecke}^{\mathrm{spec,glob}}_{\check{G},\underline{x}} \arrow[l] \arrow[r] \arrow[d, "\mathrm{ev}_{\underline{x}}"'] & \mathrm{LS}_{\check{G}} \arrow[d, "\mathrm{ev}_{\underline{x}}"] \\
\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} & \mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\underline{x}} \arrow[l] \arrow[r] & \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}},
\end{tikzcd}$$

in which both squares are Cartesian.

Remark 17.5.2. In Sect. 18.2.1 we will consider a Ran version of this action. This involves some technical difficulties, inherent to the definition of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ as a factorization category, see Sect. 1.6.4.

17.5.3. We have a natural action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$ given by pull-push along the following diagram

$$\begin{tikzcd}
\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'} \arrow[d] & \mathrm{Hecke}^{\mathrm{spec},\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'}}_{\check{G},\underline{x}} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'} \arrow[d] \\
\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}} & \mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\underline{x}} \arrow[l] \arrow[r] & \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}},
\end{tikzcd}$$

in which both squares are Cartesian, where:

- $\underline{x} \subseteq \underline{x}'$;
- $\mathrm{Hecke}^{\mathrm{spec},\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}'}}_{\check{G},\underline{x}} := \mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}'} \underset{\mathrm{LS}^{\mathrm{mer}}_{\check{G},\underline{x}'}}{\times} (\mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}'} \underset{\mathrm{LS}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{\check{G},\underline{x}\subseteq\underline{x}'}}{\times} \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}'})$;
- $\mathrm{LS}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{\check{G},\underline{x}\subseteq\underline{x}'} := \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x})$.

17.5.4. Consider the functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{Ran}_{\underline{x}}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran}_{\underline{x}}).$$

The following results by unwinding the constructions:

Lemma 17.5.5. *The functor* $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{Ran}_{\underline{x}}}$ *intertwines the actions of* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ *on the two sides.*

Remark 17.5.6. One can define an action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ as a factorization category. Furthermore, one can show that that the functor $\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},*}$ is compatible with the action of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$, where the latter is thought of as a crystal of monoidal categories over Ran.

Moreover, the above action and compatibility are in turn compatible with the unital structures. This will be performed in Sect. E.8.

17.5.7. Recall (see [AG, Sect. 12.8.2]) that the subcategory

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \subset \mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$$

can be singled out by the *temperedness* condition:

Namely, it is the maximal subcategory on which for some/any $\underline{x} \in \mathrm{Ran}$, the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ on $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$ factors via

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}} \twoheadrightarrow \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},\underline{x}},$$

see Sect. 7.1.1.

17.5.8. A basic property of the spectral Poincaré functor is the following:

Proposition 17.5.9. *The essential image of the functor*

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$$

lies in

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \subset \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}).$$

Proof. Choose some/any $\underline{x} \in \mathrm{Ran}$. By the unital property of $\underline{\mathrm{Poinc}}^{\mathrm{spec}}_{\check{G},*}$, it suffices to show that the functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{Ran}_{\underline{x}}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran}_{\underline{x}})$$

takes values in

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran}_{\underline{x}}) \subset \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran}_{\underline{x}}).$$

By Sect. 17.5.7 and Lemma 17.5.5, it suffices to show that the $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$-action on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}})$ factors via $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},\underline{x}}$.

The latter assertion can be checked strata-wise, so it is enough to show that for a fixed $\underline{x} \subseteq \underline{x}'$, the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}'}})$ factors via $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},\underline{x}}$.

Write $\underline{x}' = \underline{x} \sqcup \underline{x}''$. In terms of the factorization

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}'}}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}}}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}''}}),$$

the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}_{\underline{x}'}})$ is via the first factor.

The required assertion follows now from Proposition 7.2.4.

□

Remark 17.5.10. Note that thanks to Theorem 17.4.7, we obtain that an assertion parallel to Proposition 17.5.9 holds for the functor $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}$. (Alternatively, one can prove it by the same argument.)

17.6. **The spectral localization and global sections functors.**

17.6.1. The spectral localization functor $\underline{\mathrm{Loc}}^{\mathrm{spec}}_{\check{G}}$, i.e., the collectiion functors

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G},\mathcal{Z}} : \mathrm{Rep}(\check{G})_{\mathcal{Z}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathcal{Z})$$

for $\mathcal{Z} \to \mathrm{Ran}$, is defined as pullback along

$$\mathrm{LS}_{\check{G}} \times \mathcal{Z} \to \mathrm{LS}^{\mathrm{reg}}_{\check{G},\mathcal{Z}}, \tag{17.19}$$

where we identify

$$\mathrm{Rep}(\check{G})_{\mathcal{Z}} \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{\check{G}})_{\mathcal{Z}}.$$

The functor $\underline{\mathrm{Loc}}^{\mathrm{spec}}_{\check{G}}$ possesses a natural unital structure (see Sect. 11.3.5 for what this means).

17.6.2. The functor

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

admits a right adjoint, denoted

$$\Gamma^{\mathrm{spec}}_{\check{G}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

Explicitly, for a given $\underline{x} \in \mathrm{Ran}$, the corresponding functor

$$\Gamma^{\mathrm{spec}}_{\check{G},\underline{x}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{Rep}(\check{G})_{\underline{x}}$$

is given by *-direct image along

$$\mathrm{LS}_{\check{G}} \to \mathrm{LS}^{\mathrm{reg}}_{\check{G},\underline{x}}.$$

17.6.3. Note also that the categories $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ and $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ are both canonically self-dual, and with respect to these dualities, we have

$$(\mathrm{Loc}^{\mathrm{spec}}_{\check{G}})^{\vee} \simeq \Gamma^{\mathrm{spec}}_{\check{G}}.$$

17.6.4. By a slight abuse of notation we will denote by the same symbol $\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$ the composite functor

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \overset{\Xi_{\mathrm{LS}_{\check{G}}}}{\hookrightarrow} \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}).$$

We will denote by $\Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G}}$ the functor

$$\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \overset{\Psi_{\mathrm{LS}_{\check{G}}}}{\twoheadrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{\Gamma^{\mathrm{spec}}_{\check{G}}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

The functors

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \leftrightarrows \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) : \Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G}}$$

also form an adjoint pair.

17.6.5. Note that the category $\operatorname{IndCoh}(\operatorname{LS}_{\check{G}})$ is also self-dual by means of *Serre* duality. Under this duality and the standard self-duality of $\operatorname{QCoh}(\operatorname{LS}_{\check{G}})$, we have

$$\Psi^\vee_{\operatorname{LS}_{\check{G}}} \simeq \Upsilon_{\operatorname{LS}_{\check{G}}}.$$

However, note that $\operatorname{LS}_{\check{G}}$ is quasi-smooth and Calabi-Yau:

$$\omega_{\operatorname{LS}_{\check{G}}} = \mathcal{O}_{\operatorname{LS}_{\check{G}}} \otimes \det(\operatorname{Lie}(Z_G))^{\otimes(2-2g)}[2\delta_G].$$

Hence, we have

$$\Upsilon_{\operatorname{LS}_{\check{G}}} \simeq \Xi_{\operatorname{LS}_{\check{G}}} \otimes \det(\operatorname{Lie}(Z_G))^{\otimes(2-2g)}[2\delta_G].$$

17.6.6. Hence, we obtain that with respect to the self-duality of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$ and the Serre duality of $\operatorname{IndCoh}(\operatorname{LS}_{\check{G}})$, we have

$$(\operatorname{Loc}^{\operatorname{spec}}_{\check{G}})^\vee \simeq \Gamma^{\operatorname{spec},\operatorname{IndCoh}}_{\check{G}} \otimes \det(\operatorname{Lie}(Z_G))^{\otimes(2-2g)}[2\delta_G].$$

17.7. **Composing spectral Poincaré and global sections functors.**

17.7.1. Our current goal is to study the composite functor

$$\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})_{\operatorname{Ran}} \overset{\operatorname{Poinc}^{\operatorname{spec}}_{\check{G},*}}{\longrightarrow} \operatorname{QCoh}(\operatorname{LS}_{\check{G}}) \overset{\Gamma^{\operatorname{spec}}_{\check{G}}}{\longrightarrow} \operatorname{Rep}(\check{G})_{\operatorname{Ran}}. \tag{17.20}$$

Applying the canonical self-duality of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}}$, the datum of the functor (17.20) is equivalent to the datum of the pairing

$$\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})_{\operatorname{Ran}} \otimes \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \to \operatorname{Vect},$$

given by

$$\begin{aligned}\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})_{\operatorname{Ran}} \otimes \operatorname{Rep}(\check{G})_{\operatorname{Ran}} &\overset{\operatorname{Poinc}^{\operatorname{spec}}_{\check{G},*}\otimes\operatorname{Id}}{\longrightarrow} \operatorname{QCoh}(\operatorname{LS}_{\check{G}}) \otimes \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \to \\ &\overset{\Gamma^{\operatorname{spec}}_{\check{G}}\otimes\operatorname{Id}}{\longrightarrow} \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \otimes \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \to \operatorname{Vect}.\end{aligned} \tag{17.21}$$

We will prove (cf. Theorem 14.2.4):

Theorem 17.7.2. *The functor* (17.21) *identifies canonically with*

$$\begin{aligned}&\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})_{\operatorname{Ran}} \otimes \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \overset{\text{ins.unit}_{\operatorname{Ran}}\otimes\text{ins.unit}_{\operatorname{Ran}}}{\longrightarrow} \\ &\to \operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})_{\operatorname{Ran}^{\subseteq}} \otimes \operatorname{Rep}(\check{G})_{\operatorname{Ran}^{\subseteq}} \to \\ &\to \left(\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}}) \otimes \operatorname{Rep}(\check{G})\right)_{\operatorname{Ran}^{\subseteq} \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{\subseteq}} \overset{\mathsf{P}^{\text{loc,enh}}_{\check{G}}}{\longrightarrow} \\ &\to \mathcal{O}_{\operatorname{Op}^{\operatorname{reg}}_{\check{G}}}\text{-mod}^{\text{fact}}_{\operatorname{Ran}^{\subseteq} \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{\subseteq}} \overset{\mathrm{C}^{\text{fact}}_\cdot(X;\mathcal{O}_{\operatorname{Op}^{\operatorname{reg}}_{\check{G}}},-)_{\operatorname{Ran}^{\subseteq} \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{\subseteq}}}{\longrightarrow} \operatorname{D-mod}(\operatorname{Ran}^{\subseteq} \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{\subseteq}) \to \\ &\overset{\mathrm{C}^\cdot_c(\operatorname{Ran}^{\subseteq} \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{\subseteq},-)}{\longrightarrow} \operatorname{Vect},\end{aligned} \tag{17.22}$$

where $\mathsf{P}^{\text{loc,enh}}_{\check{G}}$ *is the functor introduced in Sect. 6.4.7.*

Remark 17.7.3. Note that the functor (17.22), appearing in Theorem 17.7.2 can also be rewritten as

$$\begin{aligned}&\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})_{\operatorname{Ran}} \otimes \operatorname{Rep}(\check{G})_{\operatorname{Ran}} \overset{\text{ins.unit}_{\operatorname{Ran}}\otimes\text{ins.unit}_{\operatorname{Ran}}}{\to} \\ &\to \operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})_{\operatorname{Ran}^{\subseteq}} \otimes \operatorname{Rep}(\check{G})_{\operatorname{Ran}^{\subseteq}} \to \\ &\left(\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}}) \otimes \operatorname{Rep}(\check{G})\right)_{\operatorname{Ran}^{\subseteq} \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{\subseteq}} \overset{\mathsf{P}^{\text{loc}}_{\check{G}}}{\longrightarrow} \operatorname{D-mod}(\operatorname{Ran}^{\subseteq} \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{\subseteq}) \overset{\mathrm{C}^\cdot_c(\operatorname{Ran}^{\subseteq} \underset{\operatorname{Ran}}{\times} \operatorname{Ran}^{\subseteq})}{\longrightarrow} \operatorname{Vect},\end{aligned}$$

i.e., instead of $\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}, -)_{\mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}$ we can use the functor $\mathbf{oblv}_{\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}, \mathrm{Ran}^{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}^{\subseteq}}$. This follows by the same manipulation as in Remark 14.2.5

17.7.4. The rest of this subsection is devoted to the proof of Theorem 17.7.2.

First, using the (non-derived) Satake action, as in the proof of Theorem 14.2.4, we obtain that the assertion of the theorem is equivalent to that of the following:

Theorem 17.7.5. *The functor*

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \xrightarrow{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{\Gamma(\mathrm{LS}_{\check{G}}, -)} \mathrm{Vect}$$

identifies canonically with

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \xrightarrow{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})_{\mathrm{Ran}} \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}, -)^{\mathrm{enh}}} \\ \to \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \xrightarrow{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}, -)} \mathrm{Vect}\,. \tag{17.23}$$

17.7.6. For expositional purposes, will replace the situation over Ran by one with a fixed $\underline{x} \in \mathrm{Ran}$. So, we want to show that the composition

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \xrightarrow{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x}}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{\Gamma(\mathrm{LS}_{\check{G}}, -)} \mathrm{Vect} \tag{17.24}$$

identifies canonically with

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \xrightarrow{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}, -)^{\mathrm{enh}}} \\ \to \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \xrightarrow{\mathrm{C}^{\mathrm{ch}}_{\cdot}(X, \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}, -)_{\underline{x}}} \mathrm{Vect}\,. \tag{17.25}$$

17.7.7. The functor (17.24) can be tautologically rewritten as the composition

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \xrightarrow{\mathrm{ev}^{*,\mathrm{IndCoh}}_{\underline{x}}} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}}, -)} \mathrm{Vect},$$

and further as

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \xrightarrow{\mathrm{ev}^{*,\mathrm{IndCoh}}_{\underline{x}}} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},\underline{x}}) \xrightarrow{(\iota^{\text{mon-free,glob}})^{\mathrm{IndCoh}}_*} \\ \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}, -)} \mathrm{Vect}\,. \tag{17.26}$$

Applying (17.14), we rewrite (17.26) as

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\underline{x}}) \xrightarrow{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \xrightarrow{\mathrm{ev}^{*,\mathrm{IndCoh}}_{\underline{x}}} \\ \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}, -)} \mathrm{Vect}\,. \tag{17.27}$$

Thus, it suffices to establish an isomorphism between

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \xrightarrow{\mathrm{ev}^{*,\mathrm{IndCoh}}_{\underline{x}}} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}, -)} \mathrm{Vect} \tag{17.28}$$

and

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}, -)^{\mathrm{enh}}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \xrightarrow{\mathrm{C}^{\mathrm{ch}}_{\cdot}(X, \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}, -)_{\underline{x}}} \mathrm{Vect}\,. \tag{17.29}$$

17.7.8. By Lemma 17.1.9, we can rewrite (17.28) as

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \overset{\Psi_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \overset{\mathrm{ev}^*_{\underline{x}}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}) \overset{\Gamma(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}},-)}{\longrightarrow} \mathrm{Vect}, \tag{17.30}$$

while (17.29) is by definition

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \overset{\Psi_{\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \overset{\Gamma(\mathrm{Op}^{\mathrm{mer}}_{\check{G}},-)^{\mathrm{enh}}}{\longrightarrow} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\underline{x}} \overset{\mathrm{C}^{\mathrm{ch}}_\cdot(X,\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}},-)_{\underline{x}}}{\longrightarrow} \mathrm{Vect}\,. \tag{17.31}$$

Hence, it suffices to establish an isomorphism between

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \overset{\mathrm{ev}^*_{\underline{x}}}{\longrightarrow} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}}) \overset{\Gamma(\mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},\underline{x}},-)}{\longrightarrow} \mathrm{Vect} \tag{17.32}$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G},\underline{x}}) \overset{\Gamma(\mathrm{Op}^{\mathrm{mer}}_{\check{G}},-)^{\mathrm{enh}}}{\longrightarrow} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\underline{x}} \overset{\mathrm{C}^{\mathrm{ch}}_\cdot(X,\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}},-)_{\underline{x}}}{\longrightarrow} \mathrm{Vect}\,. \tag{17.33}$$

However, the latter is the statement of Proposition F.4.4.

18. The Langlands functor

In this section we recall the construction of the Langlands functor, and establish the following of its properties:

- Compatibility with the functors coeff_G and $\Gamma^{\mathrm{spec}}_{\check{G}}$;
- Compatibility with the actions of Sph_G and $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$;
- Compatibility with the functors Loc_G and $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}$.

18.1. **Recollections on the Langlands functor–the coarse version.** In this and the next subsections we recall the construction of the *coarse version of the* Langlands functor

$$\mathbb{L}_{G,\mathrm{coarse}} : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}). \tag{18.1}$$

18.1.1. We consider $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\star$ as a monoidal category (see Sect. H.5.2), and

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\star \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \tag{18.2}$$

as a monoidal functor. Recall that $\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$ is a *localization*, i.e., its right adjoint is fully faithful (the proof is given, e.g., in [GLC4, Corollary C.1.8 and Sect. C.1.9]).

18.1.2. We consider $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ as acted on by $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\star$ via the action of $(\mathrm{Sph}_{G,\mathrm{Ran}})^\star$ on $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ (see Sect. H.6.8) and

$$\mathrm{Sat}^{-1,\mathrm{nv}}_G : (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\star \to (\mathrm{Sph}_{G,\mathrm{Ran}})^\star.$$

According to [Ga1, Corollary 4.5.5], the action of $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^\star$ on $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ factors through (18.2), so we obtain an action on $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$.

18.1.3. The coarse Langlands functor

$$\mathbb{L}_{G,\mathrm{coarse}} : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}),$$

as constructed in [GLC1, Sect. 1.4], is uniquely characterized by the following two properties:

- The functor $\mathbb{L}_{G,\mathrm{coarse}}$ is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$-linear;

• The diagram

$$
\begin{tikzcd}
\mathrm{Vect} \arrow[r, "\mathrm{Id}"] & \mathrm{Vect} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G^{\mathrm{Vac,glob}}"] \arrow[r, "\mathbb{L}_{G,\mathrm{coarse}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma(\mathrm{LS}_{\check{G}},-)"']
\end{tikzcd}
\tag{18.3}
$$

commutes.

18.1.4. Note that since $\mathrm{Loc}_{\check{G}}^{\mathrm{spec}}$ is a localization, the second property can be equivalently formulated as linearity with respect to $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\star}$.

By Corollary H.6.11, we obtain that the functor

$$\mathbb{L}_{G,\mathrm{coarse}} \otimes \mathrm{Id} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran})$$

intertwines the actions of $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\overset{!}{\otimes}}$ on both sides.

In fact, Corollary H.6.11 implies that the functor

$$\mathbb{L}_{G,\mathrm{coarse}} \otimes \mathrm{Id} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$$

intertwines the actions of $\mathrm{Rep}(\check{G})$, viewed as a crystal of monoidal categories over $\mathrm{Ran}^{\mathrm{untl}}$.

18.1.5. We now claim:

Proposition 18.1.6. *The following diagram commutes:*

$$
\begin{tikzcd}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[r, "\mathrm{CS}_G"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "{\mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}"] \arrow[r, "\mathbb{L}_{G,\mathrm{coarse}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}), \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G}}"']
\end{tikzcd}
\tag{18.4}
$$

where $\delta_{N_{\rho(\omega_X)}}$ is as in Sect. 9.6.5.

Proof. It suffices to construct the datum of commutativity for the diagram

$$
\begin{tikzcd}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[r, "\mathrm{CS}_G"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \arrow[u, "{\mathrm{coeff}_{G,\mathrm{Ran}}[2\delta_{N_{\rho(\omega_X)}}]}"] \arrow[r, "\mathbb{L}_{G,\mathrm{coarse}} \otimes \mathrm{Id}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran}). \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G},\mathrm{Ran}}"']
\end{tikzcd}
\tag{18.5}
$$

Consider the categories appearing in (18.5) as equipped with an action of $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\overset{!}{\otimes}}$. We will construct a datum of commutativity of (18.5) as $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\overset{!}{\otimes}}$-module categories.

Note, however, that the upper right corner, i.e.,

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} = (\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\overset{!}{\otimes}},$$

is *co-free*, when viewed as a module over itself. Hence, the datum of commutativity of (18.5) as $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\overset{!}{\otimes}}$-module categories, is equivalent to the datum of commutativity of the *outer diagram* in

$$\begin{tikzcd}
\text{D-mod}(\mathrm{Ran}) \arrow[rr, "\mathrm{Id}"] & & \text{D-mod}(\mathrm{Ran}) \\
\mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[u] \arrow[rr, "\mathrm{CS}_G"] & & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[u] \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \arrow[u, "{\mathrm{coeff}_{G,\mathrm{Ran}}[2\delta_{N_{\rho(\omega_X)}}]}"] \arrow[rr, "\mathbb{L}_{G,\mathrm{coarse}}\otimes \mathrm{Id}"] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran}), \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G},\mathrm{Ran}}"']
\end{tikzcd} \tag{18.6}$$

as D-mod(Ran)-linear categories, where:

- The upper right vertical arrow is the factorization functor $\mathrm{inv}_{\check{G}} : \mathrm{Rep}(\check{G}) \to \mathrm{Vect}$;
- The upper left vertical arrow is the factorization functor

$$\mathrm{Whit}^!(G)_{\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}) \to \mathrm{Vect},$$

where the second arrow is the functor of !-fiber at the unit.

In its turn, the datum of commutativity of the outer diagram in (18.6) is equivalent to the datum of commutativity of

$$\begin{tikzcd}
\text{D-mod}(\mathrm{Ran}) \arrow[rr, "\mathrm{Id}"] & & \text{D-mod}(\mathrm{Ran}) \\
\mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[u] & & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[u] \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \arrow[u, "{\mathrm{coeff}_{G,\mathrm{Ran}}[2\delta_{N_{\rho(\omega_X)}}]}"] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran}) \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G},\mathrm{Ran}}"'] \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "\mathrm{Id}\otimes\omega_{\mathrm{Ran}}"] \arrow[rr, "\mathbb{L}_{G,\mathrm{coarse}}"] & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, "\mathrm{Id}\otimes\omega_{\mathrm{Ran}}"']
\end{tikzcd} \tag{18.7}$$

just as DG categories.

Note, however, that the composite left vertical arrow in (18.7) is the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{coeff}^{\mathrm{Vac}}_G[2\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect} \overset{\omega_{\mathrm{Ran}}}{\longrightarrow} \text{D-mod}(\mathrm{Ran})$$

and the composite right vertical arrow in (18.7) is the functor

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \overset{\Gamma(\mathrm{LS}_{\check{G}},-)}{\longrightarrow} \mathrm{Vect} \overset{\omega_{\mathrm{Ran}}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}).$$

Now, the required commutativity is supplied by (18.3), combined with Lemma 9.6.7.

□

18.2. **Compatibility with the full spherical action.**

18.2.1. As was mentioned in Sect. 17.5.1, for a fixed $\underline{x} \in \mathrm{Ran}$, the category $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$ carries an action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$.

In Sect. E.7.1 we will extend this to an action of $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ on $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran})$. In fact, we have an action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$, viewed as a crystal of monoidal categories over $\mathrm{Ran}^{\mathrm{untl}}$ on $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$.

By Sect. H.6.8, this gives rise to an action of
$$(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\star} \twoheadrightarrow \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$$
on $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$.

18.2.2. A basic feature of the above action is that the functor
$$\Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G},\mathrm{Ran}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$
is compatible with the actions of $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ on the two sides.

18.2.3. We claim:

Lemma 18.2.4.

(a) *The action of* $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ *on* $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})$ *preserves the subcategory*
$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \overset{\Xi_{\mathrm{LS}_{\check{G}}}\otimes \mathrm{Id}}{\hookrightarrow} \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}).$$

(b) *The resulting action of* $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ *on* $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})$ *is compatible with the projection*
$$\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \overset{\Psi_{\mathrm{LS}_{\check{G}}}\otimes \mathrm{Id}}{\twoheadrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}).$$

Proof. To prove point (a), it suffices to show that the generators of $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ preserve the subcategory $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})$. We take these generators to be the essential image of the factorization functor
$$\mathrm{nv} : \mathrm{Rep}(\check{G}) \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}.$$

This makes the assertion evident: the resulting action is the natural action of $(\mathrm{Rep}(\check{G})_{\mathrm{Ran}})^{\overset{!}{\otimes}}$ on $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})$.

Point (b) of the lemma follows similarly.

$\square$

Remark 18.2.5. Note that for a fixed $\underline{x} \in \mathrm{Ran}$, the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ on $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ factors through the quotient
$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}} \twoheadrightarrow \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},\underline{x}},$$
see Sect. 7.1.1. This follows from the fact that the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\underline{x}}$ on $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ is given by t-exact functors, combined with the fact that the t-structure on $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ is separated.

We do *not* know how to formulate a parallel property for the action of $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ on the category $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})$, see Remark 7.1.2.

18.2.6. We now claim:

Proposition 18.2.7. *The functor*
$$\mathbb{L}_{G,\mathrm{coarse}} \otimes \mathrm{Id} : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})$$
intertwines the $(\mathrm{Sph}_{G,\mathrm{Ran}})^{\overset{!}{\otimes}}$ *action on the left-hand side with the* $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$*-action on the right-hand side via the functor*
$$(\mathrm{Sph}_{G,\mathrm{Ran}})^{\overset{!}{\otimes}} \to (\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}},$$
induced by the factorization functor
$$\mathrm{Sat}_G : \mathrm{Sph}_G \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}.$$

The rest of this subsection is devoted to the proof of Proposition 18.2.7.

18.2.8. By Corollaries H.6.4 and H.6.7, we can reformulate the assertion of the proposition as follows: the functor

$$\mathbb{L}_{G,\text{coarse}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{QCoh}(\text{LS}_{\check{G}})$$

intertwines the actions of

$$\text{Sph}_{G,\text{Ran}^{\text{untl}},\text{indep}}$$

on the left-hand side with the action of

$$\text{Sph}^{\text{spec}}_{\check{G},\text{Ran}^{\text{untl}},\text{indep}}$$

on the right-hand side via the functor

$$\text{Sph}_{G,\text{Ran}^{\text{untl}},\text{indep}} \to \text{Sph}^{\text{spec}}_{\check{G},\text{Ran}^{\text{untl}},\text{indep}},$$

induced by the factorization functor

$$\text{Sat}_G : \text{Sph}_G \to \text{Sph}^{\text{spec}}_{\check{G}}.$$

18.2.9. We start with the commutative diagram (18.4)

$$\begin{tikzcd}
\text{Whit}^!(G)_{\text{Ran}} \arrow[r, "\text{CS}_G"] & \text{Rep}(\check{G})_{\text{Ran}} \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \arrow[u, "{\text{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}"] \arrow[r, "\mathbb{L}_{G,\text{coarse}}"] & \text{QCoh}(\text{LS}_{\check{G}}), \arrow[u, "\Gamma^{\text{spec}}_{\check{G}}"']
\end{tikzcd} \tag{18.8}$$

and note that the vertical arrows factor as

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{Whit}^!(G)_{\text{Ran}^{\text{untl}},\text{indep}} \hookrightarrow \text{Whit}^!(G)_{\text{Ran}}$$

and

$$\text{QCoh}(\text{LS}_{\check{G}}) \to \text{Rep}(\check{G})_{\text{Ran}^{\text{untl}},\text{indep}} \hookrightarrow \text{Rep}(\check{G})_{\text{Ran}},$$

respectively, so that we obtain a commutative diagram

$$\begin{tikzcd}
\text{Whit}^!(G)_{\text{Ran}^{\text{untl}},\text{indep}} \arrow[r, "\text{CS}_G"] & \text{Rep}(\check{G})_{\text{Ran}^{\text{untl}},\text{indep}} \\
\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \arrow[u, "{\text{coeff}_{G,\text{Ran}^{\text{untl}},\text{indep}}[2\delta_{N_{\rho(\omega_X)}}]}"] \arrow[r, "\mathbb{L}_{G,\text{coarse}}"] & \text{QCoh}(\text{LS}_{\check{G}}). \arrow[u, "\Gamma^{\text{spec}}_{\check{G},\text{Ran}^{\text{untl}},\text{indep}}"']
\end{tikzcd} \tag{18.9}$$

18.2.10. Since the functor

$$\text{coeff}_{G,\text{Ran}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \otimes \text{D-mod}(\text{Ran}) \to \text{Whit}^!(G)_{\text{Ran}}$$

is compatible with the action of $(\text{Sph}_{G,\text{Ran}})^{\overset{!}{\otimes}}$, from Corollaries H.6.4 and H.6.7 we obtain that the functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \overset{-\otimes\omega_{\text{Ran}^{\text{untl}}}}{\simeq} \tag{18.10}$$

$$\simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \otimes \text{D-mod}(\text{Ran}^{\text{untl}})_{\text{indep}} \overset{\text{coeff}_{G,\text{Ran}^{\text{untl}}}}{\longrightarrow} \text{Whit}^!(G)_{\text{Ran}^{\text{untl}},\text{indep}},$$

appearing as the left vertical arrow in (18.9), is compatible with the action of $\text{Sph}_{G,\text{Ran}^{\text{untl}},\text{indep}}$.

Similarly, since the functor

$$\Gamma^{\text{spec}}_{\check{G},\text{Ran}} : \text{QCoh}(\text{LS}_{\check{G}}) \otimes \text{D-mod}(\text{Ran}) \to \text{Rep}(\check{G})_{\text{Ran}}$$

is compatible with the actions of $(\text{Sph}^{\text{spec}}_{\check{G},\text{Ran}})^{\overset{!}{\otimes}}$, we obtain that the functor

$$\text{QCoh}(\text{LS}_{\check{G}}) \overset{-\otimes\omega_{\text{Ran}^{\text{untl}}}}{\simeq} \text{QCoh}(\text{LS}_{\check{G}}) \otimes \text{D-mod}(\text{Ran}^{\text{untl}})_{\text{indep}} \overset{\Gamma^{\text{spec}}_{\check{G},\text{Ran}^{\text{untl}}}}{\longrightarrow} \text{Rep}(\check{G})_{\text{Ran}^{\text{untl}},\text{indep}}, \tag{18.11}$$

appearing as the right vertical arrow in (18.9), is compatible with the action of $\text{Sph}^{\text{spec}}_{\check{G},\text{Ran}^{\text{untl}},\text{indep}}$.

18.2.11. Recall now that the functor

$$\Gamma^{\mathrm{spec}}_{\check{G}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

is fully faithful. By Proposition H.3.2, this implies that the functor $\Gamma^{\mathrm{spec}}_{\check{G},\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$ is fully faithful.

Hence, in order to equip $\mathbb{L}_{G,\mathrm{coarse}}$ with a datum of compatibility with respect to

$$\mathrm{Sph}_{G,\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \overset{\mathrm{Sat}_G}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$$

it suffices to do so for the counter-clockwise composition in (18.9).

By the commutativity of (18.9), this is equivalent to endowing the clockwise composition in (18.9) with a datum of compatibility with the above action.

However, this follows from the compatibility for (18.10) mentioned above, combined with the compatibility of CS_G with Sat_G.

$\square$[Proposition 18.2.7]

18.3. **The actual Langlands functor.**

18.3.1. We now quote the following result established in [GLC1, Corollary 1.6.5]:

Theorem 18.3.2. *There exists a uniquely defined functor*

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}),$$

subject to the following conditions:

- $\Psi_{\mathrm{LS}_{\check{G}}} \circ \mathbb{L}_G \simeq \mathbb{L}_{G,\mathrm{coarse}}$;
- *The functor* $\mathbb{L}_G$ *sends compact objects in* $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ *to*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).$$

18.3.3. Let

$$\Xi_{0,\mathrm{Nilp}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) : \Psi_{\mathrm{Nilp},0}$$

and

$$\Xi_{\mathrm{Nilp},\mathrm{all}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \rightleftarrows \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) : \Psi_{\mathrm{all},\mathrm{Nilp}}$$

denote the resulting pairs of adjoint functors.

By a slight abuse of notation we will denote by $\Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G}}$ the functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \overset{\Xi_{\mathrm{Nilp},\mathrm{all}}}{\hookrightarrow} \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{\Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G}}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}},$$

which is the same as

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \overset{\Psi_{\mathrm{Nilp},0}}{\twoheadrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{\Gamma^{\mathrm{spec}}_{\check{G}}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

18.3.4. From Proposition 18.1.6 we formally obtain:

Corollary 18.3.5. *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[r, "\mathrm{CS}_G"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "{\mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}"] \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}). \arrow[u, "\Gamma^{\mathrm{spec},\mathrm{IndCoh}}_{\check{G}}"']
\end{tikzcd} \tag{18.12}$$

18.3.6. Consider again the action of $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ on $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})$. By the same mechanism as in Lemma 18.2.4, this action gives rise to an action of $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ on the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})$, which is compatible with the functors

$$(\Xi_{0,\mathrm{Nilp}}, \Psi_{\mathrm{Nilp},0}) \text{ and } (\Xi_{\mathrm{Nilp},\mathrm{all}}, \Psi_{\mathrm{all},\mathrm{Nilp}}).$$

In particular, we obtain an action of

$$(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\star} \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$$

on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$.

18.3.7. We now claim:

Proposition 18.3.8. *The functor $\mathbb{L}_G$ intertwines the action of $(\mathrm{Sph}_{G,\mathrm{Ran}})^{\star}$ on the left-hand side and the action of $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\star}$ on the right-hand side via*

$$\mathrm{Sat}_G : \mathrm{Sph}_G \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}.$$

Proof. Note that $(\mathrm{Sph}_{G,\mathrm{Ran}})^{\star}$ is compactly generated, and the subcategory

$$((\mathrm{Sph}_{G,\mathrm{Ran}})^{\star})^c \subset (\mathrm{Sph}_{G,\mathrm{Ran}})^{\star}$$

is closed under the monoidal operation, and its action on $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ preserves the subcategory

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \subset \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

Hence, in order to prove the proposition, it suffices to equip the functor

$$\mathbb{L}_G|_{\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c} : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$$

with a datum of compatibility with respect to the action of $((\mathrm{Sph}_{G,\mathrm{Ran}})^{\star})^c$.

By the definition of the functor $\mathbb{L}_G$, the restriction $\mathbb{L}_G|_{\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c}$ factors as

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \overset{\mathbb{L}_G}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty} \hookrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).$$

where

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$$

is also preserved by the action of

$$((\mathrm{Sph}_{G,\mathrm{Ran}})^{\star})^c \simeq ((\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\star})^c.$$

Hence, it suffices to endow the functor

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \overset{\mathbb{L}_G}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty}$$

with a datum of compatibility with respect to the action of $((\mathrm{Sph}_{G,\mathrm{Ran}})^{\star})^c$.

Next, we note that the functor

$$\Psi_{\mathrm{Nilp},0}|_{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

is compatible with the action of $((\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\star})^c$ and is fully faithful.

Hence, it suffices to endow the composition

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \overset{\mathbb{L}_G}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{>-\infty} \overset{\Psi_{\mathrm{Nilp},0}}{\to} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

with a datum of compatibility with respect to the action of $((\mathrm{Sph}_{G,\mathrm{Ran}})^{\star})^c$.

However, the latter composition is the functor

$$\mathbb{L}_{G,\mathrm{coarse}}|_{\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c},$$

and the required datum is supplied by Proposition 18.2.7. $\square$

18.3.9. Combining Proposition 18.3.8 with Corollaries H.6.4, H.6.7 and H.6.11, we obtain:

Corollary 18.3.10.

(a) *The functor* $\mathbb{L}_G$ *intertwines the action of* $\mathrm{Sph}_{G,\mathrm{Ran},\mathrm{indep}}$ *on the left-hand side and the action of* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$ *on the right-hand side.*

(b) *The functor*

$$\mathbb{L}_G \otimes \mathrm{Id} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}(\mathrm{Ran}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran})$$

intertwines the action of $(\mathrm{Sph}_{G,\mathrm{Ran}})^{\overset{!}{\otimes}}$ *on the left-hand side and the action of* $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}})^{\overset{!}{\otimes}}$ *on the right-hand side.*

(c) *The functor*

$$\mathbb{L}_G \otimes \mathrm{Id} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$$

intertwines the actions of Sph_G *and* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$, *viewed as crystals of monoidal categores over* $\mathrm{Ran}^{\mathrm{untl}}$.

18.4. **Critical localization and temperedness.**

18.4.1. Choose $\underline{x} \in \mathrm{Ran}$, and let

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp},\underline{x}} := \mathrm{Sph}_{G,\mathrm{temp},\underline{x}} \underset{\mathrm{Sph}_{G,\underline{x}}}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

The pair of adjoint functors

$$\mathrm{Sph}_{G,\mathrm{temp},\underline{x}} \rightleftarrows \mathrm{Sph}_{G,\underline{x}}$$

allows us to view $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp},\underline{x}}$ as a colocalization of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

According to [FR, Sect. 2.6.2], this colocalization is actually independent of the choice of $\underline{x}$. So from now on we will omit the subscript and denote the corresponding sub/quotient category by $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}}$. Denote by

$$\mathbf{u} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) : \mathbf{u}^R$$

the corresponding pair of adjoint functors.

18.4.2. From Proposition 18.3.8 we obtain:

Corollary 18.4.3. *There exists a uniquely defined functor*

$$\mathbb{L}_{G,\mathrm{temp}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}),$$

which makes both squares in the next diagram commute:

$$\begin{CD}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) @>{\mathbb{L}_G}>> \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \\
@V{\mathbf{u}^R}VV @VV{\Psi_{\mathrm{Nilp},0}}V \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} @>{\mathbb{L}_{G,\mathrm{temp}}}>> \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\
@V{\mathbf{u}}VV @VV{\Xi_{0,\mathrm{Nilp}}}V \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) @>{\mathbb{L}_G}>> \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).
\end{CD}$$

Furthermore,

$$\mathbb{L}_{G,\mathrm{temp}} \simeq \mathbb{L}_{G,\mathrm{coarse}} \circ \mathbf{u}.$$

18.4.4. Let

$$\mathrm{Loc}_G : \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

be as in Sect. 14.1.4.

The following assertion is a counterpart of Proposition 17.5.9:

Proposition 18.4.5. *The essential image of the functor*

$$\mathrm{Loc}_G : \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

lies in

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

Proof. Repeats the proof of Proposition 17.5.9 using Proposition 7.2.6.

□

18.5. **Compatibility of the Langlands functor with critical localization.**

18.5.1. The following theorem expresses the compatibility of the Langlands functor with critical localization:

Theorem 18.5.2. *The diagram*

$$\begin{tikzcd}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[u, "\mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}]"] \arrow[r, "\mathrm{FLE}_{G,\mathrm{crit}}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}"']
\end{tikzcd}$$

commutes, where the lines $\mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}}$ *and* $\mathfrak{l}_{N_{\rho(\omega_X)}}$ *are as in* (9.5) *and* (14.2), *respectively.*

The rest of the subsection is devoted to the proof of Theorem 18.5.2.

18.5.3. First, by Propositions 18.4.5 and 17.5.9, the commutativity of the diagram in Theorem 18.5.2 is equivalent to the commutativity of the following one:

$$\begin{tikzcd}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \arrow[r, "\mathbb{L}_{G,\mathrm{temp}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[u] \arrow[r, "\mathrm{FLE}_{G,\mathrm{crit}}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}, \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}"']
\end{tikzcd}$$

and is further equivalent to the commutativity of

$$\begin{tikzcd}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_{G,\mathrm{coarse}}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[u] \arrow[r, "\mathrm{FLE}_{G,\mathrm{crit}}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}. \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}"']
\end{tikzcd} \tag{18.13}$$

18.5.4. Since the right vertical arrow in (18.4) is fully faithful, it suffices to show that the two circuits in (18.13) become isomorphic after composing with the functor $\Gamma^{\mathrm{spec}}_{\check{G}}$.

Since the diagram (18.4) is commutative, we obtain that it suffices to establish the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[r, "\mathrm{CS}_G"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "{\mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G}}"'] \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[u, "{\mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}]}"] \arrow[r, "\mathrm{FLE}_{G,\mathrm{crit}}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}, \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}"']
\end{tikzcd}$$

or which is the same

$$\begin{tikzcd}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} \arrow[r, "\mathrm{CS}_G"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \arrow[u, "\mathrm{coeff}_G"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, "\Gamma^{\mathrm{spec}}_{\check{G}}"'] \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \arrow[u, "{\mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}}", "{[\delta_{N_{\rho(\omega_X)}}]}"'] \arrow[r, "\mathrm{FLE}_{G,\mathrm{crit}}"] & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}}. \arrow[u, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}"']
\end{tikzcd} \tag{18.14}$$

18.5.5. Applying duality, we obtain that it suffices to show that the pairing

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{Loc}_G \otimes \mathrm{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{coeff}_G \otimes \mathrm{Id}}{\longrightarrow} \\ \to \mathrm{Whit}^!(G)_{\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \mathrm{Vect} \overset{(-\otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect} \tag{18.15}$$

agrees under the FLE equivalences

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \text{ and } \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G)$$

with

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*} \otimes \mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \\ \overset{\Gamma^{\mathrm{spec}}_{\check{G}} \otimes \mathrm{Id}}{\longrightarrow} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}\,. \tag{18.16}$$

18.5.6. By Theorem 14.2.4, the functor (18.15) identifies canonically with (14.5). By Theorem 17.7.2, the functor (18.16) identifies canonically with (17.22).

The desired assertion follows now from Corollary 6.4.10.

$\square$[Theorem 18.5.2]

Part III: Appendix

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The main body of the paper relies on a lot of foundational material, which is developed in this Appendix. The main points are:

- Ind-coherent sheaves on algebrao-geometric objects of infinite type (our main, but *by far, not only* example is $\mathrm{Op}_{\check{G}}^{\text{mon-free}}$). This is developed out in Sect. A;
- The notion of *factorization category*, and associated objects (factorization module categories, factorization algebras within a factorization category, etc.). This is developed in Sect. B.
- The notion *unitality* in the factorization setting. This is developed in C;
- A result connecting the (pre dual of the) category of quasi-coherent sheaves on the loop space and the category of factorization modules over the corresponding factorization algebra D;
- The definition of the *spectral spherical category* in the factorization setting (the underlying algebro-geometric object is so unwieldy that one cannot algorithmically apply a procedure from Sect. A). This is developed in Sect. E.

The majority of this material is of *local* nature, i.e., it is needed to set up the local Langlands theory. That said, some sections in this Appendix (notably, Sects. F , H and I) consider local-to-global constructions.

Appendix A. Ind-coherent sheaves in infinite type

This section is devoted to the development of the theory of ind-coherent sheaves on algebro-geometric objects of infinite-type.

Prior to doing so, we introduce another player, which in some sense lies in between $\mathrm{QCoh}(-)$ and $\mathrm{IndCoh}(-)$. This object is denoted by

$$\mathcal{Y} \in \mathrm{PreStk} \rightsquigarrow \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}),$$

and it is defined by a *colimit* procedure (unlike $\mathrm{QCoh}(\mathcal{Y})$, which is defined as a *limit*). The category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ is a *predual* of $\mathrm{QCoh}(\mathcal{Y})$.

We now turn to the IndCoh theory. A priori, $\mathrm{IndCoh}(-)$ is defined for affine schemes almost of finite type, and by the process of right Kan extension on all prestacks that are locally almost of finite type. When S is an affine scheme that is not of finite type, one can approximate it by affine schemes of finite type S_α, but then one faces a choice: one can define $\mathrm{IndCoh}(S)$ either as the colimit of $\mathrm{IndCoh}(S_\alpha)$ with respect to !-pullbacks and as a limit of $\mathrm{IndCoh}(S_\alpha)$ with respect to *-pullbacks. This leads to two different categories, denoted $\mathrm{IndCoh}^!(S)$ and $\mathrm{IndCoh}^*(S)$, respectively. In good cases (technically, when S is *placid*), the Serre duality in finite type gives rise to a duality between $\mathrm{IndCoh}^!(S)$ and $\mathrm{IndCoh}^*(S)$.

The majority of this section is devoted to developing the $\mathrm{IndCoh}^!(-)$ and $\mathrm{IndCoh}^*(-)$ theories, and their interactions with other actors.

A.1. The category $\mathrm{QCoh}_{\mathrm{co}}(-)$.

A.1.1. In section we will work with *all* affine schemes (i.e., ones not necessarily almost of finite type). We denote the corresponding category by $\mathrm{Sch}^{\mathrm{aff}}$. We denote by PreStk the category of all functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

A.1.2. Consider the functor

$$\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{DGCat}, \quad S \mapsto \mathrm{QCoh}(S), \quad (S_1 \xrightarrow{f} S_2) \rightsquigarrow \mathrm{QCoh}(S_1) \xrightarrow{f_*} \mathrm{QCoh}(S_2). \tag{A.1}$$

Consider the left Kan extension of (A.1) along the fully faithful embedding

$$\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk};$$

this yields a functor

$$\mathrm{PreStk} \to \mathrm{DGCat}\,. \tag{A.2}$$

We will denote the value of (A.2) on a given prestack $\mathcal{Y}$ by

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}).$$

Explicitly,

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \simeq \underset{S \to \mathcal{Y}, S \in \mathrm{Sch}^{\mathrm{aff}}}{\mathrm{colim}} \mathrm{QCoh}(S), \tag{A.3}$$

where the colimit is formed using the pushforward functors.

Remark A.1.3. The above definition turns out to be useful in many contexts, but in the present paper its main application is the following (see Sect. A.2):

Let $\mathcal{Y}$ be and ind-affine ind-scheme

$$\mathcal{Y} = \text{“}\underset{i}{\mathrm{colim}}\text{”}\ \mathrm{Spec}(R_i).$$

Let R denote the topological ring $\lim\limits_i R_i$.

Then

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \simeq \underset{i}{\mathrm{colim}}\, R_i\text{-mod},$$

i.e., this formalizes the notion of “the category of discrete R-modules”.

Note that the above definition is close, but not the same, as $\mathrm{IndCoh}(\mathcal{Y})$. The difference is two-fold:

- $\mathrm{IndCoh}(\mathcal{Y})$ is a priori defined only when $\mathcal{Y}$ is locally almost of finite type (however, we will generalize that in Sect. A.5.6 below), while $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ does not require this assumption;
- When $\mathcal{Y}$ is an affine scheme Y, we have $\mathrm{QCoh}_{\mathrm{co}}(Y) = \mathrm{QCoh}(Y)$, i.e., there is no renormalization procedure involved. (The price we will have to pay for this is that, even for ind-schemes locally almost of finite type, the category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ is not necessarily dualizable.)

A.1.4. By construction, the assignment

$$\mathcal{Y} \mapsto \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$$

has a functoriality with respect to pushforwards, i.e., for a map $\mathcal{Y}_1 \to \mathcal{Y}_2$ we have the functor

$$f_* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_2).$$

A.1.5. The construction

$$\mathcal{Y} \rightsquigarrow \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$$

has a natural multiplicativity property. Namely, for a pair of prestacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$, we have a naturally defined equivalence

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1) \otimes \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_2) \simeq \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1 \times \mathcal{Y}_2). \tag{A.4}$$

Namely, we have, by definition:

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1) \otimes \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_2) \simeq \underset{S_1 \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_1}, S_2 \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_2}}{\mathrm{colim}} \mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2) \simeq$$

$$\simeq \underset{S_1 \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_1}, S_2 \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_2}}{\mathrm{colim}} \mathrm{QCoh}(S_1 \times S_2),$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1 \times \mathcal{Y}_2) \simeq \underset{S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_1 \times \mathcal{Y}_2}}{\mathrm{colim}} \mathrm{QCoh}(S).$$

Now, the functor

$$\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_1} \times \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_2} \to \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_1 \times \mathcal{Y}_2}, \quad S_1, S_2 \mapsto S_1 \times S_2$$

is cofinal.

A.1.6. Note that there is no reason for the category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ to be dualizable. However, we claim that $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ is *a pre-dual* of $\mathrm{QCoh}(\mathcal{Y})$, i.e., we have a canonical identification

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}), \mathrm{Vect}) =: \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^\vee \simeq \mathrm{QCoh}(\mathcal{Y}). \tag{A.5}$$

Indeed, using (A.3), we have

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^\vee \simeq \left(\underset{S \to \mathcal{Y}, S \in \mathrm{Sch}^{\mathrm{aff}}}{\mathrm{colim}} \mathrm{QCoh}(S) \right)^\vee \simeq$$
$$\simeq \underset{S \to \mathcal{Y}, S \in \mathrm{Sch}^{\mathrm{aff}}}{\lim} \mathrm{QCoh}(S)^\vee \simeq \underset{S \to \mathcal{Y}, S \in \mathrm{Sch}^{\mathrm{aff}}}{\lim} \mathrm{QCoh}(S) =: \mathrm{QCoh}(\mathcal{Y}),$$

where we recall that for $f : S_1 \to S_2$, with respect to the self-dualities

$$\mathrm{QCoh}(S_i)^\vee \simeq \mathrm{QCoh}(S_i),\ i = 1, 2,$$

the dual of f_* is f^*.

A.1.7. We claim that there is a natural action of $\mathrm{QCoh}(\mathcal{Y})$ on $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$. Namely, in terms of (A.3), an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ gives rise to a compatible family of endofunctors of $\mathrm{QCoh}(S)$ for $y : S \to \mathcal{Y}$, namely

$$S \rightsquigarrow y^*(\mathcal{F}) \otimes (-).$$

This action is compatible with the identification (A.5).

Furthermore, it satisfies the projection formula: for $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ we have

$$f_*(f^*(\mathcal{F}_2) \otimes \mathcal{F}_1) \simeq \mathcal{F}_2 \otimes f_*(\mathcal{F}_1), \quad \mathcal{F}_1 \in \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1),\ \mathcal{F}_2 \in \mathrm{QCoh}(\mathcal{Y}_2).$$

A.1.8. We can rewrite the canonical pairing

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \to \mathrm{Vect} \tag{A.6}$$

in terms of the above action of $\mathrm{QCoh}(\mathcal{Y})$ on $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$.

Namely, it is given by

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \overset{\mathrm{action}}{\to} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \overset{\Gamma(\mathcal{Y},-)}{\longrightarrow} \mathrm{Vect}\,.$$

A.1.9. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be affine. In this case we claim that the functor

$$f_* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_2)$$

admits a left adjoint, to be denoted f^*.

Indeed, the functor

$$\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_2} \to \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_1}, \quad S \mapsto S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$$

is cofinal, so the functor

$$\underset{S \in \mathcal{Y}_2, S \in \mathrm{Sch}^{\mathrm{aff}}}{\mathrm{colim}} \mathrm{QCoh}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1).$$

is an equivalence.

In terms of this identification, the functor f^* is given by the compatible family of the pullback functors

$$\mathrm{QCoh}(S) \to \mathrm{QCoh}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1).$$

A.1.10. Let $\mathcal{Y}$ be a prestack over an affine scheme S. Let $f : S' \to S$ be a map between affine schemes; set $\mathcal{Y}' := S' \underset{S}{\times} \mathcal{Y}$. By a slight abuse of notation we will denote by the same character f the resulting map $\mathcal{Y}' \to \mathcal{Y}$.

The category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ (resp., $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}')$) is naturally tensored over $\mathrm{QCoh}(S)$ (resp., $\mathrm{QCoh}(S')$, and the functor $f_* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}') \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ is $\mathrm{QCoh}(S)$-linear. Hence, so is its right adjoint f^*. From here we obtain a functor

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}'). \tag{A.7}$$

We claim:

Lemma A.1.11. *The functor* (A.7) *is an equivalence.*

Proof. Follows from the fact that the functor

$$\widetilde{S} \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}} \mapsto S' \underset{S}{\times} \widetilde{S} \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}'}$$

is cofinal.

□

A.1.12. Let $\mathcal{Y}$ have an affine diagonal. In this case we claim that there is a naturally defined functor

$$\Omega_{\mathcal{Y}} : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}). \tag{A.8}$$

Namely, in terms of (A.3), the functor (A.8) is given by the (compatible family) of direct image functors

$$\mathrm{QCoh}(S) \to \mathrm{QCoh}(\mathcal{Y}),$$

which are well-defined, since the morphisms $S \to \mathcal{Y}$ are affine.

Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic map. Note that, by construction, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1) & \xrightarrow{f_*} & \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_2) \\
{\scriptstyle \Omega_{\mathcal{Y}_1}}\downarrow & & \downarrow{\scriptstyle \Omega_{\mathcal{Y}_2}} \\
\mathrm{QCoh}(\mathcal{Y}_1) & \xrightarrow{f_*} & \mathrm{QCoh}(\mathcal{Y}_2).
\end{array}$$

A.1.13. The following assertion is established in [Ga5, Theorems 2.2.4 or 2.2.6 and Proposition 6.3.8]:

Theorem A.1.14. *Let $\mathcal{Y}$ be an quasi-compact algebraic stack with an affine diagonal. Suppose that one of the following conditions holds:*

(i) *$\mathcal{Y}$ can be realized as a quotient of an algebraic space by an action of a (finite-dimensional) algebraic group;*

(ii) *$\mathcal{Y}$ is eventually coconnective algebraic stack almost of finite type.*

Then the functor $\Omega_{\mathcal{Y}}$ of (A.8) *is an equivalence.*

A.1.15. Let $\mathcal{Y}$ be a (not necessarily quasi-compact) algebraic stack. Suppose that $\mathcal{Y}$ can be written as a union of quasi-compact open substacks $\mathcal{Y}_i$ that satisfy one of the conditions in Theorem A.1.14.

Corollary A.1.16. *Under the above circumstances, we have a canonical equivalence*

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \simeq \underset{i}{\mathrm{colim}}\, \mathrm{QCoh}(\mathcal{Y}_i),$$

where the colimit is taken with respect to the pushforward functors.

Proof. Note that the map
$$\operatorname{colim}_i \mathcal{Y}_i \to \mathcal{Y}$$
is an isomorphism in PreStk.

Hence, the functor
$$\operatorname{colim}_i \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_i) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$$
is an equivalence.

Now the assertion follows from Theorem A.1.14.

□

A.2. The category $\mathrm{QCoh}_{\mathrm{co}}$ on ind-schemes.

A.2.1. Let $\mathcal{Y}$ be an ind-affine ind-scheme (see [GaRo1]). According to Corollary 1.6.6 in *loc. cit.*, the map
$$\operatorname{colim}_{S \text{ closed in } \mathcal{Y}} S \to \mathcal{Y}$$
is an isomorphism in PreStk, where the index category is that of affine schemes, equipped with a closed embedding to $\mathcal{Y}$.

Hence, in this case, we have

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \simeq \operatorname{colim}_{S \text{ closed in } \mathcal{Y}} \mathrm{QCoh}(S). \tag{A.9}$$

A.2.2. Recall the following general paradigm:

Let
$$i \mapsto \mathbf{C}_i, \quad i \in I$$
be a diagram in DGCat. Denote
$$\mathbf{C} := \operatorname{colim}_{i \in I} \mathbf{C}_i,$$
where, per our conventions, the colimit is taken in DGCat (i.e., the category of cocomplete DG categories and continuous functors).

Let $\mathrm{ins}_i : \mathbf{C}_i \to \mathbf{C}$ denote the tautological functors.

A.2.3. Suppose that each $\mathbf{C}_i$ is equipped with a t-structure, compatible with filtered colimits, i.e., $\mathbf{C}_i^{\geq 0}$ is closed under filtered colimits. And suppose that the transition functors
$$\mathbf{C}_i \overset{F_{i,j}}{\to} \mathbf{C}_j$$
are t-exact.

We equip with $\mathbf{C}$ with a t-structure by declaring that $\mathbf{C}^{\leq 0}$ is generated under colimits by the essential images of $\mathrm{ins}_i(\mathbf{C}_i^{\leq 0})$.

So by construction, the functors ins_i are right t-exact.

A.2.4. We claim:

Lemma A.2.5. *Assume that I is filtered. Then:*

(a) *The t-structure on $\mathbf{C}$ is compatible with filtered colimits.*

(b) *The functors* ins_i *are t-exact.*

(c) *The category $\mathbf{C}^{\geq 0}$ is generated under filtered colimits by the essential images of $\mathbf{C}_i^{\geq 0}$ along the functors* ins_i.

Proof. The first two points are proved in [Lu3, Proposition C.3.3.5].

Namely, in the notation of loc. cit., the category $\mathrm{Groth}^{\mathrm{lex}}$ is equivalent to the category of presentable stable ∞-categories with right-complete t-structures and colimit preserving functors which are t-exact. The first two points are equivalent to the assertion that $\mathrm{Groth}^{\mathrm{lex}}$ admits filtered colimits and the forgetful functor

$$\mathrm{Groth}^{\mathrm{lex}} \to \mathrm{Pr}^{\mathrm{L}}$$

given by $(\mathbf{C}, \mathbf{C}^{\leq 0}) \mapsto \mathbf{C}$ preserves filtered colimits. By [Lu3, Proposition C.3.3.5], $\mathrm{Groth}^{\mathrm{lex}}$ admits filtered colimits and the functor $(\mathbf{C}, \mathbf{C}^{\leq 0}) \mapsto \mathbf{C}^{\leq 0}$ preserves filtered colimits. The result now follows from the fact that for any $(\mathbf{C}, \mathbf{C}^{\leq 0}) \in \mathrm{Groth}^{\mathrm{lex}}$, we have

$$\mathbf{C} \simeq \mathrm{Stab}(\mathbf{C}^{\leq 0})$$

is given by the stabilization, and the functor $\mathrm{Stab} : \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ preserves colimits.

To prove the third point we note that any $\mathbf{c} \in \mathbf{C}$ is canonically isomorphic to

$$\underset{i \in I}{\mathrm{colim}}\, \mathrm{ins}_i \circ \mathrm{ins}_i^R(\mathbf{c}).$$

If $\mathbf{c} \in \mathbf{C}^{\geq 0}$, then so are all $\mathrm{ins}_i^R(\mathbf{c})$ (since the functors ins_i^R are left t-exact, being right adjoints of (right) t-exact functors.

□

Remark A.2.6. Note that in the situation of Lemma A.2.5, we have

$$\mathbf{C} \simeq \lim_{i \mathrm{i} I^{\mathrm{op}}} \mathbf{C}_i,$$

where:

- The limit is taken in the category of ∞-categories;
- The functor $\to$ is given by the (compatible collection of) the functors ins_i^R.

Since the functors ins_i^R send $\mathbf{C}^{\geq 0}$ to $\mathbf{C}_i^{\geq 0}$, we obtain they also induce an equivalence

$$\mathbf{C}^{\geq 0} \simeq \lim_{i \mathrm{i} I^{\mathrm{op}}} \mathbf{C}_i^{\geq 0}. \tag{A.10}$$

Corollary A.2.7. *Let $\Phi : \mathbf{C} \to \mathbf{D}$ be a continuous functor, where $\mathbf{D}$ is also equipped with a t-structure.*

(a) *The functor Φ is right t-exact if and only if each $\Phi \circ \mathrm{ins}_i =: \Phi_i : \mathbf{C}_i \to \mathbf{D}$ is right t-exact.*

(b) *If Φ is left t-exact, then so is each Φ_i.*

(b') *Suppose that the t-structure on $\mathbf{D}$ is compatible with filtered colimits. Then the assertion in (b) is "if and only if".*

A.2.8. Let $\mathcal{Y}$ be an ind-affine ind-scheme. We use the presentation (A.9) and the construction in Sect. A.2.3 to equip the category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ with a t-structure:

By definition, $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\leq 0}$ is generated under colimits by the essential images of $\mathrm{QCoh}(S)^{\leq 0}$ for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$.

A.2.9. From Lemma A.2.5, we obtain:

Corollary A.2.10.

(a) *The t-structure on $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ is compatible with filtered colimits.*

(b) *For every $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$, the direct image functor $\mathrm{QCoh}(S) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ is t-exact.*

A.2.11. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between ind-affine ind-schemes. It follows by definition that the functor

$$f_* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_2)$$

is right t-exact.

However, from Corollaries A.2.10(b) and A.2.7 we obtain:

Corollary A.2.12. *The functor f_* is t-exact.*

A.2.13. Assume that f is affine, in which case we have a well-defined functor

$$f^* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_2) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1).$$

From Corollary A.2.12 we obtain:

Corollary A.2.14. *The functor f^* is right t-exact.*

Finally, assume that f is flat. In this case, unwinding the construction of f^* in Sect. A.1.9 and using Corollaries A.2.10(b) and A.2.7, we obtain:

Lemma A.2.15. *For a flat map f between ind-affine ind-schemes, the functor $f^* : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_2) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}_1)$ is t-exact.*

A.3. **A descent property of $\mathrm{QCoh}_{\mathrm{co}}(-)$.** In this subsection we will prove a certain technical assertion used in the main body of the text.

A.3.1. Let $\mathcal{Y}$ be an ind-affine ind-scheme. Let $g : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be a map of prestacks that is an affine fpqc cover, and let $\widetilde{\mathcal{Y}}^\bullet$ denote its Čech nerve.

Consider $\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^\bullet)$ as a cosimplicial category, equipped with an augmentation by $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ using *-pullbacks (they are well-defined since the maps involved are affine, see Sect. A.1.9).

Thus, we obtain a functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^\bullet)). \tag{A.11}$$

A.3.2. From now on we will perceive $\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^\bullet)$ as a *semi*-cosimplicial category, so that transition functors involved are t-exact (by the flatness assumption on g, see Lemma A.2.15). Hence, the functor (A.11) induces a functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{>-\infty} \to \mathrm{Tot}(\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^\bullet)^{>-\infty}). \tag{A.12}$$

We will prove:

Proposition A.3.3. *Suppose that $\mathcal{Y}$ can be exhibited as a filtered colimit in* PreStk *of affine schemes with transition maps that are almost finitely presented.*[58] *Then the functor* (A.12) *is an equivalence.*

The rest of this subsection is devoted to the proof of this proposition.

A.3.4. It suffices to show that

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0} \to \mathrm{Tot}(\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^\bullet)^{\geq 0}) \tag{A.13}$$

is an equivalence.

A.3.5. Let

$$\mathcal{Y} = \underset{i\in I}{\mathrm{colim}}\, Y_i$$

be the presentation of $\mathcal{Y}$ as in the statement of the proposition. Denote the map $Y_i \to Y_j$ for $(i \to j) \in I$ by $f_{i,j}$.

Set

$$\widetilde{Y}_i^\bullet := \widetilde{\mathcal{Y}}^\bullet \underset{\mathcal{Y}}{\times} Y_i.$$

Then for every m, we also have

$$\widetilde{\mathcal{Y}}^m \simeq \underset{i}{\mathrm{colim}}\, \widetilde{Y}_i^m.$$

Denote the corresponding maps $\widetilde{Y}_i^m \to \widetilde{Y}_j^m$ by $f_{i,j}^m$.

[58]I.e., finitely presented after each coconnective truncation.

A.3.6. We have

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \simeq \underset{i\in I}{\mathrm{colim}}\, \mathrm{QCoh}(Y_i)$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^m) \simeq \underset{i\in I}{\mathrm{colim}}\, \mathrm{QCoh}(\widetilde{Y}_i^m),$$

where in both cases the colimit is taken with respect to the pushforward functors (recall that by default, colimits are taken in DGCat, i.e., in the ∞-category of cocomplete DG categories and continuous functors).

We can rewrite the above colimits as *limits* (in the category of DG categories with not necessarily continuous functors)

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{QCoh}(Y_i)$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^m) \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{QCoh}(\widetilde{Y}_i^m),$$

where the transition functors are $f_{i,j}^! := (f_{i,j})_*^R$ and $(f_{i,j}^m)^! := (f_{i,j}^m)_*^R$, respectively (note that these right adjoints are indeed in general *discontinuous*).

From here we obtain:

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0} \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{QCoh}(Y_i)^{\geq 0}$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^m)^{\geq 0} \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{QCoh}(\widetilde{Y}_i^m)^{\geq 0},$$

where limits are taken in the category of ∞-categories and all functors, see (A.10).

A.3.7. For every $\phi : [m'] \to [m'']$ in[59] $\boldsymbol{\Delta}$ denote by g^ϕ the corresponding map

$$\widetilde{\mathcal{Y}}^{m''} \to \widetilde{\mathcal{Y}}^{m'}.$$

For every index i, let g_i^ϕ denote the resulting map $\widetilde{Y}_i^{m''} \to \widetilde{Y}_i^{m'}$.

For every arrow $(i \to j) \in I$, we have a Cartesian diagram of schemes

$$\begin{tikzcd}
\widetilde{Y}_i^{m''} \arrow[r, "f_{i,j}^{m''}"] \arrow[d, "g_i^\phi"'] & \widetilde{Y}_j^{m''} \arrow[d, "g_j^\phi"] \\
\widetilde{Y}_i^{m'} \arrow[r, "f_{i,j}^{m'}"] & \widetilde{Y}_j^{m'},
\end{tikzcd}$$

which gives rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{QCoh}(\widetilde{Y}_i^{m''}) \arrow[r, "(f_{i,j}^{m''})_*"] & \mathrm{QCoh}(\widetilde{Y}_j^{m''}) \\
\mathrm{QCoh}(\widetilde{Y}_i^{m'}) \arrow[r, "(f_{i,j}^{m'})_*"] \arrow[u, "(g_i^\phi)^*"] & \mathrm{QCoh}(\widetilde{Y}_j^{m'}). \arrow[u, "(g_j^\phi)^*"']
\end{tikzcd}$$

From here we obtain a natural transformation

$$(g_i^\phi)^* \circ (f_{i,j}^{m'})^! \to (f_{i,j}^{m''})^! \circ (g_j^\phi)^*. \tag{A.14}$$

Now, the assumption that the maps $f_{i,j}$ are almost of finite presentation and the maps g^ϕ are flat implies that the natural transformations (A.14) are isomorphisms when evaluated on $\mathrm{QCoh}(\widetilde{Y}_j^{m'})^{\geq 0}$.

---

[59]We remind that we only consider injective maps ϕ.

Hence, we obtain a family of commutative diagrams

$$\begin{tikzcd} \mathrm{QCoh}(\widetilde{Y}_i^{m''})^{\geq 0} & \mathrm{QCoh}(\widetilde{Y}_j^{m''})^{\geq 0} \arrow[l, "(f_{i,j}^{m''})^!"'] \\ \mathrm{QCoh}(\widetilde{Y}_i^{m'})^{\geq 0} \arrow[u, "(g_i^\phi)^*"] & \mathrm{QCoh}(\widetilde{Y}_j^{m'})^{\geq 0}. \arrow[l, "(f_{i,j}^{m'})^!"'] \arrow[u, "(g_j^\phi)^*"'] \end{tikzcd}$$

A.3.8. Thus, we obtain a well-defined functor from $\mathbf{\Delta} \times I$ to the category of ∞-categories

$$m, i \mapsto \mathrm{QCoh}(\widetilde{Y}_i^m)^{\geq 0},$$

and we can rewrite

$$\mathrm{Tot}(\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^\bullet)^{\geq 0}) := \lim_{m \in \mathbf{\Delta}} \lim_{i \in I^{\mathrm{op}}} \mathrm{QCoh}(\widetilde{Y}_i^m)^{\geq 0}$$

as

$$\lim_{(m,i) \in \mathbf{\Delta} \times I^{\mathrm{op}}} \mathrm{QCoh}(\widetilde{Y}_i^m)^{\geq 0}$$

and further as

$$\lim_{i \in I^{\mathrm{op}}} \lim_{m \in \mathbf{\Delta}} \mathrm{QCoh}(\widetilde{Y}_i^m)^{\geq 0}.$$

Unwinding the construction, we obtain that the following diagram commutes

$$\begin{tikzcd} \mathrm{Tot}(\mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}}^\bullet)^{\geq 0}) \arrow[r, "\sim"] & \lim\limits_{i \in I^{\mathrm{op}}} \lim\limits_{m \in \mathbf{\Delta}} \mathrm{QCoh}(\widetilde{Y}_i^m)^{\geq 0} \\ \mathrm{QCoh}_{\mathrm{co}}(\widetilde{\mathcal{Y}})^{\geq 0} \arrow[u] \arrow[r, "\sim"] & \lim\limits_{i \in I^{\mathrm{op}}} \mathrm{QCoh}(Y_i)^{\geq 0}, \arrow[u] \end{tikzcd} \tag{A.15}$$

where the right vertical arrow is comprised of the functors

$$\mathrm{QCoh}(Y_i)^{\geq 0} \to \mathrm{Tot}(\mathrm{QCoh}_{\mathrm{co}}(\widetilde{Y}_i^\bullet)^{\geq 0}). \tag{A.16}$$

Now, the functors (A.16) are equivalences by the usual fpqc descent. Hence, the right vertical arrow in (A.15) is an equivalence.

Hence, the left vertical arrow is also an equivalence, as required.

□[Proposition A.3.3]

A.4. The category $\mathrm{IndCoh}^!(-)$.

A.4.1. Let ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ denote the category of n-coconnective affine schemes, i.e.,

$${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} = (\mathrm{ComAlg}(\mathrm{Vect}^{\geq -n, \leq 0}))^{\mathrm{op}}.$$

We have

$$\mathrm{Sch}^{\mathrm{aff}} \simeq \lim_n ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}).$$

A.4.2. Let

$${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \subset {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$$

be the full subcategory consisting of n-coconnective affine schemes *of finite type* (see [GaRo3, Chapter 1, Sect. 1.5]).

Note that

$${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \simeq \mathrm{Pro}({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}).$$

Let $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ denote the full subcategory of $\mathrm{Sch}^{\mathrm{aff}}$ consisting of affine schemes almost of finite type, which is by definition

$$\lim_n ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}).$$

Let $\mathrm{PreStk}_{\mathrm{aft}} \subset \mathrm{PreStk}$ the full subcategory consisting of prestacks locally almost of finite type. We have

$$\mathrm{PreStk}_{\mathrm{aft}} \simeq \lim_n ({}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}),$$

where

$$ {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \simeq \mathrm{Funct}(({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}, \infty\text{-}\mathrm{Grpd}). $$

A.4.3. We define the functor

$$ {}^{\leq n}\mathrm{IndCoh}^! : ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}, $$

to be the left Kan extension of the functor

$$ \mathrm{IndCoh} : ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{DGCat} $$

along

$$ ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}. $$

Explicitly, for $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, we have

$$ \mathrm{IndCoh}^!(S) = \underset{S \to S_0, S_0 \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\mathrm{colim}} \mathrm{IndCoh}(S_0), \tag{A.17} $$

where the transition functors are given by

$$ (S \to S_0' \xrightarrow{f} S_0'') \rightsquigarrow \mathrm{IndCoh}(S_0'') \xrightarrow{f^!} \mathrm{IndCoh}(S_0'). $$

A.4.4. It is easy to see that the natural transformation from

$$ ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \xrightarrow{{}^{\leq n}\mathrm{IndCoh}^!} \mathrm{DGCat} $$

to

$$ ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow ({}^{\leq n+1}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \xrightarrow{{}^{\leq n+1}\mathrm{IndCoh}^!} \mathrm{DGCat} $$

is an isomorphism. Hence, we obtain a well-defined functor

$$ \mathrm{IndCoh}^! : ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}, \tag{A.18} $$

where

$$ {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} = \underset{n}{\mathrm{colim}}\, {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}. $$

A.4.5. We define the functor

$$ \mathrm{IndCoh}^! : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat} $$

to be the right Kan extension of (A.18) along the embedding

$$ ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}}. $$

Explicitly, for $\mathcal{Y} \in \mathrm{PreStk}$, we have

$$ \mathrm{IndCoh}^!(\mathcal{Y}) = \underset{S \to \mathcal{Y}, S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}}{\lim} \mathrm{IndCoh}^!(S), $$

where the transition functors are given by

$$ (S' \xrightarrow{f} S'' \to \mathcal{Y}) \rightsquigarrow \mathrm{IndCoh}^!(S'') \xrightarrow{f^!} \mathrm{IndCoh}^!(S'). $$

A.4.6. Thus, by definition, for a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk, we have a well-defined functor

$$ f^! : \mathrm{IndCoh}^!(\mathcal{Y}_2) \to \mathrm{IndCoh}^!(\mathcal{Y}_1). $$

In particular, taking the projection $\mathcal{Y} \to \mathrm{pt}$, we obtain that for any $\mathcal{Y}$, we have a well-defined object

$$ \omega_{\mathcal{Y}} \in \mathrm{IndCoh}^!(\mathcal{Y}). $$

A.4.7. It follows from the convergence property of the usual IndCoh functor

$$\mathrm{IndCoh} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}$$

(see [GaRo3, Chapter 4, Prop. 6.4.3]) that the natural transformation

$$\mathrm{IndCoh}^!|_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{IndCoh}$$

is an isomorphism.

I.e., the value of $\mathrm{IndCoh}^!(\mathcal{Y})$ on a prestack locally almost of finite type recovers the usual $\mathrm{IndCoh}(\mathcal{Y})$.

Remark A.4.8. The above construction gives a definition of the functor $\mathrm{IndCoh}^!(\mathcal{Y})$ for a general prestack $\mathcal{Y}$. But unless some conditions on $\mathcal{Y}$ are imposed, we will not be able to say much about the properties of this category.

For example, it is not even clear (and, probably, not true) whether for $S \in \mathrm{Sch}^{\mathrm{aff}}$, the category $\mathrm{IndCoh}(S)$ is dualizable.

A condition on $\mathcal{Y}$ that makes $\mathrm{IndCoh}^!(\mathcal{Y})$ manageable is called "placidity", to be discussed in Sect. A.9.

A.4.9. In the sequel we will need the following property of $\mathrm{IndCoh}^!$:

Let $\mathcal{Y}$ be a prestack mapping to a *smooth* affine scheme S of finite type. Let $f : S' \to S$ be a map, where $S' \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. Denote

$$\mathcal{Y}' := S' \underset{S}{\times} \mathcal{Y}.$$

By a slight abuse of notation, we will denote by the same symbol f the resulting map $\mathcal{Y}' \to \mathcal{Y}$.

The functor $f^! : \mathrm{IndCoh}^!(\mathcal{Y}) \to \mathrm{IndCoh}^!(\mathcal{Y}')$ extends to a functor

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Y}) \to \mathrm{IndCoh}^!(\mathcal{Y}'). \tag{A.19}$$

We claim:

Lemma A.4.10. *The functor* (A.19) *is fully faithful. If f is smooth, it is an equivalence.*

Proof. Follows from [Ga7, Propositions 4.4.2 and 7.5.7].

□

A.4.11. In the sequel we will need the following assertion about the behavior of $\mathrm{IndCoh}^!(-)$. Let

$$\begin{tikzcd} \mathcal{Y}_1 \arrow[r, "f_Y"] \arrow[d] & \mathcal{Y}_2 \arrow[d] \\ S_1 \arrow[r, "f_S"] & S_2 \end{tikzcd}$$

be a fiber square, where S_i are affine schemes almost of finite type, and f_S is a closed embedding of finite Tor-dimension. Unwinding the definitions, we obtain:

Lemma A.4.12. *Under the above circumstances, the functor $f_Y^! : \mathrm{IndCoh}^!(\mathcal{Y}_2) \to \mathrm{IndCoh}^!(\mathcal{Y}_1)$ admits a left adjoint, to be denoted $(f_Y)_*^{\mathrm{IndCoh}}$. Furthermore, the functor $(f_Y)_*^{\mathrm{IndCoh}}$ satisfies base change for any fiber square*

$$\begin{tikzcd} \mathcal{Y}'_1 \arrow[r, "f'_Y"] \arrow[d] & \mathcal{Y}'_2 \arrow[d] \\ \mathcal{Y}_1 \arrow[r, "f_Y"] & \mathcal{Y}_2. \end{tikzcd}$$

A.4.13. Let S be an affine scheme almost of finite type, and let $S' \subset S$ we a Zariski-closed subset. Let $S^\wedge$ denote the formal completion of S along S'. Let $\mathcal{Y}$ be a prestack over S, and set $\mathcal{Y}^\wedge := S^\wedge \underset{S}{\times} \mathcal{Y}$:

$$\begin{tikzcd} \mathcal{Y}^\wedge \arrow[r, "i_Y"] \arrow[d] & \mathcal{Y} \arrow[d] \\ S^\wedge \arrow[r, "i_S"] & S. \end{tikzcd}$$

We now claim:

Proposition A.4.14. *Under the above circumstances, the functor* $i_Y^! : \mathrm{IndCoh}^!(\mathcal{Y}) \to \mathrm{IndCoh}^!(\mathcal{Y}^\wedge)$ *is a colocalization, i.e., it admits a fully faithful left adjoint. The essential image of this left adjoint is the full subcategory of* $\mathrm{IndCoh}^!(\mathcal{Y})$ *consisting of objects with set-theoretic support on* $\mathcal{Y}' := S' \underset{S}{\times} \mathcal{Y}$*. The formation of the left adjoint satisfies base change for any fiber square*

$$\begin{tikzcd} \widetilde{\mathcal{Y}}_1^\wedge \arrow[r] \arrow[d] & \widetilde{\mathcal{Y}} \arrow[d] \\ \widetilde{\mathcal{Y}} \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

Proof. By [GaRo1, Proposition 6.7.4], we can write $S^\wedge$ as "$\underset{i}{\mathrm{colim}}$" S_i, where S_i are closed subschemes of S, and the maps $S_i \to S$ are of finite Tor-dimension.

Unwinding the definition of $\mathrm{IndCoh}^!(-)$, we reduce the assertion to the case when $\mathcal{Y}$ is an eventually connective affine scheme $\widetilde{S}$, so that

$$\widetilde{S}^\wedge \simeq \text{“}\underset{i}{\mathrm{colim}}\text{”}\, \widetilde{S}_i, \quad \widetilde{S}_i := S_i \underset{S}{\times} \widetilde{S};$$

note that all $\widetilde{S}_i$ are eventually coconnective.

Unwinding further, we reduce the assertion to the case when $\widetilde{S}$ is of finite type; in this case the assertion follows from [GaRo1, Proposition 7.4.5].

□

A.5. The category $\mathrm{IndCoh}^*(-)$.

A.5.1. We define the functor

$$^{\leq n}\mathrm{IndCoh}^* : {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{DGCat},$$

to be

$$^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \overset{(^{\leq n}\mathrm{IndCoh}^!)^{\mathrm{op}}}{\longrightarrow} (\mathrm{DGCat})^{\mathrm{op}} \to \mathrm{DGCat},$$

where:

- The first arrow is the opposite of the functor

$$S \mapsto \mathrm{IndCoh}^!(S), \quad (S_1 \xrightarrow{f} S_2) \rightsquigarrow (\mathrm{IndCoh}^!(S_2) \xrightarrow{f^!} \mathrm{IndCoh}^!(S_1));$$

- The second arrow is

$$\mathbf{D} \mapsto \mathbf{D}^\vee := \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathrm{Vect})$$

(note that in the above formula, the DG category $\mathbf{D}$ is not assumed dualizable).

A.5.2. Explicitly, for $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, we have:

$$^{\leq n}\mathrm{IndCoh}^*(S) = \lim_{S \to S_0, S_0 \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}} \mathrm{IndCoh}(S_0), \tag{A.20}$$

where the transition functors are given by

$$(S \to S_0' \xrightarrow{f} S_0'') \rightsquigarrow \mathrm{IndCoh}(S_0') \overset{f_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}(S_0'').$$

A.5.3. As in Sect. A.4.4, it is easy to see that the collection

$$n \rightsquigarrow {}^{\leq n}\mathrm{IndCoh}^*$$

gives rise to a well-defined functor

$$\mathrm{IndCoh}^* : {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{DGCat}\,. \tag{A.21}$$

A.5.4. Note that, *by construction*, for $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$, the category $\mathrm{IndCoh}^!(S)$ is naturally a pre-dual of $\mathrm{IndCoh}^*(S)$.

This will be a perfect duality if S is placid, see Sect. A.10.2 below.

A.5.5. Let $f : S_1 \to S_2$ be a morphism between eventually coconnective affine schemes. Unwinding the definitions, we obtain that with respect to the identifications

$$\mathrm{IndCoh}^!(S_i)^\vee \simeq \mathrm{IndCoh}^*(S_i), \quad i = 1, 2,$$

we have

$$(f^!)^\vee \simeq f_*^{\mathrm{IndCoh}}.$$

A.5.6. Unlike $\mathrm{IndCoh}^!$, we do not even attempt to define IndCoh^* on all prestacks. Rather, we define it on *ind-affine ind-schemes* (see [GaRo4, Chap. 3.1]).

Namely, we let the functor

$$\mathrm{IndCoh}^* : \mathrm{indSch} \to \mathrm{DGCat}$$

to be the left Kan extension of (A.21) along the embedding

$${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{indSch}^{\text{ind-aff}}\,.$$

A.5.7. Explicitly, for $\mathcal{Y} \in \mathrm{indSch}^{\text{ind-aff}}$, we have

$$\mathrm{IndCoh}^*(\mathcal{Y}) = \underset{S}{\mathrm{colim}}\, \mathrm{IndCoh}^*(S), \tag{A.22}$$

where:

- The index category is that of $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ equipped with a closed embedding $S \to \mathcal{Y}$;
- The transition functors are given by

$$(S' \xrightarrow{f} S'' \to \mathcal{Y}) \rightsquigarrow \mathrm{IndCoh}^*(S') \xrightarrow{f_*^{\mathrm{IndCoh}}} \mathrm{IndCoh}^*(S'').$$

A.5.8. Thus, by definition, for a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in $\mathrm{indSch}^{\text{ind-aff}}$, we have a well-defined functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathcal{Y}_1) \to \mathrm{IndCoh}^*(\mathcal{Y}_2).$$

In particular, taking the projection $\mathcal{Y} \to \mathrm{pt}$, we obtain that for any $\mathcal{Y} \in \mathrm{indSch}^{\text{ind-aff}}$ there is a well-defined functor

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, -) : \mathrm{IndCoh}^*(\mathcal{Y}) \to \mathrm{Vect}\,.$$

A.5.9. It follows from [GaRo1, Sect. 2.4.2] that if $\mathcal{Y} \in \mathrm{indSch}^{\text{ind-aff}}_{\mathrm{laft}}$, the naturally defined functor

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}^*(\mathcal{Y})$$

is an equivalence.

A.6. **The multiplicative structure.**

A.6.1. Note that since the index category in (A.17) is filtered (and, in particular, sifted), for $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, the category $\mathrm{IndCoh}^!(S)$ carries a naturally defined monoidal structure.

Explicitly, the corresponding binary operation is given by

$$\mathrm{IndCoh}^!(S) \otimes \mathrm{IndCoh}^!(S) \to \mathrm{IndCoh}^!(S \times S) \overset{\Delta_S^!}{\to} \mathrm{IndCoh}^!(S).$$

In other words, the functor

$$\mathrm{IndCoh}^! : ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}$$

lifts to a functor

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{ComAlg}(\mathrm{DGCat}) = \mathrm{DGCat}^{\mathrm{SymMon}}.$$

A.6.2. By construction, we obtain that the functor

$$\mathrm{IndCoh}^! : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}$$

also lifts to a functor

$$(\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{ComAlg}(\mathrm{DGCat}) = \mathrm{DGCat}^{\mathrm{SymMon}},$$

i.e., for every $\mathcal{Y} \in \mathrm{PreStk}$, the category $\mathrm{IndCoh}^!(\mathcal{Y})$ has a naturally defined symmetric monoidal structure.

Namely, the corresponding binary operation is given by

$$\mathrm{IndCoh}^!(\mathcal{Y}) \otimes \mathrm{IndCoh}^!(\mathcal{Y}) \to \mathrm{IndCoh}^!(\mathcal{Y} \times \mathcal{Y}) \overset{\Delta_{\mathcal{Y}}^!}{\to} \mathrm{IndCoh}^!(\mathcal{Y}).$$

The unit for this symmetric monoidal structure is the object $\omega_{\mathcal{Y}}$.

A.6.3. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be a pair of prestacks. The operation of pullback and tensor product gives rise to a functor

$$\mathrm{IndCoh}^!(\mathcal{Y}_1) \otimes \mathrm{IndCoh}^!(\mathcal{Y}_2) \to \mathrm{IndCoh}^!(\mathcal{Y}_1 \times \mathcal{Y}_2). \tag{A.23}$$

For general prestacks there is no reason for (A.23) to be an equivalence.

A.6.4. Let S_1 and S_2 be a pair of eventually coconnective schemes. Given an eventually coconnective scheme S of finite type and a map $S_1 \times S_2 \to S$, the category of factorizations of this map as

$$S_1 \times S_2 \to S_{1,0} \times S_{2,0} \to S$$

is contractible, where:

- $S_{i,0}$ are eventually coconnective;
- The first arrow comes from a pair of maps $S_i \to S_{i,0}$.

This implies that we have a well-defined functor

$$\mathrm{IndCoh}^*(S_1) \otimes \mathrm{IndCoh}^*(S_2) \to \mathrm{IndCoh}(S).$$

Passing to the limit over S, we obtain a functor

$$\mathrm{IndCoh}^*(S_1) \otimes \mathrm{IndCoh}^*(S_2) \to \mathrm{IndCoh}^*(S_1 \times S_2). \tag{A.24}$$

Ind-extending, we obtain a functor

$$\mathrm{IndCoh}^*(\mathcal{Y}_1) \otimes \mathrm{IndCoh}^*(\mathcal{Y}_2) \to \mathrm{IndCoh}^*(\mathcal{Y}_1 \times \mathcal{Y}_2), \tag{A.25}$$

where $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are ind-schemes.

For general ind-schemes there is no reason for (A.25) to be an equivalence.

A.6.5. By a similar principle, we obtain a symmetric monoidal functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}^!(\mathcal{Y}).$$

A.6.6. Note also that by the definition of IndCoh^*, for $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ we have a naturally defined action of $\mathrm{IndCoh}^!(S)$ on $\mathrm{IndCoh}^*(S)$.

For a map $f : S_1 \to S_2$, this action satisfies the projection formula

$$\mathcal{F}_2 \overset{!}{\otimes} f_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \simeq f_*^{\mathrm{IndCoh}}(f^!(\mathcal{F}_2) \overset{!}{\otimes} \mathcal{F}_1), \quad \mathcal{F}_1 \in \mathrm{IndCoh}^*(S_1),\ \mathcal{F}_2 \in \mathrm{IndCoh}^!(S_2).$$

This implies that for $\mathcal{Y} \in \mathrm{indSch}^{\text{ind-aff}}_{\mathrm{laft}}$, we also have a natural action of $\mathrm{IndCoh}^!(\mathcal{Y})$ on $\mathrm{IndCoh}^*(\mathcal{Y})$, and the projection formula holds.

A.6.7. For an ind-scheme $\mathcal{Y}$ we have a canonically defined pairing:

$$\mathrm{IndCoh}^*(\mathcal{Y}) \otimes \mathrm{IndCoh}^!(\mathcal{Y}) \overset{\text{action}}{\to} \mathrm{IndCoh}^*(\mathcal{Y}) \overset{\Gamma^{\mathrm{IndCoh}}(\mathcal{Y},-)}{\longrightarrow} \mathrm{Vect}\,. \tag{A.26}$$

Note, however, that unlike the case of schemes, we do not claim that the above pairing realizes $\mathrm{IndCoh}^!(\mathcal{Y})$ as the predual of $\mathrm{IndCoh}^*(\mathcal{Y})$. (It will, however, be a perfect duality, under the placidity assumption.)

A.7. **Further properties of** IndCoh^*.

A.7.1. *The functor* Ψ. Let S be an eventually coconnective scheme. The presentation in (A.20) shows that we have a canonically defined functor

$$\Psi_S : \mathrm{IndCoh}^*(S) \to \mathrm{QCoh}(S).$$

Indeed, it is given by the compatible family of functors

$$\Psi_{S_0} : \mathrm{IndCoh}(S_0) \to \mathrm{QCoh}(S_0),$$

where we use the fact that for any affine scheme S, written as a limit of other affine schemes

$$S \simeq \lim_{\alpha} S_\alpha,$$

the functor

$$\mathrm{QCoh}(S) \to \lim_{\alpha} \mathrm{QCoh}(S_\alpha),$$

given by taking direct images along $S \to S_\alpha$, is an equivalence.

For a map $f : S_1 \to S_2$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}^*(S_1) \arrow[r, "\Psi_{S_1}"] \arrow[d, "f_*^{\mathrm{IndCoh}}"'] & \mathrm{QCoh}(S_1) \arrow[d, "f_*"] \\
\mathrm{IndCoh}^*(S_2) \arrow[r, "\Psi_{S_2}"] & \mathrm{QCoh}(S_2).
\end{tikzcd}$$

A.7.2. Unwinding, we obtain that with respect to the identification

$$\mathrm{IndCoh}^*(S) \simeq \mathrm{IndCoh}^!(S)^\vee$$

of Sect. A.5.4 and the canonical self-duality on $\mathrm{QCoh}(S)$, we have

$$\Psi_S \simeq (\Upsilon_S)^\vee$$

A.7.3. For an ind-affine ind-scheme $\mathcal{Y}$, we have the functor

$$\Psi_{\mathcal{Y}} : \mathrm{IndCoh}^*(\mathcal{Y}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$$

defined in terms of the presentation (A.22) by

$$\mathrm{IndCoh}^*(\mathcal{Y}) \simeq \underset{S\to\mathcal{Y}}{\mathrm{colim}}\, \mathrm{IndCoh}^*(S) \overset{\{\Psi_S\}}{\longrightarrow} \underset{S\to\mathcal{Y}}{\mathrm{colim}}\, \mathrm{QCoh}(S) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}),$$

where the colimits are taken over the index category of eventually coconnective affine schemes equipped with a closed embedding into $\mathcal{Y}$.

Note that the functors $\Psi_{\mathcal{Y}}$ and $\Upsilon_{\mathcal{Y}}$ are mutually dual in the sense that the following diagram commutes

$$\begin{tikzcd}
\mathrm{IndCoh}^*(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \arrow[r, "\mathrm{Id}\otimes\Upsilon_{\mathcal{Y}}"] \arrow[d, "\Psi_{\mathcal{Y}}\otimes\mathrm{Id}"'] & \mathrm{IndCoh}^*(\mathcal{Y}) \otimes \mathrm{IndCoh}^!(\mathcal{Y}) \arrow[d, "\text{(A.26)}"] \\
\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \arrow[r, "\text{(A.6)}"] & \mathrm{Vect}\,.
\end{tikzcd}$$

A.7.4. Let $f : S_1 \to S_2$ be a map between eventually connective affine schemes. Assume that S_1 is finitely presented over S_2 (inside the category of n-coconnective schemes for some n) and that f is of finite Tor-dimension.

We claim that in this case the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(S_1) \to \mathrm{IndCoh}^*(S_2)$$

admits a left adjoint, to be denoted $f^{*,\mathrm{IndCoh}}$.

By Noetherian approximation (see [Lu3, Sect. 4.4.1]), the assumption on f implies that we can write S_2 as $\lim\limits_{\alpha} S_{2,\alpha}$, where $S_{2,\alpha} \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, so that there exists a compatible family of Cartesian diagrams

$$\begin{tikzcd}
S_1 \arrow[r, "f"] \arrow[d] & S_2 \arrow[d] \\
S_{1,\alpha} \arrow[r, "f_\alpha"] & S_{2,\alpha}
\end{tikzcd}$$

with $S_{1,\alpha} \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ and $S_1 \simeq \lim\limits_{\alpha} S_{1,\alpha}$.

In this case we have

$$\mathrm{IndCoh}^*(S_2) \simeq \lim_{\alpha} \mathrm{IndCoh}(S_{2,\alpha}) \text{ and } \mathrm{IndCoh}^*(S_1) \simeq \lim_{\alpha} \mathrm{IndCoh}(S_{1,\alpha}),$$

and the functor $f^{*,\mathrm{IndCoh}}$ is given by the compatible family of functors

$$\mathrm{IndCoh}(S_{2,\alpha}) \overset{f_\alpha^{*,\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}(S_{1,\alpha}),$$

which exists thanks to [Ga7, Lemma 3.5.8].

A.7.5. Let

$$\begin{tikzcd}
S_1' \arrow[r, "f'"] \arrow[d, "g_1"'] & S_2' \arrow[d, "g_2"] \\
S_1 \arrow[r, "f"] & S_2
\end{tikzcd}$$

be a Cartesian diagram of eventually connective affine schemes, where the horizontal arrows are of finite presentation and finite Tor-dimension.

We have the tautological isomorphism

$$(g_2)_*^{\mathrm{IndCoh}} \circ f'^{\mathrm{IndCoh}}_* \simeq f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}},$$

from which we obtain a natural transformation

$$f^{*,\mathrm{IndCoh}} \circ (g_2)_*^{\mathrm{IndCoh}} \to (g_1)_*^{\mathrm{IndCoh}} \circ f'^{*,\mathrm{IndCoh}}.$$

However, it is easy to see that the above natural transformation is an isomorphism, see [Ga7, Lemma 3.6.9].

A.7.6. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a a map between ind-affine ind-schemes, and assume that f is (i) affine, (ii) of finite presentation, (iii) of finite Tor-dimension (i.e., the above properties hold after base change of f by an affine scheme).

We claim that in this case the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathcal{Y}_1) \to \mathrm{IndCoh}^*(\mathcal{Y}_2)$$

admits a left adjoint (to be denoted $f^{*,\mathrm{IndCoh}}$).

Indeed, write

$$\mathrm{IndCoh}^*(\mathcal{Y}_2) \simeq \underset{S_{2,\alpha}\to\mathcal{Y}_2}{\mathrm{colim}} \mathrm{IndCoh}^*(S_{2,\alpha}),$$

where the index category is that of $S_{2,\alpha} \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ equipped with a closed embedding $S_{2,\alpha} \to \mathcal{Y}_2$.

For $S_{2,\alpha}$ as above, set

$$S_{1,\alpha} := \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} S_{2,\alpha}.$$

Then the family

$$S_{1,\alpha} \to \mathcal{Y}_1$$

is cofinal in the category of eventually coconnective affine schemes mapping to $\mathcal{Y}_1$, and hence we have

$$\mathrm{IndCoh}^*(\mathcal{Y}_1) \simeq \underset{S_{1,\alpha}\to\mathcal{Y}_1}{\mathrm{colim}} \mathrm{IndCoh}^*(S_1).$$

In terms of this presentation, the functor $f^{*,\mathrm{IndCoh}}$ is given by the (compatible) family of functors

$$f_\alpha^{*,\mathrm{IndCoh}} : \mathrm{IndCoh}^*(S_{2,\alpha}) \to \mathrm{IndCoh}^*(S_{1,\alpha}),$$

see Sect. A.7.4.

A.7.7. The following is a counterpart of Sect. A.4.9 for IndCoh^*. Let us be in the situation of *loc. cit.*, but let us assume that $\mathcal{Y}$ is an ind-scheme.

Since f is finite Tor-dimension, we can consider the functor $f^{*,\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathcal{Y}) \to \mathrm{IndCoh}^*(\mathcal{Y}')$, and it extends to a functor

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{IndCoh}^*(\mathcal{Y}) \to \mathrm{IndCoh}^*(\mathcal{Y}'). \tag{A.27}$$

We have:

Lemma A.7.8. *The functor* (A.27) *is fully faithful. If f is smooth, it is an equivalence.*

A.8. The t-structure on IndCoh^*.

A.8.1. Let S be an eventually coconnective affine scheme. The presentation in (A.20) endows the category $\mathrm{IndCoh}^*(S)$ with a t-structure. It is uniquely characterized by the property that for a map

$$f : S \to S_0, \quad S_0 \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$$

the functor

$$f_* : \mathrm{IndCoh}^*(S) \to \mathrm{IndCoh}(S_0)$$

is t-exact.

This t-structure is compatible with filtered colimits, by construction.

A.8.2. For a map $S_1 \to S_2$ between eventually coconnective affine schemes, the corresponding functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(S_1) \to \mathrm{IndCoh}^*(S_2)$$

is t-exact.

A.8.3. By construction, the functor Ψ_S is t-exact and induces an equivalence

$$\mathrm{IndCoh}^*(S)^{>-\infty} \xrightarrow{\sim} \mathrm{QCoh}(S)^{>-\infty}.$$

A.8.4. Let $\mathcal{Y}$ be an ind-affine ind-scheme. We use the presentation (A.22) and the construction in Sect. A.2.3 to equip $\mathrm{IndCoh}^*(\mathcal{Y})$ with a t-structure:

By definition, $\mathrm{IndCoh}^*(\mathcal{Y})^{\leq 0}$ is generated under colimits by the essential images of $\mathrm{IndCoh}^*(S)^{\leq 0}$ for $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ equipped with a closed embedding $S \to \mathcal{Y}$.

A.8.5. From Lemma A.2.5, we obtain:

Corollary A.8.6.

(a) *The t-structure on* $\mathrm{IndCoh}^*(\mathcal{Y})$ *is compatible with filtered colimits.*

(b) *For every* $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ *equipped with a closed embedding* $S \to \mathcal{Y}$*, the direct image functor* $\mathrm{IndCoh}^*(S) \to \mathrm{IndCoh}^*(\mathcal{Y})$ *is t-exact.*

A.8.7. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between ind-affine ind-schemes. As in Corollary A.2.12, we have:

Corollary A.8.8. *The functor*

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathcal{Y}_1) \to \mathrm{IndCoh}^*(\mathcal{Y}_2)$$

is t-exact.

A.8.9. Recall the functor

$$\Psi_{\mathcal{Y}} : \mathrm{IndCoh}^*(\mathcal{Y}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}),$$

see Sect. A.7.3. We claim:

Lemma A.8.10. *The functor* $\Psi_{\mathcal{Y}}$ *is t-exact and induces an equivalence:*

$$\mathrm{IndCoh}^*(\mathcal{Y})^{>-\infty} \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{>-\infty}.$$

Proof. The fact that Ψ is t-exact follows from from Corollaries A.2.10(b) and A.2.7.

To prove the equivalence statement, it suffices to show that for any n, the corresponding functor

$$\mathrm{IndCoh}^*(\mathcal{Y})^{\geq 0, \leq n} \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0, \leq n}$$

is an equivalence.

As in the proof of Lemma A.2.5, we can write

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0, \leq n} \simeq \underset{S \to \mathcal{Y}}{\mathrm{colim}}\, \mathrm{QCoh}(S)^{\geq 0, \leq n}$$

and

$$\mathrm{IndCoh}^*(\mathcal{Y})^{\geq 0, \leq n} \simeq \underset{S' \to \mathcal{Y}}{\mathrm{colim}}\, \mathrm{IndCoh}^*(S')^{\geq 0, \leq n},$$

where:

- The index S runs over the category affine schemes equipped with a closed embedding into $\mathcal{Y}$;
- The index S' runs over the category of eventually coconnective affine schemes equipped with a closed embedding into $\mathcal{Y}$.
- Both colimits are taken in the ∞-category of categories closed under filtered colimits and functors that preserve filtered colimits.

Now, the assertion follows from the fact that that for $m \geq n$, the direct image functor

$$\mathrm{QCoh}({}^{\leq m}S) \to \mathrm{QCoh}(S)$$

induces an equivalence

$$\mathrm{QCoh}({}^{\leq m}S)^{\geq 0, \leq n} \to \mathrm{QCoh}(S)^{\geq 0, \leq n}.$$

□

A.9. **Placidity.**

A.9.1. An affine scheme S is said to be *placid* if it can be written as a limit

$$S \simeq \lim_{\alpha \in A} S_\alpha, \tag{A.28}$$

where:

- $S_\alpha \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$;
- The transition maps $f_{\beta,\alpha} : S_\beta \to S_\alpha$ are flat.
- The category A of indices is co-filtered (i.e., the opposite category is filtered).

Remark A.9.2. There are in fact two variants of the definition of placidity. The less restricted one is what we just gave above. In the more restrictive one, one requires that the maps $f_{\beta,\alpha}$ be smooth.

The flatness condition is sufficient for our purposes, which are to ensure the compact generation of the categories IndCoh^* and $\mathrm{IndCoh}^!$ (see Sect. A.10.1). One needs smoothness when one works with D-modules.

That said, in most examples of placid schemes that we will encounter, the smoothness condition is satisfied as well.

A.9.3. Note that if S is placid, then so is any of its truncations: for a presentation (A.28), we have

$$^{\leq n}S \simeq \lim_{\alpha} {}^{\leq n}S_\alpha,$$

and the truncated maps $^{\leq n}S_\beta \to {}^{\leq n}S_\alpha$ are also flat; in fact the flatness of $f_{\beta,\alpha}$ implies that

$$^{\leq n}S_\beta \simeq S_\beta \underset{S_\alpha}{\times} {}^{\leq n}S_\alpha.$$

A.9.4. Fix an integer n. Let $R \to R'$ be a map in $\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0, \geq -n})$. Recall that R' is said to be *finitely presented* as an R-algebra if R' is compact as an object of $\mathrm{ComAlg}((R\text{-mod})^{\leq 0, \geq -n})$.

Let $R \to R'$ be a map in $\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$. We shall say that R' *almost finitely presented* as an R-algebra if for every n, the map $\tau^{\geq -n}(R) \to \tau^{\geq -n}(R')$ realizes $\tau^{\geq -n}(R')$ as a finitely presented $\tau^{\geq -n}(R)$-algebra.

We shall say that a morphism of n-coconnective affine schemes (resp., affine schemes) $\mathrm{Spec}(R') = S' \to S = \mathrm{Spec}(R)$ is of finite presentation (resp., almost of finite presentation) if R' is finitely presented (resp., almost finitely presented) as an R-algbera.

A.9.5. Let $\mathcal{Y}$ be an ind-affine ind-scheme mapping to an affine scheme S. We shall say that $\mathcal{Y}$ is locally almost of finite presentation over S if for every n, the truncation

$$^{\leq n}\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$$

can be exhibited as a filtered colimit

$$^{\leq n}\mathcal{Y} \simeq \underset{i \in I}{\mathrm{colim}}\, S_i, \quad S_i \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}},$$

such that the maps $S_i \to S$ are of finite presentation.

Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between ind-affine ind-schemes. We shall say that f is locally almost of finite presentation if the base change of f by any affine scheme S yields an ind-affine ind-scheme locally almost of finite presentation over S.

A.9.6. We have the following hereditary property of placidity:

Lemma A.9.7. *Let $S' \to S$ be a map almost of finite presentation between affine schemes. Suppose that every coconnective truncation of S is placid. Then the same is true for S'.*

Proof. Fix n and consider the corresponding map ${}^{\leq n}S' \to {}^{\leq n}S$. Write ${}^{\leq n}S$ as

$$\lim_{\alpha \in A} S_\alpha, \quad S_\alpha \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$$

as in (A.28).

Then for some index α, we have an affine scheme $S'_\alpha \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and a Cartesian square

$$\begin{tikzcd} {}^{\leq n}S' \arrow[r] \arrow[d] & {}^{\leq n}S \arrow[d] \\ S'_\alpha \arrow[r] & S_\alpha. \end{tikzcd}$$

Consider the category $A_{/\alpha}$. Since A is cofiltered, the category $A_{/\alpha}$ is also cofiltered and the opposite of the inclusion functor $A_{/\alpha} \to A$ is cofinal.

For any $\beta \in A_{/\alpha}$ denote

$$S'_\beta := S'_\alpha \underset{S_\alpha}{\times} S_\beta.$$

Then we have

$${}^{\leq n}S' \simeq \lim_{\beta \in A_{/\alpha}} S'_\beta,$$

and the maps $S'_{\beta_1} \to S'_{\beta_2}$ are flat.

□

A.9.8. Let $\mathcal{Y}$ be an ind-affine ind-scheme. We shall say that $\mathcal{Y}$ is *ind-placid* if for every n, the truncation

$${}^{\leq n}\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$$

can be exhibited as a filtered colimit

$$ {}^{\leq n}\mathcal{Y} \simeq \operatorname*{colim}_{i \in I} S_i, \quad S_i \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}, \tag{A.29}$$

where:

- The affine schemes S_i are placid;
- The transition maps $S_i \to S_j$ are of finite presentation.

A.9.9. From Lemma A.9.7 we obtain:

Corollary A.9.10. *Let $\mathcal{Y}$ be an ind-placid ind-scheme and let $f : \mathcal{Y}' \to \mathcal{Y}$ be a map locally almost of finite presentation. Then $\mathcal{Y}'$ is also an ind-placid ind-scheme.*

Proof. Fix an integer n and a presentation of ${}^{\leq n}\mathcal{Y}$ as in (A.29). For every index i, set

$$\mathcal{Y}'_i := {}^{\leq n}(\mathcal{Y}' \underset{\mathcal{Y}}{\times} S_i).$$

By assumption, this is an ind-affine ind-scheme of ind-finite presentation over S_i. Consider the category

$$\mathsf{F}_i := \{S'_i \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}},\ S'_i \overset{\text{closed emb}}{\hookrightarrow} \mathcal{Y}'_i\}.$$

By assumption, this subcategory contains a full cofinal subcategory, denoted F'_i consisting of those objects for which the map $S'_i \to S_i$ is of finite presentation.

The assignment $i \mapsto \mathsf{F}_i$ extends to a co-Cartesian fibration

$$\mathsf{F} \to I.$$

The assumption that the maps $S_i \to S_j$ are of finite presentation implies that the assignment $i \mapsto \mathsf{F}'_i$ corresponds to a full cofinal subcategory $\mathsf{F}' \subset \mathsf{F}$, and

$$\mathsf{F}' \to I$$

is also a co-Cartesian fibration. By cofinality, we obtain that the map

$$\operatorname*{colim}_{(i, S'_i) \in \mathsf{F}} S'_i \to \mathcal{Y}' \tag{A.30}$$

is an isomorphism in ${}^{\leq n}\mathrm{PreStk}$. Moreover, for a map $(i, S'_i) \to (j, S'_j)$ in F', the corresponding map $S'_i \to S'_j$ is of finite presentation (because its composition with $S'_j \to S_j$ is).

Since I is filtered and each F'_i is filtered, we obtain that F' is filtered.

Finally, by Lemma A.9.7, the affine schemes S'_i are placid. Hence, (A.30) gives the desired presentation of $\mathcal{Y}'$.

$\square$

A.10. **The categories** $\mathrm{IndCoh}^!(-)$ **and** $\mathrm{IndCoh}^*(-)$ **in the (ind)-placid case.**

A.10.1. Let $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ be placid. We claim that in this case, the category $\mathrm{IndCoh}^!(S)$ is compactly generated.

Indeed, write S as in (A.28). Since the maps $f_{\beta,\alpha}$ are flat, the functors $f^!_{\beta,\alpha}$ preserves coherence, and hence compactness. We obtain

$$\mathrm{IndCoh}^!(S) \simeq \underset{\alpha}{\mathrm{colim}}\, \mathrm{IndCoh}(S_\alpha),$$

where the terms are compactly generated, and the transition functors preserve compactness.

Hence, the images of $\mathrm{Coh}(S_\alpha) \subset \mathrm{IndCoh}(S_\alpha)$ under the !-pullback functors

$$\mathrm{IndCoh}(S_\alpha) \to \mathrm{IndCoh}^!(S)$$

provide a set of compact generators of $\mathrm{IndCoh}^!(S)$.

A.10.2. From the identification

$$\mathrm{IndCoh}^*(S) \simeq \mathrm{IndCoh}^!(S)^\vee \tag{A.31}$$

of Sect. A.5.4, we obtain that $\mathrm{IndCoh}^*(S)$ is also compactly generated, and (A.31) is a duality in DGCat.

A.10.3. Explicitly, the presentation

$$\mathrm{IndCoh}^*(S) \simeq \underset{\alpha}{\lim}\, \mathrm{IndCoh}(S_\alpha)$$

(with respect to the *-pushforward functors) implies that

$$\mathrm{IndCoh}^*(S) \simeq \underset{\alpha}{\mathrm{colim}}\, \mathrm{IndCoh}(S_\alpha),$$

with respect to the *-pullback functors (which are well-defined, due to the flatness assumption).

Thus, the compact generators of $\mathrm{IndCoh}^*(S)$ are the images of $\mathrm{Coh}(S_\alpha) \subset \mathrm{IndCoh}(S_\alpha)$ along the *-pullback functors

$$\mathrm{IndCoh}(S_\alpha) \to \mathrm{IndCoh}^*(S).$$

A.10.4. Let $f : S_1 \to S_2$ be a morphism almost of finite presentation between placid eventually coconnective affine schemes. Assume now that f is a closed embedding. We claim that in this case the functor

$$f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}^*(S_1) \to \mathrm{IndCoh}^*(S_2)$$

admits a continuous right adjoint (to be denoted $f^!$).

Indeed, we claim that f^{IndCoh}_* preserves compactness. This follows from the manipulation in the proof of Lemma A.9.7 using the fact that for a Cartesian diagram of affine schemes almost of finite type

$$\begin{CD} S'_1 @>{f'}>> S'_2 \\ @V{g_1}VV @VV{g_2}V \\ S''_1 @>{f''}>> S''_2 \end{CD}$$

with the maps g_1, g_2 flat, the natural transformation

$$g_2^{\mathrm{IndCoh},*} \circ f''^{\mathrm{IndCoh}}_* \to f'^{\mathrm{IndCoh}}_* \circ g_1^{\mathrm{IndCoh},*}$$

is an isomorphism.

