"Non, c'est normal: les dénonciateurs dénoncent, les cambrioleurs cambriolent, les assassins assassinent, les amoureux s'aiment." J.-L. Godard, À bout de souffle (Michel à Patricia).

THE GEOMETRIC LANGLANDS FUNCTOR II: EQUIVALENCE ON THE EISENSTEIN PART

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Abstract. We prove the de Rham geometric Langlands conjecture for reducible spectral parameters.

The problem reduces to calculating constant terms of geometric Eisenstein series in spectral terms, or equivalently, to proving the compatibility of the geometric Langlands functor $\mathbb{L}_G$ with geometric and spectral constant term functors. Essentially because geometric Langlands has previously been understood for irreducible local systems in the case of $G = GL_n$, we are able to deduce the full geometric Langlands conjecture in this case.

We perform this calculation using Kac-Moody localization at the critical level. Namely, the interaction of Kac-Moody localization with the Langlands functor has been well-understood. One of the main results of this paper describes the interaction of Kac-Moody localization with constant term functors. We then deduce the general compatibility of the Langlands functor with constant terms using this calculation.

Our analysis goes by reduction to a local problem, namely, calculating BRST functors on the critical level Kazhdan-Lusztig category via a sort of Miura transform on the Langlands dual side. For our applications, it is important to work at the level of factorization categories. These purely local results may be of independent interest.

Finally, a substantial part of this paper develops foundational local-to-global methods related to chiral homology; these results have been folklore in the subject for some time.

First draft

Contents

Introduction 5
0.1. What is done in this paper? 5
0.2. The logical structure: compatibility with constant terms 6
0.3. The logical structure: equivalence on Eisenstein parts 7
0.4. The business of enhancement 9
0.5. Description of the actual contents 11
0.6. Conventions and notation: generalities 13
0.7. Conventions and notation: factorization 13
0.8. Conventions and notation: Ind-coherent sheaves 15
0.9. Acknowledgements 17

Part I. Local Theory 19
1. Geometric Satake and Casselman-Shalika formula: recollections 19
1.1. The critical twist 20
1.2. A geometric twisting construction 21
1.3. The Whittaker category on the affine Grassmannian 22
1.4. The geometric Casselman-Shalika formula 23
1.5. The spherical category 24
1.6. The spectral spherical category 25
1.7. Geometric Satake equivalence 27
1.8. The curse of σ and τ 28
2. The local semi-infinite category: recollections 29

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2.1. The corrected Jacquet functor 29
2.2. The local semi-infinite category 30
2.3. A twisting procedure 31
2.4. The spectral semi-infinite category 32
2.5. The (factorization) algebra Ω^{spec} 33
2.6. Semi-infinite geometric Satake 34
2.7. The Jacquet functor 37
3. The local semi-infinite category: dualities 38
3.1. The dual of the semi-infinite category 39
3.2. The duality on the geometric side 40
3.3. The algebra Ω 41
3.4. The algebra Ω and duality 43
4. The Kazhdan-Lusztig category 44
4.1. Defintion and basic properties 44
4.2. Critical level and Feigin-Frenkel center 44
4.3. Duality 45
4.4. Twisting by $Z^0_{\check{G}}$-torsors 45
4.5. BRST and Wakimoto functors 47
4.6. Twisting by Z_G-torsors 48
4.7. The twisted BRST and Wakimoto functors 49
4.8. The enhanced functor of BRST reduction 51
4.9. The functor of Drinfeld-Sokolov reduction 52
5. Monodromy-free opers 53
5.1. IndCoh^* of monodromy-free opers 53
5.2. IndCoh^* of monodromy-free opers as factorization modules 54
5.3. Self-duality for opers 55
5.4. (Parabolic) Miura opers 56
5.5. The spectral Jacquet functor 58
5.6. The semi-infinite spectral Jacquet functor 58
5.7. The enhanced spectral Jacquet functor 60
6. A digression: factorization modules categories over $\mathrm{Rep}(\check{G})$ 61
6.1. Creating factorization modules categories 62
6.2. Factorization modules categories attached to schemes 62
6.3. Spherical vs. Whittaker 64
6.4. Spherical vs. Whittaker, continued 65
7. The critical FLE 67
7.1. The enhanced functor of Drinfeld-Sokolov reduction 67
7.2. Compatibility with the factorization module structure 68
7.3. The critical FLE 70
7.4. Coarsened versions of the FLE functor 72
7.5. Compatibility of $\mathrm{FLE}_{G,\mathrm{crit}}$ and $\mathrm{FLE}_{\check{G},\infty}$ 73
7.6. Compatibility with duality 74
7.7. Twisted version 75
8. Proof of Theorem 7.6.4 76
8.1. Recollections on the Feigin-Frenkel center 76
8.2. Feigin-Frenkel center and self-duality 77
8.3. Self-duality on opers via Kac-Moody 77
8.4. Proof of Theorem 7.6.4 79

Part II. The FLE and the Jacquet functors 82
9. Compatibility of the FLE with the Jacquet functors 82
9.1. An initial formulation 82
9.2. A dual formulation of Theorem 9.1.3 83

9.3. The dual of the left vertical arrow 84
9.4. The dual of the right vertical arrow 85
9.5. A dual formulation of Theorem 9.1.7 88
10. Proof of Theorem 9.5.3 89
10.1. Strategy of proof 89
10.2. The factorization algebra $\Omega(R_{\check{G}})^{\mathrm{spec}}$ 90
10.3. Functors to and from $\mathbf{C}$ on the spectral side 92
10.4. Functor to $\mathbf{C}$ on the geometric side 94
10.5. Functor to $\mathbf{C}$ on the geometric side, continued 95
10.6. Functor *from* $\mathbf{C}$ on the geometric side 98
10.7. A sketch of proof that $\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ is a pointwise equivalence 100
11. Engaging the Pentagon 101
11.1. Comparison of the *un*enhanced functors 101
11.2. Comparison of the further *un*enhancements 103
11.3. Identification of the $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$-action 104
11.4. Upgrading to IndCoh^* 106
11.5. Identification of the map of factorization algebras 108

Part III. Local-to-global constructions 112
12. The coefficient and Poincaré functors 112
12.1. Twisted D-modules on Bun_G 113
12.2. Restricting to (twists of) Bun_N 114
12.3. The coefficient functor 116
12.4. Poincaré functor(s) 118
12.5. Coefficient and Poincaré functors: local version: global interpretation 119
13. The localization functor 121
13.1. Basics of localization 121
13.2. Localization in the twisted setting 122
13.3. A digression: unitality 124
13.4. A digression: factorization homology 126
13.5. Localization and the forgetful functor 129
13.6. Localization and restriction 130
13.7. Localization for unipotent group-schemes 134
13.8. Integrating over (twists of) Bun_N 135
14. The composition of localization and coefficient functors 138
14.1. The vacuum case 138
14.2. Composition of coefficient and localization functors: the general case 139
14.3. Composition of coefficient and localization functors: the twisted case 141
15. Localization and constant term functors 142
15.1. The (untwisted) functor of constant term 143
15.2. Constant term functor on gerbe-twisted D-modules 143
15.3. Constant term and localization: a general level 145
15.4. Constant term and localization: the half-twisted case 147
16. Localization and constant term functors: the enhanced version 148
16.1. The enhanced recipient category 148
16.2. The enhanced constant term functor 150
16.3. The enhanced localization functor 151
16.4. Compatibility of localization with constant terms–enhanced version 152
16.5. Localization in the presence of level structure 155
16.6. Enhanced categories–twisted versions 156
17. Spectral Poincaré and global section functors 158
17.1. The spectral Poincaré functor 158

17.2. Another version of the spectral Poincaré functor 160
17.3. The spectral localization and global sections functors 162
17.4. Composing spectral Poincaré and global sections functors 163
17.5. The twisted case 165
18. Spectral Poincaré and constant terms functors 166
18.1. The spectral constant term functor 166
18.2. The spectral Poincaré vs constant term compatibility 166
18.3. The clockwise circuit 167
18.4. Morphing into the anti-clockwise circuit 168
18.5. Proof of Lemma 18.3.5 170
19. The enhanced spectral constant term functor 171
19.1. The enhanced recipient category on the spectral side 172
19.2. Partial enhancement 173
19.3. The enhanced spectral constant term functor 175
19.4. The enhanced spectral Poincaré series functor 176
19.5. The enhanced spectral Poincaré vs constant term compatibility 177

Part IV. Applications to the Langlands functor 178
20. The Langlands functor 178
20.1. Recollections on the Langlands functor–the coarse version 178
20.2. The case $G = T$ 180
20.3. The actual Langlands functor 181
20.4. The Langlands functor and Eisenstein series 182
20.5. The commutativity of (20.18) 184
20.6. Compatibility of the the Langlands functor with critical localization 186
21. Compatibility of the Langlands functor with constant terms 188
21.1. The cube 188
21.2. The statement of the (unenhanced) compatibility theorem 191
21.3. Proof of Theorem 21.2.2 for $P = B$ 192
21.4. Proof of Proposition 21.3.7 195
21.5. Proof of Theorem 21.2.2 for a general Levi 198
22. Compatibility of the Langlands functor with constant terms: enhanced version 202
22.1. Some enhanced functors 202
22.2. Enhanced version of the cube 203
22.3. Proof of Theorem 22.2.4 204
23. The left adjoint of the Langlands functor 206
23.1. The existence of the left adjoint 206
23.2. The left adjoint as a dual 207
23.3. Proof of Theorem 23.2.5 208
23.4. Proof of Lemma 23.3.9 210
23.5. Proof of Lemma 23.3.10 210
23.6. The composition $\mathbb{L}_G \circ (\mathbb{L}_G)^L$ 211
23.7. Proof of Theorem 23.6.2 212
23.8. Proof of Proposition 23.7.6 214

Part V. The Langlands functor is an equivalence on Eisenstein subcategories 216
24. Statement of the equivalence 216
24.1. Statement of the result 216
24.2. Proof of Conjecture 20.3.8 for $G = GL_n$ 217
24.3. Proof of Theorem 24.1.2: initial observations 219
24.4. Proof of Main Lemma 24.3.7 220
24.5. The partially enhanced geometric constant term functor 221
24.6. The unenhanced version of Theorem 24.5.7 223

24.7. Proof of Proposition 24.6.5 225
25. A digression: enhanced Eisenstein series functors 226
25.1. Enhanced Eisenstein series functors on the geometric side 226
25.2. Miraculous functor and enhanced Eisenstein series 229
25.3. Enhanced spectral Eisenstein series 231
25.4. What is missing for a direct proof of (24.14)? 232
25.5. Proof of Theorem 25.3.5 234
26. Proof of Theorem 24.4.2 236
26.1. Partially enhanced Eisenstein functors 236
26.2. Partial enhancement and the miraculous functor 237
26.3. Proof of Theorem 24.4.2 240
Appendix A. IndCoh^* and $\mathrm{IndCoh}^!$ 242
References 244

Introduction

0.1. **What is done in this paper?** This paper is the second in the series of four, in the course of which a proof of the geometric Langlands conjecture (stated here as Conjecture 20.3.8) will be given.

0.1.1. As far as the program of proving the geometric Langlands conjecture is concerned, in this paper the following two steps toward the proof are performed:

- It is shown (Theorem 21.2.2) that the geometric Langlands functor

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \tag{0.1}$$

  is compatible with the functors of *constant term*;

- Assuming the geometric Langlands conjecture for proper Levi subgroups, it is shown (Theorem 24.1.2) that $\mathbb{L}_G$ induces an equivalence on Eisenstein-generated subcategories

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}} \xrightarrow{\sim} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}.$$

In the special case when $G = GL_n$, it turns out that Theorem 24.1.2 already implies the full geometric Langlands conjecture (see Sect. 24.2).

Another result that concerns the global geometric Langlands program, proved here, and which is of independent interest is Theorem 23.2.5, which says that:

- The left adjoint functor of $\mathbb{L}_G$ can be obtained from the functor *dual* to $\mathbb{L}_G$, by composing with the *Miraculous functor* and Cartan involution.

0.1.2. Other results established in this paper concern the *local geometric Langlands theory*. Apart from being of independent interest, these results provide local ingredients for the proofs of global theorems mentioned above.

The two main local results are:

- The critical FLE (Theorem 7.3.4), i.e., an equivalence

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times));$$

- The compatibility of the critical FLE with Jacquet functors (Theorem 9.1.3 and its enhancement Theorem 9.1.7).

We should remark that both of the above local results are new only *at the factorization level*, i.e., when view both sides as categories over the Ran space. Namely, the poinwtise version of Theorem 7.3.4 had been established in [FG4], and the pointwise version of Theorem 9.1.3 had been (essentially) established in [FG2]. However, the proofs of both these results given in *loc.cit.* use methods that do not extend to a statement at the factorization level.

0.2. **The logical structure: compatibility with constant terms.** We will now describe the logical structure of the paper from the point of view of the geometric Langland conjecture.

0.2.1. In Sects. 20.1 and 20.3, we recall, referring to [GR1], the construction of the Langlands functor (0.1).

By design, the functor $\mathbb{L}_G$ makes the following diagram commute[1]

$$\begin{array}{ccc} \mathrm{Whit}^!(\mathrm{Gr}_{G,\mathrm{Ran}}) & \xrightarrow{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{coeff}_G}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{0.2}$$

Here:

- CS_G is the equivalence of Theorem 1.4.2, which we call the "geometric Casselman-Shalika formula";
- coeff_G is the functor of Whittaker coefficient(s);
- $\Gamma^{\mathrm{spec}}_{\check{G}}$ is the functor right adjoint to the localization functor

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \hookrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

0.2.2. The geometric Langlands conjecture (Conjecture 20.3.8) says that the functor $\mathbb{L}_G$ is an equivalence.

0.2.3. In the process of showing that $\mathbb{L}_G$ is well-defined, one proves that it is compatible with the *Eisenstein functors*, i.e., for a standard (negative) parabolic P^- with Levi quotient M, it makes the following diagram commute (again, up to a cohomological shift):

$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{0.3}$$

In the above formula, $\mathrm{Eis}^-_{!,\rho_P(\omega_X)}$ is the *translated* Eisenstein series functor, see Sect. 20.4.2.

0.2.4. Given diagram (0.3), by passing to right adjoint functors along the vertical arrows, we obtain a diagram

$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)), \\ {\scriptstyle \mathrm{CT}^-_{*,\rho_P(\omega_X)}}\uparrow & \searrow & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}_G]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{0.4}$$

that commutes *up to a natural transformation*.

However, it is entirely not obvious that the natural transformation in (0.4) is an isomorphism. Ultimately, we establish that it is an isomorphism (Corollary 24.1.4), but this comes after we prove our main result, Theorem 24.1.2.

[1] Up to a cohomological shift, which we omit in the Introduction.

0.2.5. First, prove *a priori* that there exists *some* natural transformation that makes the diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{CT}^-_{*,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \end{array} \tag{0.5}$$

commute.[2]

The existence of the commutative diagram (0.5) is one of the main results of this paper (Theorem 21.2.2), and it uses *local-to-global* methods.

0.2.6. In Part III of this paper we review the critical localization construction, which is a functor

$$\mathrm{Loc}_G : \mathrm{KL}(G)_{\mathrm{crit,Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G), \tag{0.6}$$

where $\mathrm{KL}(G)_{\mathrm{crit,Ran}}$ is the Kazhdan-Lusztig category at the critical level.

The spectral counterpart of (0.6) is the functor of *spectral Poincaré series*

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \tag{0.7}$$

In Theorem 20.6.2 we prove that the Langlands functor is compatible with the above local-to-global functors, i.e., that the diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\ {\scriptstyle \mathrm{Loc}_G}\uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{KL}(G)_{\mathrm{crit,Ran}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \end{array} \tag{0.8}$$

commutes,[3] where the bottom horizontal arrow is the *critical FLE equivalence*, given by Theorem 7.3.4.

0.2.7. We prove the existence of (0.5) by constructing a commutative cube (see diagram (21.2)) that relates the diagram (0.8) for G with a similar diagram for M, with the crucial ingredient being the compatibility of the critical FLE with Jacquet functors, given by Theorem 9.1.3 mentioned above.

0.2.8. Of course, the compatibility of $\mathbb{L}_G$ with the critical localization functor, expressed by diagram (0.8) plays a much bigger role in this project than just proving the existence of (0.5).

In the next paper, it will be used to show that the functor $\mathbb{L}_G$ is *ambidextrous* (at least on the cuspidal part), which is another crucial step towards the proof of the geometric Langlands conjecture.

Remark 0.2.9. One can say that our approach to the proof of the geometric Langlands conjecture consists of playing diagrams (0.2) and (0.8) one against the other.

Note that the approach to geometric Langlands via (0.2) was essentially the idea behind Drinfeld's founding work [Dri] (later taken up by [FGV]), and the approach via (0.8) was the idea of the Beilinson-Drinfeld approach in [BD].

Remark 0.2.10. In [Gai1, Sect. 6.7], the second author suggested a different approach to proving the compatibility between the Langlands functor and constant terms. The approach in the present paper differs substantially from the strategy outlined there, relying on statements about the Kac-Moody algebra rather than the more geometric tools suggested in [Gai1].

With that said, completing the older strategy of deducing the constant term compatibility of geometric Langlands is the subject of work-in-progress by the first author and K. Lin.

0.3. **The logical structure: equivalence on Eisenstein parts.**

[2] We do not know, and are not sure that it is true, that the natural isomorphism in (0.5) equals one of (0.4). One can show, however, that the two differ by a (non-zero) scalar.

[3] Up to a graded line, omitted in the Introduction.

0.3.1. In Theorem 23.1.2, we deduce from the diagram (0.5) that the functor $\mathbb{L}_G$ admits a left adjoint, which we denote $\mathbb{L}_G^L$. Moreover, the functor $\mathbb{L}_G^L$ makes the diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xleftarrow{\mathbb{L}_M^L} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{0.9}$$

commute.

Having both diagrams (0.3) and (0.9) implies that the functors $\mathbb{L}_G$ and $\mathbb{L}_G^L$ send the subcategories

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

and

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$$

to one another.

The main result of this paper, Theorem 24.1.2, says that the resulting adjoint functors

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{Eis}} \leftrightarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} : \mathbb{L}_G^L \tag{0.10}$$

are mutually inverse equivalences, provided that we know that the geometric Langlands conjecture holds for all proper Levi subgroups of G.

We will now explain the logic of the proof of this theorem.

0.3.2. Consider the composition

$$\mathbb{L}_G \circ \mathbb{L}_G^L,$$

viewed as a monad acting on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$.

We show (Theorem 23.6.2) that this monad is given by the action of an associative algebra object

$$\mathcal{A}_G \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$$

(we view $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ as a module category over $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$).

0.3.3. The assertion that the functor $\mathbb{L}_G^L$ is fully faithful is equivalent to the assertion that the unit

$$\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)} \to \mathcal{A}_G \tag{0.11}$$

is an isomorphism. By [FR1], we already know that $\mathbb{L}_G$ is conservative, so the fully-faithfulness of $\mathbb{L}_G^L$ is equivalent to the geometric Langlands conjecture Conjecture 20.3.8.

The assertion that $\mathbb{L}_G^L|_{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}}$ is fully faithful is equivalent to the assertion that the map

$$\mathcal{O}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)} \to \mathcal{A}_G|_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)}, \tag{0.12}$$

induced by (0.11), is an isomorphism, where $\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X) \subset \mathrm{LS}_{\check{G}}(X)$ is *any* closed substack, whose underlying subset consists of reducible local systems.

0.3.4. The latter assertion is equivalent to the map

$$\mathcal{O}_{\mathrm{LS}_{\check{P}^-}(X)} \to (\mathsf{p}^{\mathrm{glob}})^*(\mathcal{A}_G) \tag{0.13}$$

being an isomorphism for any *proper* standard (negative) parabolic $\check{P}^- \subset \check{G}$, where

$$\mathsf{p}^{\mathrm{glob}} : \mathrm{LS}_{\check{P}^-}(X) \to \mathrm{LS}_{\check{G}}(X)$$

is the canonical morphism.

Now, a simple but crucial observation is given by Proposition 24.3.8, which says that in order to prove that (0.13) is an isomorphism, it is enough to show that the object

$$(\mathsf{p}^{\mathrm{glob}})^*(\mathcal{A}_G) \in \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(X))$$

is a *line bundle*.

We will actually prove that

$$(\mathsf{p}^{\mathrm{glob}})^!(\mathcal{A}_G) \simeq (\mathsf{p}^{\mathrm{glob}})^!(\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}). \tag{0.14}$$

This will imply that $(\mathsf{p}^{\mathrm{glob}})^*(\mathcal{A}_G)$ is a line bundle, since the both stacks $\mathrm{LS}_{\check{G}}(X)$ and $\mathrm{LS}_{\check{P}^-}(X)$ are quasi-smooth.

0.3.5. Note that

$$\mathcal{A}_G \simeq \mathbb{L}_G \circ \mathbb{L}^L_G(\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}).$$

Recall also that the spectral constant term functor

$$\mathrm{CT}^{-,\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)),$$

is given by

$$(\mathsf{q}^{\mathrm{glob}})_* \circ (\mathsf{p}^{\mathrm{glob}})^!,$$

where

$$\mathsf{q}^{\mathrm{glob}} : \mathrm{LS}_{\check{P}^-}(X) \to \mathrm{LS}_{\check{M}}(X)$$

is the canonical morphism.

We will show in Theorem 24.6.2 that there exists a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xleftarrow{\mathbb{L}^L_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{CT}^-_{*,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}^L_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{0.15}$$

Combined with the commutative diagram (0.5) (and assuming the Langlands conjecture for M), this implies that there *exists an* isomorphism

$$(\mathsf{q}^{\mathrm{glob}})_*((\mathsf{p}^{\mathrm{glob}})^!(\mathcal{A}_G)) \simeq (\mathsf{q}^{\mathrm{glob}})_*((\mathsf{p}^{\mathrm{glob}})^!(\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)})). \tag{0.16}$$

However, this is not enough to prove the existence of an isomorphism (0.14) itself.

0.3.6. Thus, in order to construct (0.14), we will need to *enhance* both (0.5) and (0.15), so that on the spectral side they involve the category $\mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X))$ rather than $\mathrm{IndCoh}(\mathrm{LS}_{\check{M}}(X))$.

0.4. **The business of enhancement.**

0.4.1. The enhancement mentioned in Sect. 0.14 follows ideas initiated in [BG] and [**?**]. For us, it is constructed using the local *semi-infinite* geometric and spectral categories

$$\mathrm{I}(G, P^-)^{\mathrm{loc}} \text{ and } \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}},$$

introduced in Sect. 2.

By tensoring these categories over the spherical categories

$$\mathrm{Sph}_M \text{ and } \mathrm{Sph}_{\check{M}}^{\mathrm{spec}},$$

respectively, we obtain what we call enhancements of the corresponding local and global categories:

- $\mathrm{KL}(M)_{\mathrm{crit}} \rightsquigarrow \mathrm{KL}(M)_{\mathrm{crit}}^{-,\mathrm{enh}}$;
- $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \rightsquigarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}$;
- $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \rightsquigarrow \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))^{-,\mathrm{enh}}$;
- $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightsquigarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$.

The global constant term and local Jacquet functors all admit enhancements to functors with values in the corresponding enhanced categories.

0.4.2. An enhancement of Theorem 21.2.2, given by Theorem 22.2.4, says that the commutative diagram (0.5) can be enhanced to

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xrightarrow{\mathbb{L}_M^{-,\mathrm{enh}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\ {\scriptstyle \mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec,enh}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \end{array} \tag{0.17}$$

0.4.3. In an ideal world, we would say that diagram (0.15) also admits an enhanced version

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xleftarrow{(\mathbb{L}_M^L)^{-,\mathrm{enh}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\ {\scriptstyle \mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec,enh}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}_G^L} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{0.18}$$

For technical reasons (see Sect. 25.4), instead of (0.15) we could only produce its *partially enhanced* version

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xleftarrow{(\mathbb{L}_M^L)^{-,\mathrm{part.enh}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\ {\scriptstyle \mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec,part.enh}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}_G^L} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{0.19}$$

0.4.4. Now, combined with the partially enhanced version of (0.17), i.e.,

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{\mathbb{L}_M^{-,\mathrm{part.enh}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\ {\scriptstyle \mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec,part.enh}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)), \end{array}$$

we obtain an isomorphism of functors

$$\mathrm{CT}^{-,\mathrm{spec,part.enh}} \circ (\mathbb{L}_G \circ \mathbb{L}_G^L) \simeq (\mathbb{L}_M^{-,\mathrm{part.enh}} \circ (\mathbb{L}_M^L)^{-,\mathrm{part.enh}}) \circ \mathrm{CT}^{-,\mathrm{spec,part.enh}}. \tag{0.20}$$

0.4.5. Assuming the geometric Langlands conjecture for M, we know that

$$\mathbb{L}_M \circ \mathbb{L}_M^L \simeq \mathrm{Id},$$

which formally implies that

$$\mathbb{L}_M^{-,\mathrm{part.enh}} \circ (\mathbb{L}_M^L)^{-,\mathrm{part.enh}} \simeq \mathrm{Id}\,.$$

That is, (0.20) implies

$$\mathrm{CT}^{-,\mathrm{spec},\mathrm{part.enh}} \circ (\mathbb{L}_G \circ \mathbb{L}_G^L) \simeq \mathrm{CT}^{-,\mathrm{spec},\mathrm{part.enh}}\,. \tag{0.21}$$

0.4.6. However, the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}$ is exactly rigged so that it identifies with a full subcategory of

$$\mathrm{IndCoh}_{\check{M}-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \subset \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X)) \tag{0.22}$$

(see Proposition 19.2.3), and under this equivalence, the functor

$$\mathrm{CT}^{-,\mathrm{spec},\mathrm{part.enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}$$

corresponds to

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \overset{(\mathsf{p}^{\mathrm{glob}})^!}{\longrightarrow} \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X)) \to \mathrm{IndCoh}_{\check{M}-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)),$$

where the second arrow is the right adjoint to (0.22).

This implies the existence of an isomorphism of functors

$$(\mathsf{p}^{\mathrm{glob}})^! \circ (\mathbb{L}_G \circ \mathbb{L}_G^L)|_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))} \simeq (\mathsf{p}^{\mathrm{glob}})^!|_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))}.$$

As a special case, we obtain the formula (0.14).

0.5. **Description of the actual contents.** This paper is subdivided into five parts. We will now briefly outline the contents of each.

0.5.1. In Part I we review the local theory, which constitutes an ingredient for local-to-global constructions.

In Sect. 1 we review various categories on the geometric side associated with the affine Grassmannian of G, as well as their spectral counterparts. The main results here are the geometric Casselman-Shalika formula (Theorem 1.4.2) and (derived) geometric Satake equivalence (Theorem 1.7.2).

In Sect. 2 we review the geometric and spectral semi-infinite categories. The main result here is the the *semi-infinite geometric Satake* (Theorem 2.6.7), which establishes an equivalence between the two.

In Sect. 3 we discuss the self-duality on the geometric semi-infinite category (Theorem 3.2.2). We also introduce the (factorization, associative) algebras Ω^{spec} and Ω, which will later be used for the construction of *partial enhancements*.

In Sect. 4 we discuss the Kazhdan-Lusztig category at the critical level. We also introduce local operations associated with it, such as BRST and Drinfeld-Sokolov functors.

In Sect. 5 we introduce the space of local monodromy-free opers, and study operations associated with the category of ind-coherent sheaves on it, such as the Jacquet functor.

In Sect. 6 we make preparations for the construction of the critical FLE equivalence, by studying factorization module categories over $\mathrm{Rep}(\check{G})$. The main result is Proposition 6.4.4, which relates the spherical and Whittaker categories of a given category, equipped with an action of the loop group $\mathfrak{L}(G)$.

In Sect. 7 we prove the main result of this Part: the critical FLE, Theorem 7.3.4.

In Sect. 8 we give a proof of Theorem 7.6.4, which says that the critical FLE is compatible with the natural self-dualities of the two sides.

0.5.2. In Part II we formulate and prove the compatibility of the critical FLE with the BRST and Jacquet functors (Theorem 9.1.3), as well as its enhancement, Theorem 9.1.7.

In Sect. 9 we formulate Theorems 9.1.3 and 9.1.7, and then reformulate them in dual terms, as Theorems 9.2.4 and 9.5.3, respectively.

In Sect. 10 we reduce Theorem 9.5.3 to the construction of diagram (10.2). We construct the 1-skeleton of this diagram, and check the commutativity of the three triangles.

In Sect. 11 we prove the commutativity of the pentagon in diagram (10.2).

0.5.3. In Part III we review various local-to-global constructions.

In Sect. 12 we study the Whittaker coefficient and Poincaré series functors, which connect $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ with the Whittaker category.

In Sect. 13 we study the localization functor, which connects $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ to $\text{KL}(G)_{\text{crit}}$.

In Sect. 14 we express the composition

$$\text{KL}(G)_{\text{crit}} \overset{\text{Loc}_G}{\to} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \overset{\text{coeff}_G}{\longrightarrow} \text{Whit}^!_{G,\text{Ran}}$$

in terms of factorization homology.

In Sect. 15 we give an expression to the composition of the localization and constant term functors in terms of BRST and the localization functor for the Levi subgroup.

In Sect. 16 we introduce the global enhanced category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}}$ and generalize the results of the previous section to the enhanced setting.

In Sect. 17 we introduce the spectral Poincaré series functor, which connects the local category $\text{IndCoh}^*(\text{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))$ with $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))$. We also study the interaction of this functor with the spectral localization functor

$$\text{Loc}^{\text{spec}}_{\check{G}} : \text{Rep}(\check{G})_{\text{Ran}} \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)).$$

In Sect. 18 we give an expression to the composition of the spectral Poincaré and constant term functors in terms of spectral Jacquet and spectral Poincaré functors for the Levi subgroup.

In Sect. 19 we introduce the enhancement $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{M}}(X))^{-,\text{enh}}$, and generalize the results of the previous section to the enhanced setting.

0.5.4. In Part IV we combine the results of Parts I-III to deduce consequences for the Langlands functor.

In Sect. 20 we recall (following [GR1]) the construction of the Langlands functor, along with the commutativity of (0.2) and (0.3). We establish the compatibility of $\mathbb{L}_G$ with the localization functor, expressed by diagram (0.8).

In Sect. 21 we prove Theorem 21.2.2, which expresses the compatibility of the Langlands functor with constant term functors.

In Sect. 22 we prove Theorem 22.2.4, which is the enhanced version of Theorem 21.2.2.

In Sect. 23 we prove that the functor $\mathbb{L}_G$ admits a left adjoint, which makes the diagram (0.9) commute (Theorem 23.1.2). We relate this left adjoint to the functor *dual* to $\mathbb{L}_G$ (Theorem 23.2.5). We show that the composition $\mathbb{L}_G \circ \mathbb{L}^L_G$ is given be tensor product by an associative algebra object $\mathcal{A}_G \in \text{QCoh}(\text{LS}_{\check{G}}(X))$ (Theorem 23.6.2).

0.5.5. In Part V we prove the main result of this paper, Theorem 24.1.2.

In Sect. 24 we state Theorem 24.1.2, which is the Eisenstein part of the geometric Langlands conjecture. We show that it implies the geometric Langlands conjecture when $G = GL_n$. We reduce Theorem 24.1.2 to Theorem 24.5.7, which says that diagram (0.19) commutes.

In Sect. 25 we prepare for the proof of Theorem 24.5.7: we introduce enhanced Eisenstein series functors, and show that they are also compatible with $\mathbb{L}_G$.

In Sect. 26 we prove Theorem 24.5.7.

0.6. **Conventions and notation: generalities.**

0.6.1. *The players.* Throughout the paper we work over a fixed algebraically closed field k of characteristic 0. Thus, all algebro-geometric objects are defined over k.

In particular, X is a smooth projective curve over k, G is a reductive group over k, and $\check{G}$ is the Langlands dual of G.

0.6.2. *Categories.* When we say "category", we mean a DG category over k (as defined, e.g., in [GR2, Chapter 1, Sect. 1.10]. Unless explicitly stated otherwise, a DG category $\mathbf{C}$ is assumed cocomplete (i.e., to contain arbitrary direct sums). (An exception would be, e.g., the category of compact objects in a given $\mathbf{C}$, denoted $\mathbf{C}^c$.)

Given a pair of DG categories $\mathbf{C}_1$ and $\mathbf{C}_2$, by a functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ we will always understand a *continuous* functor, i.e., one that commutes with arbitrary direct sums (equivalently, colimits).

Conventions adopted in this paper regarding higher algebra and derived algebraic geometry follow closely those of [AGKRRV].

0.6.3. *Adjunctions and monads.* Let

$$F : \mathbf{D} \to \mathbf{C}$$

be a functor that admits a left adjoint F^L. The composition $\mathsf{M} := F \circ F^L$ has a structure of monad acting on $\mathbf{C}$, and the functor F enhances to a functor

$$F^{\mathrm{enh}} : \mathbf{D} \to \mathsf{M}\text{-mod}(\mathbf{C}).$$

We shall say that the pair (F^L, F) (or just F) is *monadic* if F^{enh} is an equivalence. The Barr-Beck-Lurie theorem says that this happens if and only if F is conservative (as our conventions presuppose F to commute with colimits).

In general, given a monad M acting on a category $\mathbf{C}$, we denote by

$$\mathbf{ind}_{\mathsf{M}} : \mathbf{C} \rightleftarrows \mathsf{M}\text{-mod}(\mathbf{C}) : \mathbf{oblv}_{\mathsf{M}}$$

the resulting monadic adjunction.

0.7. **Conventions and notation: factorization.**

0.7.1. *The Ran space.* We let Ran denote the *non-unital* Ran space of X. I.e., this is a prestack whose value on an affine test-scheme S is the set of finite non-empty subsets in $\mathrm{Hom}(S, X_{\mathrm{dR}})$.

Given a point $\underline{x} \in \mathrm{Ran}$, we denote by $\mathrm{Ran}_{\underline{x}\subseteq}$ the prestack, whose value on an affine test scheme S is the set of those subsets $I \subset \mathrm{Hom}(S, X_{\mathrm{dR}})$, for which the union of their graphs

$$\Gamma_I \subset S \times X$$

set-theoretically contains $S \times \underline{x}$.

Making $\underline{x}$ vary over Ran, we obtain a prestack denoted $\mathrm{Ran}_{\subseteq}$. We denote by

$$\mathrm{pr}_{\mathrm{small}} \text{ and } \mathrm{pr}_{\mathrm{big}}$$

the resulting two projections $\mathrm{Ran}_{\subseteq} \rightrightarrows \mathrm{Ran}$.

0.7.2. *Factorization categories and algebras.* Factorization categories play a prominent role in this paper. We refer to [Ras1] for definitions. Properly speaking, a factorization category is a sheaf of categories over Ran with extra structure. However, we will slightly abuse the terminology in the following way:

When talking about a factorization category $\mathbf{A}$, we will denote by the same symbol $\mathbf{A}$ its fiber over the closed point of the standard formal disc $\mathcal{D}$. Thus, we will see factorization as structure on a given abstract category $\mathbf{A}$.

For a space Z mapping to Ran, we will denote by $\mathbf{A}_Z$ the category of sections of $\mathbf{A}$ (viewed as a sheaf of categories) over Z. Thus, for $\underline{x} \in \mathrm{Ran}$, we obtain the category, denoted $\mathbf{A}_{\underline{x}}$.

For the identity map $\mathrm{Ran} \to \mathrm{Ran}$, we obtain the category $\mathbf{A}_{\mathrm{Ran}}$ (i.e., the category of sections of $\mathbf{A}$ over Ran viewed as a sheaf of categories over Ran).

Given a factorization category $\mathbf{A}$, we can talk about *factorization* algebras on it. We will thing of a factorization algebra $\mathcal{A}$ as an object in $\mathbf{A}$ (understood as the fiber of $\mathbf{A}$ over the closed point of the standard disc), equipped with extra structure.

We will denote by $\mathcal{A}_{\mathrm{Ran}}$ the corresponding object in $\mathbf{A}_{\mathrm{Ran}}$. More generally, for $Z \to \mathrm{Ran}$, we can consider the corresponding object

$$\mathcal{A}_Z \in \mathbf{A}_Z.$$

0.7.3. *Pointwise vs. factorizable.* Given a factorization functor between factorization categories $F : \mathbf{A}_1 \to \mathbf{A}_2$, we can talk about a certain property of this functor (such as admitting an adjoint or being an equivalence) taking place at a *pointwise* level or a *factorization* level.

The latter is obviously implies the former.

0.7.4. *Factorization modules categories.* Given a factorization category $\mathbf{A}$ one can talk about a factorization module category $\mathbf{M}$ over it *on* any space Z mapping to Ran. Typical examples are: (a) $Z = \mathrm{pt}$ mapping to the distinguished point of the standard disc; (a') $Z = \mathrm{pt}$ mapping to a point $\underline{x} \in \mathrm{Ran}$; (b) $Z = \mathrm{Ran}$; (c) $Z = \mathrm{Ran}_{\subseteq}$.

A factorization module category $\mathbf{M}$ over $\mathbf{A}$ gives rise to a sheaf of categories over

$$Z_{\subseteq} := Z_{\mathrm{dR}} \underset{\mathrm{Ran},\mathrm{pr}_{\mathrm{small}}}{\times} \mathrm{Ran}_{\subseteq}.$$

We denote by $\mathbf{M}_{Z_{\subseteq}}$ the resulting category of global sections.

We will denote by $\mathbf{M}$ the retsriction of this sheaf of categories to Z_{dR} (along the tautological map $Z \to Z_{\subseteq}$). Slightly abusing the terminology, we will think of a factorization module category as a sheaf of categories $\mathbf{M}$ over Z_{dR}, equipped with extra structure. We will denote by $\mathbf{M}_Z$ the category of sections of $\mathbf{M}$ over Z.

For $\mathbf{A}$ and Z as above, the restriction of $\mathbf{A}$ (viewed as a sheaf of categories over Ran) to Z, is naturally a factorization $\mathbf{A}$-module category. We will abuse the terminology slightly and say that "we view $\mathbf{A}$ as a factorization module over itself."

0.7.5. *Factorization modules over factorization algebras.* Given a factorization module $\mathbf{M}$ over $\mathbf{A}$, and a factorization algebra $\mathcal{A}$ in $\mathbf{A}$, we can talk about factorization $\mathcal{A}$-modules *in* $\mathbf{M}$. We denote this category by

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M}).$$

An object of $\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M})$ gives rise to an object of the category that we denoted above by $\mathbf{M}_{Z_{\subseteq}}$. Applying the restriction functor

$$\mathbf{M}_{Z_{\subseteq}} \to \mathbf{M}_Z,$$

we obtain a (conservative) forgetful functor

$$\mathbf{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M}) \to \mathbf{M}_Z.$$

Note that this functor is *not necessarily* monadic, as it *does not necessarily* admit a left adjoint.

0.7.6. *Unitality.* All factorization categories appearing in this paper will be unital. The unit in a factorization category $\mathbf{A}$ is a section

$$\mathbf{1}_{\mathbf{A},\mathrm{Ran}} \in \mathbf{A}_{\mathrm{Ran}}$$

with a natural unitality *property*.

In particular, we have a functor

$$\text{ins. unit} : \mathbf{A}_{\mathrm{Ran}} \to \mathbf{A}_{\mathrm{Ran}_\subseteq},$$

where $\mathbf{A}_{\mathrm{Ran}_\subseteq}$ is formed with respect to the map $\mathrm{pr}_{\mathrm{big}} : \mathrm{Ran}_\subseteq \to \mathrm{Ran}$.

Given $Z \to \mathrm{Ran}$, we will denote by $\mathbf{1}_{\mathbf{A},Z}$ the corresponding object of $\mathbf{A}_Z$.

In particular, we have the object denoted $\mathbf{1}_{\mathbf{A}} \in \mathbf{A}$, when we think of $\mathbf{A}$ as the fiber at the closed point of the standard disc.

For $\underline{x} \in \mathbf{A}$, we have the corresponding object $\mathbf{1}_{\mathbf{A},\underline{x}} \in \mathbf{A}_{\underline{x}}$.

0.7.7. *Enhancement.* Let $F : \mathbf{A}_1 \to \mathbf{A}_2$ be a factorization functor between factorization categories. Assume that $\mathbf{A}_1$ is unital.

Then $F(\mathbf{1}_{\mathbf{A}})$ is naturally a factorization algebra in $\mathbf{A}_2$. The functor F naturally enhances to a functor

$$F^{\mathrm{enh}} : \mathbf{A}_1 \to F(\mathbf{1}_{\mathbf{A}})\text{-mod}^{\mathrm{fact}}(\mathbf{A}_2).$$

It is difficult, however, to specify conditions that guarantee that F^{enh} is an equivalence.

0.7.8. *t-structures.* At times, we give arguments involving t-structures for factorization categories. All such arguments should be understood by first implicitly fixing a finite set I and working over some individual X^I_{dR} so that the t-structures are defined.

0.8. **Conventions and notation: Ind-coherent sheaves.**

0.8.1. If Z is an affine scheme almost of finite type, we have a well-defined category $\mathrm{IndCoh}(Z)$. The assigment

$$Z \rightsquigarrow \mathrm{IndCoh}(Z), \quad (Z_1 \xrightarrow{f} Z_2) \rightsquigarrow (\mathrm{IndCoh}(Z_1) \xrightarrow{f^!} \mathrm{IndCoh}(Z_2))$$

is a functor

$$\mathrm{IndCoh}^! : (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}, \tag{0.23}$$

see [GR2, Chapter 5, Sect. 3].

The operation of right Kan extension produces from $\mathrm{IndCoh}^!$ a functor

$$(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat},$$

see [GR2, Chapter 5, Sect. 3.4], where

$$\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$$

is the full subcategory, consisting of prestacks *locally almost of finite type*, see [GR2, Chapter 2, Sect. 1.7].

0.8.2. In this paper, we will need the theory of IndCoh for algebro-geometric objects that are not necessarily locally (almost) of finite type.

Following [GR2, Chapter 2, Sect. 1.2], let ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ denote the category of n-coconective affine schemes. We define the functor

$$\mathrm{IndCoh}^! : ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}, \tag{0.24}$$

by *left Kan extending* the restriction of (0.23) to ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \subset \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ along

$$({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}.$$

I.e., for $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, we set

$$\mathrm{IndCoh}^!(S) := \underset{S \to S_0,\, S_0 \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\mathrm{colim}} \mathrm{IndCoh}(S_0).$$

Taking the union over n, we obtain a functor

$$\mathrm{IndCoh}^! : ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}, \tag{0.25}$$

where ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ is the category of *eventually coconnective* affine schemes.

We extend (0.25) to a functor

$$(\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}$$

by *right Kan extending* (0.25) along

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk})^{\mathrm{op}}.$$

I.e., for $Z \in \mathrm{PreStk}$, we set

$$\mathrm{IndCoh}^!(Z) := \underset{S \to Z,\, S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}}{\lim} \mathrm{IndCoh}^!(S).$$

0.8.3. For any Z, there is a well-defined object

$$\omega_Z \in \mathrm{IndCoh}^!(Z).$$

Moreover, $\mathrm{IndCoh}^!(Z)$ carries a symmetric monoidal structure, given by the $\overset{!}{\otimes}$ tensor product, for which ω_Z is the unit.

0.8.4. For a scheme (or more generally an *inf-scheme*) Z locally almost of finite type, Serre duality defines a identification

$$(\mathrm{IndCoh}(Z))^\vee \simeq \mathrm{IndCoh}(Z).$$

Suppose that for a given $Z \in \mathrm{PreStk}$, the category $\mathrm{IndCoh}^!(Z)$ is dualizable, We set

$$\mathrm{IndCoh}^*(Z) := (\mathrm{IndCoh}^!(Z))^\vee.$$

When well-defined, the category $\mathrm{IndCoh}^*(Z)$ is a module over $\mathrm{IndCoh}^!(Z)$ viewed as a (symmetric) monoidal category (see Sect. 0.8.3.

In addition, $\mathrm{IndCoh}^*(Z)$ is a module over $\mathrm{QCoh}(Z)$ with the usual (symmetric) monoidal structure.

0.8.5. In some situations, one can describe the category $\mathrm{IndCoh}^*(Z)$ more explicitly. Suppose that Z can be written as

$$\lim_{\alpha} Z_\alpha,$$

where $Z_\alpha \in \mathrm{PreStk}_{\mathrm{laft}}$ with $\mathrm{IndCoh}(Z_\alpha)$ compactly generated, and the transition maps

$$f_{\alpha,\beta} : Z_\beta \to Z_\alpha$$

are schematic and of finite Tor-dimension, so that the functors

$$f^!_{\alpha,\beta} : \mathrm{IndCoh}(Z_\alpha) \to \mathrm{IndCoh}(Z_\beta)$$

preserve compactness.

Then the functors

$$f^*_{\alpha,\beta} : \mathrm{IndCoh}(Z_\alpha) \to \mathrm{IndCoh}(Z_\beta),$$

left adjoint to

$$(f_{\alpha,\beta})_* : \mathrm{IndCoh}(Z_\beta) \to \mathrm{IndCoh}(Z_\alpha)$$

are well-defined, and we have

$$\mathrm{IndCoh}^*(Z) \simeq \operatorname*{colim}_{\alpha} \mathrm{IndCoh}(Z_\alpha),$$

where in the formation of the colimit the transition functors are $f^*_{\alpha,\beta}$.

We can also write

$$\mathrm{IndCoh}^*(Z) \simeq \lim_{\alpha} \mathrm{IndCoh}(Z_\alpha),$$

where in the formation of the limit the transition functors are $f^*_{\alpha,\beta}$.

In particular, we have a well-defined functor

$$\Gamma(Z, -) : \mathrm{IndCoh}^*(Z) \to \mathrm{Vect}\,.$$

If Z_α are eventually coconnective, so that $\mathcal{O}_{Z_\alpha}$ is well-defined as an object of $\mathrm{IndCoh}(Z_\alpha)$, we have a well-defined (compact) object

$$\mathcal{O}_Z \in \mathrm{IndCoh}^*(Z),$$

so that

$$\Gamma(Z, -) \simeq \mathcal{H}om_{\mathrm{IndCoh}^*(Z)}(\mathcal{O}_Z, -).$$

0.8.6. Let us continue being in the situation of Sect. 0.8.5. Assume now that all Z_α are smooth (schemes or algebraic stacks). Then the above description of $\mathrm{IndCoh}^*(Z)$ shows that we have a canonical equivalence

$$\mathrm{IndCoh}^*(Z) \simeq \mathrm{QCoh}(Z).$$

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A crucial role in local and local-to-global constructions is played by the concept of *factorization*. Its appearance in representation theory was pioneered by M. Finkelberg, I. Mirković and V. Schechtman, and it was further subsequently elucidated by A. Beilinson, V. Drinfeld and J. Lurie.

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Part I. Local Theory

This Part consists almost entirely of a review of known results,[4] which constitute local ingredients for local-to-global constructions in Parts III and IV.

There are two types of results and constructions that we will need to review. The first type takes place either on the geometric or the spectral side separately. A typical example of such a construction is the Jacquet functor that relates a category for G with the corresponding category for its Levi subgroup M.

The second type passes from the geometric to the spectral side; such results, by definition, involve Langlands correspondence of some sort. In fact, there are exactly two sources of such results (as long as we stay at the critical level for G and level ∞ for $\check{G}$): one is the geometric Casselman-Shalika formula (Theorem 1.4.2), and the other is the Feigin-Frenkel isomorphism (given by (7.2)). The compatibility between the two is incapsulated by Theorem 7.2.5. The other results of local Langlands nature are ultimately deduced from one (or a combination) of these two.

The main result of this part is the *critical FLE*, Theorem 7.3.4, which says that the Kazhdan-Lusztig category at the critical level (for G) is equivalent to the category of ind-coherent sheaves on the space of monodromy-free opers on the punctured disc (for $\check{G}$).

1. Geometric Satake and Casselman-Shalika formula: recollections

In this section we will review the constructions of categories of *geometric nature* associated, on the geometric side to spaces of maps

$$\mathcal{D} \to G \text{ and } \mathcal{D}^\times \to G,$$

and (twisted) D-modules on these spaces, and on the spectral side to spaces of maps

$$\mathcal{D}_{\mathrm{dR}} \to \check{G} \text{ and } \mathcal{D}^\times_{\mathrm{dR}} \to \check{G}$$

and ind-coherent sheaves on these spaces.

Thus, the main players are:

- The category $\mathrm{Whit}^!(G)$ of Whittaker D-modules on the affine Grassmannian;
- Its spectral counterpart $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \simeq \mathrm{Rep}(\check{G})$;
- The equaivalence $\mathrm{Whit}^!(G) \simeq \mathrm{Rep}(\check{G})$, which we call the geometric Casselman-Shalika formula (Theorem 1.4.2);
- The local Hecke category Sph_G (which can, in a certain precise sense, be recovered from its action on $\mathrm{Whit}^!(G)$);
- Its spectral counterpart $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ (which can also be recovered from its action on $\mathrm{Rep}(\check{G})$);
- The (derived) geometric Satake equivalence $\mathrm{Sat}_G : \mathrm{Sph}_G \simeq \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ (Theorem 1.7.2).