A.10.5. Moreover, in the above situation, for a Cartesian diagram

$$\begin{tikzcd} \widetilde{S}_1 \arrow[r, "\widetilde{f}"] \arrow[d, "g_1"'] & \widetilde{S}_2 \arrow[d, "g_2"] \\ S_1 \arrow[r, "f"] & S_2, \end{tikzcd}$$

where:

- All affine schemes involved are eventually coconnective and placid;
- The maps f and $\widetilde{f}$ are closed embeddings of finite presentation,

the natural transformation

$$(g_1)_*^{\mathrm{IndCoh}} \circ \widetilde{f}^! \to f^! \circ (g_2)_*^{\mathrm{IndCoh}}, \quad \mathrm{IndCoh}^*(\widetilde{S}_2) \rightrightarrows \mathrm{IndCoh}^*(S_1),$$

obtained by adjunction from

$$f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}} \to (g_2)_*^{\mathrm{IndCoh}} \circ \widetilde{f}_*^{\mathrm{IndCoh}},$$

is an isomorphism.

A.10.6. Recall (see Sect. A.5.5) that that with respect to the dualities

$$\mathrm{IndCoh}^*(S_i) \simeq \mathrm{IndCoh}^!(S_i)^\vee, \quad i = 1, 2,$$

we have

$$f_*^{\mathrm{IndCoh}} \simeq (f^!)^\vee.$$

Hence, the existence of a continuous right adjoint of

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(S_1) \to \mathrm{IndCoh}^*(S_2)$$

implies the existence of a *left* adjoint of the functor

$$f^! : \mathrm{IndCoh}^!(S_2) \to \mathrm{IndCoh}^!(S_1).$$

We will denote this left adjoint by

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^!(S_1) \to \mathrm{IndCoh}^!(S_2).$$

A.10.7. Let now $\mathcal{Y}$ be an ind-placid ind-scheme. We claim that in this case the category $\mathrm{IndCoh}^*(\mathcal{Y})$ is compactly generated.

Indeed, writing

$$\mathrm{IndCoh}^*(\mathcal{Y}) \simeq \underset{i}{\mathrm{colim}}\, \mathrm{IndCoh}^*(S_i),$$

where S_i are eventually coconnective placid affine schemes and the transition maps $S_i \overset{f_{i,j}}{\to} S_j$ are closed embeddings almost of finite presentation, we obtain that the transition functors

$$\mathrm{IndCoh}^*(S_i) \to \mathrm{IndCoh}^*(S_j)$$

preserve compactness (see Sect. A.10.4 above).

Similarly, the category $\mathrm{IndCoh}^!(\mathcal{Y})$ can be written as

$$\underset{i}{\mathrm{colim}}\, \mathrm{IndCoh}^!(S_i),$$

where the transition functors are $(f_{i,j})_*^{\mathrm{IndCoh}}$. This implies that $\mathrm{IndCoh}^!(\mathcal{Y})$ is also compactly generated by the essential images of $\mathrm{IndCoh}^!(S_i)^c$.

A.10.8. Note also that it follows that if $\mathcal{Y}$ is placid, the pairing (A.26) is a perfect duality.

A.10.9. By a similar logic we obtain:

Lemma A.10.10. *Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be a pair of ind-placid ind-schemes. Then the functors* (A.23) *and* (A.25) *are equivalences.*

A.10.11. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between ind-placid ind-affine ind-schemes that is locally almost of finite presentation. Assume that f is an ind-closed embedding (i.e., for every closed embedding $S \to \mathcal{Y}_1$, the composite map $S \to \mathcal{Y}_1 \to \mathcal{Y}_2$ is also a closed embedding).

Similarly to the above, we obtain that in this case, the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathcal{Y}_1) \to \mathrm{IndCoh}^*(\mathcal{Y}_2)$$

preserves compactness, and hence admits a continuous right adjoint, to be denoted $f^!$.

By duality, the functor

$$f^! : \mathrm{IndCoh}^!(\mathcal{Y}_2) \to \mathrm{IndCoh}^!(\mathcal{Y}_1)$$

admits a left adjoint, to be denoted

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^!(\mathcal{Y}_1) \to \mathrm{IndCoh}^!(\mathcal{Y}_2).$$

A.10.12. Let

$$\begin{array}{ccc} \widetilde{\mathcal{Y}}_1 & \xrightarrow{\widetilde{f}} & \widetilde{\mathcal{Y}}_2 \\ {\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle g_2} \\ \mathcal{Y}_1 & \xrightarrow{f} & \mathcal{Y}_2 \end{array}$$

be a Cartesian diagram, where:

- All objects are involved are ind-placid affine ind-schemes;
- The maps f and $\widetilde{f}$ are ind-closed embeddings of finite presentation.

Unwinding, it follows from Sect. A.10.5 that in this case, the natural transformation

$$(g_1^{\mathrm{IndCoh}})_* \circ \widetilde{f}^! \to f^! \circ (g_2^{\mathrm{IndCoh}})_*, \quad \mathrm{IndCoh}^*(\widetilde{\mathcal{Y}}_2) \rightrightarrows \mathrm{IndCoh}^*(\mathcal{Y}_1), \quad \mathrm{IndCoh}^*(\widetilde{\mathcal{Y}}_2) \rightrightarrows \mathrm{IndCoh}^*(\mathcal{Y}_1),$$

obtained by adjunction from

$$f_*^{\mathrm{IndCoh}} \circ (g_1^{\mathrm{IndCoh}})_* \to (g_2^{\mathrm{IndCoh}})_* \circ \widetilde{f}_*^{\mathrm{IndCoh}},$$

is an isomorphism.

By duality, the natural transformation

$$\widetilde{f}_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}, \quad \mathrm{IndCoh}^*(\mathcal{Y}_1) \rightrightarrows \mathrm{IndCoh}^*(\widetilde{\mathcal{Y}}_2),$$

obtained by adjunction from

$$g_1^! \circ f^! \simeq \widetilde{f}^! \circ g_2^!,$$

is also an isomorphism.

A.10.13. Let now $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between ind-placid ind-schemes. Assume that f is affine and of finite Tor-dimension (but we are not assuming that f be of finite presentation).

We claim, generalizing Sect. A.7.4, that in this case the functor

$$f^{*,\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathcal{Y}_2) \to \mathrm{IndCoh}^*(\mathcal{Y}_1),$$

left adjoint to f_*^{IndCoh}, exists, and satisfies base change against functors g_*^{IndCoh} for $g : \mathcal{Y}'_1 \to \mathcal{Y}_1$, where g is another ind-placid ind-scheme.

Indeed, by Sect. A.10.11, the question reduces to the case when $\mathcal{Y}_2$ is an eventually coconnective affine scheme, to be denoted S_2. In this case, $\mathcal{Y}_1$ is also an eventually coconnective affine scheme, to be denoted S_1.

Furthermore, in this case we can further reduce to the case when S_2 is of finite type. Since S_1 is placid, we can factor the morphism f as

$$S_1 \overset{h}{\to} S_{1,0} \overset{f_0}{\to} S_2,$$

where:

- $S_{1,0}$ is an eventually coconnective scheme of finite type,
- The morphism h is flat;

- The morphism f_0 is of finite Tor-dimension.

This reduces the assertion to the case of eventually coconnective affine schemes of finite type, where it follows from [Ga7, Lemma 3.5.8].

Appendix B. Factorization patterns

The local Langlands theory considers various representation-theoretic categories $\mathbf{A}_x$ attached to the group G (or its dual $\check{G}$) and the formal disc $\mathcal{D}_x$, attached to a point $x \in X$. More generally, one is led to consider the multi-disc $\mathcal{D}_{\underline{x}}$, $\underline{x} = x_1, ..., x_n$; moreover the points $x_1, ..., x_n$ are allowed to move in families over X and collide. In this case, we shall say that $\underline{x}$ is a (scheme-theoretic) point of the *Ran* space of X.

The datum of a *factorization category* attached to such $\underline{x}$ a category $\mathbf{A}_{\underline{x}}$, such that if $\underline{x}_1$ and $\underline{x}_2$ are disjoint, we are given an isomorphism

$$\mathbf{A}_{\underline{x}_1 \sqcup \underline{x}_2} \simeq \mathbf{A}_{\underline{x}_1} \otimes \mathbf{A}_{\underline{x}_2}.$$

We develop the theory of factorization categories in this section, along with various adjoining notions (factorization spaces, factorization algebras). Can can view this section as a natural development of the theory of chiral algebras, initiated in [BD2]. The main difference with *loc. cit.* is that all our constructions take place in the world of ∞-categories, whereas in [BD2] one mainly worked at the abelian level.

In order to produce examples of factorization categories one often uses geometric objects associated to the formal (resp., formal punctured) disc, such as arcs and loop spaces. Some of the work in this section is devoted to the study of the relevant geometries.

B.1. **Factorization spaces.**

B.1.1. The Ran space of X, denoted Ran, is the prestack that assigns to an affine test scheme S the set of finite-non-empty subsets of $\mathrm{Hom}(S_{\mathrm{red}}, X)$.

Note that, by definition, the map $\mathrm{Ran} \to \mathrm{Ran}_{\mathrm{dR}}$ is an isomorphism.

We denote k-points of Ran by $\underline{x}$. By definition, these are finite non-empty collections

$$\underline{x} = \{x_1, ..., x_n\} \tag{B.1}$$

of k-points of X.

B.1.2. In what follows we will use the following notations. Let $\underline{x} : S \to \mathrm{Ran}$ be a map corresponding to $I \subset \mathrm{Hom}(S_{\mathrm{red}}, X)$.

- For $i \in I$, we will denote by x_i the corresponding map $S_{\mathrm{red}} \to X$;
- We will denote by $\mathrm{Graph}_{x_i} \subset S \times X$ the graph of x_i (viewed as a closed *subset*, i.e., we ignore its scheme-theoretic structure);
- We will denote by $\mathrm{Graph}_{\underline{x}}$ the Zariski-closed subset of $S \times X$ equal to $\underset{i}{\cup} \mathrm{Graph}_{x_i}$.

B.1.3. One can exhibit Ran explicitly as a colimit of de Rham spaces of schemes. Namely,

$$\mathrm{Ran} \simeq \underset{I \in (\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}}}{\mathrm{colim}} X^I_{\mathrm{dR}},$$

where $\mathrm{fSet}^{\mathrm{surj}}$ is the category of non-empty finite sets and surjective maps (see [Ro2, Sect. 2] for a detailed discussion).

B.1.4. The presentation (B.1.3) implies, in particular, that X_{dR} is locally almost of finite type as a prestack. Hence, it is sufficient to probe it by eventually connective affine schemes of finite type.

Hence, in the discussion below, we will be tacitly assuming that schemes and prestacks mapping to Ran are laft (locally almost of finite type).

B.1.5. Here are the two basic features of Ran that will be used in the sequel:

(i) There is a canonically defined map

$$\text{union} : \text{Ran} \times \text{Ran} \to \text{Ran},$$

given by the operation of *union* of finite subsets.

(ii) There exists an open subspace

$$(\text{Ran} \times \text{Ran})_{\text{disj}} \subset \text{Ran} \times \text{Ran},$$

corresponding to the condition that the two subsets are disjoint. Namely, for a affine test scheme S, a pair of S-points $\underline{x}_1, \underline{x}_2$ of Ran maps to $(\text{Ran} \times \text{Ran})_{\text{disj}}$ if

$$\text{Graph}_{\underline{x}_1} \cap \text{Graph}_{\underline{x}_2} = \emptyset.$$

Note that the restriction of the map union to $(\text{Ran} \times \text{Ran})_{\text{disj}}$ is étale.

B.1.6. By a factorization space $\mathcal{T}$ over X we will mean a prestack

$$\mathcal{T}_{\text{Ran}} \to \text{Ran},$$

equipped with a *factorization structure*, which is by definition the datum of an isomorphism

$$\mathcal{T}_{\text{Ran}} \underset{\text{Ran},\text{union}}{\times} (\text{Ran} \times \text{Ran})_{\text{disj}} \simeq (\mathcal{T}_{\text{Ran}} \times \mathcal{T}_{\text{Ran}}) \underset{\text{Ran} \times \text{Ran}}{\times} (\text{Ran} \times \text{Ran})_{\text{disj}}, \tag{B.2}$$

equipped with a homotopy-coherent data of associativity and commutativity (see [Ra6, Sect. 6], where this is spelled out in detail).

B.1.7. Given a map $\mathcal{Z} \to \text{Ran}$, we will denote by $\mathcal{T}_{\mathcal{Z}}$ the base change

$$\mathcal{Z} \underset{\text{Ran}}{\times} \mathcal{T}_{\text{Ran}}.$$

For $\mathcal{Z} = \text{pt}$ so that $\mathcal{Z} \to \text{Ran}$ corresponds to $\underline{x} \in \text{Ran}$, we will write $\mathcal{T}_{\underline{x}}$ for the corresponding $\mathcal{Z}$. The factorization structure on $\mathcal{T}$ implies that for $\underline{x}$ as in (B.1), we have

$$\mathcal{T}_{\underline{x}} \simeq \prod_i \mathcal{T}_{x_i}.$$

B.1.8. A basic example of a factorization space is the affine Grassmannian Gr_G. Namely, for an affine test scheme S and a map $\underline{x} : S \to \text{Ran}$, its lift to $\text{Gr}_{G,\text{Ran}}$ is a datum of

$$(\mathcal{P}_G, \alpha),$$

where $\mathcal{P}_G$ is a G-bundle on $S \times X$, and α is a trivialization of $\mathcal{P}_G$ over the open

$$S \times X - \text{Graph}_{\underline{x}}.$$

B.1.9. Let $\mathcal{T}$ be a factorization space. We can talk about its local properties, such as being a scheme, being an ind-scheme, being (ind)-placid, being formally smooth, etc.

By definition, this means that these properties hold for $\mathcal{T}_S$ relatively to S for every $S \in \text{Sch}^{\text{aff}}_{/\text{Ran}}$.

In a similar way, we can talk about local properties of a map between factorization spaces (e.g., being flat or an fpqc cover).

B.2. Factorization *module* spaces.

B.2.1. Let $\text{Ran}^{\subseteq}$ be the subfunctor of $\text{Ran} \times \text{Ran}$, such that $\text{Maps}(S, \text{Ran}^{\subseteq})$ corresponds to pairs $\underline{x} \subseteq \underline{x}'$, as subsets of $\text{Hom}(S_{\text{red}}, X)$.

Denote by pr_{small} and pr_{big} the two projections $\text{Ran}^{\subseteq} \rightrightarrows \text{Ran}$ that send a pair $(\underline{x}, \underline{x}')$ to $\underline{x}$ and $\underline{x}'$, respectively.

B.2.2. Let $\mathcal{Z}$ be a prestack equipped with a map to Ran. Denote

$$\mathcal{Z}^{\subseteq} := \mathcal{Z} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}}}{\times} \mathrm{Ran}^{\subseteq}.$$

For $\mathcal{Z} = \mathrm{pt}$, so that $\mathcal{Z} \to \mathrm{Ran}$ corresponds to $\underline{x} \in \mathrm{Ran}$, we will write $\mathrm{Ran}_{\underline{x}}$ for the corresponding space $\mathcal{Z}^{\subseteq}$.

B.2.3. Note that we have a variant of the map union:

$$\mathrm{union} : \mathrm{Ran} \times \mathcal{Z}^{\subseteq} \to \mathcal{Z}^{\subseteq}, \quad (\underline{x}, (z, \underline{x}')) \mapsto (z, \underline{x} \cup \underline{x}').$$

Denote by

$$(\mathrm{Ran} \times \mathcal{Z}^{\subseteq})_{\mathrm{disj}} \subset \mathrm{Ran} \times \mathcal{Z}^{\subseteq}$$

the open subfunctor equal to the preimage of $(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}$ under

$$\mathrm{Ran} \times \mathcal{Z}^{\subseteq} \to \mathrm{Ran} \times \mathrm{Ran}^{\subseteq} \overset{\mathrm{id} \times \mathrm{pr}_{\mathrm{big}}}{\to} \mathrm{Ran} \times \mathrm{Ran}.$$

B.2.4. Given a factorization space $\mathcal{T}$, a factorization module space $\mathcal{T}_m$ *over* $\mathcal{T}$ *at* $\mathcal{Z}$ is a prestack

$$(\mathcal{T}_m)_{\mathcal{Z}^{\subseteq}} \to \mathcal{Z}^{\subseteq},$$

equipped with a datum of factorization *against* $\mathcal{T}$:

$$(\mathcal{T}_m)_{\mathcal{Z}^{\subseteq}} \underset{\mathcal{Z}^{\subseteq},\mathrm{union}}{\times} (\mathrm{Ran} \times \mathcal{Z}^{\subseteq})_{\mathrm{disj}} \simeq (\mathcal{T}_{\mathrm{Ran}} \times (\mathcal{T}_m)_{\mathcal{Z}^{\subseteq}}) \underset{\mathrm{Ran}\times\mathcal{Z}^{\subseteq}}{\times} (\mathrm{Ran} \times \mathcal{Z}^{\subseteq})_{\mathrm{disj}}, \tag{B.3}$$

equipped with a homotopy-coherent data of associativity; see [Ra6, Sect. 6] for complete details.

B.2.5. For a factorization module space $\mathcal{T}_m$ at $\mathcal{Z}$, denote

$$(\mathcal{T}_m)_{\mathcal{Z}} := \mathcal{Z} \underset{\mathcal{Z}^{\subseteq}}{\times} (\mathcal{T}_m)_{\mathcal{Z}^{\subseteq}},$$

where $\mathcal{Z} \to \mathcal{Z}^{\subseteq}$ is the map $\mathrm{diag}_{\mathcal{Z}}$ of Sect. 11.2.6.

We will refer to $(\mathcal{T}_m)_{\mathcal{Z}}$ as the *prestack underlying* the factorization module space $\mathcal{T}_m$.

B.2.6. A basic example of a factorization module space over $\mathcal{T}$, defined for any $\mathcal{Z} \to \mathrm{Ran}$, denoted $\mathcal{T}^{\mathrm{fact}_{\mathcal{Z}}}$, is constructed as follows:

$$\mathcal{T}^{\mathrm{fact}_{\mathcal{Z}}}_{\mathcal{Z}^{\subseteq}} := \mathcal{T}_{\mathcal{Z}^{\subseteq}},$$

where $\mathcal{Z}^{\subseteq} \to \mathrm{Ran}$ is the map $\mathrm{pr}_{\mathrm{big},\mathcal{Z}}$, with the datum of (B.3) being provided by the factorization structure on $\mathcal{T}$ itself.

We refer to $\mathcal{T}^{\mathrm{fact}_{\mathcal{Z}}}$ as the *vacuum* factorization module space over $\mathcal{T}$ at $\mathcal{Z}$.

B.2.7. Here is an example of a factorization module space over Gr_G, also defined for any $\mathcal{Z} \to \mathrm{Ran}$, to be denoted

$$\mathrm{Gr}_G^{\mathrm{level}_{\mathcal{Z}}}.$$

For $(z, \underline{x}) : S \to \mathcal{Z}^{\subseteq}$, a lift of this point to $\mathrm{Gr}_G^{\mathrm{level}_{\mathcal{Z}}}$ is a lift of $\underline{x}$ to a point of $\mathrm{Gr}_{G,\mathrm{Ran}}$, and the trivialization of the restriction of the resulting G-bundle $\mathcal{P}_G$ over $S \times X$ to $\widehat{\mathcal{D}}_{\underline{x}}$ (see Sect. B.3.1 below).

B.3. **Digression: formal discs.**

B.3.1. Fix a map $\underline{x}: S \to \mathrm{Ran}$. Let $\widehat{\mathcal{D}}_{\underline{x}}$ be the formal scheme equal to the formal completion of $S \times X$ along the closed subset

$$\mathrm{Graph}_{\underline{x}} \subset S \times X,$$

i.e.,

$$\widehat{\mathcal{D}}_{\underline{x}} := (S \times X) \underset{(S\times X)_{\mathrm{dR}}}{\times} (\mathrm{Graph}_{\underline{x}})_{\mathrm{dR}},$$

where $(\mathrm{Graph}_{\underline{x}})_{\mathrm{dR}} \to (S \times X)_{\mathrm{dR}}$ is given by the embedding

$$(\mathrm{Graph}_{\underline{x}})_{\mathrm{red}} \hookrightarrow S \times X.$$

Note that $\widehat{\mathcal{D}}_{\underline{x}}$ is naturally the pullback of a relative formal scheme, denoted $\widehat{\mathcal{D}}_{\underline{x},\nabla}$, over $S \times X_{\mathrm{dR}}$. Namely,

$$\widehat{\mathcal{D}}_{\underline{x},\nabla} := (S \times X_{\mathrm{dR}}) \underset{(S\times X)_{\mathrm{dR}}}{\times} (\mathrm{Graph}_{\underline{x}})_{\mathrm{dR}}.$$

B.3.2. Let $\mathrm{PreStk}_{\text{disj-loc}} \subset \mathrm{PreStk}$ be the full subcategory of disjoint-union-local prestacks, i.e., prestacks $\mathcal{Z}$, for which for a disjoint union of affine schemes

$$S = S_1 \sqcup S_2,$$

the map

$$\mathrm{Maps}(S, \mathcal{Z}) \to \mathrm{Maps}(S_1, \mathcal{Z}) \times \mathrm{Maps}(S_2, \mathcal{Z}) \tag{B.4}$$

is an isomorphism.

Most prestacks that one encounters in practice satisfy this condition. E.g., note that if $\mathcal{Z}$ satisfies Zariski descent, then it is disjoint-union-local.

Note that the Yoneda embedding

$$\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{PreStk}_{\text{disj-loc}}$$

commutes with coproducts (this would not be true for the original Yoneda embedding into PreStk).

B.3.3. Let $\underline{x}_1$ and $\underline{x}_2$ be a pair of S-points of Ran, such that $(\underline{x}_1, \underline{x}_2)$ lands in $(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}$. Set $\underline{x} := \mathrm{union}(\underline{x}_1, \underline{x}_2)$. In this case we have

$$\widehat{\mathcal{D}}_{\underline{x}} \simeq \widehat{\mathcal{D}}_{\underline{x}_1} \sqcup \widehat{\mathcal{D}}_{\underline{x}_2} \text{ and } \widehat{\mathcal{D}}_{\underline{x},\nabla} \simeq \widehat{\mathcal{D}}_{\underline{x}_1,\nabla} \sqcup \widehat{\mathcal{D}}_{\underline{x}_2,\nabla}, \tag{B.5}$$

where $\sqcup$ is the coproduct taken in $\mathrm{PreStk}_{\text{disj-loc}}$.

B.3.4. Note that, when viewed as an ind-scheme, $\widehat{\mathcal{D}}_{\underline{x}}$ is ind-affine, i.e., of the form

$$\text{“}\underset{\alpha}{\mathrm{colim}}\text{”}\ \mathrm{Spec}(R_\alpha).$$

Let $\mathcal{D}_{\underline{x}}$ denote the affine *scheme* equal to

$$\underset{\alpha}{\mathrm{colim}}\ \mathrm{Spec}(R_\alpha),$$

where the colimit is taken in $\mathrm{Sch}^{\mathrm{aff}}$. I.e.,

$$\mathcal{D}_{\underline{x}} = \mathrm{Spec}(R), \quad R = \lim_{\alpha} R_\alpha.$$

Now, by [Bh, Theorem 1.1], the map

$$\widehat{\mathcal{D}}_{\underline{x}} \to X \tag{B.6}$$

canonically extends to a map

$$\mathcal{D}_{\underline{x}} \to X. \tag{B.7}$$

Remark B.3.5. Here we have used [Bh, Theorem 1.1] in a very elementary situation, in which the required assertion can be handled explicitly:

The obsertvation is that maps from an affine scheme $\widetilde{S}$ to X can be described as right-exact symmetric monoidal functors

$$\mathrm{Perf}(X) \to \mathrm{Perf}(\widetilde{S}),$$

and hence, this is true for $\widetilde{S}$ replaced by any prestack.

Now, the required assertion is that the restriction functor

$$\mathrm{Perf}(\mathcal{D}_{\underline{x}}) \to \mathrm{Perf}(\widehat{\mathcal{D}}_{\underline{x}})$$

is an equivalence, so the restriction map

$$\mathrm{Maps}(\mathcal{D}_{\underline{x}}, X) \to \mathrm{Maps}(\widehat{\mathcal{D}}_{\underline{x}}, X)$$

is an isomorphism.

Remark B.3.6. Note that while the assignment

$$\underline{x} \rightsquigarrow \widehat{\mathcal{D}}_{\underline{x}}$$

is compatible with Zariski localization along S, the formation of $\mathcal{D}_{\underline{x}}$ is *not*.

I.e., for an open $S' \subset S$ and $\underline{x}' := \underline{x}|_{S'}$, the square

$$\begin{tikzcd} \mathcal{D}_{\underline{x}'} \arrow[r] \arrow[d] & \mathcal{D}_{\underline{x}} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is *not* Cartesian.

B.3.7. We have an ind-closed embedding

$$\widehat{\mathcal{D}}_{\underline{x}} \to \mathcal{D}_{\underline{x}}.$$

In particular, $\mathrm{Graph}_{\underline{x}}$ is a Zariski-closed subset of $\mathcal{D}_{\underline{x}}$. Set

$$\mathcal{D}^{\times}_{\underline{x}} := \mathcal{D}_{\underline{x}} - \mathrm{Graph}_{\underline{x}}.$$

Composing with (B.7), we obtain a map

$$\mathcal{D}^{\times}_{\underline{x}} \to X.$$

B.3.8. We claim that $\mathcal{D}_{\underline{x}}$ descends to a relative affine scheme $\mathcal{D}_{\underline{x},\nabla}$ over X_{dR}. In order to construct this descent, it is enough to construct a version of $\mathcal{D}_{\underline{x}}$ over the Čech nerve of the infinitesimal groupoid

$$(X \times X)^{\wedge}$$

of X.

I.e., we have to construct a *compatible* family of affine schemes

$$\mathcal{D}'_{\underline{x}} \to X'$$

over each infinitesimal thickening X' of the main diagonal in $X^{\times n}$ for all n.

We let

$$\widehat{\mathcal{D}}'_{\underline{x}} := X' \underset{X_{\mathrm{dR}}}{\times} \widehat{\mathcal{D}}_{\underline{x},\nabla},$$

viewed as an ind-affine ind-scheme, and we let $\mathcal{D}'_{\underline{x}}$ be the colimit of $\widehat{\mathcal{D}}'_{\underline{x}}$, taken in the category of affine schemes.

The map $\widehat{\mathcal{D}}'_{\underline{x}} \to X'$ extends to a map $\mathcal{D}'_{\underline{x}} \to X'$ extends by the same principle as in the case of $\mathcal{D}_{\underline{x}}$.

B.3.9. In order to establish the compatibility of the above construction, it suffices to show that (with respect to any of the projections $X' \to X$), the map

$$\mathcal{D}'_{\underline{x}} \to X' \underset{X}{\times} \mathcal{D}_{\underline{x}} \tag{B.8}$$

is an isomorphism.

We note that $X' \to X$ has the form $\operatorname{Spec}_X(A_0)$, where is given by $A_0 \in \operatorname{Perf}(X)$, and hence $X' \underset{X}{\times} \mathcal{D}_{\underline{x}}$ has the form $\operatorname{Spec}_{\mathcal{D}_{\underline{x}}}(A)$, where $A \in \operatorname{Perf}(\mathcal{D}_{\underline{x}})$.

In other words, if R is as in Sect. B.3.4, then $X' \underset{X}{\times} \mathcal{D}_{\underline{x}} \simeq \operatorname{Spec}(R')$, where R' is compact as an object of R-mod.

Let R_α be as in Sect. B.3.4. Then

$$\widehat{\mathcal{D}}'_{\underline{x}} = \text{“}\underset{\alpha}{\operatorname{colim}}\text{”}\ \operatorname{Spec}(R' \underset{R}{\otimes} R_\alpha).$$

The assertion that (B.8) is an isomorphism is equivalent to saying that the map

$$R' \simeq R' \underset{R}{\otimes} (\lim_{\alpha} R_\alpha) \to \lim_{\alpha} (R' \underset{R}{\otimes} R_\alpha)$$

is an isomorphism. However, this follows from the fact that R' is compact as an R-module.

B.3.10. We let $\mathcal{D}^{\times}_{\underline{x},\nabla}$ be the open sub-functor of $\mathcal{D}_{\underline{x},\nabla}$, equal to

$$\mathcal{D}_{\underline{x},\nabla} - \operatorname{Graph}_{\underline{x}}.$$

It is easy to see that the prestacks

$$\mathcal{D}_{\underline{x},\nabla} \text{ and } \mathcal{D}^{\times}_{\underline{x},\nabla}, \tag{B.9}$$

when viewed as relative affine schemes over X_{dR}, are independent of the choice of X^o.

B.3.11. The prestacks (B.9) satisfy a splitting property parallel to (B.5).

B.4. **Formation of (horizontal) arc and loop spaces.** In this subsection we will discuss two ubiquitous sources of examples of factorization spaces.

B.4.1. Let $\mathcal{Y} \to X$ be a D-prestack[60] over X, i.e., $\mathcal{Y}$ is the pullback along $X \to X_{\mathrm{dR}}$ of a prestack $\mathcal{Y}_\nabla \to X_{\mathrm{dR}}$.

We will assume that $\mathcal{Y}_\nabla$ is disjoint-union-local (see Sect. B.3.2).

B.4.2. We define the factorization prestack $\mathfrak{L}^+_\nabla(\mathcal{Y})$ as follows. For an affine test scheme S and a map $\underline{x} : S \to \operatorname{Ran}$, a lift of this map to $\mathfrak{L}^+_\nabla(\mathcal{Y})$ is an X_{dR}-map

$$\widehat{\mathcal{D}}_{\underline{x},\nabla} \to \mathcal{Y}_\nabla.$$

The factorization structure on $\mathfrak{L}^+_\nabla(\mathcal{Y})$ follows from (B.5) and (B.4).

Remark B.4.3. Note that the restriction $\mathfrak{L}^+_\nabla(\mathcal{Y})|_X$ recovers the original $\mathcal{Y}$. In Sect. C.6.10 we will upgrade the assignment

$$\mathcal{Y} \mapsto \mathfrak{L}^+_\nabla(\mathcal{Y})$$

to an equivalence of categories.

[60]A.k.a., crystal of prestacks.

B.4.4. For the rest of this subsection we will assume that $\mathcal{Y}$ is affine over X. I.e., $\mathcal{Y}$ is an affine D-scheme.

There is an (obvious) equivalence between the category of D-schemes and that of commutative algebras A in D-mod(X) so that $\mathbf{oblv}^l(A)$ is connective.

The plain affine scheme underlying a given affine D-scheme is

$$\mathrm{Spec}_X(\mathbf{oblv}^l(A)).$$

We will write

$$\mathrm{Spec}_X(A)$$

when we want to emphasize the D-structure.

B.4.5. Note that the assumption that $\mathcal{Y}$ is affine over X allows us to interpret the datum of a map

$$\widehat{\mathcal{D}}_{\underline{x},\nabla} \to \mathcal{Y}_\nabla$$

as

$$\mathcal{D}_{\underline{x},\nabla} \to \mathcal{Y}_\nabla,$$

where $\mathcal{D}_{\underline{x},\nabla}$ is an in Sect. B.3.8.

B.4.6. We will now define another factorization space, denoted $\mathfrak{L}_\nabla(\mathcal{Y})$. A lift of this map to $\mathfrak{L}_\nabla(\mathcal{Y})_{\mathrm{Ran}}$ is by definition a X_{dR} map

$$\mathcal{D}^\times_{\underline{x},\nabla} \to \mathcal{Y}_\nabla.$$

The factorization structure on $\mathfrak{L}_\nabla(\mathcal{Y})_{\mathrm{Ran}}$ follows from the splitting property in Sect. B.3.11.

B.4.7. Using Sect. B.4.5 and the open embeddings

$$\mathcal{D}^\times_{\underline{x},\nabla} \hookrightarrow \mathcal{D}_{\underline{x},\nabla}$$

we obtain a map of factorization spaces:

$$\iota : \mathfrak{L}^+_\nabla(\mathcal{Y}) \to \mathfrak{L}_\nabla(\mathcal{Y}).$$

We claim:

Lemma B.4.8.

(a) $\mathfrak{L}^+_\nabla(\mathcal{Y})$ *is an factorization affine scheme.*

(b) $\mathfrak{L}_\nabla(\mathcal{Y})$ *is a factorization ind-affine ind-scheme.*

(c) *The map* ι *is a closed embedding.*[61]

B.4.9. *Proof of Lemma B.4.8.* At any level of coconnective truncation, we can write $\mathcal{Y}$ as a finite product of affine D-schemes that are (potentially infinite) products of affine D-schemes of the form

$$\mathrm{Spec}_X(A), \quad A = \mathrm{Sym}^!(\mathbf{ind}^l(\mathcal{E})), \tag{B.10}$$

where $\mathcal{E}$ is a vector bundle on X.

Since (ind-)affine (ind-)schemes are closed under finite limits and products, and the functors

$$\mathcal{Y} \mapsto \mathfrak{L}^+_\nabla(\mathcal{Y}) \text{ and } \mathcal{Y} \mapsto \mathfrak{L}_\nabla(\mathcal{Y})$$

map products to products, we are reduced to considering $\mathcal{Y}$ of the form specified in (B.10).

In the latter case, the assertions of the lemma can be (easily) checked directly (see Sect. B.5.6 below). $\square$

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[61]By a slight abuse of terminology, we call a map $Z \to \mathcal{Z}$ from a scheme Z to an ind-scheme $\mathcal{Z}$ a "closed embedding" if the map from Z to some/any of the schemes that comprise $\mathcal{Z}$ is a closed embedding. Note that such a map is *not* a closed embedding in the DAG sense, but rather an *ind-closed embedding*.

B.4.10. *Example.* Recall the factorization space Gr_G. By Beauville-Laszlo theorem, we can rewrite it as follows: for $\underline{x} : S \to \mathrm{Ran}$, a lift of this point to a point of $\mathrm{Gr}_{G,\mathrm{Ran}}$ is the datum of G-bundle on $\mathcal{D}_{\underline{x}}$ (which is equivalent to that of a G-bundle on $\widehat{\mathcal{D}}_{\underline{x}}$) and the trivialization of its restriction to $\mathcal{D}^\times_{\underline{x}}$.

From this description, we obtain a canonical projection

$$\mathfrak{L}(G) \to \mathrm{Gr}_G.$$

We claim that this projection identifies Gr_G with the étale quotient $\mathfrak{L}(G)/\mathfrak{L}^+(G)$. Indeed, this follows from the fact that a G-bundle on $\widehat{\mathcal{D}}_{\underline{x}}$ can be trivialized after an étale sheafification along S (see Sect. B.7.2).

A similar description applies to factorization Gr_G-module space $\mathrm{Gr}_G^{\mathrm{level}_{\mathfrak{Z}}}$, see Sect. B.2.7.

B.5. **Digression: the jet construction.**

B.5.1. Let $\mathcal{Y}$ be a prestack over X. Its jets construction, denoted

$$\mathrm{Jets}(\mathcal{Y}) \to X_{\mathrm{dR}}$$

is by definition restriction of scalars à la Weil of $\mathcal{Y}$ along the projection

$$X \to X_{\mathrm{dR}}.$$

B.5.2. Explicitly for $x : S \to X_{\mathrm{dR}}$, a lift of this map to a map $S \to \mathrm{Jets}(\mathcal{Y})$ is an X-map

$$\widehat{\mathcal{D}}_x \to \mathcal{Y}.$$

B.5.3. Suppose for a moment that $\mathcal{Y}$ is affine over X, i.e.,

$$\mathcal{Y} = \mathrm{Spec}_X(A_0), \quad A_0 \in \mathrm{ComAlg}(\mathrm{QCoh}(X)^{\leq 0}).$$

In this case

$$\mathrm{Jets}(\mathcal{Y}) = \mathrm{Spec}_X(A),$$

where A is the D-algebra obtained by applying to A_0 the left adjoint of the forgetful functor

$$\mathbf{oblv}^l : \mathrm{ComAlg}(\text{D-mod}(X)) \to \mathrm{ComAlg}(\mathrm{QCoh}(X)).$$

For example, when

$$A_0 = \mathrm{Sym}(\mathcal{E}), \quad \mathcal{E} \in \mathrm{QCoh}(X),$$

the above left adjoint produces

$$\mathrm{Sym}^!(\mathbf{ind}^l(\mathcal{E})).$$

B.5.4. Denote

$$\mathfrak{L}^+(\mathcal{Y}) := \mathfrak{L}^+_\nabla(\mathrm{Jets}(\mathcal{Y})).$$

One can tautologically rewrite the definition of $\mathfrak{L}^+(\mathcal{Y})$ as follows. For $\underline{x} : S \to \mathrm{Ran}$, its lift to $\mathfrak{L}^+(\mathcal{Y})_{\mathrm{Ran}}$ is an X-map

$$\widehat{\mathcal{D}}_{\underline{x}} \to \mathcal{Y}.$$

B.5.5. Assume again that $\mathcal{Y}$ is affine over X. Then $\mathrm{Jets}(\mathcal{Y})$ is affine over X_{dR} (see Sect. B.5.3), and one can consider

$$\mathfrak{L}(\mathcal{Y}) := \mathfrak{L}_\nabla(\mathrm{Jets}(\mathcal{Y})).$$

Explicitly, for an affine test scheme S and $\underline{x} : S \to \mathrm{Ran}$, its lift to a map $\mathfrak{L}(\mathcal{Y})_{\mathrm{Ran}}$ is an X-map

$$\mathcal{D}^\times_{\underline{x}} \to \mathcal{Y}.$$

Note that in this case a lift to $\mathfrak{L}^+(\mathcal{Y})_{\mathrm{Ran}}$ can also be described as an X-map

$$\mathcal{D}_{\underline{x}} \to \mathcal{Y}.$$

B.5.6. *Example.* Let $\mathcal{Y} = \mathrm{Spec}_X(\mathrm{Sym}(\mathcal{E}))$, where $\mathcal{E}$ is a vector bundle on X. Then the spaces

$$\mathfrak{L}^+(\mathcal{Y}) \text{ and } \mathfrak{L}(\mathcal{Y})$$

can be described explicitly as follows.

Fix a map $\underline{x} : S \to \mathrm{Ran}$. Fix lifts of the maps $x_i : S_{\mathrm{red}} \to X$ to maps $\widetilde{x}_i : S \to X$. Let D be the divisor on $S \times X$ equal

$$\sum_i n_i \cdot \mathrm{Graph}_{\widetilde{x}_i}$$

for some/any choice of $n_i \geq 1$.

Then a lift of $\underline{x}$ to a point of $\mathfrak{L}^+(\mathcal{Y})$ is a point of

$$\lim_n \Gamma(S \times X, \mathcal{O}_X \otimes \mathcal{E}^\vee / \mathcal{O}_X \otimes \mathcal{E}^\vee(-n \cdot D)).$$

A lift of $\underline{x}$ to a point of $\mathfrak{L}(\mathcal{Y})$ is a point of

$$\underset{m}{\mathrm{colim}} \lim_n \Gamma(S \times X, \mathcal{O}_X \otimes \mathcal{E}^\vee(m \cdot D) / \mathcal{O}_X \otimes \mathcal{E}^\vee(-n \cdot D)).$$

B.5.7. *Example.* Let H be a smooth group-scheme over X. On the one hand, we can consider the algebraic stack[62] pt/H over X, and consider the corresponding D-prestack

$$\mathrm{Jets}(\mathrm{pt}/H).$$

On the other hand, we can consider

$$\mathrm{pt}/\mathrm{Jets}(H),$$

i.e., the étale sheafification of $B(\mathrm{Jets}(H))$.

We have a tautological map

$$\mathrm{pt}/\mathrm{Jets}(H) \to \mathrm{Jets}(\mathrm{pt}/H). \tag{B.11}$$

We claim that (B.11) is an isomorphism. Indeed, this follows from the fact that if an H-bundle on $\widehat{\mathcal{D}}_{\underline{x}}$ is such that its restriction to S is trivial, then it is itself trivial.

B.6. **Digression: the notion of (almost) finite presentation in the D-sense.**

B.6.1. Fix a natural number n, and consider the category $\mathrm{ComAlg}(\text{D-mod}(X))^{\leq 0, \geq -n}$ of connective n-coconnective commutative algebras in $\text{D-mod}(X)$, which is by definition

$$\mathrm{ComAlg}(\text{D-mod}(X)) \underset{\mathrm{QCoh}(X)}{\times} \mathrm{QCoh}(X)^{\leq 0, \geq -n},$$

where the functor $\mathrm{ComAlg}(\text{D-mod}(X)) \to \mathrm{QCoh}(X)$ is

$$\mathrm{ComAlg}(\text{D-mod}(X)) \overset{\mathbf{oblv}_{\mathrm{ComAlg}}}{\longrightarrow} \text{D-mod}(X) \overset{\mathbf{oblv}^l}{\to} \mathrm{QCoh}(X).$$

We shall say that $A \in \mathrm{ComAlg}(\text{D-mod}(X))^{\leq 0, \geq -n}$ is n-D-afp if it is *compact* as an object of this category.

B.6.2. Suppose that $A \in \mathrm{ComAlg}(\text{D-mod}(X))^{\leq 0, \geq -n}$ is isomorphic to the geometric realization of a simplicial object $A_\bullet$ in $\mathrm{ComAlg}(\text{D-mod}(X))^{\leq 0, \geq -n}$ with terms of the form

$$A_n = \mathrm{Sym}^!(\mathcal{M}_n), \quad \mathcal{M}_n \in \text{D-mod}(X), \quad \mathbf{oblv}^l(\mathcal{M}_n) \in \text{D-mod}^{\heartsuit, \text{f.g.}}.$$

It is clear that such A is compact.

Furthermore, it easy to see that objects of this form generate $\mathrm{ComAlg}(\text{D-mod}(X))^{\leq 0, \geq -n}$ under filtered colimits. Furthermore, for the generation statement, we can take $\mathcal{M}_n$ to be locally free, i.e., of the form $\mathbf{ind}^l(\mathcal{E}_n)$, where $\mathcal{E}_n$ is a vector bundle on X.

From here it follows that every n-D-afp algebra can be written as a retract of such $|A_\bullet|$.

---

[62] Throughout the paper we write pt/H, where we mean the étale sheafification of $B(H)$, where the latter is the quotient of the base (in this case, X) by the trivial action of H.

B.6.3. Consider now the category $\mathrm{ComAlg}(\text{D-mod}(X))^{\leq 0}$ of connective commutative algebras in D-mod(X), i.e.,

$$\mathrm{ComAlg}(\text{D-mod}(X)) \underset{\mathrm{QCoh}(X)}{\times} \mathrm{QCoh}(X)^{\leq 0}.$$

We shall say that A is D-afp if for every n, the truncation $\tau^{\geq -n}(A) \in \mathrm{ComAlg}(\text{D-mod}(X))^{\leq 0, \geq -n}$ is n-D-afp.

We shall say that $\mathcal{Y} = \mathrm{Spec}_X(A)$ is D-afp if A is.

Here are some examples of D-afp algebras.

B.6.4. Let $\mathcal{Y}_0 \to X$ be an affine scheme almost of finite type, and take $\mathcal{Y} := \mathrm{Jets}(\mathcal{Y}_0)$. Then $\mathcal{Y}$ is D-afp.

Indeed, let $\mathcal{Y}_0 = \mathrm{Spec}_X(A_0)$, where $A_0 \in \mathrm{ComAlg}(\mathrm{QCoh}(X)^{\leq 0})$. We can write

$$A_0 = |A_{0,\bullet}|, \quad A_{0,n} = \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}_n), \quad \mathcal{E}_n \in \mathrm{Perf}(X)^{\heartsuit}.$$

Then $\mathcal{Y} = \mathrm{Spec}_X(A)$ for

$$A = |A_\bullet|, \quad A_n = \mathrm{Sym}^!(\mathbf{ind}^l(\mathcal{E}_n)).$$

B.6.5. Let now $\mathcal{Y}$ be the constant affine D-scheme with fiber $\mathcal{Y}_0$, where $\mathcal{Y}_0$ is almost of finite type. We claim that it is D-afp.

Indeed, write $\mathcal{Y}_0 = \mathrm{Spec}(A_0)$, where

$$A_0 = |A_{0,\bullet}|, \quad A_{0,\bullet} = \mathrm{Sym}(V_n), \quad V_n \in \mathrm{Vect}^{\heartsuit,\text{f.d.}}.$$

Then $\mathcal{Y} = \mathrm{Spec}(A)$ where

$$A = |A_\bullet|, \quad A_n = \mathrm{Sym}(V_n) \otimes \omega_X[1] = \mathrm{Sym}^!(V_n \otimes \omega_X[1])$$

(recall that according to our conventions in Sect. 1.1.1, the object $\omega_X[1] \in \text{D-mod}(X)$ is the dualizing sheaf on X).

B.6.6. Let $\mathcal{Y} \to \mathcal{Y}_0$ be a map of affine D-schemes. The notion of being D-afp has a straightforward analog in this situation:

If $\mathcal{Y}_0 = \mathrm{Spec}_X(A_0)$ and $\mathcal{Y} = \mathrm{Spec}_X(A)$, so that $\mathcal{Y} \to \mathcal{Y}_0$ corresponds to a map $A_0 \to A$, it makes sense to talk about A being finitely presented over A_0 in the D-sense.

The notion of being D-afp is transitive: if

$$\mathcal{Y}'' \to \mathcal{Y}' \to \mathcal{Y}$$

are maps of affine D-schemes, with $\mathcal{Y}' \to \mathcal{Y}$ and $\mathcal{Y}'' \to \mathcal{Y}'$ D-afp, then so is $\mathcal{Y}'' \to \mathcal{Y}$.

B.7. **Local systems on the formal (punctured) disc.**

B.7.1. Let H be a (finite-dimensional) algebraic group. Consider the algebraic stack pt/H. We regard it as a constant D-prestack over X, i.e.,

$$\mathcal{Y}_\nabla := \mathrm{pt}/H \times X_{\mathrm{dR}}.$$

We define the factorization space

$$\mathrm{LS}^{\mathrm{reg}}_H := \mathfrak{L}^+_\nabla(\mathrm{pt}/H).$$

B.7.2. Note that the natural map

$$\mathrm{pt}/\mathfrak{L}^+_\nabla(H) \to \mathfrak{L}^+_\nabla(\mathrm{pt}/H) =: \mathrm{LS}^{\mathrm{reg}}_H \tag{B.12}$$

is an isomorphism (as was mentioned earlier, the notation $\mathrm{pt}/\mathfrak{L}^+_\nabla(H)$ means the étale sheafification of $B(\mathfrak{L}^+_\nabla(H))$, the quotient of the base by the trivial action of the corresponding group-scheme).

Indeed, this follows from the fact that for $\underline{x} : S \to \mathrm{Ran}$, the étale topology on S generates the étale topology on $\widehat{\mathcal{D}}_{\underline{x}}$.

B.7.3. First, we claim:

Lemma B.7.4. *The map* $\mathrm{pt} \to \mathrm{LS}_H^{\mathrm{reg}}$ *is étale-locally surjective and is an fpqc cover.*

Proof. The étale surjectivity follows from the isomorphism (B.12).

Hence, to prove the lemma it remains to show that

$$\mathrm{pt} \underset{\mathrm{LS}_H^{\mathrm{reg}}}{\times} \mathrm{pt} \simeq \mathfrak{L}_\nabla^+(H)$$

is flat. I.e., we need to show that for every finite set I, the scheme $\mathfrak{L}_\nabla^+(H)_{X^I}$ is flat over X^I.

The latter is the assertion that for any flat D-algebra A on X, the factorization algebra $\mathrm{Fact}(A)$ (see (C.45)) has the property that for every I, the restriction $\mathbf{oblv}^l(\mathrm{Fact}(A)_{X^I})$ is flat over X^I (this is essentially [BD2, Lemma 3.4.12]).

□

B.7.5. One of the key facts about $\mathrm{LS}_H^{\mathrm{reg}}$ is that it can be accessed via *gauge* forms. Namely, consider the tautological map

$$\mathrm{pt}/H \to \mathrm{Jets}(\mathrm{pt}/H) \simeq \mathrm{pt}/\mathrm{Jets}(H)$$

Note that the fiber product

$$\mathrm{pt}/H \underset{\mathrm{pt}/\mathrm{Jets}(H)}{\times} \mathrm{pt}$$

identifies with the D-scheme

$$\mathrm{Jets}(\mathfrak{h} \otimes \omega_X) =: \mathrm{Conn}(\mathfrak{h})$$

of jets into $\mathfrak{h} \otimes \omega_X$ (i.e., the total space of the corresponding vector bundle, viewed as an affine scheme over X).[63] The resulting $\mathrm{Jets}(H)$-action on $\mathrm{Conn}(\mathfrak{h})$ is called *the gauge action.*

Hence, we can identify

$$\mathrm{pt}/H \simeq \mathrm{Conn}(\mathfrak{h})/\mathrm{Jets}(H)$$

as D-prestacks over X.

In particular, we obtain an action of $\mathfrak{L}^+(H)$ on $\mathfrak{L}_\nabla^+(\mathrm{Conn}(\mathfrak{h}))$ as factorization spaces, and an identification:

$$\mathrm{LS}_H^{\mathrm{reg}} \simeq \mathfrak{L}_\nabla^+(\mathrm{Conn}(\mathfrak{h}))/\mathfrak{L}^+(H). \tag{B.13}$$

B.7.6. We note:

Lemma B.7.7. *The factorization space* $\mathrm{LS}_H^{\mathrm{reg}}$ *is fomally smooth.*

Proof. This follows from (B.13).

□

Remark B.7.8. Note that the map

$$\mathrm{pt} \to \mathrm{LS}_H^{\mathrm{reg}}$$

is *not* formally smooth. Indeed, the fiber product

$$\mathrm{pt} \underset{\mathrm{LS}_H^{\mathrm{reg}}}{\times} \mathrm{pt}$$

is the affine factorization scheme $\mathfrak{L}_\nabla^+(H)$, associated to the constant D-scheme H (see Sect. 4.3.2), and one can show that $\mathfrak{L}_\nabla^+(H)_{X^2}$ is *not* formally smooth over X^2.

Namely, the cotangent space to $\mathfrak{L}_\nabla^+(H)_{X^2}$ at the unit section is the object of $\mathrm{QCoh}(X^2)$ underlying the *left* D-module

$$\mathrm{Fib}\left(j_*((\mathfrak{h}^* \oplus \mathfrak{h}^*) \otimes \mathcal{O}_{X^2-\Delta(X)}) \to \Delta_*(\mathfrak{h}^* \otimes \mathcal{O}_X)\right), \tag{B.14}$$

where the map in (B.14) is the composition

$$\mathrm{Fib}\left(j_*((\mathfrak{h}^* \oplus \mathfrak{h}^*) \otimes \mathcal{O}_{X^2-\Delta(X)}) \to \Delta_*((\mathfrak{h}^* \oplus \mathfrak{h}^*) \otimes \mathcal{O}_X)\right) \to \Delta_*(\mathfrak{h}^* \otimes \mathcal{O}_X),$$

[63]Indeed, the map $\mathrm{dlog} : \mathrm{Jets}(H) \to \mathrm{Jets}(\mathfrak{h} \otimes \omega_X)$ identies the target with $\mathrm{Jets}(H)/H$.

where the second arrow is induced by the addition map $\mathfrak{h}^* \oplus \mathfrak{h}^* \to \mathfrak{h}^*$.

The object (B.14) of $\mathrm{QCoh}(X^2)$ is flat, but not projective, implying that $\mathfrak{L}^+_\nabla(H)$ is not formally smooth.

Note that this is not in contradiction with the conclusion of Corollary 3.1.11. Let us see that $T^*_{X^2}(\mathrm{LS}^{\mathrm{reg}}_H)$ does satisfy the infinitesimal condition for formal smoothness, i.e., that

$$\mathrm{Hom}(T^*_{X^2}(\mathrm{LS}^{\mathrm{reg}}_H), \mathcal{F}) = 0 \text{ if } \mathcal{F} \in \mathrm{QCoh}(X^2)^{\leq -1}.$$

We have

$$T^*_{X^2}(\mathrm{LS}^{\mathrm{reg}}_H) \simeq T^*_{X^2}(\mathfrak{L}^+_\nabla(H))[-1].$$

Now, we claim for any flat *countably generated* object $\mathcal{E} \in \mathrm{QCoh}(X^2)$, we have

$$\mathrm{Hom}(\mathcal{E}[-1], \mathcal{F}) = 0 \text{ if } \mathcal{F} \in \mathrm{QCoh}(X^2)^{\leq -1}.$$

Indeed, any such $\mathcal{E}$ can be written as a countable filtered colimit of projective modules $\mathcal{E}_i$, and

$$\mathcal{H}om(\mathcal{E}[-1], \mathcal{F}) = \mathrm{lim.\,proj.}\ \mathcal{H}om(\mathcal{E}_i[-1], \mathcal{F}).$$

Here all $\mathcal{H}om(\mathcal{E}_i[-1], \mathcal{F})$ live in cohomological degrees ≤ -2, while lim. proj. has amplitude 1, due to the countability assumption.

B.7.9. We now proceed to defining the factorization space $\mathrm{LS}^{\mathrm{mer}}_H$. Naively, one would want to apply the functor $\mathfrak{L}_\nabla(-)$ to the constant D-prestack pt/H. But we cannot quite do this, and that is for two reasons, which already occur for $\mathrm{pt}/\mathrm{Jets}(H)$:

(i) When considering $\mathfrak{L}_\nabla(-)$, we only allow affine targets.

(ii) Even over a fixed point $x = \underline{x} \in \mathrm{Ran}$ (but an arbitrary affine scheme $S = \mathrm{Spec}(R)$ of parameters), the space of maps

$$\mathcal{D}^\times_x \to \mathrm{pt}/H$$

is the space of étale H-torsors over the affine scheme $\mathrm{Spec}(R(\!(t)\!))$, where t is a local coordinate near x. However, we do not want to consider the étale topology on $\mathrm{Spec}(R(\!(t)\!))$. Rather, we want to étale-localize with respect to $S = \mathrm{Spec}(R)$ itself, i.e., we only want to consider covers of $\mathrm{Spec}(R(\!(t)\!))$ of the form $\mathrm{Spec}(\widetilde{R}(\!(t)\!))$, where $\mathrm{Spec}(\widetilde{R}) \to \mathrm{Spec}(R)$ is étale.

B.7.10. The gauge action of $\mathrm{Jets}(H)$ on $\mathrm{Conn}(\mathfrak{h})$ gives rise to an action of $\mathfrak{L}(H)$ as a factorization group ind-scheme on $\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))$.

We define $\mathrm{LS}^{\mathrm{mer}}_H$ to be

$$\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))/\mathfrak{L}(H),$$

the étale sheafification of the non-sheafified quotient of $\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))$ by the gauge action of $\mathfrak{L}(H)$.

B.7.11. We can rephrase the definition of $\mathrm{LS}^{\mathrm{mer}}_H$ as follows:

Consider the étale quotient

$$\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))/\mathfrak{L}^+(H).$$

It carries an action of the groupoid

$$\mathfrak{L}^+(H)\backslash\mathfrak{L}(H)/\mathfrak{L}^+(H).$$

Then $\mathrm{LS}^{\mathrm{mer}}_H$ is the quotient of $\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))/\mathfrak{L}^+(H)$ by $\mathfrak{L}^+(H)\backslash\mathfrak{L}(H)/\mathfrak{L}^+(H)$, subsequent sheafified in the étale topology.

B.7.12. By construction, we have a naturally defined map

$$\mathrm{LS}_H^{\mathrm{reg}} \to \mathrm{LS}_H^{\mathrm{mer}}. \tag{B.15}$$

We claim:

Lemma B.7.13. *The map* (B.15) *is ind-affine locally almost of finite presentation*[64].

Proof. By construction, it suffices to show that the fiber product

$$\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h})) \to \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h})) \tag{B.16}$$

is ind-schematic locally almost of finite presentation.

The left-hand side in (B.16) is the space

$$\{g \in \mathfrak{L}(H), \alpha \in \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h})) \,|\, g \cdot \alpha \in \mathfrak{L}_\nabla^+(\mathrm{Conn}(\mathfrak{h}))\}/\mathfrak{L}^+(H).$$

In other words, we can rewrite it as

$$(\mathrm{Gr}_H \times \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))) \underset{\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h})/\mathfrak{L}^+(H)}{\times} \mathfrak{L}_\nabla^+(\mathrm{Conn}(\mathfrak{h}))/\mathfrak{L}^+(H)$$

and its map to the right-hand side of (B.16) is the composition

$$\begin{multline} (\mathrm{Gr}_H \times \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))) \underset{\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h})/\mathfrak{L}^+(H)}{\times} \mathfrak{L}_\nabla^+(\mathrm{Conn}(\mathfrak{h}))/\mathfrak{L}^+(H) \to \\ \to \mathrm{Gr}_H \times \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h})) \to \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h})). \end{multline} \tag{B.17}$$

Since the map $\mathfrak{L}_\nabla^+(\mathrm{Conn}(\mathfrak{h})) \to \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))$ is an ind-closed embedding locally almost of finite presentation, we obtain that the first arrow in (B.17) has this property.

The second arrow in (B.17) is ind-schematic and locally almost of finite presentation since Gr_H is an ind-scheme locally almost of finite type. Hence (B.16) is ind-schematic and locally almost of finite presentation.

Finally, let us show that (B.15) is ind-affine. Let H' denote the reductive quotient of H. Factor the map (B.15) as

$$\mathrm{LS}_H^{\mathrm{reg}} \to \mathrm{LS}_H^{\mathrm{mer}} \underset{\mathrm{LS}_{H'}^{\mathrm{mer}}}{\times} \mathrm{LS}_{H'}^{\mathrm{reg}} \to \mathrm{LS}_H^{\mathrm{mer}}, \tag{B.18}$$

and it is enough to show that both arrows in (B.18) are ind-affine.

For the first arrow in (B.18), after base-changing to the fpqc cover $\mathrm{pt} \to \mathrm{LS}_{H'}^{\mathrm{reg}}$, it suffices to show that the map

$$\mathrm{LS}_{H''}^{\mathrm{reg}} \to \mathrm{LS}_{H''}^{\mathrm{mer}}$$

is ind-affine, where H'' is the unipotent radical of H. This follows from the fact that in this case the second arrow (B.17) is ind-affine, since the affine Grassmannian for a unipotent group is ind-affine.

For the second arrow in (B.18), it suffices to show that the original map (B.15) is ind-affine when H is reductive. Note that in this case Gr_H is ind-proper, so the second arrow in (B.17) is ind-proper. Hence, the map (B.15) is ind-proper. However, since it is also injective at the level of k-points, we obtain that it is ind-affine (indeed, a proper map between schemes that is injective at k-points is affine). $\square$

[64]See Sect. A.9.5 for what this means.

B.7.14. We now discuss a global version of the factorization space $\mathrm{LS}_H^{\mathrm{mer}}$.

We define the (non-factorization!) space

$$\mathrm{LS}_{H,\mathrm{Ran}}^{\mathrm{mer,glob}} \to \mathrm{Ran}$$

as follows:

For an affine test scheme S and a map $\underline{x} \to \mathrm{Ran}$, its lift to $\mathrm{LS}_{H,\mathrm{Ran}}^{\mathrm{mer,glob}}$ is a datum of a map

$$(S \times X_{\mathrm{dR}} - \mathrm{Graph}_{\underline{x}}) \to \mathrm{pt}/H \tag{B.19}$$

such that étale-locally on S, the composite map

$$(S \times X - \mathrm{Graph}_{\underline{x}}) \to (S \times X_{\mathrm{dR}} - \mathrm{Graph}_{\underline{x}}) \to \mathrm{pt}/H$$

admits an extension to a map

$$S \times X \to \mathrm{pt}/H. \tag{B.20}$$

B.7.15. We claim that we have a naturally defined evaluation map

$$\mathrm{ev}_{\mathrm{Ran}} : \mathrm{LS}_{H,\mathrm{Ran}}^{\mathrm{mer,glob}} \to \mathrm{LS}_{H,\mathrm{Ran}}^{\mathrm{mer}}.$$

Indeed, for $\underline{x} : S \to \mathrm{Ran}$, given a map (B.19) and a lift (B.20), the restriction of the connection form to $\mathcal{D}_{\underline{x}}$ gives rise to a section of

$$\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))_S/\mathfrak{L}^+(H)_S.$$

A modification of (B.20) results in an action of the groupoid

$$\mathfrak{L}^+(H)_S\backslash\mathfrak{L}(H)_S/\mathfrak{L}^+(H)_S.$$

B.8. Sheaves of categories over the Ran space.

B.8.1. Let $\mathcal{Y}$ be a prestack. When discussing sheaves of categories over $\mathcal{Y}$, we will assume that $\mathcal{Y}$ is locally almost of finite type. In this context, when considering affine schemes S or general prestacks $\mathcal{Z}$ mapping to $\mathcal{Y}$, we will assume that they are also locally almost of finite type.

B.8.2. A sheaf of categories $\underline{\mathbf{C}}$ on $\mathcal{Y}$ is an assignment

$$(S, y) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}} \rightsquigarrow \mathbf{C}_{S,y} \in \mathrm{QCoh}(S)\text{-}\mathbf{mod},$$

equipped with identifications:

$$S' \xrightarrow{f} S, \quad \mathbf{C}_{S',y\circ f} \simeq \mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_{S,y},$$

satisfying a homotopy-coherent system of compatibilities; we refer the reader to [Ga5] for details.

Let $\mathrm{ShvCat}(\mathcal{Y})$ denote the (2-)-category of sheaves of categories over $\mathcal{Y}$. It has a natural symmetric monoidal structure given by

$$(\mathbf{C}_1 \otimes \mathbf{C}_2)_{S,y} := \mathbf{C}_{1,S,y} \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_{2,S,y}.$$

B.8.3. A basic example of a sheaf of categories over $\mathcal{Y}$ is

$$S \rightsquigarrow \mathrm{QCoh}(S);$$

we denote it by $\underline{\mathrm{QCoh}(\mathcal{Y})}$.

This is the unit for the above symmetric monoidal structure on $\mathrm{ShvCat}(\mathcal{Y})$.

B.8.4. Given a sheaf of categories $\underline{\mathbf{C}}$ on $\mathcal{Y}$, the category of its global sections, denoted $\Gamma(\mathcal{Y}, \underline{\mathbf{C}})$ is defined as

$$\lim_{(S,y)\in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}} \mathbf{C}_{S,y}.$$

This category is naturally a module over

$$\lim_{(S,y)\in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}} \mathrm{QCoh}(S) =: \mathrm{QCoh}(\mathcal{Y}).$$

Thus, the functor

$$\Gamma(\mathcal{Y}, -) : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{DGCat}$$

upgrades to a functor

$$\Gamma(\mathcal{Y}, -)^{\mathrm{enh}} : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}.$$

B.8.5. *Example.* We have

$$\Gamma(\mathcal{Y}, \underline{\mathrm{QCoh}}(\mathcal{Y})) \simeq \mathrm{QCoh}(\mathcal{Y}),$$

as a module over itself.

B.8.6. A prestack $\mathcal{Y}$ is said to be *1-affine* if the functor $\Gamma(\mathcal{Y}, -)^{\mathrm{enh}}$ is an equivalence.

In the paper [Ga5] a number of results was proved, showing that various classes of prestacks are 1-affine.

The most relevant for us are:

- The stack pt/H, where H is a (finite-dimensional) algebraic group, is 1-affine.
- The stack Z_{dR}, where Z is a scheme of finite type, is 1-affine.

B.8.7. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between prestacks. We define a functor

$$f^* : \mathrm{ShvCat}(\mathcal{Y}_2) \to \mathrm{ShvCat}(\mathcal{Y}_1)$$

by sending $\underline{\mathbf{C}} \in \mathrm{ShvCat}(\mathcal{Y}_2)$ to $f^*(\underline{\mathbf{C}}) \in \mathrm{ShvCat}(\mathcal{Y}_1)$ that assigns

$$(S, y) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_1} \rightsquigarrow \mathbf{C}_{S, f\circ y} \in \mathrm{QCoh}(S)\text{-}\mathbf{mod}.$$

The above functor f^* has a right adjoint, denoted f_*. Explicitly, f_* sends $\underline{\mathbf{C}} \in \mathrm{ShvCat}(\mathcal{Y}_1)$ to $f_*(\underline{\mathbf{C}}) \in \mathrm{ShvCat}(\mathcal{Y}_2)$ that assigns

$$(S, y) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}_2} \rightsquigarrow \Gamma(S \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_2, y'^*(\underline{\mathbf{C}})) \in \mathrm{QCoh}(S \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_2)\text{-}\mathbf{mod} \xrightarrow{f'^*} \mathrm{QCoh}(S)\text{-}\mathbf{mod},$$

where y' denotes the map

$$S \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_2 \to \mathcal{Y}_1$$

and f' denotes the map

$$S \underset{\mathcal{Y}_1}{\times} \mathcal{Y}_2 \to S.$$

B.8.8. Let $\underline{\mathbf{C}}$ be a sheaf of categories over $\mathcal{Y}$. We shall say that $\underline{\mathbf{C}}$ is compactly generated if for every $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$, the category $\mathbf{C}_S$ is compactly generated.

B.8.9. Suppose for a moment that $\underline{\mathbf{C}}$ is pulled back from a sheaf of categories on $\mathcal{Y}_{\mathrm{dR}}$. Hence, for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$ we have a well-defined category $\mathbf{C}_{S_{\mathrm{dR}}}$. Note that if $\mathbf{C}_S$ is compactly generated, then so is $\mathbf{C}_{S_{\mathrm{dR}}}$:

Indeed, with no restriction of generality we can assume that S is eventually coconnective. By the 1-affineness of S_{dR}, we have

$$\mathbf{C}_S \simeq \mathrm{QCoh}(S) \underset{\mathrm{D\text{-}mod}(S)}{\otimes} \mathbf{C}_{S_{\mathrm{dR}}},$$

and the pair of (D-mod(S)-linear) adjoint functors

$$\mathbf{ind}^l : \mathrm{QCoh}(S) \rightleftarrows \mathrm{D\text{-}mod}(S) : \mathbf{oblv}^l$$

induces an adjoint pair

$$(\mathbf{ind}^l \otimes \mathrm{Id}) : \mathbf{C}_S \rightleftarrows \mathbf{C}_{S_{\mathrm{dR}}} : (\mathbf{oblv}^l \otimes \mathrm{Id}),$$

where the essential image of the left adjoint generates the target category.

B.8.10. Let $F : \underline{\mathbf{C}}_1 \to \underline{\mathbf{C}}_2$ be a functor between sheaves of categories. Suppose that $\underline{\mathbf{C}}_1$ is compactly generated (in the above sense).

We shall say that F preserves compactness if for every $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$, the corresponding functor

$$F_S : \mathbf{C}_{1,S} \to \mathbf{C}_{2,S}$$

preserves compactness.

Then the usual argument (using the fact that for an affine schemes S, the category QCoh(S) is rigid) shows that in this case the functor F admits a right adjoint, to be denoted F^R, as a functor between sheaves of categories over Ran.

B.8.11. Let $\underline{\mathbf{C}}$ be a sheaf of categories over $\mathcal{Y}$. A t-structure on $\mathbf{C}$ is a collection of t-structures on $\mathbf{C}_S$ for any $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$ such that:

For any $f : S' \to S$, the functor

$$f_* : \mathbf{C}_{S'} \to \mathbf{C}_S$$

is t-exact.

Note that this condition can be rewritten as saying that with respect to the identification

$$\mathbf{C}_{S'} \simeq \mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S,$$

the t-structure on $\mathbf{C}_{S'}$ is the *tensor product* t-structure, i.e., $\left(\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S\right)^{\leq 0}$ is generated under colimits by the essential image of

$$\mathrm{QCoh}(S')^{\leq 0} \times \mathbf{C}_S^{\leq 0} \to \mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S.$$

B.8.12. Let $\underline{\mathbf{C}}$ be equipped with a t-structure. For $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$, consider the category

$$\mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S.$$

We equip it with the tensor product t-structure.

Assume now that the t-structure on $\mathbf{C}_S$ is right-complete and compatible with filtered colimits. Then a standard argument shows that the functor

$$\mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S \xrightarrow{\Psi_S \otimes \mathrm{Id}} \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S \simeq \mathbf{C}_S$$

is t-exact and induces an equivalence of the eventually coconnective subcategories of the two sides (see, e.g., [Lu3, Proposition C.4.6.1]).

B.8.13. Suppose again that $\underline{\mathbf{C}}$ is the pull back of a sheaf of categories on $\mathcal{Y}_{\mathrm{dR}}$. For S as above consider the category $\mathbf{C}_{S_{\mathrm{dR}}}$. Note that we can identify

$$\mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S \simeq \mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{QCoh}(S) \underset{\text{D-mod}(S)}{\otimes} \mathbf{C}_{S_{\mathrm{dR}}} \simeq \mathrm{IndCoh}(S) \underset{\text{D-mod}(S)}{\otimes} \mathbf{C}_{S_{\mathrm{dR}}}.$$

The pair of (D-mod(S)-linear) adjoint functors

$$\mathbf{ind}^r : \mathrm{IndCoh}(S) \rightleftarrows \text{D-mod}(S) : \mathbf{oblv}^r$$

gives rise to an adjucntion

$$(\mathbf{ind}^r \otimes \mathrm{Id}) : \mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S \rightleftarrows \mathbf{C}_{S_{\mathrm{dR}}} : (\mathbf{oblv}^r \otimes \mathrm{Id}).$$

We define a t-structure on $\mathbf{C}_{S_{\mathrm{dR}}}$ by letting $\mathbf{C}^{\leq 0}_{S_{\mathrm{dR}}}$ be generated under colimits by the essential image under $(\mathbf{ind}^r \otimes \mathrm{Id})$ of $(\mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S)^{\leq 0}$.

Assume for a moment that S is smooth. In this case the functor $\mathbf{ind}^r$ is t-exact, and hence the endofunctor $\mathbf{oblv}^r \circ \mathbf{ind}^r$ of $\mathrm{IndCoh}(S)$ is t-exact. This implies that the functor

$$(\mathbf{oblv}^r \otimes \mathrm{Id}) : \mathbf{C}_{S_{\mathrm{dR}}} \to \mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_S$$

is t-exact.

B.8.14. We now specialize the case of $\mathcal{Y} = \mathrm{Ran}$. We claim:

Lemma B.8.15. *The prestack* Ran *is 1-affine.*

Proof. The proof that we will give applies to any prestack $\mathcal{Y}$ that can be written as a colimit

$$\underset{i \in I}{\mathrm{colim}}\, (Z_i)_{\mathrm{dR}},$$

where Z_i are schemes and the transition maps are proper, and for which the diagonal morphism

$$\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$$

is closed (at the reduced level[65]).

We can think of an object of $\mathrm{ShvCat}(\mathcal{Y})$ as a compatible collection of categories $\{\mathbf{C}_i\}$

$$\mathbf{C}_i \in \text{D-mod}(Z_i), \quad (Z_i \overset{f_{i,j}}{\to} Z_j) \rightsquigarrow \mathbf{C}_i \simeq \text{D-mod}(Z_i) \underset{\text{D-mod}(Z_j)}{\otimes} \mathbf{C}_j.$$

The functor $\Gamma(\mathcal{Y}, -)$ sends such a collection to

$$\lim_{i \in I^{\mathrm{op}}} \mathbf{C}_i,$$

viewed as a module over

$$\mathrm{QCoh}(\mathcal{Y}) = \text{D-mod}(\mathcal{Y}) \simeq \lim_i \text{D-mod}(Z_i).$$

Given a D-mod$(\mathcal{Y})$-module category $\mathbf{D}$, we attach to it an object of $\mathrm{ShvCat}(\mathcal{Y})$ by setting

$$\mathbf{C}_i := \mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_i), \mathbf{D}).$$

For $(i \to j) \in I$, the corresponding transition functor

$$\mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_j), \mathbf{D}) \to \mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_i), \mathbf{D})$$

is given by precomposition with $(f_{i,j})_!$.

For $(i \to j) \in I$, we have, tautologically,

$$\mathrm{Funct}_{\text{D-mod}(Z_j)}(\text{D-mod}(Z_i), \mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_j), \mathbf{D})) \simeq \mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_i), \mathbf{D}).$$

However, this implies that we also have

$$\text{D-mod}(Z_i) \underset{\text{D-mod}(Z_j)}{\otimes} \mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_j), \mathbf{D}) \simeq \mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_i), \mathbf{D}),$$

[65]I.e., the base change of this morphism by an affine scheme yields a morphism of prestacks, such that the morphism of the underlying reduced prestacks is a closed embedding.

since D-mod(Z_i) is self-dual as a D-mod(Z_j)-module.

Let us establish the equivalence

$$\lim_{i\in I^{\mathrm{op}}} \mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_i), \mathbf{D}) \simeq \mathbf{D}. \tag{B.21}$$

Indeed, we can rewrite the left-hand side as

$$\mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\operatorname*{colim}_{i\in I} \text{D-mod}(Z_i), \mathbf{D}),$$

where the colimit is taken with respect to the $(f_{i,j})_!$-functors.

Now, we can rewrite

$$\operatorname*{colim}_{i\in I} \text{D-mod}(Z_i) \simeq \lim_{i\in I^{\mathrm{op}}} \text{D-mod}(Z_i),$$

where in the right-hand side the limit is taken with respect to the $f^!_{i,j}$ functors. Finally,

$$\operatorname*{colim}_{i\in I^{\mathrm{op}}} \text{D-mod}(Z_i) \simeq \text{D-mod}(\mathcal{Y}),$$

by definition. Hence, the left-hand side in (B.21) identifies with

$$\mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(\mathcal{Y}), \mathbf{D}) \simeq \mathbf{D},$$

as required.

It is easy to see that in order to show that the above two functors are mutually inverse, it remains to show that for $i_1, i_2 \in I$, and the corresponding maps

$$Z_{i_1} \overset{f_{i_1}}{\to} \mathcal{Y} \overset{f_{i_2}}{\leftarrow} Z_{i_2}$$

the naturally defined functor

$$\text{D-mod}(Z_{i_1} \underset{\mathcal{Y}}{\times} Z_{i_2}) \to \mathrm{Funct}_{\text{D-mod}(\mathcal{Y})}(\text{D-mod}(Z_{i_1}), \text{D-mod}(Z_{i_2})) \tag{B.22}$$

is an equivalence.

We calculate the right-hand side in (B.22) as the totalization of the cosimplicial category with terms

$$\text{D-mod}(Z_{i_1} \times \mathcal{Y}^m \times Z_{i_2}),$$

with terms given by !-pushforward functors along the maps in the corresponding cosimplicial prestack.

However, due to the assumption that $\mathcal{Y}$ has a closed diagonal, the face maps in this cosimplicial prestack are closed embeddings, and hence the corresponding !-pushforward functors are fully faithful. Hence, the above totalization is the equalizer of

$$\text{D-mod}(Z_{i_1} \times Z_{i_2}) \rightrightarrows \text{D-mod}(Z_{i_1} \times \mathcal{Y} \times Z_{i_2}).$$

Objects of this equalizer are supported on

$$(Z_{i_1} \times Z_{i_2}) \underset{\mathrm{Graph}_{f_{i_1}} \times \mathrm{id}, Z_{i_1} \times \mathcal{Y} \times Z_{i_2}, \mathrm{id} \times \mathrm{Graph}_{f_{i_2}}}{\times} (Z_{i_1} \times Z_{i_2}).$$

However, the above fiber product identifies with

$$Z_{i_1} \underset{\mathcal{Y}}{\times} Z_{i_2}.$$

□

Remark B.8.16. Let $\underline{\mathbf{C}}$ be a sheaf of categories over Ran. In the course of the proof of Lemma B.8.15, we have encountered another way of how one may think of the category $\Gamma(\mathrm{Ran}, \underline{\mathbf{C}})$. Namely,

$$\Gamma(\mathrm{Ran}, \underline{\mathbf{C}}) \simeq \operatorname*{colim}_{I\in(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}}} \mathbf{C}_{X^I_{\mathrm{dR}}},$$

where the colimit is formed using the !-pushforward functors, i.e., for $(I \overset{\phi}{\to} J) \in \mathrm{fSet}^{\mathrm{surj}}$ and the corresponding map

$$\Delta_\phi : X^J \to X^I,$$

the functor in question is

$$\mathbf{C}_{X^J_{\mathrm{dR}}} \simeq \text{D-mod}(X^J) \underset{\text{D-mod}(X^I)}{\otimes} \mathbf{C}_{X^I_{\mathrm{dR}}} \overset{(\Delta_\phi)_!\otimes \mathrm{Id}}{\longrightarrow} \text{D-mod}(X^I) \underset{\text{D-mod}(X^I)}{\otimes} \mathbf{C}_{X^I_{\mathrm{dR}}} \simeq \mathbf{C}_{X^I_{\mathrm{dR}}}.$$

B.8.17. *Notational convention.* We use the term *crystal of categories* for sheaves of categories on $\mathcal{Y}_{\mathrm{dR}}$ and denote

$$\mathrm{CrystCat}(\mathcal{Y}) := \mathrm{ShvCat}(\mathcal{Y}_{\mathrm{dR}}).$$

From this perspective, we refer to crystalline objects of $\underline{\mathbf{C}}$ in $\mathrm{CrystCat}(\mathcal{Y})$ to mean the objects of the category of global sections of $\underline{\mathbf{C}}$ as a sheaf of categories on $\mathcal{Y}_{\mathrm{dR}}$.

When we use such "crystalline" terminology and are given $f : Z \to \mathcal{Y}$, we use the symbol $\mathbf{C}_Z$ to denote the crystalline objects of $\underline{\mathbf{C}}$ restricted to Z, that is,

$$\mathbf{C}_{\mathcal{Z}} := \Gamma(\mathcal{Z}_{\mathrm{dR}}, (f_{\mathrm{dR}})^*(\underline{\mathbf{C}})).$$

B.9. **Factorization algebras and modules.**

B.9.1. A factorization algebra $\mathcal{A}$ on X is an object

$$\mathcal{A}_{\mathrm{Ran}} \in \text{D-mod}(\mathrm{Ran}),$$

equipped with a datum of factorization

$$\mathrm{union}^!(\mathcal{A}_{\mathrm{Ran}})|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}} \simeq \mathcal{A}_{\mathrm{Ran}} \boxtimes \mathcal{A}_{\mathrm{Ran}}|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}} \tag{B.23}$$

and with a homotopy-coherent data of associativity and commutativity, see [Ra6, Sect. 6] for details.

Let $\mathrm{FactAlg}(X)$ denote the category of factorization algebras on X.

For $\mathcal{A} \in \mathrm{FactAlg}(X)$ and $\mathcal{Z} \to \mathcal{A}$, we will denote by $\mathcal{A}_{\mathcal{Z}} \in \text{D-mod}(\mathcal{Z})$ the pullback of $\mathcal{A}$ to $\mathcal{Z}$.

B.9.2. *Example.* We let k denote the unit factorization algebra. I.e., the underlying object

$$k_{\mathrm{Ran}} \in \text{D-mod}(\mathrm{Ran})$$

is ω_{Ran}, equipped with the natural factorization structure.

B.9.3. Let $\mathcal{A}$ be a factorization algebra on X. Let $\mathcal{Z}$ be a prestack equipped with a map $\mathcal{Z} \to \mathrm{Ran}$. Recall the space $\mathcal{Z}^{\subseteq}$, see Sect. B.2.2.

A factorization $\mathcal{A}$-module $\mathcal{M}$ *at* $\mathcal{Z}$ is an object

$$\mathcal{M}_{\mathcal{Z}^{\subseteq}} \in \mathrm{QCoh}(\mathcal{Z}^{\subseteq})$$

equipped with a datum of factorization against $\mathcal{A}$:

$$\mathcal{M}_{\mathcal{Z}^{\subseteq}}|_{(\mathrm{Ran}\times\mathcal{Z}^{\subseteq})_{\mathrm{disj}}} \simeq (\mathcal{A}_{\mathrm{Ran}} \boxtimes \mathcal{M}_{\mathcal{Z}^{\subseteq}})|_{(\mathrm{Ran}\times\mathcal{Z}^{\subseteq})_{\mathrm{disj}}} \tag{B.24}$$

and a homotopy-coherent data of associativity.

We denote the category of factorization $\mathcal{A}$-modules *at* $\mathcal{Z}$ by

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}.$$

This category is naturally tensored over $\mathrm{QCoh}(\mathcal{Z})$ via the projection

$$\mathcal{Z}^{\subseteq} \overset{\mathrm{pr}_{\mathrm{small},\mathcal{Z}}}{\longrightarrow} \mathcal{Z}.$$

Remark B.9.4. For $\mathcal{Z} = X^I$, one can described the category $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{X^I}$ explicitly as chiral modules, see [Ro1, Sect. 3].

B.9.5. Let $\mathrm{diag}_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}^{\subseteq}$ be the diagonal map (see Sect. 11.2.6). For $\mathcal{M} \in \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}$ we will denote by $\mathcal{M}_{\mathcal{Z}}$ the object

$$\mathrm{diag}^*_{\mathcal{Z}}(\mathcal{M}_{\mathcal{Z}^{\subseteq}}) \in \mathrm{QCoh}(\mathcal{Z}).$$

We will refer to $\mathcal{M}_{\mathcal{Z}}$ as the *quasi-coherent sheaf on* $\mathcal{Z}$ *underlying* $\mathcal{M}$.

The resulting functor

$$\mathbf{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \mathrm{QCoh}(\mathcal{Z}) \tag{B.25}$$

is conservative and compatible with colimits. (Sometimes instead of $\mathbf{oblv}_{\mathcal{A}}$ we write $\mathbf{oblv}_{\mathcal{A},\mathcal{Z}}$ in order to emphasize the dependence on $\mathcal{Z}$.)

We will think of the datum of $\mathcal{M}$ as $\mathcal{M}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$, equipped with an additional datum of factorization against $\mathcal{A}$.

B.9.6. Take $\mathcal{A} = k$ from Sect. B.9.2. We claim that there is a naturally defined functor

$$\mathrm{QCoh}(\mathcal{Z}) \to k\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}. \tag{B.26}$$

Namely, the functor (B.26) is given by pullback along the projection

$$\mathcal{Z}^{\subseteq} \to \mathcal{Z}.$$

In Sect. C.7.7, we will upgrade the functor (B.26) to an equivalence of categories, once we replace the right-hand side by the category of *unital* factorization modules.

B.9.7. A basic example of a factorization $\mathcal{A}$-module is the *vacuum* module, denoted

$$\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}} \in \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}.$$

Namely, the corresponding object

$$\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}_{\mathcal{Z}^{\subseteq}} \in \mathrm{QCoh}(\mathcal{Z}^{\subseteq})$$

is the pullback of $\mathcal{A}_{\mathrm{Ran}}$ along the map

$$\mathcal{Z}^{\subseteq} \xrightarrow{\mathrm{pr}_{\mathrm{big}}} \mathrm{Ran}.$$

Note that

$$\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}_{\mathcal{Z}} \simeq \mathcal{A}_{\mathcal{Z}}.$$

For that reason, we will sometimes abuse the notation and write $\mathcal{A}_{\mathcal{Z}}$ instead of $\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}$. (This is similar to denoting the free object over an associative algebra A by the same symbol A as the underlying vector space.)

B.9.8. Generalizing the above construction, given $\mathcal{Z} \to \mathrm{Ran}$ and $f : \mathcal{Z}' \to \mathcal{Z}^{\subseteq}$, which we perceive as mapping to Ran by means of

$$\mathcal{Z}' \to \mathcal{Z}^{\subseteq} \overset{\mathrm{pr}_{\mathrm{big}}}{\to} \mathrm{Ran},$$

there is a naturally defined functor[66]

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}'} \tag{B.27}$$

that makes the following diagram commute

$$\begin{tikzcd}
\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}'} \arrow[r, "\mathbf{oblv}_{\mathcal{A},\mathcal{Z}'}"] & \mathrm{QCoh}(\mathcal{Z}') \\
\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \arrow[u] \arrow[r] & \mathrm{QCoh}(\mathcal{Z}^{\subseteq}). \arrow[u, "f^!"']
\end{tikzcd}$$

We will denote the functor (B.27) by

$$\mathcal{M} \mapsto \mathcal{M}|_{\mathcal{Z}'}.$$

[66]See Sect. C.11.13 for more details.

Remark B.9.9. The functor (B.27) reflects the *unital structure* on the assugnment

$$\mathcal{Z} \mapsto \mathcal{A}\text{-mod}^{\text{fact}}_{\mathcal{Z}},$$

see Sect. C.11.13.

B.9.10. Let S be an affine scheme mapping to Ran, and let $S' \xrightarrow{f} S$ be a map of affine schemes. Pullback along

$$f^{\subseteq} : S'^{\subseteq} \to S^{\subseteq}$$

defines a functor

$$f^* : \mathcal{A}\text{-mod}^{\text{fact}}_S \to \mathcal{A}\text{-mod}^{\text{fact}}_{S'}. \tag{B.28}$$

This functor is $\text{QCoh}(S)$-linear, and hence gives rise to a functor

$$\text{QCoh}(S') \underset{\text{QCoh}(S)}{\otimes} \mathcal{A}\text{-mod}^{\text{fact}}_S \to \mathcal{A}\text{-mod}^{\text{fact}}_{S'}. \tag{B.29}$$

We claim:

Lemma B.9.11. *The functor* (B.29) *is an equivalence.*

Proof. Pushforward along $f^{\subseteq}$ gives rise to a functor

$$f_* : \mathcal{A}\text{-mod}^{\text{fact}}_{S'} \to \mathcal{A}\text{-mod}^{\text{fact}}_S,$$

which is a right adjoint of (B.28). It is easy to see that it is monadic.

The functor (B.29) gives rise to a map of monads, and in order to prove the lemma, it suffices to show that this map of monads induces an isomorphism of the underlying endofunctors.

However, the latter is obvious: both endofunctors are given by tensoring by $f_*(\mathcal{O}_{S'})$, viewed as an algebra on $\text{QCoh}(S)$.

□

B.9.12. From Lemma B.9.11 we obtain that the assignment

$$(S \to \text{Ran}) \rightsquigarrow \mathcal{A}\text{-mod}^{\text{fact}}_S$$

forms a sheaf of categories over Ran.

We will denote this sheaf of categories by $\mathcal{A}\text{-}\underline{\text{mod}}^{\text{fact}}$. For any $f : \mathcal{Z} \to \text{Ran}$, we have

$$\mathcal{A}\text{-mod}^{\text{fact}}_{\mathcal{Z}} \simeq \Gamma(\mathcal{Z}, f^*(\mathcal{A}\text{-}\underline{\text{mod}}^{\text{fact}})).$$

B.9.13. We shall say that a factorization algebra $\mathcal{A}$ is *connective* if

$$\mathbf{oblv}^l(\mathcal{A}_X) \in \text{QCoh}(X)$$

is connective, where $X \to \text{Ran}$ is the tautological map.

B.9.14. Recall that for any finite non-empty set, we have a tautological map

$$X^I_{\text{dR}} \to \text{Ran}.$$

We have (by the argument in [BD2, Lemma 3.4.12]):

Lemma B.9.15. *Suppose that $\mathcal{A}$ is connective in the above sense, and* unital *(see Sect. C.7.1) for what this means. Then for any I, the object*

$$\mathbf{oblv}^l(\mathcal{A}_{X^I}) \in \text{QCoh}(X^I)$$

is connective.

Corollary B.9.16. *Under the assumptions of Lemma B.9.15, for any* scheme Z *equipped with a map to* Ran, *the object*

$$\mathcal{A}_Z \in \text{QCoh}(Z)$$

is connective.

B.9.17. We now claim:

Proposition B.9.18. *Let $\mathcal{A}$ be connective.*[67] *Then for any scheme Z mapping to* Ran*, the category $\mathcal{A}\text{-mod}^{\text{fact}}_Z$ carries a t-structure, uniquely characterized by the property that the forgetful functor* (B.25) *is t-exact. Furthermore, $\mathcal{A}\text{-mod}^{\text{fact}}_Z$ is left-complete in its t-structure.*

Proof. This follows by interpreting factorization modules as chiral modules, see [FraG]. The connectivity assumption translates into the fact that the corresponding chiral algebra is connective (for the *right* t-structure on D-mod(X)):

Namely, we consider the category $\text{QCoh}(Z^{\subseteq})$ as acted on via the map union by D-mod(Ran), equipped with the *chiral* symmetric monoidal structure (see [BD2, Sect. 3.4.10] and/or [FraG, Sect 2.2]).

Let

$$\mathcal{A}^{\text{ch}} \in \text{D-mod}(X) \subset \text{D-mod}(\text{Ran})$$

be the chiral algebra, corresponding to $\mathcal{A}$ (see Sect. D.1.1); it has a structure of Lie algebra, viewed as an object of D-mod(Ran) in the chiral symmetric monoidal structure.

We can interpret $\mathcal{A}\text{-mod}^{\text{fact}}_Z$ as the full subcategory of

$$\mathcal{A}^{\text{ch}}\text{-mod}(\text{QCoh}(Z^{\subseteq})),$$

consisting of objects with set-theoretic support on $Z \overset{\text{diag}_Z}{\to} Z^{\subseteq}$ (note that the map diag_Z identifies Z with its formal completion inside $Z^{\subseteq}$).

Now the assertion of the proposition follows from Lemma B.9.20 below.

□

B.9.19. Let $\mathbf{C}$ be a DG category with a t-structure compatible with filtered colimits, and suppose we are given an embedding $\mathbf{C} \subset \mathbf{C}'$. Let $\mathbf{A}$ be a symmetric monoidal (resp., monoidal) category acting on $\mathbf{C}'$; denote the action functor by $\star$.

Let L (resp., A) be a Lie algebra (resp., associative) in $\mathbf{A}$ with the property that for every $\mathbf{c}_1 \in \mathbf{C}^{\leq 0}$ and every $\mathbf{c}_2 \in \mathbf{C}^{>0}$, $\text{Maps}(L \star \mathbf{c}_1, \mathbf{c}_2) = 0$ (resp., $\text{Maps}(A \star \mathbf{c}_1, \mathbf{c}_2) = 0$).

Lemma B.9.20. *Under the above circumstances, the category $L\text{-mod}(\mathbf{C}) := L\text{-mod}(\mathbf{C}') \underset{\mathbf{C}'}{\times} \mathbf{C}$ (resp., $A\text{-mod}(\mathbf{C}) := A\text{-mod}(\mathbf{C}') \underset{\mathbf{C}'}{\times} \mathbf{C}$) admits a unique t-structure for which the forgetful functor to $\mathbf{C}$ is t-exact.*

Remark B.9.21. In the terminology of Sect. B.8.11, the assertion of Proposition B.9.18 is that if $\mathcal{A}$ is connective, the sheaf of categories $\mathcal{A}\text{-}\underline{\text{mod}}^{\text{fact}}$ carries a t-structure, for which the forgetful functor $\mathbf{oblv}_C A$ is conservative.

Remark B.9.22. An assertion parallel to Proposition B.9.18 holds for the category

$$\mathcal{A}\text{-mod}^{\text{fact}}_{Z_{\text{dR}}},$$

i.e., for crystalline factorization $\mathcal{A}$-modules at Z. In this case, we consider $\text{QCoh}(Z_{\text{dR}}) \simeq \text{D-mod}(Z)$ equipped with the *right* t-structure, i.e., one for which the functor $\mathbf{ind}^r$ is t-exact.

Remark B.9.23. Assume for a moment that $\mathcal{A}$ is such that $\mathbf{oblv}^l(\mathcal{A}_X)$ belongs to $\text{QCoh}(X)^{\heartsuit}$ and is flat. Then [BD2, Lemma 3.4.12] implies that the same is true for $\mathbf{oblv}^l(\mathcal{A}_{X^I})$ for all finite sets I.

This in turn implies that for any $S \to \text{Ran}$ with $S \in \text{Sch}^{\text{aff}}$, the object $\mathcal{A}_S \in \text{QCoh}(S)$ also belongs to $\text{QCoh}(S)^{\heartsuit}$ and is flat.

[67]The unitality assumption on $\mathcal{A}$ here is irrelevant, as one can add a unit to it (and then consider the corresponding category of unital modules).

B.9.24. Let now $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism of factorization algebras. Let

$$\mathcal{M}_i \in \mathcal{A}_i\text{-mod}^{\text{fact}}_{\mathcal{Z}}.$$

There is a natural notion of map $\mathcal{M}_1 \to \mathcal{M}_2$, compatible with factorization. Namely, this is a map

$$\phi_m : \mathcal{M}_{1,\mathcal{Z}^{\subseteq}} \to \mathcal{M}_{2,\mathcal{Z}^{\subseteq}}$$

in $\text{QCoh}(\mathcal{Z}^{\subseteq})$, which makes the following diagram commute,

$$\begin{tikzcd}
\mathcal{M}_{1,\mathcal{Z}^{\subseteq}}|_{(\text{Ran}\subset\mathcal{Z}^{\subseteq})} \arrow[r, "\sim"] \arrow[d, "\phi_m"'] & (\mathcal{A}_{1,\text{Ran}} \boxtimes \mathcal{M}_{1,\mathcal{Z}^{\subseteq}})|_{(\text{Ran}\subset\mathcal{Z}^{\subseteq})} \arrow[d, "\phi\boxtimes\phi_m"] \\
\mathcal{M}_{2,\mathcal{Z}^{\subseteq}}|_{(\text{Ran}\subset\mathcal{Z}^{\subseteq})} \arrow[r, "\sim"] & (\mathcal{A}_{2,\text{Ran}} \boxtimes \mathcal{M}_{2,\mathcal{Z}^{\subseteq}})|_{(\text{Ran}\subset\mathcal{Z}^{\subseteq})},
\end{tikzcd}$$

along with a homotopy-coherent system of higher compatibilities.

Denote the corresponding space of maps by

$$\text{Maps}_{\mathcal{A}_1\to\mathcal{A}_2}(\mathcal{M}_1, \mathcal{M}_2). \tag{B.30}$$

B.9.25. For a fixed $\mathcal{M}_2 \in \mathcal{A}_2\text{-mod}^{\text{fact}}_{\mathcal{Z}}$, we can consider the functor

$$(\mathcal{A}_1\text{-mod}^{\text{fact}}_{\mathcal{Z}})^{\text{op}} \to \text{Spc},$$

given by

$$\mathcal{M}_1 \mapsto \text{Maps}_{\mathcal{A}_1\to\mathcal{A}_2}(\mathcal{M}_1, \mathcal{M}_2). \tag{B.31}$$

This functor sends colimits to limits, and hence is representable. We denote the representing object by

$$\text{Res}_\phi(\mathcal{M}_2) \in \mathcal{A}_1\text{-mod}^{\text{fact}}_{\mathcal{Z}}.$$

We have a tautologically defined object in $\text{Maps}_{\mathcal{A}_1\to\mathcal{A}_2}(\text{Res}_\phi(\mathcal{M}_2), \mathcal{M}_2)$, i.e., a map

$$\text{Res}_\phi(\mathcal{M}_2) \to \mathcal{M}_2, \tag{B.32}$$

compatible with factorization.

We claim:

Lemma B.9.26. *The map* (B.32) *induces an isomorphism*

$$\text{Res}_\phi(\mathcal{M}_2)_{\mathcal{Z}} \to \mathcal{M}_{2,\mathcal{Z}}.$$

Proof. This also follows by interpreting factorization modules as chiral modules. Under this equivalence, restriction corresponds to restriction of modules along a homomorphism of chiral algebras.

□

Remark B.9.27. Let us emphasize that Lemma B.9.26 says that the operation of restriction acts as identity on the underlying object $\text{QCoh}(\mathcal{Z})$, and its content is that one can restrict the factorization action on it of $\mathcal{A}_2$ to obtain a factorization action of $\mathcal{A}_1$.

B.9.28. *Example.* One can describe the object $\text{Res}_\phi(\mathcal{M}_2)_{\mathcal{Z}^{\subseteq}}$ explicitly. Let us consider the example of $\mathcal{Z} = \text{pt}$, where $\mathcal{Z} \to \text{Ran}$ corresponds to a singleton $x \in X$, so that $\mathcal{Z}^{\subseteq} = \text{Ran}_x$.

We have a natural map

$$X \to \text{Ran}_x, \quad x' \mapsto x \cup x',$$

and let us describe the restriction of $\text{Res}_\phi(\mathcal{M}_2)$ along this map.

Let j denote the embedding $X - x \hookrightarrow X$. Then

$$\text{Res}_\phi(\mathcal{M}_2)_X \simeq \mathcal{M}_{2,X} \underset{j_*\circ j^*(\mathcal{A}_2\boxtimes\mathcal{M}_{2,x})}{\times} j_* \circ j^*(\mathcal{A}_1 \boxtimes \mathcal{M}_{2,x}),$$

where the map

$$\mathcal{M}_{2,X} \to j_* \circ j^*(\mathcal{A}_2 \boxtimes \mathcal{M}_{2,x})$$

is

$$\mathcal{M}_{2,X} \overset{\text{adjunction}}{\longrightarrow} j_* \circ j^*(\mathcal{M}_{2,X}) \overset{\text{factorization}}{\simeq} j_* \circ j^*(\mathcal{A}_2 \boxtimes \mathcal{M}_{2,x}).$$

Using a description of $\mathrm{Res}_\phi(\mathcal{M}_2)_{\mathcal{Z}^\subseteq}$ along these lines, one can prove Lemma B.9.26 without resorting to chiral algebras.

B.9.29. *Example.* Let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map of factorization algebras. For $\mathcal{Z} \to \mathrm{Ran}$, consider the resulting map

$$\mathrm{pr}^*_{\mathrm{big}}(\mathcal{A}_{1,\mathrm{Ran}}) \to \mathrm{pr}^*_{\mathrm{big}}(\mathcal{A}_{2,\mathrm{Ran}})$$

in $\mathrm{QCoh}(\mathcal{Z}^\subseteq)$.

It is easy to see that this map is compatible with factorization, i.e., gives rise to a map

$$\phi : \mathcal{A}_1^{\mathrm{fact}_\mathcal{Z}} \to \mathcal{A}_2^{\mathrm{fact}_\mathcal{Z}},$$

compatible with factorization.

In particular, we obtain a map

$$\mathcal{A}_1^{\mathrm{fact}_\mathcal{Z}} \to \mathrm{Res}_\phi(\mathcal{A}_2^{\mathrm{fact}_\mathcal{Z}})$$

in $\mathcal{A}_1\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}$.

B.10. **Commutative factorization algebras.**

B.10.1. The category $\mathrm{FactAlg}(X)$ has an evident (pointwise) symmetric monoidal structure. Consider the category

$$\mathrm{ComAlg}(\mathrm{FactAlg}(X)).$$

We will refer to its objects as *commutative factorization algebras.*

We have a tautological forgetful functor

$$\mathrm{ComAlg}(\mathrm{FactAlg}(X)) \to \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran})) \tag{B.33}$$

B.10.2. Restriction along $X \to \mathrm{Ran}$ defines a functor

$$\mathrm{ComAlg}(\mathrm{FactAlg}(X)) \to \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran})) \to \mathrm{ComAlg}(\text{D-mod}(X)). \tag{B.34}$$

This functor has a right inverse, to be denoted

$$A \mapsto \mathrm{Fact}(A)$$

.

Remark B.10.3. In Sect. C.8 we will upgrade the functor $\mathrm{Fact}(-)$ to an equivalence of categories.

B.10.4. The object

$$\mathrm{Fact}(A)_{\mathrm{Ran}} \in \text{D-mod}(\mathrm{Ran})$$

can described by an explicit colimit procedure:

Let $\mathrm{TwArr}(\mathrm{fSet}^{\mathrm{surj}})$ be the twisted arrows category of $\mathrm{fSet}^{\mathrm{surj}}$. I.e., its objects are

$$I \overset{\phi}{\to} J$$

and morphisms $(I_1 \overset{\phi_1}{\to} J_1) \to (I_2 \overset{\phi_2}{\to} J_2)$ are diagrams

$$\begin{tikzcd} I_1 \arrow[r, "\phi_1"] \arrow[d, "\psi_I"'] & J_1 \\ I_2 \arrow[r, "\phi_2"] & J_2. \arrow[u, "\psi_J"'] \end{tikzcd} \tag{B.35}$$

We define a functor

$$A_{\mathrm{TwArr}} : \mathrm{TwArr}(\mathrm{fSet}^{\mathrm{surj}}) \to \text{D-mod}(\mathrm{Ran})$$

be sending $I \overset{\phi}{\to} J$ to the direct image along

$$\Delta_{X^J,\mathrm{Ran}} : X^J_{\mathrm{dR}} \to \mathrm{Ran}$$

of

$$\underset{j\in J}{\boxtimes} A^{\otimes \phi^{-1}(j)}.$$

For a morphism (B.35), the corresponding map

$$(\Delta_{X^{J_1},\mathrm{Ran}})_!(\underset{j_1\in J_1}{\boxtimes} A^{\otimes \phi_1^{-1}(j_1)}) \to (\Delta_{X^{J_2},\mathrm{Ran}})_!(\underset{j_2\in J_2}{\boxtimes} A^{\otimes \phi_2^{-1}(j_2)})$$

is

$$\begin{aligned}(\Delta_{X^{J_1},\mathrm{Ran}})_!(\underset{j_1\in J_1}{\boxtimes} A^{\otimes \phi_1^{-1}(j_1)}) \simeq (\Delta_{X^{J_2},\mathrm{Ran}})_! \circ (\Delta_{\psi_J})_!(\underset{j_1\in J_1}{\boxtimes} A^{\otimes \phi_1^{-1}(j_1)}) \simeq \\ \simeq (\Delta_{X^{J_2},\mathrm{Ran}})_! \circ (\Delta_{\psi_J})_! \circ (\Delta_{\psi_J})^!(\underset{j_2\in J_2}{\boxtimes} A^{\otimes (\phi_2\circ\psi_I)^{-1}(j_2)}) \to (\Delta_{X^{J_2},\mathrm{Ran}})_!(\underset{j_2\in J_2}{\boxtimes} A^{\otimes (\phi_2\circ\psi_I)^{-1}(j_2)}) \simeq \\ \simeq (\Delta_{X^{J_2},\mathrm{Ran}})_!(\underset{j_2\in J_2}{\boxtimes} \underset{i_2\in\phi_2^{-1}(j_2)}{\otimes} A^{\otimes \psi_I^{-1}(i_2)}) \to (\Delta_{X^{J_2},\mathrm{Ran}})_!(\underset{j_2\in J_2}{\boxtimes} \underset{i_2\in\phi_2^{-1}(j_2)}{\otimes} A) \simeq \\ \simeq (\Delta_{X^{J_2},\mathrm{Ran}})_!(\underset{j_2\in J_2}{\boxtimes} A^{\otimes (\phi_2)^{-1}(j_2)}),\end{aligned}$$

where the arrow in the third line is the tensor product of the maps

$$A^{\otimes \psi_I^{-1}(i_2)} \to A,$$

given by the commutative algebra structure on A.

Then

$$\mathrm{Fact}(A)_{\mathrm{Ran}} \simeq \underset{\mathrm{TwArr}(\mathrm{fSet}^{\mathrm{surj}})}{\mathrm{colim}} A_{\mathrm{TwArr}}.$$

B.10.5. Let $\mathcal{A}$ be a commutative factorization algebra on X. Let $\mathcal{Z}$ be a prestack mapping to Ran. Denote

$$\mathcal{A}\text{-mod}^{\mathrm{com}}_{\mathcal{Z}} := \mathcal{A}_{\mathcal{Z}}\text{-mod}(\mathrm{QCoh}(\mathcal{Z})),$$

where we regard $\mathcal{A}_{\mathcal{Z}}$ as an object of $\mathrm{ComAlg}(\mathrm{QCoh}(\mathcal{Z}))$.

We will view the assignment

$$\mathcal{Z} \rightsquigarrow \mathcal{A}\text{-mod}^{\mathrm{com}}_{\mathcal{Z}}$$

as a sheaf of categories over Ran, which we will denote by $\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{com}}$.

B.10.6. Let $\mathcal{A}$ be of the form $\mathrm{Fact}(A)$, for $A \in \mathrm{ComAlg}(\mathrm{D\text{-}mod}(X)$. We claim that in this case there is a canonically defined functor of sheaves of categories,

$$\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{com}} \to \mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}},$$

i.e., a compatible collection of functors

$$\mathcal{A}\text{-mod}^{\mathrm{com}}_{\mathcal{Z}} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}. \tag{B.36}$$

In order to construct (B.36), it suffices to exhibit an object of $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}$, equipped with an action of $\mathcal{A}_{\mathcal{Z}} \in \mathrm{ComAlg}(\mathrm{QCoh}(\mathcal{Z}))$, where we view $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}$ as tensored over $\mathrm{QCoh}(\mathcal{Z})$.

The object in question is the vacuum module $\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}$, see Sect. B.9.7. Thus, we need to define an action of $\mathcal{A}_{\mathcal{Z}}$ on $\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}$. To do so, we can consider the universal case, i.e., $\mathcal{Z} = \mathrm{Ran}$.

Thus, we need to define an action of $\mathrm{pr}^*_{\mathrm{small}}(\mathcal{A}_{\mathrm{Ran}})$ on $\mathrm{pr}^*_{\mathrm{big}}(\mathcal{A}_{\mathrm{Ran}})$, compatible with the factorization. This map comes from the commutative algebra structure on $\mathcal{A}_{\mathrm{Ran}}$, and a homomorphism

$$\mathrm{pr}^*_{\mathrm{small}}(\mathcal{A}_{\mathrm{Ran}}) \to \mathrm{pr}^*_{\mathrm{big}}(\mathcal{A}_{\mathrm{Ran}}),$$

given by the *unital structure* on $\mathcal{A}$, see Sect. C.8.4 below.

Note that since

$$\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}_{\mathcal{Z}} \simeq \mathcal{A}_{\mathcal{Z}},$$

the functor (B.36) is compatible with the forgetful functors to $\mathrm{QCoh}(\mathcal{Z})$.

B.10.7. Let $\mathcal{A}$ be a commutative factorization algebra. Restriction along the binary operation

$$\mathcal{A} \overset{!}{\otimes} \mathcal{A} \to \mathcal{A}$$

in the sense of Sect. B.9.24 defines on $\mathcal{A}\text{-mod}^{\text{fact}}_{\mathcal{Z}}$ a structure of (symmetric) *pseudo-monoidal* category (see [BD2, Sect. 1.1]) for what this means.

Concretely, for $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N} \in \mathcal{A}\text{-mod}^{\text{fact}}_{\mathcal{Z}}$ this means that we know what it means to map

$$\mathcal{M}_1 \text{``}\otimes\text{''} \mathcal{M}_2 \to \mathcal{N}.$$

Namely, by definition, the space of such maps is

$$\text{Maps}_{\mathcal{A}\overset{!}{\otimes}\mathcal{A}\to\mathcal{A}}(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N})$$

in the notation of (B.30), where we regard $\mathcal{M}_1 \otimes \mathcal{M}_2$ as an object of $(\mathcal{A} \overset{!}{\otimes} \mathcal{A})\text{-mod}^{\text{fact}}_{\mathcal{Z}}$.

In particular, it makes sense to talk about (commutative) algebra objects in $\mathcal{A}\text{-mod}^{\text{fact}}_{\mathcal{Z}}$.

Note that by construction, the functor (B.36) is right-lax (pseudo)-monoidal. In particular, it maps (commutative) algebras to (commutative) algebras. In addition, the forgetful functor

$$\mathcal{A}\text{-mod}^{\text{fact}}_{\mathcal{Z}} \to \text{QCoh}(\mathcal{Z}^{\subseteq})$$

is also right-lax (pseudo)-monoidal; in particular, it maps (commutative) algebras to (commutative) algebras.

B.11. **Factorization categories.**

B.11.1. A factorization category $\mathbf{A}$ on X is a sheaf/crystal of categories $\underline{\mathbf{A}}$ over Ran, equipped with a datum of factorization

$$\text{union}^*(\underline{\mathbf{A}})|_{(\text{Ran}\times\text{Ran})_{\text{disj}}} \simeq \underline{\mathbf{A}} \boxtimes \underline{\mathbf{A}}|_{(\text{Ran}\times\text{Ran})_{\text{disj}}} \tag{B.37}$$

(here $-|_{-}$ denotes pullback of sheaves of categories along an open embedding) and with a homotopy-coherent datum of associativity and commutativity; see [Ra6, Sect. 6], where the definition is written out in detail.

For $f : \mathcal{Z} \to \text{Ran}$, we will denote

$$\mathbf{A}_{\mathcal{Z}} := \Gamma(\mathcal{Z}, f^*(\underline{\mathbf{A}})).$$

In particular, we denote

$$\mathbf{A}_{\text{Ran}} := \Gamma(\text{Ran}, \underline{\mathbf{A}});$$

this is a category tensored over D-mod(Ran).

Let $\text{FactCat}(X)$ denote the (2-)category of factorization categories over X. This category carries a naturally defined symmetric monoidal structure.

B.11.2. The unit object in this symmetric monoidal category is the sheaf of categories $\underline{\text{QCoh}}(\text{Ran})$.

By a slight abuse of notation, we will denote this factorization category by Vect (i.e., its fiber at any $\underline{x} \in \text{Ran}$ is $\text{Vect} \in \text{DGCat}$).

B.11.3. Note that since Ran is 1-affine (by Lemma B.8.15), the datum of the sheaf of categories $\underline{\mathbf{A}}$ is equivalent to that of the category $\mathbf{A}_{\text{Ran}}$, equipped with an action of D-mod(Ran).

Furthermore, the datum of factorization is equivalent to

$$\mathbf{A}_{\text{Ran}} \underset{\text{D-mod(Ran)},\text{union}^!}{\otimes} \text{D-mod}((\text{Ran}\times\text{Ran})_{\text{disj}}) \simeq \tag{B.38}$$
$$\simeq (\mathbf{A}_{\text{Ran}} \otimes \mathbf{A}_{\text{Ran}}) \underset{\text{D-mod(Ran)}\otimes\text{D-mod(Ran)}}{\otimes} \text{D-mod}((\text{Ran}\times\text{Ran})_{\text{disj}}),$$

equipped with a homotopy-coherent datum of associativity and commutativity.

B.11.4. Let $\mathbf{A}$ be a factorization category. A factorization algebra $\mathcal{A} \in \mathbf{A}$ is an object $\mathcal{A}_{\mathrm{Ran}} \in \mathbf{A}_{\mathrm{Ran}}$, equipped with a datum of factorization

$$\mathrm{union}^!(\mathcal{A}_{\mathrm{Ran}})|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}} \simeq (\mathcal{A}_{\mathrm{Ran}} \boxtimes \mathcal{A}_{\mathrm{Ran}})|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}},$$

as objects in the two sides of (B.37), and with a homotopy-coherent datum of associativity and commutativity.

Let $\mathrm{FactAlg}(X, \mathbf{A})$ denote the category of factorization algebras in $\mathbf{A}$.

A factorization functor $\mathbf{A}_1 \to \mathbf{A}_2$ induces a functor

$$\mathrm{FactAlg}(X, \mathbf{A}_1) \to \mathrm{FactAlg}(X, \mathbf{A}_2).$$

B.11.5. Given $\mathcal{A} \in \mathrm{FactAlg}(X, \mathbf{A})$, parallel to Sect. B.9.3, given $\mathcal{Z} \to \mathrm{Ran}$, one defines the category of factorization $\mathcal{A}$-modules *in* $\mathbf{A}$ *at* $\mathcal{Z}$, denoted

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}}.$$

The assignment

$$\mathcal{Z} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}} \tag{B.39}$$

forms a sheaf of categories over Ran that we denote by

$$\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathbf{A}).$$

B.11.6. We will see many examples of factorization categories in the sequel. However, one family of examples we can produce right away:

Let $\underline{\mathbf{A}}_X$ be a crystal of symmetric monoidal categories over X. To it we attach a (symmetric monoidal) factorization category, denoted $\mathrm{Fact}(\underline{\mathbf{A}}_X)$, see [GLys, Sect. 8.1].

Equivalently, let $\mathbf{A}_X$ be a symmetric monoidal category tensored over D-mod(X). To it we can attach a DG category, denoted $\mathrm{Fact}(\mathbf{A}_X)$, tensored over D-mod(Ran) (in fact, a commutative algebra object in D-mod(Ran)-**mod**), and equipped with a factorization structure as in Sect. B.11.3.

This can be done by directly mimicking the procedure in Sect. B.10.4.

Similarly, if $A \in \mathbf{A}_X$ is a commutative algebra object, it gives rise to a commutative algebra object

$$\mathrm{Fact}(A) \in \mathrm{FactAlg}(X, \mathrm{Fact}(\underline{\mathbf{A}}_X)).$$

B.11.7. Let A be an object in ComAlg(D-mod(X)). Denote $\mathcal{A} := \mathrm{Fact}(A) \in \mathrm{FactAlg}(X)$. It is easy to see that the sheaf of categories $\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{com}}$ (see Sect. B.10.5) has a natural factorization structure. We denote the resulting factorization category by $\mathcal{A}\text{-mod}^{\mathrm{com}}$.

Furthermore, we have:

$$\mathcal{A}\text{-mod}^{\mathrm{com}} \simeq \mathrm{Fact}(A\text{-mod}(\text{D-mod}(X))).$$

B.11.8. *Notational convention.* In the special case when $\mathbf{A}_X$ is constant, i.e., of the form $\mathbf{A} \otimes \text{D-mod}(X)$, where $\mathbf{A}$ is a symmetric monoidal category, we will denote the factorization category $\mathrm{Fact}(\mathbf{A}_X)$ simply by $\mathbf{A}$.

For example, when $\mathbf{A} = \mathrm{Rep}(\check{G})$, we will use the symbol $\mathrm{Rep}(\check{G})$ to denote the corresponding factorization category.

Note that this is in line with the notation for Vect in Sect. B.11.2.

B.11.9. Let $\mathbf{A}$ be a factorization category over X. We shall say that $\mathbf{A}$ is dualizable if for every $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$, the category $\mathbf{A}_S$ is dualizable. This is equivalent to $\underline{\mathbf{A}}$ being a dualizable object in $\mathrm{ShvCat}(\mathrm{Ran})$.

If $\mathbf{A}$ is dualizable, then the dual $\underline{\mathbf{A}}^\vee$ of $\underline{\mathbf{A}}$ as a sheaf of categories admits a natural factorization structure; we will denote the resulting factorization category by $\mathbf{A}^\vee$.

The datum of duality between two factorization categories $\mathbf{A}_1$ and $\mathbf{A}_2$ is equivalent to that of factorization functors

$$\mathrm{Vect} \to \mathbf{A}_1 \otimes \mathbf{A}_2 \text{ and } \mathbf{A}_1 \otimes \mathbf{A}_2 \to \mathrm{Vect},$$

satisfying the usual axioms.

B.11.10. Let $\mathbf{A}$ be a factorization category over X. We shall say that $\mathbf{A}$ is compactly generated if $\underline{\mathbf{A}}$ is compactly generated as a sheaf of categories over Ran.

Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a factorization functor between factorization categories. Suppose that $\mathbf{A}_1$ is compactly generated. Suppose also that Φ preserves compactness, so that Φ admits a right adjoint Φ^R, as a functor between sheaves of categories.

It follows automatically that Φ^R carries a structure of compatibility with factorization.

B.11.11. Let $\mathbf{A}$ be a factorization category over X. A t-structure on $\mathbf{A}$ is a t-structure on $\underline{\mathbf{A}}$ as a sheaf of categories that is compatible with factorization in the following sense:

Let us be given a map

$$(\underline{x}_1, \underline{x}_2) : S \to (\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}, \quad S \in \mathrm{Sch}^{\mathrm{aff}}.$$

We need that the factorization equivalence

$$\mathbf{A}_{S, \mathrm{union} \circ (\underline{x}_1, \underline{x}_2)} \simeq \mathbf{A}_{S, \underline{x}_1} \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{A}_{S, \underline{x}_2}$$

be t-exact, where the right-hand side is equipped with the tensor product t-structure.

B.11.12. A lax factorization category $\mathbf{A}$ is a sheaf of categories $\mathbf{A}_{\mathrm{Ran}}$ over Ran, equipped with *functors*

$$\underline{\mathbf{A}} \boxtimes \underline{\mathbf{A}}|_{(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}} \to \mathrm{union}^*(\underline{\mathbf{A}})|_{(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}}, \tag{B.40}$$

equipped with a homotopy-coherent datum of associativity and commutativity.

The entire discussion above equally applies to lax factorization categories.

B.11.13. In particular, we can talk about factorization algebras in a lax factorization category:

We require that the corresponding functor

$$(\mathbf{A}_{\mathrm{Ran}} \otimes \mathbf{A}_{\mathrm{Ran}}) \underset{\text{D-mod}(\mathrm{Ran}) \otimes \text{D-mod}(\mathrm{Ran})}{\otimes} \text{D-mod}((\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}) \to \tag{B.41}$$
$$\to \mathbf{A}_{\mathrm{Ran}} \underset{\text{D-mod}(\mathrm{Ran}), \mathrm{union}^!}{\otimes} \text{D-mod}((\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}),$$

maps the object

$$(\mathcal{A}_{\mathrm{Ran}} \boxtimes \mathcal{A}_{\mathrm{Ran}})|_{(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}} \in (\mathbf{A}_{\mathrm{Ran}} \otimes \mathbf{A}_{\mathrm{Ran}}) \underset{\text{D-mod}(\mathrm{Ran}) \otimes \text{D-mod}(\mathrm{Ran})}{\otimes} \text{D-mod}((\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}})$$

to the object

$$\mathrm{union}^!(\mathcal{A}_{\mathrm{Ran}})|_{(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}} \in \mathbf{A}_{\mathrm{Ran}} \underset{\text{D-mod}(\mathrm{Ran}), \mathrm{union}^!}{\otimes} \text{D-mod}((\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}).$$

B.11.14. *Example.* Let $\mathcal{A}$ be a factorization algebra on X. Note that the sheaf of categories $\mathcal{A}\text{-}\underline{\text{mod}}^{\text{fact}}$ carries a natural lax factorization structure:

Let us be given a pair of maps $\underline{x}_i : S_i \to \text{Ran}$, $i = 1, 2$ such that

$$S_1 \times S_2 \overset{\underline{x}_1 \times \underline{x}_2}{\longrightarrow} \text{Ran} \times \text{Ran}$$

lands in $(\text{Ran} \times \text{Ran})_{\text{disj}}$. Let us be given a pair of objects $\mathcal{M}_i \in \mathcal{A}\text{-mod}^{\text{fact}}_{S_i}$. We define the corresponding object

$$\mathcal{M}_1 \boxtimes \mathcal{M}_2 \in \mathcal{A}\text{-mod}^{\text{fact}}_{S_1 \times S_2}$$

as follows.

Consider the prestack

$$(S_1 \times S_2)^{\subseteq} = \{\underline{x}' \in \text{Ran}, \mid \underline{x}_1 \subseteq \underline{x}', \underline{x}_2 \subseteq \underline{x}'\}.$$

For $i = 1, 2$, let $U_i \subset (S_1 \times S_2)^{\subseteq}$ be the open sub-prestack corresponding to the condition that $(\underline{x}_i, \underline{x}')$ belongs to $(\text{Ran} \times \text{Ran})_{\text{disj}}$. The condition that $\underline{x}_1$ and $\underline{x}_2$ are disjoint implies that

$$U_1 \cup U_2 = (S_1 \times S_2)^{\subseteq}.$$

Denote $U_{1,2} := U_1 \cap U_2$.

Let pr_1 (resp., pr_2) denote the map $U_1 \to S_1 \times S_2^{\subseteq}$ (resp., $U_2 \to S_1^{\subseteq} \times S_2$) whose second (resp., first) component remembers the data of $\underline{x}$. Let $\text{pr}_{1,2}$ denote the map $U_{1,2} \to S_1 \times S_2 \times \text{Ran}$, whose last component remembers $\underline{x}'$.

We let

$$(\mathcal{M}_1 \boxtimes \mathcal{M}_2)_{(S_1 \times S_2)^{\subseteq}}|_{U_1} \simeq \text{pr}_1^*(\mathcal{M}_{1,S_1} \boxtimes \mathcal{M}_{2,S_2^{\subseteq}})$$

and

$$(\mathcal{M}_1 \boxtimes \mathcal{M}_2)_{(S_1 \times S_2)^{\subseteq}}|_{U_2} \simeq \text{pr}_2^*(\mathcal{M}_{1,S_1^{\subseteq}} \boxtimes \mathcal{M}_{2,S_2}).$$

The factorization structures on $\mathcal{M}_1$ and $\mathcal{M}_2$ imply that

$$(\mathcal{M}_1 \boxtimes \mathcal{M}_2)_{(S_1 \times S_2)^{\subseteq}}|_{U_1}|_{U_{1,2}} \simeq \text{pr}_{1,2}^*(\mathcal{M}_{1,S_1} \boxtimes \mathcal{M}_{2,S_2} \boxtimes \mathcal{A}_{\text{Ran}}) \simeq (\mathcal{M}_1 \boxtimes \mathcal{M}_2)_{(S_1 \times S_2)^{\subseteq}}|_{U_2}|_{U_{1,2}}.$$

This defines the object

$$(\mathcal{M}_1 \boxtimes \mathcal{M}_2)_{(S_1 \times S_2)^{\subseteq}} \in \text{QCoh}((S_1 \times S_2)^{\subseteq}).$$

The factorization structure on it against $\mathcal{A}$ follows from the construction.

B.11.15. Let us denote the resulting lax factorization category by $\mathcal{A}\text{-mod}^{\text{fact}}$. For a map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ between the factorization algebras, the restriction functor

$$\text{Res}_\phi : \mathcal{A}_2\text{-}\underline{\text{mod}}^{\text{fact}} \to \mathcal{A}_1\text{-}\underline{\text{mod}}^{\text{fact}}$$

upgrades to a factorization functor, denoted

$$\text{Res}_\phi : \mathcal{A}_2\text{-mod}^{\text{fact}} \to \mathcal{A}_1\text{-mod}^{\text{fact}}.$$

Denote by $\mathbf{oblv}_{\mathcal{A}}$ the forgetful functor $\mathcal{A}\text{-mod}^{\text{fact}} \to \text{Vect}$ (see Sect. B.11.8 above for the convention regarding Vect).

Note that the assignment

$$\mathcal{Z} \mapsto \mathcal{A}^{\text{fact}_{\mathcal{Z}}}$$

(see Sect. B.9.7) gives rise to a factorization algebra object in $\mathcal{A}\text{-mod}^{\text{fact}}$, to be denoted $\mathcal{A}^{\text{fact}}$, so that

$$\mathbf{oblv}_{\mathcal{A}}(\mathcal{A}^{\text{fact}}) = \mathcal{A}.$$

That said, we will sometimes abuse the notation and instead of $\mathcal{A}^{\text{fact}}$ simply write $\mathcal{A}$.

Suppose for moment that $\mathcal{A}$ is connective. Then the construction in Sect. B.9.17 equips $\mathcal{A}\text{-mod}^{\text{fact}}$ with a t-structure in the sense of Sect. B.11.11.[68]

[68]In the case of lax factorization categories, we require that the functor (B.40) be t-exact.

B.11.16. The example in Sect. B.11.14 generalizes to the situation when $\mathcal{A}$ is a factorization algebra in a given lax factorization category $\mathbf{A}$. We obtain that the sheaf of categories

$$\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathbf{A})$$

carries a structure of lax factorization category. We denote it by $\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})$.

As in Sect. B.11.15, we can consider $\mathcal{A}^{\mathrm{fact}}$ as an object of $\mathrm{FactAlg}(X, \mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathbf{A}))$. By a slight abuse of notation, we will sometimes denote this factorization algebra simply by $\mathcal{A}$.

By definition, for any $\mathcal{Z} \to \mathrm{Ran}$

$$(\mathcal{A}^{\mathrm{fact}})_{\mathcal{Z}} \simeq \mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}$$

as objects of $\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}}$.

If $\mathbf{A}$ carries a t-structure (in the sense of Sect. B.11.11) and $\mathcal{A}$ is connective (i.e., $\mathbf{oblv}^l(\mathcal{A}_X) \in \mathbf{A}_X$ is connective), then the construction in Sect. B.9.17 equips $\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})$ with a t-structure.

B.11.17. Let $\mathbf{A}$ be a lax factorization category, and let $\phi : \mathcal{A} \to \mathcal{A}'$ be a homomorphism between factorization algebras in $\mathbf{A}$. Restriction along ϕ denotes a factorization functor

$$\mathrm{Res}_\phi : \mathcal{A}'\text{-mod}^{\mathrm{fact}}(\mathbf{A}) \to \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A}).$$

In particular, we can consider

$$\mathrm{Res}_\phi(\mathcal{A}'^{\mathrm{fact}}) \in \mathrm{FactAlg}(X, \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})).$$

We have

$$\mathbf{oblv}_{\mathcal{A}}(\mathrm{Res}_\phi(\mathcal{A}'^{\mathrm{fact}})) = \mathcal{A}'.$$

We will sometimes abuse the notation, and instead of $\mathrm{Res}_\phi(\mathcal{A}'^{\mathrm{fact}})$ simply write $\mathcal{A}'$.

B.12. **Factorization module categories.**

B.12.1. Let $\mathbf{A}$ be a factorization category over X. Let $\mathcal{Z}$ be a prestack mapping to Ran. A factorization module category $\mathbf{C}$ *over* $\mathbf{A}$ *at* $\mathcal{Z}$ is a sheaf of categories $\underline{\mathbf{C}}$ on $\mathcal{Z}^{\subseteq}$, equipped with a factorization structure:

$$\mathrm{union}^*(\underline{\mathbf{C}})|_{(\mathrm{Ran}\times\mathcal{Z}^{\subseteq})_{\mathrm{disj}}} \simeq \underline{\mathbf{A}} \boxtimes \underline{\mathbf{C}}|_{(\mathrm{Ran}\times\mathcal{Z}^{\subseteq})_{\mathrm{disj}}} \tag{B.42}$$

and with a homotopy-coherent datum of associativity; see [Ra6, Sect. 6] for details.

Let $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$ denote the (2-)category of factorization module categories over $\mathbf{A}$ at $\mathcal{Z}$.

B.12.2. For an object $\mathbf{C} \in \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$ and $f : \mathcal{Z}' \to \mathcal{Z}^{\subseteq}$, we denote

$$\mathbf{C}_{\mathcal{Z}'} := \Gamma(\mathcal{Z}', f^*(\underline{\mathbf{C}})).$$

Taking $\mathcal{Z}' = \mathcal{Z}$ and $f = \mathrm{diag}_{\mathcal{Z}}$, we obtain the category $\mathbf{C}_{\mathcal{Z}}$, tensored over $\mathrm{QCoh}(\mathcal{Z})$. We will refer to $\mathbf{C}_{\mathcal{Z}}$ as *the category underlying* $\mathbf{C}$.

B.12.3. As in Sect. B.9.8, the above category $\mathbf{C}_{\mathcal{Z}'}$ is in fact the category underlying an $\mathbf{A}$-module category *at* $\mathcal{Z}'$, to be denoted $\mathbf{C}|_{\mathcal{Z}'}$.

B.12.4. *Example.* Repeating the construction in Sect. B.9.7, we obtain that for any $\mathcal{Z}$ there exists a distinguished object

$$\mathbf{A}^{\mathrm{fact}_{\mathcal{Z}}} \in \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}},$$

whose underlying category is $\mathbf{A}_{\mathcal{Z}}$.

We will refer to $\mathbf{A}^{\mathrm{fact}_{\mathcal{Z}}}$ as the *vacuum* factorization module category at $\mathcal{Z}$.

B.12.5. *Example.* Take $\mathbf{A} = \mathrm{Vect}$, and let $\underline{\mathbf{C}}_0$ be a sheaf of categories over $\mathcal{Z}$. We claim that it gives rise to a factorization module category over Vect at $\mathcal{Z}$, to be denoted $\mathbf{C}$ (cf. Sect. B.9.6).

Namely, the corresponding sheaf of categories $\underline{\mathbf{C}}$ over $\mathcal{Z}^{\subseteq}$ is the pullback of $\underline{\mathbf{C}}_0$ along the projection $\mathrm{pr}_{\mathrm{small}} : \mathcal{Z}^{\subseteq} \to \mathcal{Z}$.

B.12.6. Let $\mathcal{A}$ be a factorization algebra in $\mathbf{A}$, and let $\mathbf{C}$ be an object of $\mathbf{A}\text{-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}$.

A factorization $\mathcal{A}$-module $\mathcal{M}$ *in* $\mathbf{C}$ is an object

$$\mathcal{M}_{\mathcal{Z}^{\subseteq}} \in \mathbf{C}_{\mathcal{Z}^{\subseteq}},$$

equipped with an isomorphism

$$\text{union}^*(\mathcal{M}_{\mathcal{Z}^{\subseteq}})|_{(\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}} \simeq (\mathcal{A}_{\text{Ran}} \boxtimes \mathcal{M}_{\mathcal{Z}^{\subseteq}})|_{(\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}}, \tag{B.43}$$

where:

- $\text{union}^*(\mathcal{M}_{\mathcal{Z}^{\subseteq}})$ is an object in
$$\Gamma\left((\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}, \text{union}^*(\underline{\mathbf{C}})|_{(\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}}\right);$$
- $\mathcal{A}_{\text{Ran}} \boxtimes \mathcal{M}_{\mathcal{Z}^{\subseteq}}$ is an object in
$$\Gamma\left((\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}, \underline{\mathbf{A}} \boxtimes \underline{\mathbf{C}}|_{(\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}}\right);$$
- The isomorphisms between the two sides is understood in the sense of the identification (B.42).

The isomorphism (B.43) is required to be equipped with a homotopy-coherent datum of associativity.

B.12.7. We denote the category of factorization $\mathcal{A}$-modules in $\mathbf{C}$ by

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\mathcal{Z}}.$$

Note that when $\mathbf{C} := \mathbf{A}^{\text{fact}_{\mathcal{Z}}}$, recover the category (B.39).

B.12.8. Let $\underline{\mathbf{C}}_0$ be as in Sect. B.12.5. Denote

$$\mathbf{C}_0 := \Gamma(\mathcal{Z}, \underline{\mathbf{C}}_0).$$

Let $\mathcal{A}$ be a factorization algebra (i.e., a factorization algebra in Vect, viewed as a factorization category). Then it makes sense to consider the category

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\mathcal{Z}}.$$

Note that we have a tautologically defined functor

$$\mathcal{A}\text{-mod}^{\text{fact}}_{\mathcal{Z}} \underset{\text{QCoh}(\mathcal{Z})}{\otimes} \mathbf{C}_0 \to \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\mathcal{Z}}. \tag{B.44}$$

We claim:

Lemma B.12.9. *Assume that $\mathcal{Z} = S$ is an affine scheme, and assume that $\underline{\mathbf{C}}_0$ dualizable as a sheaf of categories. Then the functor* (B.44) *is an equivalence.*

Proof. Let $\underline{\mathbf{C}}_0^\vee$ be the dual sheaf of categories; note that $\Gamma(\mathcal{Z}, \underline{\mathbf{C}}_0^\vee)$ identifies with the dual of $\mathbf{C}_0$ as a QCoh(S)-linear category, and hence also as a plain DG category.

We have a naturally defined functor

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_S \underset{\text{QCoh}(S)}{\otimes} \mathbf{C}_0^\vee \to \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C}\otimes\mathbf{C}^\vee)_S,$$

where $\mathbf{C}^\vee \in \text{Vect-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}$ is attached to $\underline{\mathbf{C}}_0^\vee$ by the procedure of Sect. B.12.5.

Composing with the evaluation map

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C}\otimes\mathbf{C}^\vee)_S \to \mathcal{A}\text{-mod}^{\text{fact}}(\text{Vect}_S^{\text{fact}})_S \simeq \mathcal{A}\text{-mod}^{\text{fact}}_S,$$

we obtain a functor

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_S \underset{\text{QCoh}(S)}{\otimes} \mathbf{C}_0^\vee \to \mathcal{A}\text{-mod}^{\text{fact}}_S. \tag{B.45}$$

The sought-for inverse functor to (B.44) is given by

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\mathcal{Z}} \overset{\text{Id}\otimes\text{co-eval}}{\longrightarrow} \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\mathcal{Z}} \underset{\text{QCoh}(S)}{\otimes} \underline{\mathbf{C}}_0^\vee \underset{\text{QCoh}(S)}{\otimes} \underline{\mathbf{C}}_0 \overset{\text{(B.45)}}{\longrightarrow} \mathcal{A}\text{-mod}^{\text{fact}}_S \underset{\text{QCoh}(S)}{\otimes} \underline{\mathbf{C}}_0.$$

□

B.12.10. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a factorization functor between factorization categories. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be objects in $\mathbf{A}_1\text{-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}$ and $\mathbf{A}_2\text{-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}$, respectively.

A functor $\Phi_m : \mathbf{C}_1 \to \mathbf{C}_2$ compatible with factorization is a functor

$$\Phi_m : \underline{\mathbf{C}}_1 \to \underline{\mathbf{C}}_2$$

between sheaves of categories on $\mathcal{Z}^{\subseteq}$ that makes the following diagram commute

$$\begin{CD}
\text{union}^*(\underline{\mathbf{C}}_1)|_{(\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}} @>{\sim}>> \underline{\mathbf{A}}_1 \boxtimes \underline{\mathbf{C}}_1|_{(\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}} \\
@V{\Phi_m}VV @VV{\Phi\boxtimes\Phi_m}V \\
\text{union}^*(\underline{\mathbf{C}}_2)|_{(\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}} @>{\sim}>> \underline{\mathbf{A}}_2 \boxtimes \underline{\mathbf{C}}_2|_{(\text{Ran}\times\mathcal{Z}^{\subseteq})_{\text{disj}}}
\end{CD}$$

along with a homotopy-coherent system of higher compatibilities.

Let

$$\text{Funct}_{\mathbf{A}_1\to\mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2)$$

denote the category of such functors. When an ambiguity is likely to occur, we will use the notation $\text{Funct}_{\Phi:\mathbf{A}_1\to\mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2)$, i.e., we will insert Φ in the subscript.

B.12.11. Given $\mathbf{C}_2 \in \mathbf{A}_2\text{-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}$, one defines its restriction,

$$\text{Res}_\Phi(\mathbf{C}_2) \in \mathbf{A}_1\text{-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}$$

by the universal property as in Sect. B.9.25, i.e.,

$$\text{Funct}_{\mathbf{A}_1\text{-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}}(\mathbf{C}_1, \text{Res}_\Phi(\mathbf{C}_2)) \simeq \text{Funct}_{\mathbf{A}_1\to\mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2).$$

One can explicitly describe $\text{Res}_\Phi(\mathbf{C}_2)$ by a limit procedure as in Sect. B.9.28. [69]

We have a tautologically defined functor

$$\text{Res}_\Phi(\mathbf{C}_2) \to \mathbf{C}_2 \tag{B.46}$$

compatible with factorization.

Parallel to Lemma B.9.26, we have:

Lemma B.12.12. *The functor* (B.46) *induces an equivalence of the underlying categories*

$$\text{Res}_\Phi(\mathbf{C}_2)_{\mathcal{Z}} \to \mathbf{C}_{2,\mathcal{Z}}.$$

B.12.13. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a factorization functor, and let $\mathcal{A}_1$ be a factorization algebra in $\mathbf{A}_1$. Note that $\Phi(\mathcal{A}_1)$ has a natural structure of factorization algebra in $\mathbf{A}_2$.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ as in Sect. B.12.10. Given a functor $\Phi_m : \mathbf{C}_1 \to \mathbf{C}_2$ compatible with factorization, we obtain a naturally defined functor

$$\mathcal{A}_1\text{-mod}^{\text{fact}}(\mathbf{C}_1)_{\mathcal{Z}} \to \Phi(\mathcal{A}_1)\text{-mod}^{\text{fact}}(\mathbf{C}_2)_{\mathcal{Z}}. \tag{B.47}$$

Parallel with Lemma B.12.12 one proves:

Lemma B.12.14. *Assume that the functor $\Phi_m : \mathbf{C}_1 \to \mathbf{C}_2$ induces an equivalence*

$$\mathbf{C}_1 \to \text{Res}_\Phi(\mathbf{C}_2)$$

as factorization module categories over $\mathbf{A}_1$. Then the functor (B.47) *is an equivalence.*

[69] This will be written out in detail in [CFGY].

B.12.15. As in Sect. B.9.29, given a factorization functor $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$, for any $\mathcal{Z} \to \mathrm{Ran}$, we have a canonically defined functor

$$\mathbf{A}_1^{\mathrm{fact}_{\mathcal{Z}}} \to \mathrm{Res}_{\Phi}(\mathbf{A}_2^{\mathrm{fact}_{\mathcal{Z}}}).$$

In particular, given a commutative diagram of factorization categories

$$\begin{tikzcd} \mathbf{A}'_1 \arrow[r] & \mathbf{A}'_2 \\ \mathbf{A}_1 \arrow[u, "\Psi_1"] \arrow[r] & \mathbf{A}_2. \arrow[u, "\Psi_2"'] \end{tikzcd}$$

we obtain a functor

$$\mathrm{Res}_{\Psi_1}(\mathbf{A}_1'^{\,\mathrm{fact}_{\mathcal{Z}}}) \to \mathrm{Res}_{\Psi_2}(\mathbf{A}_2'^{\,\mathrm{fact}_{\mathcal{Z}}}),$$

compatible with factorization.

B.13. **Factorization categories of algebro-geometric nature.**

B.13.1. Let $\mathcal{Z}_{\mathrm{Ran}} \to \mathrm{Ran}$ be a prestack. We attach to it a sheaf of categories $\underline{\mathrm{QCoh}}(\mathcal{Z})$ over Ran, namely,

$$\underline{\mathrm{QCoh}}(\mathcal{Z}) := \pi_*(\underline{\mathrm{QCoh}}(\mathcal{Z}_{\mathrm{Ran}})),$$

where π denotes the projection $\mathcal{Z}_{\mathrm{Ran}} \to \mathrm{Ran}$.

Explicitly, for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$, we have

$$\mathrm{QCoh}(\mathcal{Z})_S = \mathrm{QCoh}(S \underset{\mathrm{Ran}}{\times} \mathcal{Z}_{\mathrm{Ran}}).$$

B.13.2. Let $\mathcal{T}$ be a factorization space over X. Consider the corresponding sheaf of categories $\underline{\mathrm{QCoh}}(\mathcal{T})$, i.e.,

$$\mathrm{QCoh}(\mathcal{T})_S = \mathrm{QCoh}(\mathcal{T}_S).$$

The factorization structure on $\mathcal{T}$ equips $\underline{\mathrm{QCoh}}(\mathcal{T})$ with a lax factorization structure. We denote the corresponding lax factorization category by $\mathrm{QCoh}(\mathcal{T})$.

Remark B.13.3. The reason that $\mathcal{T}$ is a priori only lax is that for a pair of prestacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$, the naturally defined functor

$$\mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1 \otimes \mathcal{Y}_2)$$

is not necessarily an equivalence.

This lax structure is strict, e.g., if $\underline{\mathrm{QCoh}}(\mathcal{T})$ is dualizable (as a sheaf of categories over Ran, which is equivalent to each $\mathrm{QCoh}(\mathcal{T})_S$ being dualizable). This happens, e.g., if $\mathcal{T}$ is a factorization *affine scheme*.

B.13.4. *Example.* Let $\mathcal{Y}$ be an affine D-scheme over X. On the one hand, we can consider the factorization scheme $\mathfrak{L}^+_\nabla(\mathcal{Y})$ and the corresponding factorization category $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}))$. As such, it is equipped with a symmetric monoidal structure.

On the other hand, we can consider the symmetric monoidal factorization category

$$\mathrm{Fact}(\mathrm{QCoh}(\mathcal{Y})).$$

We claim that there is a canonical equivalence:

$$\mathrm{Fact}(\mathrm{QCoh}(\mathcal{Y})) \simeq \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})).$$

Indeed, both sides identify, as factorization categories with $\mathcal{A}\text{-mod}^{\mathrm{com}}$ (see Sect. B.11.7), where

$$\mathcal{A} = \mathrm{Fact}(A), \quad \mathcal{Y} = \mathrm{Spec}_X(A),$$

see (C.44) below.

B.13.5. Let now $\mathcal{T} := \mathrm{LS}^{\mathrm{reg}}_H$. We claim that the lax factorization category $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_H)$ identifies with $\mathrm{Rep}(H)$ (see Sect. B.11.8 for the notational conventions); in particular $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_H)$ is a *factorization* category.

Indeed, on the one hand, unwinding the definitions, we obtain that $\mathrm{Rep}(H)$, viewed as a factorization category is the (factorization) category of comodules with respect to $\mathrm{Fact}(\mathcal{O}_H)$, viewed as a factorization *coalgebra*.

We can rewrite it as the totalization of the cosimplicial factorization category with terms

$$\mathrm{Fact}(\mathcal{O}_{H^\bullet})\text{-mod}^{\mathrm{com}},$$

where $H^\bullet$ is the Čech nerve of $\mathrm{pt} \to \mathrm{pt}/H$.

On the other hand since $\mathrm{pt} \to \mathrm{LS}^{\mathrm{reg}}_H$ is an fpqc cover (see Lemma B.7.4), $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_H)$ identifies with the totalization of the cosimplicial factorization category with terms $\mathrm{QCoh}(-)$ of the Čech nerve of $\mathrm{pt} \to \mathrm{LS}^{\mathrm{reg}}_H$, the latter being $\mathfrak{L}^+_\nabla(H^\bullet)$.

How the desired equivalence follows from the fact that

$$\mathcal{O}_{\mathfrak{L}^+_\nabla(H^\bullet)} \simeq \mathrm{Fact}(\mathcal{O}_{H^\bullet}),$$

see (C.44) below.

Remark B.13.6. For completeness, we remark on the following comparison with [Ra3].

In *loc. cit.*, a different construction of the factorization category associated to a symmetric monoidal category was used; in particular, $\mathrm{Rep}(H)_{\mathrm{Ran}}$ has an a priori different meaning than how it is used in this paper. One can directly compare the two constructions, but rather than doing so here, we note that the above material combined with [Ra4, Lemma 9.8.1] allows us to indirectly deduce that the two constructions coincide.

B.13.7. Let $\mathcal{Z}_{\mathrm{Ran}} \to \mathrm{Ran}$ be a prestack. We attach to it a sheaf of categories $\underline{\mathrm{QCoh}}_{\mathrm{co}}(\mathcal{Z})$ over Ran by setting for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Z})_S := \mathrm{QCoh}_{\mathrm{co}}(S \underset{\mathrm{Ran}}{\times} \mathcal{Z}_{\mathrm{Ran}}).$$

The sheaf of categories structure holds thanks to Lemma A.1.11.

B.13.8. Let $\mathcal{T}$ be factorization space over X. Consider the corresponding sheaf of categories $\underline{\mathrm{QCoh}}_{\mathrm{co}}(\mathcal{T})$, i.e.,

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})_S := \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_S).$$

The factorization structure on $\mathcal{T}$ induces a *factorization* structure on $\underline{\mathrm{QCoh}}_{\mathrm{co}}(\mathcal{T})$, see Lemma A.1.5. We will denote the resulting factorization category by $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})$.

By Sect. A.2.8, if $\mathcal{T}$ is a factorization ind-scheme, the factorization category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})$ carries a naturally defined t-structure.

B.13.9. Let $\mathcal{Z}_{\mathrm{Ran}} \to \mathrm{Ran}$ be a prestack. We will attach to it a sheaf of categories over Ran, denoted $\underline{\mathrm{IndCoh}}^!(\mathcal{Z})$. Unlike the cases of $\mathrm{QCoh}(-)$ and $\mathrm{QCoh}_{\mathrm{co}}(-)$, this will use some special features of Ran.

Namely, we will use the fact that Ran can be exhibited as a colimit of prestacks $S_{\alpha,\mathrm{dR}}$, where S_α are *smooth* schemes with transition $f_{\alpha,\beta} : S_\alpha \to S_\beta$ maps being closed embeddings. In practice, $S_\alpha = X^I$ for finite non-empty sets I, see Sect. B.1.3.

B.13.10. For S as above set
$$\mathrm{IndCoh}^!(\mathcal{Z})_{S_\alpha} := \mathrm{IndCoh}(\mathcal{Z}_{S_\alpha}).$$

For a map $S_\alpha \overset{f_{\alpha,\beta}}{\to} S_\beta$ denote by the same symbol $f_{\alpha,\beta}$ the corresponding map
$$\mathcal{Z}_{S_\alpha} \to \mathcal{Z}_{S_\beta}.$$

The functor
$$f^!_{\alpha,\beta} : \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}) \to \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha})$$
gives rise to a functor

$$\mathrm{QCoh}(S_\alpha) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}) \to \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}). \tag{B.48}$$

We claim:

Lemma B.13.11. *The functor* (B.48) *is an equivalence.*

Proof. The question is Zariski-local, so we can assume that S_α and S_β are affine. By Lemma A.4.10, the functor (B.48) is fully faithful. So we only have to show that it is essentially surjective.

Let $S^\wedge_\beta$ be the formal completion of S_β along S_α. Consider the corresponding functor

$$\mathrm{QCoh}(S^\wedge_\beta) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}) \to \mathrm{IndCoh}^!(\mathcal{Z}_{S^\wedge_\beta}). \tag{B.49}$$

We have a commutative diagram
$$\begin{CD}
\mathrm{QCoh}(S^\wedge_\beta) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}) @>>> \mathrm{IndCoh}^!(\mathcal{Z}_{S^\wedge_\beta}) \\
@AAA @AAA \\
\mathrm{QCoh}(S_\beta) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}) @>{=}>> \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}).
\end{CD}$$

The left vertical arrow is a colocalization. We claim that the right vertical arrow is also a colocalization: indeed, this follows from Proposition A.4.14. Hence, we obtain that the functor (B.49) is also a colocalization.[70] In particular, (B.49) is essentially surjective.

We have a commutative diagram

$$\begin{CD}
\mathrm{QCoh}(S_\alpha) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}) @>>> \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}) \\
@AAA @AAA \\
\mathrm{QCoh}(S^\wedge_\beta) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}) @>>> \mathrm{IndCoh}^!(\mathcal{Z}_{S^\wedge_\beta}),
\end{CD} \tag{B.50}$$

where:

- The right vertical arrow is given by !-pullback along $\mathcal{Z}_{S_\alpha} \to \mathcal{Z}_{S^\wedge_\beta}$;
- The left vertical arrows is given by the *-pullback functor along $S_\alpha \to S^\wedge_\beta$ along the $\mathrm{QCoh}(-)$ factors.

We wish to show that the top horizontal arrow in (B.50) is essentially surjective. By the above, the bottom horizontal arrow is essentially surjective. Hence, it suffices to show that the functor
$$\mathrm{IndCoh}^!(\mathcal{Z}_{S^\wedge_\beta}) \to \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha})$$
is essentially surjective.

We claim, however, that the map $\mathcal{Z}_{S_\alpha} \to \mathcal{Z}_{S^\wedge_\beta}$ admits a retraction. Indeed, since S_α is smooth, the embedding
$$S_\alpha \to S^\wedge_\beta$$

[70] In fact, the above argument shows that (B.49) is an equivalence.

admits a retraction, denote it by $g_{\beta,\alpha}$. Note that the two maps

$$S^\wedge_\beta \to \mathrm{Ran} \text{ and } S^\wedge_\beta \overset{g_{\beta,\alpha}}{\to} S_\alpha \to \mathrm{Ran}$$

agree on $(S^\wedge_\beta)_{\mathrm{red}} \simeq S_\alpha$. Hence, since $\mathrm{Ran} \to \mathrm{Ran}_{\mathrm{dR}}$ is an isomorphism, we can identify

$$\mathcal{Z}_{S^\wedge_\beta} \simeq S^\wedge_\beta \underset{g_{\beta,\alpha}, S_\alpha}{\times} \mathcal{Z}_{S_\alpha}.$$

In terms of this identification, the projection

$$S^\wedge_\beta \underset{g_{\beta,\alpha}, S_\alpha}{\times} \mathcal{Z}_{S_\alpha} \to \mathcal{Z}_{S_\alpha}$$

provides the sought-for retraction.

□

B.13.12. Let S be an affine scheme mapping to Ran. This map factors as

$$S \overset{f}{\to} S_\alpha \to \mathrm{Ran}$$

for some α.

Set

$$\mathrm{IndCoh}^!(\mathcal{Z})_{S,f} := \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}).$$

In order to show that the assignment

$$S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}} \rightsquigarrow \mathrm{IndCoh}^!(\mathcal{Z})_{S,f}$$

gives a well-defined sheaf of categories over Ran, it remains to show that for two maps f_1 and f_2 as above, for which $f_1|_{S_{\mathrm{red}}} \simeq f_2|_{S_{\mathrm{red}}}$, we have a canonical identification

$$\mathrm{IndCoh}^!(\mathcal{Z})_{S,f_1} \simeq \mathrm{IndCoh}^!(\mathcal{Z})_{S,f_2},$$

i.e.,

$$\mathrm{QCoh}(S) \underset{f_1^*, \mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}) \simeq \mathrm{QCoh}(S) \underset{f_2^*, \mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}). \tag{B.51}$$

(In addition, one needs to show that these identifications satisfy a homotopy-coherent system of compatibilities for multi-fold comparisons $f_1|_{S_{\mathrm{red}}} \simeq f_2|_{S_{\mathrm{red}}} \simeq ... \simeq f_n|_{S_{\mathrm{red}}}$, but this will be automatic from the construction explained below.)

B.13.13. Let $(S_\alpha \times S_\alpha)^\wedge$ be the formal completion of the diagonal in $S_\alpha \times S_\alpha$. Note that we have a well-defined map

$$(S_\alpha \times S_\alpha)^\wedge \to \mathrm{Ran},$$

Consider the corresponding prestack $\mathcal{Z}_{(S_\alpha \times S_\alpha)^\wedge}$ and the category

$$\mathrm{IndCoh}^!(\mathcal{Z}_{(S_\alpha \times S_\alpha)^\wedge}).$$

For $i = 1, 2$, let

$$p_i : (S_\alpha \times S_\alpha)^\wedge \to S_\alpha$$

denote the corresponding projection. We will denote by the same symbol p_i the corresponding map

$$\mathcal{Z}_{(S_\alpha \times S_\alpha)^\wedge} \to \mathcal{Z}_{S_\alpha}.$$

The functor

$$p_i^! : \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}) \to \mathrm{IndCoh}^!(\mathcal{Z}_{(S_\alpha \times S_\alpha)^\wedge})$$

gives rise to a functor

$$\mathrm{QCoh}((S_\alpha \times S_\alpha)^\wedge) \underset{p_i^*, \mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}) \to \mathrm{IndCoh}^!(\mathcal{Z}_{(S_\alpha \times S_\alpha)^\wedge}). \tag{B.52}$$

We claim:

Lemma B.13.14. *The functor* (B.52) *is an equivalence.*

Proof. Proceeds along the same lines as the proof of Lemma B.13.11. □

B.13.15. Using Lemma B.13.14, we obtain

$$\mathrm{QCoh}(S) \underset{f_i^*,\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}) =$$

$$= \mathrm{QCoh}(S) \underset{(f_1\times f_2)^*,\mathrm{QCoh}((S_\alpha\times S_\alpha)^\wedge)}{\otimes} \mathrm{QCoh}((S_\alpha \times S_\alpha)^\wedge) \underset{p_i^*,\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}) \simeq$$

$$\simeq \mathrm{QCoh}(S) \underset{(f_1\times f_2)^*,\mathrm{QCoh}((S_\alpha\times S_\alpha)^\wedge)}{\otimes} \mathrm{IndCoh}^!(\mathcal{Z}_{(S_\alpha\times S_\alpha)^\wedge}),$$

thereby establishing (B.51).

This completes the construction of $\underline{\mathrm{IndCoh}}^!(\mathcal{Z})$ as a sheaf of categories over Ran.

B.13.16. Let $\mathcal{T}$ be a factorization space over X. The multiplicative structure in Sect. A.6.3 equips the corresponding sheaf of categories $\underline{\mathrm{IndCoh}}^!(\mathcal{T})$ with a structure of lax factorization category; we will denote it by $\mathrm{IndCoh}^!(\mathcal{T})$.

Suppose now that $\mathcal{T}$ is ind-placid. In this case, from Lemma A.10.10, we obtain that the lax factorization structure on $\mathrm{IndCoh}^!(\mathcal{T})$ is a *factorization* structure.

B.13.17. Let now $\mathcal{Z}_{\mathrm{Ran}} \to \mathrm{Ran}$ be a relative ind-placid ind-scheme. In this case, we are going to define the sheaf of categories $\underline{\mathrm{IndCoh}}^*(\mathcal{Z})$.

We proceed with the same recipe as in the case of $\underline{\mathrm{IndCoh}}^!(\mathcal{Z})$. For an index α, set

$$\mathrm{IndCoh}^*(\mathcal{Z})_{S_\alpha} := \mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}).$$

For a map $f_{\alpha,\beta}$, we have a well-defined functor

$$f_{\alpha,\beta}^{*,\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathcal{Z}_{S_\beta}) \to \mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha})$$

(see Sect. A.7.4). Consider the resulting functor

$$\mathrm{QCoh}(S_\alpha) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^*(\mathcal{Z}_{S_\beta}) \to \mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}). \tag{B.53}$$

We claim:

Lemma B.13.18. *The functor* (B.53) *is an equivalence.*

Proof. We can reformulate the assertion of the lemma as saying that the right adjoint of $f_{\alpha,\beta}^{*,\mathrm{IndCoh}}$, i.e., the functor

$$(f_{\alpha,\beta})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}) \to \mathrm{IndCoh}^*(\mathcal{Z}_{S_\beta})$$

gives rise to an equivalence

$$\mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}) \to (f_{\alpha,\beta})_*(\mathcal{O}_{S_\alpha})\text{-mod}(\mathrm{IndCoh}^*(\mathcal{Z}_{S_\beta})). \tag{B.54}$$

Note that Lemma B.13.11 can be reformulated as saying that the functor

$$f_{\alpha,\beta}^! : \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta}) \to \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha})$$

admits a right adjoint, and this right adjoint identifies

$$(f_{\alpha,\beta})_*(\mathcal{O}_{S_\alpha})\text{-mod}(\mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta})) \simeq \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha}).$$

Passing to the duals, we obtain that the dual of $f_{\alpha,\beta}^!$ identifies

$$(f_{\alpha,\beta})_*(\mathcal{O}_{S_\alpha})\text{-mod}(\mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta})^\vee) \simeq \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha})^\vee. \tag{B.55}$$

We will now use the identifications

$$\mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}) \simeq \mathrm{IndCoh}^!(\mathcal{Z}_{S_\alpha})^\vee \text{ and } \mathrm{IndCoh}^*(\mathcal{Z}_{S_\beta}) \simeq \mathrm{IndCoh}^!(\mathcal{Z}_{S_\beta})^\vee,$$

see Sect. A.10.8.

Under these identifications

$$(f_{\alpha,\beta}^!)^\vee \simeq (f_{\alpha,\beta})_*^{\mathrm{IndCoh}}.$$

Unwinding the definitions, it is easy to see that the resulting identification (B.55) is the same as (B.54).

□

B.13.19. Let S be an affine scheme mapping to Ran. This map factors as

$$S \xrightarrow{f} S_\alpha \to \mathrm{Ran}$$

for some α.

Set

$$\mathrm{IndCoh}^*(\mathcal{Z})_{S,f} := \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}).$$

As in the case of $\mathrm{IndCoh}^!$, in order to complete the construction of $\underline{\mathrm{IndCoh}}^*(\mathcal{Z})$ as a sheaf of categories over Ran, it suffices to show that for a pair of maps f_1 and f_2 as above, for which $f_1|_{S_{\mathrm{red}}} \simeq f_2|_{S_{\mathrm{red}}}$, we have a canonical identification

$$\mathrm{IndCoh}^*(\mathcal{Z})_{S,f_1} \simeq \mathrm{IndCoh}^*(\mathcal{Z})_{S,f_2},$$

i.e.,

$$\mathrm{QCoh}(S) \underset{f_1^*,\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}) \simeq \mathrm{QCoh}(S) \underset{f_2^*,\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}). \tag{B.56}$$

This follows from the following:

Lemma B.13.20. *In the notations of Lemma B.13.14, the functor*

$$\mathrm{QCoh}((S_\alpha \times S_\alpha)^\wedge) \underset{p_i^*,\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^*(\mathcal{Z}_{S_\alpha}) \to \mathrm{IndCoh}^*(\mathcal{Z}_{(S_\alpha \times S_\alpha)^\wedge}).$$

is an equivalence.

The lemma follows by duality from Lemma B.13.20.

B.13.21. By construction, the sheaves of categories $\underline{\mathrm{IndCoh}}^*(\mathcal{Z})$ and $\underline{\mathrm{IndCoh}}^!(\mathcal{Z})$ are mutually dual.

By Sect. A.8, the sheaf of categories $\underline{\mathrm{IndCoh}}^*(\mathcal{Z})$ is equipped with a t-structure.

B.13.22. Let $\mathcal{T}$ be a factorization ind-placid ind-scheme over X. The multiplicative structure in Sect. A.6.4 and Lemma A.10.10 imply that in this case the sheaf of categories $\underline{\mathrm{IndCoh}}^*(\mathcal{T})$ carries a factorization structure. Denote the resulting factorization category by $\mathrm{IndCoh}^*(\mathcal{T})$.

By the construction of $\mathrm{IndCoh}^*(\mathcal{T})$ in Sect. B.13.17 and Sect. A.8, the factorization category $\mathrm{IndCoh}^*(\mathcal{T})$ carries a naturally defined t-structure.

By construction, $\mathrm{IndCoh}^*(\mathcal{T})$ is dual to $\mathrm{IndCoh}^!(\mathcal{T})$ as a factorization category.

B.13.23. Let $\mathcal{T}$ be a factorization space over X, and let $\mathcal{T}_m$ be a factorization module space over $\mathcal{T}$ at some $\mathcal{Z} \to \mathrm{Ran}$.

Suppose that $\mathcal{T}$ is such that the categories $\mathrm{QCoh}(\mathcal{T}_S)$ for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$ are dualizable. Then the sheaf of categories

$$\pi_*(\underline{\mathrm{QCoh}}((\mathcal{T}_m)_{\mathcal{Z}^\subseteq}))$$

on $\mathcal{Z}^\subseteq$ (here π denotes the structural map $\mathcal{T}_{\mathcal{Z}^\subseteq} \to \mathcal{Z}^\subseteq$) admits a natural structure of factorization module category over $\mathrm{QCoh}(\mathcal{T})$ at $\mathcal{Z}$. We will denote it by $\mathrm{QCoh}(\mathcal{T}_m)$.

For general $\mathcal{T}$ and $\mathcal{T}_m$, the assignment

$$S \to \mathcal{Z}^\subseteq, \rightsquigarrow \mathrm{QCoh}_{\mathrm{co}}((\mathcal{T}_m)_S)$$

is a sheaf of categories over $\mathcal{Z}^\subseteq$, to be denoted $\underline{\mathrm{QCoh}}_{\mathrm{co}}(\mathcal{T}_m)$. It has a natural structure of factorization module category over $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})$ at $\mathcal{Z}$. We will denote it by $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_m)$.

Assume now that $\mathcal{T}_m$ is an ind-placid ind-scheme relative to $\mathcal{Z}^\subseteq$. In this case, we can consider the sheaves of categories

$$\underline{\mathrm{IndCoh}}^!(\mathcal{T}_m) \text{ and } \underline{\mathrm{IndCoh}}^*(\mathcal{T}_m)$$

and they have natural factorization module structures over $\mathrm{IndCoh}^!(\mathcal{T})$ and $\mathrm{IndCoh}^*(\mathcal{T})$, respectively. We will denote the resulting factorization module categories by $\mathrm{IndCoh}^!(\mathcal{T}_m)$ and $\mathrm{IndCoh}^*(\mathcal{T}_m)$, respectively.

B.14. **Modules over Kac-Moody Lie algebras.** In this subsection we will show how to adapt the theory developed in [Ra5] to the factorization setting. We start be defining the factorization category $\mathrm{Rep}(\mathfrak{L}^+(G))$.

In this subsection G can be arbitrary an algebraic group (i.e., not necessarily reductive).

B.14.1. For $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$ consider the group scheme $\mathfrak{L}^+(G)_S$ over S. It is pro-smooth,

$$\mathfrak{L}^+(G)_S \simeq \lim_{\alpha\in A^{\mathrm{op}}} G^\alpha_S,$$

where G^α_S are smooth group-schemes over S of finite type, and the transition maps are smooth and surjective.

For every α we can consider the algebraic stack pt/G^α_S. Set

$$\mathrm{Rep}(G^\alpha_S) := \mathrm{QCoh}(\mathrm{pt}/G^\alpha_S).$$

Set

$$\mathrm{Rep}(\mathfrak{L}^+(G))_S := \operatorname*{colim}_{\alpha\in A} \mathrm{Rep}(G^\alpha_S), \tag{B.57}$$

where the transition functors are given by restriction:

$$G^\beta_S \twoheadrightarrow G^\alpha_S \quad \rightsquigarrow \quad \mathrm{Rep}(G^\alpha_S) \to \mathrm{Rep}(G^\beta_S).$$

Note that the above transition functors admit (continuous) right adjoints, given by

$$\mathrm{inv}_{\ker(G^\beta_S \to G^\alpha_S)}.$$

Hence, we can rewrite $\mathrm{Rep}(\mathfrak{L}^+(G))_S$ also as the limit

$$\mathrm{Rep}(\mathfrak{L}^+(G))_S := \lim_{\alpha\in A^{\mathrm{op}}} \mathrm{Rep}(G^\alpha_S),$$

with respect to the above right adjoints.

B.14.2. We claim that $\mathrm{Rep}(\mathfrak{L}^+(G))_S$ is compactly generated. In order to prove that, it suffices to show that each $\mathrm{Rep}(G^\alpha_S)$ is compactly generated. This can be proved on general grounds (the category of quasi-coherent sheaves on a smooth algebraic stack is compactly generated). What follows below is an explicit construction of compact generators.

First, it is easy to see that if an object $\mathcal{V} \in \mathrm{Rep}(G^\alpha_S)$ is such that $\mathbf{oblv}_{G^\alpha_S}(\mathcal{V}) \in \mathrm{QCoh}(S)$ is compact, then $\mathcal{V}$ itself is compact. Hence, in order prove the compact generation, it suffices to exhibit a generating collection of compact objects. We do that as follows.

We can assume that the category of indices A has an initial element α_0 for which $G^{\alpha_0}_S$ is the following explicit group-scheme:

Let I be a finite set such that the map $S \to \mathrm{Ran}$ factors as

$$S \to X^I \to \mathrm{Ran}.$$

Let $\mathrm{Graph}_I \subset X^I \times X$ be the incidence divisor. Let G_{X^I} be the group-scheme over X^I equal to the restriction of scalars à la Weil along

$$\mathrm{Graph}_I \subset X^I \to X^I$$

of the pullback of the constant group-scheme with fiber G along

$$\mathrm{Graph}_I \subset X^I \to X.$$

Note that Graph_I receives a map from the disjoint union of I many copies of X^I (i.e., the pairwise diagonals of the ith and the last coordinate in $X^I \times X$). In particular, we obtain a map

$$G_{X^I} \to (G \times X)^I \tag{B.58}$$

as group-schemes over X^I.

We take $G_S^{\alpha_0}$ to be the pullback of G_{X^I} along $S \to X^I$.

Note that for every α, the kernel of the projection

$$G_S^\alpha \to G_S^{\alpha_0} \tag{B.59}$$

is unipotent (in fact, admits a filtration with subquotients isomorphic to the constant group-scheme with fiber $\mathbb{G}_a$).

Hence, the essential image of the forgetful functor

$$\mathrm{Rep}(G_S^{\alpha_0}) \to \mathrm{Rep}(G_S^\alpha)$$

generates $\mathrm{Rep}(G_S^\alpha)$.

The map (B.58) induces a map

$$G_S^\alpha \to G^I.$$

In particular, we obtain a functor

$$\mathrm{Rep}(G)^{\otimes I} \to \mathrm{Rep}(G_S^{\alpha_0}). \tag{B.60}$$

It is easy to see that the essential image of (B.60) generates $\mathrm{Rep}(G_S^{\alpha_0})$. Hence, we obtain that the images of the compact objects in $\mathrm{Rep}(G)^{\otimes I}$ under the composition

$$\mathrm{Rep}(G)^{\otimes I} \to \mathrm{Rep}(G_S^{\alpha_0}) \to \mathrm{Rep}(G_S^\alpha) \tag{B.61}$$

provide a set of compact generators of $\mathrm{Rep}(G_S^\alpha)$.

Remark B.14.3. Note that the category $\mathrm{Rep}(\mathfrak{L}^+(G))_S$ is *not* the same as representations of the group-scheme $\mathfrak{L}^+(G)_S$, i.e.,

$$\mathcal{O}_{\mathfrak{L}^+(G)_S}\text{-comod} \simeq \mathrm{QCoh}(\mathrm{pt}/\mathfrak{L}^+(G)_S).$$

Rather, it is its renormalized version, in which we declare the compacts to be the images of the compacts in $\mathrm{Rep}(G_S^\alpha)$ under the restriction along $\mathfrak{L}^+(G)_S \to G_S^\alpha$.

In particular, the forgetful functor

$$\mathbf{oblv}_{\mathfrak{L}^+(G)_S} : \mathrm{Rep}(\mathfrak{L}^+(G))_S \to \mathrm{QCoh}(S)$$

is *not* conservative.

B.14.4. The presentation (B.57) equips $\mathrm{Rep}(\mathfrak{L}^+(G))_S$ with a t-structure, for which the forgetful functor $\mathbf{oblv}_{\mathfrak{L}^+(G)_S}$ is t-exact.

B.14.5. Let $f : S' \to S$ be a map of affine schemes. Pullback along f gives rise to a functor

$$f^* : \mathrm{Rep}(\mathfrak{L}^+(G))_S \to \mathrm{Rep}(\mathfrak{L}^+(G))_{S'},$$

which in turn gives rise to a functor

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{Rep}(\mathfrak{L}^+(G))_S \to \mathrm{Rep}(\mathfrak{L}^+(G))_{S'}. \tag{B.62}$$

We claim that the functor (B.62) is an equivalence. Indeed, this follows from the fact that for every α, the corresponding functor

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{Rep}(G_S^\alpha) \to \mathrm{Rep}(G_{S'}^\alpha)$$

is an equivalence.

This defines on the assignment

$$S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}} \rightsquigarrow \mathrm{Rep}(\mathfrak{L}^+(G))_S$$

a structure of sheaf of categories over Ran.

We will denote this sheaf of categories by $\underline{\mathrm{Rep}}(\mathfrak{L}^+(G))$.

B.14.6. Let $\underline{x}_i : S_i \to \mathrm{Ran}$, $i = 1, 2$ be points such that

$$S_1 \times S_2 \overset{\underline{x}_1, \underline{x}_2}{\longrightarrow} \mathrm{Ran} \times \mathrm{Ran}$$

lands in $(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}$.

Note that we have

$$\mathfrak{L}^+(G)_{S_1 \times S_2} \simeq \mathfrak{L}^+(G)_{S_1} \times \mathfrak{L}^+(G)_{S_2}.$$

Tensor product of representations defines a functor

$$\mathrm{Rep}(\mathfrak{L}^+(G)_{S_1}) \otimes \mathrm{Rep}(\mathfrak{L}^+(G)_{S_2}) \to \mathrm{Rep}(\mathfrak{L}^+(G)_{S_1 \times S_2}). \tag{B.63}$$

We claim that (B.63) is an equivalence.[71] Indeed, we can write $\mathfrak{L}^+(G)_{S_1 \times S_2}$ as

$$\lim_{(\alpha_1, \alpha_2) \in A_1^{\mathrm{op}} \times A_2^{\mathrm{op}}} G_{S_1}^{\alpha_1} \times G_{S_2}^{\alpha_2},$$

and for each pair of indices α_1, α_2, the functor

$$\mathrm{Rep}(G_{S_1}^{\alpha_1}) \otimes \mathrm{Rep}(G_{S_2}^{\alpha_2}) \to \mathrm{Rep}(G_{S_1}^{\alpha_1} \times G_{S_2}^{\alpha_2})$$

is an equivalence (e.g., by [Ga7, Cor. 10.3.6]).

This endows the sheaf of categories $\underline{\mathrm{Rep}}(\mathfrak{L}^+(G))$ with a factorization structure. We denote the resulting factorization category by $\mathrm{Rep}(\mathfrak{L}^+(G))$.

The t-structures in Sect. B.14.4 define a t-structure on $\mathrm{Rep}(\mathfrak{L}^+(G))$ as a factorization category.

B.14.7. Let $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$ be as above. Following [Ra5], one defines the (2-)category

$$\mathfrak{L}^+(G)_S\text{-}\mathbf{mod}^{\mathrm{weak}}$$

of $\mathrm{QCoh}(S)$-linear categories equipped with a *weak* action of $\mathfrak{L}^+(G)_S$ to be equivalent to

$$\mathrm{Rep}(\mathfrak{L}^+(G))_S\text{-}\mathbf{mod}.$$

We consider the forgetful functor

$$\mathbf{oblv}_{\mathfrak{L}^+(G)_S, \mathrm{weak}} : \mathrm{Rep}(\mathfrak{L}^+(G))_S\text{-}\mathbf{mod} \to \mathrm{QCoh}(S)\text{-}\mathbf{mod} \tag{B.64}$$

given by

$$\mathbf{C} \mapsto \mathbf{C} \underset{\mathrm{Rep}(\mathfrak{L}^+(G))_S, \mathbf{oblv}_{\mathfrak{L}^+(G)_S}}{\otimes} \mathrm{QCoh}(S).$$

Remark B.14.8. As in Remark B.14.3, the 2-category $\mathrm{Rep}(\mathfrak{L}^+(G))_S\text{-}\mathbf{mod}$ is *not* the same as $\mathrm{QCoh}(S)$-linear categories equipped with a co-action of $\mathrm{QCoh}(\mathfrak{L}^+(G)_S)$. Indeed, for the unit object

$$\mathrm{Rep}(\mathfrak{L}^+(G))_S \in \mathrm{Rep}(\mathfrak{L}^+(G))_S\text{-}\mathbf{mod},$$

its category of endofunctors in $\mathrm{Rep}(\mathfrak{L}^+(G))_S\text{-}\mathbf{mod}$ is $\mathrm{Rep}(\mathfrak{L}^+(G))_S$, while for the unit object

$$\mathrm{QCoh}(S) \in \mathrm{QCoh}(\mathfrak{L}^+(G)_S)\text{-}\mathbf{comod},$$

its category of endofunctors is the non-renormalized category

$$\mathcal{O}_{\mathfrak{L}^+(G)_S}\text{-comod} \simeq \mathrm{QCoh}(\mathrm{pt}/\mathfrak{L}^+(G)_S).$$

That said, for each individual α, the stack pt/G_S^α is 1-affine, and hence the functor

$$\mathrm{Rep}(G_S^\alpha)\text{-}\mathbf{mod} \to \mathrm{QCoh}(G_S^\alpha)\text{-}\mathbf{comod}, \quad \mathbf{C} \mapsto \mathbf{C} \underset{\mathrm{Rep}(G_S^\alpha), \mathbf{oblv}_{G_S^\alpha}}{\otimes} \mathrm{QCoh}(S) \tag{B.65}$$

is an equivalence

[71] This would not be the case (at least, not obviously so) if instead of $\mathrm{Rep}(\mathfrak{L}^+(G))_{S_i}$ we used their naive versions, i.e., the categories of representations of the group-schemes $\mathfrak{L}^+(G)_{S_i}$.

B.14.9. Parallel to [Ra5], one defines the (2-)category

$$\mathfrak{L}^+(G)_S\text{-}\mathbf{mod}$$

of QCoh(S)-linear categories equipped with a *strong* action of $\mathfrak{L}^+(G)_S$. Namely, this is the category of comodules (inside QCoh(S)-**mod**) for

$$\text{D-mod}_{\text{rel}\,/S}(\mathfrak{L}^+(G)_S),$$

where:

- $\text{D-mod}_{\text{rel}\,/S}(\mathfrak{L}^+(G)_S) := \underset{\alpha\in A}{\text{colim}}\ \text{D-mod}_{\text{rel}\,/S}(G^\alpha_S)$;
- $\text{D-mod}_{\text{rel}\,/S}(G^\alpha_S) := \text{QCoh}(G^\alpha_{S_{\text{dR}}} \underset{S_{\text{dR}}}{\times} S)$.

One shows that any object

$$\mathbf{C} \in \mathfrak{L}^+(G)_S\text{-}\mathbf{mod}$$

can be *canonically written* as

$$\underset{\alpha\in A}{\text{colim}}\, \mathbf{C}^{\ker(\mathfrak{L}^+(G)_S\to G^\alpha_S)},$$

where

$$\mathbf{C}^{\ker(\mathfrak{L}^+(G)_S\to G^\alpha_S)} \subset \mathbf{C}$$

is the full subcategory of strong invariants with resect to the (pro-unipotent) group-scheme

$$\ker(\mathfrak{L}^+(G)_S \to G^\alpha_S).$$

B.14.10. We have a forgetful functor

$$\mathbf{oblv}^{\text{strong}}_{\text{weak}} : \mathfrak{L}^+(G)_S\text{-}\mathbf{mod} \to \mathfrak{L}^+(G)_S\text{-}\mathbf{mod}^{\text{weak}}$$

that sends an object $\mathbf{C}$ to

$$\underset{\alpha\in A}{\text{colim}}\, \mathbf{oblv}^{\text{strong}}_{\text{weak}}(\mathbf{C}^{\ker(\mathfrak{L}^+(G)_S\to G^\alpha_S)}),$$

where in the right-hand side $\mathbf{oblv}^{\text{strong}}_{\text{weak}}$ denotes the family of forgetful functors

$$\text{D-mod}_{\text{rel}\,/S}(G^\alpha_S)\text{-}\mathbf{comod} \overset{\mathbf{oblv}^l}{\to} \text{QCoh}(G^\alpha_S)\text{-}\mathbf{comod} \overset{\text{(B.65)}}{\simeq} \text{Rep}(G^\alpha_S)\text{-}\mathbf{mod}.$$

This functor intertwines the natural forgetful functor

$$\mathbf{oblv}_{\mathfrak{L}^+(G)_S,\text{weak}} : \mathfrak{L}^+(G)_S\text{-}\mathbf{mod} \to \text{QCoh}(S)\text{-}\mathbf{mod}$$

with the functor $\mathbf{oblv}_{\mathfrak{L}^+(G)_S,\text{weak}}$ of (B.64).

B.14.11. Our next goal is to define the category $\mathfrak{L}^+(\mathfrak{g})$-mod of modules for the arc Lie algebra $\mathfrak{L}^+(\mathfrak{g})$. The definition that we are about to give mimics the following finite-dimensional situation:

Let H be a finite-dimensional algebraic group. Then the category $\mathfrak{h}$-mod of modules over its Lie algebra has the following structures:

- It carries a (strong) action of H;
- It is equipped with a forgetful functor $\mathbf{oblv}_\mathfrak{h} : \mathfrak{h}\text{-mod} \to \text{Vect}$;
- The functor $\mathbf{oblv}_\mathfrak{h}$ is equipped with a structure of compatibility with the induced *weak* action of H.

Moreover, the category $\mathfrak{h}$-mod is universal with respect to the above pieces of structure. I.e., for a category $\mathbf{C}$, equipped with a (strong) action of H, compositing with $\mathbf{oblv}_\mathfrak{h}$ defines an equivalence between:

- Functors $\mathbf{C} \to \mathfrak{h}$-mod, compatible with (strong) actions of H;
- Functors $\mathbf{C} \to \text{Vect}$, compatible with weak actions of H.

The above universal property can be established by realizing $\mathfrak{h}$-mod as follows

$$\mathfrak{h}\text{-mod} \simeq \text{D-mod}(H)^{H\text{-weak}}. \tag{B.66}$$

B.14.12. We define the category

$$\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$$

by the universal property as in Sect. B.14.11. I.e., this is a category, equipped with:

- A (strong) action of $\mathfrak{L}^+(G)_S$;
- A forgetful functor $\mathbf{oblv}_{\mathfrak{L}^+(\mathfrak{g})} : \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S \to \mathrm{QCoh}(S)$;
- A datum of compatibility on $\mathbf{oblv}_{\mathfrak{L}^+(\mathfrak{g})}$ with the weak action of $\mathfrak{L}^+(G)_S$.

Moreover, $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$ is universal with respect to the above pieces of structure.

As in (B.66),we can explicitly realize $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$ as follows:

$$\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S \simeq \text{D-mod}_{\mathrm{rel}\,/S}(\mathfrak{L}^+(G)_S)^{\mathfrak{L}^+(G)_S\text{-weak}}. \tag{B.67}$$

Remark B.14.13. A feature of $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$ that one has to keep in mind, and which distinguishes it from the finite-dimensional situation, is that the functor

$$\mathbf{oblv}_{\mathfrak{L}^+(\mathfrak{g})} : \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S \to \mathrm{QCoh}(S)$$

is *not* conservative.

B.14.14. Note that, by definition, we have

$$\begin{aligned}\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S^{\mathfrak{L}^+(G)_S} \simeq \mathrm{Funct}_{\mathfrak{L}^+(G)_S\text{-}\mathbf{mod}}(\mathrm{QCoh}(S), \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S) \simeq \\ \simeq \mathrm{Funct}_{\mathfrak{L}^+(G)_S\text{-}\mathbf{mod}^{\mathrm{weak}}}(\mathrm{QCoh}(S), \mathrm{QCoh}(S)) \simeq \mathrm{Rep}(\mathfrak{L}^+(G))_S.\end{aligned} \tag{B.68}$$

B.14.15. By the universal property of $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$, we have naturally defined restriction functors

$$\mathrm{Lie}(G^\alpha_S)\text{-mod} \to \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S.$$

Furthermore, the functor

$$\underset{\alpha\in A}{\mathrm{colim}}\, \mathrm{Lie}(G^\alpha_S)\text{-mod} \to \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S \tag{B.69}$$

is an equivalence.

Since the transition functors in the left-hand side of (B.67) preserve compactness, we obtain that $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$ is compactly generated.

B.14.16. The presentation of $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$ as in (B.69) equips it with a t-structure, for which the forgetful functor $\mathbf{oblv}_{\mathfrak{L}^+(\mathfrak{g})}$ is t-exact.

B.14.17. For $S' \to S$, the universal property of $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$ gives rise to a functor

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S \to \mathfrak{L}^+(\mathfrak{g})\text{-mod}_{S'}.$$

One shows (e.g., using (B.67)) that the above functor is equivalence.

This endows the assignment

$$S \rightsquigarrow \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$$

with a structure of sheaf of categories over Ran. We will denote it by $\mathfrak{L}^+(\mathfrak{g})\underline{\text{-mod}}$.

B.14.18. Let $\underline{x}_i : S_i \to \mathrm{Ran}$ be as in Sect. B.14.6. By the universal property of $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_{S_1\times S_2}$, we obtain a functor

$$\mathfrak{L}^+(\mathfrak{g})\text{-mod}_{S_1} \otimes \mathfrak{L}^+(\mathfrak{g})\text{-mod}_{S_2} \to \mathfrak{L}^+(\mathfrak{g})\text{-mod}_{S_1\times S_2}.$$

One shows (e,g., using (B.67)) that this functor is an equivalence. This endows the sheaf of categories $\mathfrak{L}^+(\mathfrak{g})\underline{\text{-mod}}$ with a factorization structure.

We denote the resulting factorization category by $\mathfrak{L}^+(\mathfrak{g})\text{-mod}$. The t-structures from Sect. B.14.16 give rise to a t-structure on $\mathfrak{L}^+(\mathfrak{g})\text{-mod}$ as a factorization category.

B.14.19. We finally consider representations of Kac-Moody algebras. First, fix $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$. Having the (2-)categories

$$\mathfrak{L}^+(G)_S\text{-}\mathbf{mod}^{\mathrm{weak}} \text{ and } \mathfrak{L}^+(G)_S\text{-}\mathbf{mod},$$

proceeding as in [Ra5, Sect. 7-8], we define the (2-)categories

$$\mathfrak{L}(G)_S\text{-}\mathbf{mod}^{\mathrm{weak}} \text{ and } \mathfrak{L}(G)_{\kappa,S}\text{-}\mathbf{mod}$$

of $\mathrm{QCoh}(S)$-linear categories, equipped with weak (resp., strong at level κ) actions of $\mathfrak{L}(G)_S$.

We define the category $\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S}$ as the category, equipped with and universal with respect to the following pieces of structure:

- A *strong* action of $\mathfrak{L}(G)_S$ at level κ;
- A functor to $\mathrm{QCoh}(S)$;
- A datum of compatibility on the above functor with respect to the *weak* action of $\mathfrak{L}(G)_S$.

We can explicitly realize the category $\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S}$ as

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S} \simeq \mathrm{D}\text{-mod}_{\kappa,\mathrm{rel}\,/S}(\mathfrak{L}(G)_S)^{\mathfrak{L}(G)_S\text{-weak}}.$$

B.14.20. By the universal property of $\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S}$, it comes equipped with a (conservative) forgetful functor

$$\mathbf{oblv}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{L}^+(\mathfrak{g})} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S} \to \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S.$$

As in [Ra5, §9.12, §11.10], one shows that this functor admits a left adjoint, to be denoted

$$\mathbf{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{L}^+(\mathfrak{g})} : \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S}.$$

In particular, the fact that $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$ is compactly generated implies that so is $\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S}$.

B.14.21. We note that the endofunctor of $\mathfrak{L}^+(\mathfrak{g})\text{-mod}_S$ underlying the monad

$$\mathbf{oblv}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{L}^+(\mathfrak{g})} \circ \mathbf{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{L}^+(\mathfrak{g})}$$

is t-exact.

This allows us to equip $\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S}$ with a t-structure for which both functors $\mathbf{oblv}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{L}^+(\mathfrak{g})}$ and $\mathbf{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{L}^+(\mathfrak{g})}$ are t-exact.

B.14.22. As in Sects. Sect. B.14.17 one endows the assignment

$$S \rightsquigarrow \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S}$$

with a structure of sheaf of categories over Ran. We denote it by $\widehat{\mathfrak{g}}\text{-}\underline{\mathrm{mod}}_\kappa$.

As in Sect. B.14.18, one endows $\widehat{\mathfrak{g}}\text{-}\underline{\mathrm{mod}}_\kappa$ with a factorization structure. We denote the resulting factorization category by $\widehat{\mathfrak{g}}\text{-mod}_\kappa$.

The t-structures in Sect. B.14.21 combine to a t-structure on $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ as a factorization category.

B.14.23. For $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$ set

$$\mathrm{KL}(G)_{\kappa,S} := (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S})^{\mathfrak{L}^+(G)_S}.$$

The adjoint functors

$$\mathbf{ind}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{L}^+(\mathfrak{g})} : \mathfrak{L}^+(\mathfrak{g})\text{-mod}_S \rightleftarrows \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S} : \mathbf{oblv}^{\widehat{\mathfrak{g}}_\kappa}_{\mathfrak{L}^+(\mathfrak{g})}$$

induce a pair of adjoint functors

$$\mathbf{ind}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)} : \mathrm{Rep}(\mathfrak{L}^+(G))_S \rightleftarrows \mathrm{KL}(G)_{\kappa,S} : \mathbf{oblv}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}_{\mathfrak{L}^+(G)}. \tag{B.70}$$

In particular, we obtain that $\mathrm{KL}(G)_{\kappa,S}$ is compactly generated.

B.14.24. Let us describe in concrete terms a set of compact generators of $\mathrm{KL}(G)_{\kappa,S}$. Let I be a finite set as in Sect. B.14.2.

For a compact object $V \in \mathrm{Rep}(G)^{\otimes I}$, let us denote by a light abuse of notation by the same character V its image under (B.61).

The objects

$$\mathbf{ind}_{\mathfrak{L}^+(G)}^{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))_\kappa}(V) \in \mathrm{KL}(G)_{\kappa,S}$$

are called Weyl modules, and they compactly generate $\mathrm{KL}(G)_{\kappa,S}$.

Remark B.14.25. Let $\underline{\mathbf{C}}$ be a sheaf of categories on S_{dR}, where S is an affine scheme of finite type. There is a stronger notion than compact generation, called *ULA generation*:

An object $\mathbf{c} \in \Gamma(S_{\mathrm{dR}}, \underline{\mathbf{C}})$ is said to be ULA if its image in $\Gamma(S, \underline{\mathbf{C}})$ is compact.

We say that $\underline{\mathbf{C}}$ is ULA-generated if it contains a collection of ULA objects, whose images in $\Gamma(S, \underline{\mathbf{C}})$ generate this category.

We will say that a factorization category $\mathbf{C}$ is ULA-generated if for every $S \to \mathrm{Ran}$, the corresponding category $\mathbf{C}_{S_{\mathrm{dR}}}$ is. This property is enough to check for $S = X^I$.

Many factorization categories that appear in geometric representation theory have this property, e.g., $\mathbf{C} = \mathrm{Sph}_G$. However, the category $\mathrm{KL}(G)_\kappa$ does not: namely, $\mathrm{KL}(G)_{\kappa,X^I}$ is not ULA-generated for $|I| \geq 2$.

B.14.26. As in Sect. B.14.21, we obtain that $\mathrm{KL}(G)_{\kappa,S}$ carries a t-structure, for which both functors (B.70) are t-exact.

B.14.27. For $S' \to S$, the equivalence

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S'}$$

induces a functor

$$\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{KL}(G)_{\kappa,S} = \mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S}^{\mathfrak{L}^+(G)_S} \to \tag{B.71}$$
$$\to (\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{KL}(G)_{\kappa,S})^{\mathfrak{L}^+(G)_S} \simeq (\mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{KL}(G)_{\kappa,S})^{\mathfrak{L}^+(G)_{S'}} \simeq \mathrm{KL}(G)_{\kappa,S'}.$$

However, the functor

$$\mathbf{C} \to \mathbf{C}^{\mathfrak{L}^+(G)_S}, \quad \mathfrak{L}^+(G)_S\text{-}\mathbf{mod} \to \mathrm{QCoh}(S)\text{-}\mathbf{mod}$$

is known to commute with colimits. Hence, the functor (B.71) is an equivalence.

This endows the assignment

$$S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}, \rightsquigarrow \mathrm{KL}(G)_{\kappa,S}$$

with a structure of sheaf of categories. We denote it by $\underline{\mathrm{KL}}(G)_\kappa$.

B.14.28. Similarly, in the situation Sect. B.14.6, the equivalence

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S_1} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S_2} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S_1 \times S_2}$$

induces an equivalence

$$\mathrm{KL}(G)_{\kappa,S_1} \otimes \mathrm{KL}(G)_{\kappa,S_2} = (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S_1})^{\mathfrak{L}^+(G)_{S_1}} \otimes (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S_1})^{\mathfrak{L}^+(G)_{S_2}} \simeq$$
$$\simeq (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S_1} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S_2})^{\mathfrak{L}^+(G)_{S_1} \times \mathfrak{L}^+(G)_{S_2}} = \mathrm{KL}(G)_{\kappa,S_1 \times S_2}.$$

This endows $\underline{\mathrm{KL}}(G)_\kappa$ with a factorization structure. The resulting factorization category, denoted $\mathrm{KL}(G)_\kappa$ is the sought-for factorization incarnation of the Kazhdan-Lusztig category.

The t-structures in Sect. B.14.26 define a t-structure on $\mathrm{KL}(G)_\kappa$ as a factorization category.

B.15. **Restriction of factorization module categories, continued.** In this subsection we will discuss some additional aspects of the operation of restriction of factorization module categories, introduced in Sect. B.12.11.

We will omit most proofs (they are elaborations of the limit construction described in Sect. B.9.28); a more detailed discussion will appear in [CFGY].

B.15.1. Fix $\mathcal{Z} \to \mathrm{Ran}$. Consider the totality of pairs $(\mathbf{A}, \mathbf{C})$, where $\mathbf{A}$ is a factorization category, and $\mathbf{C} \in \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$. We view it as a 2-category, to be denoted $\mathrm{FactCat\text{-}and\text{-}Mod}(X)$ with the above objects, and the categories of morphisms defined as follows:

For a pair of objects $(\mathbf{A}_1, \mathbf{C}_1)$ and $(\mathbf{A}_2, \mathbf{C}_2)$ of $\mathrm{FactCat\text{-}and\text{-}Mod}(X)$, the category

$$\mathrm{Maps}_{\mathrm{FactCat\text{-}and\text{-}Mod}(X)}((\mathbf{A}_1, \mathbf{C}_1), (\mathbf{A}_2, \mathbf{C}_2))$$

consists of pairs of morphisms $(\Phi : \mathbf{A}_1 \to \mathbf{A}_2, \Phi_m : \mathbf{C}_1 \to \mathbf{C}_2)$ as in Sect. B.12.11.

Consider the natural projection

$$\mathrm{FactCat\text{-}and\text{-}Mod}(X) \to \mathrm{FactCat}(X). \tag{B.72}$$

The following assertion encodes the functoriality of the assignment

$$\mathbf{A} \rightsquigarrow \mathbf{C} \in \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}.$$

Theorem B.15.2. *The functor*

$$\mathrm{FactCat\text{-}and\text{-}Mod}(X)^{2\text{-}\mathrm{op}} \to \mathrm{FactCat}(X)^{2\text{-}\mathrm{op}},$$

induced by (B.72) *is a 2-Cartesian fibration, where:*

- *The symbol* $(-)^{2\text{-}\mathrm{op}}$ *refers to reversing 2-morphisms in a given 2-category;*
- *The notion of 2-Cartesian fibration is an in* [GaRo3, Chapter 11, Sect. 1.1.].

Remark B.15.3. Note that the situation in Theorem B.15.2 is parallel to the following more familiar paradigm, when instead of $\mathrm{FactCat\text{-}and\text{-}Mod}(X)$ we consider the category of pairs $(\mathbf{A}, \mathbf{C})$, where $\mathbf{A}$ is a monoidal category, and $\mathbf{C} \in \mathbf{A}\text{-}\mathbf{mod}$, and where the category of functors

$$(\mathbf{A}_1, \mathbf{C}_1) \to (\mathbf{A}_2, \mathbf{C}_2)$$

consists of right-lax monoidal functors between the monoidal categories and compatible right-lax functors between the module categories.

B.15.4. The concrete meaning of this theorem is the following. It says that:

- For
$$\mathbf{A}_1 \overset{\Phi_{1,2}}{\to} \mathbf{A}_2 \overset{\Phi_{1,2}}{\to} \mathbf{A}_3, \quad \Phi_{1,3} = \Phi_{2,3} \circ \Phi_{1,2},$$
and $\mathbf{C}_3 \in \mathbf{A}_3\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$, the tautological functor
$$\mathrm{Res}_{\Phi_{1,2}} \circ \mathrm{Res}_{\Phi_{2,3}}(\mathbf{C}_3) \to \mathrm{Res}_{\Phi_{1,3}}$$
is an equivalence;
- For $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$, $\mathbf{C}_i \in \mathbf{A}_i\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$, $i = 1, 2$, $\Phi_m \in \mathrm{Funct}_{\Phi:\mathbf{A}_1 \to \mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2)$ and a natural transformation $\Phi \to \Phi'$, there exists an object $\Phi'_m \in \mathrm{Funct}_{\Phi':\mathbf{A}_1 \to \mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2)$ together with a compatible natural transformation $\Phi'_m \to \Phi_m$, which is universal with respect to this property.

B.15.5. According to [GaRo3, Chapter 11, Theorem 1.1.8], we can interpret Theorem B.15.2 as saying that the assignment

$$\mathbf{A} \mapsto \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$$

extends to a functor between 2-categories

$$\mathrm{FactCat}(X)^{1\text{-}\mathrm{op}, 2\text{-}\mathrm{op}} \to 2\text{-}\mathrm{Cat},$$

where:

- The symbol $(-)^{1\text{-}\mathrm{op}, 2\text{-}\mathrm{op}}$ refers to reversing both 1-morphisms and 2-morphisms in a given 2-category;
- 2-Cat is the totality of $(\infty, 2)$-categories, viewed as a 2-category.

B.15.6. We will need the following corollary of Theorem B.15.2. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be a pair of factorization categories, and let

$$\Phi : \mathbf{A}_1 \leftrightarrows \mathbf{A}_2 : \Phi^R$$

be a pair of factorization functors.

We claim:

Corollary B.15.7. *For $\mathbf{C}_i \in \mathbf{A}_i\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$, $i = 1, 2$ there is a canonical equivalence*

$$\mathrm{Funct}_{\mathbf{A}_1\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}}(\mathrm{Res}_\Phi(\mathbf{C}_2), \mathbf{C}_1) \simeq \mathrm{Funct}_{\mathbf{A}_2\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}}(\mathbf{C}_2, \mathrm{Res}_{\Phi^R}(\mathbf{C}_1)).$$

This proposition can be reformulated as saying that the functor Res_Φ is the *left* adjoint of the functor Res_{Φ^R}.

Proof. This is a formal corollary of having a 2-Cartesian fibration. To be explicit, let us exhibit the unit and the counit of the adjunction.

The unit is given by

$$\mathrm{Id} = \mathrm{Res}_{\mathrm{Id}} \overset{\Phi\circ\Phi^R\to\mathrm{Id}}{\longrightarrow} \mathrm{Res}_{\Phi\circ\Phi^R} \simeq \mathrm{Res}_{\Phi^R} \circ \mathrm{Res}_\Phi\,.$$

The counit is given by

$$\mathrm{Res}_\Phi \circ \mathrm{Res}_{\Phi^R} \simeq \mathrm{Res}_{\Phi^R\circ\Phi} \overset{\mathrm{Id}\to\Phi^R\circ\Phi}{\longrightarrow} \mathrm{Res}_{\mathrm{Id}} = \mathrm{Id}\,.$$

□

B.15.8. We will now use Corollary B.15.7 to prove the following partial converse to Lemma B.12.12:

Lemma B.15.9. *Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a factorization functor, and let $\mathbf{C}_1 \to \mathbf{C}_2$ be a functor between objects of $\mathbf{A}_i\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$, $i = 1, 2$, compatible with factorization. Assume that:*

(i) *The functor $\Phi : \underline{\mathbf{A}}_1 \to \underline{\mathbf{A}}_2$ between sheaves of categories on* Ran *admits a right adjoint;*

(ii) *The functor $\Phi_m : \underline{\mathbf{C}}_1 \to \underline{\mathbf{C}}_2$ between sheaves of categories on $\mathcal{Z}^{\subseteq}$ admits a right adjoint;*

(iii) *The induced functor $\Phi_m : \mathbf{C}_{1,\mathcal{Z}} \to \mathbf{C}_{2,\mathcal{Z}}$ is an equivalence.*

Then the resulting functor

$$\mathbf{C}_1 \to \mathrm{Res}_\Phi(\mathbf{C}_2) \tag{B.73}$$

as module categories over $\mathbf{A}_1$, is an equivalence.

Proof. We claim that the functor (B.73) admits a right adjoint. Once we prove this, the lemma will follow, because a functor between sheaves of categories that admits a right adjoint is an equivalence if and only if it is an equivalence strata-wise.

The right adjoint of $\Phi_m : \underline{\mathbf{C}}_1 \to \underline{\mathbf{C}}_2$ is a functor compatible with factorization against $\Phi^R : \underline{\mathbf{A}}_1 \to \underline{\mathbf{A}}_2$. Hence, it gives rise to an object

$$\Phi^R_m \in \mathrm{Funct}_{\Phi^R:\mathbf{A}_2\to\mathbf{A}_1}(\mathbf{C}_2, \mathbf{C}_1) \simeq \mathrm{Funct}_{\mathbf{A}_2\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}}(\mathbf{C}_2, \mathrm{Res}_{\Phi^R}(\mathbf{C}_1)).$$

(In fact, the pair (Φ^R, Φ^R_m) is the right adjoint of (Φ, Φ_m) as a 1-morphism in FactCat-and-Mod(X).)

Using Corollary B.15.7, we identify the latter category with

$$\mathrm{Funct}_{\mathbf{A}_1\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}}(\mathrm{Res}_\Phi(\mathbf{C}_2), \mathbf{C}_1).$$

Unwinding the constructions, we obtain that the resulting functor

$$\mathrm{Res}_\Phi(\mathbf{C}_2) \to \mathbf{C}_1$$

is indeed the right adjoint of (B.73).

□

Appendix C. Unital structures

In the previous section we introduced the notion of factorization category. In this section we will describe an extra structure that factorization categories often carry: the *unital* structure.

The role that the unital structure plays is two-fold. For one thing, it enables various local-to-global constructions (see Sect. 11). But it also leads to purely local constructions, which play a key role in this paper: given a lax-unital factorization functor $F : \mathbf{A}_1 \to \mathbf{A}_2$ between unital factorization categories, the image $F(\mathbf{1}_{\mathbf{A}_1})$ of the unite $\mathbf{1}_{\mathbf{A}_1} \in \mathbf{A}_2$ is a factorization algebra in $\mathbf{A}_2$, and the functor F *enhances* to a functor

$$F^{\mathrm{enh}} : \mathbf{A}_1 \to F(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathbf{A}_2).$$

Now the functor is often, if not an equivalence, but is close to be such[72], and that allows to understand the more complicated category $\mathbf{A}_1$ in terms of $\mathbf{A}_1$.

In order to talk about unitality we have to enlarge our world of algebro-geometric objects. Namely, we normally work with prestacks (i.e., spaces in algebro-geometric sense, whose functor of points takes place in (∞)-groupoids). But in in order to talk about unitality, we need to work with *categorical prestacks*; whose functors of points take place in (∞)-categories. The rudiments of categorical prestacks (D-modules and sheaves of categories on them) are also developed in this section.

C.1. **Categorical prestacks.**

C.1.1. When discussing categorical prestacks, we will work in the locally almost of finite type (laft) category. Accordingly, when we write $\mathrm{Sch}^{\mathrm{aff}}$ (resp., PreStk), we will mean affine schemes (resp., prestacks) locally almost of finite type.

By a categorical prestack we shall mean a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Cat}\,.$$

Given a categorical prestack $\mathcal{Y}$ we will denote by

$$\mathcal{Y}(S) \text{ or } \mathrm{Maps}(S, \mathcal{Y})$$

the category of its values on $S \in \mathrm{Sch}^{\mathrm{aff}}$.

We let CatPreStk denote the category of categorical prestacks.

C.1.2. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between categorical prestacks. We shall say that some categorical property of f (such as being Cartesian/co-Cartesian, cofinal, admitting an adjoint) holds *value-wise* if this property holds for the corresponding functor

$$\mathcal{Y}_1(S) \to \mathcal{Y}_2(S)$$

for every $S \in \mathrm{Sch}^{\mathrm{aff}}$.

C.1.3. Let $\mathcal{Y}$ be a categorical prestack. To it we will associate two prestacks in groupoids, by applying the right and left adjoints to the embedding

$$\infty\text{-Grpd} \hookrightarrow \infty\text{-Cat},$$

respectively.

We let $\mathcal{Y}^{\mathrm{grpd}}$ denote the prestack whose value on $S \in \mathrm{Sch}^{\mathrm{aff}}$ is the groupoid underlying the category $\mathcal{Y}(S)$.

We will denote by $\mathtt{t}$ the tautological map

$$\mathcal{Y}^{\mathrm{grpd}} \to \mathcal{Y}.$$

We will denote by $\mathcal{Y}^{\mathrm{strict}}$ the prestack in groupoids, whose value on $S \in \mathrm{Sch}^{\mathrm{aff}}$ is the groupoid obtained from $\mathcal{Y}(S)$ by inverting all 1-morphisms.

[72] E.g., in multiple instances, in the presence of a t-structure, it is an equivalence on the bounded below categories.

We will denote the tautological projection by

$$\mathrm{strict} : \mathcal{Y} \to \mathcal{Y}^{\mathrm{strict}}.$$

C.1.4. Given a categorical prestack $\mathcal{Y}$, we associate to it several DG categories of algebro-geometric nature.

Let $\mathcal{Y}_{\mathrm{Sch}^{\mathrm{aff}}}$ denote the Cartesian fibration over $\mathrm{Sch}^{\mathrm{aff}}$ that attaches to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the category $\mathcal{Y}(S)$.

Let

$$\mathrm{QCoh}_{\mathrm{Sch}^{\mathrm{aff}}}, \ \mathrm{IndCoh}_{\mathrm{Sch}^{\mathrm{aff}}} \text{ and } \mathrm{D\text{-}mod}_{\mathrm{Sch}^{\mathrm{aff}}}$$

denote the Cartesian fibrations over $\mathrm{Sch}^{\mathrm{aff}}$ that attach to S the categories

$$\mathrm{QCoh}(S), \ \mathrm{IndCoh}(S) \text{ and } \mathrm{D\text{-}mod}(S), \tag{C.1}$$

respectively.

We define the categories

$$\mathrm{QCoh}(\mathcal{Y}), \ \mathrm{IndCoh}(\mathcal{Y}) \text{ and } \mathrm{D\text{-}mod}(\mathcal{Y}) \tag{C.2}$$

as functors from $\mathcal{Y}_{\mathrm{Sch}^{\mathrm{aff}}}$ to the categories in (C.2) that map arrows that are Cartesian over $\mathrm{Sch}^{\mathrm{aff}}$ to arrows with a similar property. See [Ro2, Sect. C.3] for more details.

C.1.5. For a map of prestacks $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, precomposition with

$$(\mathcal{Y}_1)_{\mathrm{Sch}^{\mathrm{aff}}} \to (\mathcal{Y}_2)_{\mathrm{Sch}^{\mathrm{aff}}}$$

gives rise to a functor

$$f^! : \mathrm{D\text{-}mod}(\mathcal{Y}_2) \to \mathrm{D\text{-}mod}(\mathcal{Y}_1),$$

and similarly for $\mathrm{IndCoh}(\mathcal{Y})$ and $\mathrm{D\text{-}mod}(\mathcal{Y})$.

C.1.6. Categorical prestacks form an $(\infty, 2)$-category, so there is a natural notion of adjunction between morphisms. Explicitly, a morphism

$$f : \mathcal{Y}_1 \rightleftarrows \mathcal{Y}_2$$

admits a right adjoint, if for every $S \in \mathrm{Sch}^{\mathrm{aff}}$, the corresponding functor

$$f : \mathcal{Y}_1(S) \to \mathcal{Y}_2(S)$$

admits a right adjoint, and for every $S' \to S$, the natural transformation

$$\begin{tikzcd}
\mathcal{Y}_1(S') & \mathcal{Y}_2(S') \arrow[l, "f^R"'] \\
\mathcal{Y}_1(S) \arrow[u] \arrow[ur, Rightarrow] & \mathcal{Y}_2(S) \arrow[l, "f^R"] \arrow[u]
\end{tikzcd}$$

arising by adjunction from the commutative diagram

$$\begin{tikzcd}
\mathcal{Y}_1(S') \arrow[r, "f"] & \mathcal{Y}_2(S') \\
\mathcal{Y}_1(S) \arrow[u] \arrow[r, "f"] & \mathcal{Y}_2(S), \arrow[u]
\end{tikzcd}$$

is an isomorphism.

We have the following useful observation:

Lemma C.1.7. *Let*

$$f : \mathcal{Y}_1 \rightleftarrows \mathcal{Y}_2 : g$$

be mutually adjoint maps. Then the functors $(g^!, f^!)$ *form an adjoint pair.*

C.1.8. For a categorical prestack $\mathcal{Y}$, we let $\mathcal{Y}_{\mathrm{dR}}$ denote the categorical prestack defined by

$$\mathcal{Y}_{\mathrm{dR}}(S) := \mathcal{Y}(S_{\mathrm{red}}).$$

A standard manipulation shows that

$$\mathrm{IndCoh}(\mathcal{Y}_{\mathrm{dR}}) \simeq \text{D-mod}(\mathcal{Y}).$$

C.1.9. One can describe the categories (C.2) explicitly as follows. We will do this for $\text{D-mod}(\mathcal{Y})$, while the other two cases are similar.

An object $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ is an assignment:

- For every $y : S \to \mathcal{Y}$ of an object $\mathcal{F}_{S,y} \in \text{D-mod}(S)$;
- For a map $(y_1 \overset{\alpha}{\to} y_2) \in \mathcal{Y}(S)$ of a map $\mathcal{F}_{S,y_1} \overset{\mathcal{F}_\alpha}{\to} \mathcal{F}_{S,y_2}$ in $\text{D-mod}(S)$;
- For $f : S' \to S$ and $y' = y \circ f$ of an isomorphism $\mathcal{F}_{S',y'} \simeq f^!(\mathcal{F}_{S,y})$ in $\text{D-mod}(S')$;
- The datum of commutativity for the diagram

$$\begin{array}{ccc} \mathcal{F}_{S',y'_1} & \xrightarrow{\mathcal{F}_{f^*(\alpha)}} & \mathcal{F}_{S',y'_2} \\ \simeq\downarrow & & \downarrow\simeq \\ f^!(\mathcal{F}_{S,y_1}) & \xrightarrow{f^!(\mathcal{F}_\alpha)} & f^!(\mathcal{F}_{S,y_2}); \end{array}$$

- A homotopy-coherent system of compatibilities for compositions.

C.1.10. In addition to the categories (C.2), one can consider their strict versions:

$$\mathrm{QCoh}(\mathcal{Y})^{\mathrm{strict}} := \mathrm{QCoh}(\mathcal{Y}^{\mathrm{strict}}),\ \mathrm{IndCoh}(\mathcal{Y})^{\mathrm{strict}} := \mathrm{IndCoh}(\mathcal{Y}^{\mathrm{strict}})$$

and

$$\text{D-mod}(\mathcal{Y})^{\mathrm{strict}} := \text{D-mod}(\mathcal{Y}^{\mathrm{strict}}),$$

respectively.

Unwinding the definitions, we obtain that pullback along

$$\mathrm{strict} : \mathcal{Y} \to \mathcal{Y}^{\mathrm{strict}}$$

is a fully faithful embedding, with essential image described as follows:

It consists of those objects $\mathcal{F}$ in Sect. C.1.9, for which the maps

$$\mathcal{F}_{S,y_1} \overset{\mathcal{F}_\alpha}{\to} \mathcal{F}_{S,y_2}$$

are isomorphisms.

Note that we can describe $\text{D-mod}(\mathcal{Y})^{\mathrm{strict}}$ also as

$$\text{D-mod}(\mathcal{Y})^{\mathrm{strict}} = \lim_{S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}} \text{D-mod}(S).$$

The same applies also to $\mathrm{QCoh}(\mathcal{Y})^{\mathrm{strict}}$ and $\mathrm{IndCoh}(\mathcal{Y})^{\mathrm{strict}}$.

C.2. Crystals of categories on categorical prestacks.

C.2.1. Let $\mathcal{Y}$ be a categorical prestack. Combining the ideas of Sects. B.8.2 and C.1.4 we obtain the notion of *sheaf of categories* over $\mathcal{Y}$.

Thus, a sheaf of categories $\underline{\mathbf{C}}$ over $\mathcal{Y}$ is an assignment:

- For every affine scheme S and a map $y : S \to \mathcal{Y}$ of a category $\mathbf{C}_{S,y}$ tensored over $\mathrm{QCoh}(S)$;
- For a map $y_1 \overset{\alpha}{\to} y_2$ in $\mathrm{Maps}(S, \mathcal{Y})$ of a $\mathrm{D\text{-}mod}(S)$-linear functor $\mathbf{C}_{S,y_1} \xrightarrow{\mathbf{C}_{S,\alpha}} \mathbf{C}_{S,y_2}$;
- For $S' \overset{f}{\to} S$ and $y' = y \circ f$ of an identification $\mathbf{C}_{S',y'} \simeq \mathrm{QCoh}(S') \underset{\mathrm{QCoh}(S)}{\otimes} \mathbf{C}_{S,y}$;
- For $\alpha' = \alpha \circ f$ of a datum of commutativity for

$$\begin{CD}
\mathbf{C}_{S,y_1} @>{\mathbf{C}_{S,\alpha}}>> \mathbf{C}_{S,y_2} \\
@V{f^!}VV @VV{f^!}V \\
\mathbf{C}_{S',y'_1} @>{\mathbf{C}_{S',\alpha'}}>> \mathbf{C}_{S',y'_2}.
\end{CD}$$

- A homotopy-coherent system of compatibilities for compositions.

C.2.2. We shall say that $\underline{\mathbf{C}}$ is strict if the functors $\mathbf{C}_{S,\alpha}$ are equivalences. Note that $\underline{\mathbf{C}}$ is strict if it is the pullback[73] along

$$\mathcal{Y} \to \mathcal{Y}^{\mathrm{strict}}$$

of a crystal of categories on $\mathcal{Y}^{\mathrm{strict}}$.

C.2.3. One can assign to $\underline{\mathbf{C}}$ two categories, denoted

$$\Gamma^{\mathrm{lax}}(\mathcal{Y}, \underline{\mathbf{C}}) \text{ and } \Gamma^{\mathrm{strict}}(\mathcal{Y}, \underline{\mathbf{C}}),$$

respectively, defined as follows, the latter being a full subcategory of the former.

An object of $\Gamma^{\mathrm{lax}}(\mathcal{Y}, \underline{\mathbf{C}})$ assigns to every affine scheme S and a map $y : S \to \mathcal{Y}$ an object $\mathbf{c}_{S,y} \in \mathbf{C}_{S,y}$ together with the following data:

- For a map $y_1 \overset{\alpha}{\to} y_2$ in $\mathrm{Maps}(S, \mathcal{Y})$ a morphism

$$\mathbf{C}_{S,\alpha}(\mathbf{c}_{S,y_1}) \to \mathbf{c}_{S,y_2}; \tag{C.3}$$

- For a map $f : S' \to S$ and $y' = y \circ f$ an isomorphism

$$f^!(\mathbf{c}_{S,y}) \simeq \mathbf{c}_{S',y'}$$

  as objects in $\mathbf{C}_{S',y'}$.
- A homotopy-coherent datum of compatibility for the above pieces of data.

The subcategory $\Gamma^{\mathrm{strict}}(\mathcal{Y}, \underline{\mathbf{C}})$ consists of those assignments for which the maps (C.3) are isomorphisms.

Remark C.2.4. We alert the reader to the discrepancy between the notations

$$\Gamma^{\mathrm{lax}}(-,-) \text{ and } \Gamma^{\mathrm{strict}}(-,-)$$

introduced above and those used in [Ra6, Sect. 4].

Namely, what we denote $\Gamma^{\mathrm{lax}}(-,-)$ is denoted $\Gamma(-,-)$ in *loc. cit.*, and what we denote $\Gamma^{\mathrm{strict}}(-,-)$ is denoted $\Gamma^{\mathrm{naive}}(-,-)$ in *loc. cit.*.

Similarly, the notion of *functor* of sheaves of categories considered in [Ra6] corresponds to the notion of *right-lax functor* considered below.

[73] In the sense of Sect. C.2.17.

C.2.5. *Example.* Let $\underline{\mathbf{C}}$ be $\underline{\mathrm{QCoh}}(\mathcal{Y})$, the unit crystal of categories, i.e., its value for $(S,y)\in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$ is $\mathrm{QCoh}(S)$.

Then

$$\Gamma^{\mathrm{lax}}(\mathcal{Y},\underline{\mathbf{C}})=\mathrm{QCoh}(\mathcal{Y}),$$

see Sect. C.1.4 and

$$\Gamma^{\mathrm{strict}}(\mathcal{Y},\underline{\mathbf{C}})=\mathrm{QCoh}(\mathcal{Y})^{\mathrm{strict}}\simeq \mathrm{QCoh}(\mathcal{Y}^{\mathrm{strict}})$$

(see Sect. C.1.10).

C.2.6. Let $\underline{\mathbf{C}}'$ and $\underline{\mathbf{C}}''$ be two crystals of categories on $\mathcal{Y}$. In this case there is an (evident) notion of functor

$$\underline{\Phi}:\underline{\mathbf{C}}'\to\underline{\mathbf{C}}''.$$

When an ambiguity is likely to occur, we will call such functors *strict*.

C.2.7. For future reference, a (strict) functor $\underline{\Phi}$ is said to be fully faithful if for every $y:S\to\mathcal{Y}$, the resulting functor

$$\mathbf{C}'_{S,y}\to\mathbf{C}''_{S,y}$$

is fully faithful.

C.2.8. In addition, there is a notion of *right-lax* functor. A right-lax functor $\underline{\Phi}:\underline{\mathbf{C}}'\to\underline{\mathbf{C}}''$ is an assignment:

- For every (S,y) of a functor $\mathbf{C}'_{S,y}\overset{\Phi_{S,y}}{\to}\mathbf{C}''_{S,y}$;
- For every map $y_1\overset{\alpha}{\to}y_2$ in $\mathrm{Maps}(S,\mathcal{Y})$ we have a *natural transformation*

$$\mathbf{C}''_{S,\alpha}\circ\Phi_{S,y_1}\to\Phi_{S,y_2}\circ\mathbf{C}'_{S,\alpha}. \tag{C.4}$$

- For $f:\widetilde{S}\to S$ and $\widetilde{y}=y\circ f$ of an isomorphism $f^!\circ\Phi_{S,y}\simeq\Phi_{\widetilde{S},\widetilde{y}}\circ f^!$;
- A homotopy-coherent system of compatibilities for the above data.

By definition, a right-lax functor is *strict* if the natural transformations (C.4) are isomorphisms.

We denote the categories of right-lax and strict functors by

$$\mathrm{Funct}^{\mathrm{lax}}_{\mathrm{CrystCat}(\mathcal{Y})}(\underline{\mathbf{C}}',\underline{\mathbf{C}}'')\text{ and }\mathrm{Funct}^{\mathrm{strict}}_{\mathrm{CrystCat}(\mathcal{Y})}(\underline{\mathbf{C}}',\underline{\mathbf{C}}''),$$

respectively.

C.2.9. We will denote the (2-)category of sheaves of categories on $\mathcal{Y}$, with 1-morphisms being strict functors by $\mathrm{ShvCat}^{\mathrm{strict}}(\mathcal{Y})$.

We will denote the (2-)category of sheaves of categories on $\mathcal{Y}$, with 1-morphisms being right-lax functors by $\mathrm{ShvCat}^{\mathrm{lax}}(\mathcal{Y})$.

Sometimes we will simply write $\mathrm{ShvCat}(\mathcal{Y})$, when the discussion is applicable in both contexts.

Both $\mathrm{ShvCat}^{\mathrm{strict}}(\mathcal{Y})$ and $\mathrm{ShvCat}^{\mathrm{lax}}(\mathcal{Y})$ carry a natural symmetric monoidal structure with the unit being $\underline{\mathrm{QCoh}}(\mathcal{Y})$.

C.2.10. We set

$$\mathrm{CrystCat}^{\mathrm{strict}}(\mathcal{Y}):=\mathrm{ShvCat}^{\mathrm{strict}}(\mathcal{Y}_{\mathrm{dR}})\text{ and }\mathrm{CrystCat}^{\mathrm{lax}}(\mathcal{Y}):=\mathrm{ShvCat}^{\mathrm{lax}}(\mathcal{Y}_{\mathrm{dR}})$$

Terminologically, when we talk about $\underline{\mathbf{C}}$ being a *crystal of categories* over $\mathcal{Y}$, for $(S,y)\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$, we will denote by

$$\mathbf{C}_{S,y}\in\mathrm{D\text{-}mod}(S)\text{-}\mathbf{mod}$$

the corresponding category of crystalline sections.

We let $\underline{\mathrm{D\text{-}mod}}(\mathcal{Y})$ denote the unit crystal of categories on $\mathcal{Y}$, i.e., its value for $(S,y)\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$ is $\mathrm{D\text{-}mod}(S)$.

C.2.11. Let $\underline{\mathbf{C}}_1$ and $\underline{\mathbf{C}}_2$ be two crystals of categories on $\mathcal{Y}$, and let

$$\underline{\Phi} : \underline{\mathbf{C}}_1 \to \underline{\mathbf{C}}_2$$

be a strict functor.

Assume that the induced functor

$$\mathsf{t}^!(\underline{\mathbf{C}}_1) \overset{\mathsf{t}^!(\underline{\Phi})}{\to} \mathsf{t}^!(\underline{\mathbf{C}}_2)$$

admits a right adjoint[74], to be denoted $(\mathsf{t}^!(\underline{\Phi}))^R$.

In this case $(\mathsf{t}^!(\underline{\Phi}))^R$ admits a natural extension, to be denoted $\underline{\Phi}^R$ to a right-lax functor

$$\underline{\mathbf{C}}_2 \to \underline{\mathbf{C}}_1,$$

see, e.g., [AMR, Lemma B.5.9].

C.2.12. Let $\underline{\mathbf{C}}$ be a crystal of categories over $\mathcal{Y}$. Assume that $\mathsf{t}^!(\underline{\mathbf{C}})$ is dualizable. Assume moreover that for every $(y_1 \overset{\alpha}{\to} y_2) \in \mathrm{Maps}(S, \mathcal{Y})$, the functor

$$\mathbf{C}_{S,y_1} \overset{\mathbf{C}_{S,\alpha}}{\to} \mathbf{C}_{S,y_2} \tag{C.5}$$

admits a right adjoint.

In this case, we can extend the dual $(\mathsf{t}^!(\underline{\mathbf{C}}))^\vee$ to a crystal of categories $\underline{\mathbf{C}}^\vee$ over $\mathcal{Y}$ by letting

$$\mathbf{C}^\vee_{S,y_1} \overset{\mathbf{C}^\vee_{S,\alpha}}{\to} \mathbf{C}^\vee_{S,y_2}$$

be the dual of the right adjoint of (C.5).