There are three "annoyances" that will be introduced in this section, and that will plague us throughout the paper:

(1) This paper is concerned with the *classical* geometric Langlands. However, "classical" for G means the *critical* level. This means that all geometric categories involved will consist not of D-modules, but of critically or half-twisted D-modules. As a result, throughout the paper, we will have to watch carefully what happens with these twistings as we move between different spaces.

(2) Ultimately, on the geometric side, the object we need to consider is not the constant group-scheme on X with fiber G, but rather its twist by the T-torsor $\rho(\omega_X)$. This twist is analogous to the usual ρ-shift in the representation theory of the finite-dimensional G. Thus, all spaces associated with G will undergo the corresponding twist.

[4] With the exception of the proof of the critical FLE, Theorem 7.3.4 and the compatibility-with-duality theorem, Theorem 7.6.4.

(3) Both categories Sph_G and $\mathrm{Sph}_{\check{G}}^{\mathrm{spec}}$ are endowed with anti-involutions, denoted σ and σ^{spec}. A source of constant headache throughout this paper is that these anti-involutions are compatible under Sat_G, *up to the Cartan involution*, denoted τ_G on G. This can be seen as a vestige (in a rather precise sense) that the square of the usual Fourier transform is not the identity, but rather is given by the action of -1.

1.1. **The critical twist.**

1.1.1. We choose once and for all a square root $\omega_X^{\otimes \frac{1}{2}}$ of the canonical line bundle ω_X on X.

1.1.2. Consider the affine Grassmannian Gr_G as a factorization space over X, equipped with an action of the (factorization) group indscheme $\mathfrak{L}(G)$.

1.1.3. Let $\det_{\mathrm{Gr}_G}$ denote the determinant (factorization) line bundle on Gr_G. We will denote by crit the de Rham twisting equal to the half of the de Rham twisting defined by $\det_{\mathrm{Gr}_G}$.

We will denote by

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G)$$

the corresponding (factorization) category of twisted D-modules

Remark 1.1.4. According to [BD, Sect. 4], the choice of $\omega_X^{\otimes \frac{1}{2}}$ gives rise to a square root of $\det_{\mathrm{Gr}_G}$, as a line bundle over $\mathrm{Gr}_{G,\mathrm{Ran}}$. However, this square root is *incompatible* with factorization.[5]

Henceforth, we will avoid using this trivialization.

1.1.5. Consider the factorization $\mathbb{Z}/2\mathbb{Z}$-gerbe on Gr_G of square roots of $\det_{\mathrm{Gr}_G}$; we denote it by $\det_{\mathrm{Gr}_G}^{\frac{1}{2}}$.

A $\mathbb{Z}/2\mathbb{Z}$-gerbe on a space defines an étale twisting of the category of D-modules on that space. Let

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)$$

denote the (factorization) category as D-modules corresponding to $\det_{\mathrm{Gr}_G}^{\frac{1}{2}}$.

1.1.6. Note that if $\mathcal{L}$ is a line bundle on a space $\mathcal{Y}$, and n is an integer, we have a canonical identification of the corresponding twisted categories of D-modules:

$$\text{D-mod}_{\frac{1}{n}\cdot \mathrm{dlog}(\mathcal{L})}(\mathcal{Y}) \xrightarrow{\sim} \text{D-mod}_{\mathcal{L}^{\frac{1}{n}}}(\mathcal{Y}), \tag{1.1}$$

where:

- For a line bundle $\mathcal{L}$ we denote by $\mathrm{dlog}(\mathcal{L})$ the de Rham twisting defined by it. Note that tensoring by $\mathcal{L}$ defines an equivalence

$$\text{D-mod}(\mathcal{Y}) \to \text{D-mod}_{\mathrm{dlog}(\mathcal{L})}(\mathcal{Y}); \tag{1.2}$$

- For a given twisting $\mathcal{T}$ and $c \in k$, we denote by $c \cdot \mathcal{T}$ the new action corresponding to the structure of k-vector space on de Rham twistings;

- The subscript $\mathcal{L}^{\frac{1}{n}}$ denotes the étale twisting by the $\mathbb{Z}/n\mathbb{Z}$-gerbe of nth roots of $\mathcal{L}$.

For example, when $n = 1$, the identification (1.1) is the identification inverse to (1.2).

[5]More precisely, this square root exists as a factorization $\mathbb{Z}/2\mathbb{Z}$-graded line bundle, where the grading over the connected component Gr_G^{λ} of Gr_G (here $\lambda \in \Lambda_{G,G} = \pi_0(\mathrm{Gr}_G)$) equals $\langle \lambda, 2\check{\rho} \rangle \bmod 2$.

1.1.7. Applying this to $\mathcal{Y} = \mathrm{Gr}_G$ and $\mathcal{L} = \det_{\mathrm{Gr}_G}$, we obtain a canonical equivalence of (factorization) categories

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G) \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G).$$

Remark 1.1.8. According to Remark 1.1.4, we can also identify

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \simeq \text{D-mod}(\mathrm{Gr}_{G,\mathrm{Ran}}),$$

or equivalently

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \simeq \text{D-mod}(\mathrm{Gr}_{G,\mathrm{Ran}}),$$

as plain categories, but these identifications are incompatible with the factorization structures.

Remark 1.1.9. We distinguish $\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G)$ and $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)$ notationally for two reasons:
(1) The gerbe-twisted version makes sense not just in the context of D-modules, but also in other sheaf-theoretic contexts (e.g., Betti, ℓ-adic).
(2) The category $\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G)$ comes equipped with a natural forgetful functor to $\mathrm{IndCoh}(\mathrm{Gr}_G)$, while for a general gerbe, the gerbe-twisted category of D-modules does not carry such a functor.

1.1.10. We can also consider the corresponding *multiplicative* factorization $\mathbb{Z}/2\mathbb{Z}$-gerbe on $\mathfrak{L}(G)$, equipped with a multiplicative trivialization of its restriction to $\mathfrak{L}^+(G)$.

Since the group indscheme $\mathfrak{L}(N)$ is contractible, the restriction of the above gerbe to it also admits a multiplicative trivialization.

In particular, if H is a factorization subgroup of either $\mathfrak{L}^+(G)$ or $\mathfrak{L}(N)$, it makes sense to consider the (factorization) category

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^H$$

of H-equivariant D-modules.

1.2. **A geometric twisting construction.**

1.2.1. Let H be a group mapping to G, and let $\mathcal{P}_H$ be an H-torsor over X. Taking sections over the formal disc, $\mathcal{P}_H$ gives rise to a factorization torsor over $\mathfrak{L}^+(H)$; by a slight abuse of notation, we will denote this $\mathfrak{L}^+(H)$-factorization torsor by the same symbol $\mathcal{P}_H$.

Given a space $\mathcal{Y}$ over X, equipped with an action of $\mathfrak{L}^+(H)$, we can form a twist, to be denoted $\mathcal{Y}_{\mathcal{P}_H}$. If $\mathcal{Y}$ was endowed with a factorization structure compatible with the $\mathfrak{L}^+(H)$-action, then so is $\mathcal{Y}_{\mathcal{P}_H}$.

The space $\mathcal{Y}_{\mathcal{P}_H}$ is acted on by the adjoint twist $\mathfrak{L}^+(H)_{\mathcal{P}_H}$ of $\mathfrak{L}^+(H)$.

1.2.2. We will denote by the subscript $\mathcal{P}_H$ the various categories of D-modules associated with the above geometric objects, such as

$$\text{D-mod}(\mathcal{Y}) \rightsquigarrow \text{D-mod}(\mathcal{Y})_{\mathcal{P}_H} \text{ and } \text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)} \rightsquigarrow (\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)})_{\mathcal{P}_H}.$$

Note, however, that the category $(\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)})_{\mathcal{P}_H}$ is canonically equivalent to the original category $\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)}$. We will denote this equivalence by

$$\alpha_{\mathcal{P}_H,\mathrm{taut}} : \text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)} \xrightarrow{\sim} (\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)})_{\mathcal{P}_H}.$$

1.2.3. A typical example of this situation that we will consider is when $H = T$, and the T-bundle is $\rho(\omega_X)$, i.e., the bundle induced from $\omega_X^{\frac{1}{2}}$ by means of

$$2\rho : \mathbb{G}_m \to T.$$

1.2.4. We now record the following observation, to be used in the sequel:

Suppose that H is abelian and the action of $\mathfrak{L}^+(H)$ on $\mathcal{Y}$ is trivial. In this case, we have a canonical isomorphism

$$\mathcal{Y}_{\mathcal{P}_H} \simeq \mathcal{Y}.$$

In particular, we obtain an equivalence

$$\alpha_{\mathcal{P}_H,\mathrm{cent}} : \text{D-mod}(\mathcal{Y})_{\mathcal{P}_H} \xrightarrow{\sim} \text{D-mod}(\mathcal{Y})$$

and an a priori *different* identification

$$\alpha_{\mathcal{P}_H,\mathrm{cent}} : (\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)})_{\mathcal{P}_H} \xrightarrow{\sim} \text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)}.$$

We will denote by

$$(\mathrm{transl}_{\mathcal{P}_{Z_H}})^* := \alpha_{\mathcal{P}_H,\mathrm{cent}} \circ \alpha_{\mathcal{P}_H,\mathrm{taut}}$$

the resulting auto-equivalence of $\text{D-mod}(\mathcal{Y})^{\mathfrak{L}^+(H)}$.

The functor $(\mathrm{transl}_{\mathcal{P}_{Z_H}})^*$ is the pullback along the automorphism of the stack $\mathcal{Y}/\mathfrak{L}^+(H)$ given by the point $\mathcal{P}_H \in \mathrm{pt}\,/\mathfrak{L}^+(H)$ and the action map

$$\mathrm{pt}\,/\mathfrak{L}^+(H) \times \mathcal{Y}/\mathfrak{L}^+(H) \to \mathcal{Y}/\mathfrak{L}^+(H).$$

1.2.5. A typical example of the situation of Sect. 1.2.4 is when $\mathcal{Y} = \mathrm{Gr}_G$, so that $\mathcal{Y}/\mathfrak{L}^+(G)$ is the local Hecke stack

$$\mathrm{Hecke}^{\mathrm{loc}}_G := \mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G).$$

Let H map to the center of G. In this case, the action of $\mathfrak{L}^+(H)$ on Gr_G is trivial.

The above automorphism of

$$\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G) = \{\mathcal{P}'_G, \mathcal{P}''_G, \mathcal{P}'_G|_{\mathcal{D}^\times} \sim \mathcal{P}''_G|_{\mathcal{D}^\times}\}$$

is given by the procedure of tensoring the G-bundles involved by $\mathcal{P}_H$, using the canonical map

$$\mathrm{pt}\,/H \times \mathrm{pt}\,/G \to \mathrm{pt}\,/G.$$

1.3. **The Whittaker category on the affine Grassmannian.**

1.3.1. We apply the construction of Sect. 1.2.3 to $\mathcal{Y} := \mathrm{Gr}_G$, viewed as a scheme acted on by $\mathfrak{L}^+(T) \subset \mathfrak{L}^+(G)$, and the group indscheme $\mathfrak{L}(N)$.

Thus, we can form the (factorization) space $\mathrm{Gr}_{G,\rho(\omega_X)}$, which is acted on by $\mathfrak{L}(G)_{\rho(\omega_X)}$, and in particular $\mathfrak{L}(N)_{\rho(\omega_X)}$.

1.3.2. The group indscheme $\mathfrak{L}(N)_{\rho(\omega_X)}$ carries a canonical (residue) homomorphism

$$\mathfrak{L}(N)_{\rho(\omega_X)} \to N.$$

Choosing a non-degenerate character χ of N, we can consider the categories

$$\mathrm{Whit}^!(G) := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}$$

and

$$\mathrm{Whit}_*(G) := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}.$$

Remark 1.3.3. The categories $\mathrm{Whit}^!(G)$ and $\mathrm{Whit}_*(G)$ are canonically independent of the choice of χ:

Indeed, given two non-degenerate characters χ_1 and χ_2, there exists an element $t \in T$ that conjugates χ_1 and χ_2. The translation by t on $\mathrm{Gr}_{G,\rho(\omega_X)}$ defines then an equivalence between the corresponding Whittaker categories.

The choice of t is unique up to an element $z \in Z_G$. However, the translation action of z on $\mathrm{Gr}_{G,\rho(\omega_X)}$ is trivial.

1.3.4. The categories $\mathrm{Whit}^!(G)$ and $\mathrm{Whit}_*(G)$ are naturally mutually dual, up to replacing χ by its inverse. Note, however, that due to Remark 1.3.3, they are actually mutually dual.

1.3.5. Let $\omega^{\mathrm{ren}}_{\mathfrak{L}(N)_{\rho(\omega_X)}}$ be *the renormalized* dualizing sheaf on $\mathfrak{L}(N)_{\rho(\omega_X)}$, defined to be the *-pullback of the dualizing sheaf along the projection

$$\mathfrak{L}(N)_{\rho(\omega_X)} \to \mathfrak{L}(N)_{\rho(\omega_X)}/\mathfrak{L}^+(N)_{\rho(\omega_X)}.$$

The operation of *-convolution with

$$\omega^{\mathrm{ren}}_{\mathfrak{L}(N)_{\rho(\omega_X)}} \otimes \chi^*(\exp)$$

is an endofunctor of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)})$, which factors as

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}) \twoheadrightarrow \mathrm{Whit}_*(G) \to \mathrm{Whit}^!(G) \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}).$$

Denote the resulting functor $\mathrm{Whit}_*(G) \to \mathrm{Whit}^!(G)$ by

$$\Theta_{\mathrm{Whit}(G)} : \mathrm{Whit}_*(G) \to \mathrm{Whit}^!(G).$$

The following fundamental result was established in [Ras6]:

Theorem 1.3.6. *The functor $\Theta_{\mathrm{Whit}(G)}$ is an equivalence (of factorization categories).*

Remark 1.3.7. The proof of Theorem 1.3.6, as recorded in [Ras6], is given for a fixed formal disc, but the same argument applies to prove that factorization version as well.

1.4. **The geometric Casselman-Shalika formula.**

1.4.1. The following is the statement of the geometric Casselman-Shalika formula (see [Ras5, Theorem 6.36.1]:

Theorem 1.4.2. *There exists a canonically defined equivalence of factorization categories:*

$$\mathrm{CS}_G : \mathrm{Whit}^!(G) \to \mathrm{Rep}(\check{G}).$$

Remark 1.4.3. In the course of the proof of Theorem 1.4.2 one uses the *naive* (i.e., non-derived) geometric Satake to construct a functor

$$\mathrm{Rep}(\check{G}) \to \mathrm{Whit}^!(G),$$

and show that it is an equivalence, see Remark 1.7.8.

Remark 1.4.4. The functor CS_G is normalized so that it sends the standard object

$$\Delta^\lambda \in \mathrm{Whit}^!(G), \quad \lambda \in \Lambda^+_G,$$

corresponding to the $\mathfrak{L}(N)_{\rho(\omega_X)}$-orbit

$$S^\lambda := \mathfrak{L}(N)_{\rho(\omega_X)} \cdot t^\lambda$$

to the highest weight module

$$V^\lambda \in \mathrm{Rep}(\check{G}).$$

(In the above formula, t denotes the uniformizer on $\mathcal{D}$.)

This normalization is not arbitrary, but is forced by the behavior of the FLE functor off critical level.

Remark 1.4.5. For the validity of Theorem 1.4.2 at the factorization level, it is crucial that in the definition of $\mathrm{Whit}^!(G)$ we use the twisted category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)$, rather than the untwisted one, i.e., $\text{D-mod}(\mathrm{Gr}_G)$.

1.4.6. The following is a basic pattern of how the equivalence CS_G interacts with duality.

Let us denote by

$$\mathrm{FLE}_{\check{G},\infty} : \mathrm{Rep}(\check{G}) \to \mathrm{Whit}_*(G)$$

the functor equal to $\mathrm{CS}_G^\vee$, with respect to the canonical dualities:

$$\mathrm{Whit}_*(G) = (\mathrm{Whit}^!(G))^\vee \text{ and } \mathrm{Rep}(\check{G})^\vee \simeq \mathrm{Rep}(\check{G}).$$

Remark 1.4.7. The notation $\mathrm{FLE}_{\check{G},\infty}$ stems from the fact that the above functor is indeed the limiting value of the (positive level) FLE equivalence. This will be made explicit in the compatibility between $\mathrm{FLE}_{\check{G},\infty}$ and $\mathrm{FLE}_{G,\mathrm{crit}}$, see Sect. 7.3 below.

1.4.8. *Example.* Note, in particular, that the functor $\mathrm{FLE}_{\check{G},\infty}$ sends

$$V^\lambda \in \mathrm{Rep}(\check{G}) \mapsto \nabla^{-w_0(\lambda)} \in \mathrm{Whit}_*(G),$$

where for $\mu \in \Lambda^+$ we denote by

$$\nabla^\mu \in \mathrm{Whit}_*(G)$$

the object dual to $\Delta^\mu \in \mathrm{Whit}^!(G)$, i.e.,

$$\langle \mathcal{F}, \nabla^\mu \rangle = \mathcal{H}om_{\mathrm{Whit}^!(G)}(\Delta^\mu, \mathcal{F}), \quad \mathcal{F} \in \mathrm{Whit}^!(G),$$

where

$$\langle -, - \rangle : \mathrm{Whit}^!(G) \otimes \mathrm{Whit}_*(G) \to \mathrm{Vect}$$

is the canonical pairing.

1.4.9. Note that the Whittaker category is canonically attached to the triple (G, B). Hence, the group of *outer automorphisms* of G (i.e., the group of automorphisms of the polarized[6] root datum of G) acts on both versions of the Whittaker category.

Let τ_G be the Cartan involution, viewed as an outer automorphism of G. The corresponding automorphism of the polarized root datum acts as $\lambda \mapsto -w_0(\lambda)$.

We can find another representative of τ_G as an actual automorphism of G (defined up to a conjugation by an element of T_{ad}, the Cartan of the adjoint group) that swaps B and B^-, and acts as inversion on the Cartan subgroup T. This choice of τ_G preserves each standard Levi subgroup M, and induces on it the automorphism τ_M.

1.4.10. We have:

Lemma 1.4.11. *The composition*

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\longrightarrow} \mathrm{Whit}_*(G) \overset{\Theta_{\mathrm{Whit}(G)}}{\longrightarrow} \mathrm{Whit}^!(G)$$

identifies canonically with

$$\tau_G \circ (\mathrm{CS}_G)^{-1}.$$

Remark 1.4.12. As a reality check, note that both functors in (1.4.11) send

$$V^\lambda \in \mathrm{Rep}(\check{G}) \ \mapsto \ \Delta^{-w_0(\lambda)} \in \mathrm{Whit}^!(G).$$

The proof of Lemma 1.4.11 follows easily from the construction of CS_G via naive Satake.

1.5. **The spherical category.**

1.5.1. We denote by $\mathrm{Sph}_G^{\mathrm{unr}}$ the (factorization) monoidal category

$$\text{D-mod}_{\frac{1}{2}}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G)).$$

Remark 1.5.2. The superscript unr stands for *unrenormalized*, compare below.

[6]By a polarization of a root datum we mean a choice of the subset of positive roots.

1.5.3. We let Sph_G denote its *renormalized* version, which is defined as the ind-completion of the full subcategory in $\mathrm{Sph}_G^{\mathrm{unr}}$ consisting of objects whose image under (either of) the forgetful functors

$$\text{D-mod}_{\frac{1}{2}}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)) \leftarrow \text{D-mod}_{\frac{1}{2}}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G)) \rightarrow \text{D-mod}_{\frac{1}{2}}(\mathfrak{L}(G)/\mathfrak{L}^+(G))$$

is compact.

By construction, the monoidal (which is also the factorization) unit

$$\mathbf{1}_{\mathrm{Sph}_G} \in \mathrm{Sph}_G$$

is compact.

1.5.4. We have an adjoint pair of functors

$$\mathrm{ren} : \mathrm{Sph}_G^{\mathrm{unr}} \rightleftarrows \mathrm{Sph}_G : \mathrm{unr},$$

with ren being fully faithful and unr monoidal. This makes $\mathrm{Sph}_G^{\mathrm{unr}}$ into a monoidal colocalization of Sph_G.

1.5.5. Inversion on the group $\mathfrak{L}(G)$ defines an anti-involution, denoted σ, of Sph_G. We will refer to it as the "flip" anti-involution.

Henceforth, we will use σ to pass between left and right module categories over Sph_G. In light of this, we will not necessarily distinguish between left and right actions of Sph_G.

1.5.6. The fact that Gr_G is ind-proper implies that the composition of the involution σ with Verdier duality (on compact objects) defines an equivalence

$$\mathrm{Sph}_G^{\vee} \simeq \mathrm{Sph}_G, \tag{1.3}$$

which identifies both with right and left monoidal dualization.

Combined with the fact that the unit in Sph_G is compact, we obtain that Sph_G is *rigid* as a monoidal category.[7]

1.5.7. Recall the setting of Sect. 1.2. For any G-bundle $\mathcal{P}_G$ on X, we can form the twisted version

$$\mathrm{Sph}_{G,\mathcal{P}_G}$$

of $\mathcal{P}_G$.

In particular, we have a natural action of $\mathrm{Sph}_{G,\rho(\omega_X)}$ on $\mathrm{Whit}^!(G)$ and $\mathrm{Whit}_*(G)$.

However, according to Sect. 1.2.2, we can identify[8]

$$\mathrm{Sph}_G \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{Sph}_{G,\rho(\omega_X)},$$

and thus we can regard $\mathrm{Whit}^!(G)$ and $\mathrm{Whit}_*(G)$ as acted on by Sph_G itself.

These actions are compatible both with the duality

$$(\mathrm{Whit}^!(G))^{\vee} \simeq \mathrm{Whit}_*(G) \tag{1.4}$$

(see Sect. 1.5.5) and the functor $\Theta_{\mathrm{Whit}(G)}$.

1.6. **The spectral spherical category.**

[7] Being a monoidal colocalization of a rigid category, $\mathrm{Sph}_G^{\mathrm{unr}}$ is semi-rigid.

[8] In the formula below we consider $\mathfrak{L}(G)$ as acted on by $\mathfrak{L}^+(G) \times \mathfrak{L}^+(G)$.

1.6.1. We let $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ denote the spectral spherical category, i.e.,

$$\mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}),$$

where

$$\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}} := \mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}).$$

We endow $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ with a (factorization) monoidal structure via *-pull and *-push along the standard convolution diagram.

The unit object is given by direct image of the structure sheaf along

$$\mathrm{LS}_{\check{G}}(\mathcal{D}) \to \mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}.$$

Furthermore, the above construction is in fact a (factorization) monoidal functor

$$\mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}.$$

We denote the resulting functor by

$$\mathrm{nv} : \mathrm{Rep}(\check{G}) \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}},$$

where "nv" stands for "naive."

1.6.2. The flip of two factors defines an anti-involution on $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ to be denoted σ^{spec}.

We will use σ^{spec} to pass between left and right $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$-module categories.

Note that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}(\check{G}) & \xrightarrow{\mathrm{nv}} & \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \sigma^{\mathrm{spec}}} \\ \mathrm{Rep}(\check{G}) & \xrightarrow{\mathrm{nv}} & \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \end{array} \tag{1.5}$$

commutes, where Id makes sense as an anti-involution of $\mathrm{Rep}(\check{G})$, since this category is symmetric monoidal.

1.6.3. Consider the general situation of a monoidal category of the form

$$\mathbf{A} := \mathrm{IndCoh}(\mathcal{Y} \underset{\mathcal{Y}_0}{\times} \mathcal{Y}),$$

where $\mathcal{Y}$ is smooth and the projection $\mathcal{Y} \to \mathcal{Y}_0$ is proper, where the monoidal structure is given by *-pull and *-push along the convolution diagram.

In this case, the functors of right and left monoidal dualization on compact objects of $\mathbf{A}$ are given by

$$\mathcal{F} \mapsto \sigma(\mathbb{D}^{\mathrm{Serre}}(\mathcal{F})) \overset{*}{\otimes} p_1^*(\omega_{\mathcal{Y}}^{\otimes -1}) \text{ and } \sigma(\mathbb{D}^{\mathrm{Serre}}(\mathcal{F})) \overset{*}{\otimes} p_2^*(\omega_{\mathcal{Y}}^{\otimes -1}),$$

respectively, where

$$p_1, p_2 : \mathcal{Y} \underset{\mathcal{Y}_0}{\times} \mathcal{Y} \to \mathcal{Y}$$

are the two projections.

Hence, if the dualizing sheaf $\omega_{\mathcal{Y}}$ on $\mathcal{Y}$ has the property that it is pulled back from $\mathcal{Y}_0$, then the functors of right and left monoidal dualization on $\mathbf{A}$ are canonically isomorphic as monoidal anti-equivalences.

1.6.4. The pattern of Sect. 1.6.3 is realized for

$$\mathcal{Y} := \mathrm{LS}_{\check{G}}(\mathcal{D}) \text{ and } \mathcal{Y}^0 := \mathrm{LS}_{\check{G}}(\mathcal{D}^\times),$$

since in this case $\omega_{\mathrm{LS}_{\check{G}}(\mathcal{D})}$ is a constant sheaf.

Hence, we obtain that the functors of left and right dualization on $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ are canonically identified, thereby giving rise to an equivalence

$$(\mathrm{Sph}^{\mathrm{spec}}_{\check{G}})^\vee \simeq \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \tag{1.6}$$

is compatible with the $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$-bimodule structure.

In particular, the category $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ is rigid, so that (1.3) is the same identification as the one obtained from rigidity (see [GR2, Lemma 9.2.4]).

1.6.5. We have a natural action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \simeq \mathrm{Rep}(\check{G}).$$

This action is compatible with the canonical self-duality of $\mathrm{Rep}(\check{G})$.

1.7. **Geometric Satake equivalence.**

1.7.1. The following is the statement of the geometric Satake equivalence (see [CR, Theorem 6.6.1]):

Theorem 1.7.2. *There exists a unique equivalence of monoidal factorization categories*

$$\mathrm{Sat}_G : \mathrm{Sph}_G \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}},$$

compatible with the actions of Sph_G *on* $\mathrm{Whit}(G)$ *and* $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ *on* $\mathrm{Rep}(\check{G})$ *via the equivalence*

$$\mathrm{CS}_G : \mathrm{Whit}(G) \simeq \mathrm{Rep}(\check{G}).$$

Remark 1.7.3. What we denote by Sat_G and refer to as the (geometric) Satake equivalence, is often also called "the derived (geometric) Satake equivalence."

Remark 1.7.4. In the above statement of Theorem 1.7.2, the definition of the spectral side via IndCoh^* does not perfectly match the definition of the spectral side in [CR]. The same issue will occur again in Theorem 2.6.7. We will address these issues in a future draft of this text.

1.7.5. *Example.* Unwinding the construction, we obtain that Sat_G sends the object in Sph_G corresponding to the double coset of the point $t^{-\lambda}$ (for $\lambda \in \Lambda^+$) to the object

$$\mathrm{nv}(V^\lambda) \in \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}\,.$$

The above object object in Sph_G is what is usually denoted by

$$\mathrm{IC}_{\overline{\mathrm{Gr}^{-w_0(\lambda)}_G}},$$

the intersection cohomology sheaf on the closure of the $\mathfrak{L}^+(G)$-orbit $\mathrm{Gr}^{-w_0(\lambda)}$ of $t^{-w_0(\lambda)}$ (which is the same as the $\mathfrak{L}^+(G)$-orbit of $t^{-\lambda}$).

Remark 1.7.6. As in the case of Theorem 1.4.2, for the validity of Theorem 1.7.2 at the factorization level, it is crucial that we work with the twisted category

$$\text{D-mod}_{\frac{1}{2}}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G))$$

rather than with $\text{D-mod}(\mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G))$.

1.7.7. In what follows, we will denote by $\mathrm{Sat}_G^{-1,\mathrm{nv}}$ the functor

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{nv}}{\to} \mathrm{Sph}_{\check{G}}^{\mathrm{spec}} \overset{\mathrm{Sat}^{-1}}{\to} \mathrm{Sph}_G\,.$$

Remark 1.7.8. Note, for example that the functor

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\longrightarrow} \mathrm{Sph}_G \overset{\sigma}{\to} \mathrm{Sph}_G \overset{\Delta^0\star -}{\longrightarrow} \mathrm{Whit}^!(G)$$

is exactly CS^{-1}.

The functor

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}_G^{-1,\mathrm{nv}}}{\longrightarrow} \mathrm{Sph}_G \overset{\Delta^0\star -}{\longrightarrow} \mathrm{Whit}^!(G)$$

is $\mathrm{FLE}_{\check{G},\infty}$.

1.8. **The curse of σ and τ.**

1.8.1. The following statement results from the uniqueness assertion in Theorem 1.7.2 combined with Lemma 1.4.11:

Corollary 1.8.2. *The following diagram of anti-equivalences commutes:*

$$\begin{array}{ccc}
\mathrm{Sph}_G & \xrightarrow{\mathrm{Sat}_G} & \mathrm{Sph}_{\check{G}}^{\mathrm{spec}} \\
{\scriptstyle\sigma}\downarrow & & \\
\mathrm{Sph}_G & & \downarrow{\scriptstyle\sigma^{\mathrm{spec}}} \\
{\scriptstyle\tau_G}\downarrow & & \\
\mathrm{Sph}_G & \xrightarrow{\mathrm{Sat}_G} & \mathrm{Sph}_{\check{G}}^{\mathrm{spec}}\,.
\end{array}$$

1.8.3. Denote by $\mathrm{Sat}_{G,\tau}$ the (factorization) equivalence

$$\mathrm{Sph}_G \overset{\tau}{\to} \mathrm{Sph}_G \overset{\mathrm{Sat}_G}{\longrightarrow} \mathrm{Sph}_{\check{G}}^{\mathrm{spec}}\,.$$

Denote by $\mathrm{Sat}_{G,\tau}^{-1,\mathrm{nv}}$ the functor

$$\tau_G \circ \mathrm{Sat}_G^{-1,\mathrm{nv}}, \quad \mathrm{Rep}(\check{G}) \to \mathrm{Sph}_G\,.$$

Remark 1.8.4. The functor $\mathrm{Sat}_{G,\tau}^{-1,\mathrm{nv}}$ may be a more standard normalization for the geometric Satake equivalence. For example, it sends the object $V^\lambda \in \mathrm{Rep}(\check{G})$ to the object in Sph_G corresponding to the double coset of the point t^λ (for $\lambda \in \Lambda^+$), i.e., $\mathrm{IC}_{\overline{\mathrm{Gr}}_G^\lambda}$.

1.8.5. As another corollary of Lemma 1.4.11 we obtain:

Corollary 1.8.6. *The equivalence*

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G)$$

is compatible with the actions of Sph_G *and* $\mathrm{Sph}_{\check{G}}^{\mathrm{spec}}$ *via* $\mathrm{Sat}_{G,\tau}$.

1.8.7. *Warning.* As has been mentioned above, we will use σ (resp., σ^{spec}) to pass between left and right module categories for Sph_G (resp., $\mathrm{Sph}_{\check{G}}^{\mathrm{spec}}$).

Note, however, that due to Corollary 1.8.2, this procedure is compatible with the geometric Satake equivalence *up to* the Cartan involution.

In practice, this will manifest itself as follows. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ (resp., $\mathbf{C}_1^{\mathrm{spec}}$ and $\mathbf{C}_2^{\mathrm{spec}}$) of left module categories for Sph_G (resp., $\mathrm{Sph}_{\check{G}}^{\mathrm{spec}}$). Due to the above left-right passage, we can form the tensor products

$$\mathbf{C}_1 \underset{\mathrm{Sph}_G}{\otimes} \mathbf{C}_2 \text{ and } \mathbf{C}_1^{\mathrm{spec}} \underset{\mathrm{Sph}_{\check{G}}^{\mathrm{spec}}}{\otimes} \mathbf{C}_2^{\mathrm{spec}}.$$

Suppose that we have a given a functor

$$F_1 : \mathbf{C}_1 \to \mathbf{C}_1^{\mathrm{spec}},$$

which is compatible with the actions via

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_G}{\simeq} \mathrm{Sph}_{\check{G}}^{\mathrm{spec}} \tag{1.7}$$

and a functor

$$F_2 : \mathbf{C}_2 \to \mathbf{C}_2^{\mathrm{spec}},$$

which is compatible with the actions via

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_{G,\tau}}{\simeq} \mathrm{Sph}_{\check{G}}^{\mathrm{spec}}. \tag{1.8}$$

In this case, we obtain a functor

$$F_1 \otimes F_2 : \mathbf{C}_1 \underset{\mathrm{Sph}_G}{\otimes} \mathbf{C}_2 \simeq \mathbf{C}_1^{\mathrm{spec}} \underset{\mathrm{Sph}_{\check{G}}^{\mathrm{spec}}}{\otimes} \mathbf{C}_2^{\mathrm{spec}}. \tag{1.9}$$

1.8.8. *Warning.* Similarly, let $\mathbf{C}$ and $\mathbf{C}'$ are left module categories for Sph_G and $\mathrm{Sph}_{\check{G}}^{\mathrm{spec}}$, respectively. Let us view $\mathbf{C}^\vee$ (resp., $\mathbf{C}'^\vee$) again as a left module, using σ (resp., σ).

Let $\mathbf{C} \simeq \mathbf{C}'$ be an equivalence compatible with the actions via (1.7) Then the induced equivalence

$$\mathbf{C}^\vee \simeq \mathbf{C}'^\vee$$

is compatible with the actions via (1.8).

2. The local semi-infinite category: recollections

In this section we study categories, also of geometric nature, that ultimately allow one to connect representation-theoretic (or also geometric) categories associated with the group G and corresponding categories for its Levi subgroups.

The relevant category on the geometric side is the local semi-infinite category[9]

$$\mathrm{I}(G, P^-)^{\mathrm{loc}} = \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{-,\frac{\infty}{2}} := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{\mathfrak{L}(N(P^-))\cdot\mathfrak{L}^+(M)},$$

and on the spectral side

$$\mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}} := \mathrm{IndCoh}^*\left(\mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\right).$$

The main result of this section is the equivalence

$$\mathrm{I}(G, P^-)^{\mathrm{loc}} \simeq \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}},$$

given by Theorem 2.6.7.

As was mentioned in the preamble to the previous section, in order the make the theory work, we need to apply the critical twist and the ρ-shift on the geometric side. The former operation is closely linked to a cohomological shift embedded into the definition of the geometric Jacquet functors.[10]

The reader is advised to ignore these shifts and twists on the first pass (i.e., trust that all these shifts work out as they should).

2.1. **The corrected Jacquet functor.**

[9] Properly, we consider the *renormalized* version of this category, see Sect. **??**.

[10] These cohomological shifts are necessary also from other points of view, and one can reverse the logic and say that the critical (or half-) twist is necessary in order to incorporate these cohomological shifts, in order to stay consistent with the sign rules.

2.1.1. Let P^- be the (negative) standard parabolic of G with Levi quotient M. Consider the restrictions of the line bundles

$$\det_{\mathrm{Gr}_G} \text{ and } \det_{\mathrm{Gr}_M}$$

along the maps

$$\mathrm{Gr}_G \leftarrow \mathrm{Gr}_{P^-} \to \mathrm{Gr}_M, \tag{2.1}$$

respectively.

Denote their ratio by $\det_{\mathrm{Gr}_{G,M}}$; it naturally descends to Gr_M. By a slight abuse of notation, we will denote the resulting line bundle on Gr_M by the same symbol $\det_{\mathrm{Gr}_{G,M}}$.

2.1.2. We consider $\det_{\mathrm{Gr}_{G,M}}$ as an (evenly) graded line bundle on Gr_M, so that its portion over the connected component Gr_M^λ has grading

$$2\langle \lambda, 2\check{\rho}_P\rangle,$$

where $2\check{\rho}_P$ is the character of M equal to the determinant of its action on $\mathfrak{n}(P)$.

It was shown in [GL, Sect. 5.2] that $\det_{\mathrm{Gr}_{G,M}}$ admits a canonical square root,[11] to be denoted $\det^{\otimes\frac{1}{2}}_{\mathrm{Gr}_{G,M}}$, viewed as a $\mathbb{Z}$-graded and hence $\mathbb{Z}/2\mathbb{Z}$-graded (=super) factorization line bundle, so that its portion over the connected component Gr_M^λ has grading

$$\langle \lambda, 2\check{\rho}_P\rangle.$$

2.1.3. Ignoring the grading, the line bundle $\det^{\otimes\frac{1}{2}}_{\mathrm{Gr}_{G,M}}$ gives rise to an identification of the pullbacks of the gerbes

$$\det^{\frac{1}{2}}_{\mathrm{Gr}_G} \text{ and } \det^{\frac{1}{2}}_{\mathrm{Gr}_M}$$

to Gr_{P^-}.

Due to the above identification of gerbes, we have a well-defined functor

$$r^{\mathrm{nv}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_M),$$

given by !-pull and *-push along (2.1).

We will refer to r^{nv} as the "naive" Jacquet functor.

2.1.4. However, due to the fact that $\det^{\otimes\frac{1}{2}}_{\mathrm{Gr}_{G,M}}$ does not factorize as a line bundle, but only as a super line bundle, the above identification of gerbes is *incompatible* with factorization.

Hence, the functor r^{nv} is *not* compatible with factorization either.

2.1.5. We introduce the corrected Jacquet functor

$$r : \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_M),$$

as follows:

Over a connected component Gr_M^λ, we set

$$r(-) := r^{\mathrm{nv}}(-)[\langle \lambda, 2\check{\rho}_P\rangle].$$

The functor r is naturally compatible with the factorization structures, due to the sign rule.

2.2. **The local semi-infinite category.**

[11] Which depends on the choice of $\omega_X^{\otimes\frac{1}{2}}$.

2.2.1. We consider the *unrenormalized* factorization category

$$\mathrm{I}(G,P^-)^{\mathrm{loc,unr}} := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{-,\frac{\infty}{2}} := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{\mathfrak{L}(N(P^-))\cdot\mathfrak{L}^+(M)}.$$

We define the *renormalized* factorization category $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ by the same procedure as in Sect. 1.5.3, i.e., as the ind-completion of the subcategory of $\mathrm{I}(G,P^-)^{\mathrm{loc,unr}}$ consisting of those objects that become compact in $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{\mathfrak{L}(N(P^-))}$. We remark that this is exactly the form of the semi-infinite category considered in [CR].

Convolution equips $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ with a natural action of Sph_G.

2.2.2. The trivialization of the gerbe in Sect. 2.1.3 gives rise to an action of Sph_M on $\mathrm{I}(G,P^-)^{\mathrm{loc}}$. However, as in Sect. 2.1.4, this action is incompatible with factorization.

We introduce a corrected version of Sph_M-action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ by precomposing the one above with the automorphism of Sph_M that acts as the cohomological shift $[\langle\lambda,2\check{\rho}_P\rangle]$ on the connected component Gr_M^λ.

From now on, unless explicitly mentioned otherwise, we will only consider this corrected version of the Sph_M-action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}$.

The resulting Sph_M-action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ is compatible with factorization, and commutes with the Sph_G-action, making $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ into a $(\mathrm{Sph}_G,\mathrm{Sph}_M)$-bimodule.

2.2.3. By the same logic, pullback along $\mathrm{Gr}_{P^-}\to\mathrm{Gr}_G$, followed by the cohomological shift $[\langle\lambda,2\check{\rho}_P\rangle]$ on Gr_M^λ, defines a (factorization) functor

$$\mathrm{I}(G,P^-)^{\mathrm{loc,unr}}\to\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{P^-})^{\mathfrak{L}(N(P^-))\cdot\mathfrak{L}^+(M)}\simeq\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_M)^{\mathfrak{L}^+(M)}=\mathrm{Sph}_M^{\mathrm{unr}}, \tag{2.2}$$

that renormalizes to a conservative functor

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}\to\mathrm{Sph}_M \tag{2.3}$$

compatible with the Sph_M-actions.

We denote this functor by $\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}$.

2.2.4. It is shown in [Gai6, Proposition 1.5.3] (see also [Che2, Lemma 2.3.4]) that the functor (2.3) admits a (factorization) left adjoint, to be denoted $\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2}}$. Since the functor (2.3) is conservative, we obtain a monadic adjunction

$$\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2}}:\mathrm{Sph}_M\rightleftarrows\mathrm{I}(G,P^-)^{\mathrm{loc}}:\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}} \tag{2.4}$$

as Sph_M-module categories.

2.2.5. Let us consider the adjunction (2.4) as between *right* Sph_M-module categories. Due to the monadicity, there exists a canonically defined (factorization) associative algebra object

$$\widetilde{\Omega}\in\mathrm{Sph}_M$$

so that

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}\simeq\widetilde{\Omega}\text{-mod}(\mathrm{Sph}_M) \tag{2.5}$$

and the adjunction (2.4) identifies with

$$\mathbf{ind}_{\widetilde{\Omega}}:\mathrm{Sph}_M\rightleftarrows\widetilde{\Omega}\text{-mod}(\mathrm{Sph}_M):\mathbf{oblv}_{\widetilde{\Omega}}.$$

2.3. **A twisting procedure.**

2.3.1. Let $\mathcal{P}_M$ be an M-torsor on X. We can consider $\mathcal{P}_M$-twisted versions of all objects in sight, i.e.,

$$\mathrm{Gr}_{M,\mathcal{P}_M},\ \mathrm{Gr}_{G,\mathcal{P}_M},\ \mathfrak{L}(N(P^-))_{\mathcal{P}_M},$$

see Sect. 1.2.1.

2.3.2. We will denote by subscript $\mathcal{P}_M$ the categories associated with the corresponding twisted geometric objects, i.e.,

$$\mathrm{Sph}_{M,\mathcal{P}_M},\ \mathrm{Sph}_{G,\mathcal{P}_M},\ \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_M}.$$

In particular, we have a monadic adjunction

$$\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2}} : \mathrm{Sph}_{M,\mathcal{P}_M} \rightleftarrows \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_M} : \mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}},$$

and the corresponding associative (factorization) algebra object

$$\widetilde{\Omega}_{\mathcal{P}_M} \in \mathrm{Sph}_{M,\mathcal{P}_M}.$$

2.3.3. Note, however, that the local Hecke stacks for M (or G), i.e.,

$$\mathrm{Hecke}^{\mathrm{loc}}_M := \mathfrak{L}^+(M)\backslash\mathfrak{L}(M)/\mathfrak{L}^+(M) \text{ and } \mathrm{Hecke}^{\mathrm{loc}}_G := \mathfrak{L}^+(G)\backslash\mathfrak{L}(G)/\mathfrak{L}^+(G)$$

are canonically isomorphic to their twisted versions, see Sect. 1.2.2.

So, we have a canonical identification of monoidal (factorization) categories

$$\mathrm{Sph}_M \overset{\alpha_{\mathcal{P}_M,\mathrm{taut}}}{\simeq} \mathrm{Sph}_{M,\mathcal{P}_M} \text{ and } \mathrm{Sph}_G \overset{\alpha_{\mathcal{P}_M,\mathrm{taut}}}{\simeq} \mathrm{Sph}_{G,\mathcal{P}_M}.$$

Similarly, we have a canonical equivalence

$$\alpha_{\mathcal{P}_M,\mathrm{taut}} : \mathrm{I}(G,P^-)^{\mathrm{loc}} \simeq \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_M}. \tag{2.6}$$

We tautologically have:

$$\alpha_{\mathcal{P}_M,\mathrm{taut}}(\widetilde{\Omega}) \simeq \widetilde{\Omega}_{\mathcal{P}_M}.$$

2.3.4. Assume now that the M-torsor $\mathcal{P}_M$ is induced by a Z_M-torsor $\mathcal{P}_{Z_M}$. Recall (see Sect. 1.2.4) that in this case we have a *different* identification

$$\mathrm{Sph}_{M,\mathcal{P}_{Z_M}} \overset{\alpha_{\mathcal{P}_{Z_M},\mathrm{cent}}}{\simeq} \mathrm{Sph}_M$$

as monoidal (factorization) categories.

The composite

$$(\mathrm{transl}_{\mathcal{P}_{Z_M}})^* := \alpha_{\mathcal{P}_{Z_M},\mathrm{cent}} \circ \alpha_{\mathcal{P}_{Z_M},\mathrm{taut}} \tag{2.7}$$

is a monoidal (factorization) automorphism of Sph_M.

2.3.5. *Convention.* Henceforth, unless explicitly specified otherwise, when we consider $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}$ as acted on by Sph_M, we will do so using the identification $\alpha_{\mathcal{P}_{Z_M},\mathrm{cent}}$.

In particular, we will view $\widetilde{\Omega}_{\mathcal{P}_{Z_M}}$ as an associative (factorization) algebra object in Sph_M via $\alpha_{\mathcal{P}_{Z_M},\mathrm{cent}}$.

2.3.6. We can view the equivalence (2.6) as that between $(\mathrm{Sph}_G, \mathrm{Sph}_M)$-bimodule categories, where:

- Sph_G acts naturally on $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ and via $\alpha_{\mathcal{P}_{Z_M},\mathrm{taut}}$ on $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}$;
- Sph_M acts as naturally on $\mathrm{I}(G,P^-)^{\mathrm{loc}}$, and via $\alpha_{\mathcal{P}_{Z_M},\mathrm{cent}}$ on $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}$ (note that by convention both actions incorporate the shift from Sect. 2.2.2).

2.3.7. Tautologically, we have

$$\widetilde{\Omega}_{\mathcal{P}_{Z_M}} \simeq (\mathrm{transl}_{\mathcal{P}_{Z_M}})^*(\widetilde{\Omega}).$$

Remark 2.3.8. In practice we will take

$$\mathcal{P}_{Z_M} := \rho_P(\omega_X) := 2\rho_P(\omega_X^{\otimes\frac{1}{2}}).$$

2.4. **The spectral semi-infinite category.**

2.4.1. Denote by

$$\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}$$

the factorization space

$$\mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

The local spectral semi-infinite category is by definition

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} := \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}).$$

The category is $\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}$ is naturally a $(\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}, \mathrm{Sph}^{\mathrm{spec}}_{\check{M}})$-bimodule.

2.4.2. For future reference denote:

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{co}} := \mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}), \tag{2.8}$$

so that

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \text{ and } \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{co}}$$

are each other's duals, as factorization categories.

2.4.3. We have a correspondence

$$\begin{array}{ccc} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) & \longrightarrow & \mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \\ \mathfrak{q}\downarrow & & \\ \mathrm{LS}_{\check{M}}(\mathcal{D}) & & \end{array}$$

and its base change

$$\begin{array}{ccc} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}) & \longrightarrow & \mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-} \\ \mathfrak{q}\downarrow & & \\ \mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{M}} & & \end{array} \tag{2.9}$$

2.4.4. The functors of !-pull and *-push along (2.9) define a forgetful functor

$$\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}} : \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \to \mathrm{Sph}^{\mathrm{spec}}_{\check{M}},$$

which admits a left adjoint, denoted $\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2}}$, given by *-pull followed by *-push.

Thus, we obtain an adjoint pair

$$\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2}} : \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \rightleftarrows \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} : \mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}. \tag{2.10}$$

Moreover, it is easy to see that the adjunction (2.10) is monadic and respects the $\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$-actions.

2.4.5. As in Sect. 2.2.5, we can view the adjunction (2.10) as between *right* $\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$-module categories.

Let

$$\widetilde{\Omega}^{\mathrm{spec}} \in \mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$$

denote the corresponding associative (factorization) algebra object so that (2.10) identifies with the adjunction

$$\mathbf{ind}_{\widetilde{\Omega}^{\mathrm{spec}}} : \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \rightleftarrows \widetilde{\Omega}^{\mathrm{spec}}\text{-mod} : \mathbf{oblv}_{\widetilde{\Omega}^{\mathrm{spec}}}. \tag{2.11}$$

2.5. **The (factorization) algebra Ω^{spec}.**

2.5.1. Note that the adjunction

$$\mathfrak{q}^* : \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(\mathcal{D})) \rightleftarrows \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(\mathcal{D})) : \mathfrak{q}_*$$

identifies with

$$\mathrm{Res}^{\check{M}}_{\check{P}^-} : \mathrm{Rep}(\check{M}) \rightleftarrows \mathrm{Rep}(\check{P}^-) : \mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-), -), \tag{2.12}$$

where $\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-), -)$ is the functor of $\mathfrak{n}(\check{P}^-)$-invariants (a.k.a. cohomological Chevalley complex).

2.5.2. Let

$$\Omega^{\mathrm{spec}} \in \mathrm{Rep}(\check{M})$$

denote the (commutative) factorization algebra equal to

$$\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-)),$$

i.e., the cohomological Chevalley complex with coefficients in the trivial module.