Under the above circumstances we will say that $\underline{\mathbf{C}}$ is dualizable, and we will refer to the above crystal of categories $\underline{\mathbf{C}}^\vee$ as the dual of $\underline{\mathbf{C}}$.

Note that we have the natural evaluation and coevaluation that are *right-lax* functors

$$\underline{\text{D-mod}}(\mathcal{Y}) \overset{\text{co-eval}}{\to} \underline{\mathbf{C}} \otimes \underline{\mathbf{D}} \text{ and } \underline{\mathbf{D}} \otimes \underline{\mathbf{C}} \overset{\text{eval}}{\to} \underline{\text{D-mod}}(\mathcal{Y}), \tag{C.6}$$

i.e., the duality between $\underline{\mathbf{C}}$ and $\underline{\mathbf{C}}^\vee$ takes place in the symmetric monoidal category $\mathrm{ShvCat}^{\mathrm{lax}}(\mathcal{Y})$.

C.2.13. Vice versa, let us be given two crystal of categories $\underline{\mathbf{C}}$ and $\underline{\mathbf{D}}$ and right-lax functors as in (C.6). Suppose that the following conditions hold:

- For every $S \overset{y}{\to} \mathcal{Y}$, the functors

$$\text{D-mod}(S) \to \mathbf{C}_{S,y} \underset{\text{D-mod}(S)}{\otimes} \mathbf{D}_{S,y} \text{ and } \mathbf{D}_{S,y} \underset{\text{D-mod}(S)}{\otimes} \mathbf{C}_{S,y} \to \text{D-mod}(S)$$

  define a perfect pairing;
- The identification of the pullback of the composition

$$\underline{\mathbf{C}} \overset{\text{co-eval} \otimes \text{Id}}{\longrightarrow} \underline{\mathbf{C}} \otimes \underline{\mathbf{D}} \otimes \underline{\mathbf{C}} \overset{\text{Id} \otimes \text{eval}}{\longrightarrow} \underline{\mathbf{C}}$$

  along t with the identity endofunctor of $\underline{\mathbf{C}}|_{\mathcal{Y}^{\mathrm{grpd}}}$ extends to $\mathcal{Y}$;
- The identification of the pullback of the composition

$$\underline{\mathbf{D}} \overset{\text{Id} \otimes \text{eval}}{\longrightarrow} \underline{\mathbf{D}} \otimes \underline{\mathbf{C}} \otimes \underline{\mathbf{D}} \overset{\text{co-eval} \otimes \text{Id}}{\longrightarrow} \underline{\mathbf{D}}$$

  along t with the identity functor of $\underline{\mathbf{D}}|_{\mathcal{Y}^{\mathrm{grpd}}}$ extends to $\mathcal{Y}$.

Then $\underline{\mathbf{C}}$ is dualizable, and $\underline{\mathbf{D}}$ identifies canonically with the dual of $\underline{\mathbf{C}}$ in the sense of Sect. C.2.12, see [CF, Remark 11.11.17].

[74]As a functor between sheaves of categories on $\mathcal{Y}^{\mathrm{grpd}}$, i.e., it admits a continuous right adjoint value-wise.

C.2.14. Still equivalently, let us be given a pair of crystals of categories $\underline{\mathbf{C}}$ and $\underline{\mathbf{D}}$ and *either* a right-lax functor

$$\underline{\mathbf{D}} \otimes \underline{\mathbf{C}} \overset{\underline{\text{eval}}}{\to} \underline{\text{D-mod}}(\mathcal{Y})$$

or a right-lax functor

$$\underline{\text{D-mod}}(\mathcal{Y}) \overset{\underline{\text{co-eval}}}{\to} \underline{\mathbf{C}} \otimes \underline{\mathbf{D}}.$$

Suppose that:

- The pullback of $\underline{\text{eval}}$ (resp., $\underline{\text{co-eval}}$) along $\mathbf{t}^!$ is a perfect pairing;
- For every $(y_1 \overset{\alpha}{\to} y_2) \in \text{Maps}(S, \mathcal{Y})$, the resulting natural transformation $\text{Id} \to \mathbf{D}^\vee_{S,\alpha} \circ \mathbf{C}_{S,\alpha}$ (resp., $\mathbf{C}_{S,\alpha} \circ \mathbf{D}^\vee_{S,\alpha} \to \text{Id}$) is the unit (resp., counit) of an adjunction.

Then this datum extends uniquely to a datum of duality between $\underline{\mathbf{C}}$ and $\underline{\mathbf{D}}$ as crystals of categories on $\mathcal{Y}$.

C.2.15. For $\underline{\mathbf{C}}'$ and $\underline{\mathbf{C}}''$ as above, we observe that a right-lax functor $\underline{\Phi} : \underline{\mathbf{C}}' \to \underline{\mathbf{C}}''$ gives rise to a functor

$$\Phi : \Gamma^{\text{lax}}(\mathcal{Y}, \underline{\mathbf{C}}') \to \Gamma^{\text{lax}}(\mathcal{Y}, \underline{\mathbf{C}}'').$$

If $\underline{\Phi}$ is strict, then Φ induces a functor

$$\Gamma^{\text{strict}}(\mathcal{Y}, \underline{\mathbf{C}}') \to \Gamma^{\text{strict}}(\mathcal{Y}, \underline{\mathbf{C}}'').$$

C.2.16. Note that the functors

$$\underline{\mathbf{C}} \mapsto \Gamma^{\text{lax}}(\mathcal{Y}, \underline{\mathbf{C}}) \text{ and } \underline{\mathbf{C}} \mapsto \Gamma^{\text{strict}}(\mathcal{Y}, \underline{\mathbf{C}})$$

can be recovered as adjoints:

For $\mathbf{D} \in \text{DGCat}$, we have

$$\text{Funct}_{\text{cont}}(\mathbf{D}, \Gamma^{\text{lax}}(\mathcal{Y}, \underline{\mathbf{C}})) \simeq \text{Funct}^{\text{lax}}_{\text{CrystCat}(\mathcal{Y})}(\mathbf{D} \otimes \underline{\text{D-mod}}(\mathcal{Y}), \underline{\mathbf{C}})$$

and

$$\text{Funct}_{\text{cont}}(\mathbf{D}, \Gamma^{\text{strict}}(\mathcal{Y}, \underline{\mathbf{C}})) \simeq \text{Funct}^{\text{strict}}_{\text{CrystCat}(\mathcal{Y})}(\mathbf{D} \otimes \underline{\text{D-mod}}(\mathcal{Y}), \underline{\mathbf{C}}),$$

respectively.

C.2.17. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between categorical prestacks. As in Sects. B.8.7 and C.1.5, there is a naturally defined functor

$$f^* : \text{CrystCat}(\mathcal{Y}_1) \to \text{CrystCat}(\mathcal{Y}_2).$$

C.2.18. For a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ between categorical prestacks and a crystal of categories $\underline{\mathbf{C}}$ on $\mathcal{Y}_2$, we have a naturally defined functor

$$f^! : \Gamma^{\text{lax}}(\mathcal{Y}_2, \underline{\mathbf{C}}) \to \Gamma^{\text{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})), \tag{C.7}$$

which induces a functor

$$f^! : \Gamma^{\text{strict}}(\mathcal{Y}_2, \underline{\mathbf{C}}) \to \Gamma^{\text{strict}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})).$$

C.3. **Two notions of direct image of a crystal of categories.**

C.3.1. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between categorical prestacks. We shall say that f is a (co)Cartesian fibration, i.e.:

- f is a value-wise (co)Cartesian fibration;
- For $S' \to S$, the functor $\mathcal{Y}_1(S) \to \mathcal{Y}_1(S')$ sends arrows that are (co)Cartesian over $\mathcal{Y}_2(S)$ to arrows in $\mathcal{Y}_1(S')$ with the same property.

Note that the second condition is automatic if f is a value-wise (co)Cartesian fibration *in groupoids*.

C.3.2. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a Cartesian fibration.

Let $\underline{\mathbf{C}}_1$ be a crystal of categories over $\mathcal{Y}_1$. In this case one can form two sheaves of categories, denoted

$$f_*(\underline{\mathbf{C}}_1) \text{ and } f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)$$

on $\mathcal{Y}_2$, as follows.

C.3.3. For $y : S \to \mathcal{Y}_2$, the value of $f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)$ on (S, y_2) is

$$\Gamma^{\mathrm{lax}}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1})$$

and the value of $f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1)$ is

$$\Gamma^{\mathrm{strict}}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}).$$

The data of crystal of categories on $f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)$ is defined as follows.

C.3.4. For a map $y' \overset{\alpha}{\to} y''$ in $\mathrm{Maps}(S, \mathcal{Y}_2)$ we have a map

$$S \underset{y'',\mathcal{Y}_2}{\times} \mathcal{Y}_1 \overset{\alpha^*_{\mathcal{Y}_2}}{\to} S \underset{y',\mathcal{Y}_2}{\times} \mathcal{Y}_1.$$

The structure on $\underline{\mathbf{C}}_1$ of crystal of categories gives rise to a (strict) functor between crystals of categories on $S \underset{y'',\mathcal{Y}_2}{\times} \mathcal{Y}_1$

$$(\alpha^*_{\mathcal{Y}_2})^*(\underline{\mathbf{C}}_1|_{S \underset{y',\mathcal{Y}_2}{\times} \mathcal{Y}_1}) \to \underline{\mathbf{C}}_1|_{S \underset{y'',\mathcal{Y}_2}{\times} \mathcal{Y}_1}.$$

Hence, the constructions of Sects. C.2.15 and C.2.18 combine to gives rise to a functor

$$\begin{aligned} f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)_{S,y'} := \Gamma^{\mathrm{lax}}(S \underset{y',\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}) \overset{(\alpha^*_{\mathcal{Y}_2})^!}{\longrightarrow} \\ \to \Gamma^{\mathrm{lax}}(S \underset{y'',\mathcal{Y}_2}{\times} \mathcal{Y}_1, (\alpha^*_{\mathcal{Y}_2})^*(\underline{\mathbf{C}}_1|_{S \underset{y',\mathcal{Y}_2}{\times} \mathcal{Y}_1})) \to \Gamma^{\mathrm{lax}}(S \underset{y'',\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}) =: f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)_{S,y''}. \end{aligned} \tag{C.8}$$

This functor induces a functor

$$\begin{aligned} f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1)_{S,y'} := \Gamma^{\mathrm{strict}}(S \underset{y',\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}) \overset{(\alpha^*_{\mathcal{Y}_2})^!}{\longrightarrow} \\ \to \Gamma^{\mathrm{strict}}(S \underset{y'',\mathcal{Y}_2}{\times} \mathcal{Y}_1, (\alpha^*_{\mathcal{Y}_2})^*(\underline{\mathbf{C}}_1|_{S \underset{y',\mathcal{Y}_2}{\times} \mathcal{Y}_1})) \to \Gamma^{\mathrm{strict}}(S \underset{y'',\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}) =: f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1)_{S,y''}. \end{aligned}$$

C.3.5. For $g : \widetilde{S} \to S$ and $\widetilde{y} = f \circ g$, we have a map

$$g : \widetilde{S} \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1,$$

and the construction of Sect. C.2.18 gives rise to a functor

$$f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)_{S,y} := \Gamma^{\mathrm{lax}}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}) \overset{g^!}{\to} \Gamma^{\mathrm{lax}}(\widetilde{S} \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{\widetilde{S} \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}) =: f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)_{\widetilde{S},\widetilde{y}}.$$

This functor induces a functor

$$f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1)_{S,y} := \Gamma^{\mathrm{strict}}(S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}) \overset{g^!}{\to} \Gamma^{\mathrm{strict}}(\widetilde{S} \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1, \underline{\mathbf{C}}_1|_{\widetilde{S} \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1}) =: f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1)_{\widetilde{S},\widetilde{y}}.$$

C.3.6. By construction, we have a strict functor

$$f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1) \to f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1),$$

which is a value-wise fully faithful embedding.

C.3.7. The above construction is functorial in the following sense: for a lax functor $\underline{\Phi} : \underline{\mathbf{C}}_1' \to \underline{\mathbf{C}}_1''$ between crystals of categories on $\mathcal{Y}_1$ we obtain a lax functor

$$f_*(\underline{\Phi}) : f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1') \to f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1'').$$

If $\underline{\Phi}$ is strict, then so is $f_*(\underline{\Phi})$, and it also induces a strict functor

$$f_*(\Phi) : f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1') \to f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1'').$$

C.3.8. The operations

$$\underline{\mathbf{C}}_1 \mapsto f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1) \text{ and } \underline{\mathbf{C}}_1 \mapsto f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)$$

can also be realized as right adjoints. Namely, for $\underline{\mathbf{C}}_2 \in \mathrm{CrystCat}(\mathcal{Y}_2)$, we have

$$\mathrm{Funct}^{\mathrm{strict}}_{\mathrm{CrystCat}(\mathcal{Y}_1)}(f^*(\underline{\mathbf{C}}_2), \underline{\mathbf{C}}_1) \simeq \mathrm{Funct}^{\mathrm{strict}}_{\mathrm{CrystCat}(\mathcal{Y}_2)}(\underline{\mathbf{C}}_2, f_{*,\mathrm{strict}}(\underline{\mathbf{C}}_1))$$

and

$$\mathrm{Funct}^{\mathrm{lax}}_{\mathrm{CrystCat}(\mathcal{Y}_1)}(f^*(\underline{\mathbf{C}}_2), \underline{\mathbf{C}}_1) \simeq \mathrm{Funct}^{\mathrm{lax}}_{\mathrm{CrystCat}(\mathcal{Y}_2)}(\underline{\mathbf{C}}_2, f_{*,\mathrm{lax}}(\underline{\mathbf{C}}_1)).$$

In particular, we have a canonically defined (strict) functor

$$\underline{\mathbf{C}}_2 \to f_{*,\mathrm{strict}} \circ f^*(\underline{\mathbf{C}}_2). \tag{C.9}$$

C.3.9. Consider the forgeful functor

$$\mathrm{CrystCat}^{\mathrm{strict}}(\mathcal{Y}) \to \mathrm{CrystCat}^{\mathrm{lax}}(\mathcal{Y}).$$

We claim that it admits a right adjoint. Namely, consider the map

$$\mathrm{pr}_{\mathrm{source}} : \mathcal{Y}^{\to} \to \mathcal{Y}$$

is a Cartesian fibration.

The above right adjoint is given by

$$(\mathrm{pr}_{\mathrm{source}})_{*,\mathrm{lax}} \circ (\mathrm{pr}_{\mathrm{target}})^*.$$

In particular, for $\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2 \in \mathrm{CrystCat}^{\mathrm{strict}}(\mathcal{Y})$ we have a canonical identification

$$\mathrm{Funct}^{\mathrm{lax}}(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) \simeq \mathrm{Funct}^{\mathrm{strict}}(\underline{\mathbf{C}}_1, (\mathrm{pr}_{\mathrm{source}})_{*,\mathrm{lax}} \circ (\mathrm{pr}_{\mathrm{target}})^*(\mathbf{C}_2)), \tag{C.10}$$

with the inclusion

$$\mathrm{Funct}^{\mathrm{lax}}(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) \hookrightarrow \mathrm{Funct}^{\mathrm{strict}}(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2)$$

corresponding to the strict functor

$$\underline{\mathbf{C}}_2 \to (\mathrm{pr}_{\mathrm{source}})_{*,\mathrm{strict}} \circ (\mathrm{pr}_{\mathrm{target}})^*(\mathbf{C}_2) \to (\mathrm{pr}_{\mathrm{source}})_{*,\mathrm{lax}} \circ (\mathrm{pr}_{\mathrm{stirct}})^*(\mathbf{C}_2),$$

where the first arrow corresponds by adjunction to the functor

$$(\mathrm{pr}_{\mathrm{source}})^*(\underline{\mathbf{C}}_2) \to (\mathrm{pr}_{\mathrm{target}})^*(\underline{\mathbf{C}}_2),$$

of (11.27).

C.3.10. We now explain an abstract framework for the construction in Sect. 11.5.

Let $\mathcal{Y}$ be a categorical prestack, and let $\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2$ be crystals of categories on it.

Suppose that the functor

$$\underline{\mathbf{C}}_2 \to (\mathrm{pr}_{\mathrm{souce}})_{*,\mathrm{strict}}(\mathbf{C}_2) \to (\mathrm{pr}_{\mathrm{souce}})_{*,\mathrm{lax}}(\mathbf{C}_2) \tag{C.11}$$

admits a left adjoint that is a strict functor.

Then the functor

$$\mathrm{Funct}^{\mathrm{strict}}(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) \to \mathrm{Funct}^{\mathrm{lax}}(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2)$$

admits a left adjoint, given, in terms of (C.10), by composing with the left adjoint of (C.11).

C.4. **Pseudo-properness.**

C.4.1. We introduce the category of *pseudo-proper* categorical prestacks as

$$\mathrm{CatPreStk}_{\text{ps-proper}} := \mathrm{Funct}((\mathrm{Sch}^{\mathrm{proper}})^{\mathrm{op}}, \infty\text{-Cat}).$$

The embedding

$$\mathrm{Sch}^{\mathrm{proper}} \hookrightarrow \mathrm{PreStk} \hookrightarrow \mathrm{CatPreStk}$$

uniquely extends to a colimit-preserving functor

$$\mathrm{CatPreStk}_{\text{ps-proper}} \to \mathrm{CatPreStk}\,. \tag{C.12}$$

Similarly, we define the category

$$\mathrm{PreStk}_{\text{ps-proper}} := \mathrm{Funct}((\mathrm{Sch}^{\mathrm{proper}})^{\mathrm{op}}, \infty\text{-Grpd})$$

and the functor

$$\mathrm{PreStk}_{\text{ps-proper}} \to \mathrm{PreStk}\,. \tag{C.13}$$

Remark C.4.2. Note that the functors (C.12) and (C.13) are *not* fully faithful. I.e., with the above definition, pseudo-properness is not a property, but extra structure.

However, since the functor $\mathrm{Sch}^{\mathrm{proper}} \hookrightarrow \mathrm{PreStk}$ preserves fiber products, so do the functors (C.12) and (C.13).

C.4.3. Concretely, pseudo-properness means the following:

A prestack $\mathcal{Y}$ is pseud-proper when it can written as

$$\underset{i\in I}{\mathrm{colim}}\, Z_i,$$

where:

- Z_i are proper schemes;
- The colimit is taken in PreStk.

As morphisms

$$\underset{i\in I}{\mathrm{colim}}\, Z_i \to \underset{i\in I}{\mathrm{colim}}\, Z'_{i'}$$

we take

$$\lim_{i\in I^{\mathrm{op}}}\, \underset{i'\in I'}{\mathrm{colim}}\, \mathrm{Maps}(Z_i, Z'_{i'}).$$

A categorical prestack $\mathcal{Y}$ is pseudo-proper if the prestacks $\mathrm{Mor}^n(\mathcal{Y})$ classifying n-fold composition of morphisms in $\mathcal{Y}$ are pseudo-proper, and for the maps $[n_1] \to [n_2]$ in $\mathbf{\Delta}$, the corresponding maps

$$\mathrm{Mor}^{[n_2]}(\mathcal{Y}) \to \mathrm{Mor}^{[n_1]}(\mathcal{Y})$$

take place in $\mathrm{PreStk}_{\text{ps-proper}}$.

C.4.4. *Example.* The prestack Ran and the categorical prestacks $\mathrm{Ran}^{\mathrm{untl}}$ and $\mathrm{Ran}^{\mathrm{untl},*}$ (see Sect. C.5.6) are pseudo-proper.

C.4.5. We define the functors D-mod$(-)$ and CrystCat$(-)$ on $\mathrm{CatPreStk}_{\text{ps-proper}}$ precomposing the same-named functors out of CatPreStk with (C.12).

C.4.6. For $\mathcal{Y} \in \mathrm{CatPreStk}_{\text{ps-proper}}$, we can describe D-mod$(\mathcal{Y})$, CrystCat$(\mathcal{Y})$ and

$$\Gamma^{\mathrm{lax}}(\mathcal{Y}, \underline{\mathbf{C}}), \quad \mathbf{C} \in \mathrm{CrystCat}(\mathcal{Y})$$

in terms of proper schemes mapping to $\mathcal{Y}$.

I.e., in the appropriate definitions, we can replace

$$\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}} \rightsquigarrow \mathrm{Sch}^{\mathrm{proper}}_{/\mathcal{Y}},$$

where in the right-hand side the morphisms take place in $\mathrm{CatPreStk}_{\text{ps-proper}}$, C.4.7. Let $\underline{\mathbf{C}}$ be a crystal of categories over $\mathcal{Y}$, where $\mathcal{Y}$ is pseudo-proper. We claim:

Lemma C.4.8. *For $Z \xrightarrow{y} \mathcal{Y}$ with Z proper and y taking place in* $\mathrm{CatPreStk}_{\text{ps-proper}}$*, the functor of evaluation*

$$\Gamma^{\mathrm{lax}}(\mathcal{Y}, \underline{\mathbf{C}}) \to \Gamma(Z, y^*(\underline{\mathbf{C}})) =: \mathbf{C}_{Z,y}$$

commutes with limits.

Proof. We can describe the category

$$\Gamma^{\mathrm{lax}}(\mathcal{Y}, \underline{\mathbf{C}})$$

as a family of assignments

$$(Z \xrightarrow{y} \mathcal{Y}) \in \mathrm{CatPreStk}_{\text{ps-proper}} \rightsquigarrow \mathbf{c}_{Z,y} \in \mathbf{C}_{Z,y}, \quad Z \text{ is proper},$$

compatible under pullbacks:

For $f : Z' \to Z$ we are given an isomorphism

$$f^!(\mathbf{c}_{Z,y}) \simeq \mathbf{c}_{Z',y\circ f}$$

in $\mathbf{C}_{Z',y\circ f}$.

To prove the lemma, it suffices to show that the functors

$$f^! : \mathbf{C}_{Z,y} \to \mathbf{C}_{Z',y\circ f}$$

commute with limits. Indeed, this would imply that limits in $\Gamma^{\mathrm{lax}}(\mathcal{Y}, \underline{\mathbf{C}})$ are computed component-wise in terms of $\{\mathbf{c}_{Z,y}\}$.

We claim that for any proper map $f : Z' \to Z$ and a crystal of categories $\underline{\mathbf{D}}$ on Z, the functor

$$f^! : \Gamma(Z, \underline{\mathbf{D}}) \to \Gamma(Z', f^*(\underline{\mathbf{D}}))$$

commutes with limits.

Indeed, since for a scheme its de Rham space is 1-affine, we can think of $\underline{\mathbf{D}}$ as a D-mod(Z)-linear category $\mathbf{D}$, so that the functor $f^!$ is

$$\Gamma(Z, \underline{\mathbf{D}}) = \mathbf{D} \simeq \text{D-mod}(Z) \underset{\text{D-mod}(Z)}{\otimes} \mathbf{D} \overset{f^! \otimes \mathrm{Id}}{\to} \text{D-mod}(Z') \underset{\text{D-mod}(Z)}{\otimes} \mathbf{D} = \Gamma(Z', f^*(\underline{\mathbf{D}})). \tag{C.14}$$

Now, since f is proper, the functor $f^! : \text{D-mod}(Z) \to \text{D-mod}(Z')$ admits a left adjoint, namely, $f_!$, which is automatically D-mod(Z)-linear. Hence, the functor (C.14) admits a left adjoint, namely,

$$\text{D-mod}(Z') \underset{\text{D-mod}(Z)}{\otimes} \mathbf{D} \overset{f_! \otimes \mathrm{Id}}{\to} \text{D-mod}(Z) \underset{\text{D-mod}(Z)}{\otimes} \mathbf{D} \simeq \mathbf{D}.$$

This implies that $f^!$ commutes with limits.

□

C.4.9. As a consequence of Lemma C.4.8 we obtain:

Corollary C.4.10. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between pseudo-proper categorical prestacks. Then for $\underline{\mathbf{C}} \in \mathrm{CrystCat}(\mathcal{Y}_2)$, the functor*

$$f^! : \Gamma^{\mathrm{lax}}(\mathcal{Y}_2, \underline{\mathbf{C}}) \to \Gamma^{\mathrm{lax}}(\mathcal{Y}_1', f^*(\underline{\mathbf{C}}))$$

admits a left adjoint (to be denoted $f_!$).

Proof. It suffices to check that the functor $f^!$ commutes with limits. The latter follows from Lemma C.4.8.

□

Remark C.4.11. We warn the reader that although for a map f between pseudo-proper prestacks, the functor $f_!$ exists, it does not in general satisfy base change. (It does, however, if f is a value-wise co-Cartesian fibration, see Lemma C.4.17 below.)

As a particular case of Corollary C.4.14 we have:

Corollary C.4.12. *Let $\mathcal{Y}$ be pseudo-proper. Then the functor*

$$C^{\cdot}_c(\mathcal{Y}, -) : \text{D-mod}(\mathcal{Y}) \to \text{Vect},$$

left adjoint to

$$\text{Vect} \overset{\omega_{\mathcal{Y}}}{\to} \text{D-mod}(\mathcal{Y})$$

is well-defined.

C.4.13. Let $\mathcal{Y}$ be pseudo-proper. It follows formally that the prestack in groupoids $\mathcal{Y}^{\text{strict}}$ is also pseudo-proper. Hence, from Corollary C.4.10 we obtain:

Corollary C.4.14. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between pseudo-proper categorical prestacks. Then for a* strict $\underline{\mathbf{C}}$, *the functor*

$$f^! : \Gamma^{\text{strict}}(\mathcal{Y}_2, \underline{\mathbf{C}}) \to \Gamma^{\text{strict}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}}))$$

admits a left adjoint (to be denoted $f_!$).

Remark C.4.15. We warn the reader that the functors

$$f_! : \Gamma^{\text{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \to \Gamma^{\text{lax}}(\mathcal{Y}_2, \underline{\mathbf{C}})$$

and

$$f_! : \Gamma^{\text{strict}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \to \Gamma^{\text{strict}}(\mathcal{Y}_2, \underline{\mathbf{C}})$$

are in general *incompatible* with the embeddings

$$\Gamma^{\text{strict}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \hookrightarrow \Gamma^{\text{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \text{ and } \Gamma^{\text{strict}}(\mathcal{Y}_2, \underline{\mathbf{C}}) \hookrightarrow \Gamma^{\text{lax}}(\mathcal{Y}_2, \underline{\mathbf{C}}),$$

respectively.

C.4.16. We will now show how to compute the functor $f_!$ more explicitly. First, unwinding the definitions, we obtain:

Lemma C.4.17. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map in* $\text{CatPreStk}_{\text{ps-proper}}$ *that is a co-Cartesian fibration.*[75] *Then for $\underline{\mathbf{C}} \in \text{CrystCat}(\mathcal{Y}_2)$, the functor $f_!$ satisfies base change, i.e., for a pullback diagram in* $\text{CatPreStk}_{\text{ps-proper}}$

$$\begin{tikzcd} \mathcal{Y}_1' \arrow[r, "g_1"] \arrow[d, "f'"'] & \mathcal{Y}_1 \arrow[d, "f"] \\ \mathcal{Y}_2' \arrow[r, "g_2"] & \mathcal{Y}_2, \end{tikzcd}$$

the natural transformation

$$f'_! \circ g_1^! \to g_2^! \circ f_!, \quad \Gamma^{\text{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \rightrightarrows \Gamma^{\text{lax}}(\mathcal{Y}_2', g_2^*(\underline{\mathbf{C}})),$$

obtained by adjunction from

$$g_1^! \circ f^! \simeq f'^! \circ g_2^!.$$

is an isomorphism.

Corollary C.4.18. *Under the assumptions of Lemma C.4.17, for a proper scheme Z equipped with a map $Z \overset{y_1}{\to} \mathcal{Y}_1$ in* $\text{CatPreStk}_{\text{ps-proper}}$, *the composition*

$$y_1^! \circ f_! : \Gamma^{\text{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \to \mathbf{C}_{Z,y_1}$$

identifies canonically with

$$\Gamma^{\text{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \overset{\text{pullback}}{\longrightarrow} \Gamma^{\text{lax}}(\mathcal{Y}_{1,Z}, \underline{\mathbf{C}}|_{\mathcal{Y}_{1,Z}}) \overset{(f_Z)_!}{\to} \mathbf{C}_{Z,y_1},$$

where:

- $\mathcal{Y}_{1,Z} := Z \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$;
- *f_Z is the map $\mathcal{Y}_{1,Z} \to Z$.*

[75] As in Sect. C.3.1, but in the category $\text{Funct}((\text{Sch}^{\text{proper}})^{\text{op}}, \infty\text{-Cat})$.

C.4.19. Let now f be an arbitrary map in $\mathrm{CatPreStk}_{\text{ps-proper}}$. Denote $\mathcal{Y}_{2,f/}$ be the (pseudo-proper) categorical prestack given by the slice construction, i.e.,

$$\mathcal{Y}_{2,f/}(S) = \{y_1 \in \mathcal{Y}_1(S),\, y_2 \in \mathcal{Y}_2(S),\, f(y_1) \to y_2\}.$$

Let $\widetilde{f}$ and pr_f denote the projections

$$\mathcal{Y}_{2,f/} \to \mathcal{Y}_2, \quad (y_1, y_2, f(y_1) \to y_2) \mapsto y_2, \quad \mathcal{Y}_{2,f/} \to \mathcal{Y}_1, \quad (y_1, y_2, f(y_1) \to y_2) \mapsto y_1,$$

respectively.

Assume now that $\underline{\mathbf{C}}$ is strict. In this case we have a canonical equivalence

$$\widetilde{f}^*(\underline{\mathbf{C}}) \simeq \mathrm{pr}_f^* \circ f^*(\underline{\mathbf{C}}).$$

We claim:

Lemma C.4.20. *Assume that $\underline{\mathbf{C}}$ is strict. Then the functor $f_!$ identifies canonically with*

$$\Gamma^{\mathrm{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \xrightarrow{\mathrm{pr}_f^!} \Gamma^{\mathrm{lax}}(\mathcal{Y}_{2,f/}, \mathrm{pr}_f^* \circ f^*(\underline{\mathbf{C}})) \simeq \Gamma^{\mathrm{lax}}(\mathcal{Y}_{2,f/}, \widetilde{f}^*(\underline{\mathbf{C}})) \xrightarrow{\widetilde{f}_!} \Gamma^{\mathrm{lax}}(\mathcal{Y}_2, \underline{\mathbf{C}}),$$

Proof. Note that f factors as

$$\mathrm{diag}_f \circ \widetilde{f},$$

where

$$\mathrm{diag}_f : \mathcal{Y}_1 \to \mathcal{Y}_{2,f/}, \quad y_1 \mapsto (y_1, f(y_1), f(y_1) \overset{\mathrm{id}}{\to} f(y_1)).$$

Hence, we obtain

$$f_! \simeq \widetilde{f}_! \circ (\mathrm{diag}_f)_!.$$

Now we claim that

$$(\mathrm{diag}_f)_! \simeq \mathrm{pr}_f^!, \quad \Gamma^{\mathrm{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \rightrightarrows \Gamma^{\mathrm{lax}}(\mathcal{Y}_{2,f/}, \mathrm{pr}_f^* \circ f^*(\underline{\mathbf{C}})).$$

Indeed, this follows from the fact that the morphisms $(\mathrm{diag}_f, \mathrm{pr}_f)$ form an adjoint pair, cf. Lemma C.1.7. $\square$

Corollary C.4.21. *In the setting of Lemma C.4.20, for a proper scheme Z equipped with a map $Z \overset{y_1}{\to} \mathcal{Y}_1$ in* $\mathrm{CatPreStk}_{\text{ps-proper}}$, *the composition*

$$y_1^! \circ f_! : \Gamma^{\mathrm{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \to \mathbf{C}_{Z,y_1}$$

identifies canonically with

$$\Gamma^{\mathrm{lax}}(\mathcal{Y}_1, f^*(\underline{\mathbf{C}})) \overset{\mathrm{pullback}}{\longrightarrow} \Gamma^{\mathrm{lax}}(\mathcal{Y}_{2,f/,Z}, \underline{\mathbf{C}}|_{\mathcal{Y}_{2,f/,Z}}) \overset{(\widetilde{f}_Z)_!}{\longrightarrow} \mathbf{C}_{Z,y_1},$$

where:

- $\mathcal{Y}_{2,f/,Z} := Z \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_{2,f/}$;
- $\widetilde{f}_Z$ *is the map* $\mathcal{Y}_{2,f/,Z} \to Z$.

C.4.22. *Example.* Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be the map

$$\mathsf{t} : \mathcal{Y}^{\mathrm{grpd}} \to \mathcal{Y}.$$

The functor $\mathsf{t}_!$ is the left adjoint of the forgetful functor, and the formula for it, given by Proposition C.4.20, coincides with that of [Ga4, Proposition 4.4.2].

C.4.23. Consider now the commutative square

$$
\begin{tikzcd}
\mathcal{Y}^{\mathrm{grpd}} \arrow[r, "\mathsf{t}"] \arrow[d, "\mathsf{t}^{\mathrm{op}}"'] & \mathcal{Y} \arrow[d, "\mathrm{strict}_{\mathcal{Y}}"] \\
\mathcal{Y}^{\mathrm{op}} \arrow[r, "\mathrm{strict}_{\mathcal{Y}^{\mathrm{op}}}"] & \mathcal{Y}^{\mathrm{strict}}.
\end{tikzcd}
\tag{C.15}
$$

We obtain a natural transformation

$$(\mathsf{t}^{\mathrm{op}})_! \circ \mathsf{t}^! \to (\mathrm{strict}_{\mathcal{Y}^{\mathrm{op}}})^! \circ (\mathrm{strict}_{\mathcal{Y}})_!. \tag{C.16}$$

Suppose now that $\mathcal{Y}$ has the property that (C.15) is Cartesian. In particular, since the right vertical arrow is a co-Cartesian fibration, then so is the left vertical arrow. By Lemma C.4.17, we obtain that in this case (C.16) is an isomorphism.

Applying this in the case $\mathcal{Y} = \mathrm{Ran}^{\mathrm{untl}}$, this gives a conceptual explanation of the commutativity of (11.24), at least in the particular case when $\underline{\mathbf{C}}^{\mathrm{loc}} = \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$, $\mathbf{C}^{\mathrm{glob}} = \mathrm{Vect}$.

C.4.24. The above definitions and assertions admit a variant when we consider prestacks over a given affine base scheme S. In this case, one can talk about pseudo-properness relative to S, and the entire discussion applies.

C.5. **The unital Ran space.**

C.5.1. There are two versions of the unital Ran space that we will consider: $\mathrm{Ran}^{\mathrm{untl}}$ and $\mathrm{Ran}^{\mathrm{untl},*}$.

For $S \in \mathrm{Sch}^{\mathrm{aff}}$, the category $\mathrm{Ran}^{\mathrm{untl}}(S)$ is that of finite *non-empty* subsets in $\mathrm{Maps}(S_{\mathrm{dR}}, X)$, with the morphisms defined as follows:

$$\mathrm{Maps}_{\mathrm{Maps}(S_{\mathrm{dR}},X)}(\underline{x}_1, \underline{x}_2) = \begin{cases} \{*\} \text{ if } \underline{x}_1 \subseteq \underline{x}_2, \\ \emptyset \text{ otherwise.} \end{cases}$$

In the above formula, $\underline{x}_i$ denotes the finite subset of $\mathrm{Hom}(S_{\mathrm{red}}, X)$ corresponding to the same-named $S \to \mathrm{Ran}$.

In the case of $\mathrm{Ran}^{\mathrm{untl},*}$, we allow $\underline{x}$ to be empty, i.e., we add to $\mathrm{Ran}^{\mathrm{untl}}$ a point $\{\emptyset\}$, which is value-wise initial, corresponding to the empty set.

Remark C.5.2. There is a variant of the above definition, where the morphisms are defined by

$$\mathrm{Maps}_{\mathrm{Maps}(S_{\mathrm{dR}},X)}(\underline{x}_1, \underline{x}_2) = \begin{cases} \{*\} \text{ if } \mathrm{Graph}_{\underline{x}_1} \subseteq \mathrm{Graph}_{\underline{x}_2}, \\ \emptyset \text{ otherwise.} \end{cases}$$

where in the formula $\subseteq$ means containment as closed subsets of $S \times X$. Denote the resulting categorical prestack by $'\mathrm{Ran}^{\mathrm{untl}}$.

The two versions are equivalent for most practical purposes. Namely, we have a naturally defined map $\mathrm{Ran}^{\mathrm{untl}} \to {}'\mathrm{Ran}^{\mathrm{untl}}$, and we claim that it induces an equivalence between the corresponding (2-)categories of crystals of categories.

This follows from the fact that the corresponding map

$$\mathrm{Mor}^1(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{Mor}^1({}'\mathrm{Ran}^{\mathrm{untl}})$$

becomes an isomorphism after sheafification in the Grothendieck topology generated by finite surjective maps, while D-modules satisfy descent for this topology.

C.5.3. The above two versions of the unital Ran space, i.e., $\mathrm{Ran}^{\mathrm{untl}}$ and $\mathrm{Ran}^{\mathrm{untl},*}$, are convenient in slightly different situations: the $\mathrm{Ran}^{\mathrm{untl},*}$ version is more convenient for discussing factorization, while the $\mathrm{Ran}^{\mathrm{untl}}$ version is more convenient for the discussion of local-to-global functors. Yet, the next assertion says that we could use $\mathrm{Ran}^{\mathrm{untl},*}$ for the latter too.

Proposition C.5.4. *Let $\underline{\mathbf{C}}$ be a sheaf of categories over $\mathrm{Ran}^{\mathrm{untl},*}$. Then for a DG category $\mathbf{D}$, pullback along $\mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl},*}$ gives rise to an an equivalence*

$$\mathrm{Funct}^{\mathrm{strict}}_{\mathrm{CrystCat}(\mathrm{Ran}^{\mathrm{untl},*})}(\underline{\mathbf{C}}, \mathbf{D} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl},*})) \overset{\sim}{\to}$$
$$\overset{\sim}{\to} \mathrm{Funct}^{\mathrm{strict}}_{\mathrm{CrystCat}(\mathrm{Ran}^{\mathrm{untl}})}(\underline{\mathbf{C}}|_{\mathrm{Ran}^{\mathrm{untl}}}, \mathbf{D} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})).$$

Proof. The assertion follows from the fact that the inclusion

$$\mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl},*}$$

is value-wise cofinal, and the following general claim:

Lemma C.5.5. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a value-wise cofinal morphism between categorical prestacks. Then for any pair of crystals of categories $\underline{\mathbf{C}}', \underline{\mathbf{C}}''$ on $\mathcal{Y}_2$ with $\underline{\mathbf{C}}''$ strict, the functor*

$$\mathrm{Funct}^{\mathrm{strict}}_{\mathcal{Y}_2}(\underline{\mathbf{C}}', \underline{\mathbf{C}}'') \to \mathrm{Funct}^{\mathrm{strict}}_{\mathcal{Y}_1}(f^*(\underline{\mathbf{C}}'), f^*(\underline{\mathbf{C}}'')).$$

□

C.5.6. Note that the presentation of Ran as in Sect. B.1.3 shows that it is pseudo-proper.

We claim that $\mathrm{Ran}^{\mathrm{untl}}$ is also pseudo-proper. We can write

$$\mathrm{Mor}^1(\mathrm{Ran}^{\mathrm{untl}}) = ((\mathrm{Ran}^{\mathrm{untl}})^{\to})^{\mathrm{grpd}} \simeq \mathrm{Ran}^{\subseteq}$$

as the colimit

$$\underset{I_{\mathrm{small}} \subseteq I_{\mathrm{big}}}{\mathrm{colim}} \; X^{I_{\mathrm{big}}},$$

where the colimit is taken over the (opposite of the) category whose objects are pairs of non-empty finite sets $I_1 \to I_2$ and whose morphisms are commutative squares

$$\begin{array}{ccc} I_{\mathrm{small}} & \longrightarrow & I_{\mathrm{big}} \\ \downarrow & & \downarrow \\ I'_{\mathrm{small}} & \longrightarrow & I'_{\mathrm{big}} \end{array}$$

with the vertical arrows surjective.

The maps $\mathrm{pr}_{\mathrm{big}}$ and $\mathrm{pr}_{\mathrm{small}}$ send the term corresponding to $I_{\mathrm{small}} \subseteq I_{\mathrm{big}}$ to

$$X^{I_{\mathrm{big}}} \to \mathrm{Ran} \text{ and } X^{I_{\mathrm{big}}} \to X^{I_{\mathrm{small}}} \to \mathrm{Ran},$$

respectively.

The prestacks of higher-order compositions $\mathrm{Mor}^n(\mathrm{Ran}^{\mathrm{untl}})$ are described similarly.

C.5.7. By Corollary C.4.12, the functors

$$\mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}, -) : \text{D-mod}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{Vect} \text{ and } \mathrm{C}^\cdot_c(\mathrm{Ran}, -) : \text{D-mod}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{Vect},$$

left adjoint to

$$k \mapsto \omega_{\mathrm{Ran}^{\mathrm{untl}}} \text{ and } k \mapsto \omega_{\mathrm{Ran}},$$

respectively, are well-defined.

The same applies to $\mathrm{Ran}^{\mathrm{untl},*}$.

C.5.8. Being the left adjoint of a symmetric monoidal structure, the functor $\mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, -)$ carries a naturally defined left-lax symmetric monoidal structure.

We claim:

Lemma C.5.9. *The left-lax monoidal structure on* $\mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, -)$ *is strict.*

Proof. We need to show that the natural transformation

$$\mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, -) \circ (\Delta_{\mathrm{Ran}^{\mathrm{untl}}})^! \to \mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, -),$$

induced by the $((\Delta_{\mathrm{Ran}^{\mathrm{untl}}})_!, (\Delta_{\mathrm{Ran}^{\mathrm{untl}}})^!)$-adjunction, is an isomorphism.

This follows, however, from the fact that the morphism

$$\Delta_{\mathrm{Ran}^{\mathrm{untl}}} : \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}$$

is value-wise cofinal.

□

C.5.10. We now consider the relation between the functors

$$\mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, -) \text{ and } \mathrm{C}^{\cdot}_c(\mathrm{Ran}, -).$$

We have the natural transformation

$$\mathrm{C}^{\cdot}_c(\mathrm{Ran}, -) \circ \mathsf{t}^! \simeq \mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, -) \circ \mathsf{t}_! \circ \mathsf{t}^! \to \mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, -) \tag{C.17}$$

as functors $\text{D-mod}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{Vect}$.

C.5.11. We claim:

Lemma C.5.12. *The natural transformation transformation* (C.17) *is an isomorphism.*

This assertion is proved in [Ga4, Theorem 4.6.2]. We include the proof for completeness:

Recall what it means for a morphism between categorical prestacks to be *universally homologically cofinal*, see [Ga4, Sect. 3.5.1]. Using [Ga4, Corollary 3.5.12], the assertion of Lemma C.5.12 follows from the next one:

Lemma C.5.13. *The map* t *is universally homologically cofinal.*

Proof of Lemma C.5.13. Let S be an affine scheme and let us be given an S-point $\underline{x}$ of Ran. Consider the corresponding prestack

$$\mathrm{Ran}_{\underline{x}/},$$

see [Ga4, Sect. 3.5.1]. We need to show that it is *universally homologically contractible* over S (see [Ga4, Sect. 2.5.1] for what this means).

Note, however, that $\mathrm{Ran}_{\underline{x}/}$ is a prestack in groupoids isomorphic to $S^{\subseteq}_{\underline{x}}$, and its projection to S is pseudo-proper. Hence, it is enough to show that the fibers of the map

$$S^{\subseteq}_{\underline{x}} \to S$$

have trivial homology. The latter follows by the usual argument for the contractibility of the Ran space.

□

Remark C.5.14. Statements parallel to Lemmas C.5.9 and C.5.12 hold for the $\mathrm{Ran}^{\mathrm{untl},*}$ version of the unital Ran space, see [Ro2, Sect. 2.5].

C.5.15. An analog of the assertion of Lemma C.5.9 would of course fail for the usual (i.e., non-unital) Ran space. I.e., the natural transformation

$$\mathrm{C}^\cdot_c(\mathrm{Ran}, -) \circ (\Delta_{\mathrm{Ran}})^! \simeq \mathrm{C}^\cdot_c(\mathrm{Ran} \times \mathrm{Ran}, -) \circ (\Delta_{\mathrm{Ran}})_! \circ (\Delta_{\mathrm{Ran}})^! \to \mathrm{C}^\cdot_c(\mathrm{Ran} \times \mathrm{Ran}, -) \tag{C.18}$$

is not an isomorphism.

However, it admits the following variant:

Let us denote by

$$\text{D-mod}(\mathrm{Ran})^{\text{almost-untl}} \subset \text{D-mod}(\mathrm{Ran})$$

the full subcategory generated by the essential image of the forgetful functor

$$\mathfrak{t}^! : \text{D-mod}(\mathrm{Ran}^{\mathrm{untl}}) \to \text{D-mod}(\mathrm{Ran}).$$

Note this subcategory is preserved by the monoidal operation.

We claim the left-lax monoidal structure on $\mathrm{C}^\cdot_c(\mathrm{Ran}, -)$, given by (C.18), becomes strict when restricted to to $\text{D-mod}(\mathrm{Ran})^{\text{almost-untl}}$. Indeed, this follows from Lemma C.5.12.

C.5.16. By a similar token, for $\mathcal{Z} \to \mathrm{Ran}$ one defines a unital version

$$\mathcal{Z}^{\subseteq,\mathrm{untl}} := \mathcal{Z} \underset{\mathrm{Ran}^{\mathrm{untl}}}{\times} (\mathrm{Ran}^{\mathrm{untl}})^{\to}$$

of $\mathcal{Z}^{\subseteq}$.

The (categorical) prestacks $\mathcal{Z}^{\subseteq}$ and $\mathcal{Z}^{\subseteq,\mathrm{untl}}$ are pseudo-proper relative to $\mathcal{Z}$.

The assertion of Lemma C.5.12 renders to the present context, when instead of the functors

$$\mathrm{C}^\cdot_c(\mathrm{Ran}, -) \text{ and } \mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}, -)$$

we use the functors

$$(\mathrm{pr}_{\mathrm{small},\mathcal{Z}})_! : \text{D-mod}(\mathcal{Z}^{\subseteq}) \to \text{D-mod}(\mathcal{Z})$$

and

$$(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},\mathcal{Z}})_! : \text{D-mod}(\mathcal{Z}^{\subseteq,\mathrm{untl}}) \to \text{D-mod}(\mathcal{Z}),$$

respectively.

C.6. **Unital and counital factorization spaces.**

C.6.1. Let $\mathcal{Z}_{\mathrm{Ran}} \to \mathrm{Ran}$ be a prestack. A unital structure on $\mathcal{Z}_{\mathrm{Ran}}$ is its extension to a categorical prestack

$$\mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}} \to \mathrm{Ran}^{\mathrm{untl},*}, \tag{C.19}$$

such that (C.19) is a value-wise *co-Cartesian fibration in groupoids.*

Let $\mathrm{PreStk}^{\mathrm{untl}}_{/\mathrm{Ran}}$ denote the category of prestacks over Ran, equipped with a unital structure.

C.6.2. In concrete terms, an upgrade

$$\mathcal{Z}_{\mathrm{Ran}} \rightsquigarrow \mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}}$$

means that for every $\underline{x} \subseteq \underline{x}'$ we give ourselves a map

$$\mathrm{ins.\,unit}_{\underline{x}\subseteq\underline{x}'} : \mathcal{Z}_{\underline{x}} \to \mathcal{Z}_{\underline{x}'},$$

in a way compatible with compositions.

In addition, we give ourselves a space $\mathcal{Z}_\emptyset$ and a system of maps

$$\mathrm{ins.\,unit}_{\emptyset\subset\underline{x}} : \mathcal{Z}_\emptyset \to \mathcal{Z}_{\underline{x}}$$

equipped with identifications

$$\mathrm{ins.\,unit}_{\underline{x}\subseteq\underline{x}'} \circ \mathrm{ins.\,unit}_{\emptyset\subset\underline{x}} \simeq \mathrm{ins.\,unit}_{\emptyset\subset\underline{x}'}\,.$$

C.6.3. Here is an example of a prestack equipped with a unital structure (see [Ro2, Sect. 3.3]). Let $\mathcal{Y}$ be an affine D-scheme. Let

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Y})_{\mathrm{Ran}}$$

be the space over Ran that attaches to $\underline{x} \in \mathrm{Ran}$ the space

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Y})_{\underline{x}} := \mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y}).$$

This prestack has a natural unital structure: namely for $\underline{x} \subset \underline{x}'$, the corresponding map

$$\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y}) \to \mathrm{Sect}_\nabla(X - \underline{x}', \mathcal{Y})$$

is given by restriction.

Note that

$$\mathrm{Sect}_\nabla(X^{\mathrm{gen}}, \mathcal{Y})_\emptyset \simeq \mathrm{Sect}_\nabla(X, \mathcal{Y}).$$

C.6.4. Let $\mathcal{T}$ be a factorization space over X. A unital structure on it is a unital structure on $\mathcal{T}_{\mathrm{Ran}}$ (in the sense of Sect. C.6.1) and an extension of (B.2) to an isomorphism

$$\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl},*}} \underset{\mathrm{Ran}^{\mathrm{untl},*},\mathrm{union}}{\times} (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}} \simeq \tag{C.20}$$
$$\simeq (\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl},*}} \times \mathcal{T}_{\mathrm{Ran}^{\mathrm{untl},*}}) \underset{\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*}}{\times} (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}},$$

equipped with a homotopy-coherent data of associativity and commutativity, where

$$(\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}} \subset \mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*}.$$

is the corresponding open subfunctor.

In addition, we stipulate that

$$\mathcal{T}_\emptyset \simeq \mathrm{pt}, \tag{C.21}$$

and this identification behaves (homotopically coherently) as a unit for the isomorphisms (C.20), i.e., the map

$$\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl},*}} \to \mathcal{T}_\emptyset \times \mathcal{T}_{\mathrm{Ran}^{\mathrm{untl},*}},$$

obtained by base-changing (C.20) with respect to

$$\{\emptyset\} \times \mathrm{Ran}^{\mathrm{untl},*} \to (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}$$

identifies via (C.21) with the identity map.

C.6.5. A typical example of a unital factorization space is Gr_G. Namely, for $\underline{x} \subseteq \underline{x}'$ the corresponding map

$$\mathrm{Gr}_{G,\underline{x}} \to \mathrm{Gr}_{G,\underline{x}'}$$

is defined as follows.

Recall (see Sect. B.4.10) that $\mathrm{Gr}_{G,\underline{x}}$ can be described as the space of G-bundles on $\mathcal{D}_{\underline{x}}$ equipped with a trivialization on $\mathcal{D}_{\underline{x}} - \underline{x}$. We have:

Lemma C.6.6. *The map*

$$\mathcal{D}_{\underline{x}} \underset{\mathcal{D}_{\underline{x}} - \underline{x}}{\sqcup} (\mathcal{D}_{\underline{x}'} - \underline{x}) \to \mathcal{D}_{\underline{x}'}$$

is an isomorphism when the pushout is taken in the category of affine schemes.

Using this lemma, we can interpret $\mathrm{Gr}_{G,\underline{x}}$ as the space of G-bundles on $\mathcal{D}_{\underline{x}'}$ with a trivialization on $\mathcal{D}_{\underline{x}'} - \underline{x}$.

The desired map

$$\mathrm{ins.\,unit}_{\underline{x} \subseteq \underline{x}'} : \mathrm{Gr}_{G,\underline{x}} \to \mathrm{Gr}_{G,\underline{x}'}$$

is given by restricting the trivialization from $\mathcal{D}_{\underline{x}'} - \underline{x}$ to $\mathcal{D}_{\underline{x}'} - \underline{x}'$.

C.6.7. Let $\mathcal{Z}_{\mathrm{Ran}} \to \mathrm{Ran}$ be a prestack. A *counital* structure on $\mathcal{Z}_{\mathrm{Ran}}$ is its extension to a categorical prestack

$$\mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}} \to \mathrm{Ran}^{\mathrm{untl},*}, \tag{C.22}$$

such that (C.22) is a value-wise *Cartesian fibration in groupoids.*

Let $\mathrm{PreStk}^{\text{co-untl}}_{/\mathrm{Ran}}$ denote the category of prestacks over Ran, equipped with a counital structure.

C.6.8. In concrete terms, an upgrade

$$\mathcal{Z}_{\mathrm{Ran}} \rightsquigarrow \mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}}$$

means that for every $\underline{x} \subseteq \underline{x}'$ we give ourselves a map

$$\mathrm{proj.\,counit}_{\underline{x}\subseteq\underline{x}'} : \mathcal{Z}_{\underline{x}'} \to \mathcal{Z}_{\underline{x}},$$

in a way compatible with compositions.

In addition, we give ourselves a space $\mathcal{Z}_\emptyset$ and a system of maps

$$\mathrm{proj.\,counit}_{\emptyset\subset\underline{x}} : \mathcal{Z}_{\underline{x}} \to \mathcal{Z}_\emptyset$$

equipped with identifications

$$\mathrm{proj.\,counit}_{\emptyset\subset\underline{x}} \circ \mathrm{proj.\,counit}_{\underline{x}\subseteq\underline{x}'} \simeq \mathrm{proj.\,counit}_{\emptyset\subset\underline{x}'}\,.$$

C.6.9. Let $\mathcal{Y}$ be a D-prestack over X. Note that the arc space $\mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}}$ has a natural counital structure:

For $\underline{x} \subseteq \underline{x}'$, the corresponding maps

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}'} \to \mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}$$

are given by restriction along $\widehat{\mathcal{D}}_{\underline{x}} \to \widehat{\mathcal{D}}_{\underline{x}'}$.

C.6.10. We claim:

Proposition C.6.11. *The functor $\mathcal{Y} \mapsto \mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}}$ is the right adjoint to the functor*

$$\mathrm{PreStk}^{\text{co-untl}}_{/\mathrm{Ran}} \to \mathrm{PreStk}_{/\mathrm{Ran}} \to \mathrm{PreStk}_{X_{\mathrm{dR}}}, \tag{C.23}$$

where the last arrow is given by pullback along $X_{\mathrm{dR}} \to \mathrm{Ran}$.

Proof. Let us construct the unit and counit map for the adjunction. The counit is easy, and it is actually an isomorphism: we have

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_{X_{\mathrm{dR}}} \simeq \mathcal{Y}_\nabla.$$

To construct the unit, for any $\underline{x} : S \to \mathrm{Ran}$ and $Z_{\mathrm{Ran}} \in \mathrm{PreStk}^{\text{co-untl}}_{/\mathrm{Ran}}$, we need to define the map

$$\mathcal{Z}_S \to \text{Weil-Res}^{\widehat{\mathcal{D}}_{\underline{x},\nabla}}_S(\widehat{\mathcal{D}}_{\underline{x},\nabla} \underset{X_{\mathrm{dR}}}{\times} \mathcal{Z}_{X_{\mathrm{dR}}}),$$

where Weil-Res is the functor of restriction of scalars à la Weil.

By adjunction, the datum of the latter map is equivalent to that of a map

$$\widehat{\mathcal{D}}_{\underline{x},\nabla} \underset{S}{\times} \mathcal{Z}_S \to \widehat{\mathcal{D}}_{\underline{x},\nabla} \underset{X_{\mathrm{dR}}}{\times} \mathcal{Z}_{X_{\mathrm{dR}}}. \tag{C.24}$$

Note that the two sides in (C.24) are the pullbacks of $\mathcal{Z}_{\mathrm{Ran}}$ along the following two maps

$$\widehat{\mathcal{D}}_{\underline{x},\nabla} \rightrightarrows \mathrm{Ran}.$$

One is

$$\widehat{\mathcal{D}}_{\underline{x},\nabla} \to S \xrightarrow{\underline{x}} \mathrm{Ran},$$

and the other is

$$\widehat{\mathcal{D}}_{\underline{x},\nabla} \to X_{\mathrm{dR}} \to \mathrm{Ran}.$$

Now, by the definition of $\widehat{\mathcal{D}}_{\underline{x},\nabla}$, the there is a natural map from latter map to the former map inside the category

$$\mathrm{Maps}(\widehat{\mathcal{D}}_{\underline{x},\nabla}, \mathrm{Ran}^{\mathrm{untl},*}).$$

Hence, the required map is provided by the unital structure on $\mathcal{Z}_{\mathrm{Ran}}$.

The fact that the unit and counit maps constructed above satisfy the adjunction axioms is a straightforward verification.

□

C.6.12. Let $\mathcal{Z}_{\mathrm{Ran}}$ be equipped with a counital structure. Note that this structure gives rise to a map

$$\mathcal{Z}_{\mathrm{Ran}} \underset{\mathrm{Ran},\mathrm{union}}{\times} (\mathrm{Ran} \times \mathrm{Ran}) \to \mathcal{Z}_{\mathrm{Ran}} \times \mathcal{Z}_{\mathrm{Ran}}. \tag{C.25}$$

Base changing along

$$(\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}} \to \mathrm{Ran} \times \mathrm{Ran},$$

we obtain a map

$$\mathcal{Z}_{\mathrm{Ran}} \underset{\mathrm{Ran},\mathrm{union}}{\times} (\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}} \to (\mathcal{Z}_{\mathrm{Ran}} \times \mathcal{Z}_{\mathrm{Ran}}) \underset{\mathrm{Ran}\times\mathrm{Ran}}{\times} (\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}. \tag{C.26}$$

We shall say that the counital structure is *factorizable* if (C.26) is an isomorphism and the diagonal map

$$\mathcal{Z}_\emptyset \to \mathcal{Z}_\emptyset \times \mathcal{Z}_\emptyset$$

is also an isomorphism (implying that $\mathcal{Z}_\emptyset \simeq \mathrm{pt}$).

Note that the fact that the maps (C.26) are isomorphisms implies that the maps

$$\begin{aligned} \mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}} \underset{\mathrm{Ran}^{\mathrm{untl},*},\mathrm{union}}{\times} (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}} \to \\ \to (\mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}} \times \mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}}) \underset{\mathrm{Ran}^{\mathrm{untl},*}\times\mathrm{Ran}^{\mathrm{untl},*}}{\times} (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}, \end{aligned} \tag{C.27}$$

are also isomorphisms.

Let

$$(\mathrm{PreStk}^{\mathrm{co\text{-}untl}}_{/\mathrm{Ran}})^{\mathrm{factzbl}} \subset \mathrm{PreStk}^{\mathrm{co\text{-}untl}}_{/\mathrm{Ran}}$$

denote the full subcategory that consists of factorizable objects.

C.6.13. We have the following more precise version of Proposition C.6.11:

Proposition C.6.14. *The functor*

$$\mathcal{Y} \mapsto \mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}} \tag{C.28}$$

has an essential image in $(\mathrm{PreStk}^{\mathrm{co\text{-}untl}}_{/\mathrm{Ran}})^{\mathrm{factzbl}}$. *The resulting functor*

$$\mathrm{PreStk}_{/X_{\mathrm{dR}}} \to (\mathrm{PreStk}^{\mathrm{co\text{-}untl}}_{/\mathrm{Ran}})^{\mathrm{factzbl}}$$

is fully faithful and defines an equivalence between the full subcategories consisting of objects that are affine over X_{dR} *in the left-hand side and over* Ran *in the right-hand side.*

Proof. The fact that the essential image of the functor (C.28) lands in $(\mathrm{PreStk}^{\mathrm{co\text{-}untl}}_{/\mathrm{Ran}})^{\mathrm{factzbl}}$ has been established in Sect. B.4.2.

We have seen that the counit of the adjunction in Proposition C.6.11 is an isomorphism. Hence, the functor (C.28) is fully faithful.

In order to prove the proposition, it remains to show that the functor (C.23) is conservative on objects in $(\mathrm{PreStk}^{\mathrm{co\text{-}untl}}_{/\mathrm{Ran}})^{\mathrm{factzbl}}$ that are affine over Ran.

Let $\phi : \mathcal{Z}_{1,\mathrm{Ran}} \to \mathcal{Z}_{2,\mathrm{Ran}}$ be a map between two objects in $(\mathrm{PreStk}^{\mathrm{co\text{-}untl}}_{/\mathrm{Ran}})^{\mathrm{factzbl}}$, such that the map

$$\phi|_{X_{\mathrm{dR}}} : \mathcal{Z}_{1,X_{\mathrm{dR}}} \to \mathcal{Z}_{2,X_{\mathrm{dR}}}$$

is an isomorphism.

In order to check that ϕ is an isomorphism, it suffices to show that it is such when restricted to X^I_{dR} for every finite non-empty set I:

$$\mathcal{Z}_{1,X^I_{\mathrm{dR}}} \to \mathcal{Z}_{2,X^I_{\mathrm{dR}}}.$$

Since both prestacks are affine over X^I_{dR}, the question of a map being an isomorphism can be checked strata-wise. Using the diagonal stratification of X^I, it suffices to show that that the further restriction to

$$\overset{\circ}{X}{}^I_{\mathrm{dR}} \subset X^I_{\mathrm{dR}}$$

is an isomorphism. Since $\mathcal{Z}_{1,\mathrm{Ran}}$ and $\mathcal{Z}_{2,\mathrm{Ran}}$ are both factorizable, the latter map is the direct product of I copies of the map $\phi|_{X_{\mathrm{dR}}}$.

□

C.6.15. As in Sect. C.6.4, given a factorization space, we can talk about a counital structure on it.

It is easy to see, however, that if $\mathcal{T}$ is a counital factorization space, then the corresponding prestack $\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl},*}}$ is factorizable (in the sense of Sect. C.6.12). Vice versa, given an object

$$\mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}} \in (\mathrm{PreStk}^{\text{co-untl}}_{/\mathrm{Ran}})^{\mathrm{factzbl}},$$

the isomorphism (C.27) defines on it a factorization structure.

So the categories of counital factorization spaces and factorizable counital prestacks over Ran are tautologically equivalent.

C.7. **Unital factorization algebras.**

C.7.1. Let $\mathcal{A}$ be a factorization algebra on X. A unital structure on $\mathcal{A}$ is an extension of $\mathcal{A}_{\mathrm{Ran}}$ to an object

$$\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \in \text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}),$$

and an extension of the isomorphism (B.23) to an isomorphism

$$\mathrm{union}^!(\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}})|_{(\mathrm{Ran}^{\mathrm{untl},*}\times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}} \simeq \mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \boxtimes \mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}|_{(\mathrm{Ran}^{\mathrm{untl},*}\times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}}, \tag{C.29}$$

equipped with a homotopy-coherent data of associativity and commutativity.

In addition, we stipulate that

$$\mathcal{A}_\emptyset \simeq k \tag{C.30}$$

and this isomorphism behaves (homotopically coherently) as a unit for the identifications (C.29), i.e., the map

$$\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \to \mathcal{A}_\emptyset \otimes \mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}},$$

obtained by restricting (C.29) to

$$\{\emptyset\} \times \mathrm{Ran}^{\mathrm{untl},*} \to (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}$$

identifies via (C.30) with the identity map.

Let $\mathrm{FactAlg}^{\mathrm{untl}}(X)$ denote the category of unital factorization algebras on X.

Remark C.7.2. Pullback along

$$\mathfrak{t} : \mathrm{Ran} \to \mathrm{Ran}^{\mathrm{untl},*}$$

gives rise to a functor

$$\mathrm{FactAlg}^{\mathrm{untl}}(X) \to \mathrm{FactAlg}(X). \tag{C.31}$$

This functor is *not* fully faithful. However, by analogy with the topological situation, we expect that:

- The functor (C.31) induces a *monomorphism* on the mapping spaces;
- The functor (C.31) induces an isomorphism on the union of the components of the mapping spaces that correspond to *isomorphisms*.

The second property can be phrased as saying that for a factorization algebra, being unital is a property and not a structure.

We will not prove this in this paper. However, we will prove a result in this direction, see Proposition C.7.13.

C.7.3. Let us denote by k the unit factorization algebra, see Sect. B.9.2. Note that it naturally upgrades to a unital factorization algebra: namely, the corresponding object in D-mod($\mathrm{Ran}^{\mathrm{untl},*}$) is $\omega_{\mathrm{Ran}^{\mathrm{untl},*}}$.

Let $\mathcal{A}$ be a unital factorization algebra on X. Note that the initial point $\{\emptyset\} \in \mathrm{Ran}^{\mathrm{untl},*}$ gives rise to a map

$$\omega_{\mathrm{Ran}^{\mathrm{untl},*}} \to \mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}$$

in D-mod($\mathrm{Ran}^{\mathrm{untl},*}$). It follows from the axioms that this map is compatible with factorization.

I.e., we obtain a map of unital factorization algebras

$$\mathrm{vac}_{\mathcal{A}} : k \to \mathcal{A}, \tag{C.32}$$

which we will refer to it as the *vacuum* map for $\mathcal{A}$.

C.7.4. Let $\mathcal{Z}$ be a prestack mapping to Ran. Consider the corresponding categorical prestack $\mathcal{Z}^{\subseteq,\mathrm{untl}}$, see Sect. C.5.16.

Let $\mathcal{A}$ be a factorization algebra on X, and let $\mathcal{M}$ be a factorization module $\mathcal{M}$ over $\mathcal{A}$ at $\mathcal{Z}$. Let $\mathcal{A}$ be equipped with a unital structure. A unital structure on $\mathcal{M}$ is an extension of $\mathcal{M}_{\mathcal{Z}^{\subseteq}}$ to an object

$$\mathcal{M}_{\mathcal{Z}^{\subseteq}} \in \text{D-mod}(\mathcal{Z}^{\subseteq,\mathrm{untl}})$$

and an extension of the isomorphism (B.24) to an isomorphism

$$\mathcal{M}_{\mathcal{Z}^{\subseteq,\mathrm{untl}}}|_{(\mathrm{Ran}^{\mathrm{untl},*}\times\mathcal{Z}^{\subseteq,\mathrm{untl}})_{\mathrm{disj}}} \simeq (\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \boxtimes \mathcal{M}_{\mathcal{Z}^{\subseteq,\mathrm{untl}}})|_{(\mathrm{Ran}^{\mathrm{untl},*}\times\mathcal{Z}^{\subseteq,\mathrm{untl}})_{\mathrm{disj}}}. \tag{C.33}$$

In addition, we stipulate that the isomorphism

$$\mathcal{M}_{\mathcal{Z}^{\subseteq,\mathrm{untl}}} \to \mathcal{A}_{\emptyset} \otimes \mathcal{M}_{\mathcal{Z}^{\subseteq,\mathrm{untl}}},$$

obtained by restricting (C.33) along

$$\{\emptyset\} \times \mathcal{Z}^{\subseteq,\mathrm{untl}} \to (\mathrm{Ran}^{\mathrm{untl},*} \times \mathcal{Z}^{\subseteq,\mathrm{untl}})_{\mathrm{disj}},$$

identifies via (C.30) with the identity map.

The contents of Sect. B.9 apply to unital factorization modules.

C.7.5. *Notational convention.* When $\mathcal{A}$ is a unital factorization algebra, we will denote by

$$\mathcal{A}\text{-mod}_{\mathcal{Z}}^{\mathrm{fact}}$$

the category of *unital* factorization modules over $\mathcal{A}$ at $\mathcal{Z}$.

The category of modules over $\mathcal{A}$ as a plain[76] factorization algebra will be denoted by

$$\mathcal{A}\text{-mod}_{\mathcal{Z}}^{\text{fact-n.u.}}.$$

C.7.6. *Example.* Let $\mathcal{A}$ be a unital factorization algebra. For $\mathcal{Z} \to \mathrm{Ran}$ consider the factorization module $\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}$ from Sect. B.9.7.

Unwinding the definitions, we obtain that $\mathcal{A}^{\mathrm{fact}_{\mathcal{Z}}}$ carries a natural unital structure.

[76] I.e., non-unital.

C.7.7. *Example.* Take $\mathcal{A} = k$ from Sect. C.7.3. Note that pullback along

$$\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},\mathcal{Z}} : \mathcal{Z}^{\subseteq,\mathrm{untl}} \to \mathcal{Z}$$

gives rise to a functor

$$\text{D-mod}(\mathcal{Z}) \to k\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}, \tag{C.34}$$

cf. Sect. B.9.6.

In other words, the functor (C.34) is given by tensoring (over D-mod($\mathcal{Z}$)) with the object

$$k^{\mathrm{fact}_{\mathcal{Z}}} \in k\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}.$$

We claim that the functor (C.34) is an equivalence, with the inverse functor being

$$\mathcal{M} \mapsto \mathcal{M}_{\mathcal{Z}} := (\mathrm{diag}^{\mathrm{untl}}_{\mathcal{Z}})^!(\mathcal{M}).$$

Indeed, the fact that the map

$$\mathrm{diag}^{\mathrm{untl}}_{\mathcal{Z}} \to \mathcal{Z}^{\subseteq,\mathrm{untl}}$$

is initial relative to $\mathcal{Z}$, implies that for any $\mathcal{M} \in k\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}$, we have a canonically defined map

$$(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},\mathcal{Z}})^! \circ (\mathrm{diag}^{\mathrm{untl}}_{\mathcal{Z}})^!(\mathcal{M}) \to \mathcal{M}.$$

Now, the factorization condition implies that this map is actually an isomorphism.

C.7.8. Let $\mathcal{A}$ be a unital factorization algebra on X.

Restriction along

$$\mathsf{t} : \mathcal{Z}^{\subseteq} \to \mathcal{Z}^{\subseteq,\mathrm{untl}}$$

gives rise to a functor

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \mathcal{A}\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}}. \tag{C.35}$$

We have the following assertion, proved in [CR, Proposition 3.8.4]:

Proposition C.7.9. *The functor* (C.35) *is fully faithful with essential image*

$$\mathcal{A}\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}} \underset{k\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}}}{\times} k\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \simeq \mathcal{A}\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}} \underset{k\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}}}{\times} \text{D-mod}(\mathcal{Z}),$$

where

$$\mathcal{A}\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}} \to k\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}}$$

is the functor of restriction along (C.32).

C.7.10. Let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a unital map between unital factorization algebras. From Proposition C.7.9 we obtain:

Corollary C.7.11. *The restriction functor*

$$\mathrm{Res}_\phi : \mathcal{A}_2\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}} \to \mathcal{A}_1\text{-mod}^{\text{fact-n.u.}}_{\mathcal{Z}}$$

sends

$$\mathcal{A}_2\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \mathcal{A}_1\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}.$$

C.7.12. Let $\mathrm{FactAlg}^{\text{q-untl}}(X)$ be the category of pairs $(\mathcal{A}, \mathrm{vac}_{\mathcal{A}})$, where $\mathcal{A}$ is a non-unital factorization algebra, and $\mathrm{vac}_{\mathcal{A}}$ is a homomorphism $k \to \mathcal{A}$, such that the object

$$\mathrm{Res}_{\mathrm{vac}_{\mathcal{A}}}(\mathcal{A}^{\mathrm{fact}_{\mathrm{Ran}}}) \in k\text{-mod}^{\text{fact-n.u.}}_{\mathrm{Ran}}$$

belongs to

$$\mathrm{D\text{-}mod}(\mathrm{Ran}) \simeq k\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \subset k\text{-mod}^{\text{fact-n.u.}}_{\mathrm{Ran}}.$$

We will call objects of $\mathrm{FactAlg}^{\text{q-untl}}(X)$ "quasi-unital factorization algebras".

We have a tautological functor

$$\mathrm{FactAlg}^{\mathrm{untl}}(X) \to \mathrm{FactAlg}^{\text{q-untl}}(X). \tag{C.36}$$

We claim:

Proposition C.7.13. *The functor* (C.36) *is an equivalence.*

The proof will be given in Sect. C.11.20.

Note that the second assertion of the proposition says that a quasi-unital factorization algebra

$$k \overset{\mathrm{vac}_{\mathcal{A}}}{\to} \mathcal{A}$$

carries a canonical unital structure, for which $\mathrm{vac}_{\mathcal{A}}$ is the unit.

C.8. Commutative *unital* factorization algebras.

C.8.1. By the same token as in Sect. B.10.1, one can consider the category

$$\mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)).$$

It is equipped with a tautological forgetful functor

$$\mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)) \to \mathrm{ComAlg}(\mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl},*})) \tag{C.37}$$

and also with a functor

$$\mathsf{t}^! : \mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)) \to \mathrm{ComAlg}(\mathrm{FactAlg}(X)). \tag{C.38}$$

C.8.2. Let $\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}$ be an object of $\mathrm{ComAlg}(\mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl},*}))$. Note that the unital structure on $\mathcal{A}$ gives rise to the maps

$$(p_i)^!(\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}) \to \mathrm{union}^!(\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}), \quad i = 1, 2$$

where p_1 and p_2 are the two projections $\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*} \to \mathrm{Ran}^{\mathrm{untl},*}$.

Since the coproduct in ComAlg is the tensor product, we obtain a map

$$\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \boxtimes \mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \to \mathrm{union}^!(\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}). \tag{C.39}$$

The map (C.39) gives rise to a map

$$\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \boxtimes \mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}|_{(\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}} \to \mathrm{union}^!(\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}})|_{(\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}}. \tag{C.40}$$

We shall say that $\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}$ is *factorizable* if the map (C.40) is an isomorphism. Note that this automatically implies that $\mathcal{A}_\emptyset \simeq k$.

C.8.3. Let
$$\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))^{\mathrm{factzble}} \subset \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))$$
denote the full subcategory consisting of factorizable objects.

It follows from the axioms that the essential image of the functor (C.37) lands in
$$\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))^{\mathrm{factzble}}.$$
Vice versa, for an object
$$\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \in \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))^{\mathrm{factzble}},$$
the isomorphism (C.40) defines on $\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}}$ a factorization structure.

It is easy to see that the resulting two functors
$$\mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)) \leftrightarrow \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))^{\mathrm{factzble}} \tag{C.41}$$
are mutually inverse.

C.8.4. Recall the functor
$$A \mapsto \mathrm{Fact}(A),$$
see Sect. B.10.2.

Unwinding the construction, we obtain that $\mathrm{Fact}(-)$ upgrades to a functor
$$\mathrm{ComAlg}(\text{D-mod}(X)) \to \mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)).$$

By a slight abuse of notation, we will use the same symbol $\mathrm{Fact}(-)$ to denote the latter functor.

C.8.5. Consider now the functor
$$\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*})) \to \mathrm{ComAlg}(\text{D-mod}(X)), \tag{C.42}$$
given by restriction along $\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}} : X_{\mathrm{dR}} \to \mathrm{Ran}^{\mathrm{untl},*}$.

We claim:

Proposition C.8.6. *The functor* (C.42) *admits a left adjoint. Moreover, this left adjoint is fully faithful and lands in* $\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))^{\mathrm{factzble}}$.

Proof. It is enough to prove the assertion of the proposition on objects of $\mathrm{ComAlg}(\text{D-mod}(X))$ of the form
$$\mathrm{Sym}^!(\mathcal{M}), \quad \mathcal{M} \in \text{D-mod}(X).$$

The value of the left adjoint on such an object is
$$\mathrm{Sym}^!((\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}})_!(\mathcal{M})). \tag{C.43}$$

To prove that this left adjoint is fully faithful, it is enough to show that the unit of the adjunction
$$\mathcal{M} \to (\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}})^! \circ (\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}})_!(\mathcal{M})$$
is an isomorphism. However, this follows from Corollary C.4.21: indeed, the categorical prestack
$$X_{\mathrm{dR}} \underset{\mathrm{Ran}^{\mathrm{untl},*}}{\times} (\mathrm{Ran}^{\mathrm{untl},*})_{\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}}/}$$
identifies with X_{dR}.

In order to show that (C.43) belongs to $\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))^{\mathrm{factzble}}$, it suffices to show that the canonical map
$$p_1^!((\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}})_!(\mathcal{M})) \oplus p_2^!((\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}})_!(\mathcal{M})) \to \mathrm{union}^!((\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}})_!(\mathcal{M}))$$
becomes an isomorphism after restricting to $(\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}$. This is the content of [Ro2, Theorem 2.7.6]. We include the argument for completeness.

However, this follows again from Corollary C.4.21: we have a canonical isomorphism

$$\Big((\mathrm{Ran}^{\mathrm{untl},*})_{\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}}/} \times \mathrm{Ran}^{\mathrm{untl},*} \sqcup \mathrm{Ran}^{\mathrm{untl},*} \times (\mathrm{Ran}^{\mathrm{untl},*})_{\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}}/}\Big) \underset{\mathrm{Ran}^{\mathrm{untl},*}\times \mathrm{Ran}^{\mathrm{untl},*}}{\times}$$
$$\underset{\mathrm{Ran}^{\mathrm{untl},*}\times \mathrm{Ran}^{\mathrm{untl},*}}{\times} (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}} \simeq$$
$$\simeq (\mathrm{Ran}^{\mathrm{untl},*})_{\Delta_{X,\mathrm{Ran}^{\mathrm{untl},*}}/} \underset{\mathrm{Ran}^{\mathrm{untl},*},\mathrm{union}}{\times} (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}.$$

Indeed, this is just the fact that for a disjoint pair $\underline{x}_1, \underline{x}_2$ of points of Ran, and a singleton x,

$$x \subseteq \underline{x}_1 \cup \underline{x}_2 \;\Leftrightarrow\; x \subseteq \underline{x}_1 \text{ or } x \subseteq \underline{x}_2.$$

□

C.8.7. As a corollary, we obtain:

Corollary C.8.8. *The composite functor*

$$\mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)) \to \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*})) \overset{\Delta^!_{X,\mathrm{Ran}^{\mathrm{untl},*}}}{\longrightarrow} \mathrm{ComAlg}(\text{D-mod}(X))$$

is an equivalence, with the inverse given by $\mathrm{Fact}(-)$.

Proof. Given Proposition C.8.6, we only need to prove that the functor in Corollary C.8.8 is conservative. But this is immediate from the factorization.

□

Corollary C.8.9. *The functor*

$$\mathrm{ComAlg}(\text{D-mod}(X)) \overset{\mathrm{Fact}(-)}{\longrightarrow} \mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)) \to \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))$$

is the left adjoint of

$$\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*})) \overset{\Delta^!_{X,\mathrm{Ran}^{\mathrm{untl},*}}}{\longrightarrow} \mathrm{ComAlg}(\text{D-mod}(X)).$$

C.8.10. Let $\mathcal{Y}$ be a D-prestack over X. Suppose that the prestack $\mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}} \to \mathrm{Ran}$ is such that the formation of direct image of the structure sheaf along

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_S \to S, \quad S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$$

is compatible with base change, and satisfies Kunneth formula.

This happens, e.g., when $\mathcal{Y}$ is affine over X, and hence $\mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}}$ is affine over Ran.

Taking the direct image of the structure sheaf along

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl},*}} \to \mathrm{Ran}^{\mathrm{untl},*},$$

we obtain an object in $\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))$, which by a slight abuse of notation we denote by $\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y}),\mathrm{Ran}^{\mathrm{untl},*}}$. By Kunneth formula and factorization, $\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y}),\mathrm{Ran}^{\mathrm{untl},*}}$ has a natural structure of factorization algebra.

Denote the resulting object by

$$\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})} \in \mathrm{FactAlg}(X).$$

The value of $\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})}$ on X, i.e., the restriction of $\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y}),\mathrm{Ran}^{\mathrm{untl},*}}$ along $X \to \mathrm{Ran}^{\mathrm{untl},*}$, is the direct image of the structure sheaf along $\mathcal{Y} \to X$, which by a slight abuse of notation we denote by $\mathcal{O}_{\mathcal{Y}}$.

Hence, by the equivalence of Corollary C.8.8, we obtain that

$$\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})} \simeq \mathrm{Fact}(\mathcal{O}_{\mathcal{Y}}). \tag{C.44}$$

C.8.11. Assume that $\mathcal{Y}$ is affine over X, i.e., $\mathcal{Y} = \mathrm{Spec}_X(A)$ for $A \in \mathrm{ComAlg}(\text{D-mod}(X))$ with $\mathbf{oblv}^l(A)$ is connective.

Let $\mathcal{A} := \mathrm{Fact}(A)$. Then (C.44) says that for $S \to \mathrm{Ran}$,

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_S \simeq \mathrm{Spec}_S(\mathcal{A}_S). \tag{C.45}$$

C.8.12. Note that the identification (C.45) is in agreement with Proposition C.6.11.

Namely, let us be given a counital prestack $\mathcal{Z}_{\mathrm{Ran}}$ over Ran, such that the direct image of its structure sheaf satisfies base change. Let

$$\mathcal{B}_{\mathrm{Ran}^{\mathrm{untl},*}} \in \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))$$

denote the corresponding object. Set $\mathcal{B}_X := \mathcal{B}_{\mathrm{Ran}^{\mathrm{untl},*}}|_{X_{\mathrm{dR}}}$.

Then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Maps}_{\mathrm{PreStk}^{\text{co-untl}}_{/\mathrm{Ran}}}(\mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}}, \mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}^{\mathrm{untl},*}}) \arrow[r, "\text{Proposition C.6.11}", "\sim"'] \arrow[d, "(\text{C.45})"', "\sim"] & \mathrm{Maps}_{X_{\mathrm{dR}}}(X_{\mathrm{dR}} \underset{\mathrm{Ran}^{\mathrm{untl},*}}{\times} \mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}}, \mathcal{Y}_\nabla) \arrow[d, "\sim"] \\
\mathrm{Maps}_{\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))}(\mathcal{A}, \mathcal{B}_{\mathrm{Ran}^{\mathrm{untl},*}}) \arrow[r, "\text{Proposition C.8.6}"] & \mathrm{Maps}_{\mathrm{ComAlg}(\text{D-mod}(X))}(A, \mathcal{B}_X).
\end{tikzcd}$$

C.9. **Factorization homology of commutative factorization algebras.**

C.9.1. Consider the functor

$$\mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{ComAlg}(\text{D-mod}(X)), \quad R \mapsto R \otimes \mathcal{O}_X,$$

where $\mathcal{O}_X$ is perceived as a *left* D-module.

In this subsection, we will describe, following [BD2, Sect. 4.6.1], its left adjoint.

C.9.2. From Lemma C.5.9 we obtain that the functor $\mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}, -)$ gives rise to a functor

$$\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl}})) \to \mathrm{ComAlg}(\mathrm{Vect}),$$

left adjoint to

$$\mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl}})), \quad R \mapsto R \otimes \omega_{\mathrm{Ran}^{\mathrm{untl}}}.$$

Remark C.9.3. Note that by Sect. C.5.15, the restriction of $\mathrm{C}^\cdot_c(\mathrm{Ran}, -)$ to the subcategory

$$\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran})^{\text{almost-untl}}) \subset \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}))$$

defines a left adjoint to the functor

$$\mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran})^{\text{almost-untl}}), \quad R \mapsto R \otimes \omega_{\mathrm{Ran}}.$$

C.9.4. Recall now that according to Corollary C.8.9, the functor

$$\mathrm{ComAlg}(\text{D-mod}(X)) \overset{\mathrm{Fact}}{\to} \mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)) \to \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))$$

provides a left adjoint to the restriction functor

$$\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*})) \to \mathrm{ComAlg}(\text{D-mod}(X)).$$

Combined with Sect. C.9.2, we obtain:

Corollary C.9.5. *The functor*

$$\mathrm{ComAlg}(\text{D-mod}(X)) \overset{\mathrm{Fact}}{\to} \mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)) \to$$
$$\to \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*})) \overset{\mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl},*},-)}{\longrightarrow} \mathrm{ComAlg}(\mathrm{Vect})$$

is the left adjoint of

$$R \mapsto R \otimes \mathcal{O}_X, \quad \mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{ComAlg}(\text{D-mod}(X)).$$

C.9.6. For $\mathcal{A} \in \mathrm{FactAlg}^{\mathrm{untl}}(X)$ recall the object

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) = \mathrm{C}^\cdot_c(\mathrm{Ran}, \mathcal{A}_{\mathrm{Ran}}),$$

see Sect. 11.9.7.

Note that by Lemma C.5.12, we can rewrite this also as

$$\mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}, \mathcal{A}_{\mathrm{Ran}^{\mathrm{untl}}}).$$

C.9.7. Thus, Corollary C.9.5 says that the functor

$$A \mapsto \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathrm{Fact}(A)), \quad \mathrm{ComAlg}(\mathrm{D\text{-}mod}(X)) \to \mathrm{ComAlg}(\mathrm{Vect})$$

is the left adjoint of

$$R \mapsto R \otimes \mathcal{O}_X, \quad \mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{ComAlg}(\mathrm{D\text{-}mod}(X)).$$

Remark C.9.8. Note that when we think of $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A})$ as $\mathrm{C}^\cdot_c(\mathrm{Ran}, \mathcal{A}_{\mathrm{Ran}})$, the commutative algebra structure on it follows from Remark C.9.3, since

$$\mathcal{A}_{\mathrm{Ran}} \in \mathrm{D\text{-}mod}(\mathrm{D\text{-}mod}_{\mathrm{Ran}})^{\text{almost-untl}}.$$

C.9.9. Let $A \to B$ be a map in $\mathrm{ComAlg}(\mathrm{D\text{-}mod}(X))$. Denote

$$\mathcal{A} := \mathrm{Fact}(A), \ \mathcal{B} := \mathrm{Fact}(B).$$

Let R be an object of ComAlg(Vect), and fix a map

$$A \to R \otimes \mathcal{O}_X,$$

or, equivalently by Corollary C.9.5, a map

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \to R.$$

C.9.10. Denote

$$B_R := B \underset{A}{\otimes} (R \otimes \mathcal{O}_X).$$

We can view B_R as an object of

$$\mathrm{ComAlg}(\mathrm{D\text{-}mod}(X) \otimes R\text{-mod}),$$

i.e., as an R-linear object in $\mathrm{ComAlg}(\mathrm{D\text{-}mod}(X))$.

C.9.11. Consider the corresponding object

$$\mathcal{B}_R \in \mathrm{ComAlg}(\mathrm{FactAlg}(X) \otimes R\text{-mod}).$$

We can apply the construction of factorization homology in the R-linear context, and thus form

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{B}_R) \in \mathrm{ComAlg}(R\text{-mod}).$$

C.9.12. We have the naturally defined maps in ComAlg(Vect)

$$R \to \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{B}_R) \leftarrow \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{B}),$$

which fit into the commutative diagram

$$\begin{tikzcd}
\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \arrow[r] \arrow[d] & R \arrow[d] \\
\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{B}) \arrow[r] & \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{B}_R).
\end{tikzcd}$$

In particular, we obtain a map

$$R \underset{\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A})}{\otimes} \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{B}) \to \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{B}_R). \tag{C.46}$$

C.9.13. We claim:

Lemma C.9.14. *The map* (C.46) *is an isomorphism.*

Proof. Follows immediately from Corollary C.9.5. □

C.10. **Unitality in correspondenes.**

C.10.1. Let $\Phi : \mathbf{C} \to \mathbf{D}$ be a functor between ∞-categories. We shall say that Φ is a *fibration-in-correspondences*[77] if the following two conditions hold:

- For every $\mathbf{d} \in \mathbf{D}$, the fiber $\mathbf{C}_{\mathbf{d}}$ is a groupoid;
- For every composable pair of arrows $\mathbf{d}_0 \overset{\alpha_{0,1}}{\to} \mathbf{d}_1 \overset{\alpha_{1,2}}{\to} \mathbf{d}_2$ in $\mathbf{D}$, the map
$$\mathbf{C}_{\alpha_{0,1}} \underset{\mathbf{C}_{\mathbf{d}_1}}{\times} \mathbf{C}_{\alpha_{1,2}} \to \mathbf{C}_{\alpha_{1,2} \circ \alpha_{0,1}}$$
is an isomorphism, where for an arrow $\mathbf{d}' \overset{\alpha}{\to} \mathbf{d}''$, viewed as a functor $[0,1] \to \mathbf{D}$, we denote by $\mathbf{C}_\alpha$ the category of lifts of $\alpha : [0,1] \to \mathbf{D}$ to a functor $[0,1] \to \mathbf{C}$, i.e.,
$$\mathrm{Funct}_{/\mathbf{D}}([0,1], \mathbf{C}).$$

Note that the second condition can be reformulated as follows: given an arrow $\mathbf{c}_0 \overset{\beta_{0,2}}{\to} \mathbf{c}_2$ in $\mathbf{C}$ and a factorization of its image $\Phi(\beta_{0,2})$ as
$$\Phi(\mathbf{d}_0) =: \mathbf{d}_0 \overset{\alpha_{0,1}}{\to} \mathbf{d}_1 \overset{\alpha_{1,2}}{\to} \mathbf{d}_2 := \Phi(\mathbf{d}_2)$$
the space of the factorizations
$$\beta_{0,2} = \beta_{1,2} \circ \beta_{0,1}, \quad \Phi(\beta_{0,1}) = \alpha_{0,1},\ \Phi(\beta_{1,2}) = \alpha_{1,2}$$
is contractible.

Remark C.10.2. There is a version of straightening construction that attaches to a fibration-in-correspondences $\Phi : \mathbf{C} \to \mathbf{D}$ a functor from $\mathbf{D}$ to Corr(Grpd), i.e., the category, whose objects are groupoids and the morphisms are correspondences between groupoids, see [AF, Theorem 0.8(2) and Theorem 0.10(1)].

C.10.3. Note that if Φ is either a Cartesian or a co-Cartesian fibration in groupoids, then it is a fibration-in-correspondences.

C.10.4. Let $\Phi : \mathbf{C} \to \mathbf{D}$ be a fibration-in-correspondences. Suppose for a moment that $\mathbf{D}$ contains an initial object $\{\emptyset\}$. Let $\mathbf{C}^+$ denote the category
$$\{\mathbf{c}_\emptyset \in \mathbf{C}_{\{\emptyset\}},\, \mathbf{c} \in \mathbf{C},\, \mathbf{c}_\emptyset \overset{\beta}{\to} \mathbf{c}\}.$$

Note the functor
$$\Phi^+ : \mathbf{C}^+ \to \mathbf{D}, \quad (\mathbf{c}_\emptyset, \mathbf{c}, \beta) \mapsto \Phi(\mathbf{c})$$
is a Cartesian fibration in groupoids.

Indeed, given
$$\mathbf{c}_\emptyset \overset{\beta}{\to} \mathbf{c} \text{ and } \mathbf{d}' \to \Phi(\mathbf{c}),$$
we let
$$\mathbf{c}_\emptyset \overset{\beta'}{\to} \mathbf{c}' \to \mathbf{c}$$
be its unique factorization covering the canonical factorization
$$\{\emptyset\} \to \mathbf{d}' \to \Phi(\mathbf{c}).$$

Note that if $\mathbf{C} \to \mathbf{D}$ is co-Cartesian fibration in groupoids, then $\mathbf{C}^+ \simeq \mathbf{C}_\emptyset \times \mathbf{D}$. If $\mathbf{C} \to \mathbf{D}$ is Cartesian fibration in groupoids, then $\mathbf{C}^+ \to \mathbf{C}$ is an equivalence.

[77]Another name for this is “conservative exponentiable fibration”, see [AF].

C.10.5. Let $\mathcal{Z}_{\mathrm{Ran}} \to \mathrm{Ran}$ be a prestack. A unital-in-correspondences structure on it is an extension of $\mathcal{Z}_{\mathrm{Ran}}$ to a categorical prestack

$$\mathcal{Z}_{\mathrm{Ran}^{\mathrm{untl},*}} \to \mathrm{Ran}^{\mathrm{untl},*},$$

which is a value-wise fibration-in-correspondences

C.10.6. Note that the construction in Sect. C.10.4 associates to such $\mathcal{Z}_{\mathrm{Ran}}$ a prestack $\mathcal{Z}^+_{\mathrm{Ran}}$, equipped with a counital structure.

In what follows we will say that $\mathcal{Z}_{\mathrm{Ran}}$ admits a unital-in-correspondences structure *relative* to $\mathcal{Z}^+$. And we will refer to $\mathcal{Z}^+_{\mathrm{Ran}}$ as the counital prestack *underlying* $\mathcal{Z}_{\mathrm{Ran}}$.

For an arrow in $\mathrm{Ran}^{\mathrm{untl},*}$ given by $\underline{x} \subseteq \underline{x}'$ we will denote by

$$\mathcal{Z}^{\mathrm{all}\rightsquigarrow +}_{\underline{x}\subseteq\underline{x}'}$$

the prestack of its lifts to $\mathcal{Z}_{\mathrm{Ran}}$ (see Sect. C.10.1 above). It is equipped with maps

$$\mathcal{Z}_{\underline{x}} \overset{\mathrm{pr}^{\mathcal{Z}}_{\mathrm{small}}}{\longleftarrow} \mathcal{Z}^{\mathrm{all}\rightsquigarrow +}_{\underline{x}\subseteq\underline{x}'} \overset{\mathrm{pr}^{\mathcal{Z}}_{\mathrm{big}}}{\longrightarrow} \mathcal{Z}_{\underline{x}'}.$$

C.10.7. Note that by Sect. C.10.4, a unital structure on $\mathcal{Z}_{\mathrm{Ran}}$ gives rise to a unital-in-correspondences structure, with $\mathcal{Z}^+ \simeq \mathcal{Z}_\emptyset \times \mathrm{Ran}$.

A counital structure on $\mathcal{Z}_{\mathrm{Ran}}$ gives rise to a unital-in-correspondences structure with $\mathcal{Z}^+_{\mathrm{Ran}} \to \mathcal{Z}_{\mathrm{Ran}}$ being an isomorphism.

C.10.8. Let $\mathcal{T}$ be factorization space over X. There is a natural notion of unital-in-correspondences structure on $\mathcal{T}$ (i.e., $\mathcal{T}_{\mathrm{Ran}}$ has a unital-in-correspondences structure, compatible with factorization, and we stipulate $\mathcal{T}_\emptyset \simeq \mathrm{pt}$).

Let $\mathcal{T}^+$ be the corresponding counital factorization space (see Sect. C.10.4); in this case we will say that $\mathcal{T}$ has a unital-in-correspondences structure *relative* to $\mathcal{T}^+$. We will refer to $\mathcal{T}^+$ as the counital factorization space *underlying* $\mathcal{T}$.

C.10.9. Let $\mathcal{T}$ be a unital factorization space. Then $\mathcal{T}$ acquires a natural unital-in-correspondences structure, for which $\mathcal{T}^+ \to \mathrm{pt}$ is an isomorphism.

According to Sect. C.10.4, if $\mathcal{T}$ is counital as a factorization space, it acquires a natural structure of unitality-in-correspondences with $\mathcal{T}^+ \to \mathcal{T}$ being an isomorphism.

C.10.10. Let $\mathcal{Y}$ be an affine D-scheme over X. We claim that the factorization space $\mathfrak{L}_\nabla(\mathcal{Y})$ (see Sect. B.4.6) possesses a natural unital-in-correspondences structure relative to $\mathfrak{L}^+_\nabla(\mathcal{Y})$.

Namely, let $\underline{x} \subseteq \underline{x}' : S \to \mathrm{Ran}$ be a pair of S-points of Ran. Consider the corresponding prestack $\mathcal{D}_{\underline{x}',\nabla}$. It contains $\mathrm{Graph}_{\underline{x}}$ as a closed subset.

By definition, a lift of $\underline{x} \subseteq \underline{x}'$ to an S-point of $\mathfrak{L}_\nabla(\mathcal{Y})^{\mathrm{all}\rightsquigarrow +}_{\underline{x}\subseteq\underline{x}'}$ is a X_{dR}-map

$$(\mathcal{D}_{\underline{x}',\nabla} - \mathrm{Graph}_{\underline{x}}) \to \mathcal{Y}_\nabla.$$

NB: in the particular case of $\mathcal{T} = \mathfrak{L}_\nabla(\mathcal{Y})$, we use the notation $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y})_{\underline{x}\subseteq\underline{x}'}$ instead of $\mathfrak{L}_\nabla(\mathcal{Y})^{\mathrm{all}\rightsquigarrow +}_{\underline{x}\subseteq\underline{x}'}$.

In order to define the composition of morphisms, we need to establish an isomorphism

$$\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y})_{\underline{x}_1\subseteq\underline{x}_2} \underset{\mathfrak{L}_\nabla(\mathcal{Y})^{\mathrm{mer}}_{\underline{x}_2}}{\times} \mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y})_{\underline{x}_2\subseteq\underline{x}_3} \simeq \mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y})_{\underline{x}_1\subseteq\underline{x}_3}$$

for $\underline{x}_1 \subseteq \underline{x}_2 \subseteq \underline{x}_3$. This follows from the isomorphism

$$(\mathcal{D}_{\underline{x}_2,\nabla} - \underline{x}_1) \underset{\mathcal{D}_{\underline{x}_2,\nabla} - \underline{x}_2}{\sqcup} (\mathcal{D}_{\underline{x}_3,\nabla} - \underline{x}_2) \simeq (\mathcal{D}_{\underline{x}_3,\nabla} - \underline{x}_1),$$

cf. Lemma C.6.6.

C.10.11. We will now define a unital-in-correspondences structure on the factorization space $\mathrm{LS}_H^{\mathrm{mer}}$ from Sect. B.7.10.

Let $\underline{x} \subseteq \underline{x}' : S \to \mathrm{Ran}$ be as above. We consider $\mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))_{\underline{x}'}$ as acted on by $\mathfrak{L}(H)_{\underline{x}'}$. Similarly, $\mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathrm{Conn}(\mathfrak{h}))_{\underline{x}\subseteq\underline{x}'}$ is acted on by

$$\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(H)_{\underline{x}\subseteq\underline{x}'}.$$

Let $\mathrm{LS}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{H,\underline{x}\subseteq\underline{x}'}$ be the prestack equal to the étale sheafification of the (non-sheafified) quotient of $\mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathrm{Conn}(\mathfrak{h}))_{\underline{x}\subseteq\underline{x}'}$ by $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(H)_{\underline{x}\subseteq\underline{x}'}$.

The composition of morphisms is defined as in Sect. C.10.10.

Note that the underlying counital factorization space of $\mathrm{LS}_H^{\mathrm{mer}}$ is $\mathrm{LS}_H^{\mathrm{reg}}$.

C.10.12. We now define a unital-in-correspondences structure on $\mathrm{Op}_{\check{G}}^{\text{mon-free}}$. This is, however, automatic since

$$\mathrm{Op}_{\check{G}}^{\text{mon-free}} := \mathrm{Op}_{\check{G}}^{\mathrm{mer}} \underset{\mathrm{LS}_{\check{G}}^{\mathrm{mer}}}{\times} \mathrm{LS}_{\mathsf{G}}^{\mathrm{reg}},$$

so the unital-in-correspondences structure on $\mathrm{Op}_{\check{G}}^{\mathrm{mer}}$, $\mathrm{LS}_{\check{G}}^{\mathrm{mer}}$ and $\mathrm{LS}_{\check{G}}^{\mathrm{reg}}$ induces one on $\mathrm{Op}_{\check{G}}^{\text{mon-free}}$.

Note that by construction, the underlying counital factorization space of $\mathrm{Op}_{\check{G}}^{\text{mon-free}}$ is $\mathrm{Op}_{\check{G}}^{\mathrm{reg}}$.

C.11. **Unital factorization categories.**

C.11.1. Let $\mathbf{A}$ be a factorization category on X. A unital structure on $\mathbf{A}$ is an extension of the crystal of categories $\underline{\mathbf{A}}$ over Ran to a crystal of categories over $\mathrm{Ran}^{\mathrm{untl},*}$ in a way compatible with factorization, i.e., we extend the identifications (B.37) to

$$\mathrm{union}^*(\underline{\mathbf{A}})|_{(\mathrm{Ran}^{\mathrm{untl},*}\times\mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}} \simeq \underline{\mathbf{A}} \boxtimes \underline{\mathbf{A}}|_{(\mathrm{Ran}^{\mathrm{untl},*}\times\mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}}. \tag{C.47}$$

In addition, we stipulate

$$\mathbf{A}_\emptyset \simeq \mathrm{Vect}, \tag{C.48}$$

so that this identification behaves (homotopically coherently) as a unit for the identifications (C.47), i.e., the functor

$$\underline{\mathbf{A}} \to \mathbf{A}_\emptyset \otimes \underline{\mathbf{A}},$$

obtained by restricting (C.47) to

$$\{\emptyset\} \times \mathrm{Ran}^{\mathrm{untl},*} \to (\mathrm{Ran}^{\mathrm{untl},*} \times \mathrm{Ran}^{\mathrm{untl},*})_{\mathrm{disj}}$$

identifies via (C.48) with the identity functor.

We let

$$\mathbf{1}_{\mathbf{A}} \in \Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl},*}, \underline{\mathbf{A}})$$

the canonical object, whose value at any $\underline{x} \in \mathrm{Ran}^{\mathrm{untl},*}$ is

$$\mathrm{ins.\,unit}_{\emptyset\subset\underline{x}}(k),$$

where:

- $k \in \mathrm{Vect} \simeq \mathbf{A}_\emptyset$;
- $\mathrm{ins.\,unit}_{\emptyset\subset\underline{x}}$ is the functor $\mathbf{A}_\emptyset \to \mathbf{A}_{\underline{x}}$ corresponding to the unique morphism $\{\emptyset\} \to \underline{x}$.

C.11.2. Note that the factorization category Vect from Sect. B.11.2 admits a tautological unital structure. Namely, the underlying crystal of categories on $\mathrm{Ran}^{\mathrm{untl},*}$ is $\underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl},*})$.

C.11.3. Given a pair of unital factorization categories, we can talk about lax unital or strictly unital functors between them, compatible with factorization, see Sect. C.2.8.

We denote the resulting (2-)categories by

$$\mathrm{FactCat}^{\mathrm{untl,lax}}(X) \text{ and } \mathrm{FactCat}^{\mathrm{untl,strict}}(X),$$

respectively.

C.11.4. Pointwise tensor product defines a symmetric monoidal structure on the category of unital factorization categories (for both variants: lax or strictly unital functors).

The unit for the above symmetric monoidal structure is Vect.

C.11.5. Let $\mathbf{A}$ be a unital factorization category. We shall say that $\mathbf{A}$ is dualizable if it is a dualizable as an object of the above category, with *lax* unital functors as morphisms.

In this case, the evaluation and the coevaluation functors

$$\mathbf{A} \otimes \mathbf{A}^\vee \to \mathrm{Vect} \text{ and } \mathrm{Vect} \to \mathbf{A}^\vee \otimes \mathbf{A}$$

carry lax unital structures.

C.11.6. *Example.* Let $\underline{\mathbf{A}}_X$ be as in Sect. B.11.6. The explicit construction of the (symmetric monoidal) factorization category $\mathrm{Fact}(\underline{\mathbf{A}}_X)$ shows that it admits a natural unital structure.

We will denote by the same symbol $\mathrm{Fact}(\underline{\mathbf{A}}_X)$ the resulting (symmetric monoidal) unital factorization category.

C.11.7. Given a unital factorization category $\mathbf{A}$, we can talk about unital factorization algebras in it: by definition, a unital factorization algebra $\mathcal{A}$ in $\mathbf{A}$ in it a lax unital factorization functor $\mathrm{Vect} \to \mathbf{A}$.

Explicitly, the datum of $\mathcal{A}$ is an object

$$\mathcal{A}_{\mathrm{Ran}^{\mathrm{untl},*}} \in \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl},*}, \underline{\mathbf{A}}),$$

which is compatible with factorization in the natural sense (i.e., combine the ideas from Sects. C.7.1 and B.11.4).

We denote by

$$\mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A})$$

the category of unital factorization algebras in $\mathbf{A}$.

The object $\mathbf{1}_{\mathbf{A}}$ has a natural structure of unital factorization algebra in $\mathbf{A}$. Furthermore, for any $\mathcal{A} \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A})$ we have a canonically defined map

$$\mathrm{vac}_{\mathcal{A}} : \mathbf{1}_{\mathbf{A}} \to \mathcal{A},$$

which we will refer to as the *unit* or *vacuum* for $\mathcal{A}$.

C.11.8. If $\mathbf{A}$ is a unital factorization category, it admits a canonically defined (strictly) unital functor

$$\mathrm{Vac}_{\mathbf{A}} : \mathrm{Vect} \to \mathbf{A}.$$

By a slight abuse of notation, we will denote this functor by $\mathbf{1}_{\mathbf{A}}$; the image of

$$k \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathrm{Vect})$$

under $\mathrm{Vac}_{\mathbf{A}}$ is $\mathbf{1}_{\mathbf{A}}$.

C.11.9. Given $\mathcal{A} \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A})$ and $\mathcal{Z} \to \mathrm{Ran}$, we can talk about unital factorization $\mathcal{A}$-modules *at* $\mathcal{Z}$. Denote this category by

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}}.$$

When we talk about *non-unital* factorization $\mathcal{A}$-modules, we will denote the corresponding category by

$$\mathcal{A}\text{-mod}^{\text{fact-n.u.}}(\mathbf{A})_{\mathcal{Z}}.$$

We have a forgetful functor

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}} \to \mathcal{A}\text{-mod}^{\text{fact-n.u.}}(\mathbf{A})_{\mathcal{Z}}$$

and Proposition C.7.9 applies in the present context as well.

The assignment

$$\mathcal{Z} \rightsquigarrow \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A})_{\mathcal{Z}}$$

is a crystal of categories over Ran that we will denote by $\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathbf{A})$. This crystal of categories has a natural lax factorization structure.

We will denote the resulting lax factorization category by $\mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathbf{A})$.

C.11.10. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a lax unital functor between unital factorization categories. Then it naturally gives rise to a functor

$$\Phi : \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A}_1) \to \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A}_2).$$

In particular, $\Phi(\mathbf{1}_{\mathbf{A}_1})$ has a natural structure of factorization algebra in $\mathbf{A}_2$, and we have a map of factorization algebras

$$\mathbf{1}_{\mathbf{A}_2} \to \Phi(\mathbf{1}_{\mathbf{A}_1}). \tag{C.49}$$

C.11.11. Let $\underline{\mathbf{A}}_X$ be as in Sect. C.11.6. Let $A \in \mathbf{A}_X$ be a commutative algebra object. Then the object

$$\mathrm{Fact}(A) \in \mathrm{ComAlg}(\mathrm{FactAlg}(X, \mathrm{Fact}(\underline{\mathbf{A}}_X)))$$

from Sect. B.11.6 naturally lifts to an object of

$$\mathrm{ComAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X, \mathrm{Fact}(\underline{\mathbf{A}}_X))).$$

By a slight abuse of notation, we will denote it by the same symbol $\mathrm{Fact}(A)$.

C.11.12. By a similar token, one defines the notion of unital structure on a *lax* factorization category (see Sect. B.11.12). The entire preceding discussion equally applies to unital lax factorization categories.

C.11.13. *Example.* Let $\mathcal{A}$ be a (not necessarily unital) factorization algebra. Recall the lax factorization category $\mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}$ (see Sect. B.11.15). We claim that it acquires a natural unital structure.

In order to define it, we need to provide the following data: for any $\mathcal{Z} \to \mathrm{Ran}$, we need to define a functor

$$\mathrm{ins.\,unit}_{\mathcal{Z}} : \mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathcal{Z}^{\subseteq}}, \tag{C.50}$$

equipped with an appropriate associativity structure. This construction was already mentioned in Sect. B.9.8:

Let $\mathcal{M}$ be an object of $\mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathcal{Z}}$. The corresponding object $\mathrm{ins.\,unit}_{\mathcal{Z}}(\mathcal{M}) \in \mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathcal{Z}^{\subseteq}}$ is constructed as follows.

Note that there is a canonical projection

$$\mathcal{Z}^{\subseteq^2} := (\mathcal{Z}^{\subseteq})^{\subseteq} \overset{\mathrm{pr}_{\mathrm{comp},\mathcal{Z}}}{\longrightarrow} \mathcal{Z}^{\subseteq}, \quad (z, \underline{x}_1 \subseteq \underline{x}_2) \mapsto (z, \underline{x}_2).$$

We let

$$(\mathrm{ins.\,unit}_{\mathcal{Z}}(\mathcal{M}))_{\mathcal{Z}^{\subseteq^2}} := (\mathrm{pr}_{\mathrm{comp},\mathcal{Z}})^!(\mathcal{M}_{\mathcal{Z}^{\subseteq}}).$$

The factorization structure on $(\mathrm{ins.\,unit}_{\mathcal{Z}}(\mathcal{M}))_{\mathcal{Z}^{\subseteq^2}}$ against $\mathcal{A}$ is induced by that on $\mathcal{M}_{\mathcal{Z}^{\subseteq}}$.

The factorization unit in $\mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}$ is the object $\mathcal{A}^{\mathrm{fact}}$ from Sect. B.11.15.

Let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a map of (non-unital) factorization algebras. Then the functor

$$\mathrm{Res}_\phi : \mathcal{A}_2\text{-}\mathrm{mod}^{\mathrm{fact}} \to \mathcal{A}_1\text{-}\mathrm{mod}^{\mathrm{fact}}$$

carries a natural lax unital structure.

C.11.14. Let now $\mathcal{A}$ be a unital factorization algebra. The above construction applies verbatim to the category of unital factorization $\mathcal{A}$-modules.

Consider the forgetful functor

$$\mathbf{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\text{fact}} \to \text{Vect}$$

(see (B.25)) as a factorization functor (where the left-hand side is a lax factorization category). We claim that it carries a naturally defined lax unital structure.

In order to define it, we need to provide the following data: for any $\mathcal{Z} \to \text{Ran}$ and $\mathcal{M} \in \mathcal{A}\text{-mod}^{\text{fact}}_{\mathcal{Z}}$ we need to define a map

$$\text{pr}^!_{\text{small},\mathcal{Z}}(\mathcal{M}_{\mathcal{Z}}) \to (\text{ins. unit}_{\mathcal{Z}}(\mathcal{M}))_{\mathcal{Z}^{\subseteq}}.$$

We note, however, that by construction

$$(\text{ins. unit}_{\mathcal{Z}}(\mathcal{M}))_{\mathcal{Z}^{\subseteq}} := \mathcal{M}_{\mathcal{Z}^{\subseteq}}.$$

Now, the sought-for map

$$\text{pr}^!_{\text{small},\mathcal{Z}}(\mathcal{M}_{\mathcal{Z}}) \to \mathcal{M}_{\mathcal{Z}^{\subseteq}}$$

is exactly provided by the structure on $\mathcal{M}$ of unital factorization $\mathcal{A}$-module.

C.11.15. By a similar token, given a (non-unital) lax factorization category $\mathbf{A}$ and a factorization algebra $\mathcal{A}$ in it, the lax factorization category $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$ (see Sect. B.11.16) acquires a naturally defined unital structure. The object

$$\mathcal{A}^{\text{fact}} \in \text{FactAlg}(X, \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}))$$

from Sect. B.11.16 extends to an object of $\text{FactAlg}^{\text{untl}}(X, \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}))$, and equals in fact the unit in $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$, i.e., the map

$$\mathbf{1}_{\mathcal{A}\text{-mod}^{\text{fact}}} \to \mathcal{A}^{\text{fact}}$$

is an isomorphism.

For a factorization functor $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ between non-unital lax factorization categories and $\mathcal{A}_1 \in \text{FactAlg}(\mathbf{A}_1, X)$, the resulting functor

$$\Phi : \mathcal{A}_1\text{-mod}^{\text{fact}}(\mathbf{A}_1) \to \Phi(\mathcal{A}_1)\text{-mod}^{\text{fact}}(\mathbf{A}_2) \tag{C.51}$$

has a natural (stictly) unital structure.

Similarly, if $\mathbf{A}$ is unital and $\mathcal{A}$ is unital, then the lax factorization category $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$ (of *unital* $\mathcal{A}$-modules) acquires a naturally defined unital structure. Furthermore, in this case the forgetful functor

$$\mathbf{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}) \to \mathbf{A}$$

carries a naturally defined lax unital structure.

For a lax unital functor $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ between unital lax factorization categories and $\mathcal{A}_1 \in \text{FactAlg}^{\text{untl}}(\mathbf{A}_1, X)$, the resulting functor

$$\Phi : \mathcal{A}_1\text{-mod}^{\text{fact}}(\mathbf{A}_1) \to \Phi(\mathcal{A}_1)\text{-mod}^{\text{fact}}(\mathbf{A}_2) \tag{C.52}$$

has a natural (stictly) unital structure.

C.11.16. Let $\mathbf{A}$ be a (non-unital) lax factorization category, and let $\phi : \mathcal{A} \to \mathcal{A}'$ be a homomorphism between factorization (non-unital) algebras in it.

Since the functor

$$\text{Res}_\phi : \mathcal{A}'\text{-mod}^{\text{fact}}(\mathbf{A}) \to \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$$

is lax unital and $\mathcal{A}'^{\text{fact}}$ is the factorization unit in $\mathcal{A}'\text{-mod}^{\text{fact}}(\mathbf{A})$, we obtain that the factorization algebra $\text{Res}_\phi(\mathcal{A}'^{\text{fact}}) \in \text{FactAlg}(X, \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}))$ acquires a natural unital structure.

Moreover, the functor (C.52) applied to $\mathbf{A}_1 = \mathcal{A}'\text{-mod}^{\text{fact}}(\mathbf{A})$, $\mathbf{A}_2 = \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$ and $\Phi = \text{Res}_\phi$ gives rise to a unital factorization functor

$$\mathcal{A}'\text{-mod}^{\text{fact}}(\mathbf{A}) \to \text{Res}_\phi(\mathcal{A}'^{\text{fact}})\text{-mod}^{\text{fact-untl}}(\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})). \tag{C.53}$$

Unwinding the definitions, we obtain:

Lemma C.11.17. *The functor* (C.53) *is an equivalence.*

C.11.18. Let $\mathbf{A}$ be a unital factorization category, and let $\phi : \mathcal{A} \to \mathcal{A}'$ be a homomorphism between unital factorization algebras in $\mathbf{A}$.

Consider the unital lax factorization categories $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$ and $\mathcal{A}'\text{-mod}^{\text{fact}}(\mathbf{A})$, and the restriction functor

$$\text{Res}_\phi : \mathcal{A}'\text{-mod}^{\text{fact}}(\mathbf{A}) \to \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}).$$

This functor carries a natural lax unital structure. In particular, the object

$$\text{Res}_\phi(\mathcal{A}'^{\text{fact}}) \in \text{FactAlg}(\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}))$$

from Sect. B.11.17 lifts to an object of $\text{FactAlg}^{\text{untl}}(\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}))$.

Note that the forgetful functor $\mathbf{oblv}_\mathcal{A} : \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}) \to \mathbf{A}$ sends $\text{Res}_\phi(\mathcal{A}'^{\text{fact}}) \to \mathcal{A}'$. In particular, it induces a functor

$$\text{Res}_\phi(\mathcal{A}'^{\text{fact}})\text{-mod}^{\text{fact}}(\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})) \to \mathcal{A}'\text{-mod}^{\text{fact}}(\mathbf{A}). \tag{C.54}$$

The following is obtained by unwinding the definitions:

Lemma C.11.19. *The functor* (C.54) *is an equivalence.*

In fact, the inverse of the functor (C.54) is given by the functor (C.52) for $\mathbf{A}_1 = \mathcal{A}'\text{-mod}^{\text{fact}}(\mathbf{A})$, $\mathbf{A}_2 = \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A})$, $\Phi = \text{Res}_\phi$ and $\mathcal{A}_1 = \mathcal{A}'^{\text{fact}}$.

C.11.20. *Proof of Proposition C.7.13.* We will explicitly construct an inverse functor.

Let $k \overset{\text{vac}_\mathcal{A}}{\to} \mathcal{A}$ be a quasi-unital factorization algebra. Consider the unital lax factorization category[78] $\mathcal{A}\text{-mod}^{\text{fact-n.u.}}$. It is equipped with a lax unital factorization functor

$$\text{Res}_{\text{vac}_\mathcal{A}} : \mathcal{A}\text{-mod}^{\text{fact-n.u.}} \to k\text{-mod}^{\text{fact-n.u.}}.$$

Consider the fiber product

$$\mathcal{A}\text{-mod}^{\text{fact-q.u.}} := \text{Vect} \underset{k\text{-mod}^{\text{fact-n.u.}}}{\times} \mathcal{A}\text{-mod}^{\text{fact-n.u.}} \subset C\mathcal{A}\text{-mod}^{\text{fact-n.u.}}.$$

It has a natural factorization structure, and the fact that

$$\text{Res}_{\text{vac}_\mathcal{A}}(\mathcal{A}^{\text{fact}_{\text{Ran}}}) \in k\text{-mod}^{\text{fact}}_{\text{Ran}} \subset k\text{-mod}^{\text{fact-n.u.}}_{\text{Ran}}$$

implies that the unital structure on $\mathcal{A}\text{-mod}^{\text{fact-n.u.}}$ induces one on $\mathcal{A}\text{-mod}^{\text{fact-q.u.}}$. The factorization unit $\mathbf{1}_{\mathcal{A}\text{-mod}^{\text{fact-q.u.}}}$ in $\mathcal{A}\text{-mod}^{\text{fact-q.u.}}$ is $\mathcal{A}^{\text{fact}}$ (see Sect. C.11.13).

Restricting $\text{Res}_{\text{vac}_\mathcal{A}}$ to $\mathcal{A}\text{-mod}^{\text{fact-q.u.}}$, we obtain a lax unital factorization functor

$$\mathcal{A}\text{-mod}^{\text{fact-q.u.}} \to \text{Vect}\,. \tag{C.55}$$

The image of the factorization unit $\mathbf{1}_{\mathcal{A}\text{-mod}^{\text{fact-q.u.}}} = \mathcal{A}^{\text{fact}}$ under the above functor is a unital factorization algebra (whose underlying plain factorization algebra is $\mathcal{A}$ itself). This defines a functor

$$\text{FactAlg}^{\text{q-untl}}(X) \to \text{FactAlg}^{\text{untl}}(X), \tag{C.56}$$

which commutes with a forgetful functor to $\text{FactAlg}(X)$.

Let us show that the functors (C.36) and (C.56) are mutually inverse. We first show that the composition (C.56)∘(C.36) is isomorphic to the identity functor.

Indeed, when $\mathcal{A}$ is unital, by Proposition C.7.9, the unital factorization category $\mathcal{A}\text{-mod}^{\text{fact-q.u.}}$ identifies with $\mathcal{A}\text{-mod}^{\text{fact}}$, and the functor (C.56) identifies with $\mathbf{oblv}_\mathcal{A}$, equipped with its natural lax unital structure. Hence, in this case, the image of $\mathcal{A}^{\text{fact}}$ under (C.56) identifies with $\mathcal{A}$ as a unital factorization algebra.

[78] The appearance of "n.u." in the superscript in the next formula is meant to emphasize that we are dealing with non-unital factorization modules, even though since $\mathcal{A}$ is non-unital, we cannot even talk about unital modules.

Vice versa, let us start with a quasi-unital factorization algebra $k \overset{\mathrm{vac}_{\mathcal{A}}}{\to} \mathcal{A}$. We have a commutative diagram of unital lax factorization categories and lax unital functors, with the horizontal arrows being strict:

$$\begin{tikzcd} \mathcal{A}\text{-mod}^{\text{fact-q.u.}} \arrow[r] \arrow[d] & \mathcal{A}\text{-mod}^{\text{fact-n.u.}} \arrow[d] \\ \mathrm{Vect} \arrow[r] & k\text{-mod}^{\text{fact-n.u.}}. \end{tikzcd}$$

Applying each circuit to $\mathbf{1}_{\mathcal{A}\text{-mod}^{\text{fact-q.u.}}}$ we obtain a factorization algebra in $k\text{-mod}^{\text{fact-n.u.}}$, equipped with a homomorphism from k. For the clockwise circuit, we obtain the original $k \overset{\mathrm{vac}_{\mathcal{A}}}{\to} \mathcal{A}$. For the anti-clockwise circuit, we obtain (C.56)($k \overset{\mathrm{vac}_{\mathcal{A}}}{\to} \mathcal{A}$), equipped with a map from k to it, given by its unital structure.

□[Proposition C.7.13]

C.11.21. In the rest of this subsection we will focus on strict (i.e., non-lax) factorization categories.

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be a pair of unital factorization categories, and let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a strictly unital functor. Assume that Φ admits a right adjoint Φ^R as a functor between the underlying crystals of categories on Ran.

Then Φ^R, viewed as a factorization functor between factorization categories admits a natural extension to a lax unital functor between unital factorization categories, see Sect. C.2.11.

C.11.22. We claim:

Lemma C.11.23. *Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a lax unital factorization functor between unital factorization categories. Then Φ is strictly unital if and only if the map* (C.49) *is an isomorphism.*

Proof. The "only if" direction is tautological. Let us prove the "if" direction, so let us assume that (C.49) is an isomorphism.

Let $\underline{x} \subseteq \underline{x}'$ be an arrow in $\mathrm{Ran}^{\mathrm{untl},*}(S)$ for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathrm{Ran}}$. We need to show that the natural transformation

$$\text{ins. unit}_{\mathbf{A}_2,\underline{x}\subseteq\underline{x}'} \circ \Phi_{\underline{x}} \to \Phi_{\underline{x}'} \circ \text{ins. unit}_{\mathbf{A}_1,\underline{x}\subseteq\underline{x}'}, \tag{C.57}$$

given by the lax unital structure on Φ, is an isomorphism.

This asssertion can be checked strata-wise, so we can assume that $\underline{x}$ and $\underline{x}'$ are field-valued points. Write

$$\underline{x}' = \underline{x} \sqcup \underline{x}''.$$

We have

$$\mathbf{A}_{i,\underline{x}'} \simeq \mathbf{A}_{i,\underline{x}} \otimes \mathbf{A}_{i,\underline{x}''}, \quad i = 1, 2$$

and the natural transformation (C.57) is the tensor product of the identity endomorphism of the functor

$$\Phi_{\underline{x}} : \mathbf{A}_{1,\underline{x}} \to \mathbf{A}_{2,\underline{x}}$$

and the natural transformation (C.57) for $\emptyset \subset \underline{x}$.

However, the latter is exactly the map

$$\mathbf{1}_{\mathbf{A}_2,\underline{x}} \to \Phi_{\underline{x}}(\mathbf{1}_{\mathbf{A}_1,\underline{x}}).$$

□

C.12. **Examples of unital factorization categories arising from algebraic geometry.**

C.12.1. Let $\mathcal{T}$ be a counital factorization space. We claim that the lax factorization category $\mathrm{QCoh}(\mathcal{T})$ admits a natural unital structure.

Indeed, for a pair of S-points $\underline{x} \subseteq \underline{x}'$ of $\mathrm{Ran}^{\mathrm{untl},*}$, the corresponding functor

$$\mathrm{QCoh}(\mathcal{T}_{S,\underline{x}}) \to \mathrm{QCoh}(\mathcal{T}_{S,\underline{x}'})$$

is given by pullback along

$$\mathcal{T}_{S,\underline{x}'} \to \mathcal{T}_{S,\underline{x}}. \tag{C.58}$$

In particular, the unit $\mathbf{1}_{\mathrm{QCoh}(\mathcal{T})}$ is the structure sheaf

$$\mathcal{O}_{\mathcal{T}} \in \mathrm{QCoh}(\mathcal{T}).$$

C.12.2. By a similar token, using Sect. B.13.16 the (lax) factorization category $\mathrm{IndCoh}^!(\mathcal{T})$ admits a natural unital structure.

Assume now that $\mathcal{T}$ is affine and placid. Recall that according to Sect. B.13.22, we can consider the factorization category $\mathrm{IndCoh}^*(\mathcal{T})$.

Assume now that the maps

$$\mathcal{T}_{S,X^{I_2}} \to \mathcal{T}_{S,X^{I_1}}$$

for inclusions of finite sets $I_1 \hookrightarrow I_2$ are of finite Tor-dimension (e.g., they are flat). In this case, by Sect. A.10.13, the functors of *-pullback along the maps (C.58) are defined for $\mathrm{IndCoh}^*(-)$

$$\mathrm{IndCoh}^*(\mathcal{T}_{S,\underline{x}}) \to \mathrm{IndCoh}^*(\mathcal{T}_{S,\underline{x}'}).$$

Hence, we obtain that in this case, $\mathrm{IndCoh}^*(\mathcal{T})$ also admits a natural unital structure.

C.12.3. Let now $\mathcal{T}$ be a unital factorization space. We claim that in this case the factorization category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})$ has a natural unital structure.

Indeed, for a pair of S-points $\underline{x} \subseteq \underline{x}'$ of $\mathrm{Ran}^{\mathrm{untl},*}$, the corresponding functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{S,\underline{x}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{S,\underline{x}'})$$

is given by pushforward along

$$\mathcal{T}_{S,\underline{x}} \to \mathcal{T}_{S,\underline{x}'}. \tag{C.59}$$

Assume now that $\mathcal{T}$ is an ind-placid factorization ind-scheme, so that we can consider the factorization category $\mathrm{IndCoh}^*(\mathcal{T})$.

Taking the IndCoh-pushforwards along (C.59) we obtain that $\mathrm{IndCoh}^*(\mathcal{T})$ acquires a natural unital structure.

C.12.4. *Example.* We obtain that the factorization categories $\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Gr}_G)$ and $\mathrm{IndCoh}(\mathrm{Gr}_G)$ acquire a natural unital structure.

By a similar mechanism, the factorization category $\mathrm{D\text{-}mod}(\mathrm{Gr}_G)$ also acquires a unital structure.

C.12.5. Let us now in addition assume that for an injection of finite sets $I_1 \hookrightarrow I_2$, the corresponding map

$$X^{I_2} \underset{X^{I_1}}{\times} \mathcal{T}_{X^{I_1}} \to \mathcal{T}_{X^{I_2}}$$

is an ind-closed embedding locally almost of finite-presentation.

In this case, by Sect. A.10.11, the IndCoh-pushforward functors along (C.59) are defined on $\mathrm{IndCoh}^!(-)$:

$$\mathrm{IndCoh}^!(\mathcal{T}_{S,\underline{x}}) \to \mathrm{IndCoh}^!(\mathcal{T}_{S,\underline{x}'}).$$

Hence, we obtain that in this case, $\mathrm{IndCoh}^!(\mathcal{T})$ also acquires a unital structure.

C.12.6. Let $\mathcal{T}$ be a factorization space over X, equipped with a unital-in-correspondences structure. Assume that for a pair of S-points $\underline{x} \subseteq \underline{x}'$ of $\mathrm{Ran}^{\mathrm{untl},*}$, the map

$$\mathcal{T}^{\mathrm{all} \rightsquigarrow +}_{\underline{x} \subseteq \underline{x}'} \overset{\mathrm{pr}^{\mathcal{T}}_{\mathrm{small}}}{\longrightarrow} \mathcal{T}_{\underline{x}} \tag{C.60}$$

is affine.

We claim that in this case, the factorization category $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})$ acquires a unital structure, and the functor

$$\mathrm{QCoh}(\mathcal{T}^+) = \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}^+) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}),$$

given by direct image along

$$\iota : \mathcal{T}^+ \to \mathcal{T}$$

is strictly unital. In particular, the unit $\mathbf{1}_{\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T})}$ is the direct image of $\mathcal{O}_{\mathcal{T}^+}$ along ι.

Namely, for a pair of S-points $\underline{x} \subseteq \underline{x}'$ of $\mathrm{Ran}^{\mathrm{untl},*}$, the corresponding functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{S,\underline{x}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathcal{T}_{S,\underline{x}'})$$

is given by pull-push along

$$\mathcal{T}_{\underline{x}} \overset{\mathrm{pr}^{\mathcal{T}}_{\mathrm{small}}}{\longleftarrow} \mathcal{T}^{\mathrm{all} \rightsquigarrow +}_{\underline{x} \subseteq \underline{x}'} \overset{\mathrm{pr}^{\mathcal{T}}_{\mathrm{big}}}{\longrightarrow} \mathcal{T}_{\underline{x}'}. \tag{C.61}$$

C.12.7. *Example.* Thus, we obtain that for an affine D-scheme $\mathcal{Y}$, the factorization category

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))$$

acquires a unital structure, with the unit $\mathbf{1}_{\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))}$ being the direct image of $\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})}$ along

$$\mathfrak{L}^+_\nabla(\mathcal{Y}) \overset{\iota}{\to} \mathfrak{L}_\nabla(\mathcal{Y}).$$

As another example, we can take $\mathcal{T} = \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}$, and we obtain that $\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$ acquires a unital structure. The unit $\mathbf{1}_{\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})}$ is the direct image of $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}$ along

$$\mathrm{Op}^{\mathrm{reg}}_{\check{G}} \overset{\iota^{+,\mathrm{mon\text{-}free}}}{\to} \mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}.$$

C.12.8. Assume now that $\mathcal{T}$ is an ind-placid ind-affine factorization ind-scheme. Assume that the maps

$$\mathcal{T}^{\mathrm{all} \rightsquigarrow +}_{I_1 \subseteq I_2} \overset{\mathrm{pr}^{\mathcal{T}}_{\mathrm{small}}}{\longrightarrow} \mathcal{T}_{X^{I_1}}$$

for an injection of finite sets $I_1 \subseteq I_2$ are affine and of finite Tor-dimension.

In this case, by Sect. A.10.13, the functors of *-pullback along the maps (C.60) are well-defined on $\mathrm{IndCoh}^*(-)$ and satisfy base change.

We define a unital structure on the factorization category $\mathrm{IndCoh}^*(\mathcal{T})$ by (IndCoh,*)-pull followed by IndCoh-pushforward along (C.61).

Note that by construction, the functor of IndCoh-pushforward along ι

$$\mathrm{IndCoh}^*(\mathcal{T}^+) \to \mathrm{IndCoh}^*(\mathcal{T})$$

is (strictly) unital.

In particular, the unit in $\mathrm{IndCoh}^*(\mathcal{T})$ is the direct image of $\mathcal{O}_{\mathcal{T}^+} \in \mathrm{IndCoh}^*(\mathcal{T}^+)$ along ι.

C.12.9. Let us continue to assume that $\mathcal{T}$ is an ind-placid ind-affine factorization ind-scheme. Assume now that for an inclusion of finite sets $I_1 \subseteq I_2$, the map

$$\mathcal{T}^{\text{all}\rightsquigarrow+}_{I_1\subseteq I_2} \xrightarrow{\text{pr}^{\mathcal{T}}_{\text{big}}} \mathcal{T}_{X^{I_2}}$$

is an ind-closed embedding locally almost of finite-presentation.

In this case, Sect. A.10.11, the IndCoh-pushforward functors along

$$\mathcal{T}^{\text{all}\rightsquigarrow+}_{\underline{x}\subseteq\underline{x}'} \xrightarrow{\text{pr}^{\mathcal{T}}_{\text{big}}} \mathcal{T}_{\underline{x}'}$$

are defined on $\text{IndCoh}^!(-)$ and satisfy base change.

We define a unital structure on the factorization category $\text{IndCoh}^!(\mathcal{T})$ by !-pull followed by IndCoh-pushforward along (C.61).

Note that by construction, the functor of IndCoh-pushforwardforward along ι

$$\text{IndCoh}^!(\mathcal{T}^+) \to \text{IndCoh}^!(\mathcal{T})$$

is (strictly) unital.

In particular, the unit in $\text{IndCoh}^!(\mathcal{T})$ is the direct image of $\omega_{\mathcal{T}^+} \in \text{IndCoh}^!(\mathcal{T}^+)$ along ι.

C.12.10. *Example.* Let $\mathcal{Y}$ be an affine D-scheme, such that:

- $\mathfrak{L}_\nabla(\mathcal{Y})$ is ind-placid;
- The maps
$$\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{I_1\subseteq I_2} \xrightarrow{\text{pr}^{\mathcal{Y}}_{\text{small}}} \mathfrak{L}_\nabla(\mathcal{Y})_{X^{I_1}}$$
for $I_1 \subseteq I_2$ are flat;
- The maps
$$\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{I_1\subseteq I_2} \xrightarrow{\text{pr}^{\mathcal{Y}}_{\text{big}}} \mathfrak{L}_\nabla(\mathcal{Y})_{X^{I_2}}$$
for $I_1 \subseteq I_2$ are locally almost of finite-presentation.

This happens, e.g., for $\mathcal{Y} = \text{Jets}(H)$, where H is a smooth group-scheme over X.

We obtain that the factorization categories $\text{IndCoh}^!(\mathfrak{L}_\nabla(\mathcal{Y}))$ and $\text{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y}))$ acquire unital structures.

C.12.11. By a similar procedure, the factorization category $\text{D-mod}(\mathfrak{L}(H))$ acquires a unital structure.

C.12.12. As yet another example, we obtain that the categories

$$\text{IndCoh}^!(\text{Op}^{\text{mon-free}}_{\check{G}}) \text{ and } \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}})$$

acquire unital structures.

C.12.13. Let $\mathcal{T}$ be an ind-placid ind-affine factorization ind-scheme. Assume that both additional conditions in Sects. C.12.8 and C.12.9 are satisfied, so both $\text{IndCoh}^*(\mathcal{T})$ and $\text{IndCoh}^!(\mathcal{T})$ acquire unital structures.

Note that the canonical pairing

$$\text{IndCoh}^!(\mathcal{T}) \otimes \text{IndCoh}^*(\mathcal{T}) \to \text{Vect} \tag{C.62}$$

(see Sect. A.10.8) as factorization categories, admits a natural lax unital structure as a functor between unital factorization categories.

Moreover, unwinding the definitions we obtain that the condition from Sect. C.2.14 is satisfied.

Hence, we obtain that (C.62) realizes $\text{IndCoh}^!(\mathcal{T})$ and $\text{IndCoh}^*(\mathcal{T})$ as each other duals as unital factorization categories.

C.13. **Unital structure on Kac-Moody representations.**

C.13.1. Our current goal is to construct a unital structure on the factorization category $\widehat{\mathfrak{g}}\text{-mod}_\kappa$. Let us be given a pair of S-points $\underline{x}, \underline{x}'$ of Ran with $\underline{x} \subseteq \underline{x}'$. We need to construct a functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}'}. \tag{C.63}$$

The unital-in-correspondences structure on $\mathfrak{L}(G)$ gives rise to the following diagram of group ind-schemes over S:

$$\mathfrak{L}(G)_{\underline{x}} \overset{\mathrm{pr}^G_{\mathrm{small}}}{\longleftarrow} \mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}\subseteq\underline{x}'} \overset{\mathrm{pr}^G_{\mathrm{small}}}{\longrightarrow} \mathfrak{L}(G)_{\underline{x}'}.$$

Proceeding as in Sect. B.14.19, we consider the corresponding categories

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}},\ \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}'} \text{ and } \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}\subseteq\underline{x}'}.$$

The universal properties of these categories give rise to restriction functors

$$\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}'}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq\underline{x}'}} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}'} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}\subseteq\underline{x}'} \tag{C.64}$$

and

$$\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq\underline{x}'}} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}\subseteq\underline{x}'}, \tag{C.65}$$

both strongly compatible with the forgetful functors to D-mod(S) and the actions of $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}\subseteq\underline{x}'}$ at level κ.

C.13.2. One checks directly that the functor $\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq\underline{x}'}}$ admits a left adjoint.

We define the functor

$$\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}'}$$

as the composition

$$(\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}'}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq\underline{x}'}})^L \circ \mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq\underline{x}'}}. \tag{C.66}$$

C.13.3. In order to upgrade the collection of functors $\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'}$ to a unital structure, we need to construct associativity isomorphisms

$$\mathrm{ins.vac}_{\underline{x}_3\subseteq\underline{x}_2} \circ \mathrm{ins.vac}_{\underline{x}_1\subseteq\underline{x}_2} \simeq \mathrm{ins.vac}_{\underline{x}_1\subseteq\underline{x}_3} \tag{C.67}$$

for $\underline{x}_1 \subseteq \underline{x}_2 \subseteq \underline{x}_3$.

Note that the inclusions

$$(\mathcal{D}_{\underline{x}_2} - \underline{x}_1) \hookrightarrow (\mathcal{D}_{\underline{x}_3} - \underline{x}_1) \hookleftarrow (\mathcal{D}_{\underline{x}_3} - \underline{x}_2)$$

give rise to maps

$$\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}_1\subseteq\underline{x}_2} \leftarrow \mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}_1\subseteq\underline{x}_3} \to \mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}_3\subseteq\underline{x}_2}.$$

We have the corresponding functors

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}_1\subseteq\underline{x}_2} \xrightarrow{\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_2}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_3}}} \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}_1\subseteq\underline{x}_3} \xleftarrow{\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_2\subseteq\underline{x}_3}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_3}}} \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}_2\subseteq\underline{x}_2}$$

and an isomorphism

$$\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_2}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_3}} \circ \mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_2}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_2}} \simeq \mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_2\subseteq\underline{x}_3}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_3}} \circ \mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_2}}_{\widehat{\mathfrak{g}}_{\underline{x}_2\subseteq\underline{x}_3}}.$$

From here we obtain a natural transformation

$$(\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_2\subseteq\underline{x}_3}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_3}})^L \circ \mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_2}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_3}} \to \mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_2}}_{\widehat{\mathfrak{g}}_{\underline{x}_2\subseteq\underline{x}_3}} \circ (\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_2}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq\underline{x}_2}})^L. \tag{C.68}$$

We claim that (C.68) is an isomorphism. Indeed, this follows from the fact that the diagram of Lie algebras

$$\begin{tikzcd}
\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathfrak{g})_{\underline{x}_1\subseteq\underline{x}_3} \arrow[r] \arrow[d] & \mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathfrak{g})_{\underline{x}_2\subseteq\underline{x}_3} \arrow[d] \\
\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathfrak{g})_{\underline{x}_1\subseteq\underline{x}_2} \arrow[r] & \mathfrak{L}(\mathfrak{g})_{\underline{x}_2}
\end{tikzcd}$$

is Cartesian.

Now, (C.67) follows by precomposing both sides of (C.68) with $\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_1}}_{\widehat{\mathfrak{g}}_{\underline{x}_1\subseteq \underline{x}_2}}$ and post-composing with $(\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}_3}}_{\widehat{\mathfrak{g}}_{\underline{x}_2\subseteq \underline{x}_3}})^L$.

C.13.4. Recall that the functor $\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq \underline{x}'}}$ is compatible with the action of $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}\subseteq\underline{x}'}$. In particular, it is compatible with the action of $\mathfrak{L}^+(G)_{\underline{x}'}$, which acts on $\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}}$ via $\mathfrak{L}^+(G)_{\underline{x}'} \to \mathfrak{L}^+(G)_{\underline{x}}$. Hence, it induces a functor

$$\mathrm{KL}(G)_{\kappa,S,\underline{x}} = (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}})^{\mathfrak{L}^+(G)_{\underline{x}}} \to \left(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}\subseteq\underline{x}'}\right)^{\mathfrak{L}^+(G)_{\underline{x}'}}.$$

Since the functor $\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}'}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq \underline{x}'}}$ is compatible with the action of $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}\subseteq\underline{x}'}$, then so is its left adjoint. In particular, we obtain that $(\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}'}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq \underline{x}'}})^L$ is compatible with the action of $\mathfrak{L}^+(G)_{\underline{x}'}$, and hence induces a functor

$$\left(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}\subseteq\underline{x}'}\right)^{\mathfrak{L}^+(G)_{\underline{x}'}} \to (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}'})^{\mathfrak{L}^+(G)_{\underline{x}'}} = \mathrm{KL}(G)_{\kappa,S,\underline{x}'}.$$

Hence, we obtain that the functor (C.66) induces a functor

$$\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'} : \mathrm{KL}(G)_{\kappa,S,\underline{x}} \to \mathrm{KL}(G)_{\kappa,S,\underline{x}'}. \tag{C.69}$$

The functors (C.69) define a unital structure on $\mathrm{KL}(G)_\kappa$.

C.13.5. By construction, the unit object $\mathbf{1}_{\widehat{\mathfrak{g}}\text{-mod}_\kappa}$ is the vacuum module $\mathrm{Vac}(G)_\kappa$, i.e.,

$$\mathrm{Vac}(G)_{\kappa,\underline{x}} := (\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}}}_{\mathfrak{L}^+(\mathfrak{g})_{\underline{x}}})^L \circ \mathbf{oblv}^0_{\mathfrak{L}^+(\mathfrak{g})_{\underline{x}}}(k).$$

Furthermore, $\mathrm{Vac}(G)_\kappa$ naturally upgrades to an object of $\mathrm{KL}(G)_\kappa$ and coincides with its unit.

C.13.6. Let κ' be as in Sect. 2.2.

Recall that according to [Ra5, Sect. 9.16.11], we have canonical pairings

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\kappa',\underline{x}} \to \mathrm{D}\text{-mod}(S)$$

making

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa \text{ and } \widehat{\mathfrak{g}}\text{-mod}_{\kappa'}$$

mutually dual as factorization categories.

We claim that this duality extends to a duality between $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ and $\widehat{\mathfrak{g}}\text{-mod}_{\kappa'}$ as unital factorization categories (see Sect. C.2.12 for what this means).

Namely, imitating the construction in [Ra5, Sect. 9.16] we obtain a duality between

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}\subseteq\underline{x}'} \text{ and } \widehat{\mathfrak{g}}\text{-mod}_{\kappa',S,\underline{x}\subseteq\underline{x}'}.$$

Under this identification, the dual of the functor $\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq \underline{x}'}}$ of (C.65) is the *right adjoint* of

$$\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq \underline{x}'}} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa',S,\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa',S,\underline{x}\subseteq\underline{x}'},$$

and the dual of the functor $\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}'}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq \underline{x}'}}$ of (C.64) is the left adjoint of the functor

$$\mathbf{oblv}^{\widehat{\mathfrak{g}}_{\underline{x}}}_{\widehat{\mathfrak{g}}_{\underline{x}\subseteq \underline{x}'}} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa',S,\underline{x}'} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa',S,\underline{x}\subseteq\underline{x}'}.$$

From here we obtain the desired identification between the dual of

$$\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,S,\underline{x}'}$$

and the right adjoint of

$$\mathrm{ins.vac}_{\underline{x}\subseteq\underline{x}'} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa',S,\underline{x}} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa',S,\underline{x}'}.$$

C.13.7. In a similar fashion we obtain that the duality between

$$\mathrm{KL}(G)_\kappa \text{ and } \mathrm{KL}(G)_{\kappa'}$$

as factorization categories extends to a duality as unital factorization categories.

C.14. **Unital factorization module categories.**

C.14.1. Let $\mathbf{A}$ be a factorization category. Let $\mathcal{Z}$ be a prestack mapping to Ran, and let $\mathbf{C}$ be a factorization module category over $\mathbf{A}$ at $\mathcal{Z}$.

Suppose that $\mathbf{A}$ is equipped with a unital structure. Combining the ideas of Sects. C.11.1, B.12.1 and C.7.4, we define the notion of unital structure on $\mathbf{C}$.

C.14.2. Concretely, the unital structure on $\mathbf{C}$ amounts to the following: given a pair of points $(z, \underline{x})$ and $(z, \underline{x}')$ of $\mathcal{Z}^{\subseteq}$ with $\underline{x} \subseteq \underline{x}'$, we must be given a functor

$$\mathrm{ins.\,unit}_{\underline{x}\subseteq\underline{x}'} : \mathbf{C}_{(z,\underline{x})} \to \mathbf{C}_{(z,\underline{x}')},$$

compatible with factorization.

The latter compatibility means the following: for $\underline{x}' = \underline{x} \sqcup \underline{x}''$, with respect to the identification

$$\mathbf{C}_{(z,\underline{x}')} \simeq \mathbf{A}_{\underline{x}''} \otimes \mathbf{C}_{(z,\underline{x})},$$

the functor $\mathrm{ins.\,unit}_{\underline{x}\subseteq\underline{x}'}$ is

$$\mathbf{1}_{\mathbf{A}_{\underline{x}''}} \otimes \mathrm{Id}\,.$$

In what follows we will denote by $\mathrm{ins.\,unit}_{\mathcal{Z}}$ the corresponding functor

$$\mathbf{C}_{\mathcal{Z}} \to \mathbf{C}_{\mathcal{Z}^{\subseteq},\mathrm{untl}}. \tag{C.70}$$

C.14.3. For a unital factorization category $\mathbf{A}$ and any $\mathcal{Z} \to \mathrm{Ran}$, consider the factorization module category $\mathbf{A}^{\mathrm{fact}_{\mathcal{Z}}}$ from Sect. B.12.4.

Unwinding the definitions, we obtain that $\mathbf{A}^{\mathrm{fact}_{\mathcal{Z}}}$ carries a natural unital structure (cf. Sect. C.7.6).

C.14.4. Given a pair of unital factorization module categories $\mathbf{C}_1$ and $\mathbf{C}_2$ over $\mathbf{A}$ at $\mathcal{Z}$ one can a priori talk about strictly unital or lax unital functors between them, compatible with factorization. However, as in Lemma C.11.23 one shows that any lax unital functor between them is automatically strictly unital.

Thus, we can unambiguously talk about the (2-) category of unital factorization module categories over $\mathbf{A}$ at $\mathcal{Z}$.

C.14.5. *Notational convention.* When $\mathbf{A}$ is unital, we will denote the (2-) category of unital factorization module categories over $\mathbf{A}$ at $\mathcal{Z}$ by

$$\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}.$$

We will denote the category of plain (i.e., non-unital) factorization module categories over $\mathbf{A}$ at $\mathcal{Z}$ by

$$\mathbf{A}\text{-}\mathbf{mod}^{\text{fact-n.u.}}_{\mathcal{Z}}.$$

Remark C.14.6. Unlike the case of modules over factorization algebras, the forgetful functor

$$\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \mathbf{A}\text{-}\mathbf{mod}^{\text{fact-n.u.}}_{\mathcal{Z}} \tag{C.71}$$

is *not* fully faithful.

Indeed, take $\mathbf{A} = \mathrm{Vect}$ and $\mathcal{Z} = \mathrm{pt}$ corresponding to a singleton $\{x\} = \underline{x} \in \mathrm{Ran}$. Take

$$\mathbf{C} = \mathrm{Vect}^{\mathrm{fact}_x} \in \mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_x.$$

Then the category of endofunctors of $\mathbf{C}$ as an object of $\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_x$ is

$$k\text{-}\mathrm{mod}^{\mathrm{fact}}_x$$

(where k is the unit factorization algebra), and the latter identifies with Vect, see Sect. C.7.7.

By contrast, the category of endofunctors of $\mathbf{C}$ as an object of Vect-$\mathbf{mod}_x^{\text{fact-n.u.}}$ is

$$k\text{-mod}_x^{\text{fact-n.u.}}.$$

So, at the level of endofunctors of the above object, the forgetful functor (C.71) is the forgetful functor

$$k\text{-mod}_x^{\text{fact}} \to k\text{-mod}_x^{\text{fact-n.u.}},$$

which is fully faithful, but not an equivalence.

C.14.7. *Example.* Recall the construction from Sect. B.12.5. It is easy to see that the resulting Vect-factorization module category $\mathbf{C}$ is unital.

In Sect. C.16.6 we will show that the functor

$$\text{CrystCat}(\mathcal{Z}_0) \to \text{Vect-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}_0}$$

is fully faithful, and we will characterize its essential image.

Note, however, that one categorical level down, the corresponding functor was an equivalence, see Sect. C.7.7.

C.14.8. Let $\mathcal{A}$ be a unital factorization algebra in a unital factorization category $\mathbf{A}$. Let $\mathbf{C}$ be a unital factorization module category over $\mathbf{A}$ at some $\mathcal{Z} \to \text{Ran}$.

Parallel to Sects. B.12.6 and C.7.4, one defines the notion of unital factorization modules *over* $\mathcal{A}$ *in* $\mathbf{C}$.

We will denote the corresponding category by

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\mathcal{Z}}.$$

By contrast, we will denote the category of plain (i.e., non-unital) $\mathcal{A}$-modules in $\mathbf{C}$ by

$$\mathcal{A}\text{-mod}^{\text{fact-n.u.}}(\mathbf{C})_{\mathcal{Z}}.$$

C.14.9. We claim:

Lemma C.14.10. $\mathbf{1_A}$-$\text{mod}^{\text{fact}}(\mathbf{C})_{\mathcal{Z}} \simeq \mathbf{C}_{\mathcal{Z}}$.

Proof. The proof essentially repeats the contents of Sect. C.7.7:

Starting from an object $\mathcal{M}' \in \mathbf{C}_{\mathcal{Z}}$, consider

$$\text{ins.\,unit}_{\mathcal{Z}}(\mathcal{M}') \in \mathbf{C}_{\mathcal{Z}^{\subseteq},\text{untl}}.$$

This object has a tautological factorization structure against $\mathbf{1_A}$.

Vice versa, starting from $\mathcal{M} \in \mathbf{1_A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\mathcal{Z}}$, the unital structure on $\mathcal{M}$ gives rise to a map

$$\text{ins.\,unit}_{\mathcal{Z}}(\mathcal{M}_{\mathcal{Z}}) \to \mathcal{M}_{\mathcal{Z}^{\subseteq},\text{untl}}$$

in $\mathbf{C}_{\mathcal{Z}^{\subseteq},\text{untl}}$, compatible with factorization.

It suffices to show that the latter map is an isomorphism. This can be checked stratawise, in which case it follows from factorization.

$\square$

C.14.11. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a lax unital factorization functor between unital factorization categories. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be unital factorization module categories over $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively, at some $\mathcal{Z} \to$ Ran.

Mimicking Sect. B.12.10 we have the notion of lax unital functor

$$\Phi_m : \mathbf{C}_1 \to \mathbf{C}_2,$$

compatible with factorization. Denote the category of such functors

$$\text{Funct}_{\mathbf{A}_1 \to \mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2).$$

C.14.12. Let $(\Phi, \Phi_m) : (\mathbf{A}_1, \mathbf{C}_1) \to (\mathbf{A}_2, \mathbf{C}_2)$ be as above. Let $\mathcal{A}_1 \in \mathbf{A}_1$ be a unital factorization algebra, and consider its image $\Phi(\mathcal{A}_1) \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A}_2)$.

Then the functor Φ_m induces a functor

$$\Phi_m : \mathcal{A}_1\text{-mod}^{\mathrm{fact}}(\mathbf{C}_1)_{\mathcal{Z}} \to \Phi(\mathcal{A}_1)\text{-mod}^{\mathrm{fact}}(\mathbf{C}_2)_{\mathcal{Z}}. \tag{C.72}$$

C.14.13. As in Lemma C.11.23, we have:

Lemma C.14.14. *Suppose that Φ is strictly unital. Then the functor between crystals of categories on $\mathcal{Z}^{\subseteq,\mathrm{untl}}$ underlying every $\Phi_m \in \mathrm{Funct}_{\mathbf{A}_1 \to \mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2)$*

$$\underline{\mathbf{C}}_1 \to \underline{\mathbf{C}}_2$$

is strict.

C.14.15. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a strictly unital functor between unital factorization categories. Let $\mathbf{C}_2$ be a unital module category over $\mathbf{A}_2$ at some $\mathcal{Z} \to \mathrm{Ran}$.

In this case, it follows from the construction of the restriction functor Res_Φ that $\mathrm{Res}_\Phi(\mathbf{C}_2)$ possesses a natural unital structure, and the tautological functor

$$\mathrm{Res}_\Phi(\mathbf{C}_2) \to \mathbf{C}_2 \tag{C.73}$$

admits a natural lax unital structure compatible with factorization.

Furthermore, the resulting object

$$\mathrm{Res}_\Phi(\mathbf{C}_2) \in \mathbf{A}_1\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$$

has a universal property parallel to that in the non-unital case:

Lemma C.14.16. *For $\mathbf{C}_1 \in \mathbf{A}_1\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$, composition with* (C.73) *defines an equivalence*

$$\mathrm{Funct}_{\mathbf{A}_1\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}}(\mathbf{C}_1, \mathrm{Res}_\Phi(\mathbf{C}_2)) \xrightarrow{\sim} \mathrm{Funct}_{\mathbf{A}_1 \to \mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2).$$

C.14.17. Let $\mathcal{A}_1 \in \mathbf{A}_1$ be a unital factorization algebra. As in Lemma B.12.14, we have:

Lemma C.14.18. *The functor* (C.73) *induces an equivalence*

$$\mathcal{A}_1\text{-mod}^{\mathrm{fact}}(\mathrm{Res}_\Phi(\mathbf{C}_2))_{\mathcal{Z}} \to \Phi(\mathcal{A}_1)\text{-mod}^{\mathrm{fact}}(\mathbf{C}_2)_{\mathcal{Z}}.$$

Remark C.14.19. The material in Sect. B.15 is applicable in the context of unital factorization categories and strictly unital factorization functors between them.

C.14.20. Let us place ourselves momentarily in the context of Sect. B.12.15, where the factorization categories in the diagram

$$\begin{array}{ccc} \mathbf{A}'_1 & \xrightarrow{\Phi'} & \mathbf{A}'_2 \\ {\scriptstyle \Psi_1}\uparrow & & \uparrow{\scriptstyle \Psi_2} \\ \mathbf{A}_1 & \xrightarrow{\Phi} & \mathbf{A}_2. \end{array}$$

are unital, all functors are lax unital, and the vertical arrows are strictly unital.

Unwinding the constructions, one obtains that in this case the resulting functor

$$\mathrm{Res}_{\Psi_1}(\mathbf{A}'_1{}^{\mathrm{fact}_{\mathcal{Z}}}) \to \mathrm{Res}_{\Psi_2}(\mathbf{A}'_2{}^{\mathrm{fact}_{\mathcal{Z}}})$$

viewed as a functor between unital module categories over $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively, possesses a natural lax unital structure compatible with factorization.

C.15. **Restriction along lax unital functors.** In this subsection we will study the phenomenon of restriction with respect to a functor

$$\Phi : \mathbf{A}_1 \to \mathbf{A}_2,$$

which is only lax unital.

C.15.1. Consider the (unital) factorization algebra

$$\Phi(\mathbf{1}_{\mathbf{A}_1}) \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A}_2),$$

see Sect. C.14.11.

Let $\mathcal{Z}$ be a prestack mapping to Ran, and let

$$\Phi_m : \mathbf{C}_1 \to \mathbf{C}_2$$

be an object of $\mathrm{Funct}_{\mathbf{A}_1 \to \mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2)$.

Consider the induced functor

$$\Phi_m : \mathbf{C}_{1,\mathcal{Z}} \to \mathbf{C}_{2,\mathcal{Z}} \tag{C.74}$$

between the underlying DG categories.

C.15.2. We claim:

Lemma-Construction C.15.3. *The functor* (C.74) *naturally enhances to a functor*

$$\Phi_m^{\mathrm{enh}} : \mathbf{C}_{1,\mathcal{Z}} \to \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathbf{C}_2)_{\mathcal{Z}}.$$

Proof. We rewrite

$$\mathbf{C}_{1,\mathcal{Z}} \simeq \mathbf{1}_{\mathbf{A}_1}\text{-mod}^{\mathrm{fact}}(\mathbf{C}_1)_{\mathcal{Z}},$$

and now the required functor is a particular case of (C.72).

□

C.15.4. *Variant.* Let us return for a moment to the setting of Sect. C.14.20. By Lemma C.15.3 for every $\underline{x} : S \to \mathrm{Ran}$ we obtain that the functor

$$\Phi' : \mathrm{Res}_{\Psi_1}(\mathbf{A}_1'^{\,\mathrm{fact}_{\underline{x}}}) \to \mathrm{Res}_{\Psi_2}(\mathbf{A}_2'^{\,\mathrm{fact}_{\underline{x}}})$$

gives rise to a functor

$$\begin{aligned} \Phi'^{\mathrm{enh}} : \mathbf{A}'_{1,\underline{x}} \to \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathrm{Res}_{\Psi_2}(\mathbf{A}_2'^{\,\mathrm{fact}_{\underline{x}}}))_{\underline{x}} &\overset{\text{Lemma C.14.18}}{\simeq} \\ &\simeq (\Psi_2 \circ \Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathbf{A}_2'^{\,\mathrm{fact}_{\underline{x}}})_{\underline{x}}. \end{aligned} \tag{C.75}$$

Note that the left-hand side in (C.75) is the value at $\underline{x}$ of a unital factorization category, namely, $\mathbf{A}'_1$ itself.

The right-hand side in (C.75) is the value at $\underline{x}$ of a unital *lax* factorization category, namely,

$$(\Psi_2 \circ \Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathbf{A}'_2),$$

see Sect. C.11.15.

Unwinding the constructions, we obtain that (C.75) upgrades to a *unital* factorization functor

$$\Phi'^{\mathrm{enh}} : \mathbf{A}'_1 \to (\Psi_2 \circ \Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathbf{A}'_2). \tag{C.76}$$

Remark C.15.5. Note that in the setting of Sect. C.15.4, we have

$$(\Psi_2 \circ \Phi)(\mathbf{1}_{\mathbf{A}_1}) \simeq (\Phi' \circ \Psi_1)(\mathbf{1}_{\mathbf{A}_1}) \simeq \Phi'(\mathbf{1}_{\mathbf{A}'_1}).$$

So, the information contained by the functor (C.76) is completely captured by the case when Ψ_1 and Ψ_2 are the identity functors. I.e., the claim is that the lax unital factorization functor

$$\Phi : \mathbf{A}_1 \to \mathbf{A}_2$$

upgrades to a unital factorization functor

$$\Phi^{\mathrm{enh}} : \mathbf{A}_1 \to \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathbf{A}_2),$$

with the caveat that the right-hand side is a lax factorization category.

C.15.6. Let $\mathcal{Z}$ and $\mathbf{C}_2$ be as in Sect. C.15.1. Consider the contravariant functor on $\mathbf{A}_1\text{-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}$ that assigns to $\mathbf{C}_1$ the category $\text{Funct}_{\mathbf{A}_1\to\mathbf{A}_2}(\mathbf{C}_1, \mathbf{C}_2)$. One shows that this functor is representable, and let

$$\text{Res}^{\text{untl}}_{\Phi}(\mathbf{C}_2) \in \mathbf{A}_1\text{-}\mathbf{mod}^{\text{fact}}_{\mathcal{Z}}$$

denote the representing object.

By Lemma C.15.3, the tautological functor

$$\text{Res}^{\text{untl}}_{\Phi}(\mathbf{C}_2) \to \mathbf{C}_2 \tag{C.77}$$

upgrades to a functor

$$\text{Res}^{\text{untl}}_{\Phi}(\mathbf{C}_2)_{\mathcal{Z}} \to \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}(\mathbf{C}_2)_{\mathcal{Z}}. \tag{C.78}$$

C.15.7. We have the following generalization of Lemma B.12.12:

Lemma C.15.8. *The functor* (C.78) *is an equivalence.*

Remark C.15.9. Note that when Φ is strictly unital, then by Lemma C.14.16

$$\text{Res}^{\text{untl}}_{\Phi}(\mathbf{C}_2) \simeq \text{Res}_{\Phi}(\mathbf{C}_2),$$

and the assertion of Lemma C.15.8 coincides with that of Lemma B.12.12.

C.15.10. *Example.* Let $\mathcal{Z} = \text{pt}$ and let $\mathcal{Z} \to \text{Ran}$ correspond to a singleton $\{x\} \in \text{Ran}$. Recall the notations of Sect. B.9.28. Let us give an explicit description of the category

$$\text{Res}^{\text{untl}}_{\Phi}(\mathbf{C}_2)_X.$$

Namely,

$$\text{Res}^{\text{untl}}_{\Phi}(\mathbf{C}_2)_X \simeq$$
$$\simeq \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}((\mathbf{C}_2)|_X)_X \underset{\Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}((\mathbf{C}_2)|_{X-x})_{X-x}}{\times} \left(\mathbf{A}_{1,X-x} \otimes \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}(\mathbf{C}_2)_x\right),$$

where:

- The notation $(\mathbf{C}_2)|_X$ is as in Sect. B.12.3, i.e., we regard the pullback of $\underline{\mathbf{C}}_2$ along $X \to \text{Ran}_x$ as a module category over $\mathbf{A}_2$ at X;
- The functor

$$\mathbf{A}_{1,X-x} \otimes \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}(\mathbf{C}_2)_x \to \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}((\mathbf{C}_2)|_{X-x})_{X-x}$$

  is the composition

$$\mathbf{A}_{1,X-x} \otimes \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}(\mathbf{C}_2)_x \overset{\Phi^{\text{enh}}\otimes\text{Id}}{\longrightarrow} \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}(\mathbf{A}_2)_{X-x} \otimes \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}(\mathbf{C}_2)_x \to$$
$$\overset{\text{factorization of } \mathbf{C}_2}{\longrightarrow} \Phi(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\text{fact}}((\mathbf{C}_2)|_{X-x})_{X-x},$$

  where the second arrow is defined as in Sect. B.11.14.

C.15.11. We have the following analog of Lemma C.14.18

Lemma C.15.12. *The functor* (C.77) *induces an equivalence*

$$\mathcal{A}_1\text{-mod}^{\text{fact}}(\text{Res}^{\text{untl}}_{\Phi}(\mathbf{C}_2))_{\mathcal{Z}} \to \Phi(\mathcal{A}_1)\text{-mod}^{\text{fact}}(\mathbf{C}_2)_{\mathcal{Z}}.$$

C.15.13. The material in Sect. B.15 is applicable in the context of unital factorization categories and lax unital factorization functors between them. We will not need it in the full generality, except for an analog of Corollary B.15.7, formulated as Lemma C.15.16 below.

C.15.14. Let

$$\Phi : \mathbf{A}_1 \to \mathbf{A}_2$$

be a unital functor between unital factorization categories.

Suppose that Φ admits a right adjoint as a functor between the underlying crystals of categories over Ran. According to Sect. C.11.21, the right adjoint Φ^R of Φ admits a natural extension to a lax unital functor between unital factorization categories.

Let $\mathbf{C}_1$ (resp., $\mathbf{C}_2$) be a unital module category over $\mathbf{A}_1$ (resp., $\mathbf{A}_2$) at some $\mathcal{Z} \to \mathrm{Ran}$.

C.15.15. We claim:

Lemma C.15.16. *There is a canonical equivalence*

$$\mathrm{Funct}_{\mathbf{A}_1\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}}(\mathrm{Res}_\Phi(\mathbf{C}_2), \mathbf{C}_1) \simeq \mathrm{Funct}_{\mathbf{A}_2\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}}(\mathbf{C}_2, \mathrm{Res}^{\mathrm{untl}}_{\Phi^R}(\mathbf{C}_1)).$$

C.15.17. We will not give a full proof of this lemma; rather we will sketch the construction of the maps in both directions in the framework of Sect. C.15.10.

Namely, for $\mathbf{C}_2 \in \mathbf{A}_2\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$ we will construct a functor

$$\mathbf{C}_2 \to \mathrm{Res}^{\mathrm{untl}}_{\Phi^R} \circ \mathrm{Res}_\Phi(\mathbf{C}_2) \tag{C.79}$$

and for $\mathbf{C}_1 \in \mathbf{A}_1\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$ we will construct a functor

$$\mathrm{Res}_\Phi \circ \mathrm{Res}^{\mathrm{untl}}_{\Phi^R}(\mathbf{C}_1) \to \mathbf{C}_1. \tag{C.80}$$

C.15.18. By the universal property of $\mathrm{Res}^{\mathrm{untl}}_{\Phi^R}$, the datum of (C.79) is equivalent to the datum of a functor

$$\mathbf{C}_2 \to \mathrm{Res}_\Phi(\mathbf{C}_2) \tag{C.81}$$

as module categories over $\mathbf{A}_2$ and $\mathbf{A}_1$, respectively, compatible with factorization against the functor Φ^R.

We now specialize to the context of Sect. C.15.10 and construct the corresponding functor

$$\mathbf{C}_{2,X} \to \mathrm{Res}_\Phi(\mathbf{C}_2)_X. \tag{C.82}$$

We write

$$\mathrm{Res}_\Phi(\mathbf{C}_2)_X \simeq \mathbf{C}_{2,X} \underset{\mathbf{C}_{2,X-x}}{\times} (\mathbf{A}_{1,X-x} \otimes \mathbf{C}_{2,x}), \tag{C.83}$$

where

$$\mathbf{A}_{1,X-x} \otimes \mathbf{C}_{2,x} \to \mathbf{C}_{2,X-x}$$

is the functor

$$\mathbf{A}_{1,X-x} \otimes \mathbf{C}_{2,x} \overset{\Phi\otimes\mathrm{Id}}{\to} \mathbf{A}_{2,X-x} \otimes \mathbf{C}_{2,x} \overset{\mathrm{fact}}{\simeq} \mathbf{C}_{2,X-x}.$$

We write

$$\mathbf{C}_{2,X} \simeq \mathbf{C}_{2,X} \underset{\mathbf{C}_{2,X-x}}{\times} (\mathbf{A}_{2,X-x} \otimes \mathbf{C}_{2,x}), \tag{C.84}$$

where

$$\mathbf{A}_{2,X-x} \otimes \mathbf{C}_{2,x} \overset{\sim}{\to} \mathbf{C}_{2,X-x}$$

is the factorization equivalence.

We define the functor (C.82) by sending an object $\mathbf{c}_2 \in \mathbf{C}_{2,X}$ to

$$\left(\mathbf{c}_2 \underset{j_*\circ j^*(\mathbf{c}_2)}{\times} j_*(((\Phi\circ\Phi^R)\otimes\mathrm{Id})(j^*(\mathbf{c}_2))), (\Phi^R\otimes\mathrm{Id})(j^*(\mathbf{c}_2))\right) \in \mathbf{C}_{2,X} \underset{\mathbf{C}_{2,X-x}}{\times} (\mathbf{A}_{1,X-x} \otimes \mathbf{C}_{2,x}),$$

where the map

$$((\Phi\circ\Phi^R)\otimes\mathrm{Id})(j^*(\mathbf{c}_2)) \to j^*(\mathbf{c}_2)$$

is the counit of the adjunction.

C.15.19. Write

$$\mathrm{Res}_\Phi \circ \mathrm{Res}^{\mathrm{untl}}_{\Phi^R}(\mathbf{C}_1)_X \simeq \tag{C.85}$$
$$\simeq (\Phi^R \circ \Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}((\mathbf{C}_1)|_X)_X \underset{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}((\mathbf{C}_1)|_{X-x})_{X-x}}{\times}$$
$$\underset{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}((\mathbf{C}_1)|_{X-x})_{X-x}}{\times} \Big(\mathbf{A}_{1,X-x} \otimes ((\Phi^R \circ \Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathbf{C}_1)_x)\Big).$$

The sought-for functor

$$\mathrm{Res}_\Phi \circ \mathrm{Res}^{\mathrm{untl}}_{\Phi^R}(\mathbf{C}_1)_X \to \mathbf{C}_{1,X}$$

sends an object $(\mathbf{c}_1', \mathbf{c}_1'')$ in the right-hand side of (C.85), i.e.,

$$\mathbf{c}_1' \in (\Phi^R \circ \Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}((\mathbf{C}_1)|_X)_X$$

and

$$\mathbf{c}_1'' \in \mathbf{A}_{1,X-x} \otimes ((\Phi^R \circ \Phi)(\mathbf{1}_{\mathbf{A}_1})\text{-mod}^{\mathrm{fact}}(\mathbf{C}_1)_x)$$

to the object

$$\mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})}(\mathbf{c}_1') \underset{j_*\circ j^*\circ \mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})}(\mathbf{c}_1')}{\times} \Big(j_* \circ (\mathrm{Id} \otimes \mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})})(\mathbf{c}_1'')\Big),$$

where the map

$$(\mathrm{Id} \otimes \mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})})(\mathbf{c}_1'') \to j^* \circ \mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})}(\mathbf{c}_1')$$

is obtained by applying the functor $\mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})}$ to the isomorphism

$$((\Phi^R \circ \Phi)^{\mathrm{enh}} \otimes \mathrm{Id})(\mathbf{c}_1'') \simeq j^*(\mathbf{c}_1'),$$

precomposed with the unit of the (Φ, Φ^R)-adjunction

$$(\mathrm{Id} \otimes \mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})})(\mathbf{c}_1'') \to ((\Phi^R \circ \Phi) \otimes \mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})})(\mathbf{c}_1'') \simeq$$
$$\simeq \mathbf{oblv}_{(\Phi^R\circ\Phi)(\mathbf{1}_{\mathbf{A}_1})} \Big(((\Phi^R \circ \Phi)^{\mathrm{enh}} \otimes \mathrm{Id})(\mathbf{c}_1'')\Big).$$

C.16. **Tightness.**

C.16.1. Let $\mathbf{A}$ be a unital factorization category. We will say that $\mathbf{A}$ is *tight* if for every $\underline{x}, \underline{x}' : S \to \mathrm{Ran}$ with $\underline{x} \subseteq \underline{x}'$, the corresponding functor

$$\mathrm{ins}_{\underline{x}\subseteq\underline{x}'} : \mathbf{A}_{S,\underline{x}} \to \mathbf{A}_{S,\underline{x}'}$$

admits a continuous right adjoint.

C.16.2. Many of the unital factorization categories we have introduced satisfy this property. This includes representation-theoretic examples, e.g.,

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa, \ \mathrm{KL}(G)_\kappa$$

and algebro-geometric examples:

$$\mathrm{QCoh}(\mathcal{T}),$$

where $\mathcal{T}$ is an affine counital factorization space, and

$$\mathrm{IndCoh}^*(\mathcal{T}) \text{ and } \mathrm{IndCoh}^!(\mathcal{T}),$$

where $\mathcal{T}$ is a unital-in-correspondences ind-placid factorization ind-schemes, satisfying the conditions from Sects. C.12.8 and C.12.9, respectively.

C.16.3. Let $\mathbf{A}$ be a tight unital factorization category, and let $\mathbf{C}$ be a unital factorization module category over $\mathbf{C}$ at some $\mathcal{Z}$.

We shall say that $\mathbf{C}$ is *tight* if for every $(z,\underline{x}),(z,\underline{x}'):S\to\mathcal{Z}^{\subseteq}$ with $\underline{x}\subseteq\underline{x}'$, the corresponding functor

$$\mathrm{ins}_{\underline{x}\subseteq\underline{x}'}:\mathbf{C}_{S,\underline{x}}\to\mathbf{C}_{S,\underline{x}'}$$

admits a continuous right adjoint.

C.16.4. The following is immediate:

Lemma C.16.5. *Suppose that* $\mathbf{A}$ *is tight. Then for any* $\mathcal{Z}\to\mathrm{Ran}$*, the factorization module category* $\mathbf{A}^{\mathrm{fact}_{\mathcal{Z}}}$ *is tight.*

C.16.6. Take $\mathbf{A}=\mathrm{Vect}$, and recall the construction from Sect. C.14.7

$$\underline{\mathbf{C}}_0\in\mathrm{CrysCat}(\mathcal{Z}_0)\rightsquigarrow\mathbf{C}\in\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}. \tag{C.86}$$

It is clear that the essential image of (C.86) is contained in the full subcategory of $\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$ consisting of tight unital factorization module categories.

We claim:

Lemma C.16.7. *The functor* (C.86) *is an equivalence onto the full subcategory of* $\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$ *consisting of tight objects.*

Proof. The functor (C.86) admits a retraction (i.e., a left inverse), given by restricting the crystal of categories from $\mathcal{Z}^{\subseteq}$ to $\mathcal{Z}$ along $\mathrm{diag}_{\mathcal{Z}}$. We claim that this left inverse is an actual inverse when applied to tight objects.

Indeed, let $\mathbf{C}'$ be a tight unital factorization module category at $\mathcal{Z}$. Let $\underline{\mathbf{C}}'$ be the corresponding crystal of categories over $\mathcal{Z}^{\subseteq}$, and let $\underline{\mathbf{C}}'_0$ denote the restriction of $\underline{\mathbf{C}}'$ along $\mathrm{diag}_{\mathcal{Z}}$.

The unital structure on $\mathbf{C}'$ gives rise to a functor

$$\mathrm{pr}^*_{\mathrm{small}}(\underline{\mathbf{C}}'_0)\to\underline{\mathbf{C}}. \tag{C.87}$$

We have to show that (C.87) is an equivalence.

By the tightness assumption, the functor (C.87) admits a right adjoint. Hence, in order to show that it is an equivalence, it suffices to show that it is an equivalence strata-wise. However, this follows from factorization.

□

C.16.8. Consider the following situation. Let $\mathbf{A}$ be a tight unital factorization category, and let $\mathbf{C}$ be a tight unital factorization module category over it at some $\mathcal{Z}$.

Consider the (strictly) unital factorization functor

$$\mathrm{Vac}_{\mathbf{A}}:\mathrm{Vect}\to\mathbf{A},$$

see Sect. C.11.8.

C.16.9. We claim:

Lemma C.16.10. *The unital factorization module category*

$$\mathrm{Res}_{\mathrm{Vac}_{\mathbf{A}}}(\mathbf{C})\in\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\mathcal{Z}}$$

is tight.

Proof. Set $\underline{\mathbf{C}}'_0 := \underline{\mathbf{C}}_{\mathcal{Z}}$ be the sheaf of categories on $\mathcal{Z}$ underlying $\mathbf{C}$. Let $\mathbf{C}'$ be the (tight, unital) factorization module over Vect, attached $\underline{\mathbf{C}}'_0$ by the functor (C.86).

The unital structure on $\mathbf{C}$ gives to a (strictly unital) functor

$$\mathbf{C}' \to \mathbf{C},$$

compatible with factorization (in the sense of Sect. C.14.11). Hence, we obtain a functor

$$\mathbf{C}' \to \mathrm{Res}_{\mathrm{Vac}_{\mathbf{A}}}(\mathbf{C}). \tag{C.88}$$

We claim that the functor (C.88) is an equivalence. Indeed, this follows from the assumptions by applying Lemma B.15.9.

□

C.16.11. We will use Lemma C.16.10 as follows. Let $\mathbf{A}$ be a tight unital factorization category, and let R be a factorization algebra (in Vect).

Using the functor $\mathrm{Vac}_{\mathbf{A}}$, we can consider the factorization algebra

$$\mathrm{Vac}_{\mathbf{A}}(R) \simeq R \otimes \mathbf{1}_{\mathbf{A}}$$

in $\mathbf{A}$, and consider the corresponding lax factorization category

$$R\text{-mod}^{\mathrm{fact}}(\mathbf{A}) := (R \otimes \mathbf{1}_{\mathbf{A}})\text{-mod}^{\mathrm{fact}}(\mathbf{A}).$$

We have a naturally defined functor between lax factorization categories

$$R\text{-mod}^{\mathrm{fact}} \otimes \mathbf{A} \to R\text{-mod}^{\mathrm{fact}}(\mathbf{A}), \tag{C.89}$$

see (B.44).

Combining Lemmas C.16.10 and B.12.9, we obtain:

Corollary C.16.12. *Assume that* $\mathbf{A}$ *is dualizable as a factorization category. Then the functor* (C.89) *is an equivalence.*

Appendix D. Chiral modules

The main purpose of this section is to prove Theorem 4.3.9, which gives a geometric description of modules over commutative factorization/chiral algebras.

To do so, we first develop a general theory describing modules over chiral algebras in terms of modules over topological algebras (although we do not write in these exact terms), with an especially explicit understanding for "nice" Lie-* algebras.

This material largely consists of transporting [BD2, Sect. 3.6] into the derived setting. However, we will encounter a surprise: a certain equivalence that always takes place at an abelian level, in order to hold at the derived level requires a finiteness condition (see Sect. D.6).

D.1. A reminder: chiral vs factorization algebras.

D.1.1. Recall that, according to [BD2, Proposition 3.4.19] (see [FraG] for the derived version), factorization algebras are the same as chiral algebras. Given a factorization algebra $\mathcal{A}$, the corresponding chiral algebra, thought of as a D-module on X, is given by

$$\mathcal{A}^{\mathrm{ch}} := \mathcal{A}_X[-1].$$

For example, the unit factorization algebra corresponds to the chiral algebra ω_X (see our conventions in Sect. 1.1.1[79]).

D.1.2. Generalizing [BD2, Proposition 3.4.19], we have

$$\mathcal{A}\text{-mod}^{\mathrm{fact}} \simeq \mathcal{A}^{\mathrm{ch}}\text{-mod}^{\mathrm{ch}}, \tag{D.1}$$

where modules on both sides can be taken on any space that maps to Ran.

[79]This was one of the main reasons for this choice of conventions, i.e., in order to be in line with [BD2].

D.1.3. Let us recall also how the bijection between between chiral and factorization algebras plays out in the commutative case.

Let A be a commutative algebra object in D-mod(X). Then the corresponding factorization algebra $\mathcal{A} := \mathrm{Fact}(A)$ is such that

$$\mathcal{A}_X = A.$$

And the corresponding chiral algebra $\mathcal{A}^{\mathrm{ch}}$ is $A[-1]$. Note that

$$\mathbf{oblv}^r(\mathcal{A}^{\mathrm{ch}}) = \mathbf{oblv}^l(A) \otimes \omega_X.$$

D.1.4. For example, for the free commutative algebra $A = \mathrm{Sym}^!(\mathcal{M}[1])$ for $\mathcal{M} \in \text{D-mod}(X)$, we have

$$\mathcal{A}^{\mathrm{ch}} = \mathrm{Sym}^!(\mathcal{M}[1])[-1] \simeq U^{\mathrm{ch}}(\mathcal{M}),$$

where:

- In the left-hand side $\mathcal{M}$ is considered as an abelian Lie-* algebra;
- U^{ch} is the functor of chiral envelope.

Note also that in this case.

$$\mathbf{oblv}^l(A) = \mathrm{Sym}_{\mathcal{O}_X}(\mathbf{oblv}^l(\mathcal{M}[1])) \simeq \mathrm{Sym}_{\mathcal{O}_X}(\mathbf{oblv}^r(\mathcal{M}) \otimes \omega_X^{\otimes -1}).$$

So, if $\mathcal{M} = \mathbf{ind}^r(\mathcal{E}) = \mathcal{E} \otimes \mathrm{D}_X$ for a classical locally free sheaf $\mathcal{E}$ on X, then the corresponding D-scheme

$$\mathrm{Spec}_X(A)$$

is the scheme $\mathrm{Jets}(\mathcal{E}^\vee \otimes \omega_X)$ of jets into the vector bundle $\mathcal{E}^\vee \otimes \omega_X$.

D.1.5. *Convention.* In what follows, by a slight abuse of notation, for a given factorization algebra $\mathcal{A}$, we will use the same symbol $\mathcal{A}$ to denote the corresponding chiral algebra (i.e., we will not write $\mathcal{A}^{\mathrm{ch}}$).

D.2. The pro-projective generator for chiral modules.

D.2.1. Let $\mathcal{A}$ be a unital chiral algebra on X. Let $\mathcal{A}\text{-mod}^{\mathrm{ch}}_x$ denote the category of *unital* chiral $\mathcal{A}$-modules at x. We let $\mathbf{oblv}_{\mathcal{A}}$ denote the forgetful functor $\mathcal{A}\text{-mod}^{\mathrm{ch}}_x \to \mathrm{Vect}$.

For $\mathcal{M} \in \mathcal{A}\text{-mod}^{\mathrm{ch}}_x$, we will consider the action map

$$j^*(\mathcal{A}) \otimes \mathcal{M} \xrightarrow{\text{action}} i_*(\mathcal{M}),$$

as a map of chiral $\mathcal{A}$-modules on X, where:

- j denotes the open embedding $X - x \hookrightarrow X$;
- i denotes the embedding of the point x into X.

D.2.2. We will also use a short-hand notation

$$\underline{\mathcal{M}} := \mathbf{oblv}_{\mathcal{A}}(\mathcal{M}), \quad \mathcal{M} \in \mathcal{A}\text{-mod}^{\mathrm{ch}}_x.$$

In what follows we will take about "elements" of $\underline{\mathcal{M}}$:

For $V \in \mathrm{Vect}$, by an element $v \in V$ we mean a point of the space $\mathrm{Maps}_{\mathrm{Vect}}(k, V)$.

D.2.3. In what follows we will denote by $\mathcal{A}_x$ is the [1]-shifted !-fiber of $\mathcal{A}$ at x (this is the same as the !-fiber at $x \in \mathrm{Ran}$ of the factorization algebra corresponding to $\mathcal{A}$), viewed as an object[80] of $\mathcal{A}\text{-mod}^{\mathrm{ch}}_x$.

We let

$$1_{\mathcal{A},x} \in \underline{\mathcal{A}}_x$$

denote the vacuum vector, i.e., the image of $1 \in k \simeq (\omega_X)_x$ under the unit map

$$\mathrm{vac}_{\mathcal{A}} : \omega_X \to \mathcal{A}.$$

[80]In fact, $\mathcal{A}_x$ should more properly be denoted $\mathcal{A}^{\mathrm{fact}_x}$, see Sect. B.9.7.

D.2.4. Consider the category $\mathrm{Modif}(\mathcal{A})$ of unital chiral algebras $\mathcal{A}'$ equipped with an isomorphism

$$\mathcal{A}'|_{X-x} \simeq \mathcal{A}|_{X-x}.$$

This category has fiber products, and hence is cofiltered.

Note that the category $\mathcal{A}\text{-mod}_x^{\mathrm{ch}}$ only depends on $\mathcal{A}|_{X-x}$, so for any $\mathcal{A}' \in \mathrm{Modif}(\mathcal{A})$ we have a canonical identification

$$\mathcal{A}\text{-mod}_x^{\mathrm{ch}} \simeq \mathcal{A}'\text{-mod}_x^{\mathrm{ch}}.$$

D.2.5. Consider the functor

$$\mathrm{Modif}(\mathcal{A}) \to \mathcal{A}\text{-mod}_x^{\mathrm{ch}}, \quad \mathcal{A}' \mapsto \mathcal{A}'_x \in \mathcal{A}'\text{-mod}^{\mathrm{ch}} \simeq \mathcal{A}\text{-mod}^{\mathrm{ch}}.$$

Set

$$\mathsf{P}_{\mathcal{A},x} := \underset{\mathcal{A}' \in \mathrm{Modif}(\mathcal{A})}{\text{``}\lim\text{''}} \mathcal{A}'_x \in \mathrm{Pro}(\mathcal{A}\text{-mod}_x^{\mathrm{ch}}).$$

The object

$$\mathbf{oblv}_{\mathcal{A}}(\mathsf{P}_{\mathcal{A},x}) \in \mathrm{Pro}(\mathrm{Vect})$$

is equipped with a canonical vector $1_{\mathsf{P},x}$ comprised of the vacuum vectors $1_{\mathcal{A}',x} \in \underline{\mathcal{A}'_x}$.

D.2.6. Evaluation on $1_{\mathsf{P},x}$ gives rise to a natural transformation

$$\mathcal{H}om(\mathsf{P}_{\mathcal{A},x}, -) \to \mathbf{oblv}_{\mathcal{A}}, \quad \mathcal{A}\text{-mod}_x^{\mathrm{ch}} \to \mathrm{Vect}\,. \tag{D.2}$$

The following assertion is a derived version of [BD2, Proposition 3.6.16]:

Proposition D.2.7. *The natural transformation is an isomorphism.*

D.2.8. The proof of Proposition D.2.7 is based on the following observation: we claim that the category $\mathrm{Modif}(\mathcal{A})$ is equivalent to the category of pairs

$$(\mathcal{M} \in \mathcal{A}\text{-mod}_x^{\mathrm{ch}}, m \in \underline{\mathcal{M}}).$$

Namely, in one direction, to $\mathcal{A}' \in \mathrm{Modif}(\mathcal{A})$ we attach the pair $(\mathcal{A}'_x, 1_{\mathcal{A}',x})$.

Vice versa, given $(\mathcal{M}, m)$ we let $\mathcal{A}'$ be the fiber of the map

$$j_* \circ j^*(\mathcal{A}) \overset{\mathrm{id}\otimes m}{\longrightarrow} j_* \circ j^*(\mathcal{A}) \otimes \mathcal{M} \overset{\mathrm{action}}{\longrightarrow} i_*(\mathcal{M}). \tag{D.3}$$

We claim that $\mathcal{A}'$ has a natural structure of unital chiral algebra. This fits into the following general paradigm:

Let L be a Lie algebra in a symmetric pseudo-monoidal monoidal category $\mathbf{A}$, and let M be a module over it. Consider $L \oplus M$ as a split square-zero extension of M. For an element $m \in M$ (i.e., a map $\mathbf{1_A} \to M$), the action of L on M gives rise to an automorphism ϕ_m of $L \oplus M$. Then we can identify

$$\mathrm{Fib}(L \overset{\text{act on } m}{\longrightarrow} M)$$

with

$$L \underset{L\oplus M}{\times} L,$$

where the two maps $L \rightrightarrows L \oplus M$ are the compositions of the tautological embedding with: (1) the identity map, and (2) ϕ_m.

We apply this to $\mathbf{A} := \text{D-mod}(X)$, equipped with the chiral pseudo-monoidal monoidal structure, so that $\mathbf{1_A}$ is the "constant sheaf" on X. We take $L = j_* \circ j^*(\mathcal{A})$ and $M = i_*(\mathcal{M})$. This endows $\mathcal{A}'$ with a chiral algebra structure, i.e.,

$$\mathcal{A}' = j_* \circ j^*(\mathcal{A}) \underset{j_*\circ j^*(\mathcal{A})\oplus i_*(\mathcal{M})}{\times} j_* \circ j^*(\mathcal{A}).$$

In order to show that $\mathcal{A}'$ is unital, by Proposition C.7.13, it is enough to equip it with a quasi-unital structure (see Sect. C.7.12 for what this means). The above fiber product presentation defines this structure on the nose.

Proof of Proposition D.2.7. This is tautological from Sect. D.2.8: the assertion of the proposition is just the fact that the map

$$\underset{(\mathcal{M}, m\in\underline{\mathcal{M}})}{\operatorname{colim}} \mathcal{H}om(\mathcal{M},\mathcal{M}') \to \underline{\mathcal{M}}'$$

is an isomorphism. □

D.2.9. Assume now that $j^*(\mathcal{A})$ is connective. Let

$$\operatorname{Modif}_{\mathrm{conn}}(\mathcal{A}) \subset \operatorname{Modif}(\mathcal{A}) \tag{D.4}$$

be the full subcategory consisting of those $\mathcal{A}'$ that are connective.

Truncation ≤ 0 on chiral algebras defines a right adjoint to the above inclusion. Hence, the opposite of (D.4) is cofinal.

In particular, the object $\mathsf{P}_{\mathcal{A},x}$ maps isomorphically to

$$\underset{\mathcal{A}'\in\operatorname{Modif}_{\mathrm{conn}}(\mathcal{A})}{\text{“}\lim\text{”}} \mathcal{A}'_x \in \operatorname{Pro}(\mathcal{A}\text{-mod}^{\mathrm{ch}}_x).$$

D.2.10. For an integer m, let

$$\operatorname{Modif}_{\mathrm{conn},\geq\text{-m}}(\mathcal{A}) \subset \operatorname{Modif}_{\mathrm{conn}}(\mathcal{A})$$

be the full subcategory consisting of those objects $\mathcal{A}'$, for which $\mathcal{A}'_x \in \operatorname{Vect}^{\geq -m,\leq 0}$.

Let $\operatorname{Modif}_{\mathrm{conn,ev\text{-}c}}(\mathcal{A})$ be the full subcategory of $\operatorname{Modif}_{\mathrm{conn}}(\mathcal{A})$ consisting of those objects $\mathcal{A}'$, for which $\mathcal{A}'_x$ is eventually coconnective (as an object of Vect). I.e.,

$$\operatorname{Modif}_{\mathrm{conn,ev\text{-}c}}(\mathcal{A}) = \underset{m}{\operatorname{colim}} \operatorname{Modif}_{\mathrm{conn},\geq\text{-m}}(\mathcal{A}).$$

Note that if $j^*(\mathcal{A})$ is itself eventually coconnective, the above condition on $\mathcal{A}'_x$ is equivalent to $\mathcal{A}'$ being eventually coconnective.

D.2.11. Set

$$\mathsf{P}_{\mathcal{A},x,\text{ev-c}} := \underset{\mathcal{A}'\in\operatorname{Modif}_{\mathrm{conn,ev\text{-}c}}(\mathcal{A})}{\text{“}\lim\text{”}} \mathcal{A}'_x \in \operatorname{Pro}(\mathcal{A}\text{-mod}^{\mathrm{ch}}_x).$$

We have a tautological map

$$\mathsf{P}_{\mathcal{A},x} \to \mathsf{P}_{\mathcal{A},x,\text{ev-c}} \tag{D.5}$$

in $\operatorname{Pro}(\mathcal{A}\text{-mod}^{\mathrm{ch}}_x)$.

Lemma D.2.12. *The map*

$$\mathcal{H}om(\mathsf{P}_{\mathcal{A},x,\text{ev-c}},-) \to \mathcal{H}om(\mathsf{P}_{\mathcal{A},x},-),$$

defined by (D.5) *is an isomorphism, when evaluated on* $(\mathcal{A}\text{-mod}^{\mathrm{ch}}_x)^{>-\infty}$.

Proof. It suffices to show that if $\mathcal{M} \in (\mathcal{A}\text{-mod}^{\mathrm{ch}}_x)^{\geq -m}$ for some m, then

$$\underset{\mathcal{A}'\in\operatorname{Modif}_{\mathrm{conn,ev\text{-}c}}(\mathcal{A})}{\operatorname{colim}} \operatorname{Maps}(\mathcal{A}'_x,\mathcal{M}) \to \underset{\mathcal{A}'\in\operatorname{Modif}_{\mathrm{conn}}(\mathcal{A})}{\operatorname{colim}} \operatorname{Maps}(\mathcal{A}'_x,\mathcal{M})$$

is an isomorphism.

However, it is clear that the map

$$\underset{\mathcal{A}'\in\operatorname{Modif}_{\mathrm{conn},\geq\text{-m}'}(\mathcal{A})}{\operatorname{colim}} \operatorname{Maps}(\mathcal{A}'_x,\mathcal{M}) \to \underset{\mathcal{A}'\in\operatorname{Modif}_{\mathrm{conn}}(\mathcal{A})}{\operatorname{colim}} \operatorname{Maps}(\mathcal{A}'_x,\mathcal{M})$$

is an isomorphism for every $m' \geq m$. □

Corollary D.2.13. *The map*

$$\mathcal{H}om(\mathsf{P}_{\mathcal{A},x,\text{ev-c}},-) \to \mathbf{oblv}_{\mathcal{A}}$$

is an isomorphism, when evaluated on $(\mathcal{A}\text{-mod}^{\mathrm{ch}}_x)^{>-\infty}$.

D.3. **The case of Lie-* algebras.** In this subsection we will assume that the chiral algebra $\mathcal{A}$ is the chiral universal envelope $U^{\mathrm{ch}}(L)$ of a (connective) Lie-* algebra L. Recall that we can identify

$$\mathcal{A}\text{-mod}_x^{\mathrm{ch}} \simeq L\text{-mod}_x^{\mathrm{ch}}.$$

We will study how the object $\mathsf{P}_{\mathcal{A},x}$ looks like in this case.

D.3.1. Consider the categories

$$\mathrm{Modif}(L) \text{ and } \mathrm{Modif}_{\mathrm{conn}}(L)$$

defined as in the case of chiral algebras, i.e., these are Lie-* algebras equipped with an isomorphism with L over $X - x$.

We have two pairs of adjoint functors

$$U^{\mathrm{ch}} : \mathrm{Modif}(L) \rightleftarrows \mathrm{Modif}(\mathcal{A}) : \mathbf{oblv}^{\mathrm{ch}\to\mathrm{Lie}\text{-}*}$$

and

$$U^{\mathrm{ch}} : \mathrm{Modif}_{\mathrm{conn}}(L) \rightleftarrows \mathrm{Modif}_{\mathrm{conn}}(\mathcal{A}) : \mathbf{oblv}^{\mathrm{ch}\to\mathrm{Lie}\text{-}*}.$$

In particular, the corresponding functors

$$\mathrm{Modif}(L)^{\mathrm{op}} \to \mathrm{Modif}(\mathcal{A})^{\mathrm{op}} \text{ and } \mathrm{Modif}_{\mathrm{conn}}(L)^{\mathrm{op}} \to \mathrm{Modif}(\mathcal{A})^{\mathrm{op}}_{\mathrm{conn}}$$

are cofinal.

D.3.2. In particular, we obtain that we can write

$$\mathsf{P}_{\mathcal{A},x} \simeq \underset{L'\in\mathrm{Modif}(L)}{\text{“}\lim\text{”}} \mathbf{ind}^{L\text{-mod}_x^{\mathrm{ch}}}_{L'\text{-mod}_x^{\mathrm{Lie}\text{-}*}}(k)$$

and when L is connective also as

$$\mathsf{P}_{\mathcal{A},x} \simeq \underset{L'\in\mathrm{Modif}_{\mathrm{conn}}(L)}{\text{“}\lim\text{”}} \mathbf{ind}^{L\text{-mod}_x^{\mathrm{ch}}}_{L'\text{-mod}_x^{\mathrm{Lie}\text{-}*}}(k),$$

where

- $\mathbf{ind}^{L\text{-mod}_x^{\mathrm{ch}}}_{L'\text{-mod}_x^{\mathrm{Lie}\text{-}*}} : L'\text{-mod}_x^{\mathrm{Lie}\text{-}*} \to L'\text{-mod}_x^{\mathrm{ch}} \simeq L\text{-mod}_x^{\mathrm{ch}}$ is the left adjoint of the restriction functor.

D.3.3. Assume now that $j^*(L)$ is classical (i.e., is in cohomological degree 0 as a right D-module), and let

$$\mathrm{Modif}_{\mathrm{cl,flat}}(L) \subset \mathrm{Modif}_{\mathrm{conn}}(L)$$

be the full subcategory, consisting of those modifications L' that are classical and flat (as $\mathcal{O}_X$-modules).

Note that the functor of chiral universal envelope maps

$$\mathrm{Modif}_{\mathrm{cl,flat}}(L) \to \mathrm{Modif}_{\mathrm{conn,ev\text{-}c}}(\mathcal{A}). \tag{D.6}$$

Proposition D.3.4. *Assume that $j^*(L)$ is finitely generated and locally free as a D-module. Then the (opposite of the) functor* (D.6) *is cofinal.*

We note that the assertion of the proposition would be false without the finite generation assumption, see Sect. D.6.

The proposition will be proved in Sect. D.4. We will now consider some applications.

Corollary D.3.5. *Under the assumptions of Proposition D.3.4, we have*

$$\mathsf{P}_{\mathcal{A},x,\mathrm{ev\text{-}c}} \simeq \underset{L'\in\mathrm{Modif}_{\mathrm{cl,flat}}(L)}{\text{“}\lim\text{”}} \mathbf{ind}^{L\text{-mod}_x^{\mathrm{ch}}}_{L'\text{-mod}_x^{\mathrm{Lie}\text{-}*}}(k).$$

D.3.6. Take

$$L = L_{\mathfrak{g}} = \mathfrak{g} \otimes \mathrm{D}_X,$$

where $\mathfrak{g}$ is a classical finite-dimensional Lie algebra (or a central extension of $L_{\mathfrak{g}}$).

In this case, the category $\mathrm{Modif}_{\mathrm{cl,flat}}(L)^{\mathrm{op}}$ contains a cofinal family of objects of the form

$$\mathfrak{g} \otimes \mathcal{O}_X(-n \cdot x) \underset{\mathcal{O}_X}{\otimes} \mathrm{D}_X\,.$$

Note that the corresponding objects $\mathbf{ind}^{L\text{-mod}^{\mathrm{ch}}_x}_{L'\text{-mod}^{\mathrm{Lie-}*}_x}(k)$ are the images under (4.2) of

$$\mathbf{ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}_n}(k),$$

where $\mathfrak{g}_n \subset \widehat{\mathfrak{g}}$ is the nth congruence subalgebra.

Hence, combining Corollaries D.3.5 and D.2.13, we obtain:

Corollary D.3.7. *The natural transformation*

$$\underset{n}{\mathrm{colim}}\, \mathcal{H}om(\mathbf{ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}_n}(k), -) \to \mathbf{oblv}_L, \quad (L\text{-mod}^{\mathrm{ch}}_x)^{>-\infty} \to \mathrm{Vect}$$

is an isomorphism.

D.3.8. Assume now that $\mathcal{M}$ is abelian (and finitely generated and locally fee as a D-module). Note that in this case

$$\mathcal{A} = U^{ch}(\mathcal{M}) \simeq \mathrm{Sym}^!(\mathcal{M}[1])[1]$$

is a commutative chiral algebra (see Sect. D.1.4).

In this case we can talk about *commutative* chiral $\mathcal{A}$-modules: this is by definition the category of modules over the commutative algebra $\mathrm{Sym}(\mathcal{M}_x)$, and it has a natural forgetful functor to

$$\mathcal{A}\text{-mod}^{\mathrm{ch}}_x \simeq \mathcal{M}\text{-mod}^{\mathrm{ch}}_x.$$

Denote

$$\mathrm{Modif}_{\mathrm{cl,flat,f.g.}}(\mathcal{M}) := \{\mathcal{M}' \in \mathrm{D\text{-}mod}(X)^{\heartsuit,\mathrm{flat,f.g.}},\ \mathcal{M}'|_{X-x} \simeq j^*(\mathcal{M})\}.$$

From Corollaries D.3.5 and D.2.13, we obtain:

Corollary D.3.9. *The natural transformation*

$$\underset{\mathcal{M}' \in \mathrm{Modif}_{\mathrm{cl,flat,f.g.}}(\mathcal{M})}{\mathrm{colim}} \mathcal{H}om(\mathrm{Sym}(\mathcal{M}'_x), -) \to \mathbf{oblv}_{\mathcal{M}}, \quad (\mathcal{M}\text{-mod}^{\mathrm{ch}}_x)^{>-\infty} \to \mathrm{Vect}$$

is an isomorphism, where

$$\mathrm{Sym}(\mathcal{M}'_x) \in \mathrm{Sym}(\mathcal{M}'_x)\text{-mod} \to \mathcal{M}'\text{-mod}^{\mathrm{ch}}_x \simeq \mathcal{M}\text{-mod}^{\mathrm{ch}}_x.$$

D.4. **Proof of Proposition D.3.4.**

D.4.1. Let $\widetilde{L}$ be an eventually coconnective Lie-* algebra on X, and let us be given a map

$$j^*(L) \xrightarrow{\alpha} j^*(\widetilde{L}).$$

Consider the category

$$\mathbf{C}_{\widetilde{L}} := \{L' \in \mathrm{Modif}_{\mathrm{cl,flat}}(L),\, L' \xrightarrow{\alpha'} \widetilde{L} \,|\, \alpha'|_{X-x} = \alpha\}.$$

We will prove:

Proposition D.4.2. *Assume that $j^*(L)$ is finitely generated and locally free as a D-module. Then the category $\mathbf{C}_{\widetilde{L}}$ is non-empty.*

Let us show how Proposition D.4.2 implies Proposition D.3.4.

Proof of Proposition D.3.4. By adjunction, it suffices to prove that in the setup of Proposition D.4.2, the category $\mathbf{C}_{\widetilde{L}}$ is contractible. We will show that it is cofiltered.

Let

$$L'_I : I \to \mathbf{C}_{\widetilde{L}}$$

be a finite diagram. We need to show that it can be extended to a diagram

$$L'_{I^{\triangleleft}} : I^{\triangleleft} \to \mathbf{C}_{\widetilde{L}},$$

where $I^{\triangleleft}$ is a left cone over I.

Set

$$\widetilde{L}_I := \lim_I L'_I,$$

where the limit is taken in the category of Lie-* algebras over $\widetilde{L}$. Note that $\widetilde{L}_I$ is eventually coconnective. By construction, we have a map

$$\alpha_I : j^*(L) \to j^*(\widetilde{L}_I).$$

The datum of $L'_{I^{\triangleleft}}$ is equivalent to finding $L' \in \mathrm{Modif}_{\mathrm{cl,flat}}(L)$ and a map

$$\alpha'_I : L' \to \widetilde{L}_I,$$

extending α_I.

However, the existence of (L', α'_I) is guaranteed by Proposition D.4.2

□

[Proposition D.3.4]

D.4.3. The rest of this subsection is devoted to the proof of Proposition D.4.2. The starting point is the following observation ([BD2, Lemma 2.5.13]):

Lemma D.4.4. *Assume that $j^*(L)$ is classical, and the underlying D-module is finitely generated and locally free. Then the tautological map*

$$j_! \circ j^*(\mathbf{oblv}_{\text{Lie-}*}(L)) \to \underset{L' \in \mathrm{Modif}_{\mathrm{cl,flat}}(L)}{\text{“}\lim\text{”}} \mathbf{oblv}_{\text{Lie-}*}(L')$$

is an isomorphism in $\mathrm{Pro}(\text{D-mod}(X))$*, where:*

- $\mathbf{oblv}_{\text{Lie-}*}$ *is the forgetful functor from the category of Lie-* algebras to the category of D-modules on X;*
- $j_! : \text{D-mod}(X - x) \to \mathrm{Pro}(\text{D-mod}(X))$ *is the pro-left adjoint of j^*.*

D.4.5. Let $\widetilde{L}$ be concentrated in degrees $[-n, 0]$. We will argue by induction on n.

Consider first the case $n = 0$.

By Lemma D.4.4, we can find $L' \in \mathrm{Modif}_{\mathrm{cl,flat}}(L)$, so that α extends to a map

$$\alpha' : L' \to \widetilde{L}$$

as plain D-modules.

The obstruction to α' being a map of Lie-* algebras is a map

$$L' \boxtimes L' \to \Delta_*(\widetilde{L}),$$

which vanishes on $(X - x) \times (X - x)$.

The assumption that $j^*(L)$ is finitely generated implies that the naturally defined map

$$(j \times j)_! \circ (j \times j)^*(L \boxtimes L) \to j_! \circ j^*(L) \boxtimes j_! \circ j^*(L)$$

in $\mathrm{Pro}(\text{D-mod}(X \times X))$ is an isomorphism. Hence, again by Lemma D.4.4, there exists an arrow $L'' \to L'$ in $\mathrm{Modif}_{\mathrm{cl,flat}}(L)$ such that the composition

$$L'' \boxtimes L'' \to L' \boxtimes L' \to \Delta_*(\widetilde{L})$$

vanishes.

Hence, $L'' \to L' \to \widetilde{L}$ provides the desired object of $\mathbf{C}_{\widetilde{L}}$.

D.4.6. We now perform the induction step. Suppose the assertion is valid for $\tau^{\geq n-1}(\widetilde{L})$, i.e., that can find an object $L' \in \mathrm{Modif}_{\mathrm{cl,flat}}(L)$ and a lift of α to a map of Lie-* algebras

$$L' \to \tau^{\geq n-1}(\widetilde{L}).$$

Fix this map, and consider the fiber product

$$\widetilde{L} \underset{\tau^{\geq n-1}(\widetilde{L})}{\times} L' =: \widetilde{L}'.$$

We wish to find an arrow $L'' \to L'$ in $\mathrm{Modif}_{\mathrm{cl,flat}}(L)$, so that the map

$$\widetilde{L}' \underset{L'}{\times} L'' =: \widetilde{L}'' \to L''$$

admits a left inverse.

By Lemma D.4.4, after replacing L', the extension

$$\widetilde{L}' \to L'$$

is given by an $(n+2)$-cocycle, which is a map

$$(L')^{\boxtimes(n+2)} \to \Delta^n_*(\widetilde{L}),$$

which vanishes on $(X-x)^n$, where Δ^n denotes the main diagonal $X \to X^n$.

Now, by the same argument as above, using the fact that L is finitely generated, we can find an arrow $L'' \to L'$ in $\mathrm{Modif}_{\mathrm{cl,flat}}(L)$ such that the composition

$$(L'')^{\boxtimes n} \to (L')^{\boxtimes n} \to \Delta^n_*(\widetilde{L})$$

vanishes.

Hence, the extension $\widetilde{L}'' \to L''$ admits a splitting.

□[Proposition D.4.2]

D.5. **Proof of Theorem 4.3.9.** We will prove the variant of the lemma with a fixed $\underline{x} = x \in \mathrm{Ran}$. The factorization version is a variant of this in families.

D.5.1. We start with the following observation: let us regard the assignment

$$\mathcal{Y} \rightsquigarrow \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{>-\infty}, \quad (\mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2) \rightsquigarrow f_* \tag{D.7}$$

as a functor from the category of ind-affine ind-schemes to ∞-categories.

We claim:

Proposition D.5.2. *The functor* (D.7) *commutes with totalizations, in the sense that if $\mathcal{Y}^\bullet$ is a cosimplicial ind-affine ind-scheme and*

$$\mathcal{Y} \simeq \mathrm{Tot}(\mathcal{F}^\bullet),$$

where the limit is taken in PreStk*, then the functor*

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{>-\infty} \to \mathrm{Tot}(\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}^\bullet)^{>-\infty})$$

is an equivalence.

The proposition will be proved in Sect. D.8. We now proceed with the proof of Theorem 4.3.9.

D.5.3. For an affine D-scheme $\mathcal{Y}$, let A denote the corresponding commutative algebra in D-mod(X) so that $\mathbf{oblv}^l(A)$ is connective and $\mathcal{Y} = \mathrm{Spec}_X(A)$.

Let $\mathcal{A}$ denote the corresponding (commutative) chiral algebra, see Sect. D.1.3, so that $\mathcal{A}$ corresponds to the factorization algebra $\mathcal{O}_\mathcal{Y}$, and

$$\mathcal{O}_\mathcal{Y}\text{-mod}_x^{\mathrm{fact}} \simeq \mathcal{A}\text{-mod}_x^{\mathrm{ch}}.$$

Over the next few subsections we will reduce the statement of Theorem 4.3.9 to the case when $A = \mathrm{Sym}^!(\mathcal{M}[1])$ for $\mathcal{M}$ a classical locally free finitely generated D-module.

D.5.4. We interpret the functor $\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}}$ of (4.4) as

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \to \mathcal{A}\text{-mod}_x^{\mathrm{ch}}.$$

This functor is t-exact and both categories are right complete in their respective t-structures. Hence, it is in enough to show that the functor $\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}}$ induces an equivalence

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))^{\geq 0, \leq m} \to (\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^{\geq 0, \leq m}$$

for every m.

Note now that if $A_1 \to A_2$ is a map in ComAlg(D-mod(X)), such that the induced map $\tau^{\geq -m}(A_1) \to \tau^{\geq -m}(A_2)$ is an isomorphism, the corresponding functors

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}_2)) \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}_1)$$

and

$$\mathcal{A}_2\text{-mod}_x^{\mathrm{ch}} \to \mathcal{A}_1\text{-mod}_x^{\mathrm{ch}}$$

induces equivalences on the corresponding $(-)^{\geq 0, \leq m}$ categories.

In particular, we obtain that it is enough to show that the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla({}^m\mathcal{Y}))^{\geq 0, \leq m} \to ({}^m\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^{\geq 0, \leq m}$$

is an equivalence for ${}^mA := \tau^{\geq -m}(A)$, and the corresponding ${}^m\mathcal{Y}$ and ${}^m\mathcal{A}$.

D.5.5. By the assumption that A is D-afp, we can find a simplicial object $A_\bullet$ in ComAlg(D-mod(X))$^{\leq 0}$ with terms $A_n = \mathrm{Sym}^!(\mathcal{M}_n[1])$, where $\mathcal{M}_n$ is a classical[81] locally free finitely generated D-module, such that ${}^mA$ is a retract of $\tau^{\geq -m}(|A_\bullet|)$, see Sect. B.6.2.

Hence, it is enough to prove that

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}'))^{\geq 0, \leq m} \to (\mathcal{A}'\text{-mod}_x^{\mathrm{ch}})^{\geq 0, \leq m}$$

is an equivalence for $A' := \tau^{\geq -m}(|A_\bullet|)$ for $A_\bullet$ as above.

Applying Sect. D.5.4 again, we obtain that it is enough to prove that

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}''))^{\geq 0, \leq m} \to (\mathcal{A}''\text{-mod}_x^{\mathrm{ch}})^{\geq 0, \leq m}$$

is an equivalence for $A'' := |A_\bullet|$ for $A_\bullet$ as above.

Hence, we can obtain that it is enough to prove Theorem 4.3.9 for A is of the form $|A_\bullet|$ for $A_\bullet$ as above.

[81]Recall that according to our conventions, this means that $\mathbf{oblv}^r(\mathcal{M}_n)$ is classical, i.e., $\mathbf{oblv}^l(\mathcal{M}_n)[1]$ is classical.

D.5.6. For $A_\bullet$ as above set $\mathcal{Y}^n := \operatorname{Spec}_X(A_n)$, and consider the corresponding simplicial affine D-scheme $\mathcal{Y}^\bullet$, so that

$$\mathcal{Y} \simeq \operatorname{Tot}(\mathcal{Y}^\bullet).$$

It is clear that that the functor

$$\mathcal{Y} \mapsto \mathfrak{L}_\nabla(\mathcal{Y})$$

preserves limits, so that

$$\mathfrak{L}_\nabla(\mathcal{Y}) \simeq |\mathfrak{L}_\nabla(\mathcal{Y}^\bullet)|.$$

Hence, by Proposition D.5.2, the functor

$$\operatorname{QCoh}(\mathfrak{L}_\nabla(\mathcal{Y}))^{>-\infty} \to \operatorname{Tot}(\operatorname{QCoh}(\mathfrak{L}_\nabla(\mathcal{Y}^\bullet)))^{>-\infty}$$

is an equivalence.

The functor

$$\mathcal{A}\text{-mod}_x^{\mathrm{ch}} \to \operatorname{Tot}(\mathcal{A}_n\text{-mod}_x^{\mathrm{ch}})$$

is also an equivalence: indeed, this is obvious for non-unital modules (this is a general property of categories of modules over operad algebras), and this property is inherited by unital modules by [CR, Proposition 3.8.4].

Hence, it is enough to show that the functors

$$\operatorname{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}^n))^{>-\infty} \to (\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^{>-\infty}$$

are equivalences for every n.

This reduces the assertion of Theorem 4.3.9 to the case when $A = \operatorname{Sym}^!(\mathcal{M}[1])$ for $\mathcal{M}$ a classical locally free finitely generated generated D-module.

D.5.7. Let $\mathcal{Y}$ be general an affine D-scheme. Consider the functor

$$\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}} : \operatorname{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \to \mathcal{A}\text{-mod}_x^{\mathrm{ch}}$$

and its (a priori discontinuous) right adjoint. Tautologically, we have

$$\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -) \simeq \mathbf{oblv}_{\mathcal{A}} \circ \Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}},$$

hence we obtain a natural transformation

$$\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -) \circ (\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}})^R \to \mathbf{oblv}_{\mathcal{A}}. \tag{D.8}$$

We claim that it suffices to show that (D.8) is an isomorphism when evaluated on $(\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^{>-\infty}$.

Indeed, this follows from the fact that both $\mathbf{oblv}_{\mathcal{A}}$ and $\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)$ are conservative on the eventually coconnective subcategories (see Sect. D.7.1).

D.5.8. We will now specialize to the case when $\mathcal{A} = \operatorname{Sym}^!(\mathcal{M}[1])$ for $\mathcal{M} \in \text{D-mod}^l(X)^{\heartsuit,\text{loc.free,f.g.}}$, and prove that (D.8) is an isomorphism on $(\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^{>-\infty}$ by an explicit calculation.

D.5.9. Let $\operatorname{Modif}_{\mathrm{cl,flat,f.g.}}(\mathcal{M})$ be the category

$$\{\mathcal{M}' \in \text{D-mod}^l(X)^{\heartsuit,\text{flat,f.g.}},\ \mathcal{M}'|_{X-x} \simeq \mathcal{M}|_{X-x}\}.$$

We claim that the ind-scheme $\mathfrak{L}_\nabla(\mathcal{Y})$ identifies in this case with

$$\underset{\mathcal{M}' \in \operatorname{Modif}_{\mathrm{cl,f.g.,flat}}}{\text{“colim”}} \operatorname{Spec}(\operatorname{Sym}(\mathcal{M}'_x)).$$

Indeed, since the question is local around x, with no restriction of generality we can assume that X is affine. Let t be a uniformizer at x. Then for a connective commutative algebra R, we have by definition

$$\operatorname{Maps}(\operatorname{Spec}(R), \mathfrak{L}_\nabla(\mathcal{Y})) = \operatorname{Maps}_{\operatorname{ComAlg}(\text{D-mod}(X))}(\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{M}), R(\!(t)\!)) \simeq \operatorname{Maps}_{\text{D-mod}(X)}(\mathcal{M}, R(\!(t)\!))$$

Since the map

$$j_! \circ j^*(\mathcal{M}) \to \underset{\mathcal{M}' \in \operatorname{Modif}_{\mathrm{cl,flat,f.g.}}}{\text{“lim”}} \mathcal{M}'$$

is an isomorphism in Pro(D-mod(X)), we have

$$\mathrm{Maps}_{\text{D-mod}(X)}(\mathcal{M}, R(\!(t)\!)) \simeq \underset{\mathcal{M}'}{\mathrm{colim}}\ \mathrm{Maps}_{\text{D-mod}(X)}(\mathcal{M}', R[\![t]\!]) \simeq \underset{\mathcal{M}'}{\mathrm{colim}}\ \mathrm{Maps}_{\mathrm{Vect}}(\mathcal{M}'_x, R) \simeq$$
$$\simeq \underset{\mathcal{M}'}{\mathrm{colim}}\ \mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(\mathrm{Sym}(\mathcal{M}'_x), R) = \underset{\mathcal{M}'}{\mathrm{colim}}\ \mathrm{Maps}(\mathrm{Spec}(R), \mathrm{Spec}(\mathrm{Sym}(\mathcal{M}'_x))) =$$
$$=: \mathrm{Maps}\left(\mathrm{Spec}(R), \text{“}\underset{\mathcal{M}'}{\mathrm{colim}}\text{”}\ \mathrm{Spec}(\mathrm{Sym}(\mathcal{M}'_x))\right),$$

as desired.

D.5.10. For $\mathcal{M}'$ as above, let us denote by

$$\mathcal{O}_{(\mathcal{M}'_x)^\vee} \in \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))$$

the direct images of the structure sheaf along the map,

$$\mathrm{Spec}(\mathrm{Sym}(\mathcal{M}'_x)) \to \mathfrak{L}_\nabla(\mathcal{Y}).$$

The above description of $\mathfrak{L}_\nabla(\mathcal{Y})$ implies that the functor $\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)$ is isomorphic to

$$\underset{\mathcal{M}'}{\mathrm{colim}}\, \mathcal{H}om(\mathcal{O}_{(\mathcal{M}'_x)^\vee}, -).$$

The functor

$$\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)^{\mathrm{enh}} : \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \to \mathcal{A}\text{-mod}_x^{\mathrm{ch}}$$

sends $\mathcal{O}_{(\mathcal{M}'_x)^\vee}$ to

$$\mathrm{Sym}(\mathcal{M}'_x) \in \mathcal{A}\text{-mod}_x^{\mathrm{ch}}.$$

Now, the required isomorphism follows from Corollary D.3.9.

$\square$[Theorem 4.3.9]

D.6. **Failure of Theorem 4.3.9 in the non-finitely presented case.** In this subsection we will explain why Theorem 4.3.9 does *not* hold when $\mathcal{Y}$ is *not* almost finitely presented (in the D-sense).

Remark D.6.1. One can show that the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))^{>-\infty} \to (\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^{>-\infty} \tag{D.9}$$

induces an equivalence of the abelian categories

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))^{\heartsuit} \to (\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^{\heartsuit}.$$

So, the failure of (D.9) to be an equivalence occurs at the derived level.

D.6.2. We will take $\mathcal{A} = \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{M})$, where $\mathcal{M}$ is an infinitely-generated D-module (i.e., the direct sum of countably many copies of D_X). We will show that (D.9) fails to be an equivalence in this case.

Namely, we will construct two objects $\mathcal{F}_1, \mathcal{F}_2 \in (\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))^{\heartsuit}$ with images in $(\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^{\heartsuit}$ denoted $\mathcal{M}_1, \mathcal{M}_2$, respectively, and an element in $\mathrm{Ext}^2_{\mathcal{A}\text{-mod}_x^{\mathrm{ch}}}(\mathcal{M}_1, \mathcal{M}_2)$ that does not come from an element in $\mathrm{Ext}^2_{\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))}(\mathcal{F}_1, \mathcal{F}_2)$.

Remark D.6.3. It follows from the description of the category $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))^{\heartsuit}$ in Sect. D.7.5 that the category $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))^b$ is the bounded derived category of its heart. So the above inequality of the Ext^2 spaces means that $(\mathcal{A}\text{-mod}_x^{\mathrm{ch}})^b$ is *not* the bounded derived category of its heart.

D.6.4. Write
$$\mathcal{M} = \underset{i}{\operatorname{colim}} \mathcal{M}_i,$$
where $\mathcal{M}_i$ are finitely generated.

Let us call a modification $\mathcal{M}'$ "quasi-finitely generated" if all the intersections $\mathcal{M}' \cap \mathcal{M}_i$ are finitely generated. As in Sect. D.5.9, we have
$$\mathfrak{L}_\nabla(\mathcal{Y}) \simeq \underset{\mathcal{M}' \in \mathrm{Modif}_{\mathrm{cl,flat,q\text{-}f.g.}}}{\text{“colim”}} \mathrm{Spec}(\mathrm{Sym}(\mathcal{M}'_x)).$$

By the same logic as in Sect. D.5.10, the functor
$$\mathcal{F} \mapsto \underset{\mathcal{M}' \in \mathrm{Modif}_{\mathrm{cl,flat,q\text{-}f.g.}}}{\operatorname{colim}} \mathcal{H}om(\mathcal{O}_{(\mathcal{M}'_x)^\vee}, -), \quad \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \to \mathrm{Vect}$$
identifies with $\Gamma(\mathfrak{L}_\nabla(\mathcal{Y}), -)$. In particular, it is t-exact.