The (monadic) adjunction (2.12) can therefore be rewritten as

$$\mathbf{ind}_{\Omega^{\mathrm{spec}}} : \mathrm{Rep}(\check{M}) \rightleftarrows \Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Rep}(\check{M})) : \mathbf{oblv}_{\Omega^{\mathrm{spec}}}. \tag{2.13}$$

2.5.3. By a slight abuse of notation, we will denote by the same symbol Ω^{spec} its image along the functor

$$\mathrm{Rep}(\check{M}) \overset{\mathrm{ny}}{\to} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}}.$$

From diagram (2.9) we obtain that the adjunction (2.11) factors as a composition of

$$\mathbf{ind}_{\Omega^{\mathrm{spec}}} : \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \rightleftarrows \Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}) : \mathbf{oblv}_{\Omega^{\mathrm{spec}}}. \tag{2.14}$$

and

$$\mathbf{ind}_{\Omega^{\mathrm{spec}} \to \widetilde{\Omega}^{\mathrm{spec}}} : \Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}) \rightleftarrows \widetilde{\Omega}^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}) : \mathbf{oblv}_{\widetilde{\Omega}^{\mathrm{spec}} \to \Omega^{\mathrm{spec}}}. \tag{2.15}$$

2.6. **Semi-infinite geometric Satake.** We now come to the central point of this section.

2.6.1. Consider the functor

$$\begin{aligned} &\mathrm{D\text{-}mod}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Gr}_G)^{\mathfrak{L}(N(P^-))\cdot\mathfrak{L}^+(M)} \simeq \\ &\simeq \mathrm{D\text{-}mod}(\mathrm{Gr}_G) \otimes (\mathrm{D\text{-}mod}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Gr}_M))^{\mathfrak{L}(P^-)} \to \\ &\to \mathrm{D\text{-}mod}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Gr}_M) \overset{(\overset{!}{\otimes})\otimes(\mathrm{Id})}{\longrightarrow} \mathrm{D\text{-}mod}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}(\mathrm{Gr}_M) \overset{\Gamma_{\mathrm{dR}}(\mathrm{Gr}_{G,\check{1}}, -)\otimes\mathrm{Id}}{\longrightarrow} \\ &\to \mathrm{D\text{-}mod}(\mathrm{Gr}_M). \end{aligned} \tag{2.16}$$

This functor is equivariant with respect to the $\mathfrak{L}(P^-)$-actions on $\mathrm{D\text{-}mod}(\mathrm{Gr}_G)$ (via $\mathfrak{L}(P^-) \to \mathfrak{L}(G)$) and on $\mathrm{D\text{-}mod}(\mathrm{Gr}_M)$ (via $\mathfrak{L}(P^-) \to \mathfrak{L}(M)$), respectively.

2.6.2. Consider a variant of (2.16), given by

$$\begin{aligned} &\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \otimes \mathrm{I}(G, P^-)^{\mathrm{loc}} \simeq \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{\mathfrak{L}(N(P^-))\cdot\mathfrak{L}^+(M)} \simeq \\ &\simeq \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \otimes \Big(\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_M)\Big)^{\mathfrak{L}(P^-)} \to \\ &\to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_M) \overset{(\overset{!}{\otimes})\otimes(\mathrm{Id})}{\longrightarrow} \\ &\to \mathrm{D\text{-}mod}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_M) \overset{\Gamma_{\mathrm{dR}}(\mathrm{Gr}_{G,\check{1}}, -)\otimes\mathrm{Id}}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_M) \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_M), \end{aligned} \tag{2.17}$$

where:

- The category $\Big(\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_M)\Big)^{\mathfrak{L}(P^-)}$ makes sense due to the identification of the two multiplicative $\mathbb{Z}/2\mathbb{Z}$-gerbes on $\mathfrak{L}(P^-)$ (one obtained by restriction from $\mathfrak{L}(G)$, and another from $\mathfrak{L}(M)$) that results from the existence of the square root $\det^{\otimes\frac{1}{2}}_{\mathrm{Gr}_{G,M}}$;

- The last arrow is the cohomological shift by $\langle \lambda, 2\check{\rho}_P \rangle$ on Gr_M^λ.

As before, the functor (2.17) has a natural factorization functor, and is compatible with the right Sph_M-actions.

The functor (2.17) has an equivariance property with respect to $\mathfrak{L}(P^-)$, similar to that of (2.16).

In addition, the (2.17) is compatible with the Sph_M-action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ (see Sect. 2.2.2) and the natural action of Sph_M on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_M)$.

2.6.3. Given an M-bundle $\mathcal{P}_M$ on X, the functor (2.17) admits a twisted version

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\mathcal{P}_M}) \otimes \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_M} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{M,\mathcal{P}_M}). \tag{2.18}$$

Let us now be given an M-bundle $\mathcal{P}'_M$ and a Z_M-bundle $\mathcal{P}''_{Z_M}$ on X, so that

$$\mathcal{P}_M \simeq \mathcal{P}'_M \otimes \mathcal{P}''_{Z_M}.$$

We have the following version of (2.18):

$$\begin{aligned} \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\mathcal{P}_M}) \otimes \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}''_{Z_M}} &\overset{\mathrm{Id}\otimes\alpha_{\mathcal{P}'_M},\mathrm{taut}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\mathcal{P}_M}) \otimes \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_M} \overset{(2.18)}{\longrightarrow} \\ &\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{M,\mathcal{P}_M}) \overset{\alpha_{\mathcal{P}''_{Z_M}},\mathrm{cent}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{M,\mathcal{P}'_M}). \end{aligned} \tag{2.19}$$

The functor (2.19) is equivariant with respect to the natural action of $\mathfrak{L}(P^-)_{\mathcal{P}_M}$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\mathcal{P}_M})$ and the action of

$$\mathfrak{L}(P^-)_{\mathcal{P}_M} \twoheadrightarrow \mathfrak{L}(M)_{\mathcal{P}_M} \simeq \mathfrak{L}(M)_{\mathcal{P}'_M}$$

on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{M,\mathcal{P}'_M})$.

The functor (2.19) is compatible with the Sph_M-action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}''_{Z_M}}$ from Sect. 2.3.5, and the action of Sph_M on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{M,\mathcal{P}'_M})$ obtained from the identification

$$\mathrm{Sph}_M \overset{\alpha_{\mathcal{P}'_M},\mathrm{taut}}{\simeq} \mathrm{Sph}_{M,\mathcal{P}'_M},$$

(see Sect. 1.2.2).

2.6.4. We take

$$\mathcal{P}'_M := \rho_M(\omega_X),\ \mathcal{P}'_{Z_M} := \rho_P(\omega_X),$$

so that

$$\mathcal{P}_M = \rho(\omega_X).$$

Consider the resulting functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}) \otimes \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{M,\rho_M(\omega_X)}). \tag{2.20}$$

The equivariance property of (2.20) with respect to $\mathfrak{L}(N(M)) \subset \mathfrak{L}(P^-)$ implies that the functor (2.20) maps

$$\mathrm{Whit}^!(G) \otimes \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \to \mathrm{Whit}^!(M). \tag{2.21}$$

2.6.5. We now consider the functor

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}} \to \mathrm{Rep}(\check{M}), \tag{2.22}$$

defined by *-pull, $\overset{*}{\otimes}$, and push along the diagram

$$\begin{array}{ccccc} \mathrm{LS}_{\check{G}}(\mathcal{D}) & \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} & \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) & \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} & \mathrm{LS}_{\check{M}}(\mathcal{D}) \\ & \swarrow & & \searrow & \\ \mathrm{LS}_{\check{G}}(\mathcal{D}) & & & & \mathrm{LS}_{\check{M}}(\mathcal{D}). \end{array}$$

2.6.6. We are now ready to state the semi-infinite geometric Satake theorem, see [CR, Theorem 6.12.3]:

Theorem 2.6.7. *There exists a unique functor (of factorization categories)*

$$\mathrm{Sat}^{-,\frac{\infty}{2}} : \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \to \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}}$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{Whit}^!(G) \otimes \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} & \xrightarrow{(\mathrm{CS}_G \circ \tau_G) \otimes \mathrm{Sat}^{-,\frac{\infty}{2}}} & \mathrm{Rep}(\check{G}) \otimes \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}} \\ \downarrow{\scriptstyle (2.21)} & & \downarrow{\scriptstyle (2.22)} \\ \mathrm{Whit}^!(M) & \xrightarrow{\mathrm{CS}_M \circ \tau_M} & \mathrm{Rep}(\check{M}). \end{array} \tag{2.23}$$

Moreover, the functor $\mathrm{Sat}^{-,\frac{\infty}{2}}$ *is an equivalence (of factorization categories).*

Remark 2.6.8. We remind the reader here of Remark 1.7.4, and our promise to correct the discrepancy between our statement and the results of [CR].

2.6.9. The uniqueness statement in Theorem 2.6.7 implies that the functor $\mathrm{Sat}^{-,\frac{\infty}{2}}$ is compatible with the actions of

$$\mathrm{Sph}_M \overset{\mathrm{Sat}_{M,\tau}}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$$

on the two sides.

Acting on the factorization units (see Sects. 2.7.1 and 2.7.4) below, we obtain that the equivalence $\mathrm{Sat}^{-,\frac{\infty}{2}}$ makes the diagrams

$$\begin{array}{ccc} \mathrm{Sph}_M & \xrightarrow{\mathrm{Sat}_{M,\tau}} & \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \\ {\scriptstyle \mathbf{ind}_{\mathrm{Sph} \to \frac{\infty}{2}}}\downarrow & & \downarrow{\scriptstyle \mathbf{ind}_{\mathrm{Sph} \to \frac{\infty}{2}}} \\ \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} & \xrightarrow{\mathrm{Sat}^{-,\frac{\infty}{2}}} & \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}} \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Sph}_M & \xrightarrow{\mathrm{Sat}_{M,\tau}} & \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \\ {\scriptstyle \mathbf{oblv}_{\frac{\infty}{2} \to \mathrm{Sph}}}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}_{\frac{\infty}{2} \to \mathrm{Sph}}} \\ \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} & \xrightarrow{\mathrm{Sat}^{-,\frac{\infty}{2}}} & \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}} \end{array}$$

commute.

Hence, we can view the equivalence $\mathrm{Sat}^{-,\frac{\infty}{2}}$ as the statement that we have an isomorphism of associative (factorization) algebra objects

$$\mathrm{Sat}_M(\widetilde{\Omega}_{\rho_P(\omega_X)}) \simeq \widetilde{\Omega}^{\mathrm{spec}}, \tag{2.24}$$

(see Sect. 1.8 for why we have here Sat_M and not $\mathrm{Sat}_{M,\tau}$).

2.6.10. Note that the functor (2.21) naturally factors as

$$\mathrm{Whit}^!(G)\otimes \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \to \mathrm{Whit}^!(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \to \mathrm{Whit}^!(M) \tag{2.25}$$

and the functor (2.22) naturally factors as

$$\mathrm{Rep}(\check{G})\otimes \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \to \mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \to \mathrm{Rep}(\check{M}). \tag{2.26}$$

The uniqueness statement in Theorem 2.6.7 (combined with (1.9)) implies that the functor $\mathrm{Sat}^{-,\frac{\infty}{2}}$ is compatible with the actions of

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_G}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$$

on the two sides, so that the diagram (2.23) factors through a commutative diagram

$$\begin{array}{ccc} \mathrm{Whit}^!(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} & \xrightarrow{(\mathrm{CS}_G\circ\tau_G)\otimes \mathrm{Sat}^{-,\frac{\infty}{2}}} & \mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \\ \downarrow & & \downarrow \\ \mathrm{Whit}^!(M) & \xrightarrow{\mathrm{CS}_M\circ\tau_M} & \mathrm{Rep}(\check{M}) \end{array} \tag{2.27}$$

commutes.

2.7. **The Jacquet functor.**

2.7.1. Let

$$\Delta^{-,\frac{\infty}{2}} \in \mathbf{1}_{\mathrm{I}(G,P^-)^{\mathrm{loc}}} \in \mathrm{I}(G,P^-)^{\mathrm{loc}}$$

be the factorization unit.

It equals the image of the factorization unit along

$$\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2}} : \mathrm{Sph}_M \to \mathrm{I}(G,P^-)^{\mathrm{loc}}.$$

Explicitly, $\Delta^{-,\frac{\infty}{2}}$ is the image of

$$\delta_{1,\mathrm{Gr}_M} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_M)^{\mathfrak{L}^+(M)} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{M,\rho_P(\omega_X)})^{\mathfrak{L}^+(M)}$$

by !-pull and !-push along the diagram

$$\mathrm{Gr}_M \leftarrow \mathrm{Gr}_{P^-} \to \mathrm{Gr}_G.$$

2.7.2. Here is another way to describe the object $\Delta^{-,\frac{\infty}{2}}$.

Note that the category $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ is related to Sph_G by a pair of adjoint functors

$$\mathrm{Av}^{\mathfrak{L}(N(P^-))}_! : \mathrm{Sph}_G \rightleftarrows \mathrm{I}(G,P^-)^{\mathrm{loc}} : \mathrm{Av}^{\mathfrak{L}^+(G)/\mathfrak{L}^+(M)}_* .$$

Then

$$\Delta^{-,\frac{\infty}{2}} \simeq \mathrm{Av}^{\frac{\infty}{2}/\,\mathrm{Sph}}_!(\delta_{1,\mathrm{Gr}_G}), \tag{2.28}$$

where $\delta_{1,\mathrm{Gr}_G} \in \mathrm{Sph}_G$ is the unit.

2.7.3. A similar discussion applies in the $\rho_P(\omega_X)$-twisted context. We will denote the corresponding object by the same symbol

$$\Delta^{-,\frac{\infty}{2}} \in \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)}.$$

Let $J^{-,!}$ denote the (factorization) functor

$$J^{-,!} : \mathrm{Whit}^!(G) \to \mathrm{Whit}^!(M)$$

given by applying (2.21) to $\Delta^{-,\frac{\infty}{2}}$ along the second factor.

2.7.4. Let

$$\mathbf{1}_{\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}} \in \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}$$

be the factorization unit.

It equals the image of the factorization unit along

$$\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2}} : \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \to \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}.$$

Explicitly, it is the image of the trivial representation along

$$\mathrm{Rep}(\check{P}^-) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}).$$

2.7.5. Note that the (factorization) functor

$$\mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{M})$$

obtained by applying (2.22) to $\mathbf{1}_{\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}}$ along the second factor is just

$$\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-), -).$$

2.7.6. From Theorem 2.6.7 we obtain:

Corollary 2.7.7. *The following diagram of factorization categories and functors commutes:*

$$\begin{array}{ccc} \mathrm{Whit}^!(G) & \xrightarrow{\mathrm{CS}_G \circ \tau_G} & \mathrm{Rep}(\check{G}) \\ {\scriptstyle J^{-,!}}\downarrow & & \downarrow{\scriptstyle \mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-),-)} \\ \mathrm{Whit}^!(M) & \xrightarrow{\mathrm{CS}_M \circ \tau_M} & \mathrm{Rep}(\check{M}). \end{array} \tag{2.29}$$

2.7.8. For future reference, we introduce the functor

$$J^{-,*} : \mathrm{Whit}_*(G) \to \mathrm{Whit}_*(M) \tag{2.30}$$

so that the diagram

$$\begin{array}{ccc} \mathrm{Whit}_*(G) & \xrightarrow{J^{-,*}} & \mathrm{Whit}_*(M) \\ {\scriptstyle \Theta_{\mathrm{Whit}(G)}}\downarrow & & \downarrow{\scriptstyle \Theta_{\mathrm{Whit}(M)}} \\ \mathrm{Whit}^!(G) & \xrightarrow{J^{-,!}} & \mathrm{Whit}^!(M) \end{array}$$

commutes.

Taking into account Lemma 1.4.11, from (2.29) we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Whit}^*(G) & \xleftarrow{\mathrm{FLE}_{\check{G},\infty}} & \mathrm{Rep}(\check{G}) \\ {\scriptstyle J^{-,*}}\downarrow & & \downarrow{\scriptstyle \mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-),-)} \\ \mathrm{Whit}^*(M) & \xleftarrow{\mathrm{FLE}_{\check{M},\infty}} & \mathrm{Rep}(\check{M}). \end{array} \tag{2.31}$$

3. The local semi-infinite category: dualities

This section can be skipped on the first pass, and returned to when necessary.

We will study a self-duality property of the category $\mathrm{I}(G,P^-)^{\mathrm{loc}}$. This property will play a role in the proof of the main result of this paper, Theorem 24.1.2.

Unfortunately, an analog of this self-duality result on the spectral side has not been established yet (see Conjecture 25.4.6). Had it been, it would have made the proof of Theorem 24.1.2 more streamlined (see Sect. 25.4).

Instead, we introduce categories

$$\Omega\text{-mod}(\mathrm{Sph}_M) \text{ and } \Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}})$$

that "sit between"

$$(\mathrm{Sph}_M \text{ and } \mathrm{I}(G,P^-)^{\mathrm{loc}}) \text{ and } (\mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \text{ and } \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{loc,spec}}),$$

respectively.

There exists an equivalence

$$\Omega\text{-mod}(\mathrm{Sph}_M) \simeq \Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}})$$

and (easy-to-establish) self-dualities on Ω-mod(Sph_M) and Ω^{spec}-mod$(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}})$, respectively, compatible with the above equivalence. This would suffice for the proof of Theorem 24.1.2.

3.1. **The dual of the semi-infinite category.**

3.1.1. Consider the category

$$\text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}^+(M)}_{\mathfrak{L}(N(P^-))}.$$

It is proved in [Che2, Corollary 1.4.5] that it is dualizable as a factorization category. Once the dualizability is established, it follows formally that Verdier duality on Gr_G gives rise to an identification

$$(\text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}(N(P^-))\cdot\mathfrak{L}^+(M)})^\vee \simeq \text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}^+(M)}_{\mathfrak{L}(N(P^-))}.$$

3.1.2. Denote

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co}} := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(M)}_{\mathfrak{L}(N(P^-))}.$$

By similar logic, we have a canonical identification

$$\left(\mathrm{I}(G,P^-)^{\mathrm{loc}}\right)^\vee \simeq \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co}}. \tag{3.1}$$

3.1.3. Note that the identification (3.1) is compatible with the Sph_M-actions in the following sense:

- The action on $\left(\mathrm{I}(G,P^-)^{\mathrm{loc}}\right)^\vee$ is induced by the Sph_M-action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ specified in Sect. 2.2.2 (as always, we pass from right to left Sph_M-modules using σ).
- The action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co}}$ is obtained from the natural geometric action, by applying the inverse cohomological shift to the one from Sect. 2.2.2.

Since Sph_M is rigid, the duality (3.1) realizes $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co}}$ as a Sph_M-module dual of $\mathrm{I}(G,P^-)^{\mathrm{loc}}$.

3.1.4. Direct image along $\mathrm{Gr}_{P^-} \to \mathrm{Gr}_G$ defines a functor

$$\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2},\mathrm{co}} : \mathrm{Sph}_M \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co}}, \tag{3.2}$$

which, with respect to the equivalence (3.1), identifies with the dual of the functor

$$\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}} : \mathrm{I}(G,P^-)^{\mathrm{loc}} \to \mathrm{Sph}_M$$

of (2.3).

In particular, we obtain that the functor (3.2) admits a right adjoint, so that we have a monadic adjunction

$$\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2},\mathrm{co}} : \mathrm{Sph}_M \rightleftarrows \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co}} : \mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph,co}}. \tag{3.3}$$

3.1.5. Let $\mathbf{A}$ be a monoidal category and $A \in \mathbf{A}$ an associative algebra. We consider A-mod$(\mathbf{A})$ as a right $\mathbf{A}$-module category, and A-mod$^r(\mathbf{A})$ as a left $\mathbf{A}$-module category.

Tautologically, A-mod$(\mathbf{A})$ and A-mod$^r(\mathbf{A})$ are each other's duals (as right and left $\mathbf{A}$-module categories, respectively).

Note also that

$$A\text{-mod}^r(\mathbf{A}) \simeq A^o\text{-mod}(\mathbf{A}^o),$$

where:

- For a monoidal categoty $\mathbf{A}$, we denoted by $\mathbf{A}^o$ denotes the monoidal category category obtained by reversing the monoidal operation;
- For an associative algebra A in a monoidal category $\mathbf{A}$, we denote by A^o the corresponding associative algebra in $\mathbf{A}^o$.

3.1.6. Let $\widetilde{\Omega}_{\mathrm{co}}$ be the associative (factorization) algebra object in Sph_M, so that (3.3) identifies with

$$\mathbf{ind}_{\widetilde{\Omega}_{\mathrm{co}}} : \mathrm{Sph}_M \rightleftarrows \widetilde{\Omega}_{\mathrm{co}}\text{-mod} : \mathbf{oblv}_{\widetilde{\Omega}_{\mathrm{co}}}.$$

From the duality between

$$\mathrm{I}(G, P^-)^{\mathrm{loc}} \text{ and } \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathrm{co}}$$

as Sph_M-module categories, we obtain an identification

$$\widetilde{\Omega}_{\mathrm{co}} \simeq \sigma((\widetilde{\Omega})^o).$$

3.1.7. Let $\mathcal{P}_{Z_M}$ be a Z_M-bundle on X. Performing the twist as in Sect. 2.3, we can consider the category $\mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathrm{co},\mathcal{P}_{Z_M}}$.

We still have an equivalence

$$\left(\mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}\right)^\vee \simeq \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathrm{co},\mathcal{P}_{Z_M}}, \tag{3.4}$$

compatible with Sph_M-actions.

Similarly, we can consider the associative (factorization) algebra object

$$\widetilde{\Omega}_{\mathrm{co},\mathcal{P}_{Z_M}} \in \mathrm{Sph}_M,$$

and we have

$$\widetilde{\Omega}_{\mathrm{co},\mathcal{P}_{Z_M}} \simeq \sigma((\widetilde{\Omega}_{\mathcal{P}_{Z_M}})^o).$$

3.1.8. We will apply the contents of Sect. 3.1.7 to $\mathcal{P}_{Z_M} := \rho_P(\omega_X)$ and also $\mathcal{P}_{Z_M} := -\rho_P(\omega_X)$.

3.2. **The duality on the geometric side.**

3.2.1. The starting point is the following key result of [Che2, Corollary 1.3.13]:

Theorem 3.2.2. *The composite functor*

$$\text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}(N(P))\cdot\mathfrak{L}^+(M)} \hookrightarrow \text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}^+(M)} \twoheadrightarrow \text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}^+(M)}_{\mathfrak{L}(N(P^-))}$$

is an equivalence (as factorization categories).

3.2.3. From Theorem 3.2.2 we formally obtain:

Corollary 3.2.4. *The functor*

$$\mathrm{I}(G, P)^{\mathrm{loc}} := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{\mathfrak{L}(N(P))\cdot\mathfrak{L}^+(M)} \hookrightarrow \tag{3.5}$$

$$\hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(M)} \twoheadrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(M)}_{\mathfrak{L}(N(P^-))} = \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathrm{co}}$$

is an equivalence (as factorization categories).

In what follows we will denote by Υ^{loc} the functor inverse to the equivalence of (3.5).

3.2.5. Applying the Cartan involution on G (normalized so that it swaps P and P^-, and thus compatible with the Cartan involution τ_M on M, see Sect. 1.4.9), we obtain an equivalence

$$\mathrm{I}(G, P)^{\mathrm{loc}} \overset{\tau_G}{\simeq} \mathrm{I}(G, P^-)^{\mathrm{loc}}.$$

Thus, composing (3.5) with (3.1) and τ_G, we obtain a self-duality

$$(\mathrm{I}(G, P^-)^{\mathrm{loc}})^\vee \simeq \mathrm{I}(G, P^-)^{\mathrm{loc}}. \tag{3.6}$$

By construction, the equivalence (3.6) is compatible with the actions of Sph_M, where:

- The action on $(\mathrm{I}(G, P^-)^{\mathrm{loc}})^\vee$ is the one specified in Sect. 3.1.3;
- The action on $\mathrm{I}(G, P^-)^{\mathrm{loc}}$ is precomposition of the action specified in Sect. 2.2.2 with the automorphism τ_M of Sph_M.

3.2.6. As in Sect. 3.1.6, from the equivalence (3.6) we obtain an isomorphism

$$\tau_M(\widetilde{\Omega}) \simeq \sigma((\widetilde{\Omega})^o), \tag{3.7}$$

as associative (factorization) algebras in Sph_M.

3.2.7. We now consider the twisted version of the above situation. Let $\mathcal{P}_{Z_M}$ be a Z_M-bundle on X. We still have an equivalence

$$\mathrm{I}(G,P)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}} \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\mathcal{P}_{Z_M}}. \tag{3.8}$$

However, the Cartan involution induces an equivalence

$$\mathrm{I}(G,P)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}} \overset{\tau_G}{\simeq} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\tau_M(\mathcal{P}_{Z_M})}.$$

One can compose it with the equivalence

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\tau_M(\mathcal{P}_{Z_M})} \overset{(\alpha_{\mathcal{P}_{Z_M},\mathrm{taut}})\circ(\alpha_{\tau_M(\mathcal{P}_{Z_M}),\mathrm{taut}})^{-1}}{\simeq} \mathrm{I}(G,P)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}},$$

and thus obtain again a self-duality

$$(\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}})^\vee \simeq \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}. \tag{3.9}$$

Note that the equivalence (3.9) is compatible with the Sph_M-actions, where

- The action on $\left(\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}\right)^\vee$ is induced by the Sph_M-action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}$ specified in Sect. 2.3.5 (as always, we pass from right to left Sph_M-modules using σ);
- The action on $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}$ is precomposition of the action specified in Sect. 2.2.2 with the automorphisms τ_M and $(\mathrm{transl}_{\mathcal{P}_{Z_M}\otimes\tau(\mathcal{P}^{\otimes -1}_{Z_M})})_*$ of Sph_M.

The equivalence (3.9) translates into the following isomorphism of associative (factorization) algebras:

$$\tau_M(\widetilde{\Omega}_{\mathcal{P}_{Z_M}}) \simeq (\mathrm{transl}_{\mathcal{P}^{\otimes -1}_{Z_M}\otimes\tau(\mathcal{P}_{Z_M})})^*\left(\sigma((\widetilde{\Omega}_{\mathcal{P}_{Z_M}})^o)\right). \tag{3.10}$$

Note, however, that (3.10) can be equivalently obtained by applying $(\mathrm{transl}_{\mathcal{P}^{\otimes -1}_{Z_M}})^*$ to (3.7).

3.2.8. We will apply the paradigm of Sect. 3.2.7 in the case $\mathcal{P}_{Z_M} = \rho_P(\omega_X)$. Thus, we have

$$(\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)})^\vee \simeq \mathrm{I}(G,P)^{\mathrm{loc}}_{\rho_P(\omega_X)}. \tag{3.11}$$

Note that

$$\tau(\rho_P(\omega_X)) = -\rho_P(\omega_X),$$

so (3.10) amounts to

$$\tau_M(\widetilde{\Omega}_{\rho_P(\omega_X)}) \simeq (\mathrm{transl}_{-2\rho_P(\omega_X)})_*\left(\sigma((\widetilde{\Omega}_{\rho_P(\omega_X)})^o)\right). \tag{3.12}$$

3.3. **The algebra Ω.**

3.3.1. Recall the commutative (factorization) algebra

$$\Omega^{\mathrm{spec}} \in \mathrm{Rep}(\check{M}) \overset{\mathrm{ny}}{\to} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}},$$

see Sect. 2.5.2.

Let $\Omega \in \mathrm{Sph}_M$ be the image of Ω^{spec} under Sat^{-1}_M. Recall that we have an identification of associative (factorization) algebras

$$\widetilde{\Omega}_{\rho_P(\omega_X)} \simeq \mathrm{Sat}^{-1}_M(\widetilde{\Omega}^{\mathrm{spec}}),$$

see (2.24).

Hence, the homomorphism

$$\Omega^{\mathrm{spec}} \to \widetilde{\Omega}^{\mathrm{spec}}$$

gives rise to a homomorphism of associative (factorization) algebras

$$\Omega \to \widetilde{\Omega}_{\rho_P(\omega_X)}. \tag{3.13}$$

3.3.2. Let $\mathcal{P}_{Z_M}$ be a Z_M-bundle on X. Note that for any such $\mathcal{P}_{Z_M}$, the diagram

$$\begin{array}{ccccc} \mathrm{Rep}(\check{M}) & \xrightarrow{\mathrm{nv}} & \mathrm{Sph}_{\check{M}}^{\mathrm{spec}} & \xrightarrow{\mathrm{Sat}_M^{-1}} & \mathrm{Sph}_M \\ \downarrow{\scriptstyle \mathrm{Id}} & & & & \downarrow{\scriptstyle (\mathrm{transl}_{\mathcal{P}_{Z_M}})_*} \\ \mathrm{Rep}(\check{M}) & \xrightarrow{\mathrm{nv}} & \mathrm{Sph}_{\check{M}}^{\mathrm{spec}} & \xrightarrow{\mathrm{Sat}_M^{-1}} & \mathrm{Sph}_M \end{array}$$

commutes. Hence, we obtain

$$(\mathrm{transl}_{\mathcal{P}_{Z_M}})_*(\Omega) \simeq \Omega. \tag{3.14}$$

Hence, (3.13) gives rise to a homomorphism

$$\Omega \to \widetilde{\Omega}_{\mathcal{P}_{Z_M}} \tag{3.15}$$

for any $\mathcal{P}_{Z_M}$.

Remark 3.3.3. The maps (3.15) can be constructed explicitly without reference to Sat_M.

Indeed, in the factorization picture over the part of Sph_M supported over

$$\Lambda_{G,P}^{\mathrm{pos}} \subset \Lambda_{G,P} = \pi_0(\mathrm{Gr}_M),$$

the sheaf Ω is exactly the 0-th perverse cohomology of $\widetilde{\Omega}$.

Remark 3.3.4. Another property of the pair $\Omega \to \widetilde{\Omega}$ that can be used to recover it is the following:

Note that $\Omega \in \mathrm{Sph}_M$ is augmented as an algebra object, so that the functor

$$- \underset{\Omega}{\otimes} \mathbf{1}_{\mathrm{Sph}_M} : \widetilde{\Omega}\text{-mod}^r(\mathbf{C}) \to \mathbf{C}$$

makes sense for any category $\mathbf{C} \in \mathrm{Sph}_M\text{-mod}^r$.

By construction, the functor

$$\mathbf{ind}_{\mathrm{Sph} \to \frac{\infty}{2}} : \mathrm{Sph}_M \to \mathrm{I}(G, P^-)^{\mathrm{loc}}$$

upgrades to a functor

$$\mathbf{ind}^{\mathrm{enh}}_{\mathrm{Sph} \to \frac{\infty}{2}} : \mathrm{Sph}_M \to \widetilde{\Omega}\text{-mod}^r(\mathrm{I}(G, P^-)^{\mathrm{loc}}).$$

In fact, $\mathbf{ind}^{\mathrm{enh}}_{\mathrm{Sph} \to \frac{\infty}{2}}$ corresponds to the tautological functor

$$\mathbf{ind}^{\mathrm{enh}}_{\widetilde{\Omega}} : \mathrm{Sph}_M \to \widetilde{\Omega}\text{-mod}^r(\widetilde{\Omega}\text{-mod}(\mathrm{Sph}_M))$$

in terms of the equivalence

$$\widetilde{\Omega}\text{-mod}(\mathrm{Sph}_M) \simeq \mathrm{I}(G, P^-)^{\mathrm{loc}}.$$

In particular, the object

$$\mathbf{1}_{\mathrm{I}(G,P^-)^{\mathrm{loc}}} \in \mathrm{I}(G, P^-)^{\mathrm{loc}}$$

naturally upgrades to an object of[12]

$$\widetilde{\Omega}\text{-mod}^r(\mathrm{I}(G, P^-)^{\mathrm{loc}}).$$

Then, by the construction explained in [Gai3], the object

$$\mathbf{1}_{\mathrm{I}(G,P^-)^{\mathrm{loc}}} \underset{\Omega}{\otimes} \mathbf{1}_{\mathrm{Sph}_M} \in \mathrm{I}(G, P^-)^{\mathrm{loc}}$$

identifies with the *semi-infinite IC sheaf*

$$\mathrm{IC}^{-, \frac{\infty}{2}} \in \mathrm{I}(G, P^-)^{\mathrm{loc}},$$

[12]In fact, the associative algebra object $\widetilde{\Omega} \in \mathrm{Sph}_M$ can be recovered as the algebra of endomorphisms of $\mathbf{1}_{\mathrm{I}(G,P^-)^{\mathrm{loc}}} \in \mathrm{I}(G, P^-)^{\mathrm{loc}}$ relative to the action of Sph_M.

introduced in [Gai3].

3.4. **The algebra Ω and duality.**

3.4.1. Note that by Corollary 1.8.2 and (1.5), we have

$$\tau_M(\Omega) \simeq \sigma(\Omega^o). \tag{3.16}$$

The following is a basic property of the homomorphism (3.15):

Lemma 3.4.2. *The following diagram of associative (factorization) algebras in* Sph_M *commutes:*

$$\begin{array}{ccc} \tau_M(\Omega) & \xrightarrow{(3.16)} & \sigma(\Omega^o) \\ {\scriptstyle \tau_M(3.15)}\downarrow & & \downarrow{\scriptstyle \sigma(3.15)^o} \\ \tau_M(\widetilde{\Omega}) & \xrightarrow{(3.7)} & \sigma((\widetilde{\Omega})^o). \end{array}$$

3.4.3. As a formal consequence, we obtain that for any Z_M-torsor $\mathcal{P}_{Z_M}$, the diagram

$$\begin{array}{ccccc} \tau_M(\Omega) & \xrightarrow{(3.16)} & \sigma(\Omega^o) & \xrightarrow{(3.14)} & (\mathrm{transl}_{\mathcal{P}_{Z_M}^{\otimes -1}\otimes \tau(\mathcal{P}_{Z_M})})_*\left(\sigma(\Omega^o)\right) \\ {\scriptstyle \tau_M(3.15)}\downarrow & & & & \downarrow{\scriptstyle (\mathrm{transl}_{\mathcal{P}_{Z_M}^{\otimes -1}\otimes \tau(\mathcal{P}_{Z_M})})_*\circ \sigma(3.15)^o} \\ \tau_M(\widetilde{\Omega}_{\mathcal{P}_{Z_M}}) & & \xrightarrow{(3.10)} & & (\mathrm{transl}_{\mathcal{P}_{Z_M}^{\otimes -1}\otimes \tau(\mathcal{P}_{Z_M})})_*\left(\sigma(\widetilde{\Omega}^o_{\mathcal{P}_{Z_M}})\right) \end{array}$$

commutes.

In particular, taking $\mathcal{P}_{Z_M} = \rho_P(\omega_X)$, we obtain a commutative diagram

$$\begin{array}{ccccc} \tau_M(\Omega) & \xrightarrow{(3.16)} & \sigma(\Omega^o) & \xrightarrow{(3.14)} & (\mathrm{transl}_{-2\rho_P(\omega_X)})^*\left(\sigma(\Omega^o)\right) \\ {\scriptstyle \tau_M(3.15)}\downarrow & & & & \downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})^*\circ \sigma(3.15)^o} \\ \tau_M(\widetilde{\Omega}_{\rho_P(\omega_X)}) & & \xrightarrow{(3.12)} & & (\mathrm{transl}_{-2\rho_P(\omega_X)})^*\left(\sigma(\widetilde{\Omega}^o_{\rho_P(\omega_X)})\right). \end{array} \tag{3.17}$$

3.4.4. Let us translate Lemma 3.4.2 into an assertion about dualities of categories.

The map (3.15) gives rise to an adjoint pair

$$\mathbf{ind}_{\Omega\to\frac{\infty}{2}} : \Omega\text{-mod}(\mathrm{Sph}_M) \rightleftarrows \mathrm{I}(G,P^-)^{\mathrm{loc}} : \mathbf{oblv}_{\frac{\infty}{2}\to\Omega}.$$

The identification (3.16) gives rise to an equivalence

$$(\Omega\text{-mod}(\mathrm{Sph}_M))^\vee \simeq \tau_M(\Omega)\text{-mod}(\mathrm{Sph}_M). \tag{3.18}$$

Then we have a commutative diagram

$$\begin{array}{ccccc} (\Omega\text{-mod}(\mathrm{Sph}_M))^\vee & & \xrightarrow{(3.18)} & & \tau_M(\Omega)\text{-mod}(\mathrm{Sph}_M) \\ {\scriptstyle (\mathbf{oblv}_{\frac{\infty}{2}\to\Omega})^\vee}\downarrow & & & & \downarrow{\scriptstyle \tau_G\circ \mathbf{ind}_{\Omega\to\frac{\infty}{2}}} \\ (\mathrm{I}(G,P^-)^{\mathrm{loc}})^\vee & \longrightarrow & \mathrm{I}(G,P^-)^{\mathrm{loc,co}} & \xrightarrow{\Upsilon^{\mathrm{loc}}} & \mathrm{I}(G,P)^{\mathrm{loc}}. \end{array}$$

3.4.5. Similarly, in the twisted situation, we have an adjunction

$$\mathbf{ind}_{\Omega\to\frac{\infty}{2}} : \Omega\text{-mod}(\mathrm{Sph}_M) \rightleftarrows \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}} : \mathbf{oblv}_{\frac{\infty}{2}\to\Omega}$$

and a commutative diagram

$$\begin{array}{ccc}
(\Omega\text{-mod}(\mathrm{Sph}_M))^\vee & \xrightarrow{(3.18)} & \tau_M(\Omega)\text{-mod}(\mathrm{Sph}_M) \\
{\scriptstyle (\mathbf{oblv}_{\frac{\infty}{2}\to\Omega})^\vee}\downarrow & & \downarrow{\scriptstyle \tau_G\circ\mathbf{ind}_{\Omega\to\frac{\infty}{2}}} \\
(\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}})^\vee & & \mathrm{I}(G,P)^{\mathrm{loc}}_{\tau_M(\mathcal{P}_{Z_M})} \\
\downarrow & & \downarrow{\scriptstyle (\alpha_{\mathcal{P}_{Z_M},\mathrm{taut}})\circ(\alpha_{\tau_M(\mathcal{P}_{Z_M}),\mathrm{taut}})^{-1}} \\
\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\mathcal{P}_{Z_M}} & \xrightarrow{\Upsilon^{\mathrm{loc}}} & \mathrm{I}(G,P)^{\mathrm{loc}}_{\mathcal{P}_{Z_M}}.
\end{array}$$

4. The Kazhdan-Lusztig category

In this section we study the local representation-theoretic category on the geometric side, which would be connected to the global category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ by a local-to-global procedure.

The category in question is the Kazhdan-Lusztig category at the critical level,

$$\mathrm{KL}(G)_{\mathrm{crit}} := \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(G)}_{\mathrm{crit}}.$$

We will need the following aspects of the theory associated with $\mathrm{KL}(G)_{\mathrm{crit}}$:

- Self-duality;
- Action of the Feigin-Frenkel center,
- Twists by Z_G-torsors, twists by Z^0_G-torsors and the combination of the two;
- The BRST functor from $\mathrm{KL}(G)_{\mathrm{crit}}$ to a (twisted version of) $\mathrm{KL}(M)_{\mathrm{crit}}$;
- A version of the previous item *enhanced* using $\mathrm{I}(G,P^-)^{\mathrm{loc}}$;
- The functor of Drinfeld-Sokolov reduction.

4.1. Defintion and basic properties.

4.1.1. Let κ be a level for $\mathfrak{g}$. We consider

$$\widehat{\mathfrak{g}}\text{-mod}_\kappa,$$

the category of Kac-Mooy modules at level κ.

This category carries a natural action of $\mathfrak{L}(G)$ at level κ.

4.1.2. Let

$$\mathrm{KL}(G)_\kappa := \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(G)}_\kappa,$$

denote the corresponding category of spherical objects.

We have an adjunction

$$\mathbf{oblv}_{\mathfrak{L}^+(G)} : \mathrm{KL}(G)_\kappa \rightleftarrows \widehat{\mathfrak{g}}\text{-mod}_\kappa : \mathrm{Av}^{\mathfrak{L}^+(G)}_*.$$

4.1.3. We have a monadic adjunction

$$\mathbf{ind}_{\mathfrak{L}^+(G)\to(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))} : \mathrm{Rep}(\mathfrak{L}^+(G)) \rightleftarrows \mathrm{KL}(G)_\kappa : \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)}. \tag{4.1}$$

In particular, $\mathrm{KL}(G)_\kappa$ is compactly generated by the image of compact generators of $\mathrm{Rep}(\mathfrak{L}^+(G))$. Those can be taken to be the objects

$$\mathrm{Res}^G_{\mathfrak{L}^+(G)}(V^\lambda), \quad V^\lambda \in \mathrm{Irrep}(G).$$

The corresponding objects in $\mathrm{KL}(G)_\kappa$ are the standard (a.k.a. Weyl) modules, denoted

$$\mathbb{V}^\lambda_\kappa \in \mathrm{KL}(G)_\kappa.$$

4.1.4. The category has a natural factorization structure, with the factorization unit being

$$\mathrm{Vac}(G)_\kappa := \mathbb{V}^0_\kappa.$$

4.2. Critical level and Feigin-Frenkel center.

4.2.1. Our primary interest in this paper is the case when $\kappa = -\frac{1}{2} \cdot \mathrm{Kil}$, where Kil is the Killing form. We will denote the corresponding level by symbol crit.

4.2.2. Let $\mathfrak{z}_{\mathfrak{g}}$ be the FF-center of $\mathrm{Vac}(G)_{\mathrm{crit}}$, viewed as a factorization (chiral) algebra.

4.2.3. Let $\mathfrak{Z}_{\mathfrak{g}}$ be the (topological) algebra of de Rham cohomology of $\mathfrak{z}_{\mathfrak{g}}$ over the formal punctured disc. We can view "Spec"$(\mathfrak{Z}_{\mathfrak{g}})$ as an indscheme.

The categories

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \text{ and } \mathrm{KL}(G)_{\mathrm{crit}}$$

are naturally tensored over QCoh("Spec"$(\mathfrak{Z}_{\mathfrak{g}})$).

Remark 4.2.4. The last assertion is intuitively well understood, but as a precise assertion about DG categories takes some work. At a point, and when also considering the commuting $\mathfrak{L}(G)$-action at critical level, this is [Ras4, Theorem 11.18.1], the main assertion of the monograph [Ras4]. The ideas from *loc. cit.* with standard modifications to treat the factorizable version that we use here.

4.3. **Duality.**

4.3.1. For a given level κ, denote

$$\kappa' := -\kappa + 2 \cdot \mathrm{crit}\,.$$

(In particular, $\mathrm{crit}' = \mathrm{crit}$.)

4.3.2. It is known that the categories

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa} \text{ and } \widehat{\mathfrak{g}}\text{-mod}_{\kappa'}$$

are canonically dual to one another, in a way compatible with factorization.

The counit of the duality is the functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\kappa'} \overset{\otimes}{\to} \widehat{\mathfrak{g}}\text{-mod}_{-\mathrm{Kil}} \to \mathrm{Vect},$$

where the second arrow is the functor of semi-infinite cohomology.

4.3.3. The above duality induces a duality between

$$\mathrm{KL}(G)_{\kappa} \text{ and } \mathrm{KL}(G)_{\kappa'},$$

so that

$$\left(\mathbf{oblv}_{\mathfrak{L}^+(G)}\right)^{\vee} \simeq \mathrm{Av}_*^{\mathfrak{L}^+(G)} \text{ and } \left(\mathrm{Av}_*^{\mathfrak{L}^+(G)}\right)^{\vee} \simeq \mathbf{oblv}_{\mathfrak{L}^+(G)}.$$

The unit of the duality is the object

$$\mathrm{CDO}(G)_{\kappa,\kappa'} \in \mathrm{KL}(G)_{\kappa} \otimes \mathrm{KL}(G)_{\kappa'}.$$

Under this duality and the canonical self-duality of $\mathrm{Rep}(\mathfrak{L}^+(G))$, we have

$$\left(\mathbf{ind}_{\mathfrak{L}^+(G)\to(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))}\right)^{\vee} \simeq \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)} \text{ and } \left(\mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)}\right)^{\vee} \simeq \mathbf{ind}_{\mathfrak{L}^+(G)\to(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))}.$$

4.3.4. In particular, we obtain canonical self-dualities

$$\left(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}\right)^{\vee} \simeq \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \tag{4.2}$$

and

$$\left(\mathrm{KL}(G)_{\mathrm{crit}}\right)^{\vee} \simeq \mathrm{KL}(G)_{\mathrm{crit}}. \tag{4.3}$$

These dualities are compatible with the action of QCoh("Spec"$(\mathfrak{Z}_{\mathfrak{g}})$), *up to the action of the Cartan involution*[13] τ_G *on* $\mathfrak{Z}_{\mathfrak{g}}$, see (8.3).

4.4. **Twisting by Z^0_G-torsors.** In the bulk of the paper the observations of this subsection will be applied when the reductive group in question is the Levi subgroup of the original G.

[13]The action of inner automorphisms of $\mathfrak{g}$ on $\mathfrak{z}_{\mathfrak{g}}$ (and hence on $\mathfrak{Z}_{\mathfrak{g}}$) is trivial; hence we obtain a well-defined action of outer automorphisms of $\mathfrak{g}$ on $\mathfrak{z}_{\mathfrak{g}}$ (and $\mathfrak{Z}_{\mathfrak{g}}$).

4.4.1. Let $[\mathfrak{g},\mathfrak{g}]$ be the Lie algebra of the derived group of G, so that $\mathfrak{g}_{\mathrm{ab}} := \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is the cocenter of $\mathfrak{g}$.

Consider the vector space $\mathfrak{L}(\mathfrak{g}_{\mathrm{ab}})/\mathfrak{L}^+(\mathfrak{g}_{\mathrm{ab}})$. Its dual, viewed as a group, acts by automorphisms of the Kac-Moody extension, and hence also by automorphisms of the categories $\widehat{\mathfrak{g}}$-mod$_\kappa$ and $\mathrm{KL}(G)_\kappa$.

In particular, given a $(\mathfrak{L}(\mathfrak{g}_{\mathrm{ab}})/\mathfrak{L}^+(\mathfrak{g}_{\mathrm{ab}}))^*$-torsor $\mathcal{P}$, we can form the twisted versions of $\widehat{\mathfrak{g}}$-mod$_\kappa$ and $\mathrm{KL}(G)_\kappa$, denoted

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa+\mathcal{P}} \text{ and } \mathrm{KL}(G)_{\kappa+\mathcal{P}},$$

respectively.

4.4.2. Note that we have a canonical duality

$$\mathfrak{g}_{\mathrm{ab}} \simeq (z_{\check{\mathfrak{g}}})^*.$$

In particular, we can identify

$$(\mathfrak{L}(\mathfrak{g}_{\mathrm{ab}})/\mathfrak{L}^+(\mathfrak{g}_{\mathrm{ab}}))^* \simeq \mathfrak{L}^+(z_{\check{\mathfrak{g}}} \otimes \omega_X),$$

where the right-hand side is the space of sections of $z_{\check{\mathfrak{g}}}$-valued 1-forms on the formal disc.

Let $Z^0_{\check{G}}$ denote the neutral connected component of the center of $\check{G}$. Consider the homomorphism

$$\mathrm{dlog} : \mathfrak{L}^+(Z^0_{\check{G}}) \to \mathfrak{L}^+(z_{\check{\mathfrak{g}}} \otimes \omega_X).$$

Thus, starting from a $Z^0_{\check{G}}$-torsor $\mathcal{P}_{Z^0_{\check{G}}}$ on X, we can:

- Produce a $\mathfrak{L}^+(Z^0_{\check{G}})$-torsor (by restricting to the formal disc);
- Induce a $\mathfrak{L}^+(z_{\check{\mathfrak{g}}} \otimes \omega_X)$-torsor using dlog;
- Think of the latter a $(\mathfrak{L}(\mathfrak{g}_{\mathrm{ab}})/\mathfrak{L}^+(\mathfrak{g}_{\mathrm{ab}}))^*$-torsor, which by a slight abuse of notation we denote by

$$\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}}).$$

The above construction is naturally compatible with factorization.

We denote the resulting (factorization) categories of Kac-Moody modules by

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} \text{ and } \mathrm{KL}(G)_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})},$$

respectively.

4.4.3. Note that the Contou-Carrère symbol defines a bilinear pairing

$$\mathfrak{L}^+(Z^0_{\check{G}}) \times \mathfrak{L}(G) \to \mathfrak{L}^+(Z^0_{\check{G}}) \times \mathfrak{L}(G_{\mathrm{ab}}) \to \mathbb{G}_m,$$

where $G_{\mathrm{ab}} = G/[G,G]$, so that it is a torus dual to $Z^0_{\check{G}}$.

This implies that a $Z^0_{\check{G}}$-torsor $\mathcal{P}_{Z^0_{\check{G}}}$ on X gives rise to a multiplicative line bundle on $\mathfrak{L}(G)$ (a.k.a., central extension by means of $\mathbb{G}_m$), compatible with the factorization structure.