Hence, if for some $\mathcal{M}'$ and $\mathcal{F} \in \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))^\heartsuit$ we have a class
$$\alpha' \in \mathrm{Ext}^2_{\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))}(\mathcal{O}_{(\mathcal{M}'_x)^\vee}, \mathcal{F}),$$
we can find $\mathcal{M}'' \subset \mathcal{M}'$ such that the image α'' of α' in
$$\mathrm{Ext}^2_{\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))}(\mathcal{O}_{(\mathcal{M}''_x)^\vee}, \mathcal{F})$$
vanishes.

D.6.5. As in Sect. D.5.10, the image of $\mathcal{O}_{(\mathcal{M}'_x)^\vee}$ in $\mathcal{A}\text{-mod}^{\mathrm{ch}}_x$ is $\mathrm{Sym}(\mathcal{M}'_x) \in \mathcal{A}\text{-mod}^{\mathrm{ch}}_x$.

By adjunction for any $\mathcal{M} \in \mathcal{A}\text{-mod}^{\mathrm{ch}}_x$, we have
$$\mathcal{H}om_{\mathcal{A}\text{-mod}^{\mathrm{ch}}_x}(\mathrm{Sym}(\mathcal{M}'_x), \mathcal{M}) \simeq \mathcal{H}om_{\mathcal{M}'\text{-mod}^{\mathrm{Lie}\text{-}*}_x}(k, \mathcal{M}).$$

D.6.6. Hence, it suffices to find $\mathcal{F} \in \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))^\heartsuit$ with image $\mathcal{M} \in (\mathcal{A}\text{-mod}^{\mathrm{ch}}_x)^\heartsuit$ and a class
$$\beta' \in \mathrm{Ext}^2_{\mathcal{M}'\text{-mod}^{\mathrm{Lie}\text{-}*}_x}(k, \mathcal{M})$$
such that for any $\mathcal{M}'' \subset \mathcal{M}'$, the image β'' of β' in
$$\mathrm{Ext}^2_{\mathcal{M}''\text{-mod}^{\mathrm{Lie}\text{-}*}_x}(k, \mathcal{M})$$
is non-zero.

We will take $\mathcal{F}$ to be the sky-scraper at the origin of $\mathcal{Y}$, so that $\mathcal{M} = k$, with the trivial chiral action of $\mathcal{M}$.

D.6.7. We calculate
$$\mathcal{H}om_{\mathcal{M}'\text{-mod}^{\mathrm{Lie}\text{-}*}_x}(\mathcal{M}_1, \mathcal{M}_2) \simeq \lim_i \mathcal{H}om_{(\mathcal{M}' \cap \mathcal{M}_i)\text{-mod}^{\mathrm{Lie}\text{-}*}_x}(\mathcal{M}_1, \mathcal{M}_2).$$

For $\mathcal{M}_1 = \mathcal{M}_2 = k$, we obtain
$$\mathrm{Ext}^2_{\mathcal{M}'\text{-mod}^{\mathrm{Lie}\text{-}*}_x}(k, k) \simeq \lim_i \Lambda^2(\mathbb{D}(\mathcal{M}' \cap \mathcal{M}_i)_x) \simeq$$
$$\simeq \lim_i \mathrm{Hom}^{\mathrm{antisym}}_{\mathrm{D\text{-}mod}(X \times X)^\heartsuit}((\mathcal{M}' \cap \mathcal{M}_i) \boxtimes (\mathcal{M}' \cap \mathcal{M}_i), \delta_{x,x}) \simeq \mathrm{Hom}^{\mathrm{antisym}}_{\mathrm{D\text{-}mod}(X \times X)^\heartsuit}(\mathcal{M}' \boxtimes \mathcal{M}', \delta_{x,x}).$$

Now, it is clear that since $\mathcal{M}'$ is infinitely generated, we can find an element
$$\beta' \in \mathrm{Hom}^{\mathrm{antisym}}_{\mathrm{D\text{-}mod}(X \times X)^\heartsuit}(\mathcal{M}' \boxtimes \mathcal{M}', \delta_{x,x}),$$
such that for any $\mathcal{M}''$, the restriction β'' of β' to
$$\mathrm{Hom}^{\mathrm{antisym}}_{\mathrm{D\text{-}mod}(X \times X)^\heartsuit}(\mathcal{M}'' \boxtimes \mathcal{M}'', \delta_{x,x})$$
is non-zero.

Remark D.6.8. Note that the above counterexample does not work for Ext^1 instead of Ext^2 (as must be the case, since (D.9) is an equivalence at the abelian level). Indeed, for any

$$\gamma' \in \mathrm{Hom}_{\text{D-mod}(X)^\heartsuit}(\mathcal{M}', \delta_x),$$

there exists $\mathcal{M}'' \subset \mathcal{M}'$, such that the restriction γ'' of γ' to

$$\mathrm{Hom}_{\text{D-mod}(X)^\heartsuit}(\mathcal{M}'', \delta_x)$$

vanishes.

D.7. **Proof of Proposition D.5.2: preparations.**

D.7.1. First, we claim that if $\mathcal{Y}$ is an ind-affine ind-scheme, then the functor

$$\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}) \to \mathrm{Vect}$$

is conservative on $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{>-\infty}$.

Indeed, since $\Gamma(\mathcal{Y}, -)$ is t-exact and the t-structure on $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})$ is right-complete, it suffices to show that $\Gamma(\mathcal{Y}, -)$ does not annihilate objects from $\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^\heartsuit$.

Write $\mathcal{Y}$ is a filtered colimit of schemes Y_α under closed embeddings

$$f_{\alpha,\beta} : Y_\alpha \to Y_\beta.$$

An object $\mathcal{F} \in \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^\heartsuit$ amounts to a collection

$$\{\mathcal{F}_\alpha \in \mathrm{QCoh}(Y_\alpha)^\heartsuit, \mathcal{F}_\alpha \simeq H^0(f^!_{\alpha,\beta}(\mathcal{F}_\beta))\}.$$

In particular, the maps

$$\Gamma(\mathcal{Y}_\alpha, \mathcal{F}_\alpha) \to \Gamma(\mathcal{Y}_\beta, \mathcal{F}_\beta)$$

are injective.

We have

$$\Gamma(\mathcal{Y}, \mathcal{F}) \simeq \underset{\alpha}{\mathrm{colim}}\, \Gamma(\mathcal{Y}_\alpha, \mathcal{F}_\alpha)$$

and the statement is manifest.

D.7.2. We claim:

Proposition D.7.3. *The functor*

$$\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0} \to \mathrm{Vect}^{\geq 0}$$

is comonadic.

Given the conservativity, the assertion of the proposition follows from the next general observation:

Lemma D.7.4. *Let $\mathbf{C}, \mathbf{D}$ be cocomplete DG categories, equipped with t-structures, compatible with filtered colimits. Assume that $\mathbf{D}$ is right-complete in its t-structure. Let $F : \mathbf{C} \to \mathbf{D}$ be a t-exact continuous functor. Assume that F is conservative on $\mathbf{C}^{>-\infty}$. Then the induced functor*

$$\mathbf{C}^{\geq 0} \to \mathbf{D}^{\geq 0}$$

is comonadic.

Proof. By Barr-Beck-Lurie, suffices to show that F preserves totalizations of cosimplicial objects in $\mathbf{C}^{\geq 0}$. Thus, let $\mathbf{c}^\bullet$ be a cosimplicial object in $\mathbf{C}$. We have to show that the map

$$F(\mathrm{Tot}(\mathbf{c}^\bullet)) \to \mathrm{Tot}(F(\mathbf{c}^\bullet))$$

is an isomorphism.

Since $\mathbf{D}$ is right-complete in its t-structure, it suffices to show that for every n,

$$\tau^{\leq n}(F(\mathrm{Tot}(\mathbf{c}^\bullet))) \to \tau^{\leq n}(\mathrm{Tot}(F(\mathbf{c}^\bullet)))$$

is an isomorphism.

Let

$$\mathrm{Tot}^{\leq n}(\mathbf{c}^\bullet) \text{ and } \mathrm{Tot}^{\leq n}(F(\mathbf{c}^\bullet))$$

be the totalizations of the corresponding $(n+1)$-skeleta.

We have natural maps

$$\mathrm{Tot}(\mathbf{c}^\bullet) \to \mathrm{Tot}^{\leq n}(\mathbf{c}^\bullet) \text{ and } \mathrm{Tot}(F(\mathbf{c}^\bullet)) \to \mathrm{Tot}^{\leq n}(F(\mathbf{c}^\bullet)).$$

Since the terms of $\mathbf{c}^\bullet$ and $F(\mathbf{c}^\bullet)$ are in $\mathbf{C}^{\geq 0}$, the above maps induce isomorphisms between the $\tau^{\leq n}$ truncations.

Hence, it suffices to show that

$$F(\mathrm{Tot}^{\leq n}(\mathbf{c}^\bullet)) \to \mathrm{Tot}^{\leq n}(F(\mathbf{c}^\bullet))$$

is an isomorphism.

However, this is obvious, since the limit over $\mathbf{\Delta}_{\leq n}$ is a finite limit.

□

D.7.5. Note that we have a canonical equivalence

$$\mathrm{Pro}(\mathrm{Vect})^{\mathrm{op}} \simeq \mathrm{Funct}_{\mathrm{discnt}}(\mathrm{Vect}, \mathrm{Vect}), \quad \mathbf{V} \mapsto \mathcal{H}om(\mathbf{V}, -).$$

where $\mathrm{Funct}_{\mathrm{discnt}}(-,-)$ denotes the category of exact k-linear functors that are not necessarily continuous.

Under this equivalence, the monoidal structure on $\mathrm{Funct}_{\mathrm{discnt}}(-,-)$ given by composition corresponds to the $\overset{\rightarrow}{\otimes}$ monoidal structure on Pro(Vect) (see [Bei]):

For

$$\mathbf{V} = \text{“}\lim_{i\in I}\text{”}\, V_i \text{ and } \mathbf{W} = \text{“}\lim_{j\in j}\text{”}\, W_j,$$

we have

$$\mathbf{V} \overset{\rightarrow}{\otimes} \mathbf{W} = \lim_{j\in j} \left(\underset{V_j^f \subset V_j}{\mathrm{colim}} \left((\text{“}\lim_{i\in I}\text{”}\, V_i) \otimes V_j^f \right) \right),$$

where:

- V_j^f runs the category of compact objects mapping to V_j;
- In the right-hand side, the outer limit and the inner colimit are taken in Pro(Vect).

Thus, comonads on Vect correspond to algebra objects in Pro(Vect) with respect to $\overset{\rightarrow}{\otimes}$. Comonads that are left t-exact correspond to algebra objects in $\mathrm{Pro}(\mathrm{Vect}^{\leq 0})$.

D.7.6. Let $\mathsf{M}_{\mathcal{Y}}$ denote the comonad on Vect corresponding to the functor $\Gamma(\mathcal{Y}, -)$, so that

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0} \simeq \mathsf{M}_{\mathcal{Y}}\text{-comod}(\mathrm{Vect}^{\geq 0}).$$

The object of $\mathrm{Pro}(\mathrm{Vect}^{\leq 0})$ that corresponds to $\mathsf{M}_{\mathcal{Y}}$ is described as follows.

Let $\mathcal{O}_{\mathcal{Y}}$ be the object of $\mathrm{Pro}(\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}))$ associated to the ind-affine ind-scheme $\mathcal{Y}$. I.e., if

$$\mathcal{Y} = \text{“}\underset{i}{\mathrm{colim}}\text{”}\, Y_i,$$

then

$$\mathcal{O}_{\mathcal{Y}} := \text{“}\lim_{i}\text{”}\, \mathcal{O}_{Y_i}.$$

Let

$$\mathbf{oblv}^{\mathrm{Pro}}_{\mathrm{ComAlg}} : \mathrm{Pro}(\mathrm{ComAlg}(\mathrm{Vect})) \to \mathrm{Pro}(\mathrm{Vect})$$

be the pro-extension of the functor

$$\mathbf{oblv}_{\mathrm{ComAlg}} : \mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{Vect}\,.$$

The endofunctor of Vect underlying the comonad $\mathsf{M}_{\mathcal{Y}}$ is given by

$$V \mapsto \mathcal{H}om(\mathbf{oblv}^{\mathrm{Pro}}_{\mathrm{ComAlg}}(\mathcal{O}_{\mathcal{Y}}), V).$$

D.7.7. The proof of Proposition D.5.2 will be based on the following two observations:

Lemma D.7.8. *For every natural number n, the functor $\mathbf{oblv}^{\mathrm{Pro}}_{\mathrm{ComAlg}} : \mathrm{Pro}(\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})) \to \mathrm{Pro}(\mathrm{Vect}^{\leq 0})$, followed by the truncation*

$$\mathrm{Pro}(\mathrm{Vect}^{\leq 0}) \to \mathrm{Pro}(\mathrm{Vect}^{\leq 0, \geq -n})$$

commutes with geometric realizations.

Lemma D.7.9. *For every natural number n and for every natural number m, the functor*

$$\mathbf{V} \mapsto \mathbf{V}^{\overset{\rightarrow}{\otimes} m}, \quad \mathrm{Pro}(\mathrm{Vect}^{\leq 0}) \to \mathrm{Pro}(\mathrm{Vect}^{\leq 0}),$$

followed by the truncation

$$\mathrm{Pro}(\mathrm{Vect}^{\leq 0}) \to \mathrm{Pro}(\mathrm{Vect}^{\leq 0, \geq -n})$$

commutes with geometric realizations.

Let us temporarily assume these lemmas and prove Proposition D.5.2.

D.8. **Proof of Proposition D.5.2.**

D.8.1. It is enough to show that for every n, the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0, \leq n} \to \mathrm{Tot}(\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}^\bullet)^{\geq 0, \leq n}) \tag{D.10}$$

is an equivalence.

D.8.2. Note that M is a left-exact endofunctor Vect, we can create its truncation $\mathsf{M}^{\leq n}$ that acts on $\mathrm{Vect}^{\geq 0, \leq n}$, namely,

$$\mathsf{M}^{\leq n}(V) := \tau^{\leq n}(\mathsf{M}(V)).$$

This assignment is monoidal, so if M is a comonad, then $\mathsf{M}^{\leq n}$ inherits a natural comonad structure, and we have

$$\mathsf{M}\text{-comod}(\mathrm{Vect}^{\geq 0}) \underset{\mathrm{Vect}^{\geq 0}}{\times} \mathrm{Vect}^{\geq 0, \leq n} \simeq \mathsf{M}^{\leq n}\text{-comod}(\mathrm{Vect}^{\geq 0, \leq n}).$$

D.8.3. According to Proposition D.7.3, we have:

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0} \simeq \mathsf{M}_{\mathcal{Y}}\text{-comod}(\mathrm{Vect}^{\geq 0}) \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}^\bullet)^{\geq 0} \simeq \mathsf{M}_{\mathcal{Y}^\bullet}\text{-comod}(\mathrm{Vect}^{\geq 0}).$$

Hence,

$$\mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y})^{\geq 0, \leq n} \simeq \mathsf{M}^{\leq n}_{\mathcal{Y}}\text{-comod}(\mathrm{Vect}^{\geq 0, \leq n}) \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathcal{Y}^\bullet)^{\geq 0, \leq n} \simeq \mathsf{M}^{\leq n}_{\mathcal{Y}^\bullet}\text{-comod}(\mathrm{Vect}^{\geq 0, \leq n}).$$

D.8.4. We now observe:

Lemma D.8.5. *Let I be an index category and let*

$$\mathsf{M}_I : I \to \mathrm{Comonad}(\mathbf{C}), \quad i \mapsto \mathsf{M}_i$$

be an I-diagram of comonads on a category $\mathbf{C}$. Let M be another comonad, equipped with a compatible collection of maps $\mathsf{M} \to \mathsf{M}_i$. Suppose that for every natural number m, the map

$$\mathsf{M}^{\times m} \to \lim_i \mathsf{M}_i^{\times m}$$

is an isomorphism, where:

- *The notation $\mathsf{M}^{\times m}$ means an m-fold composition of M, and similarly for M_i;*
- *The limit in the right-hand side is taken in the category of endofunctors of $\mathbf{C}$ (i.e., is computed value-wise).*

Then the induced functor

$$\mathsf{M}\text{-comod}(\mathbf{C}) \to \lim_i \, (\mathsf{M}_i\text{-comod}(\mathbf{C}))$$

is an equivalence.

D.8.6. Hence, in order to prove that (D.10) is an equivalence, it suffices to show that for any m, the map

$$(\mathsf{M}^{\leq n}_{\mathcal{Y}})^{\times m} \to \mathrm{Tot}\left((\mathsf{M}^{\leq n}_{\mathcal{Y}^\bullet})^{\times m}\right) \tag{D.11}$$

is an equivalence.

D.8.7. By Sect. D.7.6, we have

$$\mathsf{M}_{\mathcal{Y}}\text{-comod}(\mathrm{Vect}^{\leq 0}) \simeq \mathcal{O}_{\mathcal{Y}}\text{-mod}(\mathrm{Vect}^{\leq 0})$$

and

$$\mathsf{M}_{\mathcal{Y}^\bullet}\text{-comod}(\mathrm{Vect}^{\leq 0}) \simeq \mathcal{O}_{\mathcal{Y}^\bullet}\text{-mod}(\mathrm{Vect}^{\leq 0}),$$

where we regard $\mathcal{O}_{\mathcal{Y}}$ as an algebra object in $(\mathrm{Pro}(\mathrm{Vect}), \overset{\rightarrow}{\otimes})$ via the natural forgetful functor

$$\mathrm{Pro}(\mathrm{ComAlg}(\mathrm{Vect})) \to \mathrm{AssocAlg}(\mathrm{Pro}(\mathrm{Vect}), \overset{\rightarrow}{\otimes}).$$

Note now that in the situation of Proposition D.5.2, the map

$$\mathcal{O}_{\mathcal{Y}} \to |\mathcal{O}_{\mathcal{Y}^\bullet}|$$

is an isomorphism, where the geometric realization is taken in Pro(ComAlg(Vect)).

Hence, D.7.8 and D.7.9 guarantee that the maps (D.11) are isomorphisms, as required.

□[Proposition D.5.2]

D.9. **Proof of Lemmas D.7.8 and D.7.9.**

D.9.1. The key input is the following:

Lemma D.9.2. *Let $\mathbf{C}$ be an n-truncated category (i.e., the mapping spaces have homotopy groups π_m vanish for $m > n$). Then the map*

$$|-|_{\leq n+1} \to |-|, \quad \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{C}) \to \mathbf{C}$$

is an isomorphism, where the left-hand side is the colimit over $\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}$.

D.9.3. *Proof Lemma D.7.9.* Since $\mathbf{\Delta}^{\mathrm{op}}$ is sifted, it suffices to show that the binary operation

$$\mathrm{Pro}(\mathrm{Vect}^{\leq 0, \geq -n}) \times \mathrm{Pro}(\mathrm{Vect}^{\leq 0, \geq -n}) \to \mathrm{Pro}(\mathrm{Vect}^{\leq 0, \geq -n}), \quad \mathbf{V}, \mathbf{W} \mapsto \mathbf{V} \overset{\rightarrow}{\otimes} \mathbf{W} \tag{D.12}$$

commutes with geometric realizations in each variable.

However, by Lemma D.9.2, the functor of geometric realization in $\mathrm{Vect}^{\leq 0, \geq -n}$ is a *finite colimit*, and it is clear that (D.12) commutes with finite colimits in each variable.

□[Lemma D.7.9]

D.9.4. *Proof of Lemma D.7.8.* By Lemma D.9.2, it suffices to show that the functor

$$\mathbf{oblv}^{\mathrm{Pro}}_{\mathrm{ComAlg}} : \mathrm{Pro}(\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0, \geq -n})) \to \mathrm{Pro}(\mathrm{Vect}^{\leq 0, \geq -n})$$

commutes with colimits over $\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}$.

The rest of the argument essentially reproduces [Lu2, Proposition 6.1.5.3]:

Note that for any finite index category I and a category $\mathbf{C}$, the naturally defined functor

$$\mathrm{Pro}\,\mathrm{Funct}(I, \mathbf{C}) \to \mathrm{Funct}(I, \mathrm{Pro}(\mathbf{C}))$$

is an equivalence.

Furthermore, for an object

$$\text{“}\lim_{\alpha}\text{”}\,(i \mapsto \mathbf{c}_{i,\alpha}) \in \mathrm{Pro}(\mathrm{Funct}(I, \mathbf{C}))$$

and the resulting object

$$i \mapsto \text{“}\lim_{\alpha}\text{”}\,\mathbf{c}_{i,\alpha} \in \mathrm{Funct}(I, \mathrm{Pro}(\mathbf{C})),$$

we have

$$\underset{i\in I}{\operatorname{colim}}\left(\text{“}\lim_{\alpha}\text{”}\,\mathbf{c}_{i,\alpha}\right)\simeq \text{“}\lim_{\alpha}\text{”}\left(\underset{i\in I}{\operatorname{colim}}\,\mathbf{c}_{i,\alpha}\right).$$

We apply this observation to $I=\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}$ and $\mathbf{C}$ being

$$\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0,\geq -n}) \text{ and } \mathrm{Vect}^{\leq 0,\geq -n}.$$

Hence, in order to prove Lemma D.7.8, it suffices to show that the functor

$$\mathbf{oblv}_{\mathrm{ComAlg}}:\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0,\geq -n})\to \mathrm{Vect}^{\leq 0,\geq -n}$$

commutes with colimits over $\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}$.

However, this follows from Lemma D.9.2 combined with the fact that the usual geometric realization functor commutes with $\mathbf{oblv}_{\mathrm{ComAlg}}$.

□[Lemma D.7.8]

Appendix E. The spectral spherical category

Throughout this section we let H be an arbitrary finite-dimensional algebraic group. Our goal is to define the factorization category

$$\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H).$$

When $H=\check{G}$, the category $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}) =: \mathrm{Spc}^{\mathrm{spec}}_{\check{G}}$ is the spectral counterpart of Sph_G, and it acts by Hecke functors on the global spectral category. This action will play a key role in the sequel to this paper.

The difficulty we face is that we have not found a way to plug $\mathrm{Hecke}^{\mathrm{spec,loc}}_H$ into one of the previously discussed constructions, i.e.,

$$\mathrm{QCoh}(-),\ \mathrm{QCoh}_{\mathrm{co}}(-),\ \text{ or } \ \mathrm{IndCoh}^*(-)$$

to obtained the desired category.

Instead, we will define it as bi-coinvariants with respect to $\mathfrak{L}^+_\nabla(H)$ inside $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$, where the latter also requires some care, as the factorization scheme $\mathfrak{L}_\nabla(H)$ is *not* ind-placid.

E.1. A 1-affineness property of $\mathrm{LS}^{\mathrm{reg}}_H$.

Throughout this subsection we fix an affine scheme S and a map $\underline{x}:S\to\mathrm{Ran}$.

E.1.1. Consider $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S$ as an $\mathrm{QCoh}(S)$-linear monoidal category with respect to convolution.

Note that we have:

$$\mathrm{Funct}_{\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S\text{-}\mathbf{mod}}(\mathrm{QCoh}(S),\mathrm{QCoh}(S))\simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,S})$$

as monoidal categories.

This gives rise to a pair of adjoint functors

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,S})\text{-}\mathbf{mod}\rightleftarrows \mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S\text{-}\mathbf{mod}, \tag{E.1}$$

$$\mathbf{C}\mapsto \mathbf{C}\underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,S})}{\otimes}\mathrm{QCoh}(S),\quad \mathbf{C}'\mapsto (\mathbf{C}')^{\mathfrak{L}^+_\nabla(H)_S}.$$

Remark E.1.2. Throughout this section, the symbols $(-)^{\mathfrak{L}^+_\nabla(H)_S}$ and $(-)_{\mathfrak{L}^+_\nabla(H)_S}$ indicate *weak*[82] invariants/coinvariants of the group-scheme $\mathfrak{L}^+_\nabla(H)_S$ acting on a $\mathrm{QCoh}(S)$-linear category.

E.1.3. We have the following assertion ([Ra4, Lemma 9.8.1]):

Proposition E.1.4. *The functors* (E.1) *are mutually inverse equivalences.*

[82] Here “weak” is as opposed to “strong”. Note that we could not even talk about strong invariants/coinvariants, because we are talking about weak actions.

E.1.5. Let $\mathbf{C}$ be a module category over $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S$. The functor of $\mathfrak{L}^+_\nabla(H)_S$-averaging

$$\mathrm{Av}_*^{\mathfrak{L}^+_\nabla(H)_S} : \mathbf{C} \to \mathbf{C}^{\mathfrak{L}^+_\nabla(H)_S}$$

naturally factors via a functor

$$\mathbf{C}_{\mathfrak{L}^+_\nabla(H)_S} \to \mathbf{C}^{\mathfrak{L}^+_\nabla(H)_S}. \tag{E.2}$$

We claim:

Corollary E.1.6. *The functor* (E.2) *is an equivalence.*

Proof. Proposition E.1.4 implies that the functor

$$\mathbf{C} \mapsto \mathbf{C}^{\mathfrak{L}^+_\nabla(H)_S}$$

commutes with colimits.

Hence, both sides in (E.2) commute with colimits. Any object in $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S$-**mod** can be written as a colimit of objects of the form $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S \otimes \mathbf{D}$, where the module structure comes from the first factor. Hence, we obtain that it is sufficient to prove that (E.2) is an equivalence for such objects.

However, the latter is obvious: the corresponding functor is the identity functor

$$\mathrm{QCoh}(S)\otimes\mathbf{D} \simeq \left(\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S\otimes\mathbf{D}\right)_{\mathfrak{L}^+_\nabla(H)_S} \to \left(\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S\otimes\mathbf{D}\right)^{\mathfrak{L}^+_\nabla(H)_S} \simeq \mathrm{QCoh}(S)\otimes\mathbf{D}.$$

□

E.1.7. We now claim:

Corollary E.1.8. *The functor*

$$\Omega_{\mathrm{LS}^{\mathrm{reg}}_{H,S}} : \mathrm{QCoh}_{\mathrm{co}}(\mathrm{LS}^{\mathrm{reg}}_{H,S}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,S})$$

is an equivalence.

Proof. The fact that $S \to \mathrm{LS}^{\mathrm{reg}}_{H,S}$ is an fpqc cover implies that the functor

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S}{\otimes} \mathrm{QCoh}(S) \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{LS}^{\mathrm{reg}}_{H,S})$$

is an equivalence (cf. [Ga5, Proposition 6.2.7]).

Hence, it remains to show that the functor

$$\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S}{\otimes} \mathrm{QCoh}(S) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,S})$$

is an equivalence.

However, the latter functor is the functor (E.2) for $\mathbf{C} = \mathrm{QCoh}(S)$.

□

E.2. **Definition of** $\mathrm{Sph}^{\mathrm{spec}}_H$.

E.2.1. Recall that the local spectral Hecke stack is by definition

$$\mathrm{Hecke}^{\mathrm{spec,loc}}_H := \mathrm{LS}^{\mathrm{reg}}_H \underset{\mathrm{LS}^{\mathrm{mer}}_H}{\times} \mathrm{LS}^{\mathrm{reg}}_H .$$

Our approach to the definition of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ is based on the following observation:

Lemma E.2.2. *The factorization prestack* $\mathrm{Hecke}^{\mathrm{spec,loc}}_H$ *identifies canonically with the double quotient*

$$\mathfrak{L}^+_\nabla(H)\backslash\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H).$$

Proof. By definition, the fiber product $\mathrm{LS}^{\mathrm{reg}}_H \underset{\mathrm{LS}^{\mathrm{mer}}_H}{\times} \mathrm{LS}^{\mathrm{reg}}_H$ identifies with

$$\mathfrak{L}^+_\nabla(H)\backslash \mathrm{Stab}_{\mathfrak{L}_\nabla(\mathrm{Jets}(H))}(0)/\mathfrak{L}^+_\nabla(H),$$

where:

- $0 \in \mathfrak{L}_\nabla(\mathrm{Conn}(\mathfrak{h}))$ is the trivial connection;
- $\mathrm{Stab}_{\mathfrak{L}_\nabla(\mathrm{Jets}(H))}(0)$ denotes the stabilizer of 0 with respect to the gauge action of $\mathfrak{L}_\nabla(\mathrm{Jets}(H)) \simeq \mathfrak{L}(H)$.

However,

$$\mathrm{Stab}_{\mathfrak{L}_\nabla(\mathrm{Jets}(H))}(0) = \mathfrak{L}_\nabla(\mathrm{Stab}_{\mathrm{Jets}(H)}(0)),$$

while

$$\mathrm{Stab}_{\mathrm{Jets}(H)}(0) \simeq H,$$

as a group D-scheme.

□

E.2.3. We also note:

Lemma E.2.4. *For $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, the quotient $(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_S$ is locally almost of finite type.*

Proof. First, we note that the unit section

$$S \to (\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_S$$

is an isomorphism at the classical level. (Indeed, for any affine Y, the map $\mathfrak{L}^+_\nabla(Y) \to \mathfrak{L}_\nabla(Y)$ is an isomorphism at the classical level.)

Hence, by [GaRo4, Chapter 1, Theorem 9.1.2], it suffices to show that the cotangent space to $(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_S$ at the unit section is laft (see [GaRo4, Chapter 1, Sect. 3.4.1] for what this means). However, this cotangent space is the dual of

$$(\mathfrak{L}_\nabla(\mathfrak{h})/\mathfrak{L}^+_\nabla(\mathfrak{h}))_S,$$

which makes the assertion manifest.

□

E.2.5. In what follows we will define the (monoidal) factorization category $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$, equipped with an action of the monoidal category $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))$ on the two sides. We will then set

$$\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H) := (\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)))_{\mathfrak{L}^+_\nabla(H)\times\mathfrak{L}^+_\nabla(H)} . \tag{E.3}$$

The caveat here is that $\mathfrak{L}^+_\nabla(H)$ is *not* placid (and hence, $\mathfrak{L}_\nabla(H)$ is not ind-placid). Yet, we will show that the construction of $\mathrm{IndCoh}^*(-)$ in B.13.17-B.13.22 is applicable in this particular case.

E.2.6. Let S_α be as in Sect. B.13.9. Consider the relative affine scheme $\mathfrak{L}^+_\nabla(H)_{S_\alpha}$ and the relative ind-affine ind-scheme $\mathfrak{L}_\nabla(H)_{S_\alpha}$.

First, we claim:

Lemma E.2.7. *The functor*

$$\Psi_{\mathfrak{L}^+_\nabla(H)_{S_\alpha}} : \mathrm{IndCoh}^*(\mathfrak{L}^+_\nabla(H))_{S_\alpha} \to \mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_{S_\alpha}$$

is an equivalence.

Proof. The assertion holds for any smooth target scheme Y. Indeed, one shows that for $S_\alpha = X^I$, the relative affine scheme $\mathfrak{L}^+_\nabla(Y)_{X^I}$ is isomorphic to the limit of a sequence of affine blow-ups with smooth centers, starting with $Y^I \to X^I$, see Sect. E.12.

In particular, $\mathfrak{L}^+_\nabla(Y)_{X^I}$ is isomorphic to a filtered limit of relative affine schemes $Y_n \to S_\alpha$ that are *smooth*.

We have:

$$\mathrm{IndCoh}^*(\mathfrak{L}^+_\nabla(Y))_{S_\alpha} \simeq \lim_n \mathrm{IndCoh}(Y_n)$$

(with respect to push-forwards) and

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(Y))_{S_\alpha} \simeq \lim_n \mathrm{QCoh}(Y_n),$$

and the functor $\Psi_{\mathfrak{L}^+_\nabla(H)_{S_\alpha}}$ corresponds to the compatible family of functors

$$\Psi_{Y_n} : \mathrm{IndCoh}(Y_n) \to \mathrm{QCoh}(Y_n),$$

all of which are equivalences, since Y_n are smooth.

□

E.2.8. According to Sect. A.5, since $\mathfrak{L}_\nabla(H)_{S_\alpha}$ is an ind-affine ind-scheme, we have a well-defined category $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha}$.

The action of $\mathfrak{L}^+_\nabla(H)_{S_\alpha} \times \mathfrak{L}^+_\nabla(H)_{S_\alpha}$ on $\mathfrak{L}_\nabla(H)_{S_\alpha}$ defines on $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha}$ a structure of bimodule with respect to $\mathrm{IndCoh}^*(\mathfrak{L}^+_\nabla(H))_{S_\alpha}$.

Hence, thanks to Lemma E.2.7, we can think of $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha}$ as a bimodule with respect to $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_{S_\alpha}$.

E.2.9. Consider $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha}$ as a module over $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_{S_\alpha}$ with respect to the action on the right.

Direct image with respect to the projection

$$\mathfrak{L}_\nabla(H)_{S_\alpha} \to (\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{S_\alpha} \tag{E.4}$$

gives rise to a functor

$$\begin{aligned}(\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha})_{\mathfrak{L}^+_\nabla(H)_{S_\alpha}} \to \mathrm{IndCoh}^*((\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{S_\alpha}) \simeq \\ \overset{\text{Lemma E.2.4}}{\simeq} \mathrm{IndCoh}((\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{S_\alpha}).\end{aligned} \tag{E.5}$$

We claim:

Lemma E.2.10. *The functor* (E.5) *is an equivalence.*

Proof. Since $S_\alpha \to (\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{S_\alpha}$ is an isomorphism at the reduced level, Zariski-locally on S_α, the map (E.4) splits as a product: indeed, the restriction of the étale $\mathfrak{L}^+_\nabla(H)$-torsor (E.4) to S_α is trivial, and hence over any open affine of $S^\circ_\alpha \subset S_\alpha$, the $\mathfrak{L}^+_\nabla(H)$-torsor (E.4) itself is trivial.

Hence, by Zariski descent, it suffices to show that for $Z_\alpha \to S_\alpha$, where Z_α is an ind-affine ind-scheme locally almost of finite type, the functor

$$\left(\mathrm{IndCoh}^*(Z_\alpha \underset{S_\alpha}{\times} \mathfrak{L}^+_\nabla(H)_{S_\alpha})\right)_{\mathfrak{L}^+_\nabla(H)_{S_\alpha}} \to \mathrm{IndCoh}(Z_\alpha)$$

is an equivalence.

To prove this, it suffices to show that the functor

$$\mathrm{IndCoh}(Z_\alpha) \underset{\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_{S_\alpha} \simeq \mathrm{IndCoh}^*(Z_\alpha \underset{S_\alpha}{\times} \mathfrak{L}^+_\nabla(H)_{S_\alpha}) \tag{E.6}$$

is an equivalence.

To prove (E.6), we can assume that Z_α is an affine scheme. Writing $\mathfrak{L}^+_\nabla(H)_{S_\alpha}$ as a limit of relative affine schemes Y_n *smooth* over S_α as in the proof of Lemma E.2.7, it suffices to show that each of the functors

$$\mathrm{IndCoh}(Z_\alpha) \underset{\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{QCoh}(Y_n) \simeq \mathrm{IndCoh}^*(Z_\alpha \underset{S_\alpha}{\times} Y_n) \tag{E.7}$$

is an equivalence.

However, this follows from Lemma A.4.10.

□

E.2.11. We are finally ready to define $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ as a factorization category. Proceeding as in B.13.17-B.13.22, we need to show that Lemmas B.13.18 and B.13.20 hold for $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$.

We will prove Lemma B.13.18; Lemma B.13.20 is proved similarly.

We need to show that the functor

$$\mathrm{QCoh}(S_\alpha) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\beta} \to \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha}. \tag{E.8}$$

is an equivalence.

We consider both sides as modules over

$$\mathrm{QCoh}(S_\alpha) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_{S_\beta} \simeq \mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_{S_\alpha}.$$

By Proposition E.1.4, it suffices to show that (E.8) becomes an equivalence after taking $\mathfrak{L}^+_\nabla(H)_{S_\alpha}$-invariants, or, equivalently, thanks to Lemma E.1.6, $\mathfrak{L}^+_\nabla(H)_{S_\alpha}$-coinvariants.

However, by Lemma E.2.10, when we take $\mathfrak{L}^+_\nabla(H)_{S_\alpha}$-coinvariants in (E.8), the resulting functor identifies with

$$\mathrm{QCoh}(S_\alpha) \underset{\mathrm{QCoh}(S_\beta)}{\otimes} \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{S_\beta} \to \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{S_\alpha}. \tag{E.9}$$

Now, (E.9) is an equivalence by Lemma B.13.18, since $\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)$ is locally almost of finite type (by Lemma E.2.4) and in particular is placid.

E.2.12. By construction, $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ is equipped, as a factorization category, with an action of $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H)) \otimes \mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))$.

We define $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ by formula (E.3).

By Proposition E.1.4, we have

$$\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_H)\otimes \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_H)}{\otimes} \mathrm{Vect} \simeq \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$$

and

$$\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H) \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_H)}{\otimes} \mathrm{Vect} \simeq \mathrm{IndCoh}(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)).$$

E.2.13. The pair of adjoint functors

$$\iota_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathfrak{L}_\nabla^+(H)) \rightleftarrows \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)) : \iota^! \tag{E.10}$$

gives rise via

$$\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}}) \overset{\text{Corollary E.1.8}}{\simeq} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{LS}_H^{\mathrm{reg}}) \simeq \mathrm{Vect}_{\mathfrak{L}_\nabla^+(H)} \simeq$$
$$\simeq \mathrm{QCoh}(\mathfrak{L}_\nabla^+(H))_{\mathfrak{L}_\nabla^+(H)\times\mathfrak{L}_\nabla^+(H)} \overset{\Psi_{\mathfrak{L}_\nabla^+(H)}}{\simeq} \mathrm{IndCoh}^*(\mathfrak{L}_\nabla^+(H))_{\mathfrak{L}_\nabla^+(H)\times\mathfrak{L}_\nabla^+(H)})$$

to an adjoint pair

$$\iota_*^{\mathrm{IndCoh}} : \mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}}) \rightleftarrows \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) : \iota^!. \tag{E.11}$$

Since the essential image of the left adjoint in (E.10) generates the essential image, the same is true for (E.11).

In particular, images of compact objects in $\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}})$ under $\iota_*^{\mathrm{IndCoh}}$ provide compact generators of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$.

E.3. **Unital structure.**

E.3.1. We claim that the factorization categories we defined above, namely,

$$\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)) \text{ and } \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$$

carry naturally defined unital structures.

Let us carry out the construction for $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$; the case of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ will follow by taking $\mathfrak{L}_\nabla^+(H) \times \mathfrak{L}_\nabla^+(H)$-coinvariants.

E.3.2. By Sect. C.10.10, the factorization space $\mathfrak{L}_\nabla(H)$ carries a natural unital-in-correspondences structure. (Note, however, that we cannot deduce from there the unital structure on $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ by applying Sect. C.12.8 directly because we are not in an ind-placid situation.)

For an injection of finite sets $I_1 \subseteq I_2$ consider the corresponding diagram

$$\mathfrak{L}_\nabla(H)_{X^{I_1}} \overset{\mathrm{pr}^H_{\mathrm{small}}}{\longleftarrow} \mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(H)_{I_1\subseteq I_2} \overset{\mathrm{pr}^H_{\mathrm{big}}}{\longrightarrow} \mathfrak{L}_\nabla(H)_{X^{I_2}}. \tag{E.12}$$

E.3.3. We claim that the functor

$$(\mathrm{pr}^H_{\mathrm{small}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(H))_{I_1\subseteq I_2} \to \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{X^{I_1}}$$

admits a left adjoint, to be denoted $(\mathrm{pr}^H_{\mathrm{small}})^{*,\mathrm{IndCoh}}$.

Factor the map $\mathrm{pr}^H_{\mathrm{small}}$ as

$$\mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(H)_{I_1\subseteq I_2} \overset{'\mathrm{pr}^H_{\mathrm{small}}}{\longrightarrow} \mathfrak{L}_\nabla(H)_{X^{I_1}} \underset{X^{I_1}}{\times} X^{I_2} \to \mathfrak{L}_\nabla(H)_{X^{I_1}},$$

and it is sufficient to prove the existence of the left adjoint for the first arrow, i.e., that the functor

$$('\mathrm{pr}^H_{\mathrm{small}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(H)_{I_1\subseteq I_2}) \to \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)_{X^{I_1}} \underset{X^{I_1}}{\times} X^{I_2}) \tag{E.13}$$

admits a left adjoint.

E.3.4. We consider the two sides of (E.13) as acted on by

$$\mathfrak{L}^+_\nabla(H)_{X^{I_2}} \text{ and } \mathfrak{L}^+_\nabla(H)_{X^{I_1}} \underset{X^{I_1}}{\times} X^{I_2},$$

respectively. These actions are compatible via the the map

$$\mathfrak{L}^+_\nabla(H)_{X^{I_2}} \to \mathfrak{L}^+_\nabla(H)_{X^{I_1}} \underset{X^{I_1}}{\times} X^{I_2},$$

corresponding to the counital structure on $\mathfrak{L}^+_\nabla(H)$.

By Proposition E.1.4, it suffices to show that the functor

$$(\text{IndCoh}^*(\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(H)_{I_1\subseteq I_2}))_{\mathfrak{L}^+_\nabla(H)_{X^{I_2}}} \to \left(\text{IndCoh}^*(\mathfrak{L}_\nabla(H)_{X^{I_1}} \underset{X^{I_1}}{\times} X^{I_2})\right)_{\mathfrak{L}^+_\nabla(H)_{X^{I_1}} \underset{X^{I_1}}{\times} X^{I_2}}, \tag{E.14}$$

induced by (E.13), admits a left adjoint.

However, by Lemma E.2.10, the latter functor is the identity endofunctor of

$$\text{IndCoh}\left((\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{X^{I_1}} \underset{X^{I_1}}{\times} X^{I_2}\right).$$

E.3.5. We define the functor

$$\text{IndCoh}^*(\mathfrak{L}_\nabla(H))_{X^{I_1}} \to \text{IndCoh}^*(\mathfrak{L}_\nabla(H))_{X^{I_2}},$$

to be denoted $\text{ins.unit}_{I_1\subseteq I_2}$, to be

$$(\text{pr}^H_{\text{big}})^{\text{IndCoh}}_* \circ (\text{pr}^H_{\text{small}})^{*,\text{IndCoh}}.$$

E.3.6. In order to promote this to a unital structure on $\text{IndCoh}^*(\mathfrak{L}_\nabla(H))$, we need to construct isomorphisms

$$\text{ins.unit}_{I_2\subseteq I_2} \circ \text{ins.unit}_{I_1\subseteq I_2} \simeq \text{ins.unit}_{I_1\subseteq I_3}$$

for $I_1 \subseteq I_2 \subseteq I_3$.

Denote the maps in (E.12) by

$$\text{pr}^H_{\text{small},I_1\subseteq I_2} \text{ and } \text{pr}^H_{\text{big},I_1\subseteq I_2}$$

to indicate the dependence on the finite sets involved.

Thus, we need to construct an isomorphism

$$\begin{aligned}(\text{pr}^H_{\text{big},I_2\subseteq I_3})^{\text{IndCoh}}_* \circ (\text{pr}^H_{\text{small},I_2\subseteq I_3})^{*,\text{IndCoh}} \circ (\text{pr}^H_{\text{big},I_1\subseteq I_2})^{\text{IndCoh}}_* \circ (\text{pr}^H_{\text{small},I_1\subseteq I_1})^{*,\text{IndCoh}} \overset{\sim}{\to} \\ \overset{\sim}{\to} (\text{pr}^H_{\text{big},I_1\subseteq I_3})^{\text{IndCoh}}_* \circ (\text{pr}^H_{\text{small},I_1\subseteq I_3})^{*,\text{IndCoh}}.\end{aligned} \tag{E.15}$$

E.3.7. Note that we have a commutative diagram,

$$\begin{tikzcd}
\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(H)_{I_1\subseteq I_3} \arrow[r, "'\text{pr}^H_{\text{big},I_1\subseteq I_2}"] \arrow[d, "'\text{pr}^H_{\text{small},I_2\subseteq I_3}"'] & \mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(H)_{I_2\subseteq I_3} \arrow[r, "\text{pr}^H_{\text{big},I_2\subseteq I_3}"] \arrow[d, "\text{pr}^H_{\text{small},I_2\subseteq I_3}"] & \mathfrak{L}_\nabla(H)_{X^{I_3}} \\
\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(H)_{I_1\subseteq I_2} \arrow[r, "\text{pr}^H_{\text{big},I_1\subseteq I_2}"] \arrow[d, "\text{pr}^H_{\text{small},I_1\subseteq I_2}"'] & \mathfrak{L}_\nabla(H)_{X^{I_2}} & \\
\mathfrak{L}_\nabla(H)_{X^{I_1}}, & &
\end{tikzcd}$$

in which the inner square is Cartesian.

We rewrite the right-hand side in (E.15) as

$$(\text{pr}^H_{\text{big},I_2\subseteq I_3})^{\text{IndCoh}}_* \circ ('\text{pr}^H_{\text{big},I_1\subseteq I_2})^{\text{IndCoh}}_* \circ ('\text{pr}^H_{\text{small},I_2\subseteq I_3})^{*,\text{IndCoh}} \circ (\text{pr}^H_{\text{small},I_1\subseteq I_2})^{*,\text{IndCoh}}.$$

The isomorphism

$$(\mathrm{pr}^H_{\mathrm{big},I_1\subseteq I_2})^{\mathrm{IndCoh}}_* \circ ('\mathrm{pr}^H_{\mathrm{small},I_2\subseteq I_3})^{\mathrm{IndCoh}}_* \simeq (\mathrm{pr}^H_{\mathrm{small},I_2\subseteq I_3})^{\mathrm{IndCoh}}_* \circ ('\mathrm{pr}^H_{\mathrm{big},I_1\subseteq I_2})^{\mathrm{IndCoh}}_*$$

induces a natural transformation

$$(\mathrm{pr}^H_{\mathrm{small},I_2\subseteq I_3})^{*,\mathrm{IndCoh}} \circ (\mathrm{pr}^H_{\mathrm{big},I_1\subseteq I_2})^{\mathrm{IndCoh}}_* \to ('\mathrm{pr}^H_{\mathrm{big},I_1\subseteq I_2})^{\mathrm{IndCoh}}_* \circ ('\mathrm{pr}^H_{\mathrm{small},I_2\subseteq I_3})^{*,\mathrm{IndCoh}}. \tag{E.16}$$

E.3.8. We claim that (E.16) is an isomorphism. Indeed, this follows by the same argument as that proving the existence of $(\mathrm{pr}^H_{\mathrm{small}})^{*,\mathrm{IndCoh}}$ in Sects. E.3.3-E.3.4.

E.3.9. Finally, we define the natural isomorphism in (E.15) by precomposing the isomorphism (E.16) with $(\mathrm{pr}^H_{\mathrm{small},I_1\subseteq I_2})^{*,\mathrm{IndCoh}}$ and post-composing with $(\mathrm{pr}^H_{\mathrm{big},I_2\subseteq I_3})^{\mathrm{IndCoh}}_*$.

The higher compatibilities are constructed by a similar procedure.

E.3.10. By construction, the functor

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H)) \overset{\Psi_{\mathfrak{L}^+_\nabla(H))}}{\simeq} \mathrm{IndCoh}^*(\mathfrak{L}^+_\nabla(H)) \overset{\iota^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$$

is unital.

In particular, the object

$$\iota^{\mathrm{IndCoh}}_*(\mathcal{O}_{\mathfrak{L}^+_\nabla(H)}) \in \mathrm{FactAlg}(X, \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)))$$

is the factorization unit in $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$.

Similarly, the functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_H) \overset{\iota^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$$

is unital, and

$$\iota^{\mathrm{IndCoh}}_*(\mathcal{O}_{\mathrm{LS}^{\mathrm{reg}}_H}) \in \mathrm{FactAlg}(X, \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H))$$

is the factorization unit in $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$.

E.4. **Duality.**

E.4.1. Let S_α be as in Sect. B.13.9.

Note that the same argument as in Lemma E.2.7 shows that the functor

$$\Upsilon_{\mathfrak{L}^+_\nabla(H))_{S_\alpha}} : \mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_{S_\alpha} \to \mathrm{IndCoh}^!(\mathfrak{L}^+_\nabla(H))_{S_\alpha}$$

is an equivalence.

As a consequence, we obtain that the pairing

$$\mathrm{IndCoh}^!(\mathfrak{L}^+_\nabla(H))_{S_\alpha} \otimes \mathrm{IndCoh}^*(\mathfrak{L}^+_\nabla(H))_{S_\alpha} \to \mathrm{Vect}$$

of (A.26) is perfect.

E.4.2. Similarly, an argument parallel to that in Lemma E.2.10 shows that the !-pullback functor along

$$\mathrm{IndCoh}(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{S_\alpha} \to \mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H))_{S_\alpha}$$

gives rise to an equivalence

$$\mathrm{IndCoh}(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H))_{S_\alpha} \simeq \left(\mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H))_{S_\alpha}\right)^{\mathfrak{L}^+_\nabla(H)_{S_\alpha}}.$$

Combining with Proposition E.1.4, we obtain that the pairing

$$\mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H))_{S_\alpha} \otimes \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha} \to \mathrm{Vect}$$

of (A.26) is perfect.

E.4.3. In particular, we obtain that Lemmas B.13.11 and B.13.14 hold for $\mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H))$. I.e., the recipe in Sects. B.13.10-B.13.15 gives rise to a well-defined factorization category $\mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H))$.

Moreover, we obtain that (A.26) defines a perfect pairing between

$$\mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H)) \text{ and } \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$$

as factorization categories.

E.4.4. By a similar logic as in Sect. E.2.12, we obtain that the assignment

$$S_\alpha \rightsquigarrow \mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}})_{S_\alpha}$$

extends to a well-defined factorization category $\mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$.

Moreover, we have

$$\mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \simeq \mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H))^{\mathfrak{L}^+_\nabla(H)\times\mathfrak{L}^+_\nabla(H)}$$

and

$$\mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \underset{\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}})}{\otimes} \mathrm{Vect} \simeq \mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)).$$

We obtain that

$$\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \text{ and } \mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$$

are mutually dual as factorization categories.

E.4.5. Finally, a procedure dual to that in Sect. E.3 defines on

$$\mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H)) \text{ and } \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$$

unital structures, and the identifications

$$\mathrm{IndCoh}^!(\mathfrak{L}_\nabla(H))^\vee \simeq \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)) \text{ and } \mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}})^\vee \simeq \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$$

extend to identifications of the corresponding unital factorization categories.

E.5. **t-structures.** In this section we will discuss an alternative approach to the definition of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$.

Namely, we can start with (the more elementary) $\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$, and obtain from it $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ by a *renormalization procedure* (i.e., ind-completion of a specified small subcategory), see Sect. E.5.6 below.

E.5.1. As in Sect. B.13.22, the factorization category

$$\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$$

carries a naturally defined t-structure.

By construction, the functor

$$\Gamma^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)) \to \mathrm{Vect}$$

is t-exact, and conservative when restricted to the eventually coconnective subcategory.

Moreover, the functor

$$\iota_*^{\mathrm{IndCoh}} : \mathrm{QCoh}(\mathfrak{L}^+_\nabla(H)) \to \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$$

is t-exact.

E.5.2. We now define a t-structure on $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$. Namely, by Proposition E.1.4 the projection

$$\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)) \to \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$$

admits a left adjoint, which is comonadic.

Moreover, the resulting comonad on $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ is t-exact. This implies that the category $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ acquires a unique t-structure for which both functors

$$\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \rightleftarrows \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$$

are t-exact.

E.5.3. Consider the untal factorization categories

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)),\ \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)) \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}}).$$

The category $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H))$ carries a natural action of $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))$, and it follows formally that the functors

$$(\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)))_{\mathfrak{L}^+_\nabla(H)\times\mathfrak{L}^+_\nabla(H)} \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \tag{E.17}$$

and

$$(\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)))_{\mathfrak{L}^+_\nabla(H)} \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)), \tag{E.18}$$

induced by the direct image functors

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)) \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)) \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)),$$

respectively, are equivalences.

In addition, we have the unital factorization functors

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H)) \xrightarrow{\iota_*} \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)) \text{ and } \mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}}) \simeq \mathrm{QCoh}_{\mathrm{co}}(\mathrm{LS}_H^{\mathrm{reg}}) \xrightarrow{\iota_*} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}}).$$

E.5.4. Let S_α be as in Sect. B.13.9. Recall (see Lemma A.8.10) that the functor

$$\Psi_{\mathfrak{L}_\nabla(H)_{S_\alpha}} : \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha} \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H))_{S_\alpha}$$

is t-exact, and induces an equivalence between the eventually coconnective subcategories on both sides.

It follows from the definition of $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ as a factorization category that the functors $\Psi_{\mathfrak{L}_\nabla(H)_{S_\alpha}}$ combine to give rise to a factorization functor

$$\Psi_{\mathfrak{L}_\nabla(H)} : \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)) \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)).$$

Moreover, the functor $\Psi_{\mathfrak{L}_\nabla(H)}$ is t-exact and induces an equivalences between the eventually coconnective subcategories on both sides.

Furthermore, the functor $\Psi_{\mathfrak{L}_\nabla(H)}$ has a naturally defined unital structure.

E.5.5. Note that the contents of Sect. E.5.4 allows us to recover $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$, as a unital factorization category, from $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H))$ with its t-structure.

Namely, for $S \to \mathrm{Ran}$, the category $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_S$ identifies with the ind-completion of the full subcategory of $\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H))_S$, generated by *finite* colimits by the essential image of $\mathrm{QCoh}(\mathfrak{L}^+_\nabla(H))_S$ along ι_*.

E.5.6. It follows formally from Sect. E.5.2 that we have a naturally defined t-exact unital factorization functor

$$\Psi_{\mathrm{Hecke}_H^{\mathrm{spec,loc}}} : \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}}),$$

which induces an equivalences between the eventually coconnective subcategories on both sides.

Furthermore, as in Sect. E.5.5, we can recover $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ (as a unital factorization category) from $\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ with its t-structure.

Namely, for $S \to \mathrm{Ran}$, the category $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})_S$ identifies with the ind-completion of the full subcategory of $(\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}})_S)^{>-\infty}$, generated by *finite* colimits by the essential image of $\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}})_S$ along ι_*.

E.6. **The monoidal structure.**

E.6.1. Let S_α be as in Sect. B.13.9. The group-scheme structure on $\mathfrak{L}_\nabla(H)_{S_\alpha}$ induces on the category $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha}$ a structure of monoidal category (under convolution).

By the construction of $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ as a factorization category, we obtain that $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ acquires a structure of monoidal factorization category.

This structure is compatible with the unital structure on $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ in the sense that the monoidal operation has a natural lax unital structure. I.e., $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ is an associative algebra object in the symmetric monoidal category of unital factorization categories with lax unital functors as morphisms.

E.6.2. Note that for S_α as above, the monoidal operation on $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha}$, viewed as a functor

$$\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha} \underset{\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha} \to \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))_{S_\alpha}$$

is t-exact.

Hence, the monoidal operation on $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$ is t-exact.

E.6.3. We will now descend the above monoidal structure to one on $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$.

Consider the correspondence

$$(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}) \times (\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}) \overset{\Delta_{\mathrm{LS}_H^{\mathrm{reg}}}}{\longleftarrow} \\ \leftarrow \mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}} \overset{\mathrm{mult}}{\to} \mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}. \tag{E.19}$$

Note that by Lemma E.2.2 we can think of this diagram also as

$$(\mathfrak{L}^+_\nabla(H)\backslash\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)) \times (\mathfrak{L}^+_\nabla(H)\backslash\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)) \leftarrow \\ \leftarrow \mathfrak{L}^+_\nabla(H)\backslash \left(\underset{\mathfrak{L}^+_\nabla(H)}{\underline{\mathfrak{L}_\nabla(H) \times \mathfrak{L}_\nabla(H)}} \right) /\mathfrak{L}^+_\nabla(H)) \to \mathfrak{L}^+_\nabla(H)\backslash\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H), \tag{E.20}$$

where:

- $\underset{\mathfrak{L}^+_\nabla(H)}{\underline{\mathfrak{L}_\nabla(H) \times \mathfrak{L}_\nabla(H)}}$ denotes the quotient with respect to the diagonal action by right multiplication along the left factor and the left multiplication along the right factor;
- The arrow $\to$ is induced by the product map

$$\underset{\mathfrak{L}^+_\nabla(H)}{\underline{\mathfrak{L}_\nabla(H) \times \mathfrak{L}_\nabla(H)}} \to \mathfrak{L}_\nabla(H).$$

E.6.4. As in Sect. E.2, we obtain a well-defined factorization category

$$\mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}),$$

equipped with factorization functors

$$\mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}) \otimes \mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}) \overset{(\Delta_{\mathrm{LS}_H^{\mathrm{reg}}})_*^{\mathrm{IndCoh}}}{\longleftarrow}$$
$$\leftarrow \mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}) \overset{(\mathrm{mult})_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}). \tag{E.21}$$

It is easy to see that the functor $(\Delta_{\mathrm{LS}_H^{\mathrm{reg}}})_*^{\mathrm{IndCoh}}$ admits a left adjoint, to be denoted $(\Delta_{\mathrm{LS}_H^{\mathrm{reg}}})^{*,\mathrm{IndCoh}}$. We define the monoidal structure on $\mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}})$ with the binary operation given by

$$(\mathrm{mult})_*^{\mathrm{IndCoh}} \circ (\Delta_{\mathrm{LS}_H^{\mathrm{reg}}})^{*,\mathrm{IndCoh}}.$$

One defines similarly n-fold compositions, and they form a compatible system thanks to the fact that the functors $(\Delta_{\mathrm{LS}_H^{\mathrm{reg}}})^{*,\mathrm{IndCoh}}$ satisfy base change against IndCoh-pushforwards.

This defines on

$$\mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}) =: \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$$

a structure of monoidal factorization category.

As in the case of $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$, it is easy to see that the monoidal operation on the category $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ is t-exact.

E.6.5. A similar procedure gives rise to an action of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ on $\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}})$.

E.6.6. By construction, the monoidal structure on $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ is compatible with the unital structure, in the sense that $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ is an associative algebra object in the symmetric monoidal category of unital factorization categories with lax unital functors as morphisms.

However, we claim that, unlike $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$, in the case of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ more is true: namely, the monoidal operation is strictly unital.

Indeed, this follows from Lemma C.11.23, since the factorization unit for $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$, namely, $\iota_*^{\mathrm{IndCoh}}(\mathcal{O}_{\mathrm{LS}_H^{\mathrm{reg}}})$ is also the monoidal unit.

The same observation applies to the action of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ on $\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}})$. .

E.6.7. A similar procedure defines the monoidal factorization categories

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H)) \text{ and } \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}}), \tag{E.22}$$

with t-exact monoidal operations.

We note that the procedure Sect. E.5.6 allows us to recover

$$\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)) \text{ and } \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \tag{E.23}$$

as *monoidal factorization categories* from those in (E.22).

Namely, these monoidal structures are uniquely determined by the requirement that that the functors

$$\Psi_{\mathfrak{L}_\nabla(H)} : \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)) \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H))$$

and

$$\Psi_{\mathrm{Hecke}_H^{\mathrm{spec,loc}}} : \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$$

are monoidal.

E.6.8. Note that the functor $(\mathrm{mult})_*^{\mathrm{IndCoh}}$, involved in the definition of the monoidal structure on $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ admits a (continuous) right adjoint, to be denoted $(\mathrm{mult})^!$.

To prove this, it suffices to show that the corresponding functor

$$\mathrm{IndCoh}^*\left(\underset{\mathfrak{L}^+_\nabla(H)}{\mathfrak{L}_\nabla(H) \times \mathfrak{L}_\nabla(H)}\right) \overset{(\mathrm{mult})_*^{\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H))$$

admits a (continuous) right adjoint.

However, we can isomorph the projection

$$\underset{\mathfrak{L}^+_\nabla(H)}{\mathfrak{L}_\nabla(H) \times \mathfrak{L}_\nabla(H)} \overset{\mathrm{mult}}{\longrightarrow} \mathfrak{L}_\nabla(H)$$

to the projection

$$\mathfrak{L}_\nabla(H) \times (\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)) \to \mathfrak{L}_\nabla(H),$$

and the assertion follows from the fact that $\mathfrak{L}_\nabla(H)/\mathfrak{L}^+_\nabla(H)$ is ind-proper.

E.6.9. In particular, we obtain that the monoidal operation on $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ admits a continuous right adjoint, namely,

$$(\Delta_{\mathrm{LS}_H^{\mathrm{reg}}})_*^{\mathrm{IndCoh}} \circ (\mathrm{mult})^!.$$

Furthermore, it easy to see that this right adjoint

$$\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \to \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \otimes \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$$

is compatible with the $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$-bimodule structure.

Since the monoidal unit in $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ is compact, we obtain that $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ is *rigid* (see [GaRo3, Chapter 1, Definition 9.1.2] for what this means), i.e., for any $S \to \mathrm{Ran}$, the monoidal category

$$\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})_S$$

is rigid.

E.6.10. Passing to duals, the monoidal structure on $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ induces a comonoidal structure on its dual, i.e., $\mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$. Since $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ is rigid, the comonoidal operation on $\mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ admits a left adjoint, i.e., $\mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ is naturally a monoidal category.

Let us describe the monoidal operation on $\mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ explicitly. In terms of (E.19), it is given by

$$(\mathrm{mult})_*^{\mathrm{IndCoh}} \circ (\Delta_{\mathrm{LS}_H^{\mathrm{reg}}})^!,$$

where $(\mathrm{mult})_*^{\mathrm{IndCoh}}$ is the left adjoint of

$$\mathrm{mult}^! : \mathrm{IndCoh}^!(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}) \to \mathrm{IndCoh}^!(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}),$$

or, which is the same as the dual of

$$(\mathrm{mult})^! : \mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}) \to \mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}}),$$

whose existence was proved above.

E.6.11. Since $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ is rigid, by [GaRo3, Sect. 9.2.1], a choice of "right" or "left" determines an equivalence

$$\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \simeq \mathrm{IndCoh}^!(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \tag{E.24}$$

as monoidal categories.

E.7. **Action on** $\mathrm{IndCoh}(\mathrm{LS}_H)$**.** In this subsection we will define a (local action) of the monoidal factorization category $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ on $\mathrm{IndCoh}(\mathrm{LS}_H)$ in the sense of Sect. H.6.1.

E.7.1. Our goal is to define an action of the monoidal category $(\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}))^{\overset{!}{\otimes}}_{\mathrm{Ran}}$ (see Sect. H.5.5 for the notation) on $\mathrm{IndCoh}(\mathrm{LS}_H) \otimes \underline{\text{D-mod}}(\mathrm{Ran})$. In other words, we need to define an action of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})_S$ on $\mathrm{IndCoh}(\mathrm{LS}_H) \otimes \mathrm{QCoh}(S)$ for any $S \to \mathrm{Ran}$.

Let $\mathrm{Hecke}^{\mathrm{spec,glob}}_{\check{G},S}$ denote the fiber product

$$(\mathrm{LS}_H \times S) \underset{\mathrm{LS}^{\mathrm{mer,glob}}_{H,S}}{\times} (\mathrm{LS}_H \times S),$$

where

$$\mathrm{LS}^{\mathrm{mer,glob}}_{H,S} := S \underset{\mathrm{Ran}}{\times} \mathrm{LS}^{\mathrm{mer,glob}}_{H,\mathrm{Ran}}$$

for $\mathrm{LS}^{\mathrm{mer,glob}}_{H,\mathrm{Ran}}$ as in Sect. B.7.14.

Restriction to the formal disc gives rise to vertical arrows in the following diagram, see Sect. B.7.15:

$$\begin{tikzcd}
\mathrm{LS}_H \times S \arrow[d, "\mathrm{ev}_S"'] & \mathrm{Hecke}^{\mathrm{spec,glob}}_{H,S} \arrow[l, "\overleftarrow{h}^{\mathrm{spec,glob}}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{spec,glob}}"] \arrow[d, "\mathrm{ev}_S"'] & \mathrm{LS}_H \times S \arrow[d, "\mathrm{ev}_S"] \\
\mathrm{LS}^{\mathrm{reg}}_{H,S} & \mathrm{Hecke}^{\mathrm{spec,loc}}_{H,S} \arrow[l, "\overleftarrow{h}^{\mathrm{spec,loc}}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{spec,loc}}"] & \mathrm{LS}^{\mathrm{reg}}_{H,S}.
\end{tikzcd} \tag{E.25}$$

Lemma E.7.2. *Both squares in* (E.25) *are Cartesian.*

Proof. First, we claim that both squares are Cartesian at the classical level. Indeed, classically, the horizontal arrows in (E.25) are isomorphisms.

Hence, in order to prove the lemma, it remains to check the Cartesian property of the tangent spaces on the unit section of $\mathrm{Hecke}^{\mathrm{spec,glob}}_{H,S}$.

Let σ be a point of LS_H. Then for any $s \in S$, the relative tangent space of

$$\mathrm{Hecke}^{\mathrm{spec,glob}}_{H,S} \to \mathrm{LS}_H \times S$$

at (the image along the unit section of) (σ, s) identifies with

$$\mathrm{Fib}\left(\mathrm{C}^{\cdot}(X, \mathfrak{h}_\sigma)[1] \to \mathrm{C}^{\cdot}(X - \underline{x}, \mathfrak{h}_\sigma)[1]\right) \simeq \mathrm{coFib}\left(\mathrm{C}^{\cdot}(X, \mathfrak{h}_\sigma) \to \mathrm{C}^{\cdot}(X - \underline{x}, \mathfrak{h}_\sigma)\right),$$

where $\underline{x} \in \mathrm{Ran}$ is the image of s.

The relative tangent space of

$$\mathrm{Hecke}^{\mathrm{spec,loc}}_{H,S} \to \mathrm{LS}^{\mathrm{reg}}_{H,S}$$

at the image of this point identifies with

$$(\mathfrak{L}_\nabla(\mathfrak{h}_\sigma)/\mathfrak{L}^+_\nabla(\mathfrak{h}_\sigma))_{\underline{x}}.$$

The required Cartesian property of the tangent spaces follows from the fact that diagram

$$\begin{tikzcd}
\mathrm{C}^{\cdot}(X, \mathfrak{h}_\sigma) \arrow[r] \arrow[d] & \mathrm{C}^{\cdot}(X - \underline{x}, \mathfrak{h}_\sigma) \arrow[d] \\
\mathfrak{L}^+_\nabla(\mathfrak{h}_\sigma)_{\underline{x}} \arrow[r] & \mathfrak{L}_\nabla(\mathfrak{h}_\sigma)_{\underline{x}}
\end{tikzcd}$$

is Cartesian.

□

E.7.3. For $S \to \mathrm{Ran}$ as above, denote

$$(\mathrm{LS}_H \times S)^{\mathrm{level}} := (\mathrm{LS}_H \times S) \underset{\mathrm{LS}^{\mathrm{reg}}_{H,S}}{\times} S,$$

where $S \to \mathrm{LS}^{\mathrm{reg}}_{H,S}$ is the unit point.

By construction, $(\mathrm{LS}_H \times S)^{\mathrm{level}}$ is acted on by $\mathfrak{L}^+_\nabla(H)_S$, so that

$$(\mathrm{LS}_H \times S)^{\mathrm{level}}/\mathfrak{L}^+_\nabla(H)_S \simeq \mathrm{LS}_H \times S.$$

Note that Lemma E.7.2 can be reformulated as saying that the above action of $\mathfrak{L}^+(H)_S$ on $(\mathrm{LS}_H \times S)^{\mathrm{level}}$ extends to an action of $\mathfrak{L}(H)_S$.

E.7.4. Let S_α be as in Sect. B.13.9. Consider the category

$$\mathrm{IndCoh}^*((\mathrm{LS}_H \times S_\alpha)^{\mathrm{level}}).$$

The action of $\mathfrak{L}_\nabla(H)_{S_\alpha}$ on $(\mathrm{LS}_H \times S_\alpha)^{\mathrm{level}}$ gives rise to an action of the monoidal category $\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H)_{S_\alpha})$ on $\mathrm{IndCoh}^*((\mathrm{LS}_H \times S_\alpha)^{\mathrm{level}})$.

As in Lemma E.2.7, the IndCoh-pushforward functor

$$\mathrm{IndCoh}^*((\mathrm{LS}_H \times S_\alpha)^{\mathrm{level}}) \to \mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha)$$

gives rise to an equivalence

$$\Big(\mathrm{IndCoh}^*((\mathrm{LS}_H \times S_\alpha)^{\mathrm{level}})\Big)_{\mathfrak{L}^+_\nabla(H)_{S_\alpha}} \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha).$$

Then by the same mechanism as in Sect. E.6.4, we obtain an action of the monoidal category $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)_{S_\alpha}$ on $\mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha)$.

E.7.5. Note that we can explicitly describe the action functor as follows:

$$\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)_{S_\alpha} \underset{\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha) \overset{(\mathrm{ev}_S \times \overrightarrow{h}^{\mathrm{spec,glob}})^{*,\mathrm{IndCoh}}}{\longrightarrow}$$
$$\to \mathrm{IndCoh}((\mathrm{Hecke}^{\mathrm{spec,glob}}_H)_{S_\alpha}) \overset{(\overleftarrow{h}^{\mathrm{spec,glob}})^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha), \tag{E.26}$$

where the first arrow is obtained by identifying

$$\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)_{S_\alpha} \underset{\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha) \simeq$$
$$\simeq \left(\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H) \underset{S_\alpha}{\times} (\mathrm{LS}_H \times S_\alpha)^{\mathrm{level}})\right)_{(\mathfrak{L}^+(H)\times\mathfrak{L}^+(H)\times\mathfrak{L}^+(H))_{S_\alpha}}$$

and

$$\mathrm{IndCoh}((\mathrm{Hecke}^{\mathrm{spec,glob}}_H)_{S_\alpha}) \simeq \left(\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(H) \underset{S_\alpha}{\times} (\mathrm{LS}_H \times S_\alpha)^{\mathrm{level}})\right)_{(\mathfrak{L}^+(H)\times\mathfrak{L}^+(H))_{S_\alpha}},$$

and the functor $(\mathrm{ev}_S \times \overrightarrow{h}^{\mathrm{spec,glob}})^{*,\mathrm{IndCoh}}$ is the left adjoint to the projection from $(\mathfrak{L}^+(H) \times \mathfrak{L}^+(H))_{S_\alpha}$-coinvariants to $(\mathfrak{L}^+(H) \times \mathfrak{L}^+(H) \times \mathfrak{L}^+(H))_{S_\alpha}$-coinvariants.

E.7.6. Having defined the action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)_{S_\alpha}$ on $\mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha)$, the procedure in Sects. B.13.17-B.13.19 defines an action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)_S$ on $\mathrm{IndCoh}(\mathrm{LS}_H \times S)$ for any $S \to \mathrm{Ran}$.

Thus, we obtain the sought-for local action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ on $\mathrm{IndCoh}(\mathrm{LS}_H)$. Furthermore, unwinding the construction, we obtain that this action has a natural Ran-unital structure (see Sect. H.6.1 for what this means).

E.7.7. Recall the functor

$$\mathrm{Loc}^{\mathrm{spec}}_{H,\mathrm{Ran}} : \mathrm{Rep}(H)_{\mathrm{Ran}} \to \mathrm{IndCoh}(\mathrm{LS}_H) \otimes \underline{\text{D-mod}}(\mathrm{Ran}).$$

We claim:

Proposition E.7.8. *The functor* $\mathrm{Loc}^{\mathrm{spec}}_{H,\mathrm{Ran}}$ *intertwines the actions of* $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)^{\overset{!}{\otimes}}_{\mathrm{Ran}}$ *on the two sides.*

Proof. Unwinding the construction, we need to construct the datum of compatibility for the functor

$$\mathrm{Loc}^{\mathrm{spec}}_{H,S_\alpha} : \mathrm{Rep}(H)_{S_\alpha} \to \mathrm{IndCoh}(\mathrm{LS}_H) \otimes \mathrm{QCoh}(S_\alpha)$$

for S_α as in Sect. B.13.9.

We identify

$$\mathrm{Rep}(H)_{S_\alpha} \simeq \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,S_\alpha}),$$

so that the functor $\mathrm{Loc}^{\mathrm{spec}}_{H,S_\alpha}$ identifies with the functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,S_\alpha}) \overset{(\mathrm{ev}_{S_\alpha})^{*,\mathrm{IndCoh}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{LS}_H \otimes S_\alpha) \simeq \mathrm{IndCoh}(\mathrm{LS}_H) \otimes \mathrm{QCoh}(S_\alpha).$$

Now the assertion of the proposition follows by unwinding the constructions, using the fact that diagram (E.25) is Cartesian (see Lemma E.7.2).

□

E.7.9. Consider now the functor

$$\Gamma^{\mathrm{spec,IndCoh}}_{H,\mathrm{Ran}} : \mathrm{IndCoh}(\mathrm{LS}_H) \otimes \text{D-mod}(\mathrm{Ran}) \to \mathrm{Rep}(H)_{\mathrm{Ran}},$$

right adjoint to $\mathrm{Loc}^{\mathrm{spec}}_{H,\mathrm{Ran}}$.

Since $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ is rigid, from Proposition E.7.8, we obtain:

Corollary E.7.10. *The functor* $\Gamma^{\mathrm{spec,IndCoh}}_{H,\mathrm{Ran}}$ *intertwines the actions of* $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)^{\overset{!}{\otimes}}_{\mathrm{Ran}}$ *on the two sides.*

E.7.11. The (local) action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ on $\mathrm{IndCoh}(\mathrm{LS}_H)$ gives rise to a (local) *right* action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ on the dual $\mathrm{IndCoh}(\mathrm{LS}_H)^\vee$ of $\mathrm{IndCoh}(\mathrm{LS}_H)$.

We identify

$$\mathrm{IndCoh}(\mathrm{LS}_H)^\vee \simeq \mathrm{IndCoh}(\mathrm{LS}_H)$$

by Serre duality. Thus, we obtain a new (local) right action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ on $\mathrm{IndCoh}(\mathrm{LS}_H)$.

Note now that we can pass between right and left modules over $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ using the anti-involution σ^{spec}, induced by the inversion operation of $\mathfrak{L}_\nabla(H)$.

Unwinding the construction, we obtain that the resulting new (local) action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ on $\mathrm{IndCoh}(\mathrm{LS}_H)$ coincides with the original one.

E.7.12. The (local) action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ on $\mathrm{IndCoh}(\mathrm{LS}_H)$ gives rise to a (local) *right coaction* of the factorization comonidal category $\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ on $\mathrm{IndCoh}(\mathrm{LS}_H)$.

Since $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$ is rigid, the coaction functor admits a left adjoint, so we obtain a (local) *right action* of $\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)$, viewed as a factorization monidal category, on $\mathrm{IndCoh}(\mathrm{LS}_H)$.

The corresponding monoidal operation is described explicitly as follows.

For S_α as in Sect. B.13.9, the action functor

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_H)_{S_\alpha} \underset{\mathrm{QCoh}(S_\alpha)}{\otimes} \mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha) \to \mathrm{IndCoh}(\mathrm{LS}_H \times S_\alpha)$$

is given by

$$\text{(E.27)}\quad \operatorname{IndCoh}(\operatorname{LS}_H \times S_\alpha) \underset{\operatorname{QCoh}(S_\alpha)}{\otimes} \operatorname{IndCoh}^!(\operatorname{Hecke}^{\text{spec,loc}}_H)_{S_\alpha} \simeq$$
$$\simeq \operatorname{IndCoh}^!\left((\operatorname{LS}_H \times S_\alpha) \underset{S_\alpha}{\times} (\operatorname{Hecke}^{\text{spec,loc}}_H)_{S_\alpha}\right) \overset{(\overleftarrow{h}^{\text{spec,glob}}\times \text{ev})^!}{\longrightarrow}$$
$$\to \operatorname{IndCoh}((\operatorname{Hecke}^{\text{spec,glob}}_H)_{S_\alpha}) \overset{(\overrightarrow{h}^{\text{spec,glob}})^{\operatorname{IndCoh}}_*}{\longrightarrow} \operatorname{IndCoh}(\operatorname{LS}_H \times S_\alpha).$$

For an arbitrary $S \to \operatorname{Ran}$, this action is extended by the mechanism of Sects. B.13.10-B.13.15.

E.7.13. Recall now that according to (E.24), we can identify $\operatorname{IndCoh}^!(\operatorname{Hecke}^{\text{spec,loc}}_H)$ as a monoidal category with $\operatorname{IndCoh}^*(\operatorname{Hecke}^{\text{spec,loc}}_H)$.

It is a formal property of actions of rigid categories that with respect to this identification, the above right action of $\operatorname{IndCoh}^!(\operatorname{Hecke}^{\text{spec,loc}}_H)$ on $\operatorname{IndCoh}(\operatorname{LS}_H)$ identifies with the right action of $\operatorname{IndCoh}^*(\operatorname{Hecke}^{\text{spec,loc}}_H)$ on $\operatorname{IndCoh}(\operatorname{LS}_H)$ from Sect. E.7.11.

E.8. **Action on monodromy-free opers.** In this subsection we take $H = \check{G}$, the Langlands dual of a reductive group G. We will construct a factorization version of the action of $\operatorname{IndCoh}^*(\operatorname{Hecke}^{\text{spec,loc}}_{\check{G}})$ on $\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})$.

E.8.1. Let $\operatorname{Hecke}^{\text{spec},\operatorname{Op}^{\text{mon-free}}_{\check{G}}}_{\check{G}}$ be the factorization ind-scheme, defined as

$$\operatorname{Hecke}^{\text{spec},\operatorname{Op}^{\text{mon-free}}_{\check{G}}}_{\check{G}} := \operatorname{Op}^{\text{mer}}_{\check{G}} \underset{\operatorname{LS}^{\text{mer}}_{\check{G}}}{\times} \operatorname{Hecke}^{\text{spec,loc}}_{\check{G}}.$$

Note that we have a commutative diagram

$$\text{(E.28)}\qquad \begin{CD} \operatorname{Op}^{\text{mon-free}}_{\check{G}} @<{\overleftarrow{h}^{\text{spec,Op}}}<< \operatorname{Hecke}^{\text{spec},\operatorname{Op}^{\text{mon-free}}_{\check{G}}}_{\check{G}} @>{\overrightarrow{h}^{\text{spec,Op}}}>> \operatorname{Op}^{\text{mon-free}}_{\check{G}} \\ @V{\mathfrak{r}}VV @VV{\mathfrak{r}}V @VV{\mathfrak{r}}V \\ \operatorname{LS}^{\text{reg}}_{\check{G}} @<{\overleftarrow{h}^{\text{spec,loc}}}<< \operatorname{Hecke}^{\text{spec,loc}}_{\check{G}} @>{\overrightarrow{h}^{\text{spec,loc}}}>> \operatorname{LS}^{\text{reg}}_{\check{G}}, \end{CD}$$

in which both arrows are Cartesian.

From here, by the same mechanism as in Sect. E.6.4, we obtain an action of the monoidal category $\operatorname{IndCoh}^*(\operatorname{Hecke}^{\text{spec,loc}}_H)$ on $\operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}})$.

We write the action functor symbolically as

$$\operatorname{IndCoh}^*(\operatorname{Hecke}^{\text{spec,loc}}_H) \otimes \operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}}) \overset{(\mathfrak{r}\times \overrightarrow{h}^{\text{spec,Op}})^{*,\operatorname{IndCoh}}}{\longrightarrow}$$
$$\to \operatorname{IndCoh}^*(\operatorname{Hecke}^{\text{spec},\operatorname{Op}^{\text{mon-free}}_{\check{G}}}_{\check{G}}) \overset{(\overleftarrow{h}^{\text{spec,Op}})^{\operatorname{IndCoh}}_*}{\longrightarrow} \operatorname{IndCoh}^*(\operatorname{Op}^{\text{mon-free}}_{\check{G}}),$$

where the functor $(\mathfrak{r} \times \overrightarrow{h}^{\text{spec,Op}})^{*,\operatorname{IndCoh}}$ is assigned a meaning as in Sect. E.7.12.

Remark E.8.2. Note that $\operatorname{Hecke}^{\text{spec},\operatorname{Op}^{\text{mon-free}}_{\check{G}}}_{\check{G}}$ is ind-placid. Indeed, this follows from the fact that $\operatorname{Op}^{\text{mon-free}}_{\check{G}}$ is placid, combined with the fact that the map $\overleftarrow{h}^{\text{spec,loc}}$ is locally almost of finite presentation (see Lemma E.2.4).

E.8.3. Recall the functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{Ran}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathrm{Ran}}) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \text{D-mod}(\mathrm{Ran}),$$

see Sect. 17.4.2.

We claim:

Proposition E.8.4. *The functor* $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{Ran}}$ *intertwines the actions of* $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ *on the two sides.*

Proof. Unwinding the construction, we need to show that the functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,S_\alpha} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})_S \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{QCoh}(S_\alpha)$$

is compatible with the action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})_{S_\alpha}$ for S_α as in Sect. B.13.9.

Recall that the functor $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,S_\alpha}$ is the composition of:

- *-pullback along
$$\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},S_\alpha} \xrightarrow{\mathrm{ev}_{S_\alpha}} \mathrm{Op}^{\text{mon-free}}_{\check{G},S_\alpha};$$
- IndCoh-pushforward along $\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},S_\alpha} \xrightarrow{\mathfrak{r}^{\mathrm{glob}}} \mathrm{LS}_{\check{G}} \times S_\alpha$.

Denote

$$\mathrm{Hecke}^{\mathrm{spec,glob,Op}^{\text{mon-free}}_{\check{G}}}_{\check{G},S_\alpha} := \mathrm{Op}^{\mathrm{mer,glob}}_{\check{G},S_\alpha} \underset{\mathrm{LS}^{\mathrm{mer,glob}}_{\check{G},S_\alpha}}{\times} \mathrm{Hecke}^{\mathrm{spec,glob}}_{H,S_\alpha}.$$

The assertion of the proposition holds by unwinding the constructions from the fact that in the following diagrams both square are Cartesian

$$\begin{tikzcd}
\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},S_\alpha} \arrow[d, "\mathrm{ev}"'] & \mathrm{Hecke}^{\mathrm{spec,glob,Op}^{\text{mon-free}}_{\check{G}}}_{\check{G},S_\alpha} \arrow[l, "\overleftarrow{h}^{\mathrm{spec,Op}}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{spec,Op}}"] \arrow[d, "\mathrm{ev}"'] & \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},S_\alpha} \arrow[d, "\mathrm{ev}"] \\
\mathrm{Op}^{\text{mon-free}}_{\check{G},S_\alpha} & \mathrm{Hecke}^{\mathrm{spec,Op}^{\text{mon-free}}_{\check{G}}}_{\check{G},S_\alpha} \arrow[l, "\overleftarrow{h}^{\mathrm{spec,glob,Op}}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{spec,glob,Op}}"] & \mathrm{Op}^{\text{mon-free}}_{\check{G},S_\alpha}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Op}^{\text{mon-free,glob}}_{\check{G},S_\alpha} \arrow[d, "\mathfrak{r}^{\mathrm{glob}}"'] & \mathrm{Hecke}^{\mathrm{spec,glob,Op}^{\text{mon-free}}_{\check{G}}}_{\check{G},S_\alpha} \arrow[l, "\overleftarrow{h}^{\mathrm{spec,Op}}"'] \arrow[r, "\overleftarrow{h}^{\mathrm{spec,Op}}"] \arrow[d, "\mathfrak{r}^{\mathrm{glob}}"'] & \mathrm{Op}^{\text{mon-free,glob}}_{\check{G},S_\alpha} \arrow[d, "\mathfrak{r}^{\mathrm{glob}}"] \\
\mathrm{LS}_{\check{G}} & \mathrm{Hecke}^{\mathrm{spec,glob}}_{\check{G}} \arrow[l, "\overleftarrow{h}^{\mathrm{spec,loc}}"'] \arrow[r, "\overrightarrow{h}^{\mathrm{spec,loc}}"] & \mathrm{LS}_{\check{G}}\,.
\end{tikzcd}$$

□

E.8.5. The action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ gives rise to a *right* action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ on the dual of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$. Identifying

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}),$$

we thus obtain a right action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ on $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$.

Applying the anti-involution σ^{spec}, we can turn this right action into a left action.

E.8.6. Passing to dual functors, from the action of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ on $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$, we obtain a right coaction of $\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ on $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$. By rigidity, the coaction functor admits a left adjoint, i.e., we obtain a right action of $\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$, viewed as a monoidal category, on $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$.

The corresponding action functor is explicitly given by

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}) \otimes \mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}) \overset{(\overleftarrow{h}^{\mathrm{spec,Op}} \times \mathfrak{r})^{*,\mathrm{IndCoh}}}{\longrightarrow}$$

$$\to \mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,Op}^{\mathrm{mon\text{-}free}}_{\check{G}}}_{\check{G}}) \overset{(\overrightarrow{h}^{\mathrm{spec,Op}})^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}).$$

Using the anti-involution σ^{spec} we can turn the above right action of $\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ on $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$ into a left action.

E.8.7. As in Sect. E.7.13, it follows formally that with respect to the identification (E.24), the above right action of $\mathrm{IndCoh}^!(\mathrm{Hecke}\,\check{G}^{\mathrm{spec,loc}})$ on $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$ identifies with the right action of $\mathrm{IndCoh}^*(\mathrm{Hecke}\,\check{G}^{\mathrm{spec,loc}})$ on $\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})$ from Sect. E.8.5.

E.8.8. In a way analogous to Proposition E.8.4, one proves:

Proposition E.8.9. *The functor* $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\mathrm{Ran}}$ *intertwines the actions of* $\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ *on the two sides.*

E.8.10. Note that Lemma 3.7.17 adapts to the factorization setting as follows:

Lemma E.8.11. *The equivalence*

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\mathrm{Ran}}) \overset{\Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\mathrm{Ran}})$$

is compatible with the actions of

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\mathrm{Ran}}) \overset{(\mathrm{E.24})}{\simeq} \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\mathrm{Ran}}).$$

E.8.12. Recall now that according to Theorem 17.4.7, we have a canonical isomorphism:

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\mathrm{Ran}} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[-\delta_G] \simeq \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{Ran}} \circ \Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}}. \tag{E.29}$$

Unwinding the constructions, we obtain:

Lemma E.8.13. *The commutative diagram*

$$\begin{tikzcd}
\mathrm{IndCoh}^!(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\mathrm{Ran}}) \arrow[r, "\Theta_{\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}}"] \arrow[d, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\mathrm{Ran}}"'] & \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G},\mathrm{Ran}}) \arrow[d, "\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\mathrm{Ran}}"] \\
\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \arrow[r, "\mathrm{Id}"] & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran})
\end{tikzcd}$$

upgrades to a commutative diagram of categories equipped with actions of

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\mathrm{Ran}}) \overset{(\mathrm{E.24})}{\simeq} \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\mathrm{Ran}}),$$

where:

- *The compatibility for the left vertical arrows is given by Proposition E.8.9;*
- *The compatibility for the right vertical arrows is given by Proposition E.8.4;*
- *The compatibility for the top horizontal arrow with* (E.24) *is given by Lemma E.8.11;*
- *The compatibility for the bottom horizontal arrow with* (E.24) *is given by Sect. E.7.13.*

E.9. **An approach to** $\mathrm{Sph}^{\mathrm{spec}}_H$ **via factorization modules.** In this subsection we will review the connection between the definition of $\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{H,\mathrm{Ran}})$ developed above, and one given in [CR].

E.9.1. The projection

$$\mathrm{Hecke}_H^{\mathrm{spec,loc}} := \mathrm{LS}_H^{\mathrm{reg}} \underset{\mathrm{LS}_H^{\mathrm{mer}}}{\times} \mathrm{LS}_H^{\mathrm{reg}} \xrightarrow{f} \mathrm{LS}_H^{\mathrm{reg}} \times \mathrm{LS}_H^{\mathrm{reg}}$$

gives rise to a lax unital functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \to \mathrm{IndCoh}^*(\mathrm{LS}_H^{\mathrm{reg}} \times \mathrm{LS}_H^{\mathrm{reg}}) \overset{\Psi_{\mathrm{LS}_H^{\mathrm{reg}} \times \mathrm{LS}_H^{\mathrm{reg}}}}{\simeq}$$
$$\simeq \mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}} \times \mathrm{LS}_H^{\mathrm{reg}}) \simeq \mathrm{Rep}_{H\times H} \simeq \mathrm{Rep}_H \otimes \mathrm{Rep}_H,$$

where the first arrow is obtained by applying the functor of $\mathfrak{L}_\nabla^+(H) \times \mathfrak{L}_\nabla^+(H)$-coinvariants to

$$\Gamma^{\mathrm{IndCoh}}(\mathfrak{L}_\nabla(H), -) \to \mathrm{Vect}\,.$$

E.9.2. Note that the image of the factorization unit

$$\mathbf{1}_{\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})} \simeq \iota_*^{\mathrm{IndCoh}}(\mathcal{O}_{\mathrm{LS}_H^{\mathrm{reg}}})$$

identifies with

$$(\Delta_{\mathrm{LS}_H^{\mathrm{reg}}})_*(\mathcal{O}_{\mathrm{LS}_H^{\mathrm{reg}}}) \in \mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}} \times \mathrm{LS}_H^{\mathrm{reg}}) \;\Leftrightarrow\; R_H \in \mathrm{Rep}_{H\times H},$$

where R_H denotes the regular representation, viewed as a commutative factorization algebra in $\mathrm{Rep}_{H\times H}$

By Lemma C.15.3, the functor f_* enhances to a unital functor

$$(f_*^{\mathrm{IndCoh}})^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \to R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}_H \otimes \mathrm{Rep}_H), \tag{E.30}$$

where the right-hand side is viewed as a unital lax factorization category.

E.9.3. We claim:

Proposition E.9.4.

(a) *The functor* (E.30) *induces an equivalences between the eventually coconnective subcategories of the two sides.*

(b) *The essential image of* $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})^c \subset \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ *under the functor* (E.30) *is contained in* $\left(R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}_H \otimes \mathrm{Rep}_H)\right)^{>-\infty}$.

Proof. The proof proceeds along the same lines as that of Proposition 4.4.7, with the following difference:

Instead of appealing to Proposition A.3.3, we claim that $\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ identifies with the totalization of the cosimplicial category

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}} \underset{\mathrm{LS}_H^{\mathrm{reg}} \times \mathrm{LS}_H^{\mathrm{reg}}}{\times} \mathrm{pt}^\bullet),$$

where $\mathrm{pt}^\bullet$ is the Čech nerve of the projection

$$\mathrm{pt} \to \mathrm{LS}_H^{\mathrm{reg}} \times \mathrm{LS}_H^{\mathrm{reg}},$$

i.e., we claim that the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H))^{\mathfrak{L}_\nabla^+(H)\times\mathfrak{L}_\nabla^+(H)} \tag{E.31}$$

is an equivalence.

Indeed, the precomposition of (E.31) with (E.17) is the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H))_{\mathfrak{L}_\nabla^+(H)\times\mathfrak{L}_\nabla^+(H)} \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(H))^{\mathfrak{L}_\nabla^+(H)\times\mathfrak{L}_\nabla^+(H)}$$

of (E.2), which is an equivalence by Corollary E.1.6.

□

E.9.5. Note that we have a commutative diagram

$$\begin{tikzcd} \mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}}) \arrow[r, "\iota_*^{\mathrm{IndCoh}}"] \arrow[d, "\sim"'] & \mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}}) \arrow[dd, "(f_*^{\mathrm{IndCoh}})^{\mathrm{enh}}"] \\ \mathrm{Rep}(H) \arrow[d, "\sim"'] & \\ R_H\text{-}\mathrm{mod}^{\mathrm{com}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)) \arrow[r] & R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)). \end{tikzcd}$$

Recall now that $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})^c$ is generated under finite colimits by the essential image of $\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}})$ along $\iota_*^{\mathrm{IndCoh}}$.

Hence, from Proposition E.9.4, we obtain that the essential image of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})^c$ under (E.30) is the full subcategory of $R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))$ generated under finite colimits by the essential image of $\mathrm{Rep}(H)^c$ under the functor

$$\mathrm{Rep}(H) \simeq R_H\text{-}\mathrm{mod}^{\mathrm{com}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)) \to R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)). \tag{E.32}$$

This allows us to recover $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ from $R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))$ by an explicit procedure:

Namely, $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ identifies with the ind-completion of the full subcategory of $R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))$ generated under finite colimits by the essential image of $\mathrm{Rep}(H)^c$ under the functor (E.32).

In the rest of this subsection we will show how to recover various pieces of structure on $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ from those on $R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))$.

E.9.6. Note that $\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)$ is naturally a factorization monoidal category under *convolution*, and R_H is the monoidal unit.

Hence, the monoidal operation

$$(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))\otimes(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)) \overset{\star}{\to} \mathrm{Rep}(H)\otimes\mathrm{Rep}(H) \tag{E.33}$$

sends the factorization algebra

$$R_H\otimes R_H \in \mathrm{FactAlg}^{\mathrm{untl}}(X, (\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))\otimes(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)))$$

to

$$R_H \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathrm{Rep}(H)\otimes\mathrm{Rep}(H)).$$

From here we obtain that the functor (E.33) induces a functor

$$\begin{multline} (R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)))\otimes(R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))) \overset{\star}{\to} \\ \to R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H)), \end{multline} \tag{E.34}$$

which naturally extends to a monoidal structure on $R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))$.

E.9.7. Unwinding the constructions, we obtain that the functor (E.30) is monoidal.

The monoidal operation (E.34) is t-exact. Hence, the procedure of recovering $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ from $R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(H))$ described in Sect. E.9.5 allows us also to recover its monoidal structure.

E.9.8. Let $\mathbf{A}$ be a (unital) factorization category, equipped with a monoidal action of $\mathrm{Rep}(H) \otimes \mathrm{Rep}(H)$. Let $\mathcal{A}$ be *any* (unital) factorization algebra in $\mathbf{A}$.

The action functor

$$(\mathrm{Rep}(H) \otimes \mathrm{Rep}(H)) \otimes \mathbf{A} \overset{\star}{\to} \mathbf{A} \tag{E.35}$$

automatically sends the factorization algebra

$$R_H \otimes \mathcal{A} \in \mathrm{FactAlg}^{(\mathrm{untl})}(X, (\mathrm{Rep}(H) \otimes \mathrm{Rep}(H)) \otimes \mathbf{A})$$

to

$$\mathcal{A} \in \mathrm{FactAlg}^{(\mathrm{untl})}(X, \mathrm{Rep}(H) \otimes \mathrm{Rep}(H)).$$

Hence, we obtain that the functor (E.35) gives rise to a functor

$$(R_H\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(H) \otimes \mathrm{Rep}(H))) \otimes (\mathcal{A}\text{-mod}^{\mathrm{fact}}\text{-mod}(\mathbf{A})) \overset{\star}{\to} \mathcal{A}\text{-mod}^{\mathrm{fact}}\text{-mod}(\mathbf{A}), \tag{E.36}$$

which extends to a monoidal action of $R_H\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(H) \otimes \mathrm{Rep}(H))$ on $\mathcal{A}\text{-mod}^{\mathrm{fact}}\text{-mod}(\mathbf{A})$.

E.9.9. Take $\mathbf{A} = \mathrm{Rep}(H)$ and $\mathcal{A} = \mathbf{1}_{\mathrm{Rep}(H)} = \mathcal{O}_{\mathrm{Rep}(H)}$, so that

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}\text{-mod}(\mathbf{A}) = \mathrm{Rep}(H).$$

Hence, we obtain an action of $R_H\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(H) \otimes \mathrm{Rep}(H))$ on $\mathrm{Rep}(H)$.

Unwinding the definitions, we obtain that the functor (E.30) intertwines the above action with the action of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ on $\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{reg}})$ from Sect. E.6.5.

Furthermore, as in Sect. E.9.7, this allows us to recover the latter action from the action of $R_H\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(H) \otimes \mathrm{Rep}(H))$ on $\mathrm{Rep}(H)$.

E.9.10. Let us now take $H = \check{G}$. Let us take again $\mathbf{A} = \mathrm{Rep}(\check{G})$, but let us take $\mathcal{A} := R_{\check{G},\mathrm{Op}}$ from (4.8).

We obtain an action of $R_{\check{G}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}))$ on $R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))$.

Unwinding the constructions, we obtain that the functors (E.30) and $(\mathfrak{r}_*^{\mathrm{IndCoh}})^{\mathrm{enh}}$ intertwine the above action with the action of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ on $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\mathrm{mon\text{-}free}})$ from Sect. E.8.1.

Recall now that according to Proposition 4.4.7 the functor

$$(\mathfrak{r}_*^{\mathrm{IndCoh}})^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\mathrm{mon\text{-}free}}) \to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))$$

induces an equivalence between the eventually coconnective subcategories of the two sides and sends compact objects to eventually coconnective ones. Combined with Proposition E.9.4, this allows us to recover the action of $\mathrm{IndCoh}^*(\mathrm{Hecke}_H^{\mathrm{spec,loc}})$ on $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\mathrm{mon\text{-}free}})$ from the above action of $R_{\check{G}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}))$ on $R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))$.

E.10. **Compatibility of the FLE with (derived) Satake.** In this subsection we continue to take $H = \check{G}$. Our goal is to prove Theorem 6.4.5 in the factorization setting.

E.10.1. As a first step, we recall the construction of the geometric equivalence functor Sat_G. Consider the factorization category

$$\mathrm{Whit}^!(G) \otimes \mathrm{Whit}_*(G)$$

and note that it is naturally a bimodule[83] for the factorization monoidal category Sph_G.

We identify

$$\mathrm{Whit}^!(G) \overset{\mathrm{CS}_G}{\simeq} \mathrm{Rep}(\check{G}),$$

and we identify

$$\mathrm{Whit}_*(G) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Rep}(\check{G}).$$

[83] Recall that according to Sect. 1.5.4 we freely pass between left and right modules for Sph_G using the anti-involution σ.

Thus, we obtain that $\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G})$ acquires a bimodule structure with respect to Sph_G. The action on the factorization algebra object $R_{\check{G}} \in \operatorname{Rep}(G) \otimes \operatorname{Rep}(\check{G})$ gives rise to a factorization functor

$$\text{pre-Sat}_G : \operatorname{Sph}_G \to \operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G}).$$

Since the monoidal unit $\delta_{1,\operatorname{Gr}_G} \in \operatorname{Sph}_G$ equals the factorization unit, the functor pre-Sat_G sends $\mathbf{1}_{\operatorname{Sph}_G}$ to $R_{\check{G}}$.

The above operations are compatible with unital structures. Hence, by Lemma C.15.3, the functor pre-Sat_G upgrades to a functor

$$(\text{pre-Sat}_G)^{\text{enh}} : \operatorname{Sph}_G \to R_{\check{G}}\text{-mod}^{\text{fact}}(\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G})).$$

Unwinding the constructions, we obtain that the above functor $(\text{pre-Sat}_G)^{\text{enh}}$ carries a naturally defined monoidal structure.

E.10.2. We observe:

Lemma E.10.3. *The functor* $(\text{pre-Sat}_G)^{\text{enh}}$ *sends the subcategory* $(\operatorname{Sph}_G)^c \subset \operatorname{Sph}_G$ *to the full subcategory of* $R_{\check{G}}\text{-mod}^{\text{fact}}(\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G}))$ *generated under finite colimits by the essential image of* $\operatorname{Rep}(\check{G})^c$ *under the functor*

$$R_{\check{G}}\text{-mod}^{\text{com}}(\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G})) \to R_{\check{G}}\text{-mod}^{\text{fact}}(\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G})).$$

Thus, combining with Sect. E.9.5, we obtain that the functor $(\text{pre-Sat}_G)^{\text{enh}}$ can be uniquely factored as

$$\operatorname{Sph}_G \to \operatorname{Sph}^{\text{spec}}_{\check{G}} := \operatorname{IndCoh}^*(\operatorname{Hecke}^{\text{spec,loc}}_{\check{G}}) \overset{(\text{E.30})}{\longrightarrow} R_{\check{G}}\text{-mod}^{\text{fact}}(\operatorname{Rep}_{\check{G}} \otimes \operatorname{Rep}_{\check{G}}),$$

where the first arrow preserves compactness.

The resulting functor

$$\operatorname{Sph}_G \to \operatorname{Sph}^{\text{spec}}_{\check{G}}$$

is the functor Sat_G, as it was constructed in [CR].

E.10.4. For the proof of Theorem 6.4.5, we will need the following output of the above construction:

Consider the functor

$$\operatorname{Sph}_G \overset{\text{pre-Sat}_G}{\longrightarrow} \operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G}) \overset{\operatorname{Id} \otimes \operatorname{FLE}_{\check{G},\infty}}{\longrightarrow} \operatorname{Rep}(\check{G}) \otimes \operatorname{Whit}_*(G). \tag{E.37}$$

By construction, it has the following properties:

- It intertwines the action of Sph_G on itself by right multiplication with the natural action of Sph_G on $\operatorname{Whit}_*(G)$;
- It intertwines the action of Sph_G on itself by left multiplication with the action of $\operatorname{Sph}^{\text{spec}}_{\check{G}}$ on $\operatorname{Rep}(\check{G})$ via Sat_G.

E.10.5. Let $\mathbf{A}$ be a factorization category, equipped with a factorization action of $\mathfrak{L}(G)_{\rho(\omega_X)}$ at the critical level. We have a naturally defined factorization functor

$$\text{D-mod}_{\text{crit}}(\operatorname{Gr}_{G,\rho(\omega_X)}) \underset{\operatorname{Sph}_G}{\otimes} \operatorname{Sph}(\mathbf{A}) \to \mathbf{A},$$

which gives rise to a factorization functor

$$\operatorname{Whit}_*(G) \underset{\operatorname{Sph}_G}{\otimes} \operatorname{Sph}(\mathbf{A}) \to \operatorname{Whit}_*(\mathbf{A}). \tag{E.38}$$

By duality, the functor (E.38) gives rise to a functor

$$\operatorname{Sph}(\mathbf{A}) \to \operatorname{Whit}^!(G) \otimes \operatorname{Whit}_*(\mathbf{A}), \tag{E.39}$$

compatible with the action of Sph_G.

E.10.6. Composing with CS_G along the first factor, from (E.39) we obtain a functor

$$\mathrm{Sph}(\mathbf{A}) \to \mathrm{Rep}(\check{G}) \otimes \mathrm{Whit}_*(\mathbf{A}). \tag{E.40}$$

We claim:

Lemma E.10.7. *The functor* (E.40) *intertwines the action of* Sph_G *on* $\mathrm{Sph}(\mathbf{A})$ *with the action of* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ *on* $\mathrm{Rep}(\check{G})$ *via* Sat_G*.*

Proof. We can interpret the functor (E.40) as follows:

$$\mathrm{Sph}(\mathbf{A}) \simeq \mathrm{Sph}_G \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{A}) \overset{(E.37)}{\longrightarrow} \big(\mathrm{Rep}(\check{G}) \otimes \mathrm{Whit}_*(G)\big) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{A}) \simeq \\ \simeq \mathrm{Rep}(\check{G}) \otimes (\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{A})) \overset{\mathrm{Id}\otimes(E.38)}{\longrightarrow} \mathrm{Rep}(\check{G}) \otimes \mathrm{Whit}_*(\mathbf{A}). \tag{E.41}$$

Now the assertion follows from the second bullet point in Sect. E.10.4.

□

E.10.8. Consider the morphism

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}} \overset{\mathfrak{r}\times\iota^{\text{mon-free}}}{\to} \mathrm{LS}^{\mathrm{reg}}_{\check{G}} \times \mathrm{Op}^{\mathrm{mer}}_{\check{G}},$$

and the corresponding factorization functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \overset{(\mathfrak{r}\times\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{LS}^{\mathrm{reg}}_{\check{G}} \times \mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \simeq \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}). \tag{E.42}$$

Unwinding the construction, we obtain that (E.42) is compatible with the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on the two sides, where $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ acts on the right-hand side via the $\mathrm{LS}^{\mathrm{reg}}_{\check{G}}$-factor.

Denote

$$R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}} := ((\mathfrak{r} \times \iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*)(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}) \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})).$$

By Lemma C.15.3, the functor (E.42) enhances to a functor

$$((\mathfrak{r} \times \iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \to R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})). \tag{E.43}$$

The functor (E.43) intertwines the $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$-action on the left-hand side and the action of $R_{\check{G}}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}))$ from Sect. E.9.8 on the right-hand side via the functor (E.30).

E.10.9. We will prove:

Proposition E.10.10.

(a) *The functor* (E.43) *induces an equivalence between the eventually coconnective subcategories of the two sides.*

(b) *The essential image of the subcategory of compact objects in* $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$ *under the functor* (E.43) *is contained in* $\left(R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}))\right)^{>-\infty}$.

The proof will be given in Sect. E.11. Let us accept this proposition temporarily and proceed with the proof of Theorem 6.4.5.

E.10.11. We now launch the proof of Theorem 6.4.5 proper.

Consider the functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \overset{(E.42)}{\longrightarrow} \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}). \tag{E.44}$$

We consider its enhancement

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \overset{(E.43)}{\longrightarrow} R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})). \tag{E.45}$$

By Propositions E.9.4 and E.10.10, in order to show that the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ intertwines the action of Sph_G on $\mathrm{KL}(G)_{\mathrm{crit}}$ with the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}})$, it suffices to show that the composite functor in (E.45) intertwines the action of Sph_G on the left-hand side with the action of $R_{\check{G}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}))$ from Sect. E.9.8 on the right-hand side via the functor $(\text{pre-Sat}_G)^{\mathrm{enh}}$.

E.10.12. By the construction of the action of $R_{\check{G}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}))$ on the right-hand side of (E.45) in Sect. E.9.8, it suffices to show that the original functor (E.44) intertwines the action of Sph_G on the left-hand side with the action of $R_{\check{G}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G}))$ on the right-hand side via the functor $(\text{pre-Sat}_G)^{\mathrm{enh}}$.

We will show that the functor (E.44) intertwines the action of Sph_G on the left-hand side with the action of $\mathrm{Spc}^{\mathrm{spec}}_{\check{G}}$ on the right-hand side via Sat_G.

E.10.13. We claim that the functor (E.44) identifies canonically with the functor

$$\text{(E.46)}\quad \mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} = \mathrm{Sph}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\text{(E.40)}}{\longrightarrow}$$
$$\to \mathrm{Rep}(\check{G}) \otimes \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\mathrm{Id}\otimes \overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}}{\longrightarrow} \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).$$

Indeed, by construction, both functors are $\mathrm{Rep}(\check{G})$-linear. Hence, since $\mathrm{Rep}(\check{G})$ is rigid as a monoidal category, it suffices to identify the compositions of (E.44) and (E.46) with the functor

$$(\mathrm{inv}_{\check{G}} \otimes \mathrm{Id}) : \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).$$

The composition involving (E.44) becomes

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \overset{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}),$$

which is by construction

$$\text{(E.47)}\quad \mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \to$$
$$\to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).$$

Unwinding, we obtain that the composition involving (E.46) also identifies with (E.47).

E.10.14. Thus, we have to show that the functor (E.46) intertwines the action of Sph_G on the left-hand side with the action of $\mathrm{Spc}^{\mathrm{spec}}_{\check{G}}$ on the right-hand side via Sat_G.

However, this follows from Lemma E.10.7.

E.10.15. It remains to establish the commutativity of (6.25). This is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{KL}(G)_{\mathrm{crit}} \arrow[r, "\alpha_{\rho(\omega_X),\mathrm{taut}}"] \arrow[d, "\mathrm{FLE}_{G,\mathrm{crit}}"'] & \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \arrow[r] & \mathrm{Whit}^!(G) \otimes \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \arrow[d, "\mathrm{CS}_G \otimes \overline{\mathrm{DS}}^{\mathrm{enh,rfnd}}"] \\
\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}) \arrow[rr, "\text{(E.42)}"] & & \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}),
\end{tikzcd}$$

compatibly with the action of Sph_G on the top row and the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on the bottom row via Sat_G.

However, this amounts to the identification between (E.44) and (E.46) established above.

$\square$[Theorem 6.4.5]

E.11. **Proof of Proposition E.10.10.**

E.11.1. Consider the functor

$$\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}, -) : \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathrm{Rep}(\check{G}). \tag{E.48}$$

It sends $R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}} \mapsto R_{\check{G},\mathrm{Op}}$ and hence induces a functor

$$R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})) \to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})) \tag{E.49}$$

The composition of the functor (E.43) with (E.49) is the functor (4.10). Hence, by Proposition 4.4.7, it suffices to show that the functor (E.49) induces an equivalence between the eventually coconnective subcategories of the two sides.

E.11.2. The functor $\mathrm{Id} \otimes \Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}, -)$ enhances to a functor

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \to \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})) \tag{E.50}$$

where $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}$ is regarded as a constant (commutative) factorization algebra in $\mathrm{Rep}(\check{G})$.

The functor (E.50) sends $R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}} \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}))$ to $R_{\check{G},\mathrm{Op}}$, regarded as a unital factorization algebra in $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))$ via the homomorphism $\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}} \to R_{\check{G},\mathrm{Op}}$, see Sect. C.11.18.

Hence, (E.50) induces a functor

$$R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})) \to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))\right). \tag{E.51}$$

The functor (E.49) is the composition of (E.51) with the forgetful functor

$$\mathbf{oblv}_{\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}} : R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))\right) \to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})). \tag{E.52}$$

E.11.3. Note now that by (a relative version of) Corollary 4.4.2, the functor (E.50) induces an equivalence

$$\left(\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})\right)^{>-\infty} \xrightarrow{\sim} \left(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))\right)^{>-\infty}. \tag{E.53}$$

Note also that $R_{\check{G},\mathrm{Op}} \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathrm{Rep}(\check{G}))$ belongs to

$$\mathrm{Rep}(\check{G})^{\heartsuit} \subset \mathrm{Rep}(\check{G})^{>-\infty}.$$

Hence, it makes sense to consider the full subcategory

$$\left(R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))\right)\right)^{>-\infty} \subset R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))\right).$$

It follows formally from the equivalence (E.53) that the functor (E.51) induces an equivalence

$$\begin{aligned}\left(R^{\mathrm{Rep}}_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}))\right)^{>-\infty} &\to \\ \to \left(R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))\right)\right)^{>-\infty}&.\end{aligned} \tag{E.54}$$

E.11.4. Hence, it suffices to show that the functor (E.52) induces an equivalence between the eventually coconnective subcategories of the two sides.

However, we claim that the functor (E.52) it itself an equivalence. Indeed, this is a particular case of Lemma C.11.19.

$\square$[Proposition E.10.10]

E.12. **Arc spaces of smooth D-schemes.** In this subsection, we will prove the following result.

Proposition E.12.1. *Let $\mathcal{Y} \to X$ be a smooth affine D-scheme over X. Then for any test scheme S and a map $\underline{x} : S \to \mathrm{Ran}$, the S-scheme $\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}$ is isomorphic to the limit of a sequence of smooth affine S-schemes.*

E.12.2. To prove Proposition E.12.1, it is enough to treat the case for the canonical map $X^I \to \mathrm{Ran}$. So our goal will be to prove the following:

Proposition E.12.3. *Let $\mathcal{Y} \to X$ be a smooth D-scheme over X. Then $\mathfrak{L}^+_\nabla(\mathcal{Y})_{X^I}$ is isomorphic to the limit of a sequence of relative smooth affine schemes over X^I.*

We will now describe $\mathfrak{L}^+_\nabla(\mathcal{Y})_{X^I}$ as a limit of affine blow-ups $\mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^I}$ (see below) for *any* $\mathcal{Y}$. When $\mathcal{Y}$ is smooth, we will show that all $\mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^I}$ are smooth.

E.12.4. We first give a brief review of the classical theory of affine blow-ups (a.k.a. dilations). From now on, we only work in *classical* algebraic geometry, i.e., schemes means classical schemes, and fiber products of schemes mean non-derived fiber products, etc.

Let S be a smooth scheme and E be an effective Cartier divisor on S. Let Z be a E-regular S-scheme, i.e., the closed subscheme $Z_E := Z \underset{S}{\times} E$ is an effective Cartier divisor on Z. Let V be a closed subscheme of Z_E. The *affine blow-up* of Z with center V (with respect to E) is defined to be

$$\mathrm{Dil}^E_V(Z) := \mathrm{Bl}_V(Z) - \widetilde{Z_E},$$

where $\mathrm{Bl}_V(Z)$ is the blow-up of Z with center V, and $\widetilde{Z_E}$ is the strict transform of Z_E.

E.12.5. More explicitly, let $\mathcal{O}_S(-E) \subset \mathcal{O}_S$ be the ideal sheaf defining E, and $\mathcal{I} \subset \mathcal{O}_Z$ be the ideal sheaf defining V. Note that $\mathcal{O}_Z(-E) := \mathcal{O}_Z \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_S(-E)$ is a subsheaf of $\mathcal{I}$. Consider the inductive colimit

$$\mathcal{D}il^E_{\mathcal{I}}(\mathcal{O}_Z) := \mathrm{colim}(\mathcal{O}_Z \to \mathcal{I}(E) \to \mathcal{I}^2(2E) \to \cdots).$$

Note that the connecting morphisms are injective because Z is E-regular. This is a quasi-coherent $\mathcal{O}_Z$-algebra with multiplication defined in the obvious way. More explicitly, if E is locally cut out by a function f of $\mathcal{O}_S$, then $\mathcal{D}il^E_{\mathcal{I}}(\mathcal{O}_Z)$ is obtained from $\mathcal{O}_Z$ by adding local sections

$$f^{-n} \cdot a_1 \cdots a_n, \quad a_k \in \mathcal{I}.$$

We have

$$\mathrm{Dil}^E_V(Z) \simeq \mathrm{Spec}_Z(\mathcal{D}il^E_{\mathcal{I}}(\mathcal{O}_Z)).$$

E.12.6. We observe:

Lemma E.12.7. *In the setting of Sect. E.12.4, let $\mathrm{Sch}_{S,E-\mathrm{reg}}$ be the category of E-regular S-schemes. Then $\mathrm{Dil}^E_V(Z)$ represents the functor*

$$\mathrm{Sch}^{\mathrm{op}}_{S,E-\mathrm{reg}} \to \mathrm{Set},\ W \mapsto \{f : W \to Z \,|\, f(W_E) \subseteq V\}.$$

Here $f(W_E) \subseteq V$ means the restriction of $f|_{W_E}$ factors through V.

Proof. Follows by unwinding the definitions. □

E.12.8. Let $S^\circ \subset S$ be an open, whose complement has codimension ≥ 2. Denote $Z^\circ := Z \underset{S}{\times} S^\circ$. Observe:

Lemma E.12.9. *Assume that Z equals the affine closure of Z°. Assume moreover that V is the closure of $V^\circ := V \underset{S}{\times} S^\circ$ in Z. Then $\mathrm{Dil}^E_V(Z)$ equals the affine closure of*

$$\mathrm{Dil}^E_V(Z)^\circ := \mathrm{Dil}^E_V(Z) \underset{S}{\times} S^\circ.$$

E.12.10. We first construct the schemes $\mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^I}$ for $I = \{1,2\}$.

Let Δ_X be the diagonal of X^2 and consider the divisor $\Delta_{X,n} := n \cdot \Delta_X$. Note that the connection on $\mathcal{Y}$ relative to X provides a closed subscheme $\Delta_{\mathcal{Y},n}$ of $\mathcal{Y}^2$:

$$\Delta_{\mathcal{Y},n} := \mathcal{Y}_\nabla \underset{X_{\mathrm{dR}}}{\times} \Delta_{x,n}, \tag{E.55}$$

viewed as a closed subscheme of $\mathcal{Y}^2_\nabla \times_{X^2_{\mathrm{dR}}} X^2$.

Set

$$\mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^2} := \mathrm{Dil}^{n\cdot\Delta_X}_{\Delta_{\mathcal{Y},n}}(\mathcal{Y}^2).$$

We claim that we have a canonical isomorphism

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_{X^2} \simeq \lim_n \mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^2}.$$

First, by the universal property of Lemma E.12.7, we have a canonically defined map

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_{X^2} \to \lim_n \mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^2}. \tag{E.56}$$

Let us show that this map is an isomorphism. Let A be the commutative algebra in left D-modules on X corresponding to $\mathcal{Y}$. Choose a local coordinate t on X, so that X^2 has coordinates t_1, t_2.

By Sect. E.12.5, the algebra of functions on the right-hand side in (E.56) is the submodule of

$$j_*(A \boxtimes A)$$

that consists of local sections of the form

$$(t_1 - t_2)^{-n} \cdot a, \quad a|_{n\cdot\Delta} \in \ker(A \underset{\mathcal{O}_X}{\otimes} A|_{n\cdot\Delta_X} \to A|_{n\cdot\Delta_X}),$$

where:

- The connection on A allows us to extend it and also $A \underset{\mathcal{O}_X}{\otimes} A$ to quasi-coherent sheaves on $n\cdot\Delta_X$, denoted $A \underset{\mathcal{O}_X}{\otimes} A|_{n\cdot\Delta_X}$ and $A|_{n\cdot\Delta_X}$, respectively.
- We identify $A \boxtimes A|_{n\cdot\Delta_X} \simeq A \underset{\mathcal{O}_X}{\otimes} A|_{n\cdot\Delta_X}$.

However, this description coincides with the description of

$$\mathcal{A}_{X^2} \subset j_*(A \boxtimes A), \quad \mathcal{A} = \mathrm{Fact}(\mathcal{A}).$$

E.12.11. We now construct $\mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^I}$ for an arbitrary finite set I. Let $\Delta_{X,I}$ be the diagonal divisor in X^I, i.e., the sum of the pairwise diagonals. We define the subscheme

$$\Delta_{\mathcal{Y},I,n} \subset \mathcal{Y}^I$$

over $n \cdot \Delta_{X,I}$ as follows:

Let

$$X^{I,\circ} \overset{j_{I,\circ}}{\hookrightarrow} X^I$$

be the open corresponding to the condition that not more than two coordinates coincide (i.e., we remove diagonals of codimension ≥ 2). Denote

$$\mathcal{Y}^{I,\circ} := \mathcal{Y}^I \underset{X^I}{\times} X^{I,\circ}.$$

Formula (E.55) defines a subscheme

$$\Delta^\circ_{\mathcal{Y},I,n} \subset \mathcal{Y}^{I,\circ}.$$

We let $\Delta_{\mathcal{Y},I,n}$ be the closure of $\Delta^\circ_{\mathcal{Y},I,n}$ in $\mathcal{Y}^I$.

E.12.12. Set

$$\mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^I} := \mathrm{Dil}^{n\cdot\Delta_{X,I}}_{\Delta_{\mathcal{Y},I,n}}(\mathcal{Y}^I).$$

By the the universal property of Lemma E.12.7, we have a map

$$\mathfrak{L}^+_\nabla(\mathcal{Y})_{X^I} \to \lim_n \mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^I}. \tag{E.57}$$

We claim that (E.57) is an isomorphism.

Indeed, the fact that (E.57) is an isomorphism over $X^{I,\circ}$ follows from the case $I = \{1, 2\}$, considered above. We now claim that both sides in (E.57) are affine closures of their respective restrictions to $X^{I,\circ}$.

Indeed, for the right-hand side, this follows from Lemma E.12.9. For the left-hand side, this follows from the fact that the map

$$\mathcal{A}_{X^I} \to (j_{I,\circ})_* \circ (j_{I,\circ})^*(\mathcal{A}_{X^I})$$

induces an isomorphism on H^0.

E.12.13. We will now show that if $\mathcal{Y}$ is smooth, then the schemes $\mathfrak{L}^+_\nabla(\mathcal{Y})^n_{X^I}$ are smooth. Note that the assertion is invariant under *formal isomorphisms*. This allows us to replace $\mathcal{Y}$ by $\mathbb{A}^k$.

To prove the smoothness, we can assume $k = 1$. Thus, from now on, we will consider the schemes $\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^I}$.

E.12.14. Let $\mathsf{H}_I \subset X^I \times X$ be the incidence divisor. Consider the correspospondence

$$\begin{tikzcd} n \cdot \mathsf{H}_I \arrow[r] \arrow[d] & X \\ X^I. & \end{tikzcd}$$

For an X-scheme $\mathcal{Y}$, let

$$\mathrm{Jets}^n_I(\mathcal{Y})$$

be the scheme of n-truncated jets into $\mathcal{Y}$, i.e., the restriction of scalars á la Weil of $(n \cdot \mathsf{H}_I) \underset{X}{\times} \mathcal{Y}$ along the map $n \cdot \mathsf{H}_I \to X^I$.

E.12.15. Consider the schemes

$$\mathrm{Jets}^n_I(\mathbf{A}^1) \text{ and } \mathrm{Jets}^{n-1}_I((\mathbf{A}^1)_{\omega_X}),$$

where $(\mathbf{A}^1)_{\omega_X}$ is the total space of the line bundle of 1-forms on X.

Both schemes are vector bundles over X^I of ranks $|I| \cdot n$ and $|I| \cdot (n-1)$, respectively. De Rham differential defines a map

$$d_{\mathrm{dR}} : \mathrm{Jets}^n_I(\mathbf{A}^1) \to \mathrm{Jets}^{n-1}_I((\mathbf{A}^1)_{\omega_X}). \tag{E.58}$$

Set

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^I} := \ker(d_{\mathrm{dR}}).$$

Since d_{dR} is fiber-wise surjective as a map of vector bundles, $'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^I}$ is also a vector bundle. In particular, it is smooth over X^I.

E.12.16. We are going to prove:

Proposition E.12.17. *There exists a canonical isomorphism*

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^I} \simeq \mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^I}.$$

E.13. Proof of Proposition E.12.17.

E.13.1. First, we note that when $I = \{*\}$, there is an obvious isomorphism

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_X = \ker(\mathrm{Jets}^n(\mathbf{A}^1) \to \mathrm{Jets}^{n-1}((\mathbf{A}^1)_{\omega_X}) \overset{\sim}{\leftarrow} \ker(\mathrm{Jets}(\mathbf{A}^1) \to \mathrm{Jets}((\mathbf{A}^1)_{\omega_X}) \simeq$$
$$\simeq \mathfrak{L}^+_\nabla(\mathbb{A}^1)_X = \mathbb{A}^1 \times X.$$

E.13.2. Note that for a map of finite sets $I' \to I$, there is a canonically defined map

$$\mathrm{Jets}^n_I(\mathcal{Y}) \to \mathrm{Jets}^n_{I'}(\mathcal{Y}),$$

covering $X^I \to X^{I'}$.

In particular, we obtain a map

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^I} \to ('\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_X)^I \simeq (\mathbb{A}^1 \times X)^I \tag{E.59}$$

over X^I. This map is an isomorphism away from the diagonal divisor.

E.13.3. We will show that the map (E.59) lifts to and defines an isomorphism

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^I} \to \mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^I} \tag{E.60}$$

over $X^{I,\circ}$.

This would imply Proposition E.12.17, since both schemes in question are affine closures of their respective restrictions to $X^{I,\circ}$.

E.13.4. In order to prove (E.60), it suffices to consider the case $I = \{1, 2\}$. The following is obtained by a straightforward calculation:

Lemma E.13.5. *The map*

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^2} \to (\mathbb{A}^1 \times X)^2 \tag{E.61}$$

has the following properties:

(i) *Its restriction to* $n \cdot \Delta_X$ *maps to* $\Delta_{\mathbf{A}^1,n}$*;*

(ii) *Its restriction to* $(n+1) \cdot \Delta_X$ does not *map to* $\Delta_{\mathbf{A}^1,n+1}$.

E.13.6. From Lemmas E.13.5(i) and (E.12.7), we obtain that (E.61) lifts to a map

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^2} \to \mathfrak{L}^+_\nabla(\mathbb{A}^2)^n_{X^I}. \tag{E.62}$$

It remains to show that (E.62) is an isomorphism.

E.13.7. The map (E.61) respects the vector bundle structures. Hence, so does the map (E.62). It is an isomorphism away from the diagonal, so it remains to show that it is surjective over the diagonal.

E.13.8. It is clear that the image of

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^2}|_{\Delta_X} \to \mathfrak{L}^+_\nabla(\mathbb{A}^2)^n_{X^I}|_{\Delta_X} \simeq \mathbb{A}^1 \times \mathbb{A}^1$$

contains the diagonal copy of $\mathbb{A}^1$.

Furthermore, if the entire image *had been* contained in the diagonal copy of $\mathbb{A}^1$, it would have meant that the map (E.62) lifts further to a map

$$'\mathfrak{L}^+_\nabla(\mathbb{A}^1)^n_{X^2} \to \mathfrak{L}^+_\nabla(\mathbb{A}^2)^{n+1}_{X^I}.$$

However, the latter contradicts point (ii) of Lemma E.13.5.

□[Proposition E.12.17]

Appendix F. Horizontal sections of affine D-schemes

Let $\mathcal{Y}$ be an affine D-prestack over X. Let $\mathrm{Sect}_\nabla(X, \mathcal{Y})$ denote the prestack of its horizontal sections, i.e.,

$$\mathrm{Maps}(\mathrm{Spec}(R), \mathrm{Sect}_\nabla(X, \mathcal{Y})) := \mathrm{Maps}_{X,\nabla}(\mathrm{Spec}(R) \times X, \mathcal{Y}).$$

In this section we will describe, following [BD2], $\mathrm{Sect}_\nabla(X, \mathcal{Y})$ explicitly in terms of (vacuum) factorization homology of the algebra of functions $\mathcal{O}_\mathcal{Y}$ of $\mathcal{Y}$. We also describe spaces of sections of quasi-coherent sheaves on $\mathrm{Sect}_\nabla(X, \mathcal{Y})$, in terms of factorization homology of $\mathcal{O}_\mathcal{Y}$ with coefficients in corresponding modules.

We then generalize this discussion, when instead of $\mathrm{Sect}_\nabla(X, \mathcal{Y})$ we consider the space $\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y})$ of sections, where we allow punctures at $\underline{x} \subset X$.

F.1. **Horizontal sections via factorization homology.**

F.1.1. We start with an arbitrary D-prestack $\mathcal{Y} \to X$ (later on in this section, we will assume that $\mathcal{Y}$ is affine over X).

Let $A \in \mathrm{ComAlg}(\text{D-mod}(X))$ be the direct image of the structure sheaf of $\mathcal{Y}$, and let $\mathcal{A}$ be the corresponding object in $\mathrm{ComAlg}(\mathrm{FactAlg}(X))$, i.e., $\mathcal{A} = \mathrm{Fact}(A)$.

Consider the evaluation map

$$\mathrm{Sect}_\nabla(X, \mathcal{Y}) \times X \to \mathcal{Y}.$$

It gives rise to a map

$$A \to \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})} \otimes \mathcal{O}_X \tag{F.1}$$

in $\mathrm{ComAlg}(\text{D-mod}(X))$, where by a slight abuse of notation we denote by $\mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}$ the algebra of global functions on $\mathrm{Sect}_\nabla(X, \mathcal{Y})$.

By the adjunction of Corollary C.8.9, from (F.1) we obtain a map

$$\mathcal{A} \to \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})} \otimes \omega_{\mathrm{Ran}^{\mathrm{untl},*}} \tag{F.2}$$

in $\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}^{\mathrm{untl},*}))$, and further, by Sect. C.9.2, a map

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \to \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})} \tag{F.3}$$

in $\mathrm{ComAlg}(\mathrm{Vect})$.

F.1.2. Suppose that the prestack $\mathcal{Y}$ satisfies the assumption of Sect. C.8.10 (e.g., $\mathcal{Y} \to X$ is affine). In this case, the map (F.2) can also be interpreted as follows:

Consider the map

$$\mathrm{Sect}_\nabla(X, \mathcal{Y}) \times \mathrm{Ran} \overset{\mathrm{ev}}{\to} \mathfrak{L}^+_\nabla(\mathcal{Y})_{\mathrm{Ran}}.$$

Pullback at the level of functions defines a map

$$\mathcal{A} \overset{\text{(C.44)}}{\simeq} \mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y}),\mathrm{Ran}} \to \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})} \otimes \omega_{\mathrm{Ran}} \tag{F.4}$$

in $\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}))$.

It follows by unwinding the definition that the map (F.2) is the same as (F.4).

F.1.3. Suppose now that $\mathcal{Y} \to X$ is affine. In this case we claim:

Proposition F.1.4. *The prestack* $\mathrm{Sect}_\nabla(X, \mathcal{Y})$ *is an affine scheme, and the map* (F.3) *is an isomorphism.*

Proof. The key fact is that for any factorization algebra $\mathcal{A}$ such that $\mathbf{oblv}^l(\mathcal{A}_X) \in \mathrm{QCoh}(X)^{\leq 0}$, we have

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \in \mathrm{Vect}^{\leq 0}.$$

This follows from the fact that

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \simeq \mathrm{C}^\cdot_c(\mathrm{Ran}, \mathcal{A}_{\mathrm{Ran}})$$

can be written as a colimit with terms

$$\mathrm{C}^\cdot(X^I, \mathcal{A}_{X^I})$$

for (non-empty) finite sets I, while each $\mathcal{A}_{X^I}$ satisfies $\mathbf{oblv}^l(\mathcal{A}_{X^I}) \in \mathrm{QCoh}(X^I)^{\leq 0}$ (see [BD2, Sect. 3.4.11]) and hence $\mathrm{C}^\cdot(X^I, \mathcal{A}_{X^I}) \in \mathrm{Vect}^{\leq 0}$.

In particular, in our case

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \in \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}).$$

Now, the assertion follows immediately from Corollary C.9.5: for $R \in \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$ we have

$$\begin{aligned} \mathrm{Maps}(\mathrm{Spec}(R), \mathrm{Sect}_\nabla(X, \mathcal{Y})) = \mathrm{Maps}_{X,\nabla}(\mathrm{Spec}(R) \times X, \mathcal{Y}) &\overset{\text{affineness of } \mathcal{Y}}{\simeq} \\ \simeq \mathrm{Maps}_{\mathrm{ComAlg}(\text{D-mod}(X))}(A, R \otimes \mathcal{O}_X) &\simeq \mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}), R), \end{aligned} \tag{F.5}$$

so $\mathrm{Sect}_\nabla(X, \mathcal{Y})$ is the affine scheme $\mathrm{Spec}(\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}))$, and by construction the map (F.3) is the map

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \to \Gamma(\mathrm{Spec}(\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A})), \mathcal{O}_{\mathrm{Spec}(\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}))})$$

resulting from (F.5).

□

F.1.5. Recall now that the functor

$$\Gamma(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}, -) : \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}) \to \mathrm{Vect}$$

factors via an equivalence

$$\Gamma(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}, -)^{\mathrm{enh}} : \mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}) \to \mathcal{A}_{\underline{x}}\text{-mod} \simeq \mathcal{A}\text{-mod}^{\mathrm{com}}_x,$$

followed by the forgetful functor

$$\mathbf{oblv}_{\mathcal{A},\underline{x}} : \mathcal{A}\text{-mod}^{\mathrm{com}}_{\underline{x}} \to \mathrm{Vect}\,.$$

We claim:

Proposition F.1.6. *There exists a canonical isomorphism between*

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{\mathrm{ev}^*_{\underline{x}}} \mathrm{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})) \xrightarrow{\Gamma(\mathrm{Sect}_\nabla(X,\mathcal{Y}),-)} \mathrm{Vect} \tag{F.6}$$

and

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}) \overset{\Gamma(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}},-)^{\mathrm{enh}}}{\simeq} \mathcal{A}\text{-mod}^{\mathrm{com}}_{\underline{x}} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \xrightarrow{\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)_{\underline{x}}} \mathrm{Vect}\,. \tag{F.7}$$

Proof. Since

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}) \simeq \mathcal{A}_{\underline{x}}\text{-mod} \simeq \mathcal{A}\text{-mod}^{\mathrm{com}}_{\underline{x}},$$

the assertion of the proposition amounts to the following:

We have a canonical identification

$$\Gamma(\mathrm{Sect}_\nabla(X, \mathcal{Y}), \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}) \simeq \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{A}_{\underline{x}})_{\underline{x}}$$

as $\mathcal{A}_{\underline{x}}$-modules, where:

- $\mathcal{A}_{\underline{x}}$ acts on the left-hand side via

$$\mathcal{A}_x \xrightarrow{\sim} \mathrm{End}(\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}}) \xrightarrow{\mathrm{ev}^*_{\underline{x}}} \mathrm{End}(\mathrm{ev}^*_{\underline{x}}(\mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}})) \simeq \mathrm{End}(\mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})});$$

- $\mathcal{A}_{\underline{x}}$ acts on the right-hand side via

$$\mathcal{A}_x \xrightarrow{\sim} \mathrm{End}_{\mathcal{A}\text{-mod}^{\mathrm{com}}_{\underline{x}}}(\mathcal{A}_x) \to \mathrm{End}_{\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}}}(\mathcal{A}_{\underline{x}}).$$

However, this follows from the fact that the isomorphism

$$\Gamma(\mathrm{Sect}_\nabla(X, \mathcal{Y}), \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})}) \overset{(F.3)}{\simeq} \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}) \overset{\mathrm{unitality}}{\simeq} \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{A}_{\underline{x}})_{\underline{x}}$$

(as commutative algebras) is, by construction, compatible with the homomorphisms

$$\mathcal{A}_{\underline{x}} \simeq \Gamma(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}, \mathcal{O}_{\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}}) \to \Gamma(\mathrm{Sect}_\nabla(X, \mathcal{Y}), \mathcal{O}_{\mathrm{Sect}_\nabla(X,\mathcal{Y})})$$

and

$$\mathcal{A}_{\underline{x}} \to \mathrm{C}_c(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}, \mathcal{A}_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}}) = \mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, \mathcal{A}_{\underline{x}})_{\underline{x}}.$$

□

Remark F.1.7. Recall that Remark 15.6.15 says that for a commutative factorization algebra $\mathcal{A}$, we have a canonical identification

$$\mathcal{A}\text{-mod}^{\text{com}}_{\text{Ran,indep}} \simeq \mathrm{C}^{\cdot}(X, \mathcal{A})\text{-mod}.$$

Hence, by Proposition F.1.4 we obtain:

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}))_{\text{Ran,indep}} \simeq \mathrm{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})).$$

We can interpret Proposition F.1.6 as saying that under the above equivalence, the corresponding functor

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})) \to \mathrm{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})) \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\text{untl}})$$

sends $\underline{x} \in \mathrm{Ran}$ to to the functor

$$\mathrm{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{\mathrm{ev}^*_{\underline{x}}} \mathrm{QCoh}(\mathrm{Sect}_\nabla(X, \mathcal{Y})).$$

Remark F.1.8. The above remark applies to D-prestacks that are "as good as affine", see Sect. 12.7. For example, for a unipotent group-scheme N' over X and $\mathcal{Y} = \mathrm{pt}/\mathfrak{L}^+(N')$, we obtain an equivalence

$$\mathrm{Rep}(\mathrm{pt}/\mathfrak{L}^+(N'))_{\text{Ran,indep}} \simeq \mathrm{QCoh}(\mathrm{Bun}_{N'}), \tag{F.8}$$

where the composite functor

$$\mathrm{Rep}(\mathrm{pt}/\mathfrak{L}^+(N'))_{\text{Ran}} \twoheadrightarrow \mathrm{Rep}(\mathrm{pt}/\mathfrak{L}^+(N'))_{\text{Ran,indep}} \simeq \mathrm{QCoh}(\mathrm{Bun}_{N'})$$

is $\mathrm{Loc}^{\mathrm{QCoh}}_{N'}$.

Similarly, if N' is equipped with a connection (e.g., N' is constant), we have

$$\mathrm{Rep}(N')_{\text{Ran,indep}} \simeq \mathrm{QCoh}(\mathrm{LS}_{N'}), \tag{F.9}$$

where the composite functor

$$\mathrm{Rep}(N')_{\text{Ran}} \twoheadrightarrow \mathrm{Rep}(N')_{\text{Ran,indep}} \simeq \mathrm{QCoh}(\mathrm{LS}_{N'})$$

is $\mathrm{Loc}^{\mathrm{spec}}_{N'}$.

Remark F.1.9. We warn the reader that the equivalences (F.8) and (F.9) are a feature of unipotent group schemes. For a reductive G (e.g., for $G = \mathbb{G}_m$), the corresponding functors are far from being equivalences.

F.2. **Variant: allowing poles.** Let A and $\mathcal{Y}$ be as in the previous subsection. For $\underline{x} \in \mathrm{Ran}$, consider the prestack

$$\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y}).$$

We will now give an explicit description of $\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y})$ in terms of factorization homology. In particular, we will show that it is an ind-affine ind-scheme.

F.2.1. Write

$$\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}} \simeq \text{“}\operatorname*{colim}_{R}\text{”}\ \mathrm{Spec}(R'), \quad R' \in \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}),$$

where each

$$\mathrm{Spec}(R') \to \mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}} \tag{F.10}$$

is a closed embedding.

Consider the fiber product

$$\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y})' := \mathrm{Spec}(R') \underset{\mathfrak{L}(\mathcal{Y})_{\underline{x}}}{\times} \mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y}).$$

We will show that $\mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y})'$ is a scheme and describe the algebra of functions on it in terms of factorization homology.

F.2.2. First, we note that as in Sect. D.2.8 the datum of a closed embedding

$$\mathrm{Spec}(R') \to \mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}} \tag{F.11}$$

is equivalent to the datum of a modification A' of A at $\underline{x}$, i.e., A' is an object of $\mathrm{ComAlg}(\text{D-mod}(X)^{\leq 0})$ equipped with an isomorphism

$$A'|_{X-\underline{x}} \simeq A|_{X-\underline{x}}.$$

Indeed, given A', set

$$\mathcal{Y}' := \mathrm{Spec}_X(A'), \tag{F.12}$$

and we recover R' as $\mathcal{A}'_{\underline{x}}$, where $\mathcal{A}' := \mathrm{Fact}(A')$, and (F.11) as

$$\mathrm{Spec}(R') \simeq \mathfrak{L}^+_\nabla(\mathcal{Y}') \hookrightarrow \mathfrak{L}_\nabla(\mathcal{Y}') \simeq \mathfrak{L}_\nabla(\mathcal{Y}').$$

F.2.3. Vice versa, given (F.11), we interpret it as a map of commutative D-algebras

$$j_* \circ j^*(A) \to R'(\!(t)\!)$$

(here t is a coordinate on the multidisc $\mathcal{D}_{\underline{x}}$ and j is the open embedding $X - \underline{x} \hookrightarrow X$), where the condition that (F.11) is a closed embedding corresponds to the condition that the map of D-modules

$$j_* \circ j^*(A) \to R'(\!(t)\!) \to R'(\!(t)\!)/R'[\![t]\!] \tag{F.13}$$

is surjective on H^0.

We recover A' as the fiber product

$$j_* \circ j^*(A) \underset{R'(\!(t)\!)}{\times} R[\![t]\!],$$

and the surjectivity of (F.13) is equivalent to the condition that A' is connective.

F.2.4. Let R' be as in (F.11), and let A' be the corresponding modification of A. Unwinding the definitions, we obtain:

$$\mathrm{Spec}(R') \underset{\mathfrak{L}(\mathcal{Y})_{\underline{x}}}{\times} \mathrm{Sect}_\nabla(X - \underline{x}, \mathcal{Y}) \simeq \mathrm{Sect}_\nabla(X - x, \mathcal{Y}').$$

Hence, by Proposition F.1.4, we obtain:

Corollary F.2.5. *The prestack* $\mathrm{Spec}(R') \underset{\mathfrak{L}(\mathcal{Y})_{\underline{x}}}{\times} \mathrm{Sect}_\nabla(X - x, \mathcal{Y})$ *is affine and the naturally defined map*

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}') \to \mathcal{O}_{\mathrm{Spec}(R') \underset{\mathfrak{L}(\mathcal{Y})_{\underline{x}}}{\times} \mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y})}$$

is an isomorphism, where $\mathcal{A}' := \mathrm{Fact}(A')$.

F.2.6. We will now rewrite $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}')$ in terms of factorization homology of $\mathcal{A}$ itself.

We can view R' as an object of $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}}$ via

$$R' \in \mathcal{A}'\text{-mod}^{\mathrm{com}}_{\underline{x}} \to \mathcal{A}'\text{-mod}^{\mathrm{fact}}_{\underline{x}} \simeq \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}}.$$

When viewed as such, we will denote it by $R'_{\mathcal{A}}$. Let $R'_{\mathcal{A},\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}}$ be the corresponding object of $\text{D-mod}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}})$.

Recall (see Sect. B.10.7) that since $\mathcal{A}$ is a *commutative* factorization algebra, the category $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}}$ has a natural symmetric pseudo-monoidal structure. The commutative algebra structure on R' endows $R'_{\mathcal{A}}$ with a structure of commutative algebra object in $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}}$. In particular, $R'_{\mathcal{A},\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}}$ acquires a structure of commutative algebra object in $\text{D-mod}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}})$.

F.2.7. The assertion of Lemma C.5.9 is valid for $\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}$, hence

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},R'_{\mathcal{A}})_{\underline{x}} := \mathrm{C}^\cdot_c(\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}, R'_{\mathcal{A},\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}})$$

acquires a structure of commutative algebra in Vect.

We claim:

Lemma F.2.8. *There is a canonical isomorphism*

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},R'_{\mathcal{A}})_{\underline{x}} \simeq \mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}') \tag{F.14}$$

as objects of ComAlg(Vect).

Proof. Note that with respect to the (symmetric monoidal) equivalence

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \simeq \mathcal{A}'\text{-mod}^{\mathrm{fact}}_{\underline{x}},$$

the object

$$R'_{\mathcal{A}} \in \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}}$$

corresponds to the vacuum object

$$(\mathcal{A}')^{\mathrm{fact}_{\underline{x}}} \in \mathcal{A}'\text{-mod}^{\mathrm{fact}}_{\underline{x}},$$

where we recall that $(\mathcal{A}')^{\mathrm{fact}_{\underline{x}}}$ denotes the vacuum factorization module at $\underline{x}$.

Moreover, this isomorphism respects the structure of commutative algebra object on both sides. In particular,

$$R'_{\mathcal{A},\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}} \simeq \mathcal{A}'_{\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}}$$

as objects of ComAlg(D-mod($\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}$)).

Now, (F.14) follows by concatenating

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},R'_{\mathcal{A}})_{\underline{x}} \simeq \mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}',(\mathcal{A}')^{\mathrm{fact}_{\underline{x}}})_{\underline{x}},$$

and

$$\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}',(\mathcal{A}')^{\mathrm{fact}_{\underline{x}}})_{\underline{x}} \overset{\text{unitality}}{\simeq} \mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}').$$

□

F.2.9. Thus, $\mathrm{Spec}(\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},R'_{\mathcal{A}})_{\underline{x}})$ gives the desired expression for

$$\mathrm{Spec}(R') \underset{\mathfrak{L}(\mathcal{Y})_{\underline{x}}}{\times} \mathrm{Sect}_\nabla(X-x,\mathcal{Y})$$

in terms of factorization homology of $\mathcal{A}$.

F.2.10. Let

$$R'_{\mathcal{A},\mathrm{Ran}_{\underline{x}}} \in \mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}_{\underline{x}}))$$

be the restriction of $R'_{\mathcal{A},\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}}$ to the non-unital Ran space.

By construction, it belongs to $\mathrm{ComAlg}(\text{D-mod}(\mathrm{Ran}_{\underline{x}})^{\text{almost-untl}})$. Hence, by Sect. C.5.15,

$$\mathrm{C}^\cdot_c(\mathrm{Ran}_{\underline{x}}, R'_{\mathcal{A},\mathrm{Ran}_{\underline{x}}})$$

acquires a commutative algebra structure.

By Lemma C.5.12 we have an isomorphism

$$\mathrm{C}^\cdot_c(\mathrm{Ran}_{\underline{x}}, R'_{\mathcal{A},\mathrm{Ran}_{\underline{x}}}) \simeq \mathrm{C}^\cdot_c(\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}, R'_{\mathcal{A},\mathrm{Ran}_{\underline{x}}^{\mathrm{untl}}}) = \mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},R')_{\underline{x}}$$

as commutative algebras.

F.3. **Factorization with poles.** In his subsection we will add a slightly different spin to the discussion in Sect. F.2.

F.3.1. Let $\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\text{Ran}^\subseteq}$ be the factorization ind-scheme over $\text{Ran}^\subseteq$ that attaches to

$$(\underline{x} \subseteq \underline{x}')$$

the space

$$\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\underline{x}\subseteq\underline{x}'} := \text{Sect}_\nabla(\mathcal{D}_{\underline{x}'} - \underline{x}, \mathcal{Y}).$$

We have the projections

$$\mathfrak{L}_\nabla(\mathcal{Y}) \overset{\text{pr}^{\mathcal{Y}}_{\text{small}}}{\longleftarrow} \mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\text{Ran}^\subseteq} \overset{\text{pr}^{\mathcal{Y}}_{\text{big}}}{\longrightarrow} \mathfrak{L}_\nabla(\mathcal{Y}) \tag{F.15}$$

given by restrictions along

$$\mathcal{D}_{\underline{x}} - \underline{x} \hookrightarrow \mathcal{D}_{\underline{x}'} - \underline{x} \hookleftarrow \mathcal{D}_{\underline{x}'} - \underline{x}',$$

respectively.

F.3.2. *Example.* When

$$\underline{x}' = \underline{x} \sqcup \underline{x}'',$$

we have

$$\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\underline{x}\subseteq\underline{x}'} \simeq \mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}} \times \mathfrak{L}^+_\nabla(\mathcal{Y})_{\underline{x}''}.$$

F.3.3. Fix $\underline{x}$, and consider the space

$$\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\text{Ran}_{\underline{x}}} := \mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\text{Ran}^\subseteq} \underset{\text{Ran}^\subseteq}{\times} \text{Ran}_{\underline{x}}.$$

It has a natural structure of factorization module space with respect to $\mathfrak{L}^+_\nabla(\mathcal{Y})$, with the underlying space being $\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}$.

Moreover, viewed as such, $\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\text{Ran}_{\underline{x}}}$ has a natural *counital* structure (see Sect. C.6.7 for what this means).

In particular, we have the map

$$\text{pr}^{\mathcal{Y}}_{\text{small},\underline{x}} : \mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\text{Ran}_{\underline{x}}} \to \mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}.$$

F.3.4. Fix a closed embedding $\text{Spec}(R') \to \mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}$, and consider the fiber product

$$\text{Spec}(R') \underset{\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}}{\times} \mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\text{Ran}_{\underline{x}}}.$$

Unwinding the definitions, we obtain:

Lemma F.3.5. *We have a canonical isomorphism*

$$\text{Spec}(R') \underset{\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}}{\times} \mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y})_{\text{Ran}_{\underline{x}}} \simeq \mathfrak{L}^+_\nabla(\mathcal{Y}')_{\text{Ran}_{\underline{x}}},$$

where $\mathcal{Y}'$ *is as in* (F.12).

F.4. **Sections of quasi-coherent sheaves, meromorphic version.**

F.4.1. Recall (see Sect. 4.3.7) that the functor

$$\Gamma(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}, -) : \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \to \text{Vect}$$

admits an an enhancement to a functor

$$\Gamma(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}, -)^{\text{enh}} : \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \to \mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}}.$$

F.4.2. Note also that since the morphism

$$\text{ev}_{\underline{x}} : \text{Sect}_\nabla(X - \underline{x}, \mathcal{Y}) \to \mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}$$

is schematic, we have a well-defined functor

$$\text{ev}^*_{\underline{x}} : \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \to \text{QCoh}_{\text{co}}(\text{Sect}_\nabla(X - \underline{x}, \mathcal{Y})).$$

F.4.3. The goal of this subsection is to prove the following:

Proposition F.4.4. *There exists a canonical isomorphism between*

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{\mathrm{ev}_{\underline{x}}^*} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y})) \xrightarrow{\Gamma(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y}),-)} \mathrm{Vect} \tag{F.16}$$

and

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{\Gamma(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}},-)^{\mathrm{enh}}} \mathcal{A}\text{-mod}_{\underline{x}}^{\mathrm{fact}} \xrightarrow{\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)_{\underline{x}}} \mathrm{Vect}\,. \tag{F.17}$$

The rest of this subsection is devoted to the proof of Proposition F.4.4.

F.4.5. First, we construct a natural transformation

$$\text{(F.17)} \to \text{(F.16)}.$$

Note that the functor

$$\mathrm{ins.\,unit}_{\underline{x}} : \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \to \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\mathrm{Ran}_{\underline{x}}})$$

is given by

$$(\mathrm{pr}^{\mathcal{Y}}_{\mathrm{big},\underline{x}})_* \circ (\mathrm{pr}^{\mathcal{Y}}_{\mathrm{small},\underline{x}})^*,$$

where

$$\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}} \xleftarrow{\mathrm{pr}^{\mathcal{Y}}_{\mathrm{small},\underline{x}}} \mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathcal{Y})_{\mathrm{Ran}_{\underline{x}}} \xrightarrow{\mathrm{pr}^{\mathcal{Y}}_{\mathrm{big},\underline{x}}} \mathfrak{L}_\nabla(\mathcal{Y})_{\mathrm{Ran}_{\underline{x}}}.$$

Hence, we obtain that the functor

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{\Gamma(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}},-)^{\mathrm{enh}}} \mathcal{A}\text{-mod}_{\underline{x}}^{\mathrm{fact}} \to \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}})$$

is given by

$$(p_{\mathrm{Ran}_{\underline{x}}})_* \circ (\mathrm{pr}^{\mathcal{Y}}_{\mathrm{small},\underline{x}})^*,$$

where

$$p : \mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathcal{Y})_{\mathrm{Ran}_{\underline{x}}} \to \mathrm{Ran}_{\underline{x}}.$$

From here we obtain a natural transformation from

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{\Gamma(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}},-)^{\mathrm{enh}}} \mathcal{A}\text{-mod}_{\underline{x}}^{\mathrm{fact}} \xrightarrow{\mathrm{ins.vac}_{\underline{x}}} \mathcal{A}\text{-mod}_{\mathrm{Ran}_{\underline{x}}}^{\mathrm{fact}} \xrightarrow{\mathbf{oblv}_{\mathcal{A},\mathrm{Ran}_{\underline{x}}}} \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}) \tag{F.18}$$

to

$$\begin{split}\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{(\mathrm{pr}^{\mathcal{Y}}_{\mathrm{small},\underline{x}})^*} \mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathcal{Y})_{\mathrm{Ran}_{\underline{x}}}) \xrightarrow{(\mathrm{ev}_{\mathrm{Ran}_{\underline{x}}})^*} \\ \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y})) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}) \xrightarrow{\Gamma(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y}),-)\otimes\mathrm{Id}} \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}).\end{split} \tag{F.19}$$

F.4.6. Note, however, that the map

$$\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y}) \times \mathrm{Ran}_{\underline{x}} \xrightarrow{\mathrm{ev}_{\mathrm{Ran}_{\underline{x}}}} \mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathcal{Y})_{\mathrm{Ran}_{\underline{x}}} \xrightarrow{\mathrm{pr}^{\mathcal{Y}}_{\mathrm{small},\underline{x}}} \mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}$$

equals

$$\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y}) \times \mathrm{Ran}_{\underline{x}} \to \mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y}) \xrightarrow{\mathrm{ev}_{\underline{x}}} \mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}.$$

Hence, we obtain that (F.19) can be rewritten as

$$\begin{split}\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{\mathrm{ev}_{\underline{x}}^*} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y})) \xrightarrow{\mathrm{Id}\otimes\omega_{\mathrm{Ran}_{\underline{x}}}} \\ \to \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y})) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}) \xrightarrow{\Gamma(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y}),-)\otimes\mathrm{Id}} \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}),\end{split} \tag{F.20}$$

which is the same as

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}}) \xrightarrow{\mathrm{ev}_{\underline{x}}^*} \mathrm{QCoh}_{\mathrm{co}}(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y})) \xrightarrow{\Gamma(\mathrm{Sect}_\nabla(X-\underline{x},\mathcal{Y}),-)} \mathrm{Vect} \xrightarrow{\omega_{\mathrm{Ran}_{\underline{x}}}} \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}}). \tag{F.21}$$

Thus, we obtain a natural transformation from (F.18) to (F.21). By adjunction, this gives rise to the desired natural transformation from (F.17) to (F.16).

F.4.7. We will now prove that the above natural transformation (F.17) → (F.16) is an isomorphism.

Write
$$\mathfrak{L}_\nabla(\mathcal{Y}) = \text{“}\underset{R'}{\operatorname{colim}}\text{”}\ \operatorname{Spec}(R') = \mathfrak{L}^+_\nabla(\mathcal{Y}')$$
as in Sect. F.2.1 and $\mathcal{Y}'$ as an (F.12), so that
$$\operatorname{Sect}_\nabla(X - \underline{x}, \mathcal{Y}) = \text{“}\underset{R'}{\operatorname{colim}}\text{”}\ \operatorname{Sect}_\nabla(X - \underline{x}, \mathcal{Y}').$$

Thus, we obtain that
$$\operatorname{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \simeq \underset{R'}{\operatorname{colim}}\, \operatorname{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}')),$$
and we obtain in order to show that (F.17) → (F.16) is an isomorphism, it suffices to show that it becomes such if we precompose both functors with
$$\operatorname{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}')) \overset{\iota'_*}{\to} \operatorname{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y}))$$
for every R' as above, where ι' denotes the corresponding map $\iota' : \mathfrak{L}^+_\nabla(\mathcal{Y}') \to \mathfrak{L}_\nabla(\mathcal{Y})$.

F.4.8. Note that for every R' as above, the functor

$$\operatorname{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}')) \overset{\iota'_*}{\to} \operatorname{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \overset{\Gamma(\mathfrak{L}_\nabla(\mathcal{Y})_{\underline{x}},-)^{\mathrm{enh}}}{\longrightarrow} \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \tag{F.22}$$

identifies with

$$\operatorname{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}')) \overset{\Gamma(\mathfrak{L}^+_\nabla(\mathcal{Y}')_{\underline{x}},-)}{\longrightarrow} \mathcal{A}'\text{-mod}^{\mathrm{com}}_{\underline{x}} \to \mathcal{A}'\text{-mod}^{\mathrm{fact}}_{\underline{x}} \simeq \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}}, \tag{F.23}$$

and the functor

$$\operatorname{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}')) \overset{\iota'_*}{\to} \operatorname{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla(\mathcal{Y})) \overset{\mathrm{ev}^*_{\underline{x}}}{\longrightarrow} \operatorname{QCoh}_{\mathrm{co}}(\operatorname{Sect}_\nabla(X-\underline{x},\mathcal{Y})) \overset{\Gamma(\operatorname{Sect}_\nabla(X-\underline{x},\mathcal{Y}),-)}{\longrightarrow} \operatorname{Vect} \tag{F.24}$$

identifies with

$$\operatorname{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}')) \overset{\mathrm{ev}^*_{\underline{x}}}{\longrightarrow} \operatorname{QCoh}_{\mathrm{co}}(\operatorname{Sect}_\nabla(X,\mathcal{Y}')) \overset{\Gamma(\operatorname{Sect}_\nabla(X,\mathcal{Y}'),-)}{\longrightarrow} \operatorname{Vect}. \tag{F.25}$$

Composing (F.22) and (F.23) with the functor $\mathrm{C}^{\mathrm{fact}}_\cdot(X, \mathcal{A}, -)_{\underline{x}}$, we obtain that the natural transformation (F.17) → (F.16), constructed above, gives rise to a natural transformation from
$$\operatorname{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}')) \overset{\Gamma(\mathfrak{L}^+_\nabla(\mathcal{Y}')_{\underline{x}},-)}{\longrightarrow} \mathcal{A}'\text{-mod}^{\mathrm{com}}_{\underline{x}} \to \mathcal{A}'\text{-mod}^{\mathrm{fact}}_{\underline{x}} \simeq \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A},-)_{\underline{x}}}{\longrightarrow} \operatorname{Vect},$$
which is the same as

$$\operatorname{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}')) \overset{\Gamma(\mathfrak{L}^+_\nabla(\mathcal{Y}')_{\underline{x}},-)}{\longrightarrow} \mathcal{A}'\text{-mod}^{\mathrm{com}}_{\underline{x}} \to \mathcal{A}'\text{-mod}^{\mathrm{fact}}_{\underline{x}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X,\mathcal{A}',-)_{\underline{x}}}{\longrightarrow} \operatorname{Vect}, \tag{F.26}$$

to (F.25).

However, unwinding the definitions, we obtain that the resulting natural transformation
$$(\mathrm{F.26}) \to (\mathrm{F.25}),$$
coincides with the natural isomorphism of Proposition F.1.6.

□[Proposition F.4.4]

F.5. **Interpretation as factorization restriction.**

F.5.1. In this section we will assume that $\mathfrak{L}_\nabla(\mathcal{Y})$ is ind-placid, and that the embeddings
$$\iota : \mathfrak{L}^+_\nabla(\mathcal{Y}) \to \mathfrak{L}_\nabla(\mathcal{Y}) \text{ and } \mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y})_{\operatorname{Ran}_{\underline{x}}} \overset{\iota^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}}{\longrightarrow} \mathfrak{L}_\nabla(\mathcal{Y})_{\operatorname{Ran}_{\underline{x}}}$$
are locally almost of finite presentation, where
$$\iota^{\mathrm{mer}\rightsquigarrow\mathrm{reg}} = \mathrm{pr}^{\mathcal{Y}}_{\mathrm{big},\underline{x}}.$$

This assumption implies that $\mathfrak{L}^+_\nabla(\mathcal{Y})$ (resp., $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y})_{\operatorname{Ran}_{\underline{x}}}$) is placid (resp., ind-placid).

F.5.2. *Example.* The above assumptions hold for $\mathcal{Y} = \mathrm{Jets}(\mathcal{E})$, where $\mathcal{E}$ is the total space of a vector bundle on X.

In particular, they hold for $\mathcal{Y} = \mathrm{Op}_{\check{G}}$.

F.5.3. We can consider the factorization categories

$$\mathrm{IndCoh}^*(\mathfrak{L}^+_\nabla(\mathcal{Y})) \text{ and } \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y}))$$

and the factorization functor

$$\iota_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathfrak{L}^+_\nabla(\mathcal{Y})) \to \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y})). \tag{F.27}$$

F.5.4. Consider the assignment

$$(\mathcal{Z} \to \mathrm{Ran}_{\underline{x}}) \mapsto \mathrm{IndCoh}^*(\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y})_{\mathrm{Ran}_{\underline{x}}} \underset{\mathrm{Ran}_{\underline{x}}}{\times} \mathcal{Z})$$

as a crystal of categories over $\mathrm{Ran}_{\underline{x}}$.

It has a natural structure of factorization module category with respect to $\mathrm{IndCoh}^*(\mathfrak{L}^+_\nabla(\mathcal{Y}))$; when viewed in this capacity we will denote it by $\mathrm{IndCoh}^*(\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y}))^{\mathrm{fact}_{\underline{x}}}$.

F.5.5. The map $\iota^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}$ gives rise to a functor

$$(\iota^{\mathrm{mer}\rightsquigarrow\mathrm{reg}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}^*(\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y}))^{\mathrm{fact}_{\underline{x}}} \to \mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y}))^{\mathrm{fact}_{\underline{x}}} \tag{F.28}$$

as factorization module categories, compatible with the factorization functor (F.27).

Hence, by Sect. B.9.25, we obtain a functor

$$\mathrm{IndCoh}^*(\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_\nabla(\mathcal{Y}))^{\mathrm{fact}_{\underline{x}}} \to \mathrm{Res}_{\iota_*^{\mathrm{IndCoh}}}(\mathrm{IndCoh}^*(\mathfrak{L}_\nabla(\mathcal{Y}))^{\mathrm{fact}_{\underline{x}}}). \tag{F.29}$$

F.5.6. We claim:

Lemma F.5.7. *The functor* (F.29) *is an equivalence.*

Proof. Repeats that of Lemma 16.2.8. □

Appendix G. From module categories over $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}}_{\check{G}})$ to factorization module categories over $\mathrm{Rep}(\check{G})$

This section is not logically necessary for the rest of the paper. Here we explain a procedure that associates to a module category over $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}}_{\check{G}})$ a factorization module category over $\mathrm{Rep}(\check{G})$, and show that this functor is fully faithful (on a certain subcategory).

Using this, we deduce an alternative proof of Proposition 7.5.7.

G.1. **Creating factorization modules categories.** Throughout this section we will work at a fixed point $x \in X$. In this subsection we let H be an arbitrary algebraic group.

G.1.1. Consider the space $\mathrm{LS}^{\mathrm{mer}}_{H,x}$ and the monoidal category $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}}_{H,x})$. Let us recall the construction of a functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}}_{H,x})\text{-}\mathbf{mod} \to \mathrm{Rep}(H)\text{-}\mathbf{mod}^{\mathrm{fact}}_x, \quad \mathbf{C} \mapsto \mathbf{C}^{\mathrm{fact}_x,\mathrm{Rep}(H)}. \tag{G.1}$$

G.1.2. Namely, we will construct an object

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_H)^{\mathrm{fact}_x,\mathrm{Rep}(H)} \in \mathrm{Rep}(H)\text{-}\mathbf{mod}^{\mathrm{fact}}_x$$

that carries a commuting action of $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}}_{H,x})$. The functor (G.1) will then be given by

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_H)^{\mathrm{fact}_x,\mathrm{Rep}(H)} \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}}_{H,x})}{\otimes} -.$$

G.1.3. The object $\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{mer}\rightsquigarrow\mathrm{reg}})^{\mathrm{fact}_x,\mathrm{Rep}(H)}$ will have the feature that its underlying DG category, equipped with an action of $\mathrm{QCoh}(\mathrm{LS}_{H,x}^{\mathrm{mer}})$, identifies with $\mathrm{QCoh}(\mathrm{LS}_{H,x}^{\mathrm{mer}})$ itself.

This will imply that the functor (G.1) has the feature that for $\mathbf{C} \in \mathrm{QCoh}(\mathrm{LS}_{H,x}^{\mathrm{mer}})\text{-}\mathbf{mod}$, the category underlying $\mathbf{C}^{\mathrm{fact}_x,\mathrm{Rep}(H)}$ identifies with the original $\mathbf{C}$.

G.1.4. The object $\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{mer}\rightsquigarrow\mathrm{reg}})^{\mathrm{fact}_x,\mathrm{Rep}(H)}$ is constructed as follows.

For our fixed point $x \in X$ and a finite subset $x \in \underline{x} \subset X$, consider the multi-disc $\mathcal{D}_{\underline{x}}$, and set

$$(\mathrm{QCoh}(\mathrm{LS}_H^{\mathrm{mer}\rightsquigarrow\mathrm{reg}})^{\mathrm{fact}_x,\mathrm{Rep}(H)})_{\underline{x}} := \mathrm{QCoh}(\mathrm{LS}_{H,x\subseteq\underline{x}}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}),$$

see Sect. C.10.11.

Remark G.1.5. The construction above is a cousin of the following construction for affine D-schemes: starting from a module category over $\mathrm{QCoh}(\mathfrak{L}_\nabla(\mathcal{Y}))$ we can associate to it a factorization module category over $\mathrm{QCoh}(\mathfrak{L}_\nabla^+(\mathcal{Y}))$ by the operation of tensoring with

$$\mathrm{QCoh}_{\mathrm{co}}(\mathfrak{L}_\nabla^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(\mathcal{Y}))^{\mathrm{fact}_{\underline{x}}},$$

see Sect. F.5.4.

Remark G.1.6. This construction also be viewed as an analog of a construction in [CFGY] that associates to a $\mathfrak{L}(G)_x$-module category a factorization module category over $\mathrm{D}\text{-}\mathrm{mod}(\mathrm{Gr}_G)$; in fact a version of the latter construction has appeared in Sect. 4.7.8.

G.1.7. In what follows we will need a variant of the above construction, when instead of $\mathrm{LS}_{H,x}^{\mathrm{mer}}$ we use its formal completion $(\mathrm{LS}_{H,x}^{\mathrm{mer}})^\wedge_{\mathrm{reg}}$ around $\mathrm{LS}_{H,x}^{\mathrm{reg}}$.

The corresponding functor

$$\mathrm{QCoh}((\mathrm{LS}_{H,x}^{\mathrm{mer}})^\wedge_{\mathrm{reg}})\text{-}\mathbf{mod} \to \mathrm{Rep}(H)\text{-}\mathbf{mod}_x^{\mathrm{fact}}, \quad \mathbf{C} \mapsto \mathbf{C}^{\mathrm{fact}_x,\mathrm{Rep}(H)} \tag{G.2}$$

is constructed using the prestack

$$(x \subseteq \underline{x}) \rightsquigarrow (\mathrm{LS}_{H,x\subseteq\underline{x}}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}})^\wedge_{\mathrm{reg}},$$

i.e., the formal completion of $\mathrm{LS}_{H,x\subseteq\underline{x}}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}$ along $(\mathrm{LS}_H^{\mathrm{reg}})_{\underline{x}}^{\mathrm{fact}_x}$.

G.1.8. *Example.* Consider $\mathrm{QCoh}(\mathrm{LS}_{H,x}^{\mathrm{reg}}) \simeq \mathrm{Rep}(H)$ as an object of $\mathrm{QCoh}((\mathrm{LS}_{H,x}^{\mathrm{mer}})^\wedge_{\mathrm{reg}})\text{-}\mathbf{mod}$, via the restriction functor

$$\mathrm{QCoh}((\mathrm{LS}_{H,x}^{\mathrm{mer}})^\wedge_{\mathrm{reg}}) \to \mathrm{QCoh}(\mathrm{LS}_{H,x}^{\mathrm{reg}}).$$

We claim that

$$\mathrm{QCoh}(\mathrm{LS}_{H,x}^{\mathrm{reg}})^{\mathrm{fact}_x,\mathrm{Rep}(H)} \simeq \mathrm{Rep}(H)^{\mathrm{fact}_x}, \tag{G.3}$$

i.e., the vacuum object of $\mathrm{Rep}(H)\text{-}\mathbf{mod}_x^{\mathrm{fact}}$.

Indeed, this follows from [GaRo3, Chapter 3, Proposition 3.5.3] using the fact that

$$(\mathrm{LS}_{H,\underline{x}\subseteq\underline{x}'}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}})^\wedge_{\mathrm{reg}} \underset{(\mathrm{LS}_{H,x}^{\mathrm{mer}})^\wedge_{\mathrm{reg}}}{\times} \mathrm{LS}_{H,x}^{\mathrm{reg}} \simeq (\mathrm{LS}_H^{\mathrm{reg}})_{\underline{x}}^{\mathrm{fact}_x}.$$

When applying [GaRo3, Chapter 3, Proposition 3.5.3] we use the fact that we can identify

$$(\mathrm{LS}_{H,x}^{\mathrm{mer}})^\wedge_{\mathrm{reg}} \simeq \mathfrak{h}_0^\wedge / \mathrm{Ad}(H),$$

so this prestack is passable.

G.1.9. We claim:

Theorem G.1.10. *The functor* (G.2) *is fully faithful.*

This theorem is proved in [Ra4, Theorem 9.13.1]. We will supply a proof for completeness.

Remark G.1.11. We conjecture that the functor (G.1) is itself fully faithful. A partial result in this direction has recently been established in [Bogd]: the composition (G.1) with the restriction functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{H,x})\text{-}\mathbf{mod} \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{mer}}_{H,x})\text{-}\mathbf{mod}$$

is fully faithful, where

$$\mathrm{LS}^{\mathrm{restr}}_{H,x} \subset \mathrm{LS}^{\mathrm{mer}}_{H,x}$$

is the stack of local systems with restricted variation (see [AGKRRV, Sect. 1.4]).

Remark G.1.12. The reason the discussion in this section is for a fixed point in the Ran space is that we do not know how to prove the analog of Theorem G.1.10 in the factorization setting (and are not confident in its validity).

G.2. **Proof of Theorem G.1.10.** The proof is a "baby version" of the argument proving the corresponding assertion in [CFGY], see Remark G.1.6.

G.2.1. *Reduction steps.* First, we note that the functor (G.2) preserves both limits and colimits.

Second, the (symmetric monoidal) restriction functor

$$\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^{\wedge}_{\mathrm{reg}}) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,x})$$

is comonadic, and its right adjoint is $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^{\wedge}_{\mathrm{reg}})$-linear.

It follows that the 2-category $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^{\wedge}_{\mathrm{reg}})\text{-}\mathbf{mod}$ is generated under colimits by the essential image of

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,x})\text{-}\mathbf{mod} \to \mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^{\wedge}_{\mathrm{reg}})\text{-}\mathbf{mod}, \tag{G.4}$$

given by restriction along

$$\mathrm{LS}^{\mathrm{reg}}_{H,x} \to (\mathrm{LS}^{\mathrm{mer}}_{H,x})^{\wedge}_{\mathrm{reg}}.$$

Moreover, by passing to right adjoints, the 2-category $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^{\wedge}_{\mathrm{reg}})\text{-}\mathbf{mod}$ is also generated under *limits* by the essential image of (G.4).

Third, $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{reg}}_{H,x})\text{-}\mathbf{mod} \simeq \mathrm{Rep}(H)\text{-}\mathbf{mod}$ is generated under colimits (and separately, limits) by objects of the form $\mathbf{C} \otimes \mathrm{Rep}(H)$, for $\mathbf{C} \in \mathrm{DGCat}$.

And fourth, since

$$\mathrm{Rep}(H) \simeq \mathrm{Vect} \underset{\mathrm{QCoh}(H)}{\otimes} \mathrm{Vect},$$

we obtain that the object $\mathrm{Rep}(H) \in \mathrm{Rep}(H)\text{-}\mathbf{mod}$ is a colimit of objects on which the action of $\mathrm{Rep}(H)$ is trivial, i.e., factors via the augmentation functor

$$\mathrm{Rep}(H) \to \mathrm{Vect},$$

corresponding to

$$\mathrm{pt} \to \mathrm{LS}^{\mathrm{reg}}_{H,x}.$$

Combining these observations, we obtain that the 2-category $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^{\wedge}_{\mathrm{reg}})\text{-}\mathbf{mod}$ is generated under colimits by trivial modules and under limits by objects of the form $\mathbf{C}_0 \otimes \mathrm{Rep}(H)$ for a DG category $\mathbf{C}_0$ with a trivial action of $\mathrm{Rep}(H)$.

Hence, it is enough to show that the functor

$$\begin{aligned} &\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^{\wedge}_{\mathrm{reg}})\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathbf{C}_0 \otimes \mathrm{Rep}(H)) \to \\ &\qquad \to \mathrm{Funct}_{\mathrm{Rep}(H)\text{-}\mathbf{mod}^{\mathrm{fact}}_x}\left(\mathrm{Vect}^{\mathrm{fact}_x, \mathrm{Rep}(H)}, \mathbf{C}_0 \otimes \mathrm{Rep}(H)^{\mathrm{fact}_x}\right) \end{aligned} \tag{G.5}$$

is an equivalence.

G.2.2. Recall that for a (unital) factorization functor $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ and an object $\mathbf{C}_2 \in \mathbf{A}_2\text{-}\mathbf{mod}_x^{\mathrm{fact}}$, we denote by $\mathrm{Res}_\Phi(\mathbf{C}_2)$ its restriction along Φ.

Consider the tautological object $\mathrm{Vect}^{\mathrm{fact}_x} \in \mathrm{Vect}\text{-}\mathbf{mod}_x^{\mathrm{fact}}$, and the corresponding object

$$\mathrm{Res}_{\mathbf{oblv}_H}(\mathrm{Vect}^{\mathrm{fact}_x}) \in \mathrm{Rep}(H)\text{-}\mathbf{mod}_x^{\mathrm{fact}},$$

where $\mathbf{oblv}_H : \mathrm{Rep}(H) \to \mathrm{Vect}$ is the forgetful functor, viewed as a (strictly unital) factorization functor.

Pullback along the unit section of $(\mathrm{LS}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}_{H,\underline{x}\subseteq\underline{x}'})^\wedge_{\mathrm{reg}}$ gives rise to a functor

$$\mathrm{Vect}^{\mathrm{fact}_x,\mathrm{Rep}(H)} \to \mathrm{Vect}^{\mathrm{fact}_x} \tag{G.6}$$

compatible with factorization (in the sense of Sect. C.14.11). Hence, we obtain a morphism

$$\mathrm{Vect}^{\mathrm{fact}_x,\mathrm{Rep}(H)} \to \mathrm{Res}_{\mathbf{oblv}_H}(\mathrm{Vect}^{\mathrm{fact}_x}) \tag{G.7}$$

as unital factorization modules over $\mathrm{Rep}(H)$.

G.2.3. We claim:

Lemma G.2.4. *The functor* (G.7) *is an equivalence.*

Proof. Follows from Lemma B.15.9. □

G.2.5. Recall now the paradigm of Lemma C.15.16. We apply it to $\mathbf{oblv}_H : \mathrm{Rep}(H) \to \mathrm{Vect}$. Combined with Lemma G.2.4 above, we obtain that for $\mathbf{C} \in \mathrm{Rep}(H)\text{-}\mathbf{mod}_x^{\mathrm{fact}}$ we have a canonical equivalence

$$\mathrm{Funct}_{\mathrm{Rep}(H)\text{-}\mathbf{mod}_x^{\mathrm{fact}}}\left(\mathrm{Vect}^{\mathrm{fact}_x,\mathrm{Rep}(H)}, \mathbf{C}\right) \simeq R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathbf{C}_x). \tag{G.8}$$

G.2.6. Taking $\mathbf{C} := \mathrm{Rep}(H)^{\mathrm{fact}_x} \otimes \mathbf{C}_0$ in (G.8) and using (G.3), we obtain that (G.5) reduces to showing that the resulting functor

$$\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^\wedge_{\mathrm{reg}})\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathrm{Rep}(H) \otimes \mathbf{C}_0) \to R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H) \otimes \mathbf{C}_0) \tag{G.9}$$

is an equivalence.

It is clear that the functor

$$\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^\wedge_{\mathrm{reg}})\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathrm{Rep}(H)) \otimes \mathbf{C}_0 \to \mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^\wedge_{\mathrm{reg}})\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathrm{Rep}(H) \otimes \mathbf{C}_0)$$

is an equivalence (since $\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^\wedge_{\mathrm{reg}}$ is a semi-rigid monoidal category, see [AGKRRV, Sect. C]).

Now the fact that R_H is holonomic implies that

$$R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)) \otimes \mathbf{C}_0 \to R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H) \otimes \mathbf{C}_0)$$

is also an equivalence (see [Ra4, Theorem 8.13.1]).

Hence, it is enough to show that the functor

$$\mathrm{Funct}_{\mathrm{QCoh}((\mathrm{LS}^{\mathrm{mer}}_{H,x})^\wedge_{\mathrm{reg}})\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathrm{Rep}(H)) \to R_H\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(H)) \tag{G.10}$$

is an equivalence.

G.2.7. We rewrite the left-hand side in (G.10) as

$$\mathrm{QCoh}(\mathrm{pt} \underset{(\mathrm{LS}^{\mathrm{mer}}_{H,x})^\wedge_{\mathrm{reg}}}{\times} \mathrm{LS}^{\mathrm{reg}}_{H,x}) \simeq \mathrm{QCoh}(\mathrm{pt} \underset{\mathrm{LS}^{\mathrm{mer}}_{H,x}}{\times} \mathrm{LS}^{\mathrm{reg}}_{H,x}).$$

G.2.8. By Sect. 4.5.4, the category $(R_H\text{-mod}^{\text{fact}}(\text{Rep}(H)))^{>-\infty}$ identifies with

$$\text{QCoh}_{\text{co}}\Big(\text{pt} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x}\Big)^{>-\infty}.$$

Since $R_H\text{-mod}^{\text{fact}}(\text{Rep}(H))$ is left-complete (see Proposition B.9.18), we obtain that it identifies with the left completion of

$$\text{QCoh}_{\text{co}}(\text{pt} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x}).$$

Since $\text{pt} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x}$ is almost of finite type, we have an equivalence

$$\Psi_{\text{pt} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x}} : \text{QCoh}_{\text{co}}\Big(\text{pt} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x}\Big)^{>-\infty} \overset{\sim}{\to} \text{IndCoh}\Big(\text{pt} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x}\Big)^{>-\infty}.$$

Hence, the above left-completion identifies with

$$\text{QCoh}(\text{pt} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x}).$$

G.2.9. Unwinding the constructions, we obtain that the endo-functor of $\text{QCoh}(\text{pt} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x})$, induced by (G.10) and the above two identifications is the identity.

$\square$[Theorem G.1.10]

G.3. **Factorization module categories attached to affine D-schemes.**

G.3.1. Let $\mathcal{Y}$ be an affine D-scheme over X, equipped with a map

$$\mathcal{Y} \to \text{pt}/H.$$

Consider the corresponding factorization spaces

$$\mathfrak{L}^+_\nabla(\mathcal{Y}) \overset{\iota}{\hookrightarrow} \mathfrak{L}_\nabla(\mathcal{Y}) \tag{G.11}$$

and the commutative (but *not necessarily* Cartesian) diagram

$$\begin{tikzcd} \mathfrak{L}^+_\nabla(\mathcal{Y}) \arrow[r] \arrow[d, "\mathfrak{r}^{\text{reg}}"'] & \mathfrak{L}_\nabla(\mathcal{Y}) \arrow[d, "\mathfrak{r}"] \\ \text{LS}^{\text{reg}}_H \arrow[r] & \text{LS}^{\text{mer}}_H \end{tikzcd}$$

G.3.2. On the one hand, we can consider $\text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_x)$ as an object of $\text{QCoh}(\text{LS}^{\text{mer}}_{H,x})$-**mod**. Consider the resulting object

$$\text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_x)^{\text{fact}_x,\text{Rep}(H)} \in \text{Rep}(H)\text{-}\mathbf{mod}^{\text{fact}}. \tag{G.12}$$

G.3.3. On the other hand, consider the $\text{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y}))$-factorization category

$$\text{QCoh}_{\text{co}}(\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y}))^{\text{fact}_x},$$

defined as in Sect. F.5.4.

Pullback along

$$\mathfrak{r}^{\text{reg}} : \mathfrak{L}^+_\nabla(\mathcal{Y}) \to \text{LS}^{\text{reg}}_H$$

defines a (strictly unital) factorizaton functor

$$\text{Rep}(H) \simeq \text{QCoh}(\text{LS}^{\text{reg}}_H) \overset{(\mathfrak{r}^{\text{reg}})^*}{\longrightarrow} \text{QCoh}(\mathfrak{L}^+_\nabla(\mathcal{Y})).$$

Consider the resulting object

$$\text{Res}_{(\mathfrak{r}^{\text{reg}})^*}\Big(\text{QCoh}_{\text{co}}(\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y}))^{\text{fact}_x}\Big) \in \text{Rep}(H)\text{-}\mathbf{mod}^{\text{fact}}_x. \tag{G.13}$$

G.3.4. Note that the natural morphism

$$\mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y})_{x\subseteq\underline{x}} \to \text{LS}^{\text{mer}\rightsquigarrow\text{reg}}_{H,x\subseteq\underline{x}} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \mathfrak{L}_\nabla(\mathcal{Y})_x \tag{G.14}$$

is affine.

Hence, pullback along (G.14) gives rise to a functor

$$\text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_x)^{\text{fact}_x,\text{Rep}(H)} \to \text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y}))^{\text{fact}_x} \tag{G.15}$$

compatible with factorization.

Hence, we obtain a functor

$$\text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla(\mathcal{Y})_x)^{\text{fact}_x,\text{Rep}(H)} \to \text{Res}_{(\mathfrak{r}^{\text{reg}})^*}\left(\text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y}))^{\text{fact}_x}\right) \tag{G.16}$$

in $\text{Rep}(H)\text{-}\mathbf{mod}^{\text{fact}}$.

G.3.5. *Variant.* Let $\mathfrak{L}_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}}$ denote the formal completion of $\mathfrak{L}_\nabla(\mathcal{Y})$ along

$$\mathfrak{L}_\nabla^{\text{mon-free}}(\mathcal{Y}) := \mathfrak{L}_\nabla(\mathcal{Y}) \underset{\text{LS}^{\text{mer}}_H}{\times} \text{LS}^{\text{reg}}_H \to \mathfrak{L}_\nabla(\mathcal{Y}),$$

or which is the same

$$\mathfrak{L}_\nabla(\mathcal{Y}) \underset{\text{LS}^{\text{mer}}_H}{\times} (\text{LS}^{\text{mer}}_H)^\wedge_{\text{reg}}.$$

Similar to the above, we can consider

$$\text{QCoh}_{\text{co}}((\mathfrak{L}_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}})_x)^{\text{fact}_x,\text{Rep}(H)} \in \text{Rep}(H)\text{-}\mathbf{mod}^{\text{fact}}$$

and

$$\text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y})^\wedge_{\text{mon-free}})^{\text{fact}_x},$$

and we obtain a functor

$$\text{QCoh}_{\text{co}}((\mathfrak{L}_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}})_x)^{\text{fact}_x,\text{Rep}(H)} \to \text{Res}_{(\mathfrak{r}^{\text{reg}})^*}\left(\text{QCoh}_{\text{co}}(\mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y})^\wedge_{\text{mon-free}})^{\text{fact}_x}\right) \tag{G.17}$$

in $\text{Rep}(H)\text{-}\mathbf{mod}^{\text{fact}}$.

G.3.6. *Variant.* Let us assume now that $\mathfrak{L}_\nabla(\mathcal{Y})$ and also $\text{pt} \underset{\text{LS}^{\text{mer}}_H}{\times} \mathfrak{L}_\nabla(\mathcal{Y})$ satisfy the assumptions of Sect. C.12.10.

This implies that the morphism

$$\text{LS}^{\text{fact}_x}_{H,x\subseteq\underline{x}} \underset{\text{LS}^{\text{mer}\rightsquigarrow\text{reg}}_{H,x\subseteq\underline{x}}}{\times} \mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y})_{x\subseteq\underline{x}} \to \text{LS}^{\text{fact}_x}_{H,x\subseteq\underline{x}} \underset{\text{LS}^{\text{reg}}_{H,x}}{\times} \text{LS}^{\text{reg}}_{H,x} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \mathfrak{L}_\nabla(\mathcal{Y})_x = \text{LS}^{\text{fact}_x}_{H,x\subseteq\underline{x}} \underset{\text{LS}^{\text{mer}}_{H,x}}{\times} \mathfrak{L}_\nabla(\mathcal{Y})_x$$

is of finite Tor-dimension.

Hence, so is the morphism

$$(\mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y})_{x\subseteq\underline{x}})^\wedge_{\text{mon-free}} \to \left(\text{LS}^{\text{mer}\rightsquigarrow\text{reg}}_{H,x\subseteq\underline{x}}\right)^\wedge_{\text{mon-free}} \underset{(\text{LS}^{\text{mer}}_{H,x})^\wedge_{\text{reg}}}{\times} (\mathfrak{L}_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}})_x. \tag{G.18}$$

Hence, by Sect. A.10.13, the $(\text{IndCoh}, *)$-pullback along (G.18) gives rise to a functor

$$\text{IndCoh}^*\left((\mathfrak{L}_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}})_x\right)^{\text{fact}_x,\text{Rep}(H)} \to \text{IndCoh}^*\left((\mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y}))^\wedge_{\text{mon-free}}\right)^{\text{fact}_x} \tag{G.19}$$

compatible with factorization.

Hence, we obtain a functor

$$\text{IndCoh}^*\left((\mathfrak{L}_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}})_x\right)^{\text{fact}_x,\text{Rep}(H)} \to \text{Res}_{(\mathfrak{r}^{\text{reg}})^*}\left(\text{IndCoh}^*\left((\mathfrak{L}_\nabla^{\text{mer}\rightsquigarrow\text{reg}}(\mathcal{Y}))^\wedge_{\text{mon-free}}\right)^{\text{fact}_x}\right) \tag{G.20}$$

in $\text{Rep}(H)\text{-}\mathbf{mod}^{\text{fact}}$.

G.3.7. We claim:

Lemma G.3.8. *The functor* (G.20) *is an equivalence.*

Proof. Follows from Lemma B.15.9. □

G.3.9. Let $\iota^{+,\text{mon-free}}$ denote the morphism

$$\mathfrak{L}^+_\nabla(\mathcal{Y}) \to \mathfrak{L}_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}}.$$

The operation of IndCoh-pushforward along $\iota^{+,\text{mon-free}}$ gives rise to a (strictly) unital factorization functor

$$(\iota^{+,\text{mon-free}})^{\text{IndCoh}}_* : \text{IndCoh}^*(\mathfrak{L}^+_\nabla(\mathcal{Y})) \to \text{IndCoh}^*(\mathfrak{L}^+_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}}).$$

It follows from Lemma F.5.7 that the naturally defined functor

$$\text{IndCoh}^*\left((\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}_\nabla(\mathcal{Y}))^\wedge_{\text{mon-free}}\right)^{\text{fact}_x} \to \text{Res}_{(\iota^{+,\text{mon-free}})^{\text{IndCoh}}_*}\left((\text{IndCoh}^*(\mathfrak{L}^+_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}}))^{\text{fact}_x}\right)$$

is an equivalence.

Hence, combining with Lemma G.3.8, we obtain:

Corollary G.3.10. *There is a canonical equivalence*

$$\text{IndCoh}^*\left((\mathfrak{L}_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}})_x\right)^{\text{fact}_x,\text{Rep}(H)} \simeq \text{Res}_{(\iota^{+,\text{mon-free}})^{\text{IndCoh}}_*\circ(\mathfrak{r}^{\text{reg}})^*}\left((\text{IndCoh}^*(\mathfrak{L}^+_\nabla(\mathcal{Y})^\wedge_{\text{mon-free}}))^{\text{fact}_x}\right)$$

in $\text{Rep}(H)\text{-}\mathbf{mod}^{\text{fact}}$.

G.4. Proof of Proposition 7.5.7 via factorization.

G.4.1. Consider the following (strictly unital) factorization functor, to be denoted Φ_1

$$\text{(G.21)}\quad \text{Rep}(\check{G}) \overset{\text{FLE}_{\check{G},\infty}}{\simeq} \text{Whit}_*(G) \overset{\text{Id}\otimes\text{Vac}(G)_{\text{crit},\rho(\omega_X)}}{\longrightarrow}$$
$$\to \text{Whit}_*(G)\otimes\text{KL}(G)_{\text{crit},\rho(\omega_X)} \to \text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)}).$$

Restricting the vacuum factorization module category

$$\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)})^{\text{fact}_x} \in \text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)})\text{-}\mathbf{mod}^{\text{fact}}_x$$

along (G.21), we obtain an object

$$\text{(G.22)}\qquad \text{Res}_{\Phi_1}(\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)})^{\text{fact}_x}) \in \text{Rep}(\check{G})\text{-}\mathbf{mod}^{\text{fact}}_x.$$

G.4.2. We claim:

Lemma G.4.3. *The object* (G.22) *identifies canonically with*

$$\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)})^{\text{fact}_x,\text{Rep}(\check{G})},$$

where we regard $\text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)})$ *as an object of* $\text{QCoh}((\text{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})\text{-}\mathbf{mod}$ *by the recipe of Sect. 7.5.1.*

Proof. We rewrite Φ_1 as

$$\text{(G.23)}\quad \text{Rep}(\check{G}) \overset{\text{FLE}_{\check{G},\infty}}{\simeq} \text{Whit}_*(G) \overset{\text{Id}\otimes\text{Vac}(G)_{\text{crit},\rho(\omega_X)}}{\longrightarrow} \text{Whit}_*(G)\otimes\text{KL}(G)_{\text{crit},\rho(\omega_X)} \to$$
$$\to \text{Whit}_*(G)\underset{\text{Sph}_G}{\otimes}\text{KL}(G)_{\text{crit},\rho(\omega_X)} \simeq \text{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)}).$$

Therefore, we can reinterpret the object (G.22) as follows. Consider

$$\text{Res}_{\text{FLE}_{\check{G},\infty}}(\text{Whit}_*(G)^{\text{fact}_x}) \in \text{Rep}(\check{G})\text{-}\mathbf{mod}^{\text{fact}}_x$$

as an object equipped with a commuting action of $\text{Sph}_{G,x}$. Then (G.22) identifies with

$$\text{Res}_{\text{FLE}_{\check{G},\infty}}(\text{Whit}_*(G)^{\text{fact}_x})\underset{\text{Sph}_{G,x}}{\otimes}\text{KL}(G)_{\text{crit},\rho(\omega_X),x}.$$

Comparing with the definition of

$$\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)}) \in \mathrm{QCoh}((\mathrm{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})\text{-}\mathbf{mod}$$

(see Sect. 7.5.1), the assertion of the lemma follows from the identification (G.3).

□

G.4.4. Consider the following (strictly unital) factorization functor, to be denoted Φ_2

$$\mathrm{Rep}(\check{G}) \overset{(\mathfrak{r}^{\text{reg}})^*}{\longrightarrow} \mathrm{QCoh}(\mathrm{Op}^{\text{reg}}_{\check{G}}) \overset{(\iota^{\text{mon-free}})^{\text{IndCoh}}_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})_{\text{mon-free}}. \tag{G.24}$$

Restricting the vacuum factorization module category

$$(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})_{\text{mon-free}})^{\text{fact}_x} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})_{\text{mon-free}}\text{-}\mathbf{mod}^{\text{fact}}_x$$

along (G.24), we obtain an object

$$\mathrm{Res}_{\Phi_2}((\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})_{\text{mon-free}})^{\text{fact}_x}) \in \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\text{fact}}_x. \tag{G.25}$$

G.4.5. We claim:

Lemma G.4.6. *The object* (G.25) *identifies canonically with*

$$(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})_{\text{mon-free}})^{\text{fact}_x,\mathrm{Rep}(\check{G})},$$

where we regard $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})_{\text{mon-free}}$ *as an object of* $\mathrm{QCoh}((\mathrm{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}})$-**mod** *via*

$$\mathfrak{r} : (\mathrm{Op}^{\text{mer}}_{\check{G}})^\wedge_{\text{mon-free}} \to (\mathrm{LS}^{\text{mer}}_{\check{G},x})^\wedge_{\text{reg}}.$$

Proof. This is a particular case of Lemma G.3.8.

□

G.4.7. We now claim:

Proposition G.4.8. *We have a canonical isomorphism of (strictly unital) factorization functors*

$$\overline{\mathrm{DS}}^{\text{enh,rfnd}} \circ \Phi_1 \simeq \Phi_2.$$

Let us accept this proposition for a moment and finish the proof of Proposition 7.5.7.

G.4.9. By Theorem G.1.10, combined with Lemmas G.4.3 and G.4.6, it suffices to show that the functor $\overline{\mathrm{DS}}^{\text{enh,rfnd}}$ appearing in Proposition 7.5.7 can be realized as the fiber at x of a functor between

$$\mathrm{Res}_{\Phi_1}(\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\text{crit},\rho(\omega_X)})^{\text{fact}_x}) \text{ and } \mathrm{Res}_{\Phi_2}((\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})_{\text{mon-free}})^{\text{fact}_x}),$$

viewed as objects of $\mathrm{Rep}(\check{G})$-$\mathbf{mod}^{\text{fact}}_x$.

The latter structure is supplied by Proposition G.4.8.

□[Proposition 7.5.7]

G.5. Proof of Proposition G.4.8.

G.5.1. It is enough to show that the two functors match after we compose them with the fully faithful embedding

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}})_{\text{mon-free}} \hookrightarrow \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}}).$$

We will first establish an isomorphism between the compositions of the two functors in question with

$$\Gamma^{\text{IndCoh}}(\mathrm{Op}^{\text{mer}}_{\check{G}}, -)^{\text{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mer}}_{\check{G}}) \to \mathcal{O}_{\mathrm{Op}^{\text{reg}}_{\check{G}}}\text{-mod}^{\text{fact}}.$$

G.5.2. We start by rewriting the corresponding composition for Φ_1. It is equal to

$$\text{(G.26)}\quad \mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G) \xrightarrow{\mathrm{Id}\otimes \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}} \mathrm{Whit}_*(G)\otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to$$
$$\to \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \xrightarrow{\overline{\mathrm{DS}}^{\mathrm{enh}}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}.$$

By Remark 1.7.7, we rewrite this as

$$\text{(G.27)}\quad \mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}_G^{-1,\mathrm{nv}}} \mathrm{Sph}_G \xrightarrow{\sigma} \mathrm{Sph}_G \xrightarrow{-\star \mathbf{1}_{\mathrm{Whit}_*(G)}} \mathrm{Whit}_*(G) \xrightarrow{\mathrm{Id}\otimes \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}}$$
$$\to \mathrm{Whit}_*(G)\otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to$$
$$\to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \xrightarrow{\overline{\mathrm{DS}}^{\mathrm{enh}}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}.$$

We can rewrite the composition in the first two lines in (G.27) as

$$\text{(G.28)}\quad \mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}_G^{-1,\mathrm{nv}}} \mathrm{Sph}_G \xrightarrow{-\star \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \xrightarrow{\mathbf{1}_{\mathrm{Whit}_*(G)}\otimes \mathrm{Id}}$$
$$\to \mathrm{Whit}_*(G)\otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}.$$

Hence, we can rewrite (G.27) as

$$\text{(G.29)}\quad \mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}_G^{-1,\mathrm{nv}}} \mathrm{Sph}_G \xrightarrow{-\star \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}}$$
$$\to \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \xrightarrow{\mathrm{DS}^{\mathrm{enh}}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}.$$

G.5.3. We now apply Theorem 5.2.5. It says that the functor

$$\mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}_G^{-1,\mathrm{nv}}} \mathrm{Sph}_G \xrightarrow{-\star \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}$$

is isomorphic to

$$\text{(G.30)}\quad \mathrm{Rep}(\check{G}) \xrightarrow{(\mathfrak{r}^{\mathrm{reg}})^*} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \xrightarrow{\Gamma(\mathrm{Op}^{\mathrm{reg}}_{\check{G}},-)} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{com}} \xrightarrow{-\underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}}{\otimes} \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}.$$

Hence, we can rewrite (G.27) as

(G.31)

$$\mathrm{Rep}(\check{G}) \xrightarrow{(\mathfrak{r}^{\mathrm{reg}})^*} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \xrightarrow{\Gamma(\mathrm{Op}^{\mathrm{reg}}_{\check{G}},-)} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{com}} \xrightarrow{-\underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}}{\otimes} \mathrm{DS}^{\mathrm{enh}}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)})} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}.$$

G.5.4. We now use Theorem 4.8.3, which says that the natural map

$$\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}} \to \mathrm{DS}^{\mathrm{enh}}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)})$$

is an isomorphism.

This allows us to rewrite (G.31) as

$$\text{(G.32)}\quad \mathrm{Rep}(\check{G}) \xrightarrow{(\mathfrak{r}^{\mathrm{reg}})^*} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \xrightarrow{\Gamma(\mathrm{Op}^{\mathrm{reg}}_{\check{G}},-)} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{com}} \to \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}.$$

G.5.5. We now consider the composition involving Φ_2:

(G.33)

$$\mathrm{Rep}(\check{G}) \xrightarrow{(\mathfrak{r}^{\mathrm{reg}})^*} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \xrightarrow{(\iota^{\text{mon-free}})^{\mathrm{IndCoh}}_*} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}},-)^{\mathrm{enh}}} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}.$$

This functor identifies with (G.32) on the nose.

G.5.6. Thus, we have identified the compositions of the two functors

$$\mathrm{Rep}(\check{G}) \rightrightarrows \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \tag{G.34}$$

with

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}) \overset{\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}},-)^{\mathrm{enh}}}{\longrightarrow} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{fact}}.$$

Let us show how to upgrade this isomorphism to an isomorphism between the two functors in (G.34).

G.5.7. It is enough to establish the isomorphism between the two functors in question on the compact generators of $\mathrm{Rep}(\check{G})$. These generators can be taken to be eventually coconnective. Hence, by Corollary 4.4.2(a), it is enough to show that both functors are t-exact (in fact, it is sufficient to know that they are t-exact on $\mathrm{Rep}(\check{G})^c$).

The t-exactness of the composition involving Φ_2 is clear. For Φ_1, we interpret it as

(G.35)

$$\mathrm{Rep}(\check{G}) \overset{(\mathfrak{r}^{\mathrm{reg}})^*}{\longrightarrow} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{reg}}_{\check{G}}) \overset{\Gamma(\mathrm{Op}^{\mathrm{reg}}_{\check{G}},-)}{\longrightarrow} \mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}\text{-mod}^{\mathrm{com}} \overset{-\underset{\mathcal{O}_{\mathrm{Op}^{\mathrm{reg}}_{\check{G}}}}{\otimes} \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to$$
$$\to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh,rfnd}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}}).$$

As was remarked already, it is enough to show that this functor is t-exact on $\mathrm{Rep}(\check{G})^c$. The composition in the first line of (G.35) is t-exact. Hence, by the construction of $\mathrm{DS}^{\mathrm{enh,rfnd}}$, the corresponding functor

$$\mathrm{Rep}(\check{G})^c \to \mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})$$

maps to $\mathrm{IndCoh}^*(\mathrm{Op}^{\mathrm{mer}}_{\check{G}})^{>-\infty}$, and its t-exactness follows from the t-exactness of its composition with $\Gamma^{\mathrm{IndCoh}}(\mathrm{Op}^{\mathrm{mer}}_{\check{G}},-)^{\mathrm{enh}}$, while the latter is the functor (G.32), which is evidently t-exact.

Appendix H. Unital local-to-global functors and monoidal actions

This sections serves as a complement to Sect. 11. Here we express the notion of a (strictly) unital local-to-global functor as a functor of what we call the *independent* category.

This will allow us to study the interaction between various global monoidal categories attached to a local unital monoidal categories. These various variants are handy when studying the pattern of the Hecke action.

H.1. The "independent" category.

H.1.1. Let $\underline{\mathbf{C}}^{\mathrm{loc,untl}}$ be a crystal of categories over $\mathrm{Ran}^{\mathrm{untl}}$. Denote

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}} := \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{C}}^{\mathrm{loc,untl}}).$$

For example, when $\underline{\mathbf{C}}^{\mathrm{loc,untl}}$ is the unit sheaf of categories, i.e., $\underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$, the above category is $\text{D-mod}(\mathrm{Ran}^{\mathrm{untl}})$.

H.1.2. Let $\mathbf{C}^{\mathrm{glob}}$ be a target DG category. On the one hand we can consider the category $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob,lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$ of lax unital local-to-global functors, i.e., right-lax functors

$$\underline{\mathsf{F}}^{\mathrm{untl}} : \underline{\mathbf{C}}^{\mathrm{loc,untl}} \to \mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$$

between sheaves of categories, see Sect. 11.3.8.

On the other hand, we can consider the category $\mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}, \mathbf{C}^{\mathrm{glob}})$ of (continuous) functors

$$\mathsf{F}^{\mathrm{untl}} : \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathbf{C}^{\mathrm{glob}}.$$

There is a naturally defined functor

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob,lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}, \mathbf{C}^{\mathrm{glob}}). \tag{H.1}$$

Namely, given $\underline{\mathsf{F}}^{\mathrm{untl}}$, we construct $\mathsf{F}^{\mathrm{untl}}$ by applying the functor $\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}},-)$, followed by

$$\mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}) \overset{\mathrm{Id}\otimes \mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}},-)}{\longrightarrow} \mathbf{C}^{\mathrm{glob}}.$$

Remark H.1.3. Note that unlike the case of the usual Ran space, the functor (H.1) is *not* an equivalence.

For example, for

$$\underline{\mathbf{C}}^{\mathrm{loc,untl}} = \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}^{\mathrm{untl}}) \text{ and } \mathbf{C}^{\mathrm{glob}} = \mathrm{Vect},$$

we have

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob,lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \simeq \mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}),$$

while $\mathrm{Funct}(\mathbf{C}_{\mathrm{Ran}^{\mathrm{untl}}}, \mathbf{C}^{\mathrm{glob}})$ identifies with $\mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}})^\vee$. In terms of this identification, the functor (H.2) is the pairing

$$\mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}) \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{Vect}, \quad \mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

However, this pairing is not perfect. The reason for this is that, although the functor $(\Delta_{\mathrm{Ran}^{\mathrm{untl}}})_!$ is defined, it does *not* satisfy the projection formula.

More generally, the category $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob,lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$ can be described explicitly as lax sections of another crystal of categories on $\mathrm{Ran}^{\mathrm{untl}}$, see Remark H.1.8.

H.1.4. From now on we will make the following assumption on $\underline{\mathbf{C}}^{\mathrm{loc,untl}}$:

For every $S \in \mathrm{Sch}^{\mathrm{aff}}$ and a map $\underline{x}_1 \overset{\alpha}{\to} \underline{x}_2$ in $\mathrm{Maps}(S, \mathrm{Ran}^{\mathrm{untl}})$, the corresponding functor

$$\mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_1} \overset{\mathrm{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}}{\to} \mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_2}$$

admits a *continuous* right adjoint.

Note that in this case, this right adjoint is automatically $\mathrm{D\text{-}mod}(S)$-linear.

Remark H.1.5. This assumption is made in order to simplify the exposition. One can make do without it, but in what follows one will have to describe $\underline{\mathbf{C}}^{\mathrm{loc,untl}}$ using *proper* (rather than affine) schemes mapping to Ran, see Sect. C.4.6.

The properness assumption would guarantee that for map $f : Z' \to Z$ (of proper schemes) and

$$(\underline{x}_1 \to \underline{x}_2) \in \mathrm{Maps}(Z, \mathrm{Ran}^{\mathrm{untl}}),$$

the diagram

$$\begin{tikzcd}
\mathbf{C}_{Z',\underline{x}_1\circ f} & \mathbf{C}_{Z',\underline{x}_2\circ f} \arrow[l, "\mathrm{ins.unit}^R_{\underline{x}_1\circ f\subseteq\underline{x}_2\circ f}"'] \\
\mathbf{C}_{Z,\underline{x}_1} \arrow[u, "f^!"] & \mathbf{C}_{Z,\underline{x}_2} \arrow[l, "\mathrm{ins.unit}^R_{\underline{x}_1\subseteq\underline{x}_2}"'] \arrow[u, "f^!"']
\end{tikzcd}$$

obtained by passing to right adjoints in the commutative diagram

$$\begin{tikzcd}
\mathbf{C}_{Z',\underline{x}_1\circ f} & \mathbf{C}_{Z',\underline{x}_2\circ f} \arrow[l, "\mathrm{ins.unit}_{\underline{x}_1\circ f\subseteq\underline{x}_2\circ f}"'] \\
\mathbf{C}_{Z,\underline{x}_1} \arrow[u, "f_!"] & \mathbf{C}_{Z,\underline{x}_2}. \arrow[l, "\mathrm{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}"'] \arrow[u, "f_!"']
\end{tikzcd}$$

commutes.

H.1.6. Under the above assumption, passage to right adjoints defines a sheaf of categories
$$(\underline{\mathbf{C}}^{\mathrm{loc,untl}})^{\mathrm{op}}$$
over $(\mathrm{Ran}^{\mathrm{untl}})^{\mathrm{op}}$.

Note that we can tautologically identify $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}} := \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{C}}^{\mathrm{loc,untl}})$ with
$$\Gamma^{\mathrm{lax}}((\mathrm{Ran}^{\mathrm{untl}})^{\mathrm{op}}, (\underline{\mathbf{C}}^{\mathrm{loc,untl}})^{\mathrm{op}}).$$

Set
$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} := \Gamma^{\mathrm{strict}}((\mathrm{Ran}^{\mathrm{untl}})^{\mathrm{op}}, (\underline{\mathbf{C}}^{\mathrm{loc,untl}})^{\mathrm{op}}) \subset \Gamma^{\mathrm{lax}}((\mathrm{Ran}^{\mathrm{untl}})^{\mathrm{op}}, (\underline{\mathbf{C}}^{\mathrm{loc,untl}})^{\mathrm{op}}).$$

H.1.7. Let us describe $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$ explicitly as a full subcategory of $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}$:

An object
$$(S \overset{\underline{x}}{\to} \mathrm{Ran}) \mapsto \mathbf{c}_{S,\underline{x}} \in \mathbf{C}^{\mathrm{loc}}_{S,\underline{x}}$$
belongs to $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$ if for every
$$(\underline{x}_1 \overset{\alpha}{\to} \underline{x}_2) \in \mathrm{Maps}(S, \mathrm{Ran}^{\mathrm{untl}})$$
the map
$$\mathbf{c}_{S,\underline{x}_1} \to (\mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2})^R(\mathbf{c}_{S,\underline{x}_2}),$$
obtained by adjunction from
$$\mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2}(\underline{x}_1) \to \underline{x}_2,$$
is an isomorphism.

Remark H.1.8. Let assume in addition that $\underline{\mathbf{C}}^{\mathrm{loc,untl}}$ is value-wise dualizable. Then passing to duals in $(\underline{\mathbf{C}}^{\mathrm{loc,untl}})^{\mathrm{op}}$, we obtain a crystal of categories $((\underline{\mathbf{C}}^{\mathrm{loc,untl}})^{\mathrm{op}})^\vee$ on $\mathrm{Ran}^{\mathrm{untl}}$.

It is easy to see that the category
$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob,lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$$
identifies with
$$((\mathbf{C}^{\mathrm{op}})^\vee)_{\mathrm{Ran}^{\mathrm{untl}}} := \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, ((\underline{\mathbf{C}}^{\mathrm{loc,untl}})^{\mathrm{op}})^\vee \otimes \mathbf{C}^{\mathrm{glob}}).$$

H.1.9. We claim:

Lemma H.1.10. *The embedding*
$$\mathrm{emb.\,indep} : \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \hookrightarrow \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}$$
admits a left adjoint.

Proof. It is sufficient to show that
$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \subset \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}$$
is closed under limits.

However, this follows from the fact that limits in $\mathbf{C}_{\mathrm{Ran}^{\mathrm{untl}}}$ exists and have the property that they commute with evaluation on every *proper* Z mapping to $\mathrm{Ran}^{\mathrm{untl}}$, see Remark H.1.5.

□

H.1.11. Let $\mathrm{emb.\,indep}^L$ denote the left adjoint of emb. indep. Thus, we can view $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$ as a localization of $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}$.

Let us view the category $\mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}, \mathbf{C}^{\mathrm{glob}})$ as a full subcategory of $\mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}, \mathbf{C}^{\mathrm{glob}})$ via precomposition with $\mathrm{emb.\,indep}^L$.

H.1.12. We claim:

Lemma H.1.13. *The functor* (H.1) *sends*

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \subset \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$$

to

$$\mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}},\mathbf{C}^{\mathrm{glob}}) \subset \mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}},\mathbf{C}^{\mathrm{glob}}),$$

and the resulting functor

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}},\mathbf{C}^{\mathrm{glob}}) \tag{H.2}$$

is an equivalence.

Proof. We will access $\mathrm{Ran}^{\mathrm{untl}}$ via proper schemes mapping to it, see Sect. C.4.6.

Let us be given an object $\underline{\mathsf{F}} \in \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$. For every proper Z equipped with a map $\underline{x}: Z \to \mathrm{Ran}$, consider the corresponding functor

$$\mathsf{F}_{\int_Z,\underline{x}} : \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}} \to \mathbf{C}^{\mathrm{glob}}$$

and its (not necessarily continuous) right adjoint

$$(\mathsf{F}_{\int_Z,\underline{x}})^R : \mathbf{C}^{\mathrm{glob}} \to \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}}.$$

For $Z' \overset{g}{\to} Z$, the diagram

$$\begin{tikzcd} \mathbf{C}^{\mathrm{loc}}_{Z',\underline{x}\circ g} \arrow[r, "\mathsf{F}_{Z',\underline{x}\circ g}"] & \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(Z') \\ \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}} \arrow[u, "g^!"] \arrow[r, "\mathsf{F}_{Z,\underline{x}}"] & \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(Z) \arrow[u, "\mathrm{id}\otimes g^!"'] \end{tikzcd} \tag{H.3}$$

is equipped with a datum of commutativity. Since g is proper, the diagram

$$\begin{tikzcd} \mathbf{C}^{\mathrm{loc}}_{Z',\underline{x}\circ g} \arrow[r, "\mathsf{F}_{Z',\underline{x}\circ g}"] \arrow[d, "g_!"'] & \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(Z') \arrow[d, "\mathrm{id}\otimes g_!"] \\ \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}} \arrow[r, "\mathsf{F}_{Z,\underline{x}}"] & \mathbf{C}^{\mathrm{glob}} \otimes \text{D-mod}(Z'), \end{tikzcd} \tag{H.4}$$

obtained from (H.3) by passing to left adjoints along the vertical arrows, also commutes.

From (H.4), we obtain a datum of commutativity for the diagram

$$\begin{tikzcd} \mathbf{C}^{\mathrm{loc}}_{Z',\underline{x}\circ g} \arrow[r, "\mathsf{F}_{\int_{Z'},\underline{x}\circ g}"] \arrow[d, "g_!"'] & \mathbf{C}^{\mathrm{glob}} \arrow[d, "\mathrm{id}"] \\ \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}} \arrow[r, "\mathsf{F}_{\int_Z,\underline{x}}"] & \mathbf{C}^{\mathrm{glob}}. \end{tikzcd} \tag{H.5}$$

Finally, by passing to right adjoints in (H.5), we obtain a datum of commutativity for the diagram

$$\begin{tikzcd} \mathbf{C}^{\mathrm{loc}}_{Z',\underline{x}\circ g} & \mathbf{C}^{\mathrm{glob}} \arrow[l, "(\mathsf{F}_{\int_{Z'},\underline{x}\circ g})^R"'] \\ \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}} \arrow[u, "g^!"] & \mathbf{C}^{\mathrm{glob}}, \arrow[l, "(\mathsf{F}_{\int_Z,\underline{x}})^R"'] \arrow[u, "\mathrm{id}"'] \end{tikzcd} \tag{H.6}$$

where the functors are $(\mathsf{F}_{\int_Z,\underline{x}})^R$ are not necessarily continuous, but limit-preserving.

The functor $\underline{\mathsf{F}}$ is determined by the data of $\mathsf{F}_{\int_Z,\underline{x}}$ plus the data of commutativity for the diagrams (H.5), which is equivalent to having the data of limit-preserving functors $(\mathsf{F}_{\int_Z,\underline{x}})^R$ plus the data of commutativity for the diagrams (H.6).

Suppose now that $\underline{\mathsf{F}}$ is upgraded to an object $\underline{\mathsf{F}}^{\mathrm{untl}}$ of $\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$.

Let us be given a map $\underline{x}_1 \to \underline{x}_2$ in $\mathrm{Maps}(Z, \mathrm{Ran}^{\mathrm{untl}})$. The unital structure on $\underline{\mathbf{C}}^{\mathrm{loc}}$ gives rise to a functor

$$\mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2} : \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}_1} \to \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}_2},$$

and the lax unital structure $\underline{\mathsf{F}}^{\mathrm{untl}}$ on $\underline{\mathsf{F}}$ gives rise to a natural transformation

$$\mathsf{F}_{\int_Z, \underline{x}_1} \to \mathsf{F}_{\int_Z, \underline{x}_2} \circ \mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2}.$$

By definition, this natural transformation is an isomorphism if and only if the above lax unital structure on $\underline{\mathsf{F}}$ is strict. By adjunction, we obtain a natural transformation

$$(\mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2})^R \circ (\mathsf{F}_{\int_Z, \underline{x}_2})^R \to (\mathsf{F}_{\int_Z, \underline{x}_1})^R,$$

which is an isomorphism if and only if $\underline{\mathsf{F}}^{\mathrm{untl}}$ is strictly unital.

Since for every $(Z, \underline{x})$, the evaluation functor

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \hookrightarrow \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathbf{C}^{\mathrm{loc}}_{Z,\underline{x}}$$

commutes with limits, we obtain that the datum of a strictly unital object $\underline{\mathsf{F}}^{\mathrm{untl}}$ is equivalent to that of a limit-preserving functor

$$\mathbf{C}^{\mathrm{glob}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}},$$

which is equivalent to that of a continuous functor

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \to \mathbf{C}^{\mathrm{glob}}.$$

Thus, we have constructed an equivalence

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \simeq \mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}, \mathbf{C}^{\mathrm{glob}}).$$

Unwinding the construction, it is easy to see that the diagram

$$\begin{tikzcd}
\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \arrow[r] \arrow[d, "\sim"'] & \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \arrow[d] \\
\mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}, \mathbf{C}^{\mathrm{glob}}) \arrow[r] & \mathrm{Funct}(\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}, \mathbf{C}^{\mathrm{glob}})
\end{tikzcd}$$

commutes.

□

Remark H.1.14. The discussion in this subsection is not specific to $\mathrm{Ran}^{\mathrm{untl}}$. It applies to any pseudo-proper categorical prestack.

H.2. **The calculation of the independent category in the vacuum case.**

H.2.1. Here is a sample calculation of the category $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$. Take $\underline{\mathbf{C}}^{\mathrm{loc},\mathrm{untl}} := \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$. Denote the corresponding independent category by

$$\mathrm{Vect}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}.$$

H.2.2. Take $\mathbf{C}^{\mathrm{glob}} = \mathrm{Vect}$. Using Lemma H.1.13, from the identity functor

$$\underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}) \to \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}),$$

we obtain a functor

$$\mathrm{Vect}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \to \mathrm{Vect}. \tag{H.7}$$

We claim:

Proposition H.2.3. *The functor* (H.7) *is an equivalence.*

Remark H.2.4. Note that by Sect. H.1.6 can be reformulated as saying that the functor

$$\mathrm{Vect} \overset{k\mapsto\omega_{\mathrm{Ran}^{\mathrm{untl}}}}{\longrightarrow} \Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})) \tag{H.8}$$

is an equivalence.

Remark H.2.5. Note that, unlike the fact that Ran is contractible, which requires X to be *connected*, the assertion of Proposition H.2.3 is valid for any X that is non-empty.

Proof of Proposition H.2.3. We will show that the functor (H.8) is an equivalence. The right adjoint of this functor is given by the restriction of $\mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}, -)$ to

$$\Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})) \subset \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})) = \text{D-mod}(\mathrm{Ran}^{\mathrm{untl}}).$$

It suffices to show that for $\mathcal{F} \in \Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}))$, the map

$$\mathcal{F} \to \mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}, \mathcal{F}) \otimes \omega_{\mathrm{Ran}^{\mathrm{untl}}}$$

is an isomorphism.

For that it suffices to show that for any $\underline{x} \in \mathrm{Ran}$, the map

$$\mathcal{F}_{\underline{x}} \to \mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}, \mathcal{F}) \tag{H.9}$$

corresponding to

$$\mathrm{pt} \xrightarrow{\underline{x}} \mathrm{Ran}^{\mathrm{untl}}, \tag{H.10}$$

is an isomorphism.

We factor (H.9) as

$$\mathrm{pt} \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}} \to \mathrm{Ran}^{\mathrm{untl}}, \tag{H.11}$$

and hence (H.10) as

$$\mathcal{F}_{\underline{x}} \to \mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}, \mathcal{F}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}}) \to \mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}, \mathcal{F}). \tag{H.12}$$

We claim that both maps in (H.12) are isomorphisms. Indeed, the first map is an isomorphism, becauase

$$\mathcal{F}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}} \in \Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}})),$$

and $\underline{x}$ is the (value-wise) initial point of $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}$.

The second arrow in (H.12) is an isomorphism because the map

$$\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}} \to \mathrm{Ran}^{\mathrm{untl}}$$

is value-wise cofinal: its value-wise left adjoint is given by $\underline{x}' \mapsto \underline{x} \cup \underline{x}'$.

□

H.2.6. Let us rerturn to Proposition C.5.4, and explain its meaning in terms of local-to-global functors.

Let us be given a strictly unital functor

$$\underline{\mathsf{F}}^{\mathrm{untl}} : \underline{\mathbf{C}}^{\mathrm{loc,untl}} \to \mathbf{C}^{\mathrm{glob}} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}}).$$

Assume now that $\underline{\mathbf{C}}^{\mathrm{loc,untl}}$ is the restriction of a sheaf of categories $\mathbf{C}^{\mathrm{loc,untl},*}$ on $\mathrm{Ran}^{\mathrm{untl},*}$ (as is the case in all our examples).

Le $\mathbf{C}^{\mathrm{loc}}_\emptyset$ be the value of this extension on the initial point. The entire datum of the extension

$$\underline{\mathbf{C}}^{\mathrm{loc,untl}} \rightsquigarrow \mathbf{C}^{\mathrm{loc,untl},*}$$

is equivalent to the datum of a functor

$$\mathbf{C}^{\mathrm{loc}}_\emptyset \overset{\mathrm{ins.unit}_\emptyset}{\longrightarrow} \Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{C}}^{\mathrm{loc,untl}}) \hookrightarrow \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{C}}^{\mathrm{loc,untl}}) = \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}. \tag{H.13}$$

Remark H.2.7. Note that in most examples, $\mathbf{C}^{\mathrm{loc}}_{\emptyset} \simeq \mathrm{Vect}$, so that datum of (H.13) is that of an object

$$\mathbf{1}_{\underline{\mathbf{C}}^{\mathrm{loc}}} \in \Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{C}}^{\mathrm{loc,untl}}).$$

I.e., for every $\underline{x} \in \mathrm{Ran}$, we have an object $\mathbf{1}_{\underline{\mathbf{C}}^{\mathrm{loc}},\underline{x}} \in \mathbf{C}^{\mathrm{loc}}_{\underline{x}}$ and for every $\underline{x} \subseteq \underline{x}'$ we have an isomorphism

$$\mathrm{ins.\,unit}_{\underline{x}\subseteq\underline{x}'}(\mathbf{1}_{\underline{\mathbf{C}}^{\mathrm{loc}},\underline{x}} \simeq \mathbf{1}_{\underline{\mathbf{C}}^{\mathrm{loc}},\underline{x}'}.$$

H.2.8. Consider the composition

$$\underline{\mathsf{F}}^{\mathrm{untl}} \circ \mathrm{ins.\,unit}_{\emptyset} : \mathbf{C}^{\mathrm{loc}}_{\emptyset} \to \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}).$$

The claim is that it factors canonically as

$$\mathbf{C}^{\mathrm{loc}}_{\emptyset} \xrightarrow{\mathsf{F}_{\emptyset}} \mathbf{C}^{\mathrm{glob}} \xrightarrow{\mathrm{Id}\otimes\omega_{\mathrm{Ran}^{\mathrm{untl}}}} \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}).$$

Indeed, the functor $\underline{\mathsf{F}}^{\mathrm{untl}} \circ \mathrm{ins.\,unit}_{\emptyset}$ factors as

$$\mathbf{C}^{\mathrm{loc}}_{\emptyset} \to \mathbf{C}^{\mathrm{glob}} \otimes \Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}^{\mathrm{untl}})) \to$$
$$\to \mathbf{C}^{\mathrm{glob}} \otimes \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}^{\mathrm{lax}})) = \mathbf{C}^{\mathrm{glob}} \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}^{\mathrm{untl}}),$$

and according to Remark H.2.4, the functor

$$\mathbf{C}^{\mathrm{glob}} \to \mathbf{C}^{\mathrm{glob}} \otimes \Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}^{\mathrm{untl}}))$$

is an equivalence.