This central extension gives rise to an action of $\mathfrak{L}(G)$ at the same level κ on the category $\widehat{\mathfrak{g}}\text{-mod}_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}$. In particular, we have the convolution functor

$$\mathrm{D}\text{-mod}_\kappa(\mathrm{Gr}_G) \otimes \mathrm{KL}(G)_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} \xrightarrow{\star} \widehat{\mathfrak{g}}\text{-mod}_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}.$$

4.4.4. *Notational convention.* Assume for a moment that the above $Z^0_{\check{G}}$-torsor is of the form $\check{\lambda}(\omega_X)$ where $\check{\lambda}: \mathbb{G}_m \to Z^0_{\check{G}}$.

In this case, we will use a short-hand notation

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa+\check{\lambda}} := \widehat{\mathfrak{g}}\text{-mod}_{\kappa+\mathrm{dlog}(\check{\lambda}(\omega_X))} \text{ and } \mathrm{KL}(G)_{\kappa+\check{\lambda}} := \mathrm{KL}(G)_{\kappa+\mathrm{dlog}(\check{\lambda}(\omega_X))}$$

for the corresponding twisted categories.

Note that the assignment

$$\check{\lambda} \rightsquigarrow \mathrm{dlog}(\check{\lambda}(\omega_X))$$

is linear. So, the resulting torsor with respect to

$$\mathfrak{L}^+(z_{\check{\mathfrak{g}}} \otimes \omega_X) \simeq (\mathfrak{L}(\mathfrak{g}_{\mathrm{ab}})/\mathfrak{L}^+(\mathfrak{g}_{\mathrm{ab}}))^*$$

makes sense for any $\lambda \in z_{\check{\mathfrak{g}}}$.

Thus, the twisted categories

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa+\check{\lambda}} \text{ and } \mathrm{KL}(G)_{\kappa+\check{\lambda}}$$

are defined for any $\check{\lambda} \in z_{\check{\mathfrak{g}}}$.

4.5. **BRST and Wakimoto functors.**

4.5.1. Let P be a standard parabolic in G, and P^- its opposite. We have the functor of BRST reduction with respect to $\mathfrak{L}(\mathfrak{n}(\mathfrak{p}^-))$:

$$\mathrm{BRST}^- : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}+\kappa} \to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\kappa-\check{\rho}_P}, \tag{4.4}$$

where:

- The subscript "crit" on both sides denotes the critical level for G and M, respectively.
- The subscript $\check{\rho}_P$ is as in Sect. 4.4.4.

The functor (4.4) naturally factors as

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}+\kappa} \twoheadrightarrow (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}+\kappa})_{\mathfrak{L}(N(P^-))} \to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\kappa-\check{\rho}_P}. \tag{4.5}$$

4.5.2. The functor BRST^- induces a functor

$$\widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa} \to \widehat{\mathfrak{m}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa-\check{\rho}_P} =: \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P}.$$

By composing with the forgetful functor

$$\mathrm{KL}(G)_{\mathrm{crit}+\kappa} := \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(G)}_{\mathrm{crit}+\kappa} \overset{\mathbf{oblv}_{\mathfrak{L}^+(G)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa},$$

we obtain a functor

$$\mathrm{KL}(G)_{\mathrm{crit}+\kappa} \to \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P}, \tag{4.6}$$

which we denote by the same character BRST^-.

4.5.3. Let

$$\mathrm{Wak}^{-,\frac{\infty}{2}} : \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}-\kappa+\check{\rho}_P} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}-\kappa} \tag{4.7}$$

be the functor dual to the functor BRST^- of (4.4).

Since BRST^- factors as (4.5), the functor $\mathrm{Wak}^{-,\frac{\infty}{2}}$ naturally takes values in

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}-\kappa})^{\mathfrak{L}(N(P^-))} \subset \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}-\kappa}.$$

In particular, (4.7) induces a functor

$$\mathrm{KL}(M)_{\mathrm{crit}-\kappa+\check{\rho}_P} := \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}-\kappa+\check{\rho}_P} \to (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}-\kappa})^{\mathfrak{L}(N(P^-))\cdot\mathfrak{L}^+(M)} =: \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}-\kappa}, \tag{4.8}$$

which we denote by the same symbol $\mathrm{Wak}^{-,\frac{\infty}{2}}$.

Remark 4.5.4. The above version of the Wakimoto functor is somewhat exotic. For example, it produces objects that are not seen by the t-structure on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}-\kappa}$ (i.e., all of their cohomologies are 0).

One recovers from it the usual Wakimito functor by composing $\mathrm{Wak}^{-,\frac{\infty}{2}}$ with the functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}-\kappa} \overset{\mathrm{Av}_*^{\mathfrak{L}^+(N)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(N)}_{\mathrm{crit}-\kappa} \hookrightarrow \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}-\kappa},$$

see Sects. 11.4.4-11.4.6.

4.5.5. Denote:

$$\mathrm{Wak}^{-,\mathrm{Sph}} := \mathrm{Av}_*^{\mathfrak{L}^+(G)/\mathfrak{L}^+(M)} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}, \quad \mathrm{KL}(M)_{\mathrm{crit}-\kappa+\check{\rho}_P} \to \mathrm{KL}(G)_{\mathrm{crit}-\kappa}. \tag{4.9}$$

This is the functor dual to (4.6).

4.5.6. The functor (4.9) can be explicitly described as follows: the corresponding object of

$$(\mathrm{KL}(M)_{\mathrm{crit}-\kappa+\check{\rho}_P})^\vee \otimes \mathrm{KL}(G)_{\mathrm{crit}-\kappa} \simeq \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P} \otimes \mathrm{KL}(G)_{\mathrm{crit}-\kappa}$$

is given by

$$(\mathrm{BRST}^- \otimes \mathrm{Id})(\mathrm{CDO}(G)_{\mathrm{crit}+\kappa,\mathrm{crit}-\kappa}).$$

4.6. **Twisting by Z_G-torsors.**

4.6.1. Let $\mathcal{P}_G$ be a G-bundle on X. As in Sect. 1.2.1, we can consider the $\mathcal{P}_G$-twists of $\widehat{\mathfrak{g}}\text{-mod}_\kappa$ and $\mathrm{KL}(G)_\kappa$, denoted

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_G} \text{ and } \mathrm{KL}(G)_{\kappa,\mathcal{P}_G},$$

respectively.

4.6.2. Note, however, that as in Sect. 1.2.2, we have a canonical equivalence

$$\alpha_{\mathcal{P}_G,\mathrm{taut}} : \mathrm{KL}(G)_\kappa \to \mathrm{KL}(G)_{\kappa,\mathcal{P}_G}. \tag{4.10}$$

This equivalence fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{KL}(G)_\kappa & \xrightarrow{\alpha_{\mathcal{P}_G,\mathrm{taut}}} & \mathrm{KL}(G)_{\kappa,\mathcal{P}_G} \\ {\scriptstyle \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)}} \\ \mathrm{Rep}(\mathfrak{L}^+(G)) & \xrightarrow{\alpha_{\mathcal{P}_G,\mathrm{taut}}} & \mathrm{Rep}(\mathfrak{L}^+(G)_{\mathcal{P}_G}). \end{array}$$

4.6.3. Assume now that $\mathcal{P}_G$ is induced from a Z_G-torsor $\mathcal{P}_{Z_G}$. In this case, we have a canonical identification

$$\alpha_{\mathcal{P}_{Z_G},\mathrm{cent}} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\mathcal{P}_{Z_G}} \xrightarrow{\sim} \widehat{\mathfrak{g}}\text{-mod}_{\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}),-)}, \tag{4.11}$$

where:

- $\mathrm{dlog}(\mathcal{P}_{Z_G})$ is the $\mathfrak{L}^+(z_{\mathfrak{g}} \otimes \omega_X)$-torsor, induced by means of the map
$$\mathrm{dlog} : \mathfrak{L}^+(Z_G) \to \mathfrak{L}^+(z_{\mathfrak{g}} \otimes \omega_X).$$
- The level $\kappa(-,-)$ is viewed as defining a map
$$\kappa(-,-) : z_{\mathfrak{g}} \to z_{\check{\mathfrak{g}}},$$
and so we can view
$$\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}),-)$$
as a torsor with respect to
$$\mathfrak{L}^+(z_{\check{\mathfrak{g}}} \otimes \omega_X) \simeq (\mathfrak{L}(\mathfrak{g}_{\mathrm{ab}})/\mathfrak{L}^+(\mathfrak{g}_{\mathrm{ab}}))^*.$$

4.6.4. The equivalence (4.11) induces an equivalence

$$\alpha_{\mathcal{P}_{Z_G},\mathrm{cent}} : \mathrm{KL}(G)_{\kappa,\mathcal{P}_{Z_G}} \xrightarrow{\sim} \mathrm{KL}(G)_{\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}),-)}. \tag{4.12}$$

Composing with (4.10), we obtain an equivalence, denoted

$$\mathrm{KL}(G)_\kappa \xrightarrow{(\mathrm{transl}_{\mathcal{P}_{Z_G}})^*} \mathrm{KL}(G)_{\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}),-)}. \tag{4.13}$$

This equivalence fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{KL}(G)_\kappa & \xrightarrow{(\mathrm{transl}_{\mathcal{P}_{Z_G}})^*} & \mathrm{KL}(G)_{\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}),-)} \\ {\scriptstyle \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)}} \\ \mathrm{Rep}(\mathfrak{L}^+(G)) & & \mathrm{Rep}(\mathfrak{L}^+(G)) \\ =\downarrow & & \downarrow= \\ \mathrm{QCoh}(\mathrm{pt}/\mathfrak{L}^+(G)) & \xrightarrow{(\mathrm{transl}_{\mathcal{P}_{Z_G}})^*} & \mathrm{QCoh}(\mathrm{pt}/\mathfrak{L}^+(G)) \end{array}$$

where the bottom horizontal arrow is the functor of pullback with respect to the automorphism $\mathrm{transl}_{\mathcal{P}_{Z_G}}$ of $\mathrm{pt}/\mathfrak{L}^+(G)$, given by tensoring with $\mathcal{P}_{Z_G}|_{\mathcal{D}}$.

4.6.5. Note that when $\kappa|_{z_\mathfrak{g}} = 0$, (e.g.,, when $k = 0$, i.e., we are at the critical level), we have

$$\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}), -) = 0.$$

So in this case, $(\mathrm{transl}_{\mathcal{P}_{Z_G}})^*$ is an endofunctor of $\mathrm{KL}(G)_\kappa$. It can be explicitly described as follows:

For every trivialization of $\mathcal{P}_{Z_G}$, we have

$$(\mathrm{transl}_{\mathcal{P}_{Z_G}})^* \simeq \mathrm{Id}\,.$$

A change of trivialization given by an element $g \in \mathfrak{L}^+(Z_G)$ corresponds to the automorphism of the identity endofunctor of $\mathrm{KL}(G)_\kappa$, given by the action of g on modules.

4.6.6. *Notational convention.* Assume for a moment that the above Z_G-torsor is of the form $\lambda(\omega_X)$, where $\lambda : \mathbb{G}_m \to Z_G$.

In this case, we will use a short-hand notation

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa-\kappa(\lambda,-)} := \widehat{\mathfrak{g}}\text{-mod}_{\kappa-\kappa(\mathrm{dlog}(\lambda(\omega_X)),-)} \text{ and } \mathrm{KL}(G)_{\kappa-\kappa(\lambda,-)} := \mathrm{KL}(G)_{\kappa-\kappa(\mathrm{dlog}(\lambda(\omega_X)),-)}.$$

Note that this agrees with the notation introduced in Sect. 4.4.4, where we regard $\kappa(\lambda, -)$ as an element of $z_{\check{\mathfrak{g}}}$.

As in *loc.cit.*, the notations

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa-\kappa(\lambda,-)} \text{ and } \mathrm{KL}(G)_{\kappa-\kappa(\lambda,-)}$$

make sense for an arbitrary element $\lambda \in z_\mathfrak{g}$.

4.6.7. Similar constructions apply, if instead of $\mathrm{KL}(G)_\kappa$ (resp., $\widehat{\mathfrak{g}}\text{-mod}_\kappa$) we start with the category of the form $\mathrm{KL}(G)_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}$ (resp., $\widehat{\mathfrak{g}}\text{-mod}_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}$).

We denote the resulting categories by

$$\mathrm{KL}(G)_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}),-)} \text{ and } \widehat{\mathfrak{g}}\text{-mod}_{\kappa+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}),-)}$$

or

$$\mathrm{KL}(G)_{\kappa+\check{\lambda}-\kappa(\lambda,-)} \text{ and } \widehat{\mathfrak{g}}\text{-mod}_{\kappa+\check{\lambda}-\kappa(\lambda,-)},$$

respectively.

4.7. **The twisted BRST and Wakimoto functors.**

4.7.1. Let $\mathcal{P}_M$ be an M-torsor. Consider the $\mathcal{P}_M$-twists of the objects appearing in Sect. 4.5, see Sect. 1.2.

In particular, we obtain the functors

$$\mathrm{BRST}^-_{\mathcal{P}_M} : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}+\kappa,\mathcal{P}_M} \to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\kappa-\check{\rho}_P,\mathcal{P}_M}.$$

4.7.2. Note, however, that as in Sect. 1.2.2, we have the equivalences

$$\alpha_{\mathcal{P}_M,\mathrm{taut}} : \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa} \to \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa,\mathcal{P}_M} \text{ and } \alpha_{\mathcal{P}_M,\mathrm{taut}} : \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P} \to \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P,\mathcal{P}_M}$$

so that the diagram

$$\begin{array}{ccc} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa} & \xrightarrow{\alpha_{\mathcal{P}_M,\mathrm{taut}}} & \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa,\mathcal{P}_M} \\ {\scriptstyle \mathrm{BRST}^-}\downarrow & & \downarrow{\scriptstyle \mathrm{BRST}^-_{\mathcal{P}_M}} \\ \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P} & \xrightarrow{\alpha_{\mathcal{P}_M,\mathrm{taut}}} & \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P,\mathcal{P}_M} \end{array} \tag{4.14}$$

commutes.

4.7.3. Assume now that $\mathcal{P}_M$ is induced by a Z_M-bundle $\mathcal{P}_{Z_M}$. In this case, we can identify

$$\widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\kappa-\check{\rho}_P,\mathcal{P}_{Z_M}} \overset{\alpha_{\mathcal{P}_{Z_M},\mathrm{cent}}}{\simeq} \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}$$

and

$$\mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P,\mathcal{P}_{Z_M}} \overset{\alpha_{\mathcal{P}_{Z_M},\mathrm{cent}}}{\simeq} \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}$$

see (4.11) and (4.12), respectively.

In this case, we will view $\mathrm{BRST}^-_{\mathcal{P}_{Z_M}}$ as a functor

$$\widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa} \to \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}$$

equal to

$$\begin{aligned} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa} &\xrightarrow{\alpha_{\mathcal{P}_{Z_M},\mathrm{taut}}} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa,\mathcal{P}_{Z_M}} \xrightarrow{\mathrm{BRST}^-_{\mathcal{P}_{Z_M}}} \\ &\to \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P,\mathcal{P}_{Z_M}} \overset{\alpha_{\mathcal{P}_{Z_M},\mathrm{cent}}}{\simeq} \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}, \end{aligned} \tag{4.15}$$

or, which is the same,

$$\widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa} \xrightarrow{\mathrm{BRST}^-} \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P} \overset{(\mathrm{transl}_{\mathcal{P}_{Z_M}})^*}{\simeq} \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}. \tag{4.16}$$

4.7.4. We will denote by the same character $\mathrm{BRST}^-_{\mathcal{P}_{Z_M}}$ the restriction of (4.15) along

$$\mathrm{KL}(G)_{\mathrm{crit}+\kappa} := \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(G)}_{\mathrm{crit}+\kappa} \to \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}+\kappa},$$

so it is a functor

$$\mathrm{BRST}^-_{\mathcal{P}_{Z_M}} : \mathrm{KL}(G)_{\mathrm{crit}+\kappa} \to \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}. \tag{4.17}$$

From (4.14) we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{KL}(G)_{\mathrm{crit}+\kappa} & \xrightarrow{\mathrm{Id}} & \mathrm{KL}(G)_{\mathrm{crit}+\kappa} \\ {\scriptstyle \mathrm{BRST}^-}\downarrow & & \downarrow{\scriptstyle \mathrm{BRST}^-_{\mathcal{P}_{Z_M}}} \\ \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P} & \xrightarrow{(\mathrm{transl}_{\mathcal{P}_{Z_M}})^*} & \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}. \end{array} \tag{4.18}$$

4.7.5. *Warning.* Note that when $\kappa = 0$ (i.e., we are at the critical level), the target category of both functors

$$\mathrm{BRST}^- \text{ and } \mathrm{BRST}^-_{\mathcal{P}_{Z_M}}$$

is $\mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P}$.

Yet, these two functors are different: namely, they differ by the automorphism of $\mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P}$ given by $(\mathrm{transl}_{\mathcal{P}_{Z_M}})^*$, see Sect. 4.6.5.

4.7.6. Similar conventions apply to the Wakiomoto functors. In particular, we obtain the functors

$$\mathrm{Wak}^{-,\frac{\infty}{2}}_{\mathcal{P}_{Z_M}} : \mathrm{KL}(M)_{\mathrm{crit}-\kappa+\check{\rho}_P+\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)} \to \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}-\kappa}$$

and

$$\mathrm{Wak}^{-,\mathrm{Sph}}_{\mathcal{P}_{Z_M}} : \mathrm{KL}(M)_{\mathrm{crit}-\kappa+\check{\rho}_P+\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)} \to \mathrm{KL}(G)_{\mathrm{crit}-\kappa}.$$

4.7.7. We will apply the above constructions mostly in the case when $\mathcal{P}_{Z_M} = \rho_P(\omega_X)$. So we obtain the functors

$$\mathrm{BRST}^-_{\rho_P(\omega_X)} : \mathrm{KL}(G)_{\mathrm{crit}+\kappa} \to \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\rho_P,-)}$$

and

$$\mathrm{Wak}^{-,\mathrm{Sph}}_{\rho_P(\omega_X)} : \mathrm{KL}(M)_{\mathrm{crit}-\kappa+\check{\rho}_P+\kappa(\rho_P,-)} \to \mathrm{KL}(G)_{\mathrm{crit}-\kappa},$$

and similarly for the semi-infinite version.

When $\kappa = 0$, these functors specialize to

$$\mathrm{BRST}^-_{\rho_P(\omega_X)} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}$$

and

$$\mathrm{Wak}^{-,\mathrm{Sph}}_{\rho_P(\omega_X)} : \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P} \to \mathrm{KL}(G)_{\mathrm{crit}},$$

respectively, and similarly for the semi-infinite version.

4.8. **The enhanced functor of BRST reduction.**

4.8.1. By the definition of the functor

$$\mathrm{BRST}^- : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}-\check{\rho}_P},$$

it naturally factors as

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})_{\mathfrak{L}(N(P^-))} \overset{\overline{\mathrm{BRST}}^-}{\to} \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}-\check{\rho}_P}.$$

Moreover, the resulting functor $\overline{\mathrm{BRST}}^-$ respects the action of $\mathfrak{L}(M)$.

4.8.2. Recall the category $\mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathrm{co}}$, see Sect. 3.1.2. Restricting $\overline{\mathrm{BRST}}^-$ along

$$\mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathrm{co}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \hookrightarrow (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^{\mathfrak{L}^+(M)}_{\mathfrak{L}(N(P^-))},$$

we obtain a functor, which we denote by the same symbol

$$\overline{\mathrm{BRST}}^- : \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathrm{co}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}. \tag{4.19}$$

The functor $\overline{\mathrm{BRST}}^-$ respects the actions of Sph_M. Hence, using the fact that Sph_M and Sph_G are rigid, it gives rise to a functor, denoted

$$\mathrm{BRST}^{-,\mathrm{enh}} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{I}(G, P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}, \tag{4.20}$$

which respects the Sph_G-actions.

4.8.3. The original functor

$$\mathrm{BRST}^- : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}$$

is obtained by composing $\mathrm{BRST}^{-,\mathrm{enh}}$ with

$$\mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P} \overset{\mathbf{oblv}_{\frac{\infty}{2}} \to \mathrm{Sph}\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Sph}_M \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P} \simeq \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}.$$

4.8.4. In what follows we will need a twisted version of (4.20):

$$\mathrm{BRST}^{-,\mathrm{enh}}_{\rho_P(\omega_X)} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}. \tag{4.21}$$

Denote

$$\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P} := \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}. \tag{4.22}$$

Thus, we can regard (4.21) as a functor

$$\mathrm{BRST}^{-,\mathrm{enh}}_{\rho_P(\omega_X)} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P}. \tag{4.23}$$

4.9. **The functor of Drinfeld-Sokolov reduction.**

4.9.1. Consider the functor of semi-infinite cohomology of $\mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}$, twisted by the character χ (see Sect. 1.3.2)

$$\mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod} \to \mathrm{Vect}. \tag{4.24}$$

4.9.2. Precomposing with

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)} \to \mathfrak{L}(\mathfrak{n})_{\rho(\omega_X)}\text{-mod},$$

we obtain a functor of Drinfeld-Sokolov reduction that we denote by

$$\mathrm{DS} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)} \to \mathrm{Vect}. \tag{4.25}$$

It follows from the construction that the functor DS factors as

$$\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)} \to (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi} \to \mathrm{Vect}, \tag{4.26}$$

We denote the resulting functor

$$(\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi} \to \mathrm{Vect}$$

by

$$\overline{\mathrm{DS}} : (\widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi} \to \mathrm{Vect}. \tag{4.27}$$

4.9.3. Precomposing further with

$$\widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\kappa} \overset{\alpha_{\rho(\omega_X)},\mathrm{taut}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\kappa,\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\rho(\omega_X)},$$

we obtain a functor, denoted by the same character

$$\mathrm{DS} : \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\kappa} \to \mathrm{Vect}. \tag{4.28}$$

By the same principle as in Remark 1.3.3, the functor DS of (4.28) is canonically independent of the choice of the non-degenerate character $\chi : N \to \mathbb{G}_a$.

By a slight abuse of notation, we will denote by the same character DS the further restriction of (4.28) along

$$\mathrm{KL}(G)_{\kappa} \to \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\kappa}$$

or

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\kappa} \hookrightarrow \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\kappa}.$$

5. Monodromy-free opers

In this section we study the local counterpart of the Kazhdan-Lusztig category on the spectral side: this is the category

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

of ind-coherent sheaves on the space of monodromy-free $\check{G}$-opers on the punctured disc. This category will be related to the global spectral category (in this caee $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$) by a local-to-global procedure.

We will study the following aspects of $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$:

- Self-duality;
- The shifting procedure by a $Z^0_{\check{G}}$-bundle and Miura opers;
- The spectral Jacquet functor, which connects $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ with the a shifted variant of this category for a Levi subgroup;
- An *enhanced* version of the spectral Jacquet functor using $\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}$.

5.1. IndCoh^* of monodromy-free opers.

5.1.1. Let $\mathrm{Op}_{\check{G}}$ denote the (affine) D-scheme of $\check{G}$-opers on X. Like for any D-scheme, its fiber over a given point of X is the scheme $\mathrm{Op}_{\check{G}}(\mathcal{D})$ of $\check{G}$-opers on the formal disc.

Let $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ be the (factorization) indscheme of $\check{G}$-opers on the formal punctured disc.

5.1.2. We have a naturally defined commutative *but non-Cartesian* diagram

$$\begin{array}{ccc} \mathrm{Op}_{\check{G}}(\mathcal{D}) & \longrightarrow & \mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \\ {\scriptstyle \mathfrak{r}}\downarrow & & \downarrow{\scriptstyle \mathfrak{r}} \\ \mathrm{LS}_{\check{G}}(\mathcal{D}) & \longrightarrow & \mathrm{LS}_{\check{G}}(\mathcal{D}^\times). \end{array}$$

We define the factorization indscheme of *monodromy-free opers* as the fiber product

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) := \mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{G}}(\mathcal{D}^\times).$$

5.1.3. Our object of study is the resulting factorization category

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)).$$

We will study it along with the factorization categories

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D})) = \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}))) \text{ and } \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

5.1.4. By construction, we have a natural action of

$$\mathrm{IndCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}) =: \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$$

on $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$.

5.1.5. The functors of direct image along the closed embeddings

$$\mathrm{Op}_{\check{G}}(\mathcal{D}) \hookrightarrow \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times). \tag{5.1}$$

and

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \hookrightarrow \mathrm{Op}_{\check{G}}(\mathcal{D}^\times). \tag{5.2}$$

define *unital* functors between the corresponding factorization categories.

In particular, the factorization unit in both

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \text{ and } \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \tag{5.3}$$

is the direct image of the structure sheaf of $\mathrm{Op}_{\check{G}}(\mathcal{D})$. By a slight abuse of notation, we will denote it by $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$, even when it is viewed as an object of either of the categories in (5.3).

By further abuse of notation, we will denote by the same symbol $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$ the space if its global sections, viewed as a commutative (and hence, factorization) algebra in Vect.

5.1.6. Let

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}} \subset \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \tag{5.4}$$

be the full subcategory of objects set-theoretically supported over the image of (5.2).

Let us regard both categories in (5.4) as modules over $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$.

Note that direct image along (5.2) upgrades to a functor

$$\begin{aligned}\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \\ \to \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}}\left(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})), \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}\right) \simeq \\ \simeq \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}}\left(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})), \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\right).\end{aligned} \tag{5.5}$$

We have the following key technical observation:

Lemma 5.1.7. *The functor* (5.5) *is a* pointwise *equivalence.*

5.2. IndCoh* of monodromy-free opers as factorization modules.

5.2.1. Consider the functor of direct image

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times), -) : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{Vect}\,.$$

It sends the factorization unit to $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})} \in \mathrm{Vect}$, and hence upgrades to a functor

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times), -)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-}\mathrm{mod}^{\mathrm{fact}}. \tag{5.6}$$

The functor (5.6) is t-exact with respect to the natural t-structures on the two sides. But, it is *not* an equivalence: indeed, the right-hand side is left-complete with respect to its t-structure, and the left-hand side is not. However, we have:

Lemma 5.2.2. *The functor* (5.6) *induces an equivalence between the corresponding eventually coconnective (a.k.a. bounded below) subcategories.*

5.2.3. We can create a similar picture for $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$:

Direct image along the projection

$$\mathfrak{r} : \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \to \mathrm{LS}_{\check{G}}(\mathcal{D})$$

defines a (factorization) functor

$$\mathfrak{r}_* : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \simeq \mathrm{Rep}(\check{G}). \tag{5.7}$$

Denote

$$R_{\check{G},\mathrm{Op}} := \mathfrak{r}_*(\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}).$$

This is naturally a commutative factorization algebra in $\mathrm{Rep}(\check{G})$.

5.2.4. Explicitly, for

$$V \in \mathrm{Rep}(\check{G}) = \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),$$

let $V_{\mathrm{Op}} := (\mathfrak{r}|_{\mathrm{Op}_{\check{G}}(\mathcal{D})})^*(V)$ be the corresponding tautological vector bundle over $\mathrm{Op}_{\check{G}}(\mathcal{D})$. Then

$$R_{\check{G},\mathrm{Op}} := \left(\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}), -) \otimes \mathrm{Id}\right)((R_{\check{G}})_{\mathrm{Op}}),$$

where

$$R_{\check{G}} \in \mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{G})$$

is the regular representation.

5.2.5. The functor (5.7) naturally upgrades to a functor

$$(\mathfrak{r}_*)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \to R_{\check{G},\mathrm{Op}}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})). \tag{5.8}$$

We have:

Lemma 5.2.6. *The functor* (5.8) *induces an equivalence between the corresponding eventually coconnective subcategories.*

5.3. **Self-duality for opers.**

5.3.1. As for any indscheme, we have an equivalence of factorization categories:

$$(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)))^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)). \tag{5.9}$$

We now claim that there is a canonically defined equivalence

$$\Theta_{\mathrm{Op}(\check{G})} : \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)), \tag{5.10}$$

compatible with the monoidal action of $\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))$ on both sides.

The construction of the functor $\Theta_{\mathrm{Op}(\check{G})}$ will occupy the majority of this subsection.

5.3.2. Tautologically, for any indscheme, the datum of a functor

$$\mathrm{IndCoh}^!(\mathcal{Y}) \to \mathrm{IndCoh}^*(\mathcal{Y}), \tag{5.11}$$

compatible with an action of $\mathrm{IndCoh}^!(\mathcal{Y})$, is equivalent to a choice of an object in $\mathrm{IndCoh}^*(\mathcal{Y})$.

If the functor (5.11) is an equivalence, we will say that the corresponding object of $\mathrm{IndCoh}^*(\mathcal{Y})$ is a *fake dualizing sheaf*, and denote it by

$$\omega_{\mathcal{Y}}^{*,\mathrm{fake}} \in \mathrm{IndCoh}^*(\mathcal{Y}).$$

Thus, in order to construct (5.10), we need to exhibit a fake dualizing sheaf on $\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)$.

5.3.3. We first consider the case when we take $\mathcal{Y}$ to be the indscheme $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ of all opers on the formal punctured disc.

Recall that the D-scheme $\mathrm{Op}_{\check{G}}$ is acted on simply transitively by the D-scheme of jets into $\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}$, where $\mathfrak{a}(\check{\mathfrak{g}}) \subset \check{\mathfrak{g}}$ is the centralizer of a regular nilpotent element, and the twist by ω_X is performed with respect to the canonical $\mathbb{G}_m$-action on $\mathfrak{a}(\check{\mathfrak{g}})$.

Hence, $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ (resp., $\mathrm{Op}_{\check{G}}(\mathcal{D})$) is acted on simply-transitively by $\mathfrak{L}(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$ (resp., $\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$).

In particular, the quotient

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$$

is acted on simply-transitively by $\mathfrak{L}(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$, and hence is a (factorization) scheme of ind-finite type. In particular, the object

$$\omega_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})} \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})) := \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}))$$

is well-defined.

5.3.4. We set

$$\omega_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}^{*,\mathrm{fake}} \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \tag{5.12}$$

to be the *-pullback of $\omega_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})}$ along the projection

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \to \mathrm{Op}_{\check{G}}(\mathcal{D}^\times)/\mathfrak{L}^+(\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}).$$

The following is easy:

Lemma 5.3.5. *The functor*

$$\Theta_{\mathrm{Op}(\check{G})} : \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

defined by the object $\omega_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}^{*,\mathrm{fake}} \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ *of* (5.12) *is an equivalence.*

5.3.6. We now consider the closed embedding (5.2).

We let

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$$

be the !-pullback of $\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ along (5.2).

We claim:

Lemma 5.3.7. *The functor*

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$$

defined by the object $\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)} \in \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$ *is an equivalence.*

This gives rise to the sought-for functor $\Theta_{\mathrm{Op}(\check{G})}$ in (5.10). Note that by construction, the functor $\Theta_{\mathrm{Op}(\check{G})}$ respects the $\mathrm{Sph}_{\check{G}}$-actions.

Remark 5.3.8. By construction, the functor $\Theta_{\mathrm{Op}(\check{G})}$ in (5.10) is rigged so that the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) & \longrightarrow & \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \\ {\scriptstyle \Theta_{\mathrm{Op}(\check{G})}}\downarrow & & \downarrow{\scriptstyle \Theta_{\mathrm{Op}(\check{G})}} \\ \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) & \longrightarrow & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \end{array}$$

commutes, where the horizontal arrows are direct image functors along (5.2).

Remark 5.3.9. Note also that the !-pullback of $\omega^{*,\mathrm{fake}}_{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)}$ along (5.1) identifies with $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$. This implies that the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D})) & \longrightarrow & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \\ {\scriptstyle \Theta_{\mathrm{Op}(\check{G})}}\downarrow & & \downarrow{\scriptstyle \Theta_{\mathrm{Op}(\check{G})}} \\ \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D})) & \longrightarrow & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \end{array}$$

commutes as well, where the horizontal arrows are direct image functors along (5.1), and the left vertical arrow is defined using

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D})} := \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})} \in \mathrm{QCoh}(\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D})).$$

This shows that the functor (5.10) sends the factorization unit

$$\mathbf{1}_{\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))} \simeq \omega_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$$

to the factorization unit

$$\mathbf{1}_{\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))} \simeq \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}.$$

In other words, the functor (5.10) is unital (which is must be, since it is an equivalence).

5.3.10. Just like $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$, the category $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$ acquires a natural action of

$$\mathrm{IndCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}) \twoheadrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}) =: \mathrm{Sph}^{\mathrm{spec}}_{\check{G}},$$

and it follows from the construction that the functor $\Theta_{\mathrm{Op}(\check{G})}$ respects these actions.

5.4. **(Parabolic) Miura opers.**

5.4.1. *Translated opers.* Let $\check{G}$ be a reductive group, and let $\mathcal{P}_{Z^0_{\check{G}}}$ be a $\check{G}$-bundle on X. We let

$$\mathrm{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}$$

be the following variant of the D-scheme $\mathrm{Op}_{\check{G}}$:

In the definition of opers, instead of requiring that the induced $\check{T}$-bundle be $\check{\rho}(\omega_X)$, we require that it be

$$\check{\rho}(\omega_X) \otimes \mathcal{P}_{Z^0_{\check{G}}}. \tag{5.13}$$

We let

$$\mathrm{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}) \subset \mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times) \subset \mathrm{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)$$

denote the corresponding factorization spaces.

The material from the previous subsection transfers verbatim to the present context.

5.4.2. We now take the reductive group in question to be M, the Levi subgroup of a standard parabolic P. We take

$$\mathcal{P}_{Z_{\check{M}}} := \check{\rho}_P(\omega_X).$$

Note that in this case, the $\check{T}$-bundle (5.13) is

$$\check{\rho}_M(\omega_X) \otimes \check{\rho}_P(\omega_X) = \check{\rho}(\omega_X),$$

where $\check{\rho}$ in the right-hand side is the $\check{\rho}$ for G.

We will use a short-hand notation

$$\mathrm{Op}_{\check{M},\check{\rho}_P} := \mathrm{Op}_{\check{M},\check{\rho}_P(\omega_X)}.$$

5.4.3. *Example.* When $P = B$, we obtain that the scheme

$$\mathrm{Op}_{\check{T},\check{\rho}}$$

that classifies connections on the $\check{T}$-bundle $\check{\rho}(\omega_X)$.

5.4.4. Let $\mathrm{MOp}_{\check{G},\check{P}^-}$ denote the D-scheme of P^--Miura opers, i.e.,

$$\mathrm{MOp}_{\check{G},\check{P}^-} := (\mathrm{Op}_{\check{G}} \underset{\mathrm{LS}_{\check{G}}}{\times} \mathrm{LS}_{\check{P}^-})^{\mathrm{trans}},$$

where the superscript "trans" refers to the condition that the $\check{B}$-reduction of the $\check{G}$-bundles involved in the oper structure is transversal to the $\check{P}^-$-reduction.

We have the natural forgeful map

$$\mathsf{p}^{\mathrm{Miu}} : \mathrm{MOp}_{\check{G},\check{P}^-} \to \mathrm{Op}_{\check{G}} \tag{5.14}$$

5.4.5. Note also that we have a map

$$\mathsf{q}^{\mathrm{Miu}} : \mathrm{MOp}_{\check{G},\check{P}^-} \to \mathrm{Op}_{\check{M},\check{\rho}_P}, \tag{5.15}$$

constructed as follows:

The $\check{M}$-bundle with a connection are induced from the $\check{P}^-$-bundle. The reduction to $\check{B}(\check{M})$ comes from the reduction of the original $\check{G}$-bundle to $\check{B}$.

The following is fundamental, albeit immediate:

Lemma 5.4.6. *The map* (5.15) *is an isomorphism.*

Remark 5.4.7. The composition

$$\mathsf{p}^{\mathrm{Miu}} \circ (\mathsf{q}^{\mathrm{Miu}})^{-1}$$

is a map

$$\mathrm{Op}_{\check{M},\check{\rho}_P} \to \mathrm{Op}_{\check{G}},$$

called the (parabolic) Miura transform.

5.5. **The spectral Jacquet functor.**

5.5.1. Consider the fiber product

$$\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}).$$

Note that the maps

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \leftarrow \mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \to \mathrm{Op}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times),$$

induced by $\mathsf{p}^{\mathrm{Miu}}$ and $\mathsf{q}^{\mathrm{Miu}}$, respectively, naturally factor via maps

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \overset{\mathsf{p}^{\mathrm{Miu,mon\text{-}free}}}{\longleftarrow} \mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \overset{\mathsf{q}^{\mathrm{Miu,mon\text{-}free}}}{\longrightarrow} \mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times). \tag{5.16}$$

5.5.2. Let

$$J^{-,\mathrm{spec},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))$$

denote the functor

$$J^{-,\mathrm{spec},!} := (\mathsf{q}^{\mathrm{Miu,mon\text{-}free}})_* \circ (\mathsf{p}^{\mathrm{Miu,mon\text{-}free}})^!.$$

5.5.3. Denote by $J^{-,\mathrm{spec},*}$ the functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)),$$

so that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) & \xrightarrow{J^{-,\mathrm{spec},!}} & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\ {\scriptstyle \Theta_{\mathrm{Op}(\check{G})}}\downarrow\sim & & \sim\downarrow{\scriptstyle \Theta_{\mathrm{Op}(\check{M})}} \\ \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) & \xrightarrow{J^{-,\mathrm{spec},*}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)), \end{array}$$

where $\Theta_{\mathrm{Op}(\check{G})}$ and $\Theta_{\mathrm{Op}(\check{M})}$ are the identifications of Sect. 5.3.

5.5.4. We will refer to $J^{-,\mathrm{spec},*}$ as the "spectral Jacquet functor." The main theorem in Part I of the paper will establish its relationship with the BRST^- functor at the critical level.

5.6. **The semi-infinite spectral Jacquet functor.**

5.6.1. Consider the (non-affine) D-scheme

$$\mathrm{Op}_{\check{G},\check{P}^-} := \mathrm{Op}_{\check{G}} \underset{\mathrm{LS}_{\check{G}}}{\times} \mathrm{LS}_{\check{P}^-},$$

so that

$$\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \simeq \mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times),$$

Consider the fiber product

$$\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

Let:

- i denote the map

$$\mathrm{LS}_{\check{P}^-}(\mathcal{D}) \to \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}),$$

and also its base change

$$\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \to \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D});$$

- p denote the projection
$$\mathrm{LS}_{\check{P}^-}(\mathcal{D}) \to \mathrm{LS}_{\check{G}}(\mathcal{D}),$$
and also its base change my means of $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \to \mathrm{LS}_{\check{G}}(\mathcal{D})$, which is
$$\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \to \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times);$$
- j denote the map
$$\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \to \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times);$$
and also its base changes
$$\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}) \to \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})$$
and
$$\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \to \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}).$$

Note that the above maps give rise to a Cartesian diagram

$$\begin{array}{ccc}
\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) & \longrightarrow & \mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}) \\
\downarrow{\scriptstyle \mathsf{j}} & & \downarrow{\scriptstyle \mathsf{j}} \\
\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) & \xrightarrow{\mathsf{i}} & \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}) \\
\downarrow{\scriptstyle \mathsf{p}} & & \\
\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}) & &
\end{array} \tag{5.17}$$

5.6.2. Consider the (factorization) category
$$\mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big).$$

Let
$$J^{-,\mathrm{spec},!,\frac{\infty}{2}} : \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \tag{5.18}$$
denote the functor of pullback along
$$\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times) \simeq \mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}) \xrightarrow{\mathsf{j}} \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

5.6.3. Let
$$\mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \subset \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)$$
be the full subcategory consisting of objects set-theoretically supported over
$$\mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \subset \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times).$$

It follows by base change along (5.17) that the functor $J^{-,\mathrm{spec},!}$ factors as

$$\begin{aligned}
\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) &\xrightarrow{\mathsf{p}^!} \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D})\Big) \xrightarrow{\mathsf{i}_*} \\
&\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \hookrightarrow \\
&\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \xrightarrow{J^{-,\mathrm{spec},!,\frac{\infty}{2}}} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)).
\end{aligned} \tag{5.19}$$

5.7. **The enhanced spectral Jacquet functor.**

5.7.1. Recall the stack

$$\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-} := \mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

Note that the operations of !-pullback and $\overset{!}{\otimes}$ give rise to a functor

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \otimes \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big). \tag{5.20}$$

5.7.2. In fact, we have a canonically defined functor

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}}, \tag{5.21}$$

so that (5.20) factors as

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \otimes \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \to \mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{(5.21)}{\longrightarrow} \tag{5.22}$$

$$\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \hookrightarrow \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big).$$

5.7.3. Moreover, the partial composition

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\mathrm{i}_*\circ\mathrm{p}^!}{\to} \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}}$$

in (5.19) identifies with

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\mathbf{1}_{\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-})} \otimes \mathrm{Id}}{\longrightarrow} \to \mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{(5.21)}{\longrightarrow}$$

$$\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}}.$$

5.7.4. Consider the functor

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{(5.21)}{\longrightarrow} \tag{5.23}$$

$$\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \hookrightarrow$$

$$\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \overset{J^{-,\mathrm{spec},!,\frac{\infty}{2}}}{\longrightarrow} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)).$$

The functor (5.23) is compatible with the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$. By rigidity, it gives rise to a functor, denoted

$$J^{-,\mathrm{spec},!,\mathrm{enh}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \to \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) =: \tag{5.24}$$

$$= \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)).$$

Note that the functor $J^{-,\mathrm{spec},!,\mathrm{enh}}$ of (5.24) respects the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$.

5.7.5. It follows from Sect. 5.7.3 that the composition of $J^{-,\mathrm{spec},!,\mathrm{enh}}$ with

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \overset{\mathbf{oblv}_{\frac{\infty}{2}}\to\mathrm{Sph}\otimes\mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))$$

identifies with $J^{-,\mathrm{spec},!}$.

5.7.6. Denote

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}} := \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)),$$

so that we can regard $J^{-,\mathrm{spec},!,\mathrm{enh}}$ as a functor

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}.$$

5.7.7. Denote also

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}} := \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)).$$

Since the equivalence $\Theta_{\mathrm{Op}(\check{M})}$ is compatible with the Hecke actions, it gives rise to an equivalence

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}} \overset{\Theta^{\mathrm{enh}}_{\mathrm{Op}(\check{M})}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}.$$

5.7.8. We define the functor

$$J^{-,\mathrm{spec},*,\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}} \tag{5.25}$$

so that

$$\Theta^{\mathrm{enh}}_{\mathrm{Op}(\check{M})} \circ J^{-,\mathrm{spec},!,\mathrm{enh}} \simeq J^{-,\mathrm{spec},*,\mathrm{enh}} \circ \Theta_{\mathrm{Op}(\check{G})}.$$

Since the equivalence $\Theta_{\mathrm{Op}(\check{G})}$ is compatible with the Hecke actions, the functor $J^{-,\mathrm{spec},*,\mathrm{enh}}$ respects the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$.

5.7.9. By construction, the composition of $J^{-,\mathrm{spec},*,\mathrm{enh}}$ with

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \overset{\mathbf{oblv}_{\frac{\infty}{2}}\to\mathrm{Sph}\otimes\mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))$$

identifies with $J^{-,\mathrm{spec},*}$.

6. A digression: factorization modules categories over $\mathrm{Rep}(\check{G})$

Constructions in this section will play an auxiliary role for the analysis of the critical FLE functor in the next section.

We will explain a procedure that attaches to a module category $\mathbf{C}$ over $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$ a *factorization module category* over $\mathrm{Rep}(\check{G})$, denoted $\mathbf{C}^{\mathrm{fact},\mathrm{Rep}(\check{G})}$. Conjecturally, the assignment $\mathbf{C} \rightsquigarrow \mathbf{C}^{\mathrm{fact},\mathrm{Rep}(\check{G})}$ is fully faithful as a functor between the corresponding 2-categories; we cannot prove this at the moment, but we can make do with a particular case of this assertion, Lemma 6.1.5.

The key result of this section is the following. Let $\mathbf{C}$ be a category acted on by $\mathfrak{L}(G)$, and we wish to relate the categories $\mathrm{Whit}_*(\mathbf{C})$ and $\mathrm{Sph}(\mathbf{C})$. The first observation is that $\mathrm{Whit}_*(\mathbf{C})$ can be promoted to a factorization module category over $\mathrm{Rep}(\check{G})$, to be denoted $\mathrm{Whit}_*(\mathbf{C})^{\mathrm{fact},\mathrm{Rep}(\check{G})}$.

Now the claim (Proposition 6.4.4) is that (the tempered quotient of) $\mathrm{Sph}(\mathbf{C})$ can be recovered as

$$\mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{Whit}_*(\mathbf{C})^{\mathrm{fact},\mathrm{Rep}(\check{G})}\right),$$

where $\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}$ denotes $\mathrm{Rep}(\check{G})$ when viewed as a factorization module category over itself.

6.1. **Creating factorization modules categories.**

6.1.1. Consider the space $\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)$, and the monoidal category $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$. Let us recall the construction of a functor

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod} \to \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}, \quad \mathbf{C} \mapsto \mathbf{C}^{\mathrm{fact},\mathrm{Rep}(\check{G})} \tag{6.1}$$

Namely, we will create an object

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))^{\mathrm{fact},\mathrm{Rep}(\check{G})} \in \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}$$

that carries a commuting action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$. The functor (6.1) will then be given by

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))^{\mathrm{fact},\mathrm{Rep}(\check{G})} \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))}{\otimes} -.$$

6.1.2. The object $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))^{\mathrm{fact},\mathrm{Rep}(\check{G})}$ will have the feature that its underlying DG category, equipped with an action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$, identifies with $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$ itself.

This will imply that the functor (6.1) has the feature that for $\mathbf{C} \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}$, the category underlying $\mathbf{C}^{\mathrm{fact},\mathrm{Rep}(\check{G})}$ identifies with the original $\mathbf{C}$.

6.1.3. The object $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))^{\mathrm{fact},\mathrm{Rep}(\check{G})}$ is constructed as follows.

For our fixed point $x \in X$ and a finite subset $x \in \underline{x} \subset X$, consider the multi-disc $\mathcal{D}_{\underline{x}}$, and set

$$(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))^{\mathrm{fact},\mathrm{Rep}(\check{G})})_{\underline{x}} := \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}} - x)).$$

6.1.4. In what follows, we will need the following assertion from [Ras3, Theorem 9.13.1]:

Lemma 6.1.5. *The functor* (6.1) *is* pointwise *fully faithful when restricted to the full subcategory*

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}_{\mathrm{LS}_{\check{G}}(\mathcal{D})} \subset \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod},$$

consisting of module categories set-theoretically supported over

$$\mathrm{LS}_{\check{G}}(\mathcal{D}) \subset \mathrm{LS}_{\check{G}}(\mathcal{D}^\times).$$

Remark 6.1.6. We conjecture that the functor (6.1) is pointwise fully faithful on all of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}$. A partial result in this direction has recently been established in [Bog]: the restriction of (6.1) to

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}(\mathcal{D}^\times)} \subset \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}$$

is pointwise fully faithful, where

$$\mathrm{LS}^{\mathrm{restr}}_{\check{G}}(\mathcal{D}^\times) \subset \mathrm{LS}_{\check{G}}(\mathcal{D}^\times)$$

is the stack of local systems with restricted variation (see [AGKRRV, Sect. 1.4]). We are not confident that Lemma 6.1.5 or its extension [Bog] remain true factorizably.

6.2. **Factorization modules categories attached to schemes.**

6.2.1. Let $\mathcal{Y}$ be an affine D-scheme over X equipped with a map

$$\mathcal{Y} \to \mathrm{pt}/\check{G}.$$

Consider the corresponding factorization spaces

$$\mathfrak{L}^+(\mathcal{Y}) \subset \mathfrak{L}(\mathcal{Y}) \tag{6.2}$$

and a commutative (but *not necessarily* Cartesian) diagram

$$\begin{array}{ccc} \mathfrak{L}^+(\mathcal{Y}) & \longrightarrow & \mathfrak{L}(\mathcal{Y}) \\ {\scriptstyle \mathfrak{r}}\downarrow & & \downarrow{\scriptstyle \mathfrak{r}} \\ \mathrm{LS}_{\check{G}}(\mathcal{D}) & \longrightarrow & \mathrm{LS}_{\check{G}}(\mathcal{D}^\times). \end{array}$$

6.2.2. On the one hand, we can consider $\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))$ as an object of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}$. Consider the resulting object

$$\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))^{\mathrm{fact},\mathrm{Rep}(\check{G})} \in \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}. \tag{6.3}$$

6.2.3. On the other hand, consider

$$\mathrm{QCoh}(\mathfrak{L}^+(\mathcal{Y})) = \mathrm{IndCoh}^*(\mathfrak{L}^+(\mathcal{Y})) \text{ and } \mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))$$

as factorization categories. Direct image along (6.2) defines a (unital) factorizaton functor

$$\mathrm{IndCoh}^*(\mathfrak{L}^+(\mathcal{Y})) \to \mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y})).$$

Furthermore, pullback along $\mathfrak{r}$ defines a (unital) factorizaton functor

$$\mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \to \mathrm{QCoh}(\mathfrak{L}^+(\mathcal{Y})).$$

Therefore, the operation of restriction of factorization modules defines an object

$$\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y})) \in \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}. \tag{6.4}$$

6.2.4. The following is obtained by unwinding the definitions:

Lemma 6.2.5. *The objects of* $\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}$*, given by* (6.3) *and* (6.4)*, respectively, are canonically isomorphic.*

6.2.6. Denote

$$\mathfrak{L}(\mathcal{Y})^{\text{mon-free}} := \mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathfrak{L}(\mathcal{Y}).$$

We have the closed embeddings

$$\mathfrak{L}^+(\mathcal{Y}) \to \mathfrak{L}(\mathcal{Y})^{\text{mon-free}} \to \mathfrak{L}(\mathcal{Y}),$$

and the corresponding unital factorization functors

$$\mathrm{IndCoh}^*(\mathfrak{L}^+(\mathcal{Y})) \to \mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y})^{\text{mon-free}}) \to \mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y})).$$

Let

$$\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))_{\text{mon-free}} \subset \mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))$$

be the full subcategory consisting of objects set-theoretically supported on $\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y})^{\text{mon-free}})$.

From Lemmas 6.1.5 and 6.2.5 we obtain:

Corollary 6.2.7. *There exists a canonical* pointwise *equivalence*

$$\mathrm{Funct}_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod}}\left(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})), \mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))\right) \simeq$$
$$\simeq \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))_{\text{mon-free}}\right),$$

where:

- *In the left-hand side,* $\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))$ *is viewed as a module category over* $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$ *via the functor* $\mathfrak{r}^*$*;*

- *In the right-hand side,* $\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))$ *(and its full subcategory* $\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))_{\text{mon-free}}$*) is viewed as a factorization module category over* $\mathrm{Rep}(\check{G})$*, by the procedure of Sect. 6.2.3.*
- *In the right-hand side,* $\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}$ *denotes the factorization category, corresponding to* $\mathrm{Rep}(\check{G})$*, viewed as a factorization module over itself.*

Remark 6.2.8. It should be possible to replace $\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))_{\text{mon-free}}$ with $\mathrm{IndCoh}^*(\mathfrak{L}(\mathcal{Y}))$ in the above corollary. Our inability to do so at the present moment is responsible for some additional, innocuous technical complications.

6.2.9. We apply the above discussion to the case when $\mathcal{Y} = \mathrm{Op}_{\check{G}}$. Combining Corollary 6.2.7 with Lemma 5.1.7, we obtain:

Corollary 6.2.10. *The functor of direct image along* (5.2) *upgrades canonically to a pointwise equivalence*

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \xrightarrow{\sim} \tag{6.5}$$
$$\to \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}\right).$$

6.2.11. In the factorizable setting, we do not know if Corollary 6.2.10 is true. However, we can reconstruct

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))$$

from

$$\mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\right)$$

as follows.

Lemma 6.2.12. *Let* S *be an affine scheme with a map to* Ran.

(a) *The category*

$$\mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\right)_S$$

has a unique t-structure for which the (conservative) forgetful functor to $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ *is t-exact. The subcategory* $\mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}\right)_S$ *is preserved under truncations, so also inherits a t-structure.*

(b) *The functor*

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))_S \to \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}\right)_S$$

is t-exact and an equivalence on eventually coconnective subcategories.

(c) *The category* $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))_S$ *is compactly generated by eventually coconnective objects. Moreover, an object of* $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))_S$ *is compact if and only if its image in* $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_S$ *is compact.*

6.3. **Spherical vs. Whittaker.**

6.3.1. Let $\mathbf{C}$ be a category equipped with a $\mathfrak{L}(G)_{\rho(\omega_X)}$-action at the critical level. Denote

$$\mathrm{Sph}(\mathbf{C}) := \mathbf{C}^{\mathfrak{L}^+(G)_{\rho(\omega_X)}} \text{ and } \mathrm{Whit}_*(\mathbf{C}) := \mathbf{C}_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi}.$$

6.3.2. We claim that $\mathrm{Whit}_*(\mathbf{C})$ can be naturally upgraded to an object

$$\mathrm{Whit}_*(\mathbf{C})^{\mathrm{fact},\mathrm{Rep}(\check{G})} \in \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}. \tag{6.6}$$

Indeed, the construction of [CFGY, Sect. 1.1] upgrades $\mathbf{C}$ to an object of

$$\mathbf{C}^{\mathrm{fact},\mathrm{Gr}} \in \text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_{G,\rho(\omega_X)})\text{-}\mathbf{mod}^{\mathrm{fact}}.$$

The categories comprising $\mathbf{C}^{\mathrm{fact},\mathrm{Gr}}$ carry an action of the (factorization) version of $\mathfrak{L}(G)_{\rho(\omega_X)}$. Applying the functor of $(\mathfrak{L}(N)_{\rho(\omega_X)},\chi)$-coinvariants, we obtain that $\mathbf{C}^{\mathrm{fact},\mathrm{Gr}}$ gives rise to an object

$$\mathrm{Whit}(\mathbf{C}^{\mathrm{fact},\mathrm{Gr}}) \in \mathrm{Whit}_*(G)\text{-}\mathbf{mod}^{\mathrm{fact}}.$$

Finally identifying

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G)$$

as factorization categories, we transform $\mathrm{Whit}(\mathbf{C}^{\mathrm{fact},\mathrm{Gr}})$ to the desired object (6.6).

6.3.3. We now claim that there exists a canonically defined functor

$$\mathrm{Sph}(\mathbf{C}) \to \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{Whit}_*(\mathbf{C})^{\mathrm{fact},\mathrm{Rep}(\check{G})}\right), \tag{6.7}$$

where

$$\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})} \in \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}$$

is the object corresponding to $\mathrm{Rep}(\check{G})$, viewed as a factorization module over itself.

6.3.4. Namely, we start with the tautological functor

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_{G,\rho(\omega_X)}) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{C}) \to \mathbf{C}.$$

By construction, this functor is a morphism of categories acted on by $\mathfrak{L}(G)_{\rho(\omega_X)}$, so a morphism of factorization $\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_{G,\rho(\omega_X)})$-module categories.

Passing to Whittaker coinvariants, we obtain a functor

$$\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{C}) \to \mathrm{Whit}_*(\mathbf{C}) \tag{6.8}$$

that is a morphism of factorization $\mathrm{Whit}_*(G)$-module categories.

We now apply $\mathrm{FLE}_{\check{G},\infty}$ to obtain a pairing

$$\mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{C}) \to \mathrm{Whit}_*(\mathbf{C})$$

that is a morphism of factorization $\mathrm{Rep}(\check{G})$-module categories.

We obtain the morphism (6.7) by tensor-Hom adjunction.

6.4. **Spherical vs. Whittaker, continued.** *For the remainder Sect. 6, we work in the pointwise context, not over all of Ran space.*

6.4.1. Let

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp}} \hookrightarrow \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \tag{6.9}$$

be the *tempered subcategory*.

I.e., this is the full subcategory generated by the essential image of

$$\mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D})) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}))$$

by the !-pullback functor along

$$\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}} \to \mathrm{LS}_{\check{G}}(\mathcal{D})$$

along either of the projections.

The embedding (6.9) admits a right adjoint, whose kernel is a monoidal ideal. This allows us to view $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp}}$ as a monoidal colocalization of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$.

6.4.2. Let $\mathbf{C}$ be as in Sect. 6.3.1. Set

$$\mathrm{Sph}(\mathbf{C})_{\mathrm{temp}} := \mathrm{Sph}_{G,\mathrm{temp}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{C}),$$

where

$$\mathrm{Sph}_G \to \mathrm{Sph}_{G,\mathrm{temp}} \tag{6.10}$$

is the colocalization corresponding to

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp}}$$

(we can use either Sat_G or Sat_G to identify Sph_G with $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$; the resulting colocalizations are the same).

The functor (6.10) gives rise to a functor

$$\mathrm{Sph}(\mathbf{C}) \to \mathrm{Sph}(\mathbf{C})_{\mathrm{temp}}, \tag{6.11}$$

and since the former is a colocalization, so is the functor (6.11).

6.4.3. We now claim:

Proposition 6.4.4.

(a) *The functor* (6.7) *factors as*

$$\mathrm{Sph}(\mathbf{C}) \to \mathrm{Sph}(\mathbf{C})_{\mathrm{temp}} \to \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{Whit}_*(\mathbf{C})^{\mathrm{fact},\mathrm{Rep}(\check{G})}\right).$$

(b) *Suppose that* $\mathbf{C}$ *is generated, as a category acted on by* $\mathfrak{L}(G)_{\rho(\omega_X)}$*, by the essential image of the forgetful functor*

$$\mathrm{Sph}(\mathbf{C}) \to \mathbf{C}.$$

Then the above functor

$$\mathrm{Sph}(\mathbf{C})_{\mathrm{temp}} \to \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{Whit}_*(\mathbf{C})^{\mathrm{fact},\mathrm{Rep}(\check{G})}\right) \tag{6.12}$$

is an equivalence.

(c) *More generally, let define*

$$\mathbf{C}^{\text{Sph-gen}} \subset \mathbf{C} \tag{6.13}$$

to be the essential image of the fully faithful embedding

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{C}) \to \mathbf{C}.$$

Then the functor

$$\mathrm{Sph}(\mathbf{C})_{\mathrm{temp}} \to \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{Whit}_*(\mathbf{C}^{\text{Sph-gen}})^{\mathrm{fact},\mathrm{Rep}(\check{G})}\right) \tag{6.14}$$

is an equivalence.

Proof. The assertion (a) follows as the pairing (6.8) factors through a similar expression with $\mathrm{Sph}(\mathbf{C})_{\mathrm{temp}}$ in place of $\mathrm{Sph}(\mathbf{C})$.

In the setting of (b), note that the right hand side of (6.14) commutes with colimits in the variable $\mathbf{C}$ and commutes with tensoring by DG categories; this follows from Lemma 6.1.5. This reduces the assertion to the case when $\mathbf{C} := \text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_{G,\rho(\omega_X)})$, where it amounts to the assertion that

$$\mathrm{Sph}_{G,\mathrm{temp}} \to \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}\right) \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}}) = \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp}}$$

is an equivalence, where the displayed isomorphism follows from Lemma 6.1.5. This assertion is immediate from the construction of the derived Satake isomorphism in [CR].

Assertion (c) follows immediately from (b) applied to $\mathbf{C}^{\text{Sph-gen}}$.

□

Remark 6.4.5. Parallel to Remark 6.2.8, we actually expect (c) to hold with $\mathbf{C}$ in place of $\mathbf{C}^{\text{Sph-gen}}$.

7. The critical FLE

In this section we prove the main result of Part I, namely, the *critical FLE*, Theorem 7.3.4, which says that there exists a canonical equivalence of factorization categories

$$\text{FLE}_{G,\text{crit}} : \text{KL}(G)_{\text{crit}} \xrightarrow{\sim} \text{IndCoh}^*(\text{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)), \tag{7.1}$$

The functor in one direction in (7.1) is a variation on the theme of the functor DS; essentially $\text{FLE}_{G,\text{crit}}$ is obtained by decorating DS using Proposition 6.4.4 from the previous section.

Once we the equivalence (7.1) is established, we proceed to the study of its properties. The key ones are:

- The compatibility of $\text{FLE}_{G,\text{crit}}$ with the equivalence $\text{FLE}_{\check{G},\infty} : \text{Rep}(\check{G}) \xrightarrow{\sim} \text{Whit}_*(G)$, expressed by Corollary 7.5.2, which says that the naturally constructed pairings

  $$\text{Whit}_*(G) \otimes \text{KL}(G)_{\text{crit}} \to \text{Vect} \text{ and } \text{Rep}(\check{G}) \otimes \text{IndCoh}^*(\text{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \to \text{Vect}$$

  match up;

- Compatibility of $\text{FLE}_{G,\text{crit}}$ with the self-dualities on the two sides, expressed by Theorem 7.6.4.

7.1. The enhanced functor of Drinfeld-Sokolov reduction.

7.1.1. Consider the functor

$$\text{DS} : \widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X)} \to \text{Vect}$$

of (4.25).

Consider the factorization unit

$$\mathbf{1}_{\widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X)}} \in \widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X)},$$

which is the vacuum module

$$\text{Vac}(G)_{\text{crit},\rho(\omega_X)}.$$

7.1.2. It is a basic fact in the theory of representations at the critical level that there exists a canonical isomorphism of factorization algebras[14]

$$\mathcal{O}_{\text{Op}_{\check{G}}(\mathcal{D})} \overset{\text{FF}^{\mathcal{W}}}{\simeq} \text{DS}(\text{Vac}(G)_{\text{crit},\rho(\omega_X)}) \tag{7.2}$$

in Vect.

Hence, the functor DS can be enhanced to a functor

$$\text{DS}^{\text{enh,coarse}} : \widehat{\mathfrak{g}}\text{-mod}_{\text{crit},\rho(\omega_X)} \to \mathcal{O}_{\text{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\text{fact}}. \tag{7.3}$$

[14]The is the Feigin-Frenkel isomorphism for $\mathcal{W}$-algebras at the critical level.

7.1.3. Recall the functor

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}),-)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})) \to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}$$

of (5.6).

We claim:

Proposition 7.1.4. *The functor* $\mathrm{DS}^{\mathrm{enh,coarse}}$ *can be lifted to a (factorization) functor*

$$\mathrm{DS}^{\mathrm{enh}} : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})),$$

so that

$$\mathrm{DS}^{\mathrm{enh,coarse}} \simeq \Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}),-)^{\mathrm{enh}} \circ \mathrm{DS}^{\mathrm{enh}}.$$

Such a lifting is unique subject to the following conditions:

- $\mathrm{DS}^{\mathrm{enh}}$ *is continuous;*
- $\mathrm{DS}^{\mathrm{enh}}$ *sends compact objects in* $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$ *to eventually coconnective (i.e., bounded below) objects in* $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}))$.

Proof. It is enough to show that the restriction of $\mathrm{DS}^{\mathrm{enh,coarse}}$ to the subcategory

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^c \subset \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$$

can be uniquely lifted to a functor

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^c \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}))^{>-\infty}.$$

However, this follows from Lemma 5.2.2, using the fact that the initial functor DS sends

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^c \to \mathrm{Vect}^{>-\infty}.$$

□

Remark 7.1.5. Note that Proposition 7.1.4 gets us pretty close to the construction of the sought-for functor $\mathrm{FLE}_{G,\mathrm{crit}}$ of (7.1). Namely, the composition

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})) \tag{7.4}$$

is *almost* we want. In order to genuinely construct (7.1), we need to show that (7.4) factors via

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^{\times})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})). \tag{7.5}$$

One way to do this is to resort to abelian categories (at the abelian level, the functor (7.5) is fully faithful). This is essentially how this is done in [FG4]. However, the method by which it was proved that (7.1) is an equivalence at the pointwise level, does not seem to extend to prove that it is an equivalence at the factorization level. So in this paper, we will take a different approach, which is based on the construction in Sect. 6.3 and Theorem 7.1.7 below.

7.1.6. Recall now that the functor DS factors via a functor

$$\overline{\mathrm{DS}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{Vect}.$$

It follows formally that the functor $\mathrm{DS}^{\mathrm{enh,coarse}}$ also factors via a functor, denoted

$$\overline{\mathrm{DS}}^{\mathrm{enh,coarse}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}.$$

We now quote the following fundamental result of [Ras6]:

Theorem 7.1.7. *The functor* $\mathrm{DS}^{\mathrm{enh}}$ *factors via a functor*

$$\overline{\mathrm{DS}}^{\mathrm{enh}} : \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})), \tag{7.6}$$

and the resulting functor $\overline{\mathrm{DS}}^{\mathrm{enh}}$ *is an equivalence of factorization categories.*

7.2. **Compatibility with the factorization module structure.** Consider the functor $\overline{\mathrm{DS}}^{\mathrm{enh}}$, whose existence is guaranteed by Theorem 7.1.7. In this subsection we will endow it with a structure of functor between factorization module categories with respect to $\mathrm{Rep}(\check{G})$.

7.2.1. We consider $\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$ as a factorization module category wit respect to $\mathrm{Rep}(\check{G})$ by the procedure of Sect. 6.3.2.

We consider $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ as a factorization module category over $\mathrm{Rep}(\check{G})$ via the factorization functor

$$\mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \overset{\mathfrak{r}^*}{\to} \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D})) = \mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)). \tag{7.7}$$

We claim that the factorization functor $\overline{\mathrm{DS}}^{\mathrm{enh}}$ is compatible with this structure.

7.2.2. Indeed, we can rewrite the factorization module structure on $\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$ with respect to $\mathrm{Rep}(\check{G})$ specified above as follows:

Consider the factorization functor

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_{G,\rho(\omega_X)}) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{\star}{\to} \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}, \tag{7.8}$$

given by convolution.

Passing to $\mathfrak{L}(N)_{\rho(\omega_X),\chi}$-coinvariants, from (7.8) we obtain a functor

$$\mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{\star}{\to} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}). \tag{7.9}$$

Finally, applying the above functor to

$$\mathbf{1}_{\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}} = \mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)} \in \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}$$

we obtain a (factorization) functor

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\longrightarrow} \mathrm{Whit}_*(G) \overset{-\star\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}}{\longrightarrow} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}). \tag{7.10}$$

Unwinding the definitions, we obtain that the factorization module structure on the (factorization) category $\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$ with respect to $\mathrm{Rep}(\check{G})$ from Sect. 6.3.2 identifies with one given by restriction along the functor (7.10).

7.2.3. We are now ready to show that the functor $\overline{\mathrm{DS}}^{\mathrm{enh}}$ is compatible with the factorization module structures. This amounts to establishing an isomorphism between the following two (factorization) functors: one is

$$\mathrm{Rep}(\check{G}) \overset{(7.10)}{\longrightarrow} \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\overline{\mathrm{DS}}^{\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \tag{7.11}$$

and the other is (7.7).

Note that the functor (7.11) can be rewritten as

$$\begin{aligned}\mathrm{Rep}(\check{G}) \overset{\sigma\circ\mathrm{Sat}^{-1,\mathrm{nv}}_{G,\tau}}{\longrightarrow} \mathrm{Sph}_G \overset{-\star\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \\ \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),\end{aligned} \tag{7.12}$$

where $\mathrm{Sat}^{-1,\mathrm{nv}}_{G,\tau}$ is as in Sect. 1.8.3, ans using Corollary 1.8.2 further as

$$\begin{aligned}\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}^{-1,\mathrm{nv}}_{G}}{\longrightarrow} \mathrm{Sph}_G \overset{-\star\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \\ \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),\end{aligned} \tag{7.13}$$

7.2.4. We will first establish an isomorphism between the compositions of (7.13) and (7.7) with the functor

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}$$

of (5.6).

By the construction of the enhancement

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-) \rightsquigarrow \Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-)^{\mathrm{enh}},$$

it is enough to construct an isomorphism between the compositions of (7.13) and (7.7) with $\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-)$, as factorization functors.

However, this is given by the following result: essentially, [BD, Theorem 5.5.3] (see also [Ras2]), combined with (7.2): [15]

Theorem 7.2.5. *The composition*

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{Sat}^{-1,\mathrm{nv}}_{G,\tau}}{\to} \mathrm{Sph}_G \overset{-\star\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}}{\to} \mathrm{Vect}$$

identifies canonically with

$$\mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \overset{\mathfrak{r}^*}{\to} \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D})) \overset{\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}),-)}{\longrightarrow} \mathrm{Vect}$$

as factorization functors.

7.2.6. We now upgrade the above isomorphism of the two compositions

$$\mathrm{Rep}(\check{G}) \rightrightarrows \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-)^{\mathrm{enh}}}{\longrightarrow} \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}} \tag{7.14}$$

to an isomorphism with target $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$ itself.

It is enough to establish the isomorphism between the two functors in question on the compact generators of $\mathrm{Rep}(\check{G})$. These generators can be taken to be eventually coconnective. Hence, by Lemma 5.2.2, it is enough to show that both functors are t-exact.

7.2.7. The t-exactness is clear for (7.7).

From the isomorphism (7.14), it follows that the composition of (7.13) with $\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-)^{\mathrm{enh}}$ is t-exact.

Now the t-exactness assertion follows from the construction of the upgrade

$$\mathrm{DS}^{\mathrm{enh,coarse}} \rightsquigarrow \mathrm{DS}^{\mathrm{enh}}$$

in Proposition 7.1.4.

7.3. **The critical FLE.** The construction of the sought-for functor (7.1) will be based on the construction in Sect. 6.3.

[15] Our convention for the isomorphism (7.2) differs from one in [BD] by the Cartan involution. This convention determines one for the functor $\mathrm{FLE}_{G,\mathrm{crit}}$. The convention adopted in this paper is compatible with Jacquet functors, see Theorem 9.1.3.

7.3.1. Consider $\mathbf{C} = \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$ as a category acted on by $\mathfrak{L}(G)_{\rho(\omega_X)}$.

By (6.7), we obtain

$$\begin{aligned}(7.15)\quad \mathrm{FLE}^{\mathrm{coarse}}_{G,\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit}} = \mathrm{Sph}(\mathbf{C}) \to \\ \to \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{Whit}_*(\mathbf{C})^{\mathrm{fact},\mathrm{Rep}(\check{G})}\right) = \\ = \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\right).\end{aligned}$$

Per the pointwise statement Corollary 6.2.10, the right hand side of (7.15) can be thought of as a stand-in for $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$.[16] We refine the target as follows.

It is known[17] that the functor

$$\mathrm{DS}^{\mathrm{enh}} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

is t-exact and lands in $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}$. Moreover, as the functor $\mathrm{Sph}(\mathbf{C}) \to \mathrm{Whit}_*(\mathbf{C})$ always admits a right adjoint, the functor $\mathrm{DS}^{\mathrm{enh}}|_{\mathrm{KL}(G)_{\mathrm{crit}}}$ admits a factorizable right adjoint.

The following now formally results from Lemma 6.2.12.

Lemma 7.3.2. (a) *There exists a unique functor*

$$\mathrm{FLE}_{\mathrm{crit}} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$$

that is t-exact and fits into a commutative diagram

$$\begin{array}{ccc} & \overset{\mathrm{FLE}_{\mathrm{crit}}}{\dashrightarrow} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \\ & & \downarrow \\ \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}^{\mathrm{coarse}}_{\mathrm{crit}}} & \mathrm{Funct}_{\mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}}\left(\mathrm{Rep}(\check{G})^{\mathrm{fact},\mathrm{Rep}(\check{G})}, \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\right) \end{array}$$

(b) *The functor* $\mathrm{FLE}_{\mathrm{crit}}$ *admits a factorizable right adjoint.*

7.3.3. We now claim:

Main Theorem 7.3.4. *The functor* $\mathrm{FLE}_{G,\mathrm{crit}}$ *of* (7.1)*, constructed above, is an equivalence.*

By standard arguments (cf. [Ras5] Appendix A), Lemma 7.3.2 reduces us to proving the pointwise assertion. This is essentially known by [FG4], although the functor was presented differently there. For completeness, we present another argument of the pointwise assertion below.

7.3.5. *Proof of the pointwise assertion.* We now prove that the FLE functor is a pointwise equivalence, which yields Theorem 7.3.4 by the above. *For this subsubsection, all our categories are understood pointwise.*

By Corollary 6.2.10, it suffices to show that $\mathrm{FLE}^{\mathrm{coarse}}_{\mathrm{crit}}$ is a pointwise equivalence. In the setting of Proposition 6.4.4(c), note that

$$\mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}^{\text{Sph-gen}}_{\mathrm{crit},\rho(\omega_X)}) \subset \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

is the subcategory $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}$ consisting of objects set-theoretically supported on $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)$; for example, this follows from the calculation of Drinfeld-Sokolov reductions of Weyl modules in [FG2].

By Proposition 6.4.4, we are reduced to showing that $\mathrm{KL}(G)_{\mathrm{crit}} = (\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{temp}}$, or equivalently, that $\mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})$ is conservative.

---

[16] Per Remark 6.2.8, it might be better to replace $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$ with its subcategory $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}$ to better match our present state of knowledge.

[17] As a pointwise statement, this is standard, cf. [Ras6, Corollary 7.2.2] for example. This formally implies left t-exactness over X^I by factorization. But right t-exactness follows easily from the adolescent Whittaker formalism of [Ras6] – the subtleties in [Ras6, Appendix B] are only related to *left* t-exactness.

By [FR1, Proposition 7.3.0.1], there is a canonical t-structure on $(\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{temp}}$ for which the functor $\mathrm{KL}(G)_{\mathrm{crit}} \to (\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{temp}}$ is t-exact. By [FG4], DS is conservative on $\mathrm{KL}(G)_{\mathrm{crit}}^{>-\infty}$, so the same is true of $\mathrm{KL}(G)_{\mathrm{crit}} \to (\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{temp}}$, so this quotient functor is an equivalence on eventually coconnective subcategories.

As compact objects in $\mathrm{KL}(G)_{\mathrm{crit}}$ are eventually coconnective, it suffices to show that $\mathrm{KL}(G)_{\mathrm{crit}} \to (\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{temp}}$ preserves compact objects. We identify the latter with $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))$ as above. Then the calculation of DS reductions of Weyl modules in [FG2] yields the claim.

Remark 7.3.6. As a corollary of the proof, we observe that the action of Sph_G on $\mathrm{KL}(G)_{\mathrm{crit}}$ factors through $\mathrm{Sph}_{G,\mathrm{temp}}$.

7.4. **Coarsened versions of the FLE functor.**

7.4.1. By the construction of the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ we have the following explicit descriptions of its compositions with various forgetful functors out of $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))$:

- The composition with the functor

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

  is the functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-}\mathrm{mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times));$$

- The composition with the functor

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \overset{\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-)^{\mathrm{enh}}}{\longrightarrow} \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-}\mathrm{mod}^{\mathrm{fact}}$$

  is the functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-}\mathrm{mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh,coarse}}}{\to} \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-}\mathrm{mod}^{\mathrm{fact}};$$

- The composition with the functor

$$\Gamma(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times), -) : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \to \mathrm{Vect}$$

  is the functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-}\mathrm{mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}}{\to} \mathrm{Vect}; \tag{7.16}$$

In this subsection we will describe explicitly the composition of $\mathrm{FLE}_{G,\mathrm{crit}}$ with the functor

$$(\mathfrak{r}_*)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \to R_{\check{G},\mathrm{Op}}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}))$$

of (5.8).

7.4.2. In order to describe the composition

$$\mathrm{FLE}_{G,\mathrm{crit}}^{\mathrm{coarse}} := (\mathfrak{r}_*)^{\mathrm{enh}} \circ \mathrm{FLE}_{G,\mathrm{crit}}, \quad \mathrm{KL}(G)_{\mathrm{crit}} \to R_{\check{G},\mathrm{Op}}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G})),$$

it suffices to describe the composition

$$\text{pre-}\mathrm{FLE}_{G,\mathrm{crit}} := \mathfrak{r}_* \circ \mathrm{FLE}_{G,\mathrm{crit}}, \quad \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{Rep}(\check{G}) \tag{7.17}$$

as a factorization functor. Since $\mathrm{FLE}_{G,\mathrm{crit}}$ is unital, it will follow automatically that the image of the factorization unit

$$\mathbf{1}_{\mathrm{KL}(G)_{\mathrm{crit}}} = \mathrm{Vac}(G)_{\mathrm{crit}} \in \mathrm{KL}(G)_{\mathrm{crit}}$$

under $\text{pre-}\mathrm{FLE}_{G,\mathrm{crit}}$ identifies with $R_{\check{G},\mathrm{Op}}$.

7.4.3. Using the self-duality of $\mathrm{Rep}(\check{G})$ as a factorization category, the datum of a functor (7.17) is equivalent to that of a factorization functor

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{Vect}\,. \tag{7.18}$$

Unwinding the definitions, we obtain that the functor (7.18) equals

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{\check{G},\infty} \otimes \alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{(7.9)}{\longrightarrow}$$
$$\to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-}\mathrm{mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\overline{\mathrm{DS}}}{\to} \mathrm{Vect},$$

Remark 7.4.4. The above procedure can be used to give an alternative construction of the functor $\mathrm{FLE}_{G,\mathrm{crit}}$:

We can *define* the functor pre-$\mathrm{FLE}^{\mathrm{coarse}}_{G,\mathrm{crit}}$ by the procedure described above, then checked explicitly using Theorem 7.2.5 that it sends $\mathbf{1}_{\mathrm{KL}(G)_{\mathrm{crit}}}$ to $R_{\check{G},\mathrm{Op}}$, and thus *define* the corresponding functor $\mathrm{FLE}^{\mathrm{coarse}}_{G,\mathrm{crit}}$.

One can then lift it to a functor

$$\mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$$

using Lemma 5.2.6.

Remark 7.4.5. It is known that the functor (7.16) is t-exact. This implies that the functor $\mathrm{FLE}^{\mathrm{coarse}}_{G,\mathrm{crit}}$ is t-exact.

Arguing as in Sect. 7.2.7, one can deduce from this that the functor $\mathrm{FLE}_{G,\mathrm{crit}}$ itself is t-exact.

7.5. **Compatibility of $\mathrm{FLE}_{G,\mathrm{crit}}$ and $\mathrm{FLE}_{\check{G},\infty}$.** We now record the following compatibility property of the functors $\mathrm{FLE}_{G,\mathrm{crit}}$ and $\mathrm{FLE}_{\check{G},\infty}$, to be used in the sequel.

7.5.1. Recall that we have the equivalences

$$\mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}} \tag{7.19}$$

and

$$\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \simeq \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-}\mathrm{mod}_{\mathrm{crit},\rho(\omega_X)})^{\text{Sph-gen}} \overset{\overline{\mathrm{DS}}^{\mathrm{enh}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}. \tag{7.20}$$

The following was embedded into the construction of the $\mathrm{FLE}_{G,\mathrm{crit}}$ functor (see (1.9)):

Corollary 7.5.2. *The functors* (7.20) *and* (7.19) *match under the equivalences*

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)),\ \mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G).$$

Remark 7.5.3. Note that the actions on Sph_G and $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on $\mathrm{KL}(G)_{\mathrm{crit}}$ and $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$, respectively match under Sat_G, and on $\mathrm{Whit}_*(G)$ and $\mathrm{Rep}(\check{G})$ under $\mathrm{Sat}_{\check{G}}$. This is in line with the curse in Sect. 1.8.7.

7.5.4. Denote by

$$\mathsf{P}^{\mathrm{loc,enh}}_G : \mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \tag{7.21}$$

the resulting pairing

$$\mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{(7.20)}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}} \hookrightarrow$$
$$\hookrightarrow \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Explicitly, it is given by

$$\mathrm{Whit}_*(G)\otimes \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{Id}\otimes \alpha_{\rho(\omega_X)},\mathrm{taut}}{\longrightarrow} \mathrm{Whit}_*(G)\otimes \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{(7.9)}{\longrightarrow} \tag{7.22}$$
$$\to \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-}\mathrm{mod}_{\mathrm{crit},\rho(\omega_X)}) \overset{\overline{\mathrm{DS}}^{\mathrm{enh}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Let $\mathsf{P}^{\mathrm{loc}}_G$ and $\mathsf{P}^{\mathrm{loc,enh,coarse}}_G$ denote the compositions of $\mathsf{P}^{\mathrm{loc,enh}}_{\check{G}}$ with the forgetful functors

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-):\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\to \mathrm{Vect} \tag{7.23}$$

and

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-)^{\mathrm{enh}}:\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-}\mathrm{mod}^{\mathrm{fact}}, \tag{7.24}$$

respectively. (These two functors are obtained by replacing the last arrow in (7.22) by $\overline{\mathrm{DS}}$ and $\overline{\mathrm{DS}}^{\mathrm{enh,coarse}}$, respectively.)

7.5.5. Denote by

$$\mathsf{P}^{\mathrm{loc,enh}}_{\check{G}}:\mathrm{Rep}(\check{G})\otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \tag{7.25}$$

the resulting pairing

$$\mathrm{Rep}(\check{G})\otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))\to \mathrm{Rep}(\check{G})\underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes}\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{(7.19)}{\simeq}$$
$$\simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\hookrightarrow \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Explicitly, it is given by

$$\mathrm{Rep}(\check{G})\otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\mathfrak{r}^*\otimes \mathrm{Id}}{\to} \tag{7.26}$$
$$\to \mathrm{QCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))\otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\otimes}{\to} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))\to$$
$$\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Let $\mathsf{P}^{\mathrm{loc}}_{\check{G}}$ and $\mathsf{P}^{\mathrm{loc,enh,coarse}}_{\check{G}}$ denote the compositions of $\mathsf{P}^{\mathrm{loc,enh}}_{\check{G}}$ with the forgetful functors (7.23) and (7.24), respectively.

7.5.6. From Corollary 7.5.2 we immediately obtain:

Corollary 7.5.7. *The functors* $\mathsf{P}^{\mathrm{loc,enh}}_G$ *and* $\mathsf{P}^{\mathrm{loc,enh}}_{\check{G}}$ *match under the equivalences*

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \text{ and } \mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G).$$

And hence:

Corollary 7.5.8. *The functors* $\mathsf{P}^{\mathrm{loc}}_G$ *and* $\mathsf{P}^{\mathrm{loc}}_{\check{G}}$ *(resp.,* $\mathsf{P}^{\mathrm{loc,enh,coarse}}_G$ *and* $\mathsf{P}^{\mathrm{loc,enh,coarse}}_{\check{G}}$*) match under the equivalences*

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \text{ and } \mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G).$$

7.6. **Compatibility with duality.**

7.6.1. Recall that according to (4.3), we have a canonical identification

$$(\mathrm{KL}(G)_{\mathrm{crit}})^\vee \simeq \mathrm{KL}(G)_{\mathrm{crit}}. \tag{7.27}$$

By construction, the equivalence (7.27) respects the actions of Sph_G.

7.6.2. In addition, we have an equivalence

$$\Big(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))\Big)^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\Theta_{\mathrm{Op}(\check{G})}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)). \tag{7.28}$$

This equivalence respects the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$.

7.6.3. We claim:

Theorem 7.6.4. *With respect to the identifications* (7.27) *and* (7.28)*, the functor*

$$(\mathrm{FLE}_{G,\mathrm{crit}})^\vee : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \to \mathrm{KL}(G)_{\mathrm{crit}}$$

identifies with

$$\tau_G \circ (\mathrm{FLE}_{G,\mathrm{crit}})^{-1}.$$

Moreover, this identification of functors respects the compatibility with the actions of

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_{G,\tau}}{\sim} \mathrm{Sph}_{\check{G}}^{\mathrm{spec}}.$$

Remark 7.6.5. Note the similarity between the statement of Theorem 7.6.4 and Lemma 1.4.11: in both cases a non-tautological self-equivalence of the Whittaker side makes the FLE inverse to its dual, up to the Cartan involution.

Remark 7.6.6. Note again that the appearance of the Cartan involution in Theorem 7.6.4 is in line with the curse in Sect. 1.8.8.

7.7. **Twisted version.**

7.7.1. Let $\mathcal{P}_{Z^0_{\check{G}}}$ be a $Z^0_{\check{G}}$-bundle on X. We consider the following variants of the two sides of the FLE:

On the Kac-Moody side, we consider the category

$$\mathrm{KL}(G)_{\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})},$$

see Sect. 4.4.1.

On the oper side, we consider the category

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)),$$

see Sect. 5.4.1.

Taking into account Sect. 4.4.3, the construction in Sect. 7.3 applies and we obtain a functor

$$\mathrm{FLE}_{G,\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} : \mathrm{KL}(G)_{\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)). \tag{7.29}$$

Since the assertion of Theorem 7.3.4 is local, it formally implies that the functor (7.29) is also an equivalence.

7.7.2. Again using Sect. 4.4.3, we can repeat the construction of Sect. 7.5.4 and obtain a functor

$$\mathsf{P}^{\mathrm{loc,enh}}_G : \mathrm{Whit}_*(G) \otimes \mathrm{KL}(G)_{\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)). \tag{7.30}$$

As in Sect. 7.5.5, we obtain a functor

$$\mathsf{P}^{\mathrm{loc,enh}}_{\check{G}} : \mathrm{Rep}(\check{G}) \otimes \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)). \tag{7.31}$$

As in Corollary 7.5.7, we obtain that the above functors $\mathsf{P}^{\mathrm{loc,enh}}_G$ and $\mathsf{P}^{\mathrm{loc,enh}}_{\check{G}}$ match under the equivalences

$$\mathrm{KL}(G)_{\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} \overset{\mathrm{FLE}_{G,\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times))$$

and

$$\mathrm{Rep}(\check{G}) \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(G).$$

7.7.3. Finally, note that we have the equivalences

$$(\mathrm{KL}(G)_{\mathrm{crit},-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})})^\vee \simeq \mathrm{KL}(G)_{\mathrm{crit}+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}. \tag{7.32}$$

In addition, we have an equivalence

$$\left(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times))\right)^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)) \overset{\Theta_{\mathrm{Op}(\check{G})}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)). \tag{7.33}$$

Note also that the Cartan involution τ_G induces an equivalence

$$\mathrm{KL}(G)_{\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} \overset{\tau_G}{\simeq} \mathrm{KL}(G)_{\mathrm{crit}+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}.$$

It follows formally from Theorem 7.6.4 that, with respect to the identifications (7.32) and (7.33), the functor

$$(\mathrm{FLE}_{G,\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})})^\vee : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)) \to \mathrm{KL}(G)_{\mathrm{crit}+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}$$

identifies with

$$\tau_G \circ (\mathrm{FLE}_{G,\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})})^{-1}.$$

7.7.4. In practice, we will take the reductive group in question to be the Levi subgroup M of a standard parabolic P, and $Z_{\check{M}} := \check{\rho}_P(\omega_X)$.

So in this case, the equivalence (7.29) specializes to

$$\mathrm{FLE}_{M,\mathrm{crit}-\check{\rho}_P} : \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P} \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)), \tag{7.34}$$

see Sect. 4.4.4 for the notational conventions.

8. Proof of Theorem 7.6.4

The idea of the proof is the following: we reduce the assertion of the theorem to the fact that the natural self-duality of $\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ is compatible under

$$\overline{\mathrm{DS}}^{\mathrm{enh}} : (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)}} \simeq \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

with a self-duality of the left-hand side.

The latter assertion may be hard to see explicitly, but it follows immediately from factorization: any two self-dualities of $\mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ differ by a line bundle, which is automatically constant (i.e., is a line over k), since $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ is an affine space. However, factorization implies that this line comes from a factorization line bundle on the Ran space, and the latter is necessarily trivial.

8.1. **Recollections on the Feigin-Frenkel center.** In order to prove Theorem 7.6.4, we will use an additional piece of structure that exists on the category $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$, namely, the Feigin-Frenkel center.

8.1.1. Let $\mathfrak{z}_{\mathfrak{g}}$ denote the Feigin-Frenkel center of $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$, thought of as a factorization algebra mapping to $\mathrm{inv}_{\mathfrak{L}^+(G)}(\mathrm{Vac}(G)_{\mathrm{crit}})$.

In fact, at the pointwise level, $\mathfrak{z}_{\mathfrak{g}}$ is the 0-th cohomology of $\mathrm{inv}_{\mathfrak{L}^+(G)}(\mathrm{Vac}(G)_{\mathrm{crit}})$.

8.1.2. By construction, $\mathfrak{z}_{\mathfrak{g}}$ is insensitive to twists by $\mathfrak{L}^+(G)$-torsors. So, we can equivalently view $\mathfrak{z}_{\mathfrak{g}}$ as mapping to

$$\mathrm{inv}_{\mathfrak{L}^+(G)_{\rho(\omega_X)}}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}).$$

A basic fact in representation theory at the critical level is that the composite map

$$\mathfrak{z}_{\mathfrak{g}} \to \mathrm{Vac}(G)^{\mathfrak{L}^+(G)_{\rho(\omega_X)}}_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{Vac}(G)^{\mathfrak{L}^+(N)_{\rho(\omega_X)}}_{\mathrm{crit},\rho(\omega_X)} \to \mathrm{DS}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}) \tag{8.1}$$

is an isomorphism.

The composition

$$\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})} \xrightarrow{\mathrm{FF}^{\mathcal{W}}} \mathrm{DS}_G(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}) \overset{(8.1)^{-1}}{\simeq} \mathfrak{z}_{\mathfrak{g}}, \tag{8.2}$$

where $\mathrm{FF}^{\mathcal{W}}$ is as in (7.2) is the Feigin-Frenkel isomorphism at level of centers for G.

We will denote the map (8.2) by $\mathrm{FF}^{\mathfrak{z}}$.

8.1.3. A crucial piece of structure that we will use that arises from the identification (8.2) and Sect. 4.2.3 is an action of the (symmetric) monoidal category $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$.

8.1.4. By construction, the functor

$$\mathrm{DS}^{\mathrm{enh}} : \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

intertwines the above $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$-action on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$ and the natural action of $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ on $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$.

8.1.5. The action of $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ on $\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$ gives rise to an action of $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ on

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)}}.$$

It follows formally that the equivalence

$$\overline{\mathrm{DS}}^{\mathrm{enh}} : (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

is compatible with the $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$-actions.

8.2. **Feigin-Frenkel center and self-duality.**

8.2.1. Recall that the unit for the self-duality on $\mathrm{KL}(G)_{\mathrm{crit}}$ is given by the (factorization algebra) object

$$\mathrm{CDO}(G)_{\mathrm{crit},\mathrm{crit}} \in \mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{KL}(G)_{\mathrm{crit}}.$$

It is proved in [FG1, Theorem 5.4] that the following diagram of factorization algebras (in Vect) commutes

$$\begin{array}{ccc} \mathfrak{z}_{\mathfrak{g}} & \xrightarrow{\tau_G} & \mathfrak{z}_{\mathfrak{g}} \\ \downarrow & & \downarrow \\ \mathrm{Vac}(G)_{\mathrm{crit}} & & \mathrm{Vac}(G)_{\mathrm{crit}} \\ {\scriptstyle\mathrm{left}}\downarrow & & {\scriptstyle\mathrm{right}}\downarrow \\ \mathrm{CDO}(G)_{\mathrm{crit},\mathrm{crit}} & \xrightarrow{=} & \mathrm{CDO}(G)_{\mathrm{crit},\mathrm{crit}}, \end{array} \tag{8.3}$$

where left and right are the two maps corresponding to the structure on $\mathrm{CDO}(G)_{\mathrm{crit},\mathrm{crit}}$ of factorization algebra object in $\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{KL}(G)_{\mathrm{crit}}$.

8.2.2. This implies that the self-duality

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^\vee \simeq \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$$

of (4.2) is compatible with the $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$-actions up to τ_G.

8.3. **Self-duality on opers via Kac-Moody.**

8.3.1. Let $\mathbf{C}$ be a category acted on by $\mathfrak{L}(G)_{\rho(\omega_X)}$. The construction of Sect. 1.3.5 applies, and gives rise to a functor, to be denoted $\Theta_{\mathrm{Whit}_*(\mathbf{C})}$:

$$\mathrm{Whit}_*(\mathbf{C}) := \mathbf{C}_{\mathfrak{L}(N)_{\rho(\omega_X)},\chi} \to \mathbf{C}^{\mathfrak{L}(N)_{\rho(\omega_X)},\chi} =: \mathrm{Whit}^!(\mathbf{C}). \tag{8.4}$$

Theorem 1.3.6 applies in this general situation and implies that the functor (8.4) is an equivalence.

In particular, we obtain that if $\mathbf{C}$ is dualizable, then $\mathrm{Whit}_*(\mathbf{C})$ is dualizable and we obtain an identification

$$\mathrm{Whit}_*(\mathbf{C})^\vee \simeq \mathrm{Whit}^!(\mathbf{C}^\vee) \overset{\Theta^{-1}_{\mathrm{Whit}_*(\mathbf{C})}}{\longrightarrow} \mathrm{Whit}_*(\mathbf{C}^\vee). \tag{8.5}$$

8.3.2. We apply (8.4) to $\mathbf{C} := \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$. Combining with the identification (4.2) we obtain a self-duality

$$\Big((\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)}}\Big)^\vee \simeq (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)}}. \tag{8.6}$$

8.3.3. Combining with Theorem 7.1.7, the identification (8.6) gives rise to an identification

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))^\vee \simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)). \tag{8.7}$$

8.3.4. Since the functor (8.4) is given by averaging with respect to a subgroup of $\mathfrak{L}(G)_{\rho(\omega_X)}$, the identification $\Theta_{\mathrm{Whit}(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})}$ is compatible with the actions of $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$.

Combining with Sect. 8.2.2 we obtain that the identification (8.6) respects the actions of $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$, up to τ_G.

Combining further with Sect. 8.1.5, we obtain that the identification (8.7) is compatible with the natural action of $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ on $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$, up to τ_G.

8.3.5. Using the tautological identification

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),$$

we can interpret (8.7) as an equiavalence

$$\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)). \tag{8.8}$$

Thus, we obtain that the identification (8.8) is compatible, up to τ_G, with the natural actions of $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ on $\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ and $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$, respectively.

Hence, by Sect. 5.3.2, the identification (8.8) is given by an object

$${}'\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)} \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \tag{8.9}$$

We claim:

Theorem 8.3.6. *There exists a canonical isomorphism between* ${}'\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ *and the object* $\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ *of* (5.12)*, compatible with factorization.*

Remark 8.3.7. As we shall see below, Theorem 8.3.6 is actually easy. However, it can be seen as a particular case of a conjecture, proposed by G. Dhillon, which says that at any level κ, the self-dualities of the (renormalized) categories of factorization modules

$$\mathcal{W}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}} \simeq \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}\text{-mod}^{\mathrm{fact}}$$

that come from the identifications

$$\mathcal{W}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}} = \mathrm{Whit}_*(\widehat{\mathfrak{g}}\text{-mod}_\kappa) \text{ and } \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}\text{-mod}^{\mathrm{fact}} = \mathrm{Whit}_*(\widehat{\check{\mathfrak{g}}}\text{-mod}_{\check{\kappa}})$$

and (8.5), respectively, agree.

For non-critical κ, this conjecture is completely open. What makes it tractable at the critical level is precisely the interpretation of $\mathcal{W}_{\mathfrak{g},\mathrm{crit}}$ as the Feigin-Frenkel center.

8.3.8. *Proof of Theorem 8.3.6.* Since both objects

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)} \text{ and } {}'\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$$

define equivalences

$$\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),$$

a priori, one is obtained from another by tensoring with a line bundle.

Since $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ is fibered over Ran into affine spaces, the above line bundle is canonically pulled back from a line bundle on Ran.

Since all objects and identifications in sight are compatible with factorization, the above line bundle on Ran is equipped with a factorization structure. Furthermore, the constructions involved are *unital*, so the line bundle in question is equipped with a connection, i.e., it is a one-dimensional factorization local system on Ran.

However, it is easy to see that any such object is canonically trivial.

□[Theorem 8.3.6]

8.4. **Proof of Theorem 7.6.4.**

8.4.1. Let $\mathbf{C}$ be a category, acted on by $\mathfrak{L}(G)_{\rho(\omega_X)}$. Assume that $\mathbf{C}$ is spherically-generated, i.e., that the embedding

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Sph}(\mathbf{C}) \to \mathbf{C}$$

is an equivalence.

Note that in this case, the object

$$\mathrm{Whit}_*(\mathbf{C})^{\mathrm{fact},\mathrm{Rep}(\check{G})} \in \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}$$

lies in the essential image of the functor

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))\text{-}\mathbf{mod} \to \mathrm{Rep}(\check{G})\text{-}\mathbf{mod}^{\mathrm{fact}}, \quad \mathbf{C} \mapsto \mathbf{C}^{\mathrm{fact},\mathrm{Rep}(\check{G})}$$

of (6.1). In other words, $\mathrm{Whit}_*(\mathbf{C})$ is naturally a module category over $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}$.

Further, by Proposition 6.4.4, we have a canonical equivalence

$$\mathrm{Sph}(\mathbf{C})_{\mathrm{temp}} \xrightarrow{\sim} \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}}}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})), \mathrm{Whit}_*(\mathbf{C})). \tag{8.10}$$

8.4.2. Assume that $\mathbf{C}$ is dualizable. Note that in this case we have a canonical identification

$$\mathrm{Sph}(\mathbf{C})^\vee \simeq \mathrm{Sph}(\mathbf{C}^\vee), \tag{8.11}$$

so that the functors dual to

$$\mathbf{oblv}^{\mathfrak{L}^+(G)_{\rho(\omega_X)}} : \mathrm{Sph}(\mathbf{C}) \rightleftarrows \mathbf{C} : \mathrm{Av}_*^{\mathfrak{L}^+(G)_{\rho(\omega_X)}}$$

identify with

$$\mathbf{oblv}^{\mathfrak{L}^+(G)_{\rho(\omega_X)}} : \mathrm{Sph}(\mathbf{C}^\vee) \rightleftarrows \mathbf{C}^\vee : \mathrm{Av}_*^{\mathfrak{L}^+(G)_{\rho(\omega_X)}},$$

respectively.

Furthermore, the identification (8.11) gives rise to a uniquely defined identification

$$(\mathrm{Sph}(\mathbf{C})_{\mathrm{temp}})^\vee \simeq \mathrm{Sph}(\mathbf{C}^\vee)_{\mathrm{temp}}, \tag{8.12}$$

so that the functors dual to

$$\mathrm{Sph}(\mathbf{C}) \leftrightarrows \mathrm{Sph}(\mathbf{C})_{\mathrm{temp}}$$

identify with

$$\mathrm{Sph}(\mathbf{C}^\vee) \leftrightarrows \mathrm{Sph}(\mathbf{C}^\vee)_{\mathrm{temp}},$$

respectively.

8.4.3. By (8.5), we can identify

$$(\mathrm{Whit}_*(\mathbf{C}))^\vee \simeq \mathrm{Whit}(\mathbf{C}^\vee).$$

Since the monoidal category $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$ is *semi-rigid* (see [AGKRRV, Appendix C]), for any $\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$-module category $\mathbf{D}$, dualizable as a plain DG category, we have a canonical identification

$$\Big(\mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),\mathbf{D})\Big)^\vee \simeq \tag{8.13}$$
$$\simeq \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),\mathbf{D}^\vee).$$

Combining, we obtain an equivalence

$$\Big(\mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),\mathrm{Whit}_*(\mathbf{C}))\Big)^\vee \simeq \tag{8.14}$$
$$\simeq \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),\mathrm{Whit}(\mathbf{C}^\vee)).$$

8.4.4. Unwinding the construction, we obtain that the equivalence (8.10) and a similar equivalence for $\mathbf{C}^\vee$ are compatible with the identifications (8.12) and (8.14).

8.4.5. Let in the context of (8.13)

$$\mathbf{D} := \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),$$

so that

$$\mathbf{D}^\vee := \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$
$$\mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),\mathbf{D}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))$$

and

$$\mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),\mathbf{D}^\vee) \simeq \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)).$$

In this case, we obtain that the identification (8.13) gives back the canonical identification

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)).$$

Furthermore, for a $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$-linear functor

$$\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),$$

given by an object $\mathcal{F} \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$, the induced functor

$$\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \simeq \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),\mathbf{D}^\vee) \to$$
$$\to \mathrm{Funct}_{\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))}(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})),\mathbf{D}) \simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))$$

is given by the !-pullback of $\mathcal{F}$ along

$$\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times) \to \mathrm{Op}_{\check{G}}(\mathcal{D}^\times). \tag{8.15}$$

8.4.6. We apply this to

$$\mathbf{C} := (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})^{\text{Sph-gen}}.$$

We obtain that with respect to the equivalence

$$\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)),$$

the identification

$$\mathrm{KL}(G)_{\mathrm{crit}}^\vee \simeq \mathrm{KL}(G)_{\mathrm{crit}}$$

of (4.3) corresponds to the identification

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))^\vee \simeq \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)) \overset{{}'\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)}\circ\tau_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)),$$

where ${}'\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)}$ is the !-restriction of ${}'\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ along (8.15).

8.4.7. Thus, in order to construct the identification of functors in Theorem 7.6.4, we have to construct an isomorphism

$$'\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)} \simeq \omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}.$$

However, this follows from (8.3.6).

The compatibility with the Hecke actions follows by unwinding the construction.

□[Theorem 7.6.4]

Part II. The FLE and the Jacquet functors

In Part I of this paper, we studied operations that take place on the geometric side or the spectral side of the local Langlands theory separately, and we connected the two sides in four ways:

- Geometric Satake equivalence $\mathrm{Sph}_G \overset{\mathrm{Sat}_G}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$;
- The geometric Casselman-Shalika formula, i.e. the equivalence $\mathrm{Whit}^!(G) \overset{\mathrm{CS}_G}{\simeq} \mathrm{Rep}(\check{G})$;
- The critical FLE $\mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}))$;
- The semi-infinite geometric Satake $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \overset{\mathrm{Sat}^-,\frac{\infty}{2}}{\simeq} \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}$.

In this part, we will prove a theorem to the effect that a certain operation on the geometric side corresponds to a particular operation on the spectral side. There will be two versions of this result: "as-is", i.e., unenhanced and an enhanced one.

The unenhanced version (Theorem 9.1.3) says that the BRST functor from $\mathrm{KL}(G)_{\mathrm{crit}}$ to the (twisted version of) $\mathrm{KL}(M)_{\mathrm{crit}}$ corresponds to the spectral Jacquet functor from $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}))$ to the shifted version of $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M}}(\mathcal{D}))$. The enhanced version (Theorem 9.1.7) is more involved, and it uses the enhanced BRST and spectral Jacquet functors.

The unenhanced version will be used in the proof of the (global) Theorem 21.2.2, which expresses the compatibility of the (global) Langlands functor $\mathbb{L}_G$ with the operation of constant term. Accordingly, the enhanced version will be used in the proof of Theorem 22.2.4, which is an enhanced version of Theorem 21.2.2.

Theorem 22.2.4 is the main result that will be needed for application to the proof of Theorem 24.1.2.

9. Compatibility of the FLE with the Jacquet functors

In this section we first formulate the theorem that expresses the compatibility of the critical FLE with the BRST and the spectral Jacquet functors (Theorem 9.1.3), as well as its enhanced version (Theorem 9.1.7).

However, in order to prove both these theorems, we will reformulate them in dual terms. Thus, we will formulate Theorems 9.2.4 and 9.5.3, which are equivalent to Theorems 9.1.3 and 9.1.7, respectively.

A feature of the present situation is that although Theorem 9.2.4 looks simpler than its enhanced version, namely, Theorem 9.5.3, we will have to prove the latter in order to prove the former. I.e., the enhanced statements ends up being more accessible than the unenhanced one.

9.1. An initial formulation.

9.1.1. Recall the functor

$$\mathrm{BRST}^-_{\rho_P(\omega_X)} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P},$$

see Sect. 4.7.7.

Recall also the functor

$$J^{-,\mathrm{spec},*} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)),$$

see Sect. 5.5.3.

9.1.2. The following theorem, which is one of the main results of this paper, expresses the compatibility of the FLE with the Jacquet functors:

Main Theorem 9.1.3. *The following diagram of functors commutes*

$$\begin{array}{ccc} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P} & \xrightarrow{\mathrm{FLE}_{M,\mathrm{crit}-\check{\rho}_P}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\ {\scriptstyle \mathrm{BRST}^-_{\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle J^{-,\mathrm{spec},*}} \\ \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)). \end{array}$$

Remark 9.1.4. Note that the statement of Theorem 9.1.3 bears a similarity with that of Corollary 2.7.7.

9.1.5. Recall the functors

$$\mathrm{BRST}^{-,\mathrm{enh}}_{\rho_P(\omega_X)} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P}$$

(see (4.23)) and

$$J^{-,\mathrm{spec},*,\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M}}(\mathcal{D}^\times))^{-,\mathrm{enh}}.$$

(see (5.25)).

Since the FLE respects the Hecke actions, from (7.34) and Theorem 2.6.7 (see also (1.9)) we obtain an equivalence

$$\mathrm{Sat}^{-,\frac{\infty}{2}} \otimes \mathrm{FLE}_{M,\mathrm{crit}-\check{\rho}_P} : \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P} \xrightarrow{\sim} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M}}(\mathcal{D}^\times))^{-,\mathrm{enh}},$$

to be denoted

$$\mathrm{FLE}^{-,\mathrm{enh}}_{M,\mathrm{crit}-\check{\rho}_P}.$$

9.1.6. The following is an enhancement of Theorem 9.1.3:

Main Theorem 9.1.7. *The following diagram of functors commutes*

$$\begin{array}{ccc} \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P} & \xrightarrow{\mathrm{FLE}^{-,\mathrm{enh}}_{M,\mathrm{crit}-\check{\rho}_P}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}} \\ {\scriptstyle \mathrm{BRST}^{-,\mathrm{enh}}_{\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle J^{-,\mathrm{spec},*,\mathrm{enh}}} \\ \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}^{-,\mathrm{enh}}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)). \end{array}$$

Note that the statement of Theorem 9.1.3 can be obtained from that of Theorem 9.1.7 by concatenating with the commutative diagram

$$\begin{array}{ccc} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P} & \xrightarrow{\mathrm{FLE}_{M,\mathrm{crit}-\check{\rho}_P}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\ =\uparrow & & \uparrow= \\ \mathrm{Sph}_M \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P} & & \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\ {\scriptstyle \mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}\otimes\mathrm{Id}}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}\otimes\mathrm{Id}} \\ \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P} & & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\ =\uparrow & & \uparrow= \\ \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P} & \xrightarrow{\mathrm{FLE}^{-,\mathrm{enh}}_{M,\mathrm{crit}-\check{\rho}_P}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}. \end{array}$$

Remark 9.1.8. Note the similarity between the statement of Theorem 9.1.7 and the commutation of diagram (2.27).

9.2. **A dual formulation of Theorem 9.1.3.** In order to prove Theorems 9.1.3 and 9.1.7, we will reformulate them in dual terms. We start with Theorem 9.1.3.

9.2.1. Let

$$coJ^{-,\mathrm{spec},*} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$$

be the functor dual to

$$J^{-,\mathrm{spec},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)).$$

Explicitly, $coJ^{-,\mathrm{spec},*}$ is given by

$$(\mathfrak{p}^{\text{Miu,mon-free}})_* \circ (\mathfrak{q}^{\text{Miu,mon-free}})^!,$$

where the morphisms are

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \xleftarrow{\mathfrak{p}^{\text{Miu,mon-free}}} \mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \xrightarrow{\mathfrak{q}^{\text{Miu,mon-free}}} \mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times),$$

see diagram (5.16).

9.2.2. Consider the diagram

$$\begin{array}{ccc} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P} & \xleftarrow{\tau_M \circ (\mathrm{FLE}_{M,\mathrm{crit}-\check{\rho}_P})^{-1}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\ \downarrow{\scriptstyle \mathrm{Wak}^{-,\mathrm{Sph}}_{\rho_P(\omega_X)}} & & \downarrow{\scriptstyle coJ^{-,\mathrm{spec},*}} \\ \mathrm{KL}(G)_{\mathrm{crit}} & \xleftarrow{\tau_G \circ (\mathrm{FLE}_{G,\mathrm{crit}})^{-1}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)). \end{array}$$

According to Theorem 7.6.4 and Sect. 7.7.3, it can be identified with the diagram, obtained from the diagram in Theorem 9.1.3 by passing to the dual functors.

9.2.3. Hence, we obtain that the statement of Theorem 9.1.3 is equivalent to the following:

Theorem 9.2.4. *The following diagram of (factorization) functors commutes:*

$$\begin{array}{ccc} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P} & \xrightarrow{\mathrm{FLE}_{M,\mathrm{crit}-\check{\rho}_P} \circ \tau_M} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\ \downarrow{\scriptstyle \mathrm{Wak}^{-,\mathrm{Sph}}_{\rho_P(\omega_X)}} & & \downarrow{\scriptstyle coJ^{-,\mathrm{spec},*}} \\ \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}} \circ \tau_G} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)). \end{array} \tag{9.1}$$

9.2.5. We will next give a dual formulation of Theorem 9.1.7.

9.3. **The dual of the left vertical arrow.**

9.3.1. Consider the functor

$$\mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \to (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(M)} =: \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}. \tag{9.2}$$

It is a fully faithful embedding; we denote its essential image by

$$\left(\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}\right)^{\text{Sph-gen}} \subset \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}.$$

It is easy to see that (9.2) admits a right adjoint (as a factorization functor):

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}}. \tag{9.3}$$

9.3.2. The following is straightforward:

Lemma 9.3.3. *The functor* (9.3) *identifies with the dual of the functor*

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \hookrightarrow (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^{\mathfrak{L}^+(M)}_{\mathfrak{L}(N)}.$$

9.3.4. Recall the functor

$$\mathrm{Wak}^{-,\frac{\infty}{2}} : \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \widehat{\mathfrak{g}}\text{-}\mathrm{mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}.$$

Let $\mathrm{Wak}^{-,\frac{\infty}{2},\text{Sph-gen}}$ denote the composition of $\mathrm{Wak}^{-,\frac{\infty}{2}}$ with (9.3)

$$\mathrm{Wak}^{-,\frac{\infty}{2},\text{Sph-gen}} : \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}}.$$

From Lemma 9.3.3 we obtain:

Corollary 9.3.5. *The functor* $\mathrm{Wak}^{-,\frac{\infty}{2},\text{Sph-gen}}$ *is the dual of the functor* $\overline{\mathrm{BRST}}^-$ *of* (4.19).

9.3.6. Note that the composition

$$\widehat{\mathfrak{g}}\text{-}\mathrm{mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \overset{(9.3)}{\longrightarrow} \mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Sph}_G \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \simeq \mathrm{KL}(G)_{\mathrm{crit}}$$

identifies with the functor

$$\mathrm{Av}_*^{\mathfrak{L}^+(G)/\mathfrak{L}^+(M)} : \widehat{\mathfrak{g}}\text{-}\mathrm{mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \mathrm{KL}(G)_{\mathrm{crit}}.$$

Hence, the composition of $\mathrm{Wak}^{-,\frac{\infty}{2},\text{Sph-gen}}$ with

$$\mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}\otimes\mathrm{Id}}{\longrightarrow} \mathrm{Sph}_G \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \simeq \mathrm{KL}(G)_{\mathrm{crit}}$$

identifies with the functor

$$\mathrm{Wak}^{-,\mathrm{Sph}} : \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \mathrm{KL}(G)_{\mathrm{crit}}.$$

9.3.7. Consider the functor

$$\mathrm{BRST}^{-,\mathrm{enh}} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X)}.$$

Its dual is a functor

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co}} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \mathrm{KL}(G)_{\mathrm{crit}}. \tag{9.4}$$

The functor (9.4) is compatible with the actions of Sph_G. By rigidity, the datum of (9.4) is equivalent to the datum of a Sph_M-linear functor

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}}. \tag{9.5}$$

From Corollary 9.3.5 we obtain:

Corollary 9.3.8. *The functor* (9.5) *identifies canonically with* $\mathrm{Wak}^{-,\frac{\infty}{2},\text{Sph-gen}}$.

9.3.9. In the sequel we will need $\rho_P(\omega_X)$-twisted versions of the above constructions. In particular, we will continue the functor

$$\mathrm{Wak}^{-,\frac{\infty}{2},\text{Sph-gen}}_{\rho_P(\omega_X)} : \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}},$$

which identifies with the functor obtained by rigidity from the dual of

$$\mathrm{BRST}^{-,\mathrm{enh}}_{\rho_P(\omega_X)} : \mathrm{KL}(G)_{\mathrm{crit}} \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X)} =: \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P(\omega_X)}.$$

9.4. **The dual of the right vertical arrow.** In this subsection we will perform constructions on the spectral side, parallel to ones in Sect. 9.3.

9.4.1. Recall the functor

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \tag{9.6}$$

$$\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}},$$

see (5.21).

By a similar token, we have a functor

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) := \tag{9.7}$$

$$= \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to$$

$$\to \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}},$$

The following is straightforward:

Lemma 9.4.2. *The functors* (9.6) *and* (9.7) *are equivalences.*

9.4.3. It is easy to see that the embedding

$$\mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \to \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)$$

admits a right adjoint (as a factorization functor).

Thanks to Lemma 9.4.2, we will view the resulting right adjoint functor

$$\mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \to \tag{9.8}$$

$$\to \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \simeq$$

$$\simeq \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$$

as a right adjoint to

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \xrightarrow{\sim} \tag{9.9}$$

$$\to \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \hookrightarrow$$

$$\hookrightarrow \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big).$$

9.4.4. The following is straightforward:

Lemma 9.4.5. *The functor* (9.8) *identifies with the dual of the functor*

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \xrightarrow{\sim} \tag{9.10}$$

$$\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \hookrightarrow$$

$$\hookrightarrow \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big).$$

9.4.6. Let $coJ^{-,\mathrm{spec},*,\frac{\infty}{2}}$ denote the functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))) = \mathrm{IndCoh}^*(\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \overset{\mathsf{j}_*}{\to} \tag{9.11}$$
$$\to \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big).$$

Let $coJ^{-,\mathrm{spec},*,\frac{\infty}{2}\text{-mon-free}}$ denote the composition of

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))) \overset{coJ^{-,\mathrm{spec},*,\frac{\infty}{2}}}{\longrightarrow} \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \overset{(9.8)}{\longrightarrow} \tag{9.12}$$
$$\to \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)).$$

From Lemma 9.4.5 we obtain:

Corollary 9.4.7. *The functor $coJ^{-,\mathrm{spec},*,\frac{\infty}{2}\text{-mon-free}}$ identifies with the dual of the functor*

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\sim}{\to}$$
$$\to \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)_{\text{mon-free}} \hookrightarrow$$
$$\hookrightarrow \mathrm{IndCoh}^!\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \overset{J^{-,\mathrm{spec},!,\frac{\infty}{2}}}{\longrightarrow} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))),$$

where $J^{-,\mathrm{spec},!,\frac{\infty}{2}}$ is the functor from (5.18).

9.4.8. Note that the composition

$$\mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \overset{(9.8)}{\longrightarrow} \tag{9.13}$$
$$\to \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}\otimes\mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$$

identifies with the functor $\mathsf{p}_* \circ \mathsf{i}^!$ (see diagram (5.17) for the notations).

Hence, the composition of $coJ^{-,\mathrm{spec},*,\frac{\infty}{2}\text{-mon-free}}$ with

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}\otimes\mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \simeq \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$$

identifies with the functor $coJ^{-,\mathrm{spec},*}$.

9.4.9. Consider the functor

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{J^{-,\mathrm{spec},!,\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) =$$
$$= \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)).$$

Its dual is a functor

$$\mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)). \tag{9.14}$$

The functor (9.14) is compatible with actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$. By rigidity, the datum of the functor (9.14) is equivalent to the datum of a $\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$-linear functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \simeq \tag{9.15}$$
$$\simeq \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)).$$

From Corollary 9.4.7 we obtain:

Corollary 9.4.10. *The functor* (9.15) *identifies with* $coJ^{-,\mathrm{spec},*,\frac{\infty}{2}\text{-mon-free}}$.

9.5. **A dual formulation of Theorem 9.1.7.**

9.5.1. Denote by $\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}$ the equivalence (see (1.9))

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \xrightarrow{\mathrm{Sat}^{-,\frac{\infty}{2}} \otimes (\mathrm{FLE}_{G,\mathrm{crit}} \circ \tau_G)} \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)).$$

9.5.2. Applying Corollaries 9.3.8 and 9.4.10, we obtain that Theorem 9.1.7 is equivalent to the following:

Theorem 9.5.3. *The following diagram of (factorization) functors commutes:*

(9.16)
$$\begin{array}{ccc} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P} & \xrightarrow{\mathrm{FLE}_{M,\mathrm{crit}-\check{\rho}_P} \circ \tau_M} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\ \downarrow{\scriptstyle \mathrm{Wak}^{-,\infty,\text{Sph-gen}}_{\rho_P(\omega_X)}} & & \downarrow{\scriptstyle coJ^{-,\mathrm{spec},*,\frac{\infty}{2}\text{-mon-free}}} \\ \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}} & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)). \end{array}$$

9.5.4. Note that Theorem 9.2.4 follows formally from Theorem 9.5.3 by concatenating with the diagram

$$\begin{array}{ccc} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}} & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \\ \downarrow{\scriptstyle \mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}\otimes\mathrm{Id}} \\ \mathrm{Sph}_G \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} & & \mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \\ \downarrow{\scriptstyle =} & & {\scriptstyle =}\downarrow \\ \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}})\circ\tau_G} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)). \end{array}$$

Remark 9.5.5. In fact, we conjecture that there exists an equivalence

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \xrightarrow{\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}} \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big)$$

that fits into the commutative diagram

$$\begin{array}{ccc}
\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P} & \xrightarrow{\mathrm{FLE}_{M,\mathrm{crit}-\check{\rho}_P}\circ\tau_M} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \\
{\scriptstyle \mathrm{Wak}^{-,\infty}_{\rho_P(\omega_X)}}\downarrow & & \downarrow{\scriptstyle coJ^{-,\mathrm{spec},*,\frac{\infty}{2}}} \\
\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}} & \mathrm{IndCoh}^*\Big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times)\underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times}\mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \\
{\scriptstyle (9.3)}\downarrow & & \downarrow{\scriptstyle (9.8)} \\
\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)}\underset{\mathrm{Sph}_G}{\otimes}\mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}} & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}\underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes}\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)).
\end{array}$$

In fact, we know that such an equivalence exists at the pointwise level: this is essentially what is proved in [FG3, Main Theorem 3]. See also Remark 10.2.7

10. Proof of Theorem 9.5.3

In this section, we will begin the proof of Theorem 9.5.3. By construction, both vertical arrows in diagram (9.16) are composites, in which the middle terms are categories of "semi-infinite" nature, namely,

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} := (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^{\mathfrak{L}(N(P^-))\cdot\mathfrak{L}^+(M)} \text{ and } \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times)\underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times}\mathrm{LS}_{\check{M}}(\mathcal{D})\big), \tag{10.1}$$

respectively. Therefore, a natural approach to the proof would be to complete diagram (9.16) to one that contains an arrow (in one direction) between the two categories in (10.1).

We conjecture that such an arrow exists, and that it is moreover an equivalence. Furthermore, we know that this is the case at the pointwise level (i.e., over a specific point in Ran). However, we cannot prove this, or even construct a functor at the factorization level (i.e., as categories over Ran).

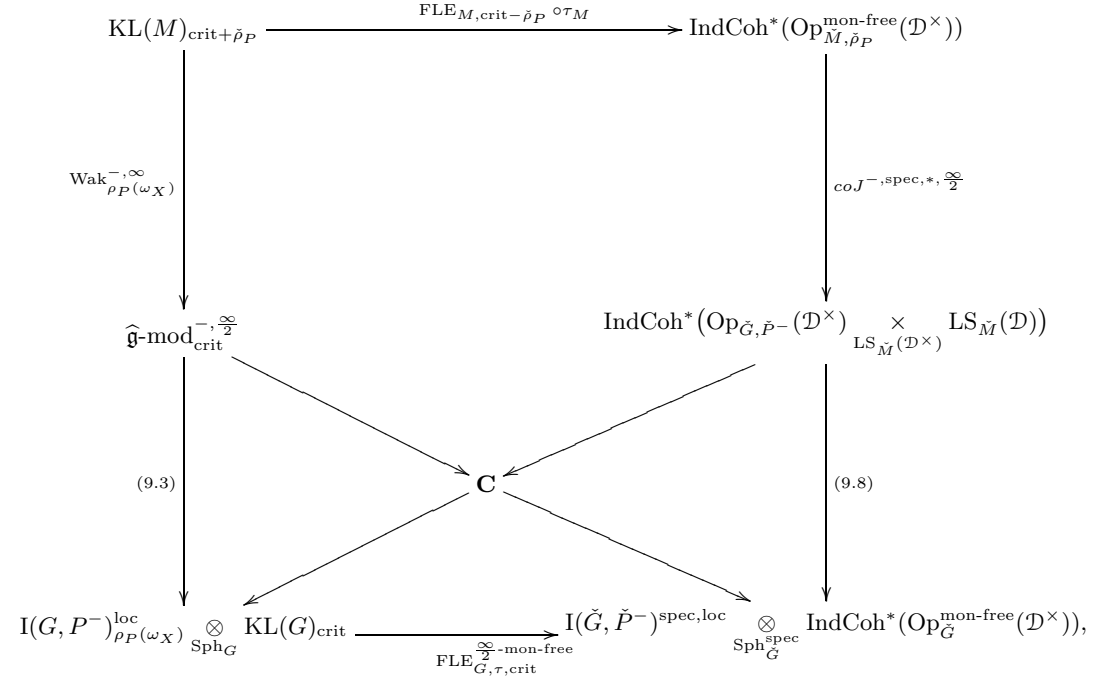
Instead, we will find a category $\mathbf{C}$ that receives functors from both categories in (10.1), and our strategy will be to show that the resulting diagram (i.e., diagram (10.2)) commutes.

In this section we will construct the 1-skeleton of (10.2), and establish the commutativity of the three triangles. The commutativity of the pentagon will be established in the next one.

10.1. **Strategy of proof.**

10.1.1. As was explained above, our method of proof of Theorem 9.2.4 will consist of the following. We will construct a category $\mathbf{C}$ (the definition is in Sect. 10.2.8) and a diagram

(10.2)



in which the upper pentagon and all three triangles commute. An existence of such a diagram will imply Theorem 9.5.3.

10.2. The factorization algebra $\Omega(R_{\check{G}})^{\mathrm{spec}}$.

10.2.1. Consider the (commutative) factorization category

$$\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{M}) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}) \times \mathrm{LS}_{\check{M}}(\mathcal{D})).$$

Let $\Omega(R_{\check{G}})^{\mathrm{spec}}$ denote the (commutative) factorization algebra in this category equal to the direct image of the unit (i.e., the structure sheaf) under the (factorization) functor

$$\mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(\mathcal{D})) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))$$

given by direct image along the map

$$(\mathsf{p} \times \mathsf{q}) : \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \to \mathrm{LS}_{\check{G}}(\mathcal{D}) \times \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

In other words, $\Omega(R_{\check{G}})^{\mathrm{spec}}$ corresponds to the commutative algebra object in $\mathrm{Rep}(\check{G}) \otimes \mathrm{Rep}(\check{M})$ equal to

$$\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-), R_{\check{G}}),$$

where

$$R_{\check{G}} \in \mathrm{Rep}(\check{G}) \times \mathrm{Rep}(\check{G})$$

is the regular representation.

10.2.2. Since the morphism $(\mathsf{p} \times \mathsf{q})$ is quasi-affine, the functor $(\mathsf{p} \times \mathsf{q})_*$ induces a (factorization) equivalence

$$(\mathsf{p} \times \mathsf{q})^{\mathrm{enh}}_* : \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(\mathcal{D})) \xrightarrow{\sim} \Omega(R_{\check{G}})^{\mathrm{spec}}\text{-mod}^{\mathrm{com}}\left(\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right). \tag{10.3}$$

10.2.3. *A notational remark.* We denote the above factorization algebra by $\Omega(R_{\check{G}})^{\mathrm{spec}}$, and not simply by Ω^{spec}, because the latter symbol is reserved for the factorization algebra from Sect. 2.5.2.

Tautologically, Ω^{spec} is obtained from $\Omega(R_{\check{G}})^{\mathrm{spec}}$ by applying the direct image functor along

$$\mathrm{LS}_{\check{G}}(\mathcal{D}) \times \mathrm{LS}_{\check{M}}(\mathcal{D}) \to \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

As was noted before, the functor

$$\mathsf{q}^{\mathrm{enh}}_* : \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(\mathcal{D})) \to \Omega\text{-mod}^{\mathrm{com}}(\mathrm{QCoh}(\mathrm{LS}_{\check{M}}(\mathcal{D})))$$

is also an equivalence; this is due to the fact that the map q is also co-affine.

10.2.4. Let us denote by $\mathsf{p}^\times$ the map

$$\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \to \mathrm{LS}_{\check{G}}(\mathcal{D}^\times).$$

We have a commutative square of factorization functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(\mathcal{D})) & \longrightarrow & \mathrm{IndCoh}^*(\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \\ \downarrow{\scriptstyle (\mathsf{p}\times\mathsf{q})_*} & & \downarrow{\scriptstyle (\mathsf{p}^\times\times\mathrm{id})_*} \\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) & \longrightarrow & \mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})), \end{array}$$

given by direct image, where the horizontal arrows are unital.

In particular, we obtain that the above functor $(\mathsf{p}^\times \times \mathrm{id})_*$ upgrades to a (factorization) functor

$$(\mathsf{p}^\times \times \mathrm{id})^{\mathrm{enh}}_* : \mathrm{IndCoh}^*(\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \\ \to \Omega(R_{\check{G}})^{\mathrm{spec}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right). \tag{10.4}$$

Remark 10.2.5. We conjecture that the functor (10.4) is actually an equivalence. We will see shortly that the equivalence statement does hold at the pointwise level, i.e., non-factorizably.

10.2.6. Let $\Omega(R_{\check{G}})^{\mathrm{Op}}$ denote the (commutative) factorization algebra in the (commutative) factorization category

$$\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}) \otimes \mathrm{LS}_{\check{M}}(\mathcal{D}))$$

obtained by taking the pullback of $\Omega(R_{\check{G}})^{\mathrm{spec}}$ along the map

$$\mathrm{Op}_{\check{G}}(\mathcal{D}) \otimes \mathrm{LS}_{\check{M}}(\mathcal{D}) \to \mathrm{LS}_{\check{G}}(\mathcal{D}) \otimes \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

Let us denote by the same symbol $\mathsf{p}^\times$ the map

$$\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \to \mathrm{Op}_{\check{G}}(\mathcal{D}^\times).$$

Consider the map

$$(\mathsf{p}^\times \times \mathrm{id}) : \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}) \to \mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \otimes \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

Similar to the above, the functor

$$(\mathsf{p}^\times \times \mathrm{id})_* : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \otimes \mathrm{LS}_{\check{M}}(\mathcal{D}))$$

upgrades naturally to a functor

$$(\mathsf{p}^\times \times \mathrm{id})^{\mathrm{enh}}_* : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \\ \to \Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \otimes \mathrm{LS}_{\check{M}}(\mathcal{D}))\right) \tag{10.5}$$

Remark 10.2.7. Similar to Remark 10.2.5, we conjecture that the functor (10.5) is an equivalence. Combining, this leads to an equivalence

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\text{crit}} \xrightarrow{\text{FLE}^{\frac{\infty}{2}}_{G,\tau,\text{crit}}} \text{IndCoh}^*\Big(\text{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\text{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \text{LS}_{\check{M}}(\mathcal{D})\Big),$$

see Remark 9.5.5.

In Sect. 10.7 we will sketch a proof that this the functor (10.5) is an equivalence at the pointwise level.

10.2.8. We define the category $\mathbf{C}$ from Sect. 10.1.1 to be

$$\mathbf{C} := \Omega(R_{\check{G}})^{\text{Op}}\text{-mod}^{\text{fact}}\left(\text{IndCoh}^*(\text{Op}_{\check{G}}(\mathcal{D}^\times) \otimes \text{LS}_{\check{M}}(\mathcal{D}))\right).$$

10.3. **Functors to and from C on the spectral side.**

10.3.1. We define the functor

$$\text{IndCoh}^*(\text{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\text{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \text{LS}_{\check{M}}(\mathcal{D})) \to \mathbf{C} \tag{10.6}$$

to be (10.5).

Our current goal is to construct a functor

$$\mathbf{C} \to \text{I}(\check{G},\check{P}^-)^{\text{spec,loc}} \underset{\text{Sph}^{\text{spec}}_{\check{G}}}{\otimes} \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)), \tag{10.7}$$

and to establish the commutativity of the right triangle in (10.2).

10.3.2. We will first construct a functor

$$\text{I}(\check{G},\check{P}^-)^{\text{spec,loc}} \underset{\text{Sph}^{\text{spec}}_{\check{G}}}{\otimes} \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathbf{C}. \tag{10.8}$$

Namely, we let (10.8) be the composition

$$\text{I}(\check{G},\check{P}^-)^{\text{spec,loc}} \underset{\text{Sph}^{\text{spec}}_{\check{G}}}{\otimes} \text{IndCoh}^*(\text{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \xrightarrow{(9.9)}$$

$$\to \text{IndCoh}^*(\text{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\text{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \text{LS}_{\check{M}}(\mathcal{D})) \xrightarrow{(10.6)} \mathbf{C}.$$

Lemma 10.3.3. *The functor* (10.8) *preserves compactness.*

10.3.4. We let the sought-for functor (10.7) be the right adjoint of (10.8). The isomorphism

$$(10.8) \simeq (10.6) \circ (9.9)$$

gives rise to a natural transformation

$$(9.8) \to (10.7) \circ (10.6). \tag{10.9}$$

We now claim:

Proposition 10.3.5. *The natural transformation* (10.9) *is an isomorphism.*

Once Proposition 10.3.5 is proved, we will have established the commutativity of the right triangle in (10.2).

The rest of this subsection is devoted to the proof of Proposition 10.3.5.

Since all functors and natural transformations in (10.9) are equipped with a factorization structure, in order to prove that (10.9) is an isomorphism, it is enough to do so at the pointwise level.

For the latter, it suffices to prove:

Proposition 10.3.6. *The functor* (10.5) *is a pointwise equivalence.*

10.3.7. Note that the functor (10.5) is obtained by (formally smooth) base change

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}) \to \mathrm{LS}_{\check{G}}(\mathcal{D}^{\times})$$

from the functor (10.4). Hence, the assertion of Proposition 10.3.6 follows from the corresponding assertion at the level of $\mathrm{LS}_{\check{G}}$:

Proposition 10.3.8. *The functor* (10.4) *is a pointwise equivalence.*

10.3.9. *Proof of Proposition 10.3.8.* Let $\mathsf{q}^{\times}$ denote the map

$$\mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \to \mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})$$

and also its base change

$$\mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}) \to \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

Consider the (factorization) functor

$$(\mathsf{q}^{\times})_* : \mathrm{IndCoh}^*(\mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{LS}_{\check{M}}(\mathcal{D})),$$

and its enhancement

$$(\mathsf{q}^{\times})^{\mathrm{enh}}_* : \mathrm{IndCoh}^*(\mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \Omega\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{LS}_{\check{M}}(\mathcal{D}))\right). \tag{10.10}$$

It is easy to see that the functor (10.10) is an equivalence at the pointwise level. The same is true for the functor

$$(\mathrm{id} \times \mathsf{q}^{\times})^{\mathrm{enh}}_* : \mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D}^{\times}) \times \mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \\ \to \Omega\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D}^{\times}) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right). \tag{10.11}$$

Hence, if $\mathcal{A}$ is a unital factorization algebra in

$$\mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D}^{\times}) \times \mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})),$$

the functor

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D}^{\times}) \times \mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))\right) \to \\ \to \left((\mathrm{id} \times \mathsf{q}^{\times})^{\mathrm{enh}}_*(\mathcal{A})\right)\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D}^{\times}) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right), \tag{10.12}$$

induced by (10.11) is also a pointwise equivalence.

Take $\mathcal{A}$ to be the direct image of the structure sheaf along the factorization functor given by direct image along

$$\mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}) \xrightarrow{\mathsf{p}^{\times} \times \mathrm{id}} \mathrm{LS}_{\check{G}}(\mathcal{D}^{\times}) \times \mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

Hence, it suffices to show that the resulting functor

$$(\mathsf{p}^{\times} \times \mathrm{id})^{\mathrm{enh}}_* : \mathrm{IndCoh}^*\left(\mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\right) \to \\ \to \mathcal{A}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*\left(\mathrm{LS}_{\check{G}}(\mathcal{D}^{\times}) \times \mathrm{LS}_{\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\right)\right)$$

is a (pointwise) equivalence.

Since the map $\mathfrak{q}^\times$ is formally smooth, it suffices to show that the functor

$$(\mathfrak{p}^\times \times \mathrm{id})^{\mathrm{enh}}_* : \mathrm{IndCoh}^*\Big(\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)\Big) \to \\ \to \Big((\mathfrak{p}^\times \times \mathrm{id})^{\mathrm{enh}}_*(\mathcal{O}_{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)})\Big)\text{-mod}^{\mathrm{fact}}\Big(\mathrm{IndCoh}^*\Big(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times)\Big)\Big)$$

is a (pointwise) equivalence.

However, the latter follows from the fact that

$$\mathrm{pt}/\check{P}^- \to \mathrm{pt}/\check{G} \times \mathrm{pt}/\check{P}^-$$

is an affine morphism.

□[Proposition 10.3.8]

10.4. **Functor to C on the geometric side.**

10.4.1. Our next goal is to define a functor

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \mathbf{C}. \tag{10.13}$$

We will denote it by

$$\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}} : \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right). \tag{10.14}$$

Remark 10.4.2. If we knew that the functor (10.6) was an equivalence, we could interpret $\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ as a functor

$$\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}} : \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big),$$

see Remark 9.5.5.

10.4.3. We will construct a factorization functor

$$\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}} : \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})),$$

and we will show that the image of the unit identifies, as a factorization algebra, with $\Omega(R_{\check{G}})^{\mathrm{Op}}$.

This will give rise to the desired functor (10.14).

10.4.4. We consider the category $\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}$ as acted on by

$$\mathrm{Rep}(\check{M}) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(\mathcal{D}))$$

via the functor $\mathrm{Sat}^{-1,\mathrm{nv}}_{M,\tau}$.

Hence, in order to construct the functor $\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$, it suffices to construct its composition with the direct image functor

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),$$

i.e., as a functor

$$\text{pre-pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}} : \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)). \tag{10.15}$$

10.4.5. We let the functor $\text{pre-pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ of (10.15) be the composition

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \hookrightarrow \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}} \to \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\mathrm{crit}} \xrightarrow{\alpha_{\rho(\omega_X),\mathrm{taut}}} \\ \to \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \xrightarrow{\mathrm{DS}^{\mathrm{enh}}} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \xrightarrow{\tau_{\check{G}}} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)), \tag{10.16}$$

where $\tau_{\check{G}}$ in the Cartan involution on $\check{G}$, viewed as an outer automorphism, and this inducing an automorphism of $\mathrm{Op}_{\check{G}}$.

10.4.6. Our next goal is to show that the image of the factorization unit along

$$\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\text{crit}}$$

identifies canonically with the image of $\Omega(R_{\check{G}})^{\mathrm{Op}}$ along

$$\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \tag{10.17}$$
$$\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})).$$

10.5. **Functor to C on the geometric side, continued.**

10.5.1. Consider the functor

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha^{-1}_{\rho_P(\omega_X),\mathrm{taut}} \otimes \mathrm{Id}}{\longrightarrow} \mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{(9.2)}{\longrightarrow} \tag{10.18}$$
$$\to \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \overset{\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})).$$

We will show that it identifies canonically with the composition

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}}{\longrightarrow} \tag{10.19}$$
$$\to \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{(9.9)}{\longrightarrow}$$
$$\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \overset{(\mathfrak{p}^\times \times \mathrm{id})_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})).$$

This will achieve two goals:

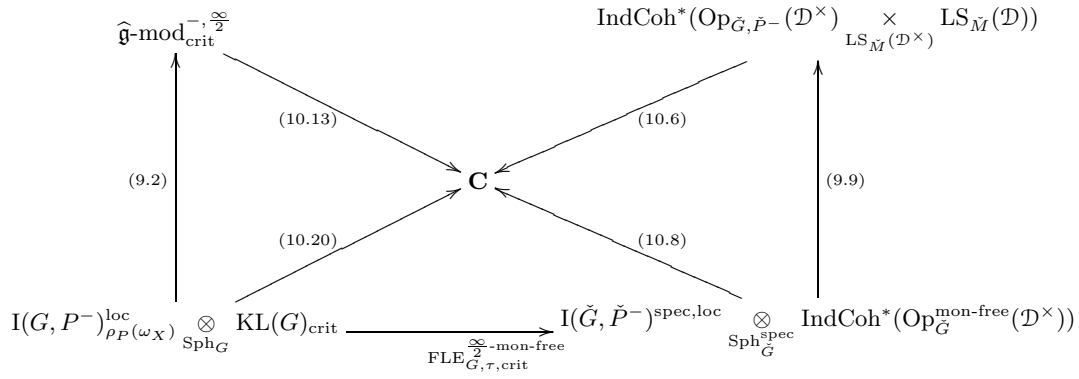
(a) This will imply that the factorization unit in $\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}$, which equals the image of the factorization unit under (9.2), gets sent by $\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ to $\Omega(R_{\check{G}})^{\mathrm{Op}}$, as promised in Sect. 10.4.6, thereby completing the construction of the functor (10.13).

(b) This will show that the functor (10.18) can be naturally enhanced to a functor

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \to \mathbf{C}, \tag{10.20}$$

so that the diagram

(10.21)



commutes.

10.5.2. Since both functors (10.18) and (10.19) respect the actions of

$$\mathrm{Sph}_M \overset{\mathrm{Sat}_{M,\tau}}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}},$$

using

$$\mathrm{Rep}(\check{M}) \overset{\mathrm{nv}}{\to} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}},$$

it suffices to construct an isomorphism between their compositions with the forgetful functor

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Thus, we have to compare

$$\begin{aligned}(10.22)\quad & \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha^{-1}_{\rho_P(\omega_X),\mathrm{taut}}\otimes \mathrm{Id}}{\longrightarrow} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{(9.2)}{\longrightarrow} \\ & \to \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \hookrightarrow \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(M)}_{\mathrm{crit}} \to \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(T)}_{\mathrm{crit},\rho(\omega_X)} \to \\ & \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \overset{\tau_{\check{G}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))\end{aligned}$$

and the composition of $\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}$ with

$$\begin{aligned}(10.23)\quad & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{(9.9)}{\longrightarrow} \\ & \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \overset{(\mathfrak{p}^\times\times \mathrm{id})^*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \\ & \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).\end{aligned}$$

10.5.3. We first rewrite (10.22). As a first step, we rewrite it as

$$\begin{aligned}(10.24)\quad & \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}\otimes\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \\ & \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \to \\ & \to (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)}} \overset{\overline{\mathrm{DS}}^{\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \overset{\tau_{\check{G}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).\end{aligned}$$

10.5.4. Consider the functor

$$(10.25)\qquad \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho(\omega_X)} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X)}) \to \mathrm{Whit}_*(G),$$

where the last arrow is the tautological projection. This functor is compatible with the action of Sph_G.

We can further rewrite (10.24) as

$$\begin{aligned}(10.26)\quad & \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{(10.25)\otimes\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \overset{\star}{\to} \\ & \to (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)}} \overset{\overline{\mathrm{DS}}^{\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \overset{\tau_{\check{G}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).\end{aligned}$$

10.5.5. We now rewrite (10.23).

Consider the functor of direct image

$$\begin{aligned}(10.27)\quad & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} = \mathrm{IndCoh}^*\Big(\mathrm{LS}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\Big) \to \\ & \to \mathrm{IndCoh}^*(\mathrm{LS}_{\check{G}}(\mathcal{D})) = \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})).\end{aligned}$$

It is compatible with the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$. Hence, it gives rise to a functor

$$\begin{aligned}(10.28)\quad \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{(10.27)\otimes \mathrm{Id}}{\longrightarrow} \\ \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\sim}{\to} \\ \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}} \hookrightarrow \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).\end{aligned}$$

It is easy to see that the functor (10.23) is isomorphic to (10.28).

10.5.6. Note now that the functor (10.25) identifies with the functor obtained by duality from

$$\mathrm{Whit}^!(G)\otimes \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \overset{(2.21)}{\longrightarrow} \mathrm{Whit}^!(M)\to \mathrm{Vect},$$

where the last arrow is the functor of fiber at $1\in \mathrm{Gr}_{M,\rho_M(\omega_X)}$, or equivalently, the functor of pairing with

$$\mathbf{1}_{\mathrm{Whit}_*(M)}\in \mathrm{Whit}_*(M).$$

Similarly, the functor (10.27) is obtained by duality from

$$\mathrm{Rep}(\check{G})\otimes \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \overset{(2.22)}{\longrightarrow} \mathrm{Rep}(\check{M})\to \mathrm{Vect},$$

where the last arrow is the functor of $\check{M}$-invariants.

Hence, from the commutativity of (2.27) it follows that the diagram

$$\begin{array}{ccc} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} & \xrightarrow{\mathrm{Sat}^-,\frac{\infty}{2}} & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \\ {\scriptstyle (10.25)}\downarrow & & \downarrow{\scriptstyle (10.27)} \\ \mathrm{Whit}_*(G) & \xrightarrow{\mathrm{FLE}^{-1}_{\check{G},\infty}\circ \tau_G} & \mathrm{Rep}(\check{G}) \end{array}$$

commutes, in a way compatible with the actions of $\mathrm{Sph}_G\simeq \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$.

10.5.7. Hence, we can rewrite the composition of $\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}$ with (10.28) as

(10.29)

$$\begin{aligned}&\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{(10.25)\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{(\mathrm{FLE}^{-1}_{\check{G},\infty}\circ\tau_G)\otimes(\mathrm{FLE}_{G,\mathrm{crit}}\circ\tau_G)}{\longrightarrow} \\ &\to \mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) = \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\sim}{\to} \\ &\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}} \hookrightarrow \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).\end{aligned}$$

10.5.8. Thus, it remains to identify

$$\begin{aligned}&\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{(\mathrm{FLE}^{-1}_{\check{G},\infty}\circ\tau_G)\otimes(\mathrm{FLE}_{G,\mathrm{crit}}\circ\tau_G)}{\longrightarrow} \mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) = \\ &\quad = \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \overset{\sim}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\text{mon-free}} \hookrightarrow \\ &\qquad \hookrightarrow \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \overset{\tau_{\check{G}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),\end{aligned}$$

which is the same as

$$\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{FLE}^{-1}_{\check{G},\infty} \otimes \mathrm{FLE}_{G,\mathrm{crit}}}{\longrightarrow} \mathrm{Rep}(\check{G}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times})) =$$

$$= \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D})) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times})) \xrightarrow{\sim} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}))_{\text{mon-free}} \hookrightarrow$$

$$\hookrightarrow \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})),$$

with

$$\mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \overset{\mathrm{Id} \otimes \alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{Whit}_*(G) \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \xrightarrow{\star}$$

$$\to (\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)})_{\mathfrak{L}(N)_{\rho(\omega_X)}} \overset{\overline{\mathrm{DS}}^{\mathrm{enh}}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})).$$

However, this is precisely the assertion of Corollary 7.5.2.

10.6. **Functor *from* C on the geometric side.**

10.6.1. Our current goal is to construct a functor

$$\mathbf{C} \to \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}}, \tag{10.30}$$

and prove the commutativity of the left and lower triangles in (10.2).

10.6.2. Since the functor (10.8) preserves compactness and $\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}$ is an equivalence, the commutativity of the lower triangle in (10.21) implies that (10.20) also preserves compactness.

We define the functor (10.30) to be the right adjoint of (10.20). The commutativity of the lower triangle in (10.2) is the automatic: we started from a commutative triangle (10.21), and replaced both legs by their respective right adjoints, while the base is an equivalence.

In order to prove that the left triangle in (10.2) commutes, by the same logic as in Sect. 10.3, it suffices to prove that it commutes at the pointwise level.

Remark 10.6.3. In fact, one can deduce from [FG3, Main Theorem 3] that the functor $\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ of (10.14) is a pointwise equivalence (see Sect. 10.7), and thus repeat the logic of Sect. 10.3 verbatim.

In the argument given below we will make do with less information than the full equivalence.

10.6.4. By Proposition 10.3.8, at the pointwise level, we can interpret the functor $\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ as a functor

$$\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}} : \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^{\times}) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^{\times})}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big), \tag{10.31}$$

and the left triangle in (10.2) as the square

(10.32)

$$\begin{array}{ccc}
\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} & \xrightarrow{\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}} & \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big) \\
\Big\downarrow (9.3) & \nearrow & \Big\downarrow (9.8) \\
\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow[\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}\text{-mon-free}]{} & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*\big(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(\mathcal{D}^\times),
\end{array}$$

in which the vertical arrows are obtained by passing to the right adjoints in the commutative diagram

(10.33)

$$\begin{array}{ccc}
\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} & \xrightarrow[\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}]{} & \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big) \\
\Big\uparrow (9.2) & & \Big\uparrow (9.9) \\
\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} & \xrightarrow[\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}\text{-mon-free}]{} & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*\big(\mathrm{Op}^{\mathrm{mon\text{-}free}}_{\check{G}}(\mathcal{D}^\times).
\end{array}$$

We need to show that the natural transformation in (10.32) is an isomorphism.

10.6.5. Recall that the functor (9.2) is fully faithful with essential image denoted

$$\Big(\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}\Big)^{\text{Sph-gen}} \subset \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}. \tag{10.34}$$

Similarly, the functor (9.9) is a fully faithful embedding with essential image

$$\mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big)_{\text{mon-free}} \subset \big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big),$$

i.e., full subcategory consisting of objects set-theoretically supported over the preimage of

$$\mathrm{LS}_{\check{G}}(\mathcal{D}) \subset \mathrm{LS}_{\check{G}}(\mathcal{D}^\times).$$

10.6.6. Using the equivalence

$$\widehat{\mathfrak{g}}\text{-mod}^{I}_{\mathrm{crit}} \simeq \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}, \tag{10.35}$$

and the action of the Iwahori-Hecke category on $\widehat{\mathfrak{g}}\text{-mod}^{I}_{\mathrm{crit}}$, we obtain the category $\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}$ is also tensored over $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$.

Unwinding the definitions, one obtains that the functor $\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}$ is compatible with the actions of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}^\times))$ on both sides.

Therefore, in order to prove that the natural transformation in (10.32) is an isomorphism it suffices to show that the subcategory

$$(\widehat{\mathfrak{g}}\text{-mod}^I_{\mathrm{crit}})^{\text{Sph-gen}} \simeq \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}},$$

corresponding under (10.35) to (10.34), equals

$$\left(\widehat{\mathfrak{g}}\text{-mod}^I_{\mathrm{crit}}\right)_{\text{mon-free}} \subset \widehat{\mathfrak{g}}\text{-mod}^I_{\mathrm{crit}},$$

i.e., the full subcategory consisting of objects set-theoretically supported over $\mathrm{LS}_{\check{G}}(\mathcal{D}) \subset \mathrm{LS}_{\check{G}}(\mathcal{D}^\times)$.

However, this follows from [FG3, Main Theorem 4].

10.7. **A sketch of proof that $\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ is a pointwise equivalence.** The material in this subsection is not needed in the remainder of the paper. But for the sake of completeness, we will sketch a proof of the following assertion:

Theorem 10.7.1. *The functor $\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ of (10.14) is a pointwise equivalence.*

Remark 10.7.2. As was mentioned above, we conjecture that the functor $\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ is actually an equivalence as a factorization functor.

10.7.3. The rest of this subsection is devoted to the proof of Theorem 10.7.1. In the paper [FG3, Main Theorem 3] an equivalence

$$\mathrm{FLE}^I_{G,\mathrm{crit}} : \widehat{\mathfrak{g}}\text{-mod}^I_{\mathrm{crit}} \to \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big) \tag{10.36}$$

was established.

Applying the equivalence (10.35), from (10.36) we obtain a (pointwise) equivalence

$$\widetilde{\mathrm{FLE}}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}} : \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \to \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big). \tag{10.37}$$

We will show that the functor $\widetilde{\mathrm{FLE}}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}$ is isomorphic to the functor $\mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}$ of (10.31). This would imply the assertion of Theorem 10.7.1.

10.7.4. By definition, the datum of an isomorphism

$$\widetilde{\mathrm{FLE}}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}} \simeq \mathrm{FLE}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}$$

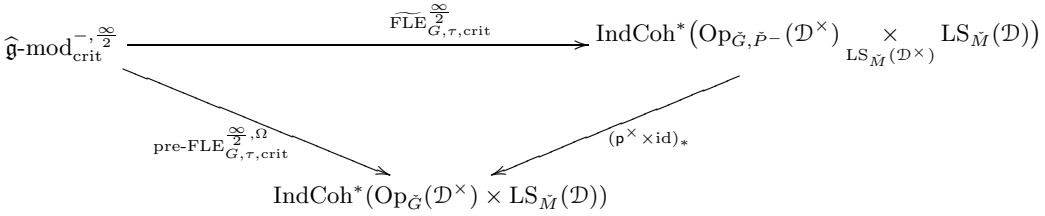
amounts to an identification between

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \xrightarrow{\widetilde{\mathrm{FLE}}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}} \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})\big) \xrightarrow{(10.5)}$$
$$\to \Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\Big(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))\Big)$$

and the functor $\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}$ of (10.14).

10.7.5. By unwinding the construction of (10.37), one obtains that the triangle

(10.38)



commutes.

We need to enhance the isomorphism of functors given by (10.38) to one with values in

$$\Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\Big(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D}))\Big).$$

10.7.6. The required enhancement is constructed as follows.

The two sides of (10.37) are factorization module categories with respect to

$$\mathrm{Sph}_G \xrightarrow{\mathrm{Sat}_{G,\tau}} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$$

respectively, and the functor $\widetilde{\mathrm{FLE}}^{\frac{\infty}{2}}_{G,\tau,\mathrm{crit}}$ is compatible with these functors.

Now, the required enhancement follows from the isomorphism between (10.18) and (10.19) using the fact that the functor $\text{pre-FLE}^{\frac{\infty}{2},\tau,\Omega}_{G,\mathrm{crit}}$ in (10.38) is compatible with the Sph_G-module structures via

$$\mathrm{Sph}_G \xrightarrow{-\star\mathrm{Vac}(G)_{\mathrm{crit}}} \mathrm{KL}(G)_{\mathrm{crit}} \xrightarrow{\mathrm{Av}^{\mathfrak{L}(N)}_!} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}} \xrightarrow{\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}}} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D})),$$

and the functor $(\mathsf{p}^\times\times\mathrm{id})_*$ is compatible with the $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$-module structures via

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G}} \xrightarrow{-\star\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \xrightarrow{\mathsf{i}_*\circ\mathsf{p}^*}$$

$$\to \mathrm{IndCoh}^*\big(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times)\underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times}\mathrm{LS}_{\check{M}}(\mathcal{D})\big) \xrightarrow{(\mathsf{p}^\times\times\mathrm{id})_*} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D})).$$

□[Theorem 10.7.1]

11. Engaging the Pentagon

In order to finish the proof of Theorem 9.5.3, it remains to establish the commutativity of the pentagon in (10.2). I.e., we need to show that the two functors

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \rightrightarrows \Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right) \tag{11.1}$$

are canonically isomorphic.

We will do so by showing that the two functors become isomorphic after composing the two functors in question with various forgetful functors from

$$\Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right), \tag{11.2}$$

which increasingly less information.

Namely, we will first consider the following sequence of forgetful functors:

$$\Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right) \to$$

$$\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}} \to \mathrm{Vect}\,.$$

The structure of the argument will be as follows:

(1) The fact that the compositions of the two functors in (11.1) with the forgetful functor to Vect are isomorphic will be a reflection of the basic fact about the action of the Feigin-Frenkel center on Wakimoto modules;
(2) The fact that we can lift this isomorphism to one with values in $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$-mod$^{\mathrm{fact}}$ will follow immediately from unitality;
(3) The fact that this isomorphism lifts further to $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ is a question of homological algebra, which we deal with explicitly;
(4) The further lift to $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D}))$ is automatic, thanks to the $\mathrm{Rep}(\check{M})$-action on both sides;
(5) The final lift to (11.1) itself is the most substantial step of the proof. We will reduce the assertion to a pointwise statement, and there we will deduce it from a basic calculation performed in the paper [FG2].

11.1. Comparison of the *un*enhanced functors.

11.1.1. We will first show that the compositions of the two functors in (11.1) with the forgetful functor

$$\Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right)\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D})) \tag{11.3}$$

are canonically isomorphic.

The two functors

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}\rightrightarrows \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D})) \tag{11.4}$$

commute with the action of

$$\mathrm{Rep}(\check{M})\simeq \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(\mathcal{D})).$$

Hence, it is enough to show that the compositions of the functors in (11.4) with the direct image functor

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)\times \mathrm{LS}_{\check{M}}(\mathcal{D}))\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

are canonically isomorphic.

11.1.2. The clockwise composition is the functor

$$\begin{aligned}\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} &\overset{\tau_M}{\to} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X)} \overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X),\rho_M(\omega_X)}\to\\ &\to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}-\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_M}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))\simeq\\ &\simeq \mathrm{IndCoh}^*(\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times)) \overset{(\mathfrak{p}^{\mathrm{Miu}})_*}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),\end{aligned} \tag{11.5}$$

where $\mathfrak{p}^{\mathrm{Miu}}$ is the Miura map

$$\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times)\to \mathrm{Op}_{\check{G}}(\mathcal{D}^\times).$$

The counter-clockwise composition is the functor

$$\begin{aligned}\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} &\overset{\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow}\\ &\to \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)}\to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \overset{\tau_{\check{G}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),\end{aligned} \tag{11.6}$$

which is the same as

$$\begin{aligned}\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} &\overset{\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow}\\ &\to \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)}\to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_G\circ\tau_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).\end{aligned} \tag{11.7}$$

Remark 11.1.3. Note that we can rewrite the functor (11.7) also as

$$\begin{aligned}\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} &\overset{(\alpha_{\check{\rho}_P(\omega_X),\mathrm{cent}})^{-1}}{\longrightarrow} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_P(\omega_X)} \overset{\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}}{\longrightarrow}\\ \to \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho_P(\omega_X)} &\overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)}\to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_G\circ\tau_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),\end{aligned} \tag{11.8}$$

where $\alpha_{\check{\rho}_P(\omega_X),\mathrm{cent}}$ is an in (4.12), or

$$\begin{aligned}\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} &\overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{(\alpha_{\check{\rho}_P(\omega_X),\mathrm{cent}})^{-1}}{\longrightarrow}\\ &\to \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho(\omega_X)} \overset{\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho(\omega_X)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)}\to\\ &\to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_G\circ\tau_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).\end{aligned} \tag{11.9}$$

11.1.4. We will first show that the two functors (11.5) and (11.7) become isomorphic after composing with

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}),-)^{\mathrm{enh}} : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})) \to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}.$$

This amounts to showing that they become isomorphic as (factorization) functors when composed with the forgetful functor

$$\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}} \to \mathrm{Vect}, \tag{11.10}$$

and that the resulting two maps of factorization algebras

$$\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})} \rightrightarrows \text{Image of}(\mathbf{1}_{\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}}) \tag{11.11}$$

agree.

11.2. **Comparison of the further *un*enhancements.**

11.2.1. The composition of $\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}),-)^{\mathrm{enh}}$ with (11.10) is the functor $\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}),-)$.

The composition of (11.5) with $\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}),-)$ is the functor

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\tau_M}{\to} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X)} \overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \to \tag{11.12}$$
$$\to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}-\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{\mathrm{DS}_M}{\longrightarrow} \mathrm{Vect}\,.$$

The composition of (11.7) with $\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}),-)$ is the functor

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \tag{11.13}$$
$$\to \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}_G\circ\tau_G}{\longrightarrow} \mathrm{Vect}\,.$$

11.2.2. We rewrite (11.12) as

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \to \tag{11.14}$$
$$\to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{\tau_M}{\to} \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}-\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{\mathrm{DS}_M}{\longrightarrow} \mathrm{Vect}\,.$$

We rewrite (11.13) as

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{(\alpha_{\check{\rho}_P(\omega_X),\mathrm{cent}})^{-1}}{\longrightarrow} \tag{11.15}$$
$$\to \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho(\omega_X)} \overset{\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho(\omega_X)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \to$$
$$\to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}_G\circ\tau_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})).$$

11.2.3. Note, however, that

$$\mathrm{DS}_G\circ\tau_G \simeq \mathrm{DS}_G \;\text{ and }\; \mathrm{DS}_M\circ\tau_M \simeq \mathrm{DS}_M\,.$$

So we have to construct an isomorphism between

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{\mathrm{DS}_M}{\longrightarrow} \mathrm{Vect} \tag{11.16}$$

and

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{(\alpha_{\check{\rho}_P(\omega_X),\mathrm{cent}})^{-1}}{\longrightarrow} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho(\omega_X)} \overset{\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho(\omega_X)}}{\longrightarrow} \tag{11.17}$$
$$\to \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})).$$

Remark 11.2.4. As we have seen above, the Cartan involutions play no role for the functors DS_G and DS_M, respectively. They will, however, play a role, once we will consider the action of $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$, i.e. in showing that the two morphisms (11.11) agree.

11.2.5. Note also that the diagram

$$\begin{array}{ccccc} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} & \longrightarrow & \widehat{\mathfrak{m}}\text{-}\mathrm{mod}_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} & \xrightarrow{\mathrm{DS}_M} & \mathrm{Vect} \\ \downarrow{\scriptstyle (\alpha_{\check{\rho}_P(\omega_X),\mathrm{cent}})^{-1}} & & \downarrow{\scriptstyle (\alpha_{\check{\rho}_P(\omega_X),\mathrm{cent}})^{-1}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho(\omega_X)} & \longrightarrow & \widehat{\mathfrak{m}}\text{-}\mathrm{mod}_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho(\omega_X)} & \xrightarrow{\mathrm{DS}_M} & \mathrm{Vect} \end{array}$$

commutes.

Thus, we have to construct an isomorphism between

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_P(\omega_X)} \to \widehat{\mathfrak{m}}\text{-}\mathrm{mod}_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho(\omega_X)} \xrightarrow{\mathrm{DS}_M} \mathrm{Vect} \tag{11.18}$$

and

$$\begin{aligned} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho(\omega_X)} &\xrightarrow{\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho(\omega_X)}} \\ &\to \widehat{\mathfrak{g}}\text{-}\mathrm{mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-}\mathrm{mod}_{\mathrm{crit},\rho(\omega_X)} \xrightarrow{\mathrm{DS}_G} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)). \end{aligned} \tag{11.19}$$

11.2.6. Finally, the isomorphism between (11.18) and (11.19) is evident:

The functor $\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}$ creates a module that is "semi-infinite free" with respect to $\mathfrak{L}(\mathfrak{n}(P))_{\rho_P(\omega_X)}$, see, e.g., [Gai5, Sect. 2.2].

11.3. Identification of the $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$-action.

11.3.1. The two circuits of the pentagon in (10.2) define maps of factorization algebras

$$\begin{aligned} \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})} &\xrightarrow{\mathrm{FF}^{\mathcal{W}}_G} \mathrm{DS}_G(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}) \xrightarrow{\tau_G} \mathrm{DS}_G(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}) \to \\ &\to \mathrm{DS}_G \circ \alpha_{\rho(\omega_X),\mathrm{taut}} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}) \end{aligned} \tag{11.20}$$

and

$$\begin{aligned} \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})} &\xrightarrow{(\mathfrak{p}^{\mathrm{Miu}})^*} \mathcal{O}_{\mathrm{Op}_{\check{M},\check{\rho}_P(\omega_X)}(\mathcal{D})} \xrightarrow{\mathrm{FF}^{\mathcal{W}}_M} \mathrm{DS}_M(\mathrm{Vac}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X),\rho_M(\omega_X)}) \xrightarrow{\tau_M} \\ &\to \mathrm{DS}_M(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)}) \simeq \mathrm{DS}_M \circ \alpha_{\rho_M(\omega_X),\mathrm{taut}}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}), \end{aligned} \tag{11.21}$$

where $\mathrm{FF}^{\mathcal{W}}_G$ is the isomorphism (7.2) for G, and $\mathrm{FF}^{\mathcal{W}}_M$ is the corresponding isomorphism for M, and

$$\mathrm{Op}_{\check{M},\check{\rho}_P(\omega_X)}(\mathcal{D}) \simeq \mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}) \xrightarrow{\mathfrak{p}^{\mathrm{Miu}}} \mathrm{Op}_{\check{G}}(\mathcal{D}).$$

We need to show that the homomorphisms (11.20) and (11.21) coincide under the identification

$$\begin{aligned} \mathrm{DS}_G \circ \tau_G \circ \alpha_{\rho(\omega_X),\mathrm{taut}} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}) \simeq \\ \simeq \mathrm{DS}_M \circ \alpha_{\rho_M(\omega_X),\mathrm{taut}} \circ \tau_M(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}), \end{aligned} \tag{11.22}$$

constructed in Sect. 11.2.

Equivalently, we have to show that the two structures of factorization (a.k.a. chiral) $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$-modules on the two sides of (11.22) coincide.

11.3.2. Note that the factorization algebras in (11.22) are *classical* chiral algebras, i.e., at the pointwise level they belong to $\mathrm{Vect}^\heartsuit$. Hence, it is enough to establish the above assertion about the chiral $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$-action on the two sides of (11.22) at the pointwise level.

We will think of a structure of chiral $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$-module as a (discrete) action of the (topological) commutative algebra $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$.

(In the process of proof, we will see that the chiral action(s) in question are/is *commutative*, i.e., the action of $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ factors via $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}$.)

11.3.3. Recall that $\mathfrak{z}_{\mathfrak{g}}$ denotes the Feigin-Frenkel center of $\widehat{\mathfrak{g}}$-mod$_{\mathrm{crit}}$, see Sect. 8.1.1. Let $\mathfrak{z}_{\mathfrak{m}}$ be the corresponding object for M, and let $\mathfrak{z}_{\mathfrak{m},\check{\rho}_P(\omega_X)}$ denote its twisted version.

Note also that $\mathfrak{z}_{\mathfrak{g}}$ is insensitive to twists by $\mathfrak{L}^+(G)$-torsors. So, we can equivalently view $\mathfrak{z}_{\mathfrak{g}}$ as mapping to

$$\mathrm{inv}_{\mathfrak{L}^+(G)_{\rho(\omega_X)}}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}),$$

and similarly for M.

Finally, recall (see Sect. 8.1.2) that the composite map

$$\begin{aligned}\mathfrak{z}_{\mathfrak{g}} \to \mathrm{inv}_{\mathfrak{L}^+(G)_{\rho(\omega_X)}}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}) \to \\ \to \mathrm{inv}_{\mathfrak{L}^+(N)_{\rho(\omega_X)}}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)}) \to \mathrm{DS}(\mathrm{Vac}(G)_{\mathrm{crit},\rho(\omega_X)})\end{aligned} \tag{11.23}$$

is an isomorphism, and similarly for M.

11.3.4. Let $\mathfrak{Z}_{\mathfrak{g}}$ be the topological commutative algebra corresponding to $\mathfrak{z}_{\mathfrak{g}}$. It maps to the Bernstein center of the category $\widehat{\mathfrak{g}}$-mod$_{\mathrm{crit}}$, i.e., it acts functorially on every object of $\widehat{\mathfrak{g}}$-mod$_{\mathrm{crit}}$.

Let $\mathfrak{Z}_{\mathfrak{m}}$ be the corresponding algebra for M. We will also consider its twisted versions

$$\mathfrak{Z}_{\mathfrak{m},-\check{\rho}_P(\omega_X)} \text{ and } \mathfrak{Z}_{\mathfrak{m},\check{\rho}_P(\omega_X)}.$$

The isomorphism (8.2) gives rise to an isomorphism

$$\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)} \overset{\mathrm{FF}^{\mathfrak{Z}}_G}{\simeq} \mathfrak{Z}_{\mathfrak{g}}, \tag{11.24}$$

and similarly for M.

11.3.5. The isomorphism (11.23) implies that in order to show that the actions of $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ on the two sides of (11.22) are equal, it is enough to show that the action of $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ on

$$\mathrm{DS}_G \circ \tau_G \circ \alpha_{\rho(\omega_X),\mathrm{taut}} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})$$

obtained via

$$\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)} \overset{\mathrm{FF}^{\mathfrak{Z}}_G}{\simeq} \mathfrak{Z}_{\mathfrak{g}} \overset{\tau_G}{\simeq} \mathfrak{Z}_{\mathfrak{g}} \tag{11.25}$$

and the action of $\mathfrak{Z}_{\mathfrak{g}}$ on $\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})$ identifies under (11.22) with the action of $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ on

$$\mathrm{DS}_M \circ \alpha_{\rho_M(\omega_X),\mathrm{taut}} \circ \tau_M(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})$$

obtained via

$$\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)} \overset{(\mathfrak{p}^{\times,\mathrm{Miu}})^*}{\to} \mathcal{O}_{\mathrm{Op}_{\check{M},\check{\rho}_P(\omega_X)}(\mathcal{D}^\times)} \overset{\mathrm{FF}^{\mathfrak{Z}}_M}{\simeq} \mathfrak{Z}_{\mathfrak{m},-\check{\rho}_P(\omega_X)} \overset{\tau_M}{\simeq} \mathfrak{Z}_{\mathfrak{m},\check{\rho}_P(\omega_X)} \tag{11.26}$$

and the action of $\mathfrak{Z}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}$ on $\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}$, where $\mathfrak{p}^{\times,\mathrm{Miu}}$ denotes the map

$$\mathrm{Op}_{\check{M},\check{\rho}_P(\omega_X)}(\mathcal{D}^\times) \simeq \mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \overset{\mathfrak{p}^{\times,\mathrm{Miu}}}{\longrightarrow} \mathrm{Op}_{\check{G}}(\mathcal{D}^\times).$$

11.3.6. For the latter, it is enough to prove the following: for any object

$$\mathcal{M} \in \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\check{\rho}_P(\omega_X)},$$

the $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$-action on

$$\mathrm{Wak}^{-,\frac{\infty}{2}}(\mathcal{M})$$

obtained via (11.25) and the action of $\mathfrak{Z}_{\mathfrak{g}}$ on $\mathrm{Wak}^{-,\frac{\infty}{2}}(\mathcal{M})$ agrees with the action, obtained via (11.26) and the action of $\mathfrak{Z}_{\mathfrak{m},\check{\rho}_P(\omega_X)}$ on $\mathcal{M}$.

11.3.7. Recall now that the duality

$$(\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}})^\vee \simeq \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$$

is compatible with the action of $\mathfrak{Z}_{\mathfrak{g}}$, up to τ_G, see Sect. 8.2.2.

Similarly, the duality

$$(\widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}-\check{\rho}_P(\omega_X)})^\vee \simeq \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\check{\rho}_P(\omega_X)}$$

is compatible with the action of

$$\mathfrak{Z}_{\mathfrak{m},-\check{\rho}_P(\omega_X)} \overset{\tau_M}{\simeq} \mathfrak{Z}_{\mathfrak{m},\check{\rho}_P(\omega_X)}.$$

11.3.8. Hence, by duality, the assertion in Sect. 11.3.6 is equivalent to the following: for $\mathcal{M}' \in \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$, the action of $\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)}$ on

$$\mathrm{BRST}^-(\mathcal{M}')$$

obtained from

$$\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)} \overset{\mathrm{FF}_G^{\mathfrak{Z}}}{\simeq} \mathfrak{Z}_{\mathfrak{g}}$$

and the $\mathfrak{Z}_{\mathfrak{g}}$-action on $\mathcal{M}'$ and the functoriality of BRST^-, agrees with the action, obtained via

$$\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)} \overset{(\mathfrak{p}^\times,\mathrm{Miu})^*}{\to} \mathcal{O}_{\mathrm{Op}_{\check{M},\check{\rho}_P(\omega_X)}(\mathcal{D}^\times)} \overset{\mathrm{FF}_M^{\mathfrak{Z}}}{\simeq} \mathfrak{Z}_{\mathfrak{m},-\check{\rho}_P(\omega_X)}$$

and the $\mathfrak{Z}_{\mathfrak{m},-\check{\rho}_P(\omega_X)}$ on $\mathrm{BRST}^-(\mathcal{M}')$ as an object of $\widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}-\check{\rho}_P(\omega_X)}$).

However, the latter is the basic property of the homomorphism $\mathrm{FF}_G^{\mathfrak{Z}}$ and $\mathrm{FF}_M^{\mathfrak{Z}}$.

11.4. **Upgrading to** IndCoh^***.** We have established that the two functors

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \rightrightarrows \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)), \tag{11.27}$$

obtained from (11.4) by composing with the direct image functor

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)),$$

become isomorphic, after we apply thefunctor

$$\Gamma(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times), -)^{\mathrm{enh}} \otimes \mathrm{Id} : \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)) \to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}.$$

We will now deduce from this that the two functors in (11.27) are themselves isomorphic.

11.4.1. It is enough to construct an isomorphism between the values of the functors in (11.27) on compact objects in $\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}$.

Using Lemma 5.2.2, it is enough to show that both functors in (11.27) send $(\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})^c$ to $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))^{>-\infty}$.

Since compact objects of $(\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})^c$ are bounded below, it is enough to check that the two functors in (11.27) are of bounded cohomological amplitude.

The latter assertion can be checked at the pointwise level.

11.4.2. Consider first the functor corresponding to the clockwise circuit in the pentagon in (10.2). We will show that the corresponding functor

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

is t-exact.

This functor is

$$(\mathfrak{p}^{\times,\mathrm{Miu}})_* \circ \tau_M \circ \mathrm{DS}^{\mathrm{enh}}_M \circ \alpha_{\rho_M(\omega_X),\mathrm{taut}}.$$

This functor is t-exact since the morphism $\mathfrak{p}^{\times,\mathrm{Miu}}$ is ind-affine, and the functor

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\alpha_{\rho_M(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \to$$
$$\to \widehat{\mathfrak{m}}\text{-mod}_{\mathrm{crit}+\check{\rho}_P(\omega_X),\rho_M(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_M}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{M},\check{\rho}_M(\omega_X)}(\mathcal{D}^\times))$$

identifies with

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\mathrm{FLE}_{M,\mathrm{crit}+\check{\rho}_P(\omega_X)}}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_M(\omega_X)}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{M},\check{\rho}_M(\omega_X)}(\mathcal{D}^\times)),$$

and the functor $\mathrm{FLE}_{M,\mathrm{crit}+\check{\rho}_P(\omega_X)}$ is t-exact (see Remark 7.4.5).

11.4.3. We now consider the functor corresponding to the counter-clockwise cicruit. The functor in question is

$$\tau_{\check{G}} \circ \mathrm{DS}^{\mathrm{enh}}_G \circ \alpha_{\rho(\omega_X),\mathrm{taut}} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}. \tag{11.28}$$

We will now rewrite it, replacing $\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}$ by the *usual* Wakimoto functor.

11.4.4. Note that the composition

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\mathrm{Wak}^{-,\frac{\infty}{2}}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}} \overset{\mathrm{Av}^{\mathfrak{L}^+(N(P))}_*}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(P)}_{\mathrm{crit}} \tag{11.29}$$

is the *usual* Wakimoto functor, to be denoted Wak.

Let $\mathrm{Wak}_{\rho(\omega_X)}$ denote its $\rho(\omega_X)$-twist, i.e., the composition

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}(\omega_X)} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}\circ\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{Av}^{\mathfrak{L}^+(N(P))_{\rho(\omega_X)}}_*}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(P)_{\rho(\omega_X)}}_{\mathrm{crit},\rho(\omega_X)}. \tag{11.30}$$

11.4.5. Note also that the composition

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

can be rewritten as

$$\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{Av}^{\mathfrak{L}^+(N(P))_{\rho(\omega_X)}}_*}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(P)_{\rho(\omega_X)}}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_G}{\longrightarrow} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

Hence, we obtain that the composition

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}\circ\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_G}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$$

can be rewritten as

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \overset{\mathrm{Wak}_{\rho(\omega_X)}}{\longrightarrow} \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(P)_{\rho(\omega_X)}}_{\mathrm{crit},\rho(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}^{\mathrm{enh}}_G}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

11.4.6. We will now describe the functor Wak in more familiar terms.

Let $\mathrm{CDO}(G)_{\mathrm{crit},\mathrm{crit}}$ be the CDO at the critical level; and let $\mathrm{CDO}(\overset{\circ}{G})_{\mathrm{crit},\mathrm{crit}}$ be its localization, corresponding to the *parabolic* big Bruhat cell

$$\overset{\circ}{G} = P \cdot P^- \subset G.$$

We have:

$$\mathrm{CDO}(\overset{\circ}{G})_{\mathrm{crit},\mathrm{crit}} \in \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}^{\mathfrak{L}^+(P)} \otimes \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}^{\mathfrak{L}^+(P^-)}.$$

Consider the object

$$(\mathrm{Id} \otimes \mathrm{BRST}^-)(\mathrm{CDO}(\overset{\circ}{G})_{\mathrm{crit},\mathrm{crit}}) \in \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}^{\mathfrak{L}^+(P)} \otimes \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X)}. \tag{11.31}$$

Then the object (11.31) defines the functor Wak using the duality

$$(\mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P(\omega_X)})^\vee \simeq \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}.$$

11.4.7. From the above description of the functor Wak we obtain:

Lemma 11.4.8. *The functor* Wak*, viewed as a functor*

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}^{\mathfrak{L}^+(P)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}$$

is t-exact.

Hence, we obtain that the functor $\mathrm{Wak}_{\rho(\omega_X)}$, viewed as a functor

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}$$

is also t-exact.

11.4.9. Note also, that from the above description of Wak, we obtain that at the pointwise level, it factors via a functor

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}^{I},$$

where $I \subset \mathfrak{L}^+(G)$ is the Iwahori subgroup.

Hence, the same is true for $\mathrm{Wak}_{\rho(\omega_X)}$.

11.4.10. Taking into account Sect. 11.4.5, we obtain that in order to show that the functor (11.28) has a bounded cohomological amplitude, it is enough to show that the functor

$$\widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)}^{I} \to \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit},\rho(\omega_X)} \overset{\mathrm{DS}_{\check{G}}^{\mathrm{enh}}}{\to} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)).$$

has this property.

However, this follows from [Ras6].

11.5. **Identification of the map of factorization algebras.** We have proved that the compositions of the two functors

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \rightrightarrows \Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right) \tag{11.32}$$

with the forgetful functor

$$\Omega(R_{\check{G}})^{\mathrm{Op}}\text{-mod}^{\mathrm{fact}}\left(\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))\right) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))$$

are canonically isomorphic.

We will now show that the functors (11.32) are canonically isomorphic.

11.5.1. Let Ω^{MOp} denote the factorization algebra in $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))$ equal to the image of

$$\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} = \mathbf{1}_{\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}} \in \mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}$$

along the two isomorphic functors

$$\mathrm{KL}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)} \rightrightarrows \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})).$$

The two functors (11.32) give rise to two homomorphisms

$$\Omega(R_{\check{G}})^{\mathrm{Op}} \rightrightarrows \Omega^{\mathrm{MOp}}. \tag{11.33}$$

We need to show that the two maps in (11.33) are isomorphic.

11.5.2. The two factorization algebras in $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))$ are

$$\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})$$

and

$$((\mathsf{p} \times \mathsf{q}) \circ \mathsf{j})_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})}),$$

respectively, where $(\mathsf{p} \circ \mathsf{j}) \times \mathsf{q}$ is the map

$$\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}) \xrightarrow{\mathsf{j}} \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}) = \mathrm{Op}_{\check{G}}(\mathcal{D}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D})}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}) \xrightarrow{\mathsf{p}\times\mathsf{q}} \mathrm{Op}_{\check{G}}(\mathcal{D}) \times \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

We will first construct *an* isomorphism

$$\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}) \simeq ((\mathsf{p} \times \mathsf{q}) \circ \mathsf{j})_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})}), \tag{11.34}$$

compatible with the maps from $\Omega(R_{\check{G}})^{\mathrm{Op}}$ to each.

We will then show that the isomorphism (11.34) equals the already constructed identification

$$\text{pre-FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}) \simeq \Omega^{\mathrm{MOp}} \simeq ((\mathsf{p} \times \mathsf{q}) \circ \mathsf{j})_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})}). \tag{11.35}$$

11.5.3. Note the datum of an isomorphism (11.34) compatible with the maps from $\Omega(R_{\check{G}})^{\mathrm{Op}}$ amounts to an isomorphism

$$\mathrm{FLE}^{\frac{\infty}{2},\Omega}_{G,\tau,\mathrm{crit}} \circ \mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}) \simeq (10.6) \circ \mathsf{j}_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})}) \tag{11.36}$$

as objects of $\mathbf{C}$, where we think of $\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})})$ as an object of

$$\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}).$$

11.5.4. Note that the object

$$\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)}) \in \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}$$

belongs to

$$\mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{KL}(G)_{\mathrm{crit}} \xrightarrow{\sim} \left(\widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}\right)^{\text{Sph-gen}} \subset \widehat{\mathfrak{g}}\text{-mod}^{-,\frac{\infty}{2}}_{\mathrm{crit}}.$$

Similarly, the object

$$\mathsf{j}_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})}) \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))$$

belongs to

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \simeq$$

$$\simeq \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))_{\text{mon-free}} \subset$$

$$\subset \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})).$$

We will construct an isomorphism

$$\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}\left(\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})\right) \simeq \mathsf{j}_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})}) \tag{11.37}$$

taking place in

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))_{\text{mon-free}}.$$

It would give rise to an isomorphism (11.36) by applying the commutative diagram (10.21).

11.5.5. The construction of (11.37) will take the following input from the paper [FG2, Theorem 4.11]:

Theorem 11.5.6. *There exists a* pointwise *isomorphism between the objects*

$$\mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}\left(\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})\right) \text{ and } \mathsf{j}_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})})$$

taking place in

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))_{\text{mon-free}}.$$

Furthermore, this isomorphism is compatible with:

- *The maps into both sides of* (11.37) *from*

$$\mathbf{1}_{\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))_{\text{mon-free}}} \simeq \mathcal{O}_{\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D})};$$

- *The identification of the images of both sides under the functor*

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))_{\text{mon-free}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D})) \overset{(\mathsf{p}^\times\times\mathrm{id})_*}{\longrightarrow}$$
$$\to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{LS}_{\check{M}}(\mathcal{D})) = \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(\mathcal{D})),$$

induced by the isomorphism of the two functors in (11.4).

We now proceed to the construction of the sought-for isomorphism (11.37).

11.5.7. Note that for an object

$$\mathcal{F} \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))_{\text{mon-free}}$$

and an open

$$U \subset \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}),$$

one can talk about the localization of $\mathcal{F}$ on U; to be denoted $\mathcal{F}_U$. It comes equipped with a universal map

$$\mathcal{F} \to \mathcal{F}_U.$$

11.5.8. Tautologically, the map

$$\mathcal{O}_{\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D})} \overset{\sim}{\to} \mathbf{1}_{\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))_{\text{mon-free}}} \to \mathsf{j}_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})})$$

identifies $\mathsf{j}_*(\mathcal{O}_{\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D})})$ with the localization of $\mathcal{O}_{\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D})}$ along the open

$$\mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}) \subset \mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}). \tag{11.38}$$

11.5.9. Hence, in order to construct the isomorphism in (11.37), it suffices to show that the map

$$\mathbf{1}_{\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G},\check{P}^-}(\mathcal{D}^\times) \underset{\mathrm{LS}_{\check{M}}(\mathcal{D}^\times)}{\times} \mathrm{LS}_{\check{M}}(\mathcal{D}))_{\text{mon-free}}} \to \tag{11.39}$$
$$\to \mathrm{FLE}^{\frac{\infty}{2}\text{-mon-free}}_{G,\tau,\mathrm{crit}}\left(\mathrm{Wak}^{-,\frac{\infty}{2}}_{\rho_P(\omega_X)}(\mathrm{Vac}(M)_{\mathrm{crit}+\check{\rho}_P(\omega_X)})\right)$$

also identifies the right-hand side with the localization of the source along the open (11.38).

The property of a map to be a localization along a given open can be checked strata-wise. Hence, since the map (11.39) is compatible with factorization, it being a localization is a pointwise property. The fact that this property holds follows from Theorem 11.5.6.

11.5.10. Thus, we have constructed the the isomorphism (11.37), and hence an isomorphism (11.34), compatible with the maps from $\Omega(R_{\check{G}})^{\mathrm{Op}}$. We will now show that it equals the identification (11.35).

Note that both objects in (11.35) belong to the heart of the natural t-structure, i.e., they can be thought of as classical chiral algebras. Hence, in order to show that two given morphisms between them are equal, it is enough to do so at the pointwise level.

At the pointwise level, in order to show that two given maps between objects of

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D}))^\heartsuit$$

are equal, it is enough to show that this is the case after applying the direct image functor

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \times \mathrm{LS}_{\check{M}}(\mathcal{D})) \to \mathrm{IndCoh}^*(\mathrm{LS}_{\check{M}}(\mathcal{D})) = \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(\mathcal{D})).$$

The required assertion follows now from the second point in Theorem 11.5.6.

□[Theorem 9.5.3]

Part III. Local-to-global constructions

Part III again mainly consists of a review of previously known results. In this part, we study the interactions of various local categories introduced in Part I with their global counterparts, which on the geometric and spectral sides are

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \text{ and } \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)),$$

respectively.

A feature of this part is that the constructions take place either purely on the geometric side, or on the spectral side, but we do not study Langlands-type interactions between them (the latter will be the subject of Part IV).

The main constructions studied in this Part as the following. On the geometric side we will see:

- Poincaré and Whattaker coefficient functors that connect $\mathrm{Whit}^!(G)$ and $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$;
- The localization functor that connects $\mathrm{KL}(G)_{\mathrm{crit}}$ and $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$;
- The relation between the above two constructions;
- The functor of *constant term*, from $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ to $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$, and its enhanced version;
- The relation between the constant term functors and localization.

Logically, we should have also included a section that studies the relation between constant term and Poincaré functors, but in order to avoid the tedium, that topic has been delegated to Parts IV (Sect. 20.5.4).

On the spectral side we will study the following constructions:

- The spectral localization and global sections functors, which relate the categories $\mathrm{Rep}(\check{G})$ and $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ (but in fact, only $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ is involved);
- The spectral Poncaré functors, which relate $\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))$ and $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ (but again, only $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ is involved);
- The relation between the above two constructions;
- The functor of *spectral constant term*, from $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ to $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$, and its enhanced version;
- The relation between spectral constant term and spectral Poncaré functors.

Again, logically, we should have also included a section that studies the relation between spectral constant term and localization functors, but that also has been delegated to Part IV (Sect. 20.5.3).

In this Part a new source of annoyance sets in: when studying relations between pairs of constructions mentioned above, various cohomological shifts and determinant lines pop up. The reader may choose to ignore them on the first pass.

12. The coefficient and Poincaré functors

This section begins by introducing our main object of study: the critically twisted category of D-modules on Bun_G. We rather thank of it as half-twisted D-modules, $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$. The reason being that the latter version makes sense also in sheatftheoretic contexts other than de Rham (i.e., Betti, ℓ-adic).

The focus of this sections is Poincaré and Whattaker coefficient functors. In fact, there are two Poincaré functors

$$\mathrm{Poinc}_{G,!} : \mathrm{Whit}^!(G)_{\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \text{ and } \mathrm{Poinc}_{G,*} : \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}},$$

where $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}$ is the dual category of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$. These two functors are Verdier-conjugate: the dual functor of $\mathrm{Poinc}_{G,*}$ is the same as the right adjoint of $\mathrm{Poinc}_{G,!}$; this is the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\mathrm{Ran}}.$$

But in fact, the functors $\mathrm{Poinc}_{G,!}$ and $\mathrm{Poinc}_{G,*}$ are also related in a much more non-trivial way: they are intertwined by the Miraculous functor

$$\mathrm{Mir}_{\mathrm{Bun}_G} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

see Theorem 23.3.6.

One can also give a global interpretation of the above functors, where instead of the affine Grassmannian, one uses the *twisted Drinfeld compactification*

$$\overline{\mathrm{Bun}}_{N,\rho(\omega_X)} \to \mathrm{Bun}_G\,.$$

This is how the global geometric Whittaker model had been mostly approached so far (see, e.g., [Gai1]). The two approaches are, however, equivalent (see [Gai4]).

For the purposes of this paper, we will only explicitly need the global interpretation of the *vacuum* cases of the above functors, see Sect. 12.5.

12.1. **Twisted D-modules on** Bun_G**.**

12.1.1. Let $\det_{\mathrm{Bun}_G}$ be the determinant line bundle on Bun_G, normalized so that it sends a G-bundle $\mathcal{P}_G$ to

$$\det\left(\Gamma(X, \mathfrak{g}_{\mathcal{P}_G})\right) \otimes \det\left(\Gamma(X, \mathfrak{g}_{\mathcal{P}^0_G})\right)^{\otimes -1},$$

where $\mathcal{P}^0_G$ is the trivial bundle.

12.1.2. Note that we have

$$\pi^*(\det_{\mathrm{Bun}_G}) \simeq \det_{\mathrm{Gr}_{G,\mathrm{Ran}}},$$

where π denotes the projection

$$\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Bun}_G\,. \tag{12.1}$$

12.1.3. Note also that up to the (constant) line $\det\left(\Gamma(X, \mathfrak{g}_{\mathcal{P}^0_G})\right)$, the line bundle $\det_{\mathrm{Bun}_G}$ identifies with the canonical line bundle on Bun_G.

12.1.4. We let crit denote the de Rham twisting on Bun_G, equal to the half of the de Rham twisting defined by $\det_{\mathrm{Bun}_G}$, i.e.,

$$\mathrm{crit} = \frac{1}{2} \cdot \mathrm{dlog}(\det_{\mathrm{Bun}_G}).$$

We will denote by

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

the corresponding category of twisted D-modules.

Note that by Sect. 12.1.3, the critical twisting on Bun_G is canonically isomorphic to the half-canonical twisting.

12.1.5. Pullback along π defines a functor

$$\pi^! : \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

Remark 12.1.6. According to [BD, Sect. 4], the choice of $\omega_X^{\otimes \frac{1}{2}}$ gives rise to a choice of the square of $\det_{\mathrm{Bun}_G}$ as a line bundle. This allows us to identify $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$ with the usual category $\text{D-mod}(\mathrm{Bun}_G)$.

However, we will avoid using this identification.

12.1.7. As in Sect. 1.1.7, we obtain a *canonical* identification

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \xrightarrow{\sim} \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G). \tag{12.2}$$

12.1.8. Pullback along π defines a functor

$$\pi^! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\text{Ran}}),$$

so that the diagram

$$\begin{array}{ccc} \text{D-mod}_{\text{crit}}(\text{Gr}_{G,\text{Ran}}) & \longrightarrow & \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\text{Ran}}) \\ {\scriptstyle \pi^!}\uparrow & & \uparrow{\scriptstyle \pi^!} \\ \text{D-mod}_{\text{crit}}(\text{Bun}_G) & \longrightarrow & \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \end{array}$$

commutes.

12.2. **Restricting to (twists of)** Bun_N**.**

12.2.1. Let $\mathcal{P}_T$ be any T-bundle. Consider the stack

$$\text{Bun}_{N,\mathcal{P}_T} \simeq \text{Bun}_B \underset{\text{Bun}_T}{\times} \text{pt},$$

where $\text{pt} \to \text{Bun}_T$ is the point $\mathcal{P}_T$.

Denote by $\mathfrak{p}$ the map

$$\text{Bun}_{N,\mathcal{P}_T} \to \text{Bun}_G\,.$$

Note that the pullback of $\det_{\text{Bun}_G}$ along this map is canonically constant. Denote the resulting line by

$$\mathfrak{l}_{G,N_{\mathcal{P}_T}}.$$

12.2.2. We obtain that $\mathfrak{p}$ gives rise to well-defined functors

$$\mathfrak{p}^!_{\text{crit}} : \text{D-mod}_{\text{crit}}(\text{Bun}_G) \to \text{D-mod}_{\text{dlog}(\mathfrak{l}_{G,N_{\mathcal{P}_T}})}(\text{Bun}_{N,\mathcal{P}_T}) \xrightarrow{\sim} \text{D-mod}(\text{Bun}_{N,\mathcal{P}_T})$$

(the second identification is due to the fact that the dlog map over pt is trivial), and

$$\mathfrak{p}^! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{D-mod}_{\mathfrak{l}^{\frac{1}{2}}_{G,N_{\mathcal{P}_T}}}(\text{D-mod}(\text{Bun}_{N,\mathcal{P}_T})),$$

where the subscript $\mathfrak{l}^{\frac{1}{2}}_{G,N_{\mathcal{P}_T}}$ means the twist by the constant gerbe of square roots of the line $\mathfrak{l}_{G,N_{\mathcal{P}_T}}$.

We have a commutative diagram

$$\begin{array}{ccc} \text{D-mod}(\text{Bun}_{N,\mathcal{P}_T}) & \xrightarrow{\sim} & \text{D-mod}_{\mathfrak{l}^{\frac{1}{2}}_{G,N_{\mathcal{P}_T}}}(\text{D-mod}(\text{Bun}_{N,\mathcal{P}_T})) \\ {\scriptstyle \mathfrak{p}^!_{\text{crit}}}\uparrow & & \uparrow{\scriptstyle \mathfrak{p}^!} \\ \text{D-mod}_{\text{crit}}(\text{Bun}_G) & \xrightarrow{(12.2)} & \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G), \end{array} \tag{12.3}$$

where the top horizontal row comes from the identification

$$\text{Vect} = \text{D-mod}(\text{pt}) \simeq \text{D-mod}_{\frac{1}{2}\cdot\text{dlog}(\mathfrak{l}_{G,N_{\mathcal{P}_T}})}(\text{pt}) \overset{(1.1)}{\simeq} \text{D-mod}_{\mathfrak{l}^{\frac{1}{2}}_{G,N_{\mathcal{P}_T}}}(\text{pt}).$$

12.2.3. We take $\mathcal{P}_T = \rho(\omega_X)$. We claim:

Proposition 12.2.4. *The line $\mathfrak{l}_{G,N_{\rho(\omega_X)}}$ admits a canonical square root.*

12.2.5. *Proof of Proposition 12.2.4.* Decompose $\mathfrak{g}$ with respect to the action of the principal SL_2

$$\mathfrak{g} \simeq \underset{e}{\oplus} V^e.$$

By definition, the line $\mathfrak{l}_{G,N_{\rho(\omega_X)}}$ is

$$\underset{e}{\otimes} \left(\det(\Gamma(X,(V^e)_{\rho(\omega_X)})) \otimes \det(\Gamma(X,V^e\otimes \mathcal{O}_X))^{\otimes -1}\right).$$

Decompose V^e into its weight spaces

$$V^e = \underset{n}{\oplus} V^e(n),$$

where each $V^e(n)$ is 1-dimensional.

We can write:

$$\begin{aligned}(12.4)\quad \det(\Gamma(X,(V^e)_{\rho(\omega_X)})) \otimes \det(\Gamma(X,V^e\otimes \mathcal{O}_X))^{\otimes -1} \simeq \\ \simeq \underset{n>0}{\otimes} \left(\det(\Gamma(X,\omega_X^{\otimes n})) \otimes \det(\Gamma(X,\omega_X^{\otimes -n})) \otimes \det(\Gamma(X,\mathcal{O}_X))^{\otimes -2}\right) \bigotimes \\ \bigotimes \underset{n>0}{\otimes} \left(V^e(n)^{\otimes n(2g-2)+(1-g)} \otimes V^e(-n)^{\otimes -n(2g-2)+(1-g)}\right).\end{aligned}$$

We claim that each term of the form

$$(12.5)\qquad \det(\Gamma(X,\omega_X^{\otimes n})) \otimes \det(\Gamma(X,\omega_X^{\otimes -n})) \otimes \det(\Gamma(X,\mathcal{O}_X))^{\otimes -2}.$$

admits a canonical square root.

Recall the formlula

$$(12.6)\qquad \det(\Gamma(X,\mathcal{L}_1\otimes\mathcal{L}_2)) \otimes \det(\Gamma(X,\mathcal{O}_X)) \simeq \det(\Gamma(X,\mathcal{L}_1)) \otimes \det(\Gamma(X,\mathcal{L}_2)) \otimes \mathrm{Weil}(\mathcal{L}_1,\mathcal{L}_2),$$

where $\mathrm{Weil}(-,-)$ is the Weil pairing.

We obtain that (12.5) is isomorphic to

$$(12.7)\qquad \mathrm{Weil}(\omega_X^{\otimes n},\omega_X^{\otimes n}).$$

Recall now that we have chosen a square root $\omega_X^{\otimes \frac{1}{2}}$ of ω_X. Then the expression in (12.7) is

$$\left(\mathrm{Weil}(\omega_X^{\otimes \frac{1}{2}},\omega_X^{\otimes \frac{1}{2}})\right)^{\otimes 4n^2},$$

which is manifestly a tensor square.

We now claim that the tensor product

$$(12.8)\qquad \underset{e}{\otimes}\ \underset{n>0}{\otimes} \left(V^e(n)^{\otimes n(2g-2)+(1-g)} \otimes V^e(-n)^{\otimes -n(2g-2)+(1-g)}\right)$$

admits a canonical square root.

Indeed, up to squares, the expression in (12.8) is isomorphic to

$$\left(\underset{e}{\otimes}\ \underset{n>0}{\otimes} (V^e(n)\otimes V^e(-n))\right)^{\otimes(1-g)} \simeq \left(\det(\mathfrak{n})\otimes\det(\mathfrak{n}^-)\right)^{\otimes(1-g)}.$$

Now, the Killing form identifies $\mathfrak{n}^-$ with the dual of $\mathfrak{n}$, and hence trivializes the line $\det(\mathfrak{n})\otimes\det(\mathfrak{n}^-)$.

□[Proposition 12.2.4]

12.2.6. Let

$$\mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes\frac{1}{2}} \tag{12.9}$$

denote the square root of the line $\mathfrak{l}_{G,N_{\rho(\omega_X)}}$ constructed in Proposition 12.2.4.

From Proposition 12.2.4 we obtain that there exists an a priori identification

$$\text{D-mod}_{\mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\frac{1}{2}}}(\mathrm{Bun}_{N,\rho(\omega)}) \simeq \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega)}). \tag{12.10}$$

Denote by

$$\mathfrak{p}^!_{\frac{1}{2}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega)})$$

the functor equal to the composition

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \xrightarrow{\mathfrak{p}^!} \text{D-mod}_{\mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\frac{1}{2}}}(\mathrm{Bun}_{N,\rho(\omega_X)}) \xrightarrow{(12.10)} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

Note that we have a commutative diagram

$$\begin{array}{ccc} \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) & \xrightarrow{-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes -\frac{1}{2}}} & \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \\ {\scriptstyle \mathfrak{p}^!_{\mathrm{crit}}}\uparrow & & \uparrow{\scriptstyle \mathfrak{p}^!_{\frac{1}{2}}} \\ \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) & \xrightarrow{(12.2)} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G). \end{array} \tag{12.11}$$

12.3. **The coefficient functor.** In this subsection we will recall the definition of the functor of Whittaker coefficient(s).

12.3.1. The functor of Whittaker coefficient(s), denoted coeff_G maps

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\mathrm{Ran}},$$

and is defined as follows.

To simplify the notation, we will work over a particular point $\underline{x} \in \mathrm{Ran}$. So we need to define the functor

$$\mathrm{coeff}_{G,\underline{x}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\underline{x}}.$$

12.3.2. Consider of the $\rho(\omega_X)$-twisted version of the map (12.1)

$$\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}} \to \mathrm{Bun}_G\,.$$

By a slight abuse of notation, we will denote it by the same symbol π. By further abuse of notation, we will keep the same notation for the restriction of this map to

$$\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}} \to \mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}. \tag{12.12}$$

12.3.3. Due to the trivialization of the $\mathbb{Z}/2\mathbb{Z}$-gerbe $\mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\frac{1}{2}}$ given by Proposition 12.2.4, the map π give rise to a well-defined functor

$$\pi^!_{\frac{1}{2}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}).$$

12.3.4. For a group-subscheme of $N^\alpha \subset \mathfrak{L}(N)_{\rho(\omega_X),\underline{x}}$, consider the functor

$$\mathrm{Av}_*^{(N^\alpha,\chi)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}})^{N^\alpha,\chi} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}).$$

For $N^\alpha \subset N^{\alpha'}$, we have a canonically defined natural transformation

$$\mathrm{Av}_*^{(N^{\alpha'},\chi)} \to \mathrm{Av}_*^{(N^\alpha,\chi)}. \tag{12.13}$$

We have the following (elementary) observation:

Lemma 12.3.5. *The natural transformation*

$$\mathrm{Av}_*^{(N^{\alpha'},\chi)} \circ \pi^!_{\frac{1}{2}} \to \mathrm{Av}_*^{(N^\alpha,\chi)} \circ \pi^!_{\frac{1}{2}}$$

induced by (12.13)*, is an isomorphism when N^α is large enough.*[18]

12.3.6. By Lemma 12.13, for N^α large enough, the functor

$$\mathrm{Av}_*^{(N^\alpha,\chi)} \circ \pi^!_{\frac{1}{2}} \tag{12.14}$$

does not depend on the choice of N^α. In particular, its essential image is contained in

$$\underset{N^\alpha}{\cap} \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}})^{N^\alpha,\chi} = \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}})^{\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}},\chi} = \mathrm{Whit}^!(G)_{\underline{x}}.$$

Thus, we let $\mathrm{coeff}_{G,\underline{x}}$ be the functor (12.14) for N^α large enough.

12.3.7. By construction, the functor $\mathrm{coeff}_{G,\underline{x}}$ is compatible with the action of $\mathrm{Sph}_{G,\underline{x}}$.

12.3.8. The functor coeff_G (i.e., the totality of the functors $\mathrm{coeff}_{G,\underline{x}}$) has the following unitality property:

For $\underline{x} \subset \underline{x'}$ consider the natural embedding

$$\mathrm{incl}_{\underline{x}\subset\underline{x'}} : \mathrm{Gr}_{G,\rho(\omega_X),\underline{x}} \hookrightarrow \mathrm{Gr}_{G,\rho(\omega_X),\underline{x'}}.$$

Then

$$\mathrm{coeff}_{G,\underline{x}} \simeq \mathrm{incl}^!_{\underline{x}\subset\underline{x'}} \circ \mathrm{coeff}_{G,\underline{x'}}. \tag{12.15}$$

12.3.9. Let

$$\mathrm{coeff}_G^{\mathrm{Vac}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Vect}$$

denote the composition of $\mathrm{coeff}_{G,\underline{x}}$ with the functor

$$\mathrm{Whit}^!(G)_{\underline{x}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \mathrm{Vect},$$

where the second arrow is the functor of !-fiber at the unit point.

By (12.15), the above definition of $\mathrm{coeff}_G^{\mathrm{Vac}}$ is canonically independent of the choice of $\underline{x}$.

Equivalently, $\mathrm{coeff}_G^{\mathrm{Vac}}$ is the unique functor $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Vect}$ so that the diagram commutes

$$\begin{array}{ccccc}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} & \longrightarrow & \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}) & \xrightarrow{(\mathbf{1}_{\mathrm{Gr}_{G,\mathrm{Ran}}})^!} & \text{D-mod}(\mathrm{Ran}) \\
{\scriptstyle \mathrm{coeff}_G}\uparrow & & & & \uparrow \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \xrightarrow{\mathrm{coeff}_G^{\mathrm{Vac}}} & & \mathrm{Vect}.
\end{array}$$

(In the above diagram the left vertical arrow is the !-pullback along $\mathrm{Ran} \to \mathrm{pt}$, which is fully-faithful, by the contractibility of the Ran space.)

12.3.10. Note that using the the equivalence CS_G, the functor coeff_G can be recovered from $\mathrm{coeff}_G^{\mathrm{Vac}}$ via the Hecke action of

$$\mathrm{Rep}(\check{G}) \xrightarrow{\mathrm{Sat}_G^{-1,\mathrm{nv}}} \mathrm{Sph}_G.$$

[18] The size of N^α depends on the genus of X and the cardinality of $\underline{x}$.

12.4. **Poincaré functor(s).**

12.4.1. The Poincaré functor

$$\mathrm{Poinc}_{G,!} : \mathrm{Whit}^!(G)_{\mathrm{Ran}} \to \text{D-mod}(\mathrm{Bun}_G)$$

is by definition the left adjoint to coeff_G.

It explicitly given by

$$\mathrm{Whit}^!(G)_{\mathrm{Ran}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}) \overset{\pi_!}{\to} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

It is easy to see (using the action of Sph_G) that the partially defined functor

$$\pi_! : \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}) \overset{\pi_{!,\frac{1}{2}}}{\to} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is actually defined on the essential image of

$$\mathrm{Whit}^!(G)_{\mathrm{Ran}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\mathrm{Ran}}).$$

(The issue here is that the "lower-!" functors are not necessarily defined on non-holonomic objects.) See also Remark 12.4.6, below.

12.4.2. We let

$$\mathrm{Poinc}_{G,!,\underline{x}} : \mathrm{Whit}^!(G)_{\underline{x}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

denote the restriction of $\mathrm{Poinc}_{G,!}$ along (12.12).

It is also given as

$$\mathrm{Whit}^!(G)_{\underline{x}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\rho(\omega_X),\underline{x}}) \overset{\pi_{!,\frac{1}{2}}}{\to} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

12.4.3. It is easy to see (say, by rigidity) that the functor $\mathrm{Poinc}_{G,!,\underline{x}}$ is also compatible with the action of $\mathrm{Sph}_{G,\underline{x}}$.

12.4.4. Let

$$\mathbf{1}_{\mathrm{Whit}^!(G),\underline{x}} \in \mathrm{Whit}^!(G)_{\underline{x}}$$

be the factorization unit.

It follows formally from Sect. 12.3.8 that the object

$$\mathrm{Poinc}_{G,!,\underline{x}}(\mathbf{1}_{\mathrm{Whit}^!(G),\underline{x}}) \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is canonically independent of the choice of $\underline{x}$.

We will denote it by

$$\mathrm{Poinc}^{\mathrm{Vac}}_{G,!} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

We also have

$$\mathrm{Poinc}^{\mathrm{Vac}}_{G,!} \simeq \mathrm{Poinc}_{G,!}(\mathbf{1}_{\mathrm{Whit}^!(G),\mathrm{Ran}}),$$

where

$$\mathbf{1}_{\mathrm{Whit}^!(G),\mathrm{Ran}} \in \mathrm{Whit}^!(G)_{\mathrm{Ran}}$$

is the factorization unit spread over the Ran space.

12.4.5. By the same token as in Sect. 12.3.10, we can recover the functor $\mathrm{Poinc}_{G,!}$ from the object $\mathrm{Poinc}^{\mathrm{Vac}}_{G,!}$ using the Hecke action.

Remark 12.4.6. Since the object $\mathbf{1}_{\mathrm{Whit}^!(G),\underline{x}}$ is ind-holonomic, it is clear that

$$\pi_{!,\frac{1}{2}}(\mathbf{1}_{\mathrm{Whit}^!(G),\underline{x}}) \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is well-defined.

One can prove that $\mathrm{Poinc}_{G,!,\underline{x}}$ on all of $\mathrm{Whit}^!(G)_{\underline{x}}$ using the Hecke action, by the same principle as in Sect. 12.4.5.

12.4.7. Recall that along with the category $\text{D-mod}(\text{Bun}_G)$, one can consider its version $\text{D-mod}(\text{Bun}_G)_{\text{co}}$, and similarly for gerbe-twisted versions $\text{D-mod}_{\mathcal{G}}(\text{Bun}_G)$.

In the untwisted case, we have the identification

$$(\text{D-mod}(\text{Bun}_G))^\vee \simeq \text{D-mod}(\text{Bun}_G)_{\text{co}}.$$

In the twisted case, this becomes

$$(\text{D-mod}_{\mathcal{G}}(\text{Bun}_G))^\vee \simeq \text{D-mod}_{\mathcal{G}^{\otimes -1}}(\text{Bun}_G)_{\text{co}}. \tag{12.16}$$

For $\mathcal{G} = \det^{\frac{1}{2}}_{\text{Bun}_G}$, the identification (12.16) becomes a self-duality

$$(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G))^\vee \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}. \tag{12.17}$$

12.4.8. Let

$$\text{Poinc}_{G,*} : \text{Whit}_*(G)_{\text{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

be the functor *dual* to coeff_G.

Let

$$\text{Poinc}_{G,*,\underline{x}} : \text{Whit}_*(G)_{\underline{x}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

be the functor dual to $\text{coeff}_{G,\underline{x}}$. It is easy to see that the functor $\text{Poinc}_{G,*,\underline{x}}$ is obtained from $\text{Poinc}_{G,*}$ by restriction along (12.12).

The functor $\text{Poinc}_{G,*,\underline{x}}$ is also compatible with the Hecke action.

12.4.9. Let

$$\text{Poinc}^{\text{Vac}}_{G,*} \in \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

be the factorization unit.

It follows formally that the pairing with $\text{Poinc}^{\text{Vac}}_{G,*}$, viewed as a functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{Vect}$$

is the functor $\text{coeff}^{\text{Vac}}_G$.

12.4.10. The functor $\text{Poinc}_{G,*,\underline{x}}$ can be explicitly described as follows. For N^α as in Sect. 12.3.4, consider the composition

$$\text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}}) \overset{\text{Av}^{(N^\alpha,\chi)}_*}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}}) \overset{\pi_{*,\frac{1}{2}}}{\to} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}.$$

For $N^\alpha \subset N^{\alpha'}$, we have a canonically defined natural transformation

$$\pi_{*,\frac{1}{2}} \circ \text{Av}^{(N^{\alpha'},\chi)}_* \to \pi_{*,\frac{1}{2}} \circ \text{Av}^{(N^\alpha,\chi)}_*, \tag{12.18}$$

and it follows follows from Lemma 12.3.5 that the maps (12.18) are isomorphisms for N^α large enough.

It follows formally that the resulting functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

factors via the projection

$$\text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}}) \to \left(\text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}})\right)_{\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}},\chi} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}.$$

The resulting functor

$$\text{Whit}_*(G)_{\underline{x}} := \left(\text{D-mod}_{\frac{1}{2}}(\text{Gr}_{G,\rho(\omega_X),\underline{x}})\right)_{\mathfrak{L}(N)_{\rho(\omega_X),\underline{x}},\chi} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{co}}$$

is the functor $\text{Poinc}_{G,*,\underline{x}}$.

12.5. **Coefficient and Poincaré functors: local version: global interpretation.**

12.5.1. Consider the stack $\mathrm{Bun}_{N,\rho(\omega_X)}$ and the map

$$\mathfrak{p} : \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathrm{Bun}_G .$$

Recall that by Sect. 12.2.6, we have a well-defined functor

$$\mathfrak{p}^!_{\frac{1}{2}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}). \tag{12.19}$$

12.5.2. The character χ has a global counterpart, which is a map

$$\chi^{\mathrm{glob}} : \mathrm{Bun}_{N,\rho(\omega_X)} \to \mathbb{G}_a.$$

12.5.3. We let

$$\mathrm{coeff}^{\mathrm{Vac,glob}}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Vect}$$

denote the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_{\frac{1}{2}}}{\to} \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{-\overset{*}{\otimes}(\chi^{\mathrm{glob}})^*(\exp)}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N,\rho(\omega_X)},-)}{\longrightarrow} \mathrm{Vect} .$$

12.5.4. Let

$$\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

be the object

$$\mathfrak{p}_{!,\frac{1}{2}} \circ (\chi^{\mathrm{glob}})^*(\exp).$$

It is the left adjoint of $\mathrm{coeff}^{\mathrm{Vac,glob}}_G$, viewed as a functor.

Let

$$\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,*} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}$$

be the object

$$\mathfrak{p}_{*,\frac{1}{2}} \circ (\chi^{\mathrm{glob}})^*(\exp).$$

12.5.5. Denote

$$\delta_{N_{\rho(\omega_X)}} := \dim(\mathrm{Bun}_{N,\rho(\omega_X)}).$$

We have

$$\mathbb{D}^{\mathrm{Verdier}}(\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!}) = \mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,*}[2\delta_{N_{\rho(\omega_X)}}],$$

where $\mathbb{D}^{\mathrm{Verdier}}$ is the usual Verdier dualization functor

$$(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c)^{\mathrm{op}} \to (\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}})^c.$$

In other words, the object $\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,*}[2\delta_{N_{\rho(\omega_X)}}]$, viewed as a functor

$$\mathrm{Vect} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}$$

is the dual of $\mathrm{coeff}^{\mathrm{Vac,glob}}_G$.

12.5.6. It is easy to see that

$$\mathrm{coeff}^{\mathrm{Vac,glob}}_G \simeq \mathrm{coeff}^{\mathrm{Vac}}_G[2\delta_{N_{\rho(\omega_X)}}],$$

$$\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!} \simeq \mathrm{Poinc}^{\mathrm{Vac}}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]$$

and

$$\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,*} \simeq \mathrm{Poinc}^{\mathrm{Vac}}_{G,*} .$$

13. The localization functor

In this section we introduce and study the localization functor

$$\mathrm{Loc}_G : \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

We do so in a more general context, when the group G in question is not even reductive, and the level κ is not necessarily critical. The topics here include:

- The behavior of localization when we introduce a twisting by a Z_G- and Z^0_G-torsors;
- The composition of the composition of the localization functor $\mathrm{Loc}_{G,\kappa} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G)$ with the forgetful functor $\text{D-mod}_\kappa(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{Bun}_G)$;
- The composition of the localization functor with the pullback functor $\text{D-mod}_\kappa(\mathrm{Bun}_G) \to \text{D-mod}_\kappa(\mathrm{Bun}_{G'})$ corresponding to a group homomorphism $G' \to G$;
- For a unipotent group-scheme N', the composition of the localization functor and the de Rham cohomology functor $\text{D-mod}(\mathrm{Bun}_{N'}) \to \mathrm{Vect}$.

The pattern in the three composite functors mentioned above is that they can all be expressed via a local operation, followed by another localization functor:

- Restriction $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \mathrm{Rep}(\mathfrak{L}^+(G))$, followed by $\mathcal{O}$-module localization $\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{Bun}_G)$;
- Restriction $\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \mathrm{KL}(G')_{\kappa,\mathrm{Ran}}$, followed by

$$\mathrm{Loc}_{G',\kappa} : \mathrm{KL}(G')_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_{G'});$$

- The functor of BRST reduction $\mathrm{KL}(N')_{\mathrm{Ran}} \to \mathrm{Vect}$.

However, there is a caveat, common to all three of these situations: we must precompose the corresponding local functor with the functor of *inserting the factorization unit*, which maps the corresponding category $\mathbf{C}_{\mathrm{Ran}}$ to its version $\mathbf{C}_{\mathrm{Ran}\subseteq}$, see Sect. 13.3. This operation is closely related to the functor of factorization (a.k.a., chiral) homology, which is reviewed in Sect. 13.4.

13.1. Basics of localization.

13.1.1. We let

$$\mathrm{Loc}_{G,\kappa} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G) \tag{13.1}$$

be the naturally defined localization functor.

It is normalized so that the following diagram commutes

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Bun}_G) & \xrightarrow{\mathbf{ind}^l_\kappa} & \text{D-mod}_\kappa(\mathrm{Bun}_G) \\ {\scriptstyle \mathrm{Loc}^{\mathrm{QCoh}}_G}\uparrow & & \uparrow{\scriptstyle \mathrm{Loc}_{G,\mathrm{crit}}} \\ \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} & \xrightarrow{\mathbf{ind}_{\mathfrak{L}^+(G)\to(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))}} & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}, \end{array} \tag{13.2}$$

where:

- The (factorization) functor $\mathbf{ind}_{\mathfrak{L}^+(G)\to(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))} : \mathrm{Rep}(\mathfrak{L}^+(G)) \to \mathrm{KL}(G)_\kappa$ is the left adjoint to the forgetful (factorization) functor $\mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)} : \mathrm{KL}(G)_\kappa \to \mathrm{Rep}(\mathfrak{L}^+(G))$;
- $\mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} \overset{\mathrm{Loc}^{\mathrm{QCoh}}_G}{\longrightarrow} \mathrm{QCoh}(\mathrm{Bun}_G)$ is the functor of pull-push along the diagram

$$B(\mathfrak{L}^+(G))_{\mathrm{Ran}} \leftarrow \mathrm{Bun}_G \times \mathrm{Ran} \to \mathrm{Bun}_G;$$

- $\mathbf{ind}^l_\kappa$ is the functor of induction for left D-modules, i.e.,

$$\mathcal{M} \mapsto \mathrm{Diff}(\mathrm{Bun}_G)_\kappa \underset{\mathcal{O}_{\mathrm{Bun}_G}}{\otimes} \mathcal{M}.$$

13.1.2. In particular, the functor $\mathbf{ind}^l_\kappa$ sends the vacuum object

$$\mathrm{Vac}(G)_{\kappa,\mathrm{Ran}} \in \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}$$

to

$$\mathrm{Diff}(\mathrm{Bun}_G)_\kappa \in \text{D-mod}_\kappa(\mathrm{Bun}_G).$$

13.1.3. We let $\mathrm{Loc}_{G,\kappa,\underline{x}}$ denote the restriction of $\mathrm{Loc}_{G,\kappa}$ along

$$\mathrm{KL}(G)_{\kappa,\underline{x}} \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}.$$

13.1.4. Let us specialize to the case when $\kappa = \mathrm{crit}$. In this case, the functor $\mathrm{Loc}_{G,\mathrm{crit},\underline{x}}$ is compatible with the actions of $\mathrm{Sph}_{G,\underline{x}}$.

13.1.5. We let

$$\mathrm{Loc}_G : \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \tag{13.3}$$

denote the composition of $\mathrm{Loc}_{G,\mathrm{crit}}$ with the equivalence (12.2).

The corresponding functors $\mathrm{Loc}_{G,\underline{x}}$ inherit the compatibility structure with the action of $\mathrm{Sph}_{G,\underline{x}}$.

13.2. **Localization in the twisted setting.** We now consider two types of twisted situations, and also their combination.

13.2.1. Let $\mathcal{P}$ be a torsor with respect to $(\mathfrak{g}_{\mathrm{ab}})^* \otimes \omega_X$ on X. It gives gives rise to a multiplicative de Rham twisting on $\mathrm{Bun}_{G_{\mathrm{ab}}}$, denoted by the same character.

By a slight abuse of notation, we will keep the same symbol $\mathcal{P}$ to denote the pullback of this twisting along the projection

$$\mathrm{Bun}_G \to \mathrm{Bun}_{G_{\mathrm{ab}}}. \tag{13.4}$$

13.2.2. Given a level κ, we will denote by $\kappa + \mathcal{P}$ the Baer sum of the two de Rham twisting on Bun_G. Consider the twisted Kazhdan-Lustig category $\mathrm{KL}(G)_{\kappa+\mathcal{P}}$, see Sect. 4.4.1.

In this case, the localization functor maps

$$\mathrm{Loc}_{G,\kappa+\mathcal{P}} : \mathrm{KL}(G)_{\kappa+\mathcal{P},\mathrm{Ran}} \to \text{D-mod}_{\kappa+\mathcal{P}}(\mathrm{Bun}_G).$$

making a diagram parallel to (13.2) commute.

13.2.3. Assume for a moment that $\mathcal{P}$ is of the form $\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})$ for a $Z^0_{\check{G}}$-torsor $\mathcal{P}_{Z_{\check{G}}}$ on X.

Using the Weil pairing

$$\mathrm{Bun}_{Z^0_{\check{G}}} \otimes \mathrm{Bun}_{G_{\mathrm{ab}}} \to \mathrm{pt}/\mathbb{G}_m,$$

the $Z^0_{\check{G}}$-torsor $\mathcal{P}_{Z_{\check{G}}}$ gives rise to a line bundle, denoted, $\mathcal{L}_{\mathcal{P}_{Z^0_{\check{G}}}}$ on $\mathrm{Bun}_{G_{\mathrm{ab}}}$. By a slight abuse of notation, we will denote by the same symbol $\mathcal{L}_{\mathcal{P}_{Z^0_{\check{G}}}}$ its pullback along (13.4).

In this case, the twisting $\mathcal{P}$ is the de Rham twisting given by $\mathcal{L}_{\mathcal{P}_{Z^0_{\check{G}}}}$; we will denote it by $\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})$.

Parallel to Sect. 4.4.4, if $\mathcal{P}_{Z^0_{\check{G}}}$ is of the form $\check{\lambda}(\omega_X)$ for $\check{\lambda} : \mathbb{G}_m \to Z_{\check{G}}$, we will use the short-hand notation

$$\kappa + \check{\lambda} := \kappa + \mathrm{dlog}(\check{\lambda}(\omega_X)).$$

Note that by linearity, the twisting $\kappa + \check{\lambda}$ makes sense for any $\check{\lambda} \in z_{\check{\mathfrak{g}}}$.

13.2.4. Suppose that in the setting of Sect. 13.2.3, $\kappa = 0$. In this case we will denote by Loc_G the composite functor

$$\begin{aligned} \mathrm{KL}(G)_{\mathrm{crit}+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}}),\mathrm{Ran}} &\overset{\mathrm{Loc}_{G,\mathrm{crit}+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}}{\longrightarrow} \text{D-mod}_{\mathrm{crit}+\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}(\mathrm{Bun}_G) \overset{-\otimes\mathcal{L}^{\otimes -1}_{\mathcal{P}_{Z^0_{\check{G}}}}}{\longrightarrow} \\ &\to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \overset{(12.2)}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G). \end{aligned} \tag{13.5}$$

13.2.5. Let now $\mathcal{P}_G$ be a G-torsor on X. We can consider the $\mathcal{P}_G$-twist of the entire situation, and in particular the localization functor

$$\mathrm{Loc}_{G,\kappa,\mathcal{P}_G} : \mathrm{KL}(G)_{\kappa,\mathcal{P}_G} \to \text{D-mod}_\kappa(\mathrm{Bun}_{G,\mathcal{P}_G}),$$

which makes the diagram

$$\begin{array}{ccc} \text{D-mod}_\kappa(\mathrm{Bun}_G) & \xrightarrow{\alpha_{\mathcal{P}_G,\mathrm{taut}}} & \text{D-mod}_\kappa(\mathrm{Bun}_{G,\mathcal{P}_G}) \\ {\scriptstyle \mathrm{Loc}_{G,\kappa}}\uparrow & & \uparrow{\scriptstyle \mathrm{Loc}_{G,\kappa,\mathcal{P}_G}} \\ \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} & \xrightarrow{\alpha_{\mathcal{P}_G,\mathrm{taut}}} & \mathrm{KL}(G)_{\kappa,\mathcal{P}_G,\mathrm{Ran}}, \end{array} \tag{13.6}$$

commute, where in the top row, $\alpha_{\mathcal{P}_G,\mathrm{taut}}$ is the tautological identification

$$\mathrm{Bun}_G \xrightarrow{\alpha_{\mathcal{P}_G,\mathrm{taut}}} \mathrm{Bun}_{G,\mathcal{P}_G}.$$

13.2.6. Assume now that $\mathcal{P}_G$ is induced from a Z_G-torsor $\mathcal{P}_{Z_G}$. In this case, we have a canonical identification

$$\mathrm{Bun}_{G,\mathcal{P}_{Z_G}} \xrightarrow{\alpha_{\mathcal{P}_{Z_G},\mathrm{cent}}} \mathrm{Bun}_G.$$

Lemma 13.2.7. *The isomorphism $\alpha_{\mathcal{P}_{Z_G},\mathrm{cent}}$ identifies the twisting κ on $\mathrm{Bun}_{G,\mathcal{P}_{Z_G}}$ with the twisting*

$$\kappa - \kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}, -)$$

on[19] Bun_G. *Furthermore, the diagram*

$$\begin{array}{ccc} \text{D-mod}_\kappa(\mathrm{Bun}_{G,\mathcal{P}_{Z_G}}) & \xrightarrow{\alpha_{\mathcal{P}_{Z_G},\mathrm{cent}}} & \text{D-mod}_{\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G},-)}(\mathrm{Bun}_G) \\ {\scriptstyle \mathrm{Loc}_{G,\kappa,\mathcal{P}_{Z_G}}}\uparrow & & \uparrow{\scriptstyle \mathrm{Loc}_{G,\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G},-)}} \\ \mathrm{KL}(G)_{\kappa,\mathcal{P}_{Z_G},\mathrm{Ran}} & \xrightarrow{\alpha_{\mathcal{P}_{Z_G},\mathrm{cent}}} & \mathrm{KL}(G)_{\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G},-),\mathrm{Ran}}. \end{array} \tag{13.7}$$

13.2.8. Note that the composite map

$$\mathrm{Bun}_G \xrightarrow{\alpha_{\mathcal{P}_G,\mathrm{taut}}} \mathrm{Bun}_{G,\mathcal{P}_{Z_G}} \xrightarrow{\alpha_{\mathcal{P}_{Z_G},\mathrm{cent}}} \mathrm{Bun}_G$$

is the inverse of the automorphism

$$\mathrm{transl}_{\mathcal{P}_{Z_G}} : \mathrm{Bun}_G \to \mathrm{Bun}_G,$$

given by tensoring with $\mathcal{P}_{Z_G}$.

From Lemma 13.2.7 we obtain that the pullback of the twisting of the twisting κ on Bun_G with respect to $\mathrm{transl}_{\mathcal{P}_{Z_G}}$ identifies canonically with the twisting $\kappa - \kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}, -)$.

By concatenating diagrams (13.6) and (13.7), we obtain a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_\kappa(\mathrm{Bun}_G) & \xrightarrow{(\mathrm{transl}_{\mathcal{P}_{Z_G}})^*} & \text{D-mod}_{\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G},-)}(\mathrm{Bun}_G) \\ {\scriptstyle \mathrm{Loc}_{G,\kappa}}\uparrow & & \uparrow{\scriptstyle \mathrm{Loc}_{G,\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G},-)}} \\ \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} & \xrightarrow{(\mathrm{transl}_{\mathcal{P}_{Z_G}})^*} & \mathrm{KL}(G)_{\kappa-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G},-),\mathrm{Ran}}, \end{array} \tag{13.8}$$

where the functor $(\mathrm{transl}_{\mathcal{P}_{Z_G}})^*$ in the bottom line is as in Sect. 4.13.

13.2.9. Note in particular, that from Lemma 13.2.7 we obtain that the critical twisting is invariant under the automorphism $\mathrm{transl}_{\mathcal{P}_{Z_G}}$.

Hence, the operation $(\mathrm{transl}_{\mathcal{P}_{Z_G}})^*$ is well-defined as a functor

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \to \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \text{ and } \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

[19] See Sect. 4.6.3, where the notation $\kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}, -)$ is introduced.

13.2.10. Parallel to Sect. 4.6.6 for $\mathcal{P}_{Z_G}$ of the form $\lambda(\omega_X)$ for $\lambda : \mathbb{G}_m \to Z_G$, we will use a short-hand notation

$$\kappa - \kappa(\lambda, -) := \kappa - \kappa(\mathrm{dlog}(\lambda(\omega_X), -).$$

Note that this notation is consistent with one in Sect. 13.2.3, i.e., we can treat $\kappa(\lambda, -)$ as a bona fide element of $z_{\check{\mathfrak{g}}}$.

13.2.11. The content of Sects. 13.2.5-13.2.10 renders as-is, if instead of the initial twisting given by κ, we start with one of the form $\kappa + \mathcal{P}$ for $\mathcal{P}$ as in Sect. 13.2.1.

In particular, we will have twistings of the form

$$\kappa + \mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}}) - \kappa(\mathrm{dlog}(\mathcal{P}_{Z_G}), -) \text{ and } \kappa + \check{\lambda} - \kappa(\lambda, -), \quad \check{\lambda} \in z_{\check{\mathfrak{g}}}, \lambda \in z_{\mathfrak{g}}.$$

13.3. **A digression: unitality.**

13.3.1. Let $\mathbf{C}^{\mathrm{loc}}$ be one of the factorization categories

$$\mathrm{Whit}^{!}(G),\ \mathrm{Whit}_{*}(G),\ \mathrm{KL}(G)_{\kappa},\ \text{etc.}$$

Let $\mathbf{C}^{\mathrm{glob}}$ be the corresponding global category, which in the above three examples is

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \text{ and } \text{D-mod}_{\kappa}(\mathrm{Bun}_G),$$

respectively.

Let F be a functor

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \to \mathbf{C}^{\mathrm{glob}}$$

be a functor, which in the above three examples is

$$\mathrm{Poinc}_{G,!},\ \mathrm{Poinc}_{G,*} \text{ and } \mathrm{Loc}_{G,\kappa},$$

respectively.

13.3.2. Note that in each case, the functor F factors naturally as

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \xrightarrow{\mathsf{F}_{\mathrm{Ran}}} \text{D-mod}(\mathrm{Ran}) \otimes \mathbf{C}^{\mathrm{glob}} \xrightarrow{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran},-)\otimes \mathrm{Id}} \mathbf{C}^{\mathrm{glob}}.$$

13.3.3. Let $\mathcal{Z}$ be an arbitrary space mapping to Ran. We can then consider the base change $\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}}$ and the corresponding functor

$$\mathsf{F}_{\mathcal{Z}} : \mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \to \text{D-mod}(\mathcal{Z}) \otimes \mathbf{C}^{\mathrm{glob}}.$$

For example, for $\mathcal{Z} = \mathrm{pt}$ and the map $\mathrm{pt} \to \mathrm{Ran}$ corresponds to $\underline{x}$, we recover the functor $\mathsf{F}_{\underline{x}}$.

By a slight abuse of notation, we will denote by the same symbol F the composition

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} \xrightarrow{\mathsf{F}_{\mathcal{Z}}} \text{D-mod}(\mathcal{Z}) \otimes \mathbf{C}^{\mathrm{glob}} \xrightarrow{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathcal{Z},-)\otimes \mathrm{Id}} \mathbf{C}^{\mathrm{glob}}.$$

13.3.4. For a fixed point $\underline{x}$, let $\mathrm{Ran}_{\underline{x}\subseteq}$ be the relative Ran space, i.e., the moduli space of collections of points $\underline{x}'$ that contain $\underline{x}$. It is equipped with a natural forgetful map

$$\mathrm{pr}_{\mathrm{big}} : \mathrm{Ran}_{\underline{x}\subseteq} \to \mathrm{Ran}$$

that remembers the ambient finite set.

13.3.5. Consider the category

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}_{\underline{x}\subseteq}}.$$

The structure of *unital* factorization category defines a functor

$$\text{ins. unit} : \mathbf{C}^{\mathrm{loc}}_{\underline{x}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}_{\underline{x}\subseteq}}.$$

13.3.6. Note also that $\mathrm{Ran}_{\underline{x}\subseteq}$ is equipped with a distinguished point

$$\mathrm{diag} : \mathrm{pt} \to \mathrm{Ran}_{\underline{x}\subseteq}$$

corresponding to $\underline{x}' = \underline{x}$.

Direct image along this embedding defines a functor

$$(\mathrm{diag})_* : \mathbf{C}^{\mathrm{loc}}_{\underline{x}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}_{\underline{x}\subseteq}},$$

so that the diagram

$$\begin{array}{ccccc} \mathbf{C}^{\mathrm{loc}}_{\underline{x}} & & \xrightarrow{(\mathrm{diag})_*} & & \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}_{\underline{x}\subseteq}} \\ {\scriptstyle \mathsf{F}_{\underline{x}}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{F}_{\mathrm{Ran}_{\underline{x}\subseteq}}} \\ \mathrm{D\text{-}mod}(\mathrm{pt}) \otimes \mathbf{C}^{\mathrm{glob}} & \xrightarrow{\sim} & \mathrm{D\text{-}mod}(\mathrm{pt}) \otimes \mathbf{C}^{\mathrm{glob}} & \xrightarrow{(\mathrm{diag})_* \otimes \mathrm{Id}} & \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}\subseteq}) \otimes \mathbf{C}^{\mathrm{glob}} \end{array} \tag{13.9}$$

commutes.

The functor $(\mathrm{diag})_*$ has a right adjoint, denoted $(\mathrm{diag})^!$ and we have a canonical identification

$$(\mathrm{diag})^! \circ \mathrm{ins.\,unit} \simeq \mathrm{Id}\,. \tag{13.10}$$

Passing to adjoints, the datum of commutativity of (13.9) and the isomorphism (13.10) give rise to a natural transformation

$$(\mathrm{diag})_* \circ \mathsf{F}_{\underline{x}} \to \mathsf{F}_{\mathrm{Ran}_{\underline{x}\subseteq}} \circ \mathrm{ins.\,unit} \tag{13.11}$$

as functors

$$\mathbf{C}^{\mathrm{loc}}_{\underline{x}} \rightrightarrows \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}\subseteq}) \otimes \mathbf{C}^{\mathrm{glob}}$$

13.3.7. In the examples that we consider the functor $\mathsf{F}_{\mathrm{Ran}}$ has the following unitality feature: the natural transformation

$$\mathsf{F}_{\underline{x}} \simeq \mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_{\underline{x}\subseteq}, -) \circ (\mathrm{diag})_* \circ \mathsf{F}_{\underline{x}} \overset{(13.11)}{\to} \mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_{\underline{x}\subseteq}, -) \circ \mathsf{F}_{\mathrm{Ran}_{\underline{x}\subseteq}} \circ \mathrm{ins.\,unit} = \mathsf{F} \circ \mathrm{ins.\,unit} \tag{13.12}$$

of functors

$$\mathbf{C}^{\mathrm{loc}}_{\underline{x}} \rightrightarrows \mathbf{C}^{\mathrm{glob}},$$

is an isomorphism.

Furthermore, the natural transformation

$$\mathsf{F}_{\mathrm{Ran}_{\underline{x}\subseteq}} \circ \mathrm{ins.\,unit} \to \omega_{\mathrm{Ran}_{\underline{x}\subseteq}} \boxtimes \mathsf{F}_{\underline{x}}, \tag{13.13}$$

as functors

$$\mathbf{C}^{\mathrm{loc}}_{\underline{x}} \rightrightarrows \mathrm{D\text{-}mod}(\mathrm{Ran}_{\underline{x}\subseteq}) \otimes \mathbf{C}^{\mathrm{glob}},$$

obtained by the $(\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_{\underline{x}\subseteq}, -), \omega_{\mathrm{Ran}_{\underline{x}\subseteq}} \otimes -)$-adjuntion from

$$\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_{\underline{x}\subseteq}, -) \circ \mathsf{F}_{\mathrm{Ran}_{\underline{x}\subseteq}} \circ \mathrm{ins.\,unit} \underset{\sim}{\overset{(13.12)}{\to}} \mathsf{F}_{\underline{x}},$$

is also an isomorphism.

Remark 13.3.8. Informally, the isomorphism (13.13) reads as follows: for $\underline{y} \in \mathrm{Ran}$ with support disjoint from that of $\underline{x}$, the diagram

$$\begin{array}{ccc} \mathbf{C}^{\mathrm{loc}}_{\underline{x}} & \xrightarrow{\mathrm{Id} \otimes \mathbf{1}_{\mathbf{C}^{\mathrm{loc}}_{\underline{y}}}} & \mathbf{C}^{\mathrm{loc}}_{\underline{x}} \otimes \mathbf{C}^{\mathrm{loc}}_{\underline{y}} \\ {\scriptstyle \mathsf{F}_{\underline{x}}}\downarrow & & \downarrow{\scriptstyle \mathsf{F}_{\underline{x}\sqcup\underline{y}}} \\ \mathbf{C}^{\mathrm{glob}} & \xrightarrow{=} & \mathbf{C}^{\mathrm{glob}} \end{array} \tag{13.14}$$

commutes.

Remark 13.3.9. In the case

$$\mathbf{C}^{\mathrm{loc}} = \mathrm{Whit}^!(G),\ \mathbf{C}^{\mathrm{glob}} = \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G),\ \mathsf{F} = \mathrm{Poinc}_{G,!}$$

the commutativity of (13.14) is equivalent to the isomorphism (12.15).

13.3.10. We now let the point $\underline{x}$ vary along Ran, we obtain the space $\mathrm{Ran}_{\subseteq}$, which is the moduli space of pairs

$$(\underline{x}, \underline{x}' \,|\, \underline{x} \subset \underline{x}').$$

We still have the maps

$$\mathrm{pr}_{\mathrm{small}}, \mathrm{pr}_{\mathrm{big}} : \mathrm{Ran}_{\subseteq} \rightrightarrows \mathrm{Ran}$$

that remember $\underline{x}$ and $\underline{x}'$, respectively.

In addition, we have the map

$$\mathrm{diag} : \mathrm{Ran} \to \mathrm{Ran}_{\subseteq}, \quad \underline{x} \mapsto (\underline{x}, \underline{x}),$$

so that

$$\mathrm{pr}_{\mathrm{small}} \circ \mathrm{diag} = \mathrm{pr}_{\mathrm{big}} \circ \mathrm{diag} = \mathrm{id}\,.$$

As above, we have the functor

$$\text{ins. unit} : \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \to \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}_{\subseteq}},$$

which is D-mod(Ran)-linear with respect to

$$(\mathrm{pr}_{\mathrm{small}})^! : \text{D-mod}(\mathrm{Ran}) \to \text{D-mod}(\mathrm{Ran}_{\subseteq}).$$

In addition, we have an adjoint pair

$$\mathrm{diag}_* : \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \rightleftarrows \mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}_{\subseteq}} : \mathrm{diag}^!,$$

so that

$$\mathrm{diag}^! \circ \text{ins. vac.} \simeq \mathrm{Id}\,.$$

13.3.11. As in (13.11) we obtain a natural transformation

$$(\mathrm{diag})_* \circ \mathsf{F}_{\mathrm{Ran}} \to \mathsf{F}_{\mathrm{Ran}_{\subseteq}} \circ \text{ins. unit} \tag{13.15}$$

as functors

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \rightrightarrows \text{D-mod}(\mathrm{Ran}_{\subseteq}) \otimes \mathbf{C}^{\mathrm{glob}}.$$

As in (13.12), the induced natural transformation

$$\mathsf{F}_{\mathrm{Ran}} \simeq ((\mathrm{pr}_{\mathrm{small}})_* \otimes \mathrm{Id}) \circ (\mathrm{diag})_* \circ \mathsf{F}_{\mathrm{Ran}} \overset{(13.15)}{\to} ((\mathrm{pr}_{\mathrm{small}})_* \otimes \mathrm{Id}) \circ \mathsf{F}_{\mathrm{Ran}_{\subseteq}} \circ \text{ins. unit} \tag{13.16}$$

of functors

$$\mathbf{C}^{\mathrm{loc}}_{\mathrm{Ran}} \rightrightarrows \text{D-mod}(\mathrm{Ran}) \otimes \mathbf{C}^{\mathrm{glob}},$$

is an isomorphism.

Furthermore, the natural transformation

$$\mathsf{F}_{\mathrm{Ran}_{\subseteq}} \circ \text{ins. unit} \to ((\mathrm{pr}_{\mathrm{small}})^! \otimes \mathrm{Id}) \circ \mathsf{F}_{\mathrm{Ran}}, \tag{13.17}$$

obtained by the $((\mathrm{pr}_{\mathrm{small}})_*, (\mathrm{pr}_{\mathrm{small}})^!)$-adjunction from

$$((\mathrm{pr}_{\mathrm{small}})_* \otimes \mathrm{Id}) \circ \mathsf{F}_{\mathrm{Ran}_{\subseteq}} \circ \text{ins. unit} \overset{(13.16)}{\longrightarrow} \mathsf{F}_{\mathrm{Ran}}$$

is also an isomorphism.

13.4. **A digression: factorization homology.** Let $\mathcal{A}$ be a factorization algebra (in Vect).

13.4.1. For any finite set I, we can consider the category

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\text{D-mod}(X^I)).$$

This is a category tensored over D-mod(X^I).

13.4.2. For a surjection of finite sets $I' \twoheadrightarrow I$ we have a tautological identification

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\text{D-mod}(X^{I'})) \underset{\text{D-mod}(X^{I'})}{\otimes} \text{D-mod}(X^I) \simeq \mathcal{A}\text{-mod}^{\mathrm{fact}}(\text{D-mod}(X^I)).$$

This allows us to pass to the limit/colimit and consider the category

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}}.$$

Remark 13.4.3. Note that $\mathcal{A}\text{-mod}^{\mathrm{fact}}$ does not necessarily form a factorization category, unless we perform some renormalization procedure.

Namely, for a pair of disjoint collections of points $\underline{x}^1$ and $\underline{x}^2$, the naturally defined functor

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}^1} \otimes \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}^2} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}^1 \cup \underline{x}^2}$$

is not necessarily an equivalence.

13.4.4. Note that for any space $\mathcal{Z}$ mapping to Ran, one can consider the base change $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}$ of $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}}$.

Denote by $\mathbf{oblv}_{\mathcal{A},\mathcal{Z}}$ the tautological forgetful functor

$$\mathbf{oblv}_{\mathcal{A},\mathcal{Z}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \text{D-mod}(\mathcal{Z}).$$

The discussion in Sect. 13.3.10 applies verbatim in the present situation with

$$\mathbf{C}^{\mathrm{loc}}_{\mathcal{Z}} := \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}.$$

In particular, we have the functor

$$\text{ins. vac.} : \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}};$$

the adjunction

$$(\mathrm{diag})_* : \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \rightleftarrows \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} : (\mathrm{diag})^!;$$

and an isomorphism

$$(\mathrm{diag})^! \circ \text{ins. vac.} \simeq \mathrm{Id}\,.$$

13.4.5. We reproduce the setting of Sect. 13.3.1 with

$$\mathbf{C}^{\mathrm{glob}} = \mathrm{Vect},$$

and the functor

$$\mathsf{F}_{\mathrm{Ran}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \to \text{D-mod}(\mathrm{Ran})$$

being

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \mathcal{A}, -)_{\mathrm{Ran}} := (\mathrm{pr}_{\mathrm{small}})_* \circ \mathbf{oblv}_{\mathcal{A},\mathrm{Ran}_{\subseteq}} \circ \text{ins. vac.}\,.$$

13.4.6. Base changing along $\mathcal{Z} \to \mathrm{Ran}$, we obtain a variant of the above functor

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \mathcal{A}, -)_{\mathcal{Z}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \text{D-mod}(\mathcal{Z}).$$

In particular, for $\underline{x} \in \mathrm{Ran}$, we obtain a functor

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \mathcal{A}, -)_{\underline{x}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \to \mathrm{Vect}\,.$$

Let $\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \mathcal{A}, -)$ denote the functor $\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \mathrm{Vect}$ equal to the composition

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \xrightarrow{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathcal{Z}}} \text{D-mod}(\mathcal{Z}) \xrightarrow{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathcal{Z},-)} \mathrm{Vect}\,.$$

13.4.7. As in Sect. 13.3.11, we have natural transformations

$$(\mathrm{diag})_* \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\underline{x}} \to \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}_{\underline{x}\subset}} \circ \text{ins. vac.}, \quad \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \rightrightarrows \text{D-mod}(\mathrm{Ran}_{\underline{x}\subset}) \tag{13.18}$$

and

$$(\mathrm{diag})_* \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}} \to \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}_{\subseteq}} \circ \text{ins. vac.}, \quad \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \rightrightarrows \text{D-mod}(\mathrm{Ran}_{\subseteq}). \tag{13.19}$$

We claim:

Lemma 13.4.8. *Assume that $\mathcal{A}$ is unital. Then the natural transformation*

$$\begin{aligned} \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\underline{x}} \simeq \mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_{\underline{x}\subset},-) \circ (\mathrm{diag})_* \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\underline{x}} \overset{(13.18)}{\longrightarrow} \\ \to \mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_{\underline{x}\subset},-) \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}_{\underline{x}\subset}} \circ \text{ins. vac.} = \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-) \circ \text{ins. vac.} \end{aligned} \tag{13.20}$$

as functors

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \rightrightarrows \mathrm{Vect}$$

and

$$\begin{aligned} \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}} \simeq (\mathrm{pr}_{\mathrm{small}})_* \circ (\mathrm{diag})_* \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}} \overset{(13.19)}{\longrightarrow} \\ \to (\mathrm{pr}_{\mathrm{small}})_* \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}_{\subseteq}} \circ \text{ins. vac.} \end{aligned} \tag{13.21}$$

as functors

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \rightrightarrows \text{D-mod}(\mathrm{Ran})$$

are isomorphisms. Furthermore, the natural transformations

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}_{\underline{x}\subset}} \circ \text{ins. vac.} \to \omega_{\mathrm{Ran}_{\underline{x}\subset}} \otimes \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\underline{x}} \tag{13.22}$$

and

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}_{\subseteq}} \circ \text{ins. vac.} \to (\mathrm{pr}_{\mathrm{small}})^! \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-)_{\mathrm{Ran}} \tag{13.23}$$

are also isomorphisms.

13.4.9. We will now discuss a generalization of Lemma 13.4.8, which will be repeatedly used in the sequel.

Let $\mathcal{B}$ be another factorization algebra, equipped with a homomorphism $\phi : \mathcal{A} \to \mathcal{B}$. Restriction along ϕ defines a functor

$$\mathrm{res}^{\phi} : \mathcal{B}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathcal{Z}}$$

for any $\mathcal{Z} \to \mathrm{Ran}$.

Consider the following two functors

$$\mathcal{B}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \rightrightarrows \text{D-mod}(\mathrm{Ran}).$$

One is just

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{B},-)_{\mathrm{Ran}}. \tag{13.24}$$

The other is

$$\begin{aligned} \mathcal{B}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\text{ins.vac.}}{\longrightarrow} \mathcal{B}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{res}^{\phi}}{\to} \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{A},-))_{\mathrm{Ran}_{\subseteq}}}{\longrightarrow} \\ \to \text{D-mod}(\mathrm{Ran}_{\subseteq}) \overset{(\mathrm{pr}_{\mathrm{small}})_*}{\longrightarrow} \text{D-mod}(\mathrm{Ran}). \end{aligned} \tag{13.25}$$

The identification $(\mathrm{diag})^! \circ \text{ins. unit} \simeq \mathrm{Id}$ gives rise to a natural transformation

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{B},-)_{\mathrm{Ran}} \to (13.25). \tag{13.26}$$

The map ϕ gives rise to a natural transformation

$$(13.25) \to (\mathrm{pr}_{\mathrm{small}})_* \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{B},-)_{\mathrm{Ran}_{\subseteq}} \circ \text{ins. vac.}. \tag{13.27}$$

We claim:

Lemma 13.4.10. *Assume that both $\mathcal{A}$, $\mathcal{B}$ and ϕ are unital. Then the composition*

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{B},-)_{\mathrm{Ran}} \overset{(13.26)}{\to} (13.25) \overset{(13.27)}{\to} (\mathrm{pr}_{\mathrm{small}})_* \circ \mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{B},-)_{\mathrm{Ran}_\subseteq} \circ \text{ins. vac.}$$

equals the map (13.21) *for $\mathcal{B}$. Moreover, the natural transformations* (13.26) *and* (13.27) *are isomorphisms.*

13.5. **Localization and the forgetful functor.**

13.5.1. Note that by adjunction, the commutative diagram (13.2) gives rise to a natural transformation

(13.28)
$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Bun}_G) & \xleftarrow{\quad\mathbf{oblv}^l_\kappa\quad} & \text{D-mod}_\kappa(\mathrm{Bun}_G) \\ \uparrow{\scriptstyle \mathrm{Loc}_G^{\mathrm{QCoh}}} & \nearrow & \uparrow{\scriptstyle \mathrm{Loc}_{G,\kappa}} \\ \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}} & \xleftarrow[\mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)}]{} & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}, \end{array}$$

where:

- $\mathbf{oblv}^l_\kappa : \text{D-mod}_\kappa(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{Bun}_G)$ is the "left" forgetful functor, i.e., the right adjoint to ind^l_κ;
- $\mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)} : \mathrm{KL}(G)_\kappa \to \mathrm{Rep}(\mathfrak{L}^+(G))$ is the natural forgetful functor.

13.5.2. The natural transformation in (13.28) is *not* an isomorphism (unless $G = 1$). We will now draw another diagram, in which a natural transformation will be an isomorphism, which encodes another basic property of the localization functor.

13.5.3. The diagram (13.28) gives rise to a diagram

(13.29)
$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Bun}_G) & \xleftarrow{\quad\mathbf{oblv}^l_\kappa\quad} & \text{D-mod}_\kappa(\mathrm{Bun}_G) \\ \uparrow{\scriptstyle \mathrm{Loc}_G^{\mathrm{QCoh}}} & \nearrow & \uparrow{\scriptstyle