H.3. **Non-unitality vs independence.** Let $\underline{\mathbf{C}}^{\mathrm{loc,untl}}$ be as in the previous subsection.

In this subsection we will utilize the contractibility of the Ran space to explain the relation between the "indepndent" category $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$ and the non-unital version $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}$.

H.3.1. We claim:

Proposition H.3.2. *The composite functor*

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \overset{\mathrm{emb.indep}}{\hookrightarrow} \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}} \xrightarrow{\mathrm{t}^!} \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}}$$

is fully faithful.

Proof. The assertion of the proposition amounts to the following. Let us be given two objects

$$\mathbf{c}', \mathbf{c}'' \in \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \subset \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}},$$

and a map

$$\mathrm{t}^!(\mathbf{c}') \xrightarrow{\phi} \mathrm{t}^!(\mathbf{c}'').$$

We need to show that this map can be uniquely upgraded to a map

$$\mathbf{c}' \xrightarrow{\phi_{\mathrm{untl}}} \mathbf{c}''.$$

This amounts to the following. Fix $S \in \mathrm{Sch}^{\mathrm{aff}}$ and let us be given a map

$$(\underline{x}_1 \to \underline{x}_2) \in \mathrm{Maps}(S, \mathrm{Ran}^{\mathrm{untl}}).$$

We need to equip the following diagram (taking place in in $\mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_2}$)

$$\begin{tikzcd}
\mathbf{c}'_{S,\underline{x}_2} \arrow[r, "\phi_{S,\underline{x}_2}"] & \mathbf{c}''_{S,\underline{x}_2} \\
\mathrm{ins.\,unit}_{\underline{x}_1\subseteq\underline{x}_2}(\mathbf{c}'_{S,\underline{x}_1}) \arrow[u] \arrow[r, "\mathrm{ins.unit}_{\underline{x}_1\subseteq\underline{x}_2}(\phi_{S,\underline{x}_1})"] & \mathrm{ins.\,unit}_{\underline{x}_1\subseteq\underline{x}_2}(\mathbf{c}''_{S,\underline{x}_1}) \arrow[u]
\end{tikzcd}$$

(where the vertical arrows are given by the structure of objects of $\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}}}$ on $\mathbf{c}'$ and $\mathbf{c}''$, respectively) with a datum of commutativity.

The datum of commutativity of the above diagram is equivalent to the datum of commutativity of the following diagram (taking place in in $\mathbf{C}_{S,\underline{x}_1}$):

$$
\begin{tikzcd}
\mathrm{ins.\,unit}^R_{\underline{x}_1\subseteq\underline{x}_2}(\mathbf{c}'_{S,\underline{x}_2}) \arrow[r, "\mathrm{ins.unit}^R_{\underline{x}_1\subseteq\underline{x}_2}(\phi_{S,\underline{x}_2})"] & \mathrm{ins.\,unit}^R_{\underline{x}_1\subseteq\underline{x}_2}(\mathbf{c}''_{S,\underline{x}_2}) \\
\mathbf{c}'_{S,\underline{x}_1} \arrow[u] \arrow[r, "\phi_{S,\underline{x}_1}"] & \mathbf{c}''_{S,\underline{x}_1}. \arrow[u]
\end{tikzcd}
\tag{H.14}
$$

Note the vertical arrows in (H.14) are isomorphisms, by the assumption that $\mathbf{c}',\mathbf{c}''\in\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$. Thus, we can view both circuits in (H.14) as an object of $\mathcal{H}om_{\mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_1}}(\mathbf{c}'_{S,\underline{x}_1},\mathbf{c}''_{S,\underline{x}_1})$.

Thus, letting $\underline{x}_2$ vary, we can view the clockwise circuit in (H.14) as a map

$$(\mathrm{Id}\otimes\mathrm{pr}^!_{\mathrm{small},S})(\mathbf{c}'_{S,\underline{x}_1})\to(\mathrm{Id}\otimes\mathrm{pr}_{\mathrm{small},S})^!(\mathbf{c}'_{S,\underline{x}_1}) \tag{H.15}$$

in

$$\mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_1}\underset{\mathrm{D\text{-}mod}(S)}{\otimes}\mathrm{D\text{-}mod}(S^{\subseteq}_{\underline{x}_1}).$$

Now, the universal homological contractibility of the map

$$\mathrm{pr}_{\mathrm{small},S}:S^{\subseteq}_{\underline{x}_1}\to S$$

implies that the functor

$$\mathrm{pr}^!_{\mathrm{small},S}:\mathrm{D\text{-}mod}(S)\to\mathrm{D\text{-}mod}(S^{\subseteq}_{\underline{x}_1})$$

is fully faithful, and hence so is

$$\mathrm{Id}\otimes\mathrm{pr}^!_{\mathrm{small},S}:\mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_1}\to\mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_1}\underset{\mathrm{D\text{-}mod}(S)}{\otimes}\mathrm{D\text{-}mod}(S^{\subseteq}_{\underline{x}_1}).$$

Hence, the space

$$\mathcal{H}om_{\mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_1}\underset{\mathrm{D\text{-}mod}(S)}{\otimes}\mathrm{D\text{-}mod}(S^{\subseteq}_{\underline{x}_1})}((\mathrm{Id}\otimes\mathrm{pr}^!_{\mathrm{small},S})(\mathbf{c}'_{S,\underline{x}_1}),(\mathrm{Id}\otimes\mathrm{pr}^!_{\mathrm{small},S})(\mathbf{c}''_{S,\underline{x}_1}))$$

is isomorphic to $\mathcal{H}om_{\mathbf{C}^{\mathrm{loc}}_{S,\underline{x}_1}}(\mathbf{c}'_{S,\underline{x}_1},\mathbf{c}''_{S,\underline{x}_1})$, with the mutually inverse isomorphisms given by the functors $\mathrm{Id}\otimes\mathrm{pr}^!_{\mathrm{small},S}$ and $\mathrm{Id}\otimes\mathrm{diag}^!_S$, respectively.

This equips the family of diagrams (H.14) with a unique datum of commutativity as $\underline{x}_2$ varies over $S^{\subseteq}_{\underline{x}_1}$.

$\square$

H.3.3. Combining Proposition H.3.2 and Lemma H.1.13, we obtain:

Corollary H.3.4. *The functor from the category of* strict *functors of crystals of categories*

$$\underline{\mathsf{F}}^{\mathrm{untl}}:\underline{\mathbf{C}}^{\mathrm{loc,untl}}\to\mathbf{C}^{\mathrm{glob}}\otimes\underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}^{\mathrm{untl}})$$

to the category of functors

$$\underline{\mathsf{F}}:\underline{\mathbf{C}}^{\mathrm{loc}}\to\mathbf{C}^{\mathrm{glob}}\otimes\underline{\mathrm{D\text{-}mod}}(\mathrm{Ran}),$$

given by restriction along $\mathrm{Ran}\to\mathrm{Ran}^{\mathrm{untl}}$*, is fully faithful.*

Unwinding the definitions, we obtain that the essential image of the above fully faithful functor

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob,untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\to\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob,lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\to\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$$

consists of objects that have a global unitality property.

I.e., this proves Proposition 11.8.8.

H.3.5. As another immediate corollary of Proposition H.3.2 we obtain:

Corollary H.3.6. *The natural transformation*

$$\mathrm{emb.\,indep}^L \circ \mathrm{t}_! \circ \mathrm{t}^! \to \mathrm{emb.\,indep}^L$$

is an isomorphism.

H.3.7. Finally, we record:

Corollary H.3.8. *The composite functor*

$$\mathbf{C}_{\mathrm{Ran}} \overset{\mathrm{t}_!}{\to} \mathbf{C}_{\mathrm{Ran}^{\mathrm{untl}}} \overset{\mathrm{emb.indep}^L}{\to} \mathbf{C}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$$

is a localization.

H.4. **Sheaves of monoidal categories on the unital Ran space.** Let $\mathbf{A}^{\mathrm{loc,untl}}$ be a sheaf of unital monoidal categories over $\mathrm{Ran}^{\mathrm{untl}}$. We will assume that $\mathbf{A}^{\mathrm{loc,untl}}$ satisfies the condition from Sect. H.1.4.

We will also assume that the monoidal operation admits a right adjoint, which is a strict functor between sheaves of categories.

In this subsection we will study the categories

$$\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} := \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}}) \text{ and } \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl,indep}}}. \tag{H.16}$$

We will equip them with monoidal structures, and study the interactions between them.

H.4.1. *Example.* An example of such $\mathbf{A}^{\mathrm{loc,untl}}$ is the sheaf of unital monoidal categories over $\mathrm{Ran}^{\mathrm{untl}}$ attached to a unital monoidal factorization category $\mathbf{A}$.

An important example of such an $\mathbf{A}$ is Sph_G.

H.4.2. Another example is the *symmetric* monoidal factorization category $\mathbf{A}$ associated to a crystal of symmetric monoidal categories over X (this is a categorical counterpart of the procedure from Sect. C.8).

An example of this is the constant crystal of symmetric monoidal categories over X with fiber $\mathrm{Rep}(\check{G})$.

Remark H.4.3. Note that being a sheaf of *unital* monoidal categories over $\mathrm{Ran}^{\mathrm{untl}}$ automatically imposes a condition on the compatibility between the monoidal unit and a structure of sheaf of categories:

The monoidal unit

$$\mathbf{1}_{\mathbf{A},\mathrm{Ran}} \in \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}}) = \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$$

belongs to

$$\Gamma^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}}) \subset \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}}).$$

H.4.4. Consider the map

$$\mathrm{union} : \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}}.$$

We have the 1-morphisms

$$p_1 \to \mathrm{union} \leftarrow p_2$$

in the category of maps from $\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}$ to $\mathrm{Ran}^{\mathrm{untl}}$. From here we obtain functors

$$p_1^*(\mathbf{A}^{\mathrm{loc,untl}}) \to \mathrm{union}^*(\mathbf{A}^{\mathrm{loc,untl}}) \leftarrow p_2^*(\mathbf{A}^{\mathrm{loc,untl}})$$

as crystals of categories over $\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}$, and hence a functor

$$\mathbf{A}^{\mathrm{loc,untl}} \boxtimes \mathbf{A}^{\mathrm{loc,untl}} \to \mathrm{union}^*(\mathbf{A}^{\mathrm{loc,untl}} \otimes \mathbf{A}^{\mathrm{loc,untl}}). \tag{H.17}$$

Combining with the monoidal operation on $\mathbf{A}^{\mathrm{loc,untl}}$, we obtain a functor

$$\mathbf{A}^{\mathrm{loc,untl}} \boxtimes \mathbf{A}^{\mathrm{loc,untl}} \to \mathrm{union}^*(\mathbf{A}^{\mathrm{loc,untl}}). \tag{H.18}$$

Applying $\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, -)$, from (H.18) we obtain a functor

$$\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} \otimes \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} \to \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, \mathrm{union}^*(\mathbf{A}^{\mathrm{loc,untl}})). \tag{H.19}$$

Finally, composing (H.19) with the functor

$$\mathrm{union}_! : \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, \mathrm{union}^*(\mathbf{A}^{\mathrm{loc,untl}})) \to \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}}),$$

left adjoint to $\mathrm{union}^!$ (it exists thanks to Corollary C.4.10), we obtain a functor

$$\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} \otimes \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}. \tag{H.20}$$

We will denote the resulting binary operation on $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$ by $\star$, and will refer to it as "convolution".

H.4.5. The above binary operation extends to a monoidal structure on $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$, which we will refer to us the *convolution* monoidal structure. We will denote $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$, viewed as a monoidal category equipped with the convolution structure by $\mathbf{A}^{\star}_{\mathrm{Ran}^{\mathrm{untl}}}$.

H.4.6. Note now that the monoidal operation on $\mathbf{A}^{\mathrm{loc,untl}}$ defines a *pointwise* monoidal structure on $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$.

We will denote $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$, viewed as a monoidal category equipped with the pointwise structure by $\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$.

Note that $\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$ is unital: its unit is the object $\mathbf{1}_{\mathrm{Ran}^{\mathrm{untl}}}$.

H.4.7. Unwinding the definitions, one obtains:

Lemma H.4.8. *The pointwise monoidal structure on* $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$ *descends to the quotient*

$$\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} \twoheadrightarrow \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}.$$

In what follows we will consider $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$ as a monoidal category, with the monoidal structure furnished by Lemma H.4.8.

Since $\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$ is unital, we obtain that so is $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$.

H.4.9. Note now that the natural transformation

$$(\Delta_{\mathrm{Ran}^{\mathrm{untl}}})^! \simeq \mathrm{union}_! \circ (\Delta_{\mathrm{Ran}^{\mathrm{untl}}})_! \circ (\Delta_{\mathrm{Ran}^{\mathrm{untl}}})^! \to \mathrm{union}_!$$

defines on the identity functor on $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$ a structure of *right-lax* monoidal functor

$$\mathbf{A}^{\star}_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}. \tag{H.21}$$

H.4.10. However, we claim:

Lemma H.4.11. *The right-lax monoidal structure on the functor* (H.21) *is strict.*

Proof. Let $\underline{\mathbf{C}}$ denote the crystal of categories

$$\mathrm{union}^*(\mathbf{A}^{\mathrm{loc,untl}} \otimes \mathbf{A}^{\mathrm{loc,untl}})$$

on $\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}$.

Let ins. union denote the functor (H.17); we will use the same notation for the induced functor

$$\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} \otimes \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} \simeq \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}} \boxtimes \mathbf{A}^{\mathrm{loc,untl}}) \to \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{C}}).$$

We have to show that for $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F} \in \mathbf{A}^{\star}_{\mathrm{Ran}^{\mathrm{untl}}}$, the map

$$\begin{aligned} &\mathcal{H}om_{\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{C}})}(\mathrm{ins.\,union}(\mathcal{F}_1 \boxtimes \mathcal{F}_2), \mathrm{union}^!(\mathrm{mult}^R(\mathcal{F}))) \to \\ &\qquad \to \mathcal{H}om_{\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}})}(\mathrm{mult}(\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2), \mathcal{F}), \end{aligned} \tag{H.22}$$

is an isomorphism, where:

- mult denotes the functor
$$\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}} \otimes \mathbf{A}^{\mathrm{loc,untl}}) \to \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}})$$
induced by the monoidal operation
$$\mathbf{A}^{\mathrm{loc,untl}} \otimes \mathbf{A}^{\mathrm{loc,untl}} \overset{\mathrm{mult}}{\to} \mathbf{A}^{\mathrm{loc,untl}};$$
- mult^R denotes the functor
$$\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}}) \to \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}} \otimes \mathbf{A}^{\mathrm{loc,untl}})$$
induced by the functor
$$\mathbf{A}^{\mathrm{loc,untl}} \overset{\mathrm{mult}^R}{\to} \mathbf{A}^{\mathrm{loc,untl}} \otimes \mathbf{A}^{\mathrm{loc,untl}},$$
right adjoint to the monoidal operation;
- $\mathrm{union}^!$ denotes the functor
$$\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}} \otimes \mathbf{A}^{\mathrm{loc,untl}}) \to \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{C}});$$
- The map (H.22) is given by the composition

$$\begin{gathered}\mathcal{H}om_{\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}, \mathbf{C})}(\mathrm{ins.\,union}(\mathcal{F}_1 \boxtimes \mathcal{F}_2), \mathrm{union}^!(\mathrm{mult}^R(\mathcal{F}))) \overset{\Delta^!_{\mathrm{Ran}^{\mathrm{untl}}}}{\longrightarrow} \\ \mathcal{H}om_{\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \Delta^*_{\mathrm{Ran}^{\mathrm{untl}}}(\underline{\mathbf{C}}))}(\Delta^!_{\mathrm{Ran}^{\mathrm{untl}}}(\mathrm{ins.\,union}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)), \Delta^!_{\mathrm{Ran}^{\mathrm{untl}}}(\mathrm{union}^!(\mathrm{mult}^R(\mathcal{F})))) \simeq \\ \simeq \mathcal{H}om_{\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}} \otimes \mathbf{A}^{\mathrm{loc,untl}})}(\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2, \mathrm{mult}^R(\mathcal{F})) \simeq \\ \simeq \mathcal{H}om_{\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}})}(\mathrm{mult}(\mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2), \mathcal{F}).\end{gathered}$$

Now the isomorphism assertion holds for any crystal of categories $\underline{\mathbf{C}}$ on $\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}$, since the map
$$\Delta_{\mathrm{Ran}^{\mathrm{untl}}} : \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}$$
is cofinal.

□

H.4.12. Thanks to Lemma H.4.11, we do not need to distinguish between the two monoidal structures on $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$. Thus we will use the symbol $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$ unambiguously to refer to a tensor structure on $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$.

Yet we will sometimes use the symbols $\mathbf{A}^{\star}_{\mathrm{Ran}^{\mathrm{untl}}}$ or $\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$ to emphasize that we are thinking of the monoidal structure as convolution or pointwise tensor product, respectively.

H.4.13. Let $\mathbf{A}^{\mathrm{loc,untl}}$ be as in Sect. H.4.2, i.e., it is associated to a crystal $\underline{\mathbf{A}}_X$ of symmetric monoidal categories on X. In this case, one can describe the corresponding (symmetric) monoidal category $\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$ explicitly.

Namely, as a DG category it is isomorphic to the colimit over the twisted arrows category of the category of finite non-empty sets (and arbitrary maps) of the functor that associates to
$$I_1 \overset{\phi}{\to} I_2$$
the category
$$\Gamma(X^{I_2}, \underset{i \in I_2}{\boxtimes} \underline{\mathbf{A}}_X^{\otimes \phi^{-1}(i_2)}). \tag{H.23}$$

The symmetric monoidal structure is given by the operation of disjoint union of finite sets, see [FraG, Sect. 2.2.1].

H.5. **Sheaves of monoidal categories on the *non*-unital Ran space.** We now consider the usual (i.e., non-unital) Ran space. For $\mathbf{A}^{\mathrm{loc,untl}}$ as above, let $\mathbf{A}^{\mathrm{loc}}$ denote its restriction along the map $\mathsf{t} : \mathrm{Ran} \to \mathrm{Ran}^{\mathrm{untl}}$.

Denote

$$\mathbf{A}_{\mathrm{Ran}} := \Gamma(\mathrm{Ran}, \mathbf{A}^{\mathrm{loc}}).$$

In this subsection we will endow $\mathbf{A}_{\mathrm{Ran}}$ with monoidal structure(s) and study its interactions with the unital counterparts.

H.5.1. By a slight abuse of notation, we will use the same symbol union to denote the corresponding map

$$\mathrm{Ran} \times \mathrm{Ran} \to \mathrm{Ran}.$$

Restricting (H.18) along

$$\mathsf{t} : \mathrm{Ran} \to \mathrm{Ran}_{\mathrm{untl}}$$

we obtain a map of crystals of categories

$$\mathbf{A}^{\mathrm{loc}} \boxtimes \mathbf{A}^{\mathrm{loc}} \to \mathrm{union}^*(\mathbf{A}^{\mathrm{loc}}) \tag{H.24}$$

on $\mathrm{Ran} \times \mathrm{Ran}$.

Since the map union is pseudo-proper, the functor (H.24) induces a functor

$$\mathbf{A}_{\mathrm{Ran}} \otimes \mathbf{A}_{\mathrm{Ran}} \overset{\star}{\to} \mathbf{A}_{\mathrm{Ran}}.$$

H.5.2. The above binary operation extends to a monoidal structure on $\mathbf{A}_{\mathrm{Ran}}$, which we will refer to us the *convolution* monoidal structure. We will denote $\mathbf{A}_{\mathrm{Ran}}$, viewed as a monoidal category equipped with the convolution monoidal structure by $\mathbf{A}^{\star}_{\mathrm{Ran}}$.

H.5.3. By construction, the functor

$$\mathsf{t}_! : \mathbf{A}_{\mathrm{Ran}} \to \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$$

has a monoidal structure, when we consider both as equipped with the convolution monoidal structure.

In particular, we obtain that the functor

$$\mathrm{emb.\,indep}^L \circ \mathsf{t}_! : \mathbf{A}^{\star}_{\mathrm{Ran}} \to \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \tag{H.25}$$

acquires a monoidal structure.

Combining with Corollary H.3.8, we obtain that the functor (H.25) is a monoidal localization.

H.5.4. Consider now the functor

$$\mathsf{t}^! : \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathbf{A}_{\mathrm{Ran}}.$$

Being the right adjoint of a monoidal functor, the functor $\mathsf{t}^!$ acquires a right-lax monoidal structure as a functor

$$\mathbf{A}^{\star}_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathbf{A}^{\star}_{\mathrm{Ran}}.$$

Note that the natural transformation

$$\mathrm{emb.\,indep}^L \circ \mathsf{t}_! \circ \mathsf{t}^! \to \mathrm{emb.\,indep}^L \tag{H.26}$$

has a natural right-lax monoidal structure. However, from Corollary H.3.6 we obtain that the natural transformation (as right-lax monoidal functors) is an isomorphism.

In particular, we obtain that the right-lax monoidal structure on (H.26) is strict.

H.5.5. As in Sect. H.4.6, we can also consider the $\overset{!}{\otimes}$-monoidal structure on $\mathbf{A}_{\mathrm{Ran}}$. We will denote $\mathbf{A}_{\mathrm{Ran}}$, viewed as a monoidal category equipped with the pointwise structure by $\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}}$.

As in Sect. H.4.6, the identity functor on $\mathbf{A}_{\mathrm{Ran}}$ has a right-lax monoidal structure, when viewed as a functor

$$\mathbf{A}^{\star}_{\mathrm{Ran}} \to \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}}.$$

H.5.6. The functor $\mathsf{t}^!$ is monoidal, when viewed as a functor

$$\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}}.$$

Hence, the functor

$$\mathsf{t}_! : \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}} \to \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$$

acquires a left-lax monoidal structure, when viewed as a functor

$$\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}} \to \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}.$$

H.5.7. Let

$$\mathbf{A}^{\text{almost-untl}}_{\mathrm{Ran}} \subset \mathbf{A}_{\mathrm{Ran}}$$

be the full subcategory generated by the essential image of the functor $\mathsf{t}^!$.

It is easy to see that it is preserved by the $\overset{!}{\otimes}$ monoidal operation, and hence it acquires a monoidal structure.

H.5.8. We claim:

Lemma H.5.9. *The left-lax monoidal structure on the functor*

$$\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}} \overset{\mathsf{t}_!}{\to} \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}} \overset{\text{emb.indep}^L}{\to} \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$$

becomes strict when restricted to $\mathbf{A}^{\text{almost-untl}}_{\mathrm{Ran}}$.

Proof. It suffices to show that the left-lax monoidal structure on the functor

$$\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}} \overset{\mathsf{t}^!}{\to} \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}} \overset{\mathsf{t}_!}{\to} \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}} \overset{\text{emb.indep}^L}{\to} \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$$

is strict.

The assertion follows now from Corollary H.3.8, which implies that the above composition is isomorphic to

$$\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}} \overset{\text{emb.indep}^L}{\to} \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$$

as a left-lax monoidal functor.

□

H.5.10. Let $\mathbf{A}$ be as in Sect. H.4.13. In this case, we can also describe the category $\mathbf{A}^{\star}_{\mathrm{Ran}}$ explicitly.

It is given by the colimit of the functor (H.23), with the only difference that we take the twisted arrows category of the category of finite non-empty sets and *surjective* maps.

The functor

$$\mathsf{t}_! : \mathbf{A}^{\star}_{\mathrm{Ran}} \to \mathbf{A}^{\star}_{\mathrm{Ran}^{\mathrm{untl}}}$$

is given by embedding the index categories one into the other.

H.6. **Local and integrated monoidal actions.**

H.6.1. Let $\mathbf{A}^{\mathrm{loc,untl}}$ be as above. Let $\mathbf{A}^{\mathrm{loc}}$ denote the restriction of $\mathbf{A}^{\mathrm{loc,untl}}$ along $\mathsf{t} : \mathrm{Ran} \to \mathrm{Ran}^{\mathrm{untl}}$.

Let $\mathbf{D}$ be a DG category. We give the following definitions:

- A local action of $\mathbf{A}^{\mathrm{loc}}$ on $\mathbf{D}$ is a (unital) action of the crystal of monoidal categories $\mathbf{A}^{\mathrm{loc}}$ on $\mathbf{D} \otimes \underline{\text{D-mod}}(\mathrm{Ran})$;
- A local lax-Ran-unital action of $\mathbf{A}^{\mathrm{loc,untl}}$ on $\mathbf{D}$ is a (unital) action of the crystal of monoidal categories $\mathbf{A}^{\mathrm{loc,untl}}$ on $\mathbf{D} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$, in the 2-category of crystals of categories and right-lax functors between them;
- A local Ran-unital action of $\mathbf{A}^{\mathrm{loc,untl}}$ on $\mathbf{D}$ is a (unital) action of the crystal of monoidal categories $\mathbf{A}^{\mathrm{loc,untl}}$ on $\mathbf{D} \otimes \underline{\text{D-mod}}(\mathrm{Ran}^{\mathrm{untl}})$, in the 2-category of crystals of categories and strict functors between them.

H.6.2. At the poinwtise level, a local action of $\mathbf{A}^{\mathrm{loc}}$ on $\mathbf{D}$ yields an action of the monoidal category $\mathbf{A}_{\underline{x}}$ on $\mathbf{D}$ for every $\underline{x} \in \mathrm{Ran}$, denoted

$$\mathbf{a}_{\underline{x}}, \mathbf{d} \mapsto \mathbf{a} \cdot \mathbf{d}, \quad \mathbf{a}_{\underline{x}} \in \mathbf{A}_{\underline{x}},\ \mathbf{d} \in \mathbf{D}.$$

A lax-Ran-unital structure on such an action is a natural transformation

$$\mathbf{a}_{\underline{x}_1} \cdot \mathbf{d} \to \mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2}(\mathbf{a}_{\underline{x}_1}) \cdot \mathbf{d}, \quad \mathbf{a}_{\underline{x}_1} \in \mathbf{A}_{\underline{x}_1},\ \mathbf{d} \in \mathbf{D}.$$

A lax-Ran-unital structure is strict if the above natural transformation is an isomorphism.

H.6.3. Thus, we obtain the 2-categories

$$\mathbf{A}^{\mathrm{loc}}\text{-}\mathbf{mod},\ \mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}} \text{ and } \mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}.$$

We have a fully faithful functor

$$\mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod} \hookrightarrow \mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}}$$

and a forgetful functor

$$\mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}} \to \mathbf{A}^{\mathrm{loc}}\text{-}\mathbf{mod}.$$

Note, however, that from Corollary H.3.4, we obtain:

Corollary H.6.4. *The composite functor*

$$\mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod} \hookrightarrow \mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}} \to \mathbf{A}^{\mathrm{loc}}\text{-}\mathbf{mod} \tag{H.27}$$

is fully faithful.

H.6.5. Consider now the (unital) monoidal category $\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$. We claim that given a local lax-Ran-unital action of $\mathbf{A}^{\mathrm{loc,untl}}$ on $\mathbf{D}$, we can construct a (unital) action of $\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$ on $\mathbf{D}$.

Indeed, this follows from the fact that the functor

$$\mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, -) : \mathrm{D}\text{-mod}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathrm{Vect}$$

is symmetric monoidal.

Explicitly, given $\mathbf{a} \in \mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}}}$, the action is given by

$$\mathbf{D} \overset{-\otimes \omega_{\mathrm{Ran}^{\mathrm{untl}}}}{\longrightarrow} \mathbf{D} \otimes \mathrm{D}\text{-mod}(\mathrm{Ran}^{\mathrm{untl}})^{\mathrm{strict}} \hookrightarrow$$
$$\to \mathbf{D} \otimes \mathrm{D}\text{-mod}(\mathrm{Ran}^{\mathrm{untl}}) \overset{\mathbf{a}\cdot -}{\to} \mathbf{D} \otimes \mathrm{D}\text{-mod}(\mathrm{Ran}^{\mathrm{untl}}) \overset{\mathrm{Id} \otimes \mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}, -)}{\longrightarrow} \mathbf{D}.$$

H.6.6. From Lemma H.1.13 we obtain:

Corollary H.6.7. *The composite functor*

$$\mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod} \hookrightarrow \mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}} \to \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}\text{-}\mathbf{mod}$$

is fully faithful with essential image being

$$\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}\text{-}\mathbf{mod} \subset \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}\text{-}\mathbf{mod}.$$

H.6.8. Precomposing the construction in Sect. H.6.5 with the monoidal functor

$$\mathbf{A}^{\star}_{\mathrm{Ran}} \overset{\mathfrak{t}_!}{\to} \mathbf{A}^{\star}_{\mathrm{Ran}^{\mathrm{untl}}} \simeq \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$$

we obtain that for

$$\mathbf{D} \in \mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}}$$

we have an action of $\mathbf{A}^{\star}_{\mathrm{Ran}}$ on $\mathbf{D}$.

Remark H.6.9. Unwinding the definitions, for $\mathbf{a} \in \mathbf{A}_{\mathrm{Ran}}$, its action on $\mathbf{D}$ is given by the composition

$$\mathbf{D} \overset{-\otimes\omega_{\mathrm{Ran}}}{\longrightarrow} \mathbf{D} \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \overset{\mathbf{a}\cdot-}{\to} \mathbf{D} \otimes \mathrm{D\text{-}mod}(\mathrm{Ran}) \overset{\mathrm{Id}\otimes \mathrm{C}^{\cdot}_c(\mathrm{Ran},-)}{\longrightarrow} \mathbf{D}.$$

I.e., the binary operation only depends on structure on $\mathbf{D}$ of object of $\mathbf{A}^{\mathrm{loc}}$-**mod**.

However, one would not be able to define either left-lax or right-lax structure on this binary operation without $\mathbf{D}$ being extended to an object of $\mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}}$.

H.6.10. From Sect. H.5.3 and Corollary H.6.7 we obtain:

Corollary H.6.11. *The composite functor*

$$\mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod} \hookrightarrow \mathbf{A}^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}} \to \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}\text{-}\mathbf{mod} \simeq \mathbf{A}^{\star}_{\mathrm{Ran}^{\mathrm{untl}}}\text{-}\mathbf{mod} \to \mathbf{A}^{\star}_{\mathrm{Ran}}\text{-}\mathbf{mod}$$

is fully faithful with the essential image being

$$\mathbf{A}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}\text{-}\mathbf{mod} \subset \mathbf{A}^{\star}_{\mathrm{Ran}}\text{-}\mathbf{mod}.$$

H.7. **Local actions with parameters.**

H.7.1. Let now S be an affine scheme mapping to Ran. The discussion in Sects. H.4-H.6 applies when we replace Ran (resp., $\mathrm{Ran}^{\mathrm{untl}}$) by $S^{\subseteq}$ (resp., $S^{\subseteq,\mathrm{untl}}$).

Note that $S^{\subseteq}$ and $S^{\subseteq,\mathrm{untl}}$ are pseudo-proper *relative* to S, so Sect. C.4.24 applies.

H.7.2. In particular, one can consider the D-mod(S)-linear monoidal categories

$$\mathbf{A}^{\star}_{S^{\subseteq,\mathrm{untl}}}\ \mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}}},\ \mathbf{A}^{\star}_{S^{\subseteq}},\ \mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq}} \text{ and } \mathbf{A}_{S^{\subseteq,\mathrm{untl}},\mathrm{indep}},$$

equipped with the monoidal functors

$$\mathbf{A}^{\star}_{S^{\subseteq,\mathrm{untl}}} \overset{\sim}{\to} \mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}}},$$

$$\mathbf{A}^{\star}_{S^{\subseteq}} \overset{\mathfrak{t}_!}{\to} \mathbf{A}^{\star}_{S^{\subseteq,\mathrm{untl}}},$$

$$\mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}}} \overset{\mathfrak{t}^!}{\to} \mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}}},$$

$$\mathbf{A}^{\star}_{S^{\subseteq,\mathrm{untl}}} \overset{\mathrm{embed.indep}^L}{\twoheadrightarrow} \mathbf{A}_{S^{\subseteq,\mathrm{untl}},\mathrm{indep}},$$

$$\mathbf{A}^{\star}_{S^{\subseteq}} \overset{\mathrm{embed.indep}^L\circ\mathfrak{t}_!}{\twoheadrightarrow} \mathbf{A}_{S^{\subseteq,\mathrm{untl}},\mathrm{indep}},$$

where the last two functors are monoidal localizations.

H.7.3. We also have the corresponding notions of local action, so we have the 2-categories

$$\mathbf{A}^{\mathrm{loc}}_S\text{-}\mathbf{mod},\ \mathbf{A}^{\mathrm{loc,untl}}_S\text{-}\mathbf{mod}^{\mathrm{lax}} \text{ and } \mathbf{A}^{\mathrm{loc,untl}}_S\text{-}\mathbf{mod},$$

the fully faithful embedding

$$\mathbf{A}^{\mathrm{loc,untl}}_S\text{-}\mathbf{mod} \hookrightarrow \mathbf{A}^{\mathrm{loc,untl}}_S\text{-}\mathbf{mod}^{\mathrm{lax}}$$

and a forgetful functor

$$\mathbf{A}^{\mathrm{loc,untl}}_S\text{-}\mathbf{mod}^{\mathrm{lax}} \to \mathbf{A}^{\mathrm{loc}}_S\text{-}\mathbf{mod},$$

so that composition

$$\mathbf{A}^{\mathrm{loc,untl}}_S\text{-}\mathbf{mod} \hookrightarrow \mathbf{A}^{\mathrm{loc,untl}}_S\text{-}\mathbf{mod}^{\mathrm{lax}} \to \mathbf{A}^{\mathrm{loc}}_S\text{-}\mathbf{mod}$$

is fully faithful.

H.7.4. In addition, we have the commutative diagrams

$$\begin{tikzcd} \mathbf{A}_S^{\mathrm{loc,untl}}\text{-}\mathbf{mod} \arrow[r] \arrow[d, "\sim"'] & \mathbf{A}_S^{\mathrm{loc,untl}}\text{-}\mathbf{mod}^{\mathrm{lax}} \arrow[d] \\ \mathbf{A}_{S^{\subseteq,\mathrm{untl}},\mathrm{indep}}\text{-}\mathbf{mod} \arrow[r] & \mathbf{A}^{\star}_{S^{\subseteq,\mathrm{untl}}}\text{-}\mathbf{mod} \end{tikzcd}$$

H.7.5. We now claim:

Proposition H.7.6. *The functor*

$$\mathrm{D\text{-}mod}(S) \otimes \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \overset{(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})^! \otimes (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^!}{\longrightarrow} \mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}},\mathrm{indep}}$$

is an equivalence.

Proof. Note that operation of union defines a map

$$S \times \mathrm{Ran}^{\mathrm{untl}} \overset{\mathrm{union}_S}{\to} S^{\subseteq,\mathrm{untl}}.$$

Hence, we obtain a functor

$$\mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}}} = \Gamma^{\mathrm{lax}}(S^{\subseteq,\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}}) \overset{\mathrm{union}_S^!}{\longrightarrow} \Gamma^{\mathrm{lax}}(S \times \mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}^{\mathrm{loc,untl}}) \simeq \mathrm{D\text{-}mod}(S) \otimes \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}},$$

and it is easy to see that it sends

$$\mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}},\mathrm{indep}} \to \mathrm{D\text{-}mod}(S) \otimes \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}.$$

The composition

$$S^{\subseteq,\mathrm{untl}} \overset{\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S} \times \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}}{\longrightarrow} S \times \mathrm{Ran}^{\mathrm{untl}} \overset{\mathrm{union}_S}{\to} S^{\subseteq,\mathrm{untl}}$$

is the identity map. Hence, the composition

$$\mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}},\mathrm{indep}} \overset{\mathrm{union}_S^!}{\longrightarrow} \mathrm{D\text{-}mod}(S) \otimes \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}} \overset{(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S})^! \otimes (\mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^!}{\longrightarrow} \mathbf{A}^{\overset{!}{\otimes}}_{S^{\subseteq,\mathrm{untl}},\mathrm{indep}}$$

is the identity functor.

The composition

$$S \times \mathrm{Ran}^{\mathrm{untl}} \overset{\mathrm{union}_S}{\to} S^{\subseteq,\mathrm{untl}} \overset{\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S} \times \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}}}{\longrightarrow} S \times \mathrm{Ran}^{\mathrm{untl}}$$

receives a 1-morphism from the identity map. This 1-morphism defines a natural transformation from the identity endofunctor on $\mathrm{D\text{-}mod}(S) \otimes \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$ to the composition

$$\mathrm{D\text{-}mod}(S) \otimes \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}} \overset{(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small},S} \times \mathrm{pr}^{\mathrm{untl}}_{\mathrm{big}})^!}{\longrightarrow} \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}} \overset{\mathrm{union}_S^!}{\longrightarrow} \mathrm{D\text{-}mod}(S) \otimes \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}.$$

However, this natural transformation is an isomorphism when restricted to $\mathrm{D\text{-}mod}(S) \otimes \mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}}}$, by the definition of this subcategory.

□

H.7.7. From Proposition H.7.6 we obtain:

Corollary H.7.8. *For a* $\mathrm{D\text{-}mod}(S)$*-module category* $\mathbf{D}$*, pullback along* $\mathrm{pr}_{\mathrm{big}} : S^{\subseteq,\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}}$ *defines an equivalence between the following data:*

(i) *A local Ran-unital action of* $\mathbf{A}_S^{\mathrm{loc,untl}}$ *on* $\underline{\mathrm{D\text{-}mod}}(S^{\subseteq,\mathrm{untl}}) \underset{\mathrm{D\text{-}mod}(S)}{\otimes} \mathbf{D}$*;*

(ii) *A local Ran-unital action of* $\mathbf{A}^{\mathrm{loc,untl}}$ *on* $\mathbf{D}$*, compatible with the* $\mathrm{D\text{-}mod}(S)$*-action.*

(ii') *An action on* $\mathbf{D}$ *of the monoidal category* $\mathbf{A}^{\overset{!}{\otimes}}_{\mathrm{Ran}^{\mathrm{untl}},\mathrm{indep}}$*, compatible with the* $\mathrm{D\text{-}mod}(S)$*-action.*

Appendix I. The integration functor in the non-unital setting

In this appendix, we give a categorical meaning to the functor (11.15)

$$\int \text{ins. unit} : \text{Funct}^{\text{loc}\to\text{glob}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}}) \to \text{Funct}^{\text{loc}\to\text{glob}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}}).$$

using the notion of *left-lax* (a.k.a. left-lax) unital structure on a local-to-global functor $\underline{\mathsf{F}}$. As a byproduct, we provide a proof to Proposition 11.8.8.

I.1. A left-lax unital structure on a local-to-global functor.

I.1.1. In this subsection, we describe the notion of a left-lax unital structure on a local-to-global functor $\underline{\mathsf{F}}$ in concrete words. The precise definition will be given in Sect. I.4. Also, we explain why left-lax unital structures provide categorical meaning to the functor $\int$ ins. unit.

I.1.2. Let $\mathcal{Z}$ be a space, and let $\underline{x}_1 \overset{\alpha}{\to} \underline{x}_2$ be a morphism in the category $\text{Maps}(\mathcal{Z}, \text{Ran}^{\text{untl}})$. Recall (see Sect. 11.3.2) that a *lax* unital structure on $\underline{\mathsf{F}}$ means there is a natural transformation

$$\mathsf{F}_{\mathcal{Z},\underline{x}_1} \to \mathsf{F}_{\mathcal{Z},\underline{x}_2} \circ \mathbf{C}^{\text{loc}}_{\alpha}$$

as functors $\mathbf{C}^{\text{loc}}_{\mathcal{Z},\underline{x}_1} \to \mathbf{C}^{\text{glob}} \otimes \text{D-mod}(\mathcal{Z})$. Then an *left-lax* unital structure on $\underline{\mathsf{F}}$ means there is a natural transformation of the opposite direction, i.e.,

$$\mathsf{F}_{\mathcal{Z},\underline{x}_2} \circ \mathbf{C}^{\text{loc}}_{\alpha} \to \mathsf{F}_{\mathcal{Z},\underline{x}_1}.$$

I.1.3. As in Sect. 11.3.4, using the prestack $\mathcal{Z}^{\subseteq}$, we can rewrite the datum of these natural transformations as a natural transformation

$$\mathsf{F}_{\mathcal{Z}^{\subseteq}} \circ \text{ins. unit}_{\mathcal{Z}} \to (\text{Id} \otimes (\text{pr}_{\text{small},\mathcal{Z}})^!) \circ \mathsf{F}_{\mathcal{Z}} \tag{I.1}$$

as functors

$$\mathbf{C}^{\text{loc}}_{\mathcal{Z}} \to \mathbf{C}^{\text{glob}} \otimes \text{D-mod}(\mathcal{Z}^{\subseteq})$$

which is supposed to be equipped with a datum of associativity.

I.1.4. Using the adjunction $((\text{pr}_{\text{small},\mathcal{Z}})_!, (\text{pr}_{\text{small},\mathcal{Z}})^!)$, knowing (I.1) is equivalent to knowing a natural transformation

$$(\text{Id} \otimes (\text{pr}_{\text{small},\mathcal{Z}})_!) \circ \mathsf{F}_{\mathcal{Z}^{\subseteq}} \circ \text{ins. unit}_{\mathcal{Z}} \to \mathsf{F}_{\mathcal{Z}}.$$

Note that the LHS is exactly the functor $\mathsf{F}_{\mathcal{Z}}^{\int \text{ins.unit}}$ (see (11.15)). Hence a left-lax unital structure on $\underline{\mathsf{F}}$ means a natural transformation

$$\underline{\mathsf{F}}^{\int \text{ins.unit}} \to \underline{\mathsf{F}}$$

equipped with a datum of associativity. As we will see in the proof of Proposition I.1.7, the endofunctor

$$\int \text{ins. unit} : \text{Funct}^{\text{loc}\to\text{glob}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}}) \to \text{Funct}^{\text{loc}\to\text{glob}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}})$$

has a natural monad structure, and this datum of associativity says exactly that $\underline{\mathsf{F}}$ is a module for this monad.

I.1.5. In particular, for *any* local-to-global functor $\underline{\mathsf{F}}$, the functor $\underline{\mathsf{F}}^{\int \text{ins.unit}}$ is an induced (a.k.a. free) module for this monad. Therefore we obtain a left-lax unital structure on $\underline{\mathsf{F}}^{\int \text{ins.unit}}$, and this left-lax unital structure satisfies the following universal property. For any local-to-global functor $\underline{\mathsf{G}}$ equipped with a left-lax unital structure, knowing a (plain) natural transformation $\underline{\mathsf{F}} \to \underline{\mathsf{G}}$ is equivalent to knowing a natural transformation $\underline{\mathsf{F}}^{\int \text{ins.unit}} \to \underline{\mathsf{G}}$ compatible with the left-lax unital structure.

I.1.6. Let

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{left-lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$$

be the category of left-lax unital local-to-global functors. The above observation suggests the following result, which will be proved in I.5 (after we give a precise definition to left-lax unital functors).

Proposition I.1.7. *The forgetful functor*

$$\iota_{\text{left-lax}\to\mathrm{all}}:\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{left-lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\to\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$$

has a left adjoint

$$\iota^L_{\text{left-lax}\to\mathrm{all}}:\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\to\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{left-lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$$

that sends

$$\underline{\mathsf{F}}\mapsto\underline{\mathsf{F}}^{\int\mathrm{ins.unit}}.$$

In particular, the latter has a natural left-lax unital structure.

I.2. Comparison with the integration functor in the lax unital setting.

I.2.1. Tautologically, there is a commutative square

$$\begin{tikzcd}
\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \arrow[r, "\subset", "\iota_{\mathrm{st}\to\mathrm{lax}}"'] \arrow[d, "\iota_{\mathrm{st}\to\text{left-lax}}"', "\subset"] & \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \arrow[d, "\iota_{\mathrm{lax}\to\mathrm{all}}"] \\
\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{left-lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \arrow[r, "\iota_{\text{left-lax}\to\mathrm{all}}"'] & \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})
\end{tikzcd} \tag{I.2}$$

where each $\iota_?$ is a forgetful functor from a category of local-to-global functors equipped with certain unital structures to a category of such functors equipped with coarser structures. Here only $\iota_{\mathrm{st}\to\mathrm{lax}}$ and $\iota_{\mathrm{st}\to\text{left-lax}}$ are fully faithful.

I.2.2. Recall in Sect. 11.5, we also constructed an adjoint pair:

$$\iota^L_{\mathrm{st}\to\mathrm{lax}}:\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\rightleftarrows\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}):\iota_{\mathrm{st}\to\mathrm{lax}}$$

such that the left adjoint sends

$$\underline{\mathsf{F}}^{\mathrm{untl}}\mapsto\underline{\mathsf{F}}^{\mathrm{untl},\int\mathrm{ins.unit}}.$$

The following result, which will be proved in I.6, says this adjoint pair is compatible with that in Proposition I.1.7.

Proposition I.2.3. *The commutative square* (I.2) *is left adjointable along the horizontal direction, i.e., the Bech–Chevalley natural transformation from the clockwise circuit in the diagram below to the counter-clockwise circuit is invertible:*

$$\begin{tikzcd}
\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \arrow[d, "\iota_{\mathrm{st}\to\text{left-lax}}"', "\subset"] & \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) \arrow[l, "\iota^L_{\mathrm{st}\to\mathrm{lax}}"'] \arrow[d, "\iota_{\mathrm{lax}\to\mathrm{all}}"] \\
\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{left-lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}) & \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}}). \arrow[l, "\iota^L_{\text{left-lax}\to\mathrm{all}}"']
\end{tikzcd} \tag{I.3}$$

Moreover,

- *The monad* $\iota_{\mathrm{st}\to\mathrm{lax}}\circ\iota^L_{\mathrm{st}\to\mathrm{lax}}$ *can be identified with* (11.23).
- *The underlying endofunctor of the monad* $\iota_{\text{left-lax}\to\mathrm{all}}\circ\iota^L_{\text{left-lax}\to\mathrm{all}}$ *can be identified with* (11.15), *and the unit of the monad can be identified with* (11.16).
- *The combination of* (I.3) *and* (I.2) *gives the commutative diagram* (11.24).

I.3. **Proof of Proposition 11.8.8.** In this subsection, we deduce Proposition 11.8.8 from Proposition I.1.7. This is essentially a formal diagram chase.

I.3.1. We need to show the functor

$$\iota_{\text{st}\to\text{all}} := \iota_{\text{lax}\to\text{all}} \circ \iota_{\text{st}\to\text{lax}} \simeq \iota_{\text{left-lax}\to\text{all}} \circ \iota_{\text{st}\to\text{left-lax}}$$

is fully faithful and identify its essential image.

Given

$$\underline{\mathsf{F}}^{\text{untl}}, \underline{\mathsf{G}}^{\text{untl}} \in \text{Funct}^{\text{loc}\to\text{glob},\text{untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}}),$$

we have

$$\begin{aligned}
& \text{Maps}(\iota_{\text{st}\to\text{all}}(\underline{\mathsf{F}}^{\text{untl}}), \iota_{\text{st}\to\text{all}}(\underline{\mathsf{G}}^{\text{untl}})) \simeq \\
\simeq\ & \text{Maps}(\iota_{\text{st}\to\text{all}}(\underline{\mathsf{F}}^{\text{untl}}), \iota_{\text{left-lax}\to\text{all}} \circ \iota_{\text{st}\to\text{left-lax}}(\underline{\mathsf{G}}^{\text{untl}})) \simeq \\
\simeq\ & \text{Maps}(\iota^L_{\text{left-lax}\to\text{all}} \circ \iota_{\text{st}\to\text{all}}(\underline{\mathsf{F}}^{\text{untl}}), \iota_{\text{st}\to\text{left-lax}}(\underline{\mathsf{G}}^{\text{untl}})) \simeq \\
\simeq\ & \text{Maps}(\iota^L_{\text{left-lax}\to\text{all}} \circ \iota_{\text{lax}\to\text{all}} \circ \iota_{\text{st}\to\text{lax}}(\underline{\mathsf{F}}^{\text{untl}}), \iota_{\text{st}\to\text{left-lax}}(\underline{\mathsf{G}}^{\text{untl}})) \simeq \\
\simeq\ & \text{Maps}(\iota_{\text{st}\to\text{left-lax}} \circ \iota^L_{\text{st}\to\text{lax}} \circ \iota_{\text{st}\to\text{lax}}(\underline{\mathsf{F}}^{\text{untl}}), \iota_{\text{st}\to\text{left-lax}}(\underline{\mathsf{G}}^{\text{untl}})) \simeq \\
\simeq\ & \text{Maps}(\iota^L_{\text{st}\to\text{lax}} \circ \iota_{\text{st}\to\text{lax}}(\underline{\mathsf{F}}^{\text{untl}}), (\underline{\mathsf{G}}^{\text{untl}})) \simeq \\
\simeq\ & \text{Maps}(\iota_{\text{st}\to\text{left-lax}}(\underline{\mathsf{F}}^{\text{untl}}), \iota_{\text{st}\to\text{left-lax}}(\underline{\mathsf{G}}^{\text{untl}})) \simeq \\
\simeq\ & \text{Maps}(\underline{\mathsf{F}}^{\text{untl}}, \underline{\mathsf{G}}^{\text{untl}}),
\end{aligned}$$

where

- The fourth equivalence is due to Proposition I.1.7;
- The sixth equivalence is because $\iota_{\text{st}\to\text{lax}}$ is fully faithful;
- The seventh equivalence is because $\iota_{\text{st}\to\text{left-lax}}$ is fully faithful.

We leave it to the reader to check the resulting composition is the inverse to the obvious map from the RHS to the LHS. This proves $\iota_{\text{st}\to\text{all}}$ is fully faithful.

I.3.2. Let $\underline{\mathsf{F}}$ be a local-to-global functor. By definition, the Global Unitality Axiom (i) means exactly the unit adjunction

$$\underline{\mathsf{F}} \to \iota_{\text{left-lax}\to\text{all}} \circ \iota^L_{\text{left-lax}\to\text{all}}(\underline{\mathsf{F}})$$

is invertible. In particular, $\underline{\mathsf{F}}$ can be naturally lifted to the object

$$\iota^L_{\text{left-lax}\to\text{all}}(\underline{\mathsf{F}}) \in \text{Funct}^{\text{loc}\to\text{glob},\text{left-lax-untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}}).$$

By the proof of Proposition I.1.7, the Global Unitality Axiom (ii) means exactly the left-lax structure on $\iota^L_{\text{left-lax}\to\text{all}}(\underline{\mathsf{F}})$ is strict (see Remark I.5.8 below). It follows that $\underline{\mathsf{F}}$ is contained in the essential image of $\iota_{\text{st}\to\text{all}}$ if and only if it satisfies the Global Unitality Axioms.

$\square$[Proposition 11.8.8]

I.4. **Definition of left-lax unital structures.** To give a homotopy-coherent definition of a left-lax unital structure on a local-to-global functor, we need some higher algebra. This will also complete the omitted higher datum in Sect. 11.2.5, e.t.c..

I.4.1. For any $[n] \in \Delta^{\text{op}}$, let $\text{Ran}^{\subseteq,[n]}$ be the moduli space of chains $\underline{x}_0 \subseteq \underline{x}_1 \subseteq \cdots \subseteq \underline{x}_n$. In particular, we have

$$\text{Ran}^{\subseteq,[0]} = \text{Ran},\ \text{Ran}^{\subseteq,[1]} = \text{Ran}^{\subseteq}.$$

We obtain a simplicial prestack $\text{Ran}^{\subseteq,\bullet}$ which is a *(∞-)categocial object* in PreStk. Indeed, $\text{Ran}^{\subseteq,[0]} =$ Ran is the "prestack of objects" and $\text{Ran}^{\subseteq,[1]} = \text{Ran}^{\subseteq}$ is the "prestack of 1-morphisms". The projections

$$\text{pr}_{\text{small}}, \text{pr}_{\text{big}} : \text{Ran}^{\subseteq} \to \text{Ran}$$

remember respectively the source and the target of a 1-morphism, while

$$\text{diag} : \text{Ran} \to \text{Ran}^{\subseteq}$$

sends an object to the identity 1-morphism at it. The higher categorical structure on $\text{Ran}^{\subseteq,\bullet}$ is provided by its simplicial structure.

I.4.2. Let $\mathcal{Y}^\bullet$ be any categorical object in PreStk. Write $\mathcal{O}bj := \mathcal{Y}^0$, $\mathcal{M}or_1 := \mathcal{Y}^1$ and $\mathcal{M}or_2 := \mathcal{Y}^2$. In general, there is a *comonad* on the 2-category

$$\mathrm{CrystCat}(\mathcal{O}bj)$$

of sheaves of categories on $\mathcal{O}bj$, whose underlying endofunctor is the composition

$$\mathrm{CrystCat}(\mathcal{O}bj) \xrightarrow{p_t^*} \mathrm{CrystCat}(\mathcal{M}or_1) \xrightarrow{p_{s,*}} \mathrm{CrystCat}(\mathcal{O}bj),$$

where $p_s, p_t : \mathcal{M}or_1 \to \mathcal{O}bj$ are the projections that remember respectively the sources and the targets.

I.4.3. The operation of composition on $p_{s,*} \circ p_t^*$ can be obtained as follows. View

$$\mathcal{O}bj \xleftarrow{p_s} \mathcal{M}or_1 \xrightarrow{p_t} \mathcal{O}bj \tag{I.4}$$

as an endomorphism ϕ on $\mathcal{O}bj$ in the 2-category $\mathbf{Corr}(\mathrm{PreStk})^{\mathrm{all}}_{\mathrm{all,all}}$ of correspondences. The categorical structure gives a monad structure on this endomorphism. Namely, $\phi \circ \phi$ can be identified with the correspondence

$$\mathcal{O}bj \xleftarrow{p_s} \mathcal{M}or_2 \xrightarrow{p_t} \mathcal{O}bj \tag{I.5}$$

Then $\phi \circ \phi \to \phi$ is induced by the 2-morphism from (I.5) to (I.4) given by the projection

$$p_{02} : \mathcal{M}or_2 \to \mathcal{M}or_1.$$

that corresponds to the map $[1] \to [2]$, $0 \mapsto 0, 1 \mapsto 2$. We have a functor between 2-categories[84]

$$\mathrm{CrystCat} : \mathbf{Corr}(\mathrm{PreStk})^{\mathrm{all,2-op}}_{\mathrm{all,all}} \to \mathbf{2-Cat}$$

that sends the endomorphism ϕ to the endomorphism $p_{s,*} \circ p_t^*$. It follows that the monad structure on ϕ gives a comonad structure on $p_{s,*} \circ p_t^*$.

I.4.4. Applying to $\mathrm{Ran}^{\subseteq,\bullet}$, we obtain a comonad

$$\mathsf{P} : \mathrm{CrystCat}(\mathrm{Ran}) \to \mathrm{CrystCat}(\mathrm{Ran})$$

whose underlying endofunctor is $(\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^*$. In other words, we have (see Sect. 11.2.5)

$$\mathsf{P}(\underline{\mathbf{C}}^{\mathrm{loc}}) \simeq \mathbf{C}^{\mathrm{loc},\subseteq}.$$

As explained in *loc.cit.*, we have

Lemma I.4.5. *A local unital structure on* $\underline{\mathbf{C}}^{\mathrm{loc}}$ *is the same as a* P*-comodule structure on it, where*

$$\mathsf{P} = (\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{pr}_{\mathrm{big}})^* : \mathrm{CrystCat}(\mathrm{Ran}) \to \mathrm{CrystCat}(\mathrm{Ran})$$

is a comonad acting on $\mathrm{CrystCat}(\mathrm{Ran})$.

I.4.6. From now on, whenever $\underline{\mathbf{C}}^{\mathrm{loc}}$ is equipped with a local unital structure, we view it as a P-comodule via the above lemma. Note that the coaction functor fits into the following commutative diagram

$$\begin{tikzcd}
\underline{\mathbf{C}}^{\mathrm{loc}} \arrow[r, "\mathrm{coact}"] \arrow[d, "\mathrm{Id}"'] & \mathsf{P}(\underline{\mathbf{C}}^{\mathrm{loc}}) \arrow[d, "\simeq"] \\
\underline{\mathbf{C}}^{\mathrm{loc}} \arrow[r, "\mathrm{ins.unit}"] & \mathbf{C}^{\mathrm{loc},\subseteq}.
\end{tikzcd}$$

[84] See [GaRo3, Chapter 7] for the notation.

I.4.7. *Example.* The constant sheaf of categories

$$\mathbf{D}^{\text{cons}} := \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}) \in \text{CrystCat}(\text{Ran})$$

has an obvious local unital structure. Note that

$$\mathsf{P}(\mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran})) \simeq \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}^{\subseteq}),$$

where the RHS is viewed as a sheaf of categories over Ran via the small projection $\text{pr}_{\text{small}} : \text{Ran}^{\subseteq} \to \text{Ran}$. The corresponding P-comodule structure is given by the functor

$$\text{Id} \otimes (\text{pr}_{\text{small}})^! : \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}) \to \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}^{\subseteq}).$$

Note that we have an adjunction

$$\text{coact}^L : \mathsf{P}(\mathbf{D}^{\text{cons}}) \rightleftarrows \mathbf{D}^{\text{cons}} : \text{coact}$$

in the 2-category CrystCat(Ran), where the left adjoint coact^L is given by the functor $\text{Id} \otimes (\text{pr}_{\text{small}})_!$. Also, the right adjoint coact is fully faithful, i.e.

$$\text{coact}^L \circ \text{coact} \xrightarrow{\sim} \text{Id}\,.$$

Indeed, this follows from the contractibility of the map $\text{pr}_{\text{small}} : \text{Ran}^{\subseteq} \to \text{Ran}$.

I.4.8. *A general paradigm.* Let P be a comonad acting on a 2-category $\mathbb{S}$. Let $\mathbf{c}, \mathbf{d}$ be two P-comodules. For any morphism $f : \mathbf{c} \to \mathbf{d}$ in $\mathbb{S}$, we can talk about (co)lax P-linear structures on f. Namely, a lax P-linear structure on f is a 2-morphism

$$\begin{tikzcd}
\mathbf{c} \arrow[r, "\text{coact}"] \arrow[d, "f"'] & \mathsf{P}(\mathbf{c}) \arrow[d, "\mathsf{P}(f)"] \\
\mathbf{d} \arrow[r, "\text{coact}"'] \arrow[ur, Rightarrow] & \mathsf{P}(\mathbf{d}),
\end{tikzcd}$$

i.e.,

$$\alpha : \text{coact} \circ f \to \mathsf{P}(f) \circ \text{coact},$$

equipped with a datum of associativity.

Recall any comonad P has a counit natural transformation $\epsilon : \mathsf{P} \to \text{Id}$, and for any comodule $\mathbf{c}$, the composition $\mathbf{c} \xrightarrow{\text{coact}} \mathsf{P}(\mathbf{c}) \xrightarrow{\epsilon} \mathbf{c}$ is isomorphic to the identity morphism. Then the above datum of associativity in particular says the outer square in the following diagram commutes:

$$\begin{tikzcd}
\mathbf{c} \arrow[r, "\text{coact}"] \arrow[d, "f"'] & \mathsf{P}(\mathbf{c}) \arrow[r, "\epsilon"] \arrow[d, "\mathsf{P}(f)"] & \mathbf{c} \arrow[d, "f"] \\
\mathbf{d} \arrow[r, "\text{coact}"'] \arrow[ur, Rightarrow] & \mathsf{P}(\mathbf{d}) \arrow[r, "\epsilon"'] & \mathbf{d}.
\end{tikzcd}$$

In other words, α becomes invertible after composing with the counit $\mathsf{P}(\mathbf{d}) \xrightarrow{\epsilon} \mathbf{d}$.

Dually, a colax P-linear structure on f is a 2-morphism

$$\beta : \mathsf{P}(f) \circ \text{coact} \to \text{coact} \circ f$$

equipped with a datum of associativity. Also, β becomes invertible after composing with ϵ.

Given a (co)lax P-linear structure on f, we say it is *strict*, or equivalently f is P-linear, if the above 2-morphism α (resp. β) is invertible.

I.4.9. Now for $\mathbb{S} := \text{CrystCat}(\text{Ran})$, $\mathbf{c} := \underline{\mathbf{C}}^{\text{loc}}$ and $\mathbf{d} := \mathbf{D}^{\text{const}} := \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran})$, a morphism $f : \mathbf{c} \to \mathbf{d}$ in $\mathbb{S}$ is just a local-to-global functor

$$\underline{\mathsf{F}} : \underline{\mathbf{C}}^{\text{loc}} \to \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}).$$

Suppose $\underline{\mathbf{C}}^{\text{loc}}$ is equipped with a local unital structure, i.e., $\mathbf{c}$ is equipped with a P-comodule structure, where recall $\mathsf{P}(\underline{\mathbf{C}}^{\text{loc}}) \simeq \mathbf{C}^{\text{loc},\subseteq}$. As explained in Sect. 11.3.4, we have

Lemma I.4.10. *In the above notations, knowing a lax unital structure on* $\underline{\mathsf{F}}$ *is equivalent to knowing a lax* P*-linear structure on* f*. Via this correspondence, the natural transformation*

$$\alpha : \text{coact} \circ f \to \mathsf{P}(f) \circ \text{coact},$$

is given by (11.12).

I.4.11. Now we *define* a left-lax unital structure on $\underline{\mathsf{F}}$ to be a left-lax P-linear structre on f. This is the homotopically sound definition promised in Sect. I.1.

I.5. Proof of Proposition I.1.7.

I.5.1. Using the notations in Sect. I.4.8, the forgetful functor $\iota_{\text{left-lax}\to\text{all}}$ is given by

$$\iota_{\text{left-lax}\to\text{all}} : \text{Funct}_{\text{left-lax-}\mathsf{P}}(\mathbf{c}, \mathbf{d}) \to \text{Funct}(\mathbf{c}, \mathbf{d})$$

which sends a left-lax P-linear morphism $f : \mathbf{c} \to \mathbf{d}$ to its underlying morphism. Now Proposition I.1.7 is a particular case of the following general result.

Lemma I.5.2. *Let* $(\mathbb{S}, \mathsf{P}, \mathbf{c}, \mathbf{d})$ *be as in Sect. I.4.8. Suppose:*

(*) *The morphism* $\text{coact} : \mathbf{d} \to \mathsf{P}(\mathbf{d})$ *has a left adjoint* $\text{coact}^L : \mathsf{P}(\mathbf{d}) \to \mathbf{d}$.

Then the forgetful functor

$$\text{Funct}_{\text{left-lax-}\mathsf{P}}(\mathbf{c}, \mathbf{d}) \to \text{Funct}(\mathbf{c}, \mathbf{d}) \tag{I.6}$$

has a left adjoint given by

$$\text{Funct}(\mathbf{c}, \mathbf{d}) \to \text{Funct}_{\text{left-lax-}\mathsf{P}}(\mathbf{c}, \mathbf{d}),\ f \mapsto \text{coact}^L \circ \mathsf{P}(f) \circ \text{coact}. \tag{I.7}$$

I.5.3. *Proof.* The rest of this subsection is devoted to the proof of the lemma. We first define the desired left-lax P-linear structure on the composition

$$\mathbf{c} \xrightarrow{\text{coact}} \mathsf{P}(\mathbf{c}) \xrightarrow{\mathsf{P}(f)} \mathsf{P}(\mathbf{d}) \xrightarrow{\text{coact}^L} \mathbf{d}. \tag{I.8}$$

It is the composition of the following three left-lax P-linear structures:

- The coaction morphism $\mathbf{c} \xrightarrow{\text{coact}} \mathsf{P}(\mathbf{c})$ always has a natural strict P-linear structure;
- The morphism $\mathsf{P}(f) : \mathsf{P}(\mathbf{c}) \to \mathsf{P}(\mathbf{d})$ is a strictly P-linear morphism between cofree P-comodules;
- As the left adjoint of the strictly P-linear morphism $\mathbf{d} \xrightarrow{\text{coact}} \mathsf{P}(\mathbf{d})$, the morphism $\mathsf{P}(\mathbf{d}) \xrightarrow{\text{coact}^L} \mathbf{d}$ has a natural left-lax P-linear structure.

I.5.4. To show (I.7) is indeed left adjoint to (I.6), we provide the unit and counit natural transformations for this adjunction.

I.5.5. *Unit natural transformation.* Given $f \in \text{Funct}(\mathbf{c}, \mathbf{d})$, we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{c} \arrow[r, "\text{coact}"] & \mathsf{P}(\mathbf{c}) \arrow[r, "\epsilon"] \arrow[d, "\mathsf{P}(f)"] & \mathbf{c} \arrow[d, "f"] \\
\mathbf{d} \arrow[r, "\text{coact}"] & \mathsf{P}(\mathbf{d}) \arrow[r, "\epsilon"] & \mathbf{d}.
\end{tikzcd}$$

Then the desired morphism $f \to \text{coact}^L \circ \mathsf{P}(f) \circ \text{coact}$ is obtained by passing to left adjoints along the bottom line and using the fact that $\epsilon \circ \text{coact} \simeq \text{Id}$. More precisely, we have a morphism

$$\epsilon \to \epsilon \circ \text{coact} \circ \text{coact}^L \simeq \text{coact}^L,$$

where the first morphism is given by the unit adjunction of $(\mathrm{coact}^L, \mathrm{coact})$. Hence we obtain a morphism

$$f \simeq f \circ \epsilon \circ \mathrm{coact} \simeq \epsilon \circ \mathsf{P}(f) \circ \mathrm{coact} \to \mathrm{coact}^L \circ \mathsf{P}(f) \circ \mathrm{coact}\,. \tag{I.9}$$

which is the value of the desired unit natural transformation at f.

I.5.6. *Counit natural transformation.* Given $g \in \mathrm{Funct}_{\text{left-lax-}\mathsf{P}}(\mathbf{c}, \mathbf{d})$, the left-lax P-linear structure on it provides a morphism in $\mathrm{Funct}_{\text{left-lax-}\mathsf{P}}(\mathbf{c}, \mathbf{d})$:

$$\mathsf{P}(g) \circ \mathrm{coact} \to \mathrm{coact} \circ g.$$

Using the adjunction $(\mathrm{coact}^L, \mathrm{coact})$, we obtain a morphism

$$\mathrm{coact}^L \circ \mathsf{P}(g) \circ \mathrm{coact} \to g$$

which is the value of the desired counit natural transformation at g.

I.5.7. It is a routine exercise to verify these natural transformations indeed satisfy the axioms of an adjunction. We leave it to the readers.

□

I.5.8. *Remark.* Note that Proposition I.1.7 is indeed a particular case of Lemma I.5.2 because of Sect. I.4.7. Also, unwinding the definitions, for a local-to-global functor $\underline{\mathsf{F}}$ and the corresponding $f : \mathbf{c} \to \mathbf{d}$, the unit adjunction (I.9) is exactly (11.16).

Moreover, by definition, $\underline{\mathsf{F}}$ satisfies the Global Unitality Axioms iff the corresponding f satisfies

(i) The unit adjunction (I.9) is invertible;

(ii) The natural transformation

$$\mathsf{P}(f) \circ \mathrm{coact} \to \mathrm{coact} \circ f \tag{I.10}$$

obtained from the inverse of (I.9) is invertible.

Now (i) implies f has a natural left-lax P-linear structure given by that of $\mathrm{coact}^L \circ \mathsf{P}(f) \circ \mathrm{coact}$. A direct diagram chasing shows this left-lax P-linear structure is exhibited by the natural transformation (I.10). Hence (ii) says this left-lax P-linear structure on f is strict. This was used in the proof of Proposition 11.8.8 in Sect. I.3.2.

I.6. **Proof of Proposition I.2.3.**

I.6.1. Unlike Proposition I.1.7, the proof of Proposition I.2.3 is *not* completely formal. Instead, we need the following particular feature of the setting of local-to-global functors.

Lemma I.6.2. *The composition*

$$\iota^L_{\text{left-lax}\to\text{all}} \circ \iota_{\text{lax}\to\text{all}} : \mathrm{Funct}^{\text{loc}\to\text{glob,lax-untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}}) \to \mathrm{Funct}^{\text{loc}\to\text{glob,left-lax-untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}})$$

takes image in $\mathrm{Funct}^{\text{loc}\to\text{glob,untl}}(\underline{\mathbf{C}}^{\text{loc}}, \mathbf{C}^{\text{glob}})$.

I.6.3. *Remark.* The claim of the lemma is a priori *stronger* than the results listed in Sect. 11.5. Namely, given a local-to-global functor $\underline{\mathsf{F}}$ equipped with a lax unital structure, Sect. 11.5 says there is *a* strictly unital structure on $\underline{\mathsf{F}}^{\int \text{ins.unit}}$, while the lemma says the natural left-lax unital structure on it (provided by Proposition I.1.7) is strict. Nevertheless, it is easy to see the proof in loc.cit. actually implies this stronger claim. For completeness, we repeat this proof.

Proof. We will deduce the claim from Lemma 11.5.5. Let $\underline{\mathsf{F}}$ be a local-to-global functor equipped with a lax unital structure, and $f : \mathbf{c} \to \mathbf{d}$ be the corresponding lax P-linear morphism. We need to show that the left-lax P-linear structure on $\mathrm{coact}^L \circ \mathsf{P}(f) \circ \mathrm{coact}$ is strict. Recall that $\mathsf{P}(f) \circ \mathrm{coact}$, i.e.,

$$\mathbf{C}^{\text{loc}}_{\text{Ran}} \xrightarrow{\text{ins.unit}} \mathbf{C}^{\text{loc},\subseteq}_{\text{Ran}} \xrightarrow{\underline{\mathsf{F}}^{\subseteq}} \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}^{\subseteq}) \tag{I.11}$$

is naturally P-linear. On the other hand, $\mathrm{coact}^L : \mathsf{P}(\mathbf{d}) \to \mathbf{d}$, i.e.,

$$\mathrm{Id} \otimes (\mathrm{pr}_{\text{small}})_! : \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}^{\subseteq}) \to \mathbf{C}^{\text{glob}} \otimes \underline{\text{D-mod}}(\text{Ran}). \tag{I.12}$$

has a natural left-lax P-linear structure because its right adjoint is naturally P-linear.

Consider the categorical prestack

$$\mathrm{Ran} \underset{\mathrm{Ran}^{\mathrm{untl}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{untl}}$$

where $\mathrm{Ran}^{\subseteq,\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}}$ is the small projection $\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}}$. Viewing it as a categorical prestack over Ran, we obtain a crystal of categories over Ran

$$\underline{\text{D-mod}}(\mathrm{Ran} \underset{\mathrm{Ran}^{\mathrm{untl}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{untl}})$$

and a functor

$$\mathsf{t}^! : \underline{\text{D-mod}}(\mathrm{Ran} \underset{\mathrm{Ran}^{\mathrm{untl}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{untl}}) \to \underline{\text{D-mod}}(\mathrm{Ran}^{\subseteq}) \tag{I.13}$$

given by pullback along $\mathsf{t} : \mathrm{Ran}^{\subseteq} \to \mathrm{Ran} \underset{\mathrm{Ran}^{\mathrm{untl}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{untl}}$. Note that both sides of (I.13) have natural local unital structures and the functor is obviously unital. Now Lemma 11.5.5, combined with a variant of Lemma C.5.12, says the composition

$$\underline{\text{D-mod}}(\mathrm{Ran} \underset{\mathrm{Ran}^{\mathrm{untl}}}{\times} \mathrm{Ran}^{\subseteq,\mathrm{untl}}) \to \underline{\text{D-mod}}(\mathrm{Ran}^{\subseteq}) \xrightarrow{(\mathrm{pr}_{\mathrm{small}})_!} \underline{\text{D-mod}}(\mathrm{Ran}),$$

which is a priori left-lax unital, is strictly unital. The statement remains true if we tensor it with the DG category $\mathbf{C}^{\mathrm{glob}}$. Then we finish the proof because the composition (I.11) factors through (I.13) by the lax unital structure on $\underline{\mathsf{F}}$.

$\square$

I.6.4. By Lemma I.6.2, there is a unique functor $\tau_{\mathrm{lax}\to\mathrm{st}}$ making the following diagram commute

$$\begin{tikzcd}
\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \arrow[d, "\iota_{\mathrm{st}\to\mathrm{left\text{-}lax}}"', "\subset"] & \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \arrow[l, "\tau_{\mathrm{lax}\to\mathrm{st}}"'] \arrow[d, "\iota_{\mathrm{lax}\to\mathrm{all}}"] \\
\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{left\text{-}lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) & \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}). \arrow[l, "\iota^L_{\mathrm{left\text{-}lax}\to\mathrm{all}}"']
\end{tikzcd} \tag{I.14}$$

In particular, we obtained:

- A lifting of the endofunctor
$$\iota_{\mathrm{left\text{-}lax}\to\mathrm{all}} \circ \iota^L_{\mathrm{left\text{-}lax}\to\mathrm{all}} : \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}})$$
to an endofunctor
$$\iota_{\mathrm{st}\to\mathrm{lax}} \circ \tau_{\mathrm{lax}\to\mathrm{st}} : \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}});$$
- A lifting of the endofunctor
$$\iota^L_{\mathrm{left\text{-}lax}\to\mathrm{all}} \circ \iota_{\mathrm{left\text{-}lax}\to\mathrm{all}} : \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{left\text{-}lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{left\text{-}lax\text{-}untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}))$$
to an endofunctor
$$\tau_{\mathrm{lax}\to\mathrm{st}} \circ \iota_{\mathrm{st}\to\mathrm{lax}} : \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}}) \to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}}, \mathbf{C}^{\mathrm{glob}});$$

I.6.5. To finish the proof, we only need to lift the unit and counit adjunctions

$$\mathrm{Id} \to \iota_{\mathrm{left\text{-}lax}\to\mathrm{all}} \circ \iota^L_{\mathrm{left\text{-}lax}\to\mathrm{all}},\ \iota^L_{\mathrm{left\text{-}lax}\to\mathrm{all}} \circ \iota_{\mathrm{left\text{-}lax}\to\mathrm{all}} \to \mathrm{Id}$$

to

$$\mathrm{Id} \to \iota_{\mathrm{st}\to\mathrm{lax}} \circ \tau_{\mathrm{lax}\to\mathrm{st}},\ \tau_{\mathrm{lax}\to\mathrm{st}} \circ \iota_{\mathrm{st}\to\mathrm{lax}} \to \mathrm{Id}$$

and verify they satisfy the axioms of an adjunction. Indeed, these will induce an equivalence $\tau_{\mathrm{lax}\to\mathrm{st}} \simeq \iota^L_{\mathrm{st}\to\mathrm{lax}}$ such that (I.14) can be identified with the Bech–Chevalley natural transformation (I.3). Then the other claims in Proposition I.2.3 follow from definitions.

I.6.6. We will lift the counit and unit natural transformations, and leave it to the readers to verify they satisfy the axioms of an adjunction.

I.6.7. *Lift the counit.* The lifting of the counit is obvious because the forgetful functor

$$\mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\mathrm{untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})\to \mathrm{Funct}^{\mathrm{loc}\to\mathrm{glob},\text{left-lax-untl}}(\underline{\mathbf{C}}^{\mathrm{loc}},\mathbf{C}^{\mathrm{glob}})$$

is fully faithful and the any morphism in the RHS (such as the counit $\iota^L_{\text{left-lax}\to\mathrm{all}}\circ\iota_{\text{left-lax}\to\mathrm{all}}\to \mathrm{Id}$) has a unique lifting to the LHS as long as its source and target are contained in the LHS.

I.6.8. *Lift the unit.* The rest of this subsection is devoted to lift the unit. Recall its definition in Sect. I.5.5. By definition, the morphism $\epsilon:\mathsf{P}(\mathbf{d})\to\mathbf{d}$ is

$$\mathrm{Id}\otimes\mathrm{diag}^!:\mathbf{C}^{\mathrm{glob}}\otimes\underline{\text{D-mod}}(\mathrm{Ran}^{\subseteq})\to\mathbf{C}^{\mathrm{glob}}\otimes\underline{\text{D-mod}}(\mathrm{Ran})$$

and the morphism

$$\epsilon\to\mathrm{coact}^L$$

is induced by the natural transformation

$$\mathrm{diag}^!\to\mathrm{diag}^!\circ\mathrm{pr}^!_{\mathrm{small}}\circ(\mathrm{pr}_{\mathrm{small}})_!\simeq(\mathrm{pr}_{\mathrm{small}})_! \tag{I.15}$$

provided by the isomorphism $\mathrm{pr}_{\mathrm{small}}\circ\mathrm{diag}\simeq\mathrm{Id}$.

I.6.9. Recall the restriction functor

$$\mathsf{t}^!:\underline{\text{D-mod}}(\mathrm{Ran}\underset{\mathrm{Ran}^{\mathrm{untl}}}{\times}\mathrm{Ran}^{\subseteq,\mathrm{untl}})\to\underline{\text{D-mod}}(\mathrm{Ran}^{\subseteq}).$$

We claim the (horizontal) composition of (I.15) with $\mathsf{t}^!$, i.e.,

$$\mathrm{diag}^!\circ\mathsf{t}^!\to(\mathrm{pr}_{\mathrm{small}})_!\circ\mathsf{t}^! \tag{I.16}$$

can be naturally lifted to a natural transformation between lax unital functors

$$\underline{\text{D-mod}}(\mathrm{Ran}\underset{\mathrm{Ran}^{\mathrm{untl}}}{\times}\mathrm{Ran}^{\subseteq,\mathrm{untl}})\to\underline{\text{D-mod}}(\mathrm{Ran}).$$

Here $\underline{\text{D-mod}}(\mathrm{Ran}\underset{\mathrm{Ran}^{\mathrm{untl}}}{\times}\mathrm{Ran}^{\subseteq,\mathrm{untl}})$ and $\underline{\text{D-mod}}(\mathrm{Ran})$ are equipped with the local unital structures given respectively by $\mathrm{Ran}^{\subseteq,\mathrm{untl}}$ and $\mathrm{Ran}^{\mathrm{untl}}$.

I.6.10. Consider the map

$$\mathrm{diag}^{\mathrm{untl}}:\mathrm{Ran}^{\mathrm{untl}}\to\mathrm{Ran}^{\subseteq,\mathrm{untl}}$$

and its left inverse

$$\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}}:\mathrm{Ran}^{\subseteq,\mathrm{untl}}\to\mathrm{Ran}^{\mathrm{untl}}.$$

Similar to (I.15), we have a natural transformation

$$(\mathrm{diag}^{\mathrm{untl}})^!\to(\mathrm{diag}^{\mathrm{untl}})^!\circ(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})^!\circ(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_!\simeq(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_!. \tag{I.17}$$

Note that $(\mathrm{diag}^{\mathrm{untl}})^!$ has an obvious lax unital structure, i.e., is a lax functor between sheaves of categories on $\mathrm{Ran}^{\mathrm{untl}}$, while $(\mathrm{pr}^{\mathrm{untl}}_{\mathrm{small}})_!$ has a strictly unital structure by Lemma 11.5.5. The composition (I.17) is compatible with the lax unital structures on both sides because each natural transformation is.

By (a variant of) Lemma C.5.12, when restricted along $\mathrm{Ran}\to\mathrm{Ran}^{\mathrm{untl}}$, (I.17) gives exactly (I.16). In other words, we have proved the claim in Sect. I.6.9.

I.6.11. Recall that $\mathsf{P}(f)\circ\mathrm{coact}$ (which is just (I.11)) factors through $\mathrm{Id}\otimes\mathsf{t}^!$. It follows that (I.16) induces a natural transformation

$$\epsilon\circ\mathsf{P}(f)\circ\mathrm{coact}\to\mathrm{coact}^L\circ\mathsf{P}(f)\circ\mathrm{coact}$$

compatible with the lax unital structures on both sides.

I.6.12. On the other hand, we have

$$f\simeq f\circ\epsilon\circ\mathrm{coact}\simeq\epsilon\circ\mathsf{P}(f)\circ\mathrm{coact}$$

because ϵ is the counit of the comonad P. Moreover, this isomorphism is obviously compatible with the lax unital structures on both sides.

I.6.13. Combining the above two subections, we obtain that

$$f \to \mathrm{coact}^L \circ \mathsf{P}(f) \circ \mathrm{coact}$$

is naturally compatible with the lax unital structures on both sides. In other words, we have found the desired lifting of the unit adjunction.

I.6.14. We leave it to the readers to check the above liftings indeed satisfy the axioms of an adjunction. □[Proposition I.1.7]

Appendix J. A homotopical device for coaction

The goal of this section is to introduce a homotopical device that will help us carry out the constructions in Sects. 4.6 and 5.3 up to *coherent* homotopy.

J.1. **Associative algebras via mock-simplicial sets.**

J.1.1. Let us recall the following device of encoding the structure of associative algebra (resp., module over a given associative algebra) in a monoidal category (see [Lu2, Sect. 2.2.4]).

Let $\mathbf{\Delta}^{\mathrm{op,mock}}$ be the category of (possibly empty) finite ordered sets. The operation of (ordered) union defines on $\mathbf{\Delta}^{\mathrm{op,mock}}$ a structure of monoidal category. Its monoidal unit is $\emptyset$.

In what follows we will denote[85]

$$\mathbf{\Delta}^{\mathrm{mock}} := (\mathbf{\Delta}^{\mathrm{op,mock}})^{\mathrm{op}}.$$

J.1.2. The datum of a unital associative algebra in a monoidal category $\mathfrak{A}$ is equivalent to that of a monoidal functor

$$\mathbf{\Delta}^{\mathrm{op,mock}} \to \mathfrak{A}.$$

Under this correspondence, for a given functor $F : \mathbf{\Delta}^{\mathrm{op,mock}} \to \mathfrak{A}$, the corresponding algebra object $\mathfrak{a} \in \mathfrak{A}$ is $F(\{*\})$. The unit in $\mathfrak{a}$ is given by the map

$$\mathbf{1}_{\mathfrak{A}} = F(\emptyset) \to F(\{*\}) = \mathfrak{a},$$

corresponding to the (unique) map $\emptyset \to \{*\}$.

The binary operation on $\mathfrak{a}$ corresponds to the map

$$\mathfrak{a} \otimes \mathfrak{a} = F(\{*\}) \otimes F(\{*\}) \simeq F(\{1,2\}) \to F(\{*\}) = \mathfrak{a},$$

where the arrows is given by the (unique) map $\{1,2\} \to \{*\}$ in $\mathbf{\Delta}^{\mathrm{op,mock}}$.

J.1.3. Let $\mathbf{\Delta}^{\mathrm{op,mock}}_*$ be the category of non-empty finite ordered sets, pointed by their maximal element. The category $\mathbf{\Delta}^{\mathrm{op,mock}}_*$ is naturally a module over $\mathbf{\Delta}^{\mathrm{op,mock}}$.

J.1.4. Let us be given a monoidal category $\mathfrak{A}$. Let $\mathfrak{a}$ be an associative algebra in $\mathfrak{A}$, thought of as a monoidal functor

$$F_a : \mathbf{\Delta}^{\mathrm{op,mock}} \to \mathfrak{A}.$$

Then the category

$$\mathfrak{a}\text{-mod}(\mathfrak{A})$$

of $\mathfrak{a}$-modules in $\mathfrak{A}$ is equivalent to that of functors of left $\mathbf{\Delta}^{\mathrm{op,mock}}$-module categories

$$\mathbf{\Delta}^{\mathrm{op,mock}}_* \to \mathfrak{A},$$

where $\mathfrak{A}$ is a left $\mathbf{\Delta}^{\mathrm{op,mock}}$-module via F_a.

[85]The category $\mathbf{\Delta}^{\mathrm{mock}}$ is equivalent to the subcategory of $\mathbf{\Delta}$ consisting of *active* morphisms, i.e. those that preserve the maximal and minimal element. For our purposes, the nonstandard "mock" terminology will be more convenient. We caution the reader that the usual simplex category, denoted $\mathbf{\Delta}$, is that of non-empty finite ordered sets. Note that arrows in $\mathbf{\Delta}$ and $\mathbf{\Delta}^{\mathrm{mock}}$ go in opposite directions.

J.1.5. Under this correspondence, given a functor $F_m : \mathbf{\Delta}_*^{\mathrm{op,mock}} \to \mathfrak{A}$, the object of $\mathfrak{A}$ underlying the corresponding $\mathfrak{a}$-module is

$$\mathfrak{m} := F_m(\{*\}).$$

The action map $\mathfrak{a} \otimes M \to M$ is given by

$$\mathfrak{a} \otimes \mathfrak{m} \simeq F_a(\{0\}) \otimes F_m(\{*\}) \simeq F_m(\{0,*\}) \to F_m(\{*\}) \simeq \mathfrak{m}.$$

J.1.6. As above, in what follows we will denote

$$\mathbf{\Delta}_*^{\mathrm{mock}} := (\mathbf{\Delta}_*^{\mathrm{op,mock}})^{\mathrm{op}}.$$

We will refer to functors $\mathbf{\Delta}^{\mathrm{op,mock}} \to \mathfrak{A}$ (resp., $\mathbf{\Delta}_*^{\mathrm{op,mock}} \to \mathfrak{A}$, $\mathbf{\Delta}^{\mathrm{mock}} \to \mathfrak{A}$, $\mathbf{\Delta}_*^{\mathrm{mock}} \to \mathfrak{A}$) as *mock-simplicial* (resp., pointed mock-simplicial, mock cosimplicial, pointed mock cosimplicial) objects of $\mathfrak{A}$.

J.2. **Factorization categories attached to *associative* factorization algebras.** Recall that our goal is to carry out the constructions in Sects. 4.6 and 5.3. We will achieve this by introducing appropriate objects to feed into the machine in Sect. J.1.

J.2.1. Let $\mathfrak{A}_1$ be a monoidal category and let $\mathfrak{C}_1$ be a module category over it. Note that given two such pairs, we can talk about *strictly* and *right-lax* monoidal functors of pairs

$$(F_a, F_c) : (\mathfrak{A}_1, \mathfrak{C}_1) \to (\mathfrak{A}_1, \mathfrak{C}_2),$$

i.e., $F_a : \mathfrak{A}_1 \to \mathfrak{A}_2$ is righ-lax monoidal, and $F_c : \mathfrak{C}_1 \to \mathfrak{C}_2$ is right-lax compatible with the actions.

We introduce several such pairs now.

J.2.2. Take

$$\mathfrak{A}_1 = \mathbf{\Delta}^{\mathrm{mock}},\ \mathfrak{A}_2 = (\mathrm{FactAlg}^{\mathrm{untl}}(X))^{\mathrm{op}},\ \mathfrak{A}_3 = \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X),$$

where the subscript "lax" means that we are considering lax factorization categories.

J.2.3. By Sect. J.1, an object

$$\mathcal{R} \in \mathrm{AssAlg}(\mathrm{FactAlg}^{\mathrm{untl}}(X)) \tag{J.1}$$

can be viewed as a *stricty* monoidal functor

$$\mathbf{\Delta}^{\mathrm{op,mock}} \to \mathrm{FactAlg}^{\mathrm{untl}}(X), \tag{J.2}$$

and hence also as a *stricty* monoidal functor

$$\mathbf{\Delta}^{\mathrm{mock}} \to (\mathrm{FactAlg}^{\mathrm{untl}}(X))^{\mathrm{op}}. \tag{J.3}$$

J.2.4. In addition, we have a naturally defined *right-lax* monoidal functor

$$(\mathrm{FactAlg}^{\mathrm{untl}})^{\mathrm{op}} \to \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X), \quad \mathcal{R} \to \mathcal{R}\text{-mod}^{\mathrm{fact}}, \quad (\mathcal{R}_1 \xrightarrow{\phi} \mathcal{R}_2) \rightsquigarrow \mathrm{Res}_\phi\,. \tag{J.4}$$

Composing, for $\mathcal{R}$ as above, we obtain a *right-lax* monoidal functor

$$\mathbf{\Delta}^{\mathrm{mock}} \to \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X), \quad n \mapsto \mathcal{R}^{\otimes n}\text{-mod}^{\mathrm{fact}}. \tag{J.5}$$

J.2.5. Take

$$\mathfrak{C}_1 = \mathbf{\Delta}_*^{\mathrm{mock}},\ \mathfrak{C}_2 = \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X),\ \mathfrak{C}_3 = \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X),$$

where:

- $\mathbf{\Delta}^{\mathrm{mock}}$ acts on $\mathbf{\Delta}_*^{\mathrm{mock}}$ as in Sect. J.1.3;
- $\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$ acts on itself via the (symmetric) monoidal structure on $\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$;
- $(\mathrm{FactAlg}^{\mathrm{untl}}(X))^{\mathrm{op}}$ acts on $\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$ by

$$\mathcal{R}, \mathbf{A} \mapsto \mathcal{R}\text{-mod}^{\mathrm{fact}}(\mathbf{A}) \simeq (\mathcal{R} \otimes \mathbf{1}_{\mathbf{A}})\text{-mod}^{\mathrm{fact}}(\mathbf{A}).$$

We note that the above action of $(\mathrm{FactAlg}^{\mathrm{untl}})^{\mathrm{op}}$ on $\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$ is monoidal thanks to Lemma C.11.19.

J.2.6. Let $\mathcal{R}$ be as in (J.1) and suppose that it acts on the unit $\mathbf{1}_{\mathbf{C}} \in \mathbf{C}$ for some $\mathbf{C} \in \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$.

Then the functor (J.2) extends to a *strictly* monoidal functor of pairs

$$(\boldsymbol{\Delta}^{\mathrm{op,mock}}, \boldsymbol{\Delta}^{\mathrm{op,mock}}_*) \to (\mathrm{FactAlg}^{\mathrm{untl}}(X), (\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X))^{\mathrm{op}}), \quad * \mapsto \mathbf{C}. \tag{J.6}$$

Hence, we obtain a *strictly* monoidal functor of pairs

$$(\boldsymbol{\Delta}^{\mathrm{mock}}, \boldsymbol{\Delta}^{\mathrm{mock}}_*) \to ((\mathrm{FactAlg}^{\mathrm{untl}}(X))^{\mathrm{op}}, \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)), \quad * \mapsto \mathbf{C}. \tag{J.7}$$

J.2.7. In addition, the functor (J.4) extends to a *right-lax* monoidal functor

$$((\mathrm{FactAlg}^{\mathrm{untl}}(X))^{\mathrm{op}}, \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)) \to (\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X), \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)), \tag{J.8}$$

which acts as identity on the module component.

J.2.8. Combining, we obtain that for $(\mathcal{R}, \mathbf{C})$ as in Sect. J.2.6 we obtain that (J.5) extends to a *right-lax* monoidal functor of pairs

$$(\boldsymbol{\Delta}^{\mathrm{mock}}_*, \boldsymbol{\Delta}^{\mathrm{mock}}_*) \to (\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X), \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)) \tag{J.9}$$

that sends

$$(* \in \boldsymbol{\Delta}^{\mathrm{mock}}_*) \rightsquigarrow \mathbf{C}.$$

J.3. The renormalization step.

J.3.1. We apply Sect. J.2.8 to

$$\mathcal{R} = \mathfrak{z}_{\mathfrak{g}} \text{ and } \mathbf{C} = \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}},$$

and consider the resulting right-lax monoidal functor of pairs (J.9); denote it

$$(F_{\mathfrak{z}_{\mathfrak{g}}}, F_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}}). \tag{J.10}$$

J.3.2. We now introduce several more (symmetric) monoidal categories. Let

$$\mathfrak{A} = \mathfrak{A}_3 := \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$$

be as above.

We let $\mathfrak{A}'$ be the following 1-full subcategory of $\mathfrak{A}$, to be denoted

$$\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)^{\text{t-str}}.$$

Its objects are lax unital factorization categories, equipped with a t-structure. For a pair of objects $\mathbf{C}_1, \mathbf{C}_2$, we let

$$\mathrm{Maps}_{\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)^{\text{t-str}}}(\mathbf{C}_1, \mathbf{C}_2) \subset \mathrm{Maps}_{\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)}(\mathbf{C}_1, \mathbf{C}_2)$$

be the full subcategory consisting of *left t-exact* functors.

J.3.3. We let $\mathfrak{A}''$ be the full subcategory of the category of arrows in $\mathfrak{A}'$, whose objects are those pairs

$$\Phi : \mathbf{C}^{\mathrm{ren}} \to \mathbf{C},$$

for which:

- Φ induces an equivalence between the eventually coconnective subcategories;
- $\mathbf{C}^{\mathrm{ren}}$ is compactly generated by objects that are eventually coconnective.

Note that the tautological forgetful functor

$$\mathfrak{A}'' \to \mathfrak{A}', \quad (\Phi : \mathbf{C}^{\mathrm{ren}} \to \mathbf{C}) \mapsto \mathbf{C}$$

is 1-fully faithful, i.e., induces a fully faithful functor on spaces of morphisms.

Explicitly, given two pairs $\mathbf{C}^{\mathrm{ren}}_1 \overset{\Phi_1}{\to} \mathbf{C}_1$ and $\mathbf{C}^{\mathrm{ren}}_2 \overset{\Phi_1}{\to} \mathbf{C}_2$, a functor $\Psi : \mathbf{C}_1 \to \mathbf{C}_2$ in $\mathfrak{A}'$ lifts to $\mathfrak{A}''$ if and only if the ind-extension of

$$(\mathbf{C}^{\mathrm{ren}}_1)^c \subset (\mathbf{C}^{\mathrm{ren}}_1)^{>-\infty} \simeq \mathbf{C}_1^{>-\infty} \overset{\Psi}{\to} \mathbf{C}_2^{>-\infty} \simeq (\mathbf{C}^{\mathrm{ren}}_2)^{>-\infty}$$

is left t-exact (equivalently, has a bounded cohomological amplitude on the left).

J.3.4. Finally, we let $\mathfrak{A}'''$ be again $\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$, and we consider the forgetful functor

$$\mathfrak{A}'' \to \mathfrak{A}''', \quad (\Phi : \mathbf{C}^{\mathrm{ren}} \to \mathbf{C}) \mapsto \mathbf{C}^{\mathrm{ren}}. \tag{J.11}$$

J.3.5. Note now that given a lax monoidal functor

$$(\mathbf{\Delta}^{\mathrm{mock}}, \mathbf{\Delta}^{\mathrm{mock}}_*) \to (\mathfrak{A}, \mathfrak{A}),$$

in order to lift to a lax monoidal functor

$$(\mathbf{\Delta}^{\mathrm{mock}}, \mathbf{\Delta}^{\mathrm{mock}}_*) \to (\mathfrak{A}'', \mathfrak{A}''),$$

it suffices to do so at the level of objects and 1-morphisms, i.e., it is sufficient to do so *at the homotopy level* (moreover, the lift at the level of objects defines it completely).

We start with the functor $(F_{\mathfrak{z}_\mathfrak{g}}, F_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}})$ of (J.10). We lift to a functor

$$(F_{\mathfrak{z}_\mathfrak{g}}, F_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}})'' : (\mathbf{\Delta}^{\mathrm{mock}}, \mathbf{\Delta}^{\mathrm{mock}}_*) \to (\mathfrak{A}'', \mathfrak{A}'')$$

at the level of objects by sending

$$(n \in \mathbf{\Delta}^{\mathrm{mock}}) \rightsquigarrow \Big(\mathrm{IndCoh}^*(\text{“}\mathrm{Spec}\text{”}(\mathfrak{Z}_\mathfrak{g}))^{\otimes n} \to \mathfrak{z}_\mathfrak{g}^{\otimes n}\text{-mod}^{\mathrm{fact}}\Big)$$

$$(\{n, *\} \in \mathbf{\Delta}^{\mathrm{mock}}_*) \rightsquigarrow \big(\mathrm{IndCoh}^*(\text{“}\mathrm{Spec}\text{”}(\mathfrak{Z}_\mathfrak{g}))^{\otimes n} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to \mathfrak{z}_\mathfrak{g}^{\otimes n}\text{-mod}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})\big).$$

The existence of the lift at the level of 1-morphisms is guaranteed by Lemma 4.6.12.

J.3.6. We compose $(F_{\mathfrak{z}_\mathfrak{g}}, F_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}})''$ with (J.11) to obtain a right-lax monoidal functor of pairs

$$(F_{\mathfrak{z}_\mathfrak{g}}, F_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}})''' : (\mathbf{\Delta}^{\mathrm{mock}}, \mathbf{\Delta}^{\mathrm{mock}}_*) \to (\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X), \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)).$$

However, by construction, the latter functor is strictly monoidal. I.e., the functor

$$F'''_{\mathfrak{z}_\mathfrak{g}} : \mathbf{\Delta}^{\mathrm{mock}} \to \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$$

is strictly monoidal, and

$$F'''_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}} : \mathbf{\Delta}^{\mathrm{mock}}_* \to \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$$

is a functor between $\mathbf{\Delta}^{\mathrm{mock}}$-module categories.

Applying Sect. J.1 to $(F_{\mathfrak{z}_\mathfrak{g}}, F_{\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}})'''$ we obtain the desired coaction of $\mathrm{IndCoh}^*(\text{“}\mathrm{Spec}\text{”}(\mathfrak{Z}_\mathfrak{g}))$ on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$.

This completes the construction from Sect. 4.6.

J.4. **Adding another monoidal category.** In order to carry out the construction in Sect. 5.3, we need to enhance the setting of Sect. J.2.

J.4.1. We modify the setting of Sect. J.2.1, and we now take $\mathfrak{A}_2$ to be the category, denoted

$$(\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X),$$

whose objects are pairs $(\mathbf{A}, \mathcal{R})$, where:

- $\mathbf{A} \in \mathrm{FactCat}^{\mathrm{untl}}_{\mathrm{lax}}(X)$;
- $\mathcal{R} \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A})$.

The space of morphisms

$$\mathrm{Maps}_{(\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)}((\mathbf{A}_1, \mathcal{R}_1), (\mathbf{A}_2, \mathcal{R}_2))$$

consists of pairs: (Φ, ϕ), where:

- Φ is a lax unital functor $\mathbf{A}_1 \to \mathbf{A}_2$;
- ϕ is a map of unital factorization algebras in $\mathbf{A}_2$

$$\mathcal{R}_2 \to \Phi(\mathcal{R}_1).$$

Remark J.4.2. Note that we can interpret $(\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)$, equipped with the forgetful functor to $\mathrm{FactCat}^{\mathrm{untl}}_{\mathrm{lax}}(X)$ as the co-Cartesian fibration corresponding to the functor

$$\mathrm{FactCat}^{\mathrm{untl}}_{\mathrm{lax}}(X) \to \infty\text{-Cat}, \quad \mathbf{A} \mapsto (\mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A}))^{\mathrm{op}}.$$

The monoidal structure on $(\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)$ corresponds to the right-lax monoidal structure on the above functor.

J.4.3. We let $\mathfrak{A}_3$ be the same as in Sect. J.2.5, i.e., $\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)$.

Note that we have a lax monoidal functor

$$(\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X) \to \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X), \quad (\mathbf{A}, \mathcal{R}) \mapsto \mathcal{R}\text{-mod}^{\mathrm{fact}}(\mathbf{A}). \tag{J.12}$$

J.4.4. We take $\mathfrak{C}_2 = \mathfrak{A}_2$ and $\mathfrak{C}_3 = \mathfrak{A}_3$. The functor (J.12) gives rise to a right-lax monoidal functor of pairs

$$\Big((\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X), (\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)\Big) \to \\ \to (\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X), \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)). \tag{J.13}$$

J.4.5. We now explain a procedure that gives rise to strictly monoidal functors

$$\mathbf{\Delta}^{\mathrm{mock}} \to (\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)$$

and strictly monoidal functors of pairs

$$(\mathbf{\Delta}^{\mathrm{mock}}, \mathbf{\Delta}^{\mathrm{mock}}_*) \to \Big((\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X), (\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)\Big).$$

J.4.6. Let $(\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)$ denote the category defined as follows. It objects are pairs $(\mathbf{A}, \mathcal{R})$, where:

- $\mathbf{A} \in \mathrm{FactCat}^{\mathrm{untl}}_{\mathrm{lax}}(X)$;
- $\mathcal{R} \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A})$.

The space of morphisms

$$\mathrm{Maps}_{(\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)}((\mathbf{A}_1, \mathcal{R}_1), (\mathbf{A}_2, \mathcal{R}_2))$$

consists of pairs: (Φ, ϕ), where:

- Φ is a strictly unital functor $\mathbf{A}_1 \to \mathbf{A}_2$;
- ϕ is a map of unital factorization algebras in $\mathbf{A}_2$

$$\Phi(\mathcal{R}_1) \to \mathcal{R}_2.$$

Remark J.4.7. As in Remark J.4.2, the category $(\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)$, equipped with the forgetful functor to $\mathrm{FactCat}^{\mathrm{untl}}_{\mathrm{lax}}(X)$, is the co-Cartesian fibration corresponding to the functor

$$\mathrm{FactCat}^{\mathrm{untl}}_{\mathrm{lax}}(X) \to \infty\text{-Cat}, \quad \mathbf{A} \mapsto \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A}).$$

J.4.8. Let $\mathbf{A}$ a monoidal unital lax factorization category $\mathbf{A}$ and let $\mathcal{R} \in \mathrm{FactAlg}^{\mathrm{untl}}(X, \mathbf{A})$ be an object, equipped with a structure of *associative algebra*, in the sense of the monoidal structure on $\mathbf{A}$.

We can think of $(\mathbf{A}, \mathcal{R})$ as an associative algebra object in $(\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)$. Hence, by Sect. J.1, it gives rise to a (strictly) monoidal functor

$$F_{\mathbf{A},\mathcal{R}} : \mathbf{\Delta}^{\mathrm{op,mock}} \to (\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X).$$

Let $\mathbf{C}$ be a unital lax factorization category, equipped with an action of $\mathbf{A}$ as a monoidal factorization category. Suppose, moreover, that $\mathcal{R}$ acts on $\mathbf{1}_{\mathbf{A}}$ in the sense of the action of $\mathbf{A}$ on $\mathbf{C}$.

Then we can consider $(\mathbf{C}, \mathbf{1}_{\mathbf{C}})$ as a module over $(\mathbf{A}, \mathcal{R})$ in $(\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)$. Hence, by Sect. J.1, the functor $F_{\mathbf{A},\mathcal{R}}$ extends to a (strictly) monoidal functor of pairs

$$(F_{\mathbf{A},\mathcal{R}}, F_{\mathbf{C}}) : (\mathbf{\Delta}^{\mathrm{op,mock}}, \mathbf{\Delta}^{\mathrm{op,mock}}_*) \to \Big((\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X), (\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)\Big).$$

J.4.9. Let

$$((\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X))_{\mathrm{adj}} \subset (\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)$$

be a 1-full subcategory, where we take the same objects, but as 1-morphisms we let

$$\mathrm{Maps}_{((\mathrm{FactAlg}\in\mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X))_{\mathrm{adj}}}((\mathbf{A}_1, \mathcal{R}_1), (\mathbf{A}_2, \mathcal{R}_2)) \subset$$
$$\subset \mathrm{Maps}_{(\mathrm{FactAlg}\in\mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)}((\mathbf{A}_1, \mathcal{R}_1), (\mathbf{A}_2, \mathcal{R}_2))$$

be the full subcategory consisting of those pairs (Φ, ϕ), for which Φ admits a factorization right adjoint, which then automatically acquires a lax unital structure (see Sect. C.11.21).

J.4.10. Note that the operation of passage to the right adjoint functor defines a (1-fully faithful, symmetric) monoidal functor

$$((\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X))_{\mathrm{adj}} \to ((\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X))^{\mathrm{op}}, \tag{J.14}$$

which acts as identity on objects, and sends

$$(\Phi : \mathbf{A}_1 \to \mathbf{A}_2, \phi(\mathcal{R}_1) \to \mathcal{R}_2) \rightsquigarrow (\Phi^R : \mathbf{A}_2 \to \mathbf{A}_1, \psi : \mathcal{R}_1 \to \Phi^R(\mathcal{R}_2)),$$

where ψ is obtained from ϕ by adjunction.

J.4.11. Therefore, given a (strictly) monoidal functor of pairs

$$(\mathbf{\Delta}^{\mathrm{op,mock}}, \mathbf{\Delta}_*^{\mathrm{op,mock}}) \to \Big((\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X), (\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)\Big),$$

which at the level of 1-morphisms lands in

$$\Big(((\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X))_{\mathrm{adj}}, ((\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X))_{\mathrm{adj}}\Big), \tag{J.15}$$

by passing to right adjoints, we can create from it a (strictly) monoidal functor of pairs

$$(\mathbf{\Delta}^{\mathrm{mock}}, \mathbf{\Delta}_*^{\mathrm{mock}}) \to \Big((\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X), (\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)\Big).$$

J.5. **Applying the paradigm.** We are finally ready to complete the construction from Sect. 5.3.

J.5.1. In the context of Sect. J.4.8, we take

$$\mathbf{A} := \mathrm{Rep}(\check{G}), \ \mathbf{C} := \mathrm{KL}(G)_{\mathrm{crit}}, \ \mathcal{R} = R_{\check{G},\mathrm{Op}}.$$

The data of action from Sect. J.4.8 is provided by Sect. 5.2.9.

Denote the resulting monoidal functor of pairs by

$$(F_{\mathrm{Rep}(\check{G}),R_{\check{G},\mathrm{Op}}}, F_{\mathrm{KL}(G)_{\mathrm{crit}}}) : (\mathbf{\Delta}^{\mathrm{op,mock}}, \mathbf{\Delta}_*^{\mathrm{op,mock}}) \to$$
$$\to \Big((\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X), (\mathrm{FactAlg} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)\Big).$$

J.5.2. Note now that since $\mathrm{Rep}(\check{G})$ is rigid, the above functor $(F_{\mathrm{Rep}(\check{G}),R_{\check{G},\mathrm{Op}}}, F_{\mathrm{KL}(G)_{\mathrm{crit}}})$ lands in the subcategory (J.15).

Hence, by Sect. J.4.11, we can produce from it a (strictly) monoidal functor of pairs

$$(F_{\mathrm{Rep}(\check{G}),R_{\check{G},\mathrm{Op}}}, F_{\mathrm{KL}(G)_{\mathrm{crit}}})_{\mathrm{adj}} : (\mathbf{\Delta}^{\mathrm{mock}}, \mathbf{\Delta}_*^{\mathrm{mock}}) \to$$
$$\to \Big((\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X), (\mathrm{FactAlg}^{\mathrm{op}} \in \mathrm{FactCat}_{\mathrm{lax}})^{\mathrm{untl}}(X)\Big).$$

J.5.3. Composing with (J.13), we obtain a right-lax monoidal functor of pairs

$$(F_{\mathrm{Rep}(\check{G}),R_{\check{G},\mathrm{Op}}}, F_{\mathrm{KL}(G)_{\mathrm{crit}}})^{\sim}_{\mathrm{adj}} : (\mathbf{\Delta}^{\mathrm{mock}}, \mathbf{\Delta}_*^{\mathrm{mock}}) \to (\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X), \mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)).$$

J.5.4. The final step consists of lifting the functor $(F_{\mathrm{Rep}(\check{G}),R_{\check{G},\mathrm{Op}}},F_{\mathrm{KL}(G)_{\mathrm{crit}}})^{\sim}_{\mathrm{adj}}$ to a (strictly) monoidal functor of pairs

$$(F_{\mathrm{Rep}(\check{G}),R_{\check{G},\mathrm{Op}}},F_{\mathrm{KL}(G)_{\mathrm{crit}}})'_{\mathrm{adj}}:(\mathbf{\Delta}^{\mathrm{mock}},\mathbf{\Delta}^{\mathrm{mock}}_{*})\to(\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X),\mathrm{FactCat}^{\mathrm{untl,lax}}_{\mathrm{lax}}(X)),$$

which at the level of objects sends

$$n\mapsto\mathrm{IndCoh}^{*}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})^{\otimes n},\quad (n\sqcup *)\mapsto\mathrm{IndCoh}^{*}(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}})^{\otimes n}\otimes\mathrm{KL}(G)_{\mathrm{crit}}.$$

This is achieved by repeating the procedure in Sect. J.3.6 using Lemma 5.3.9.

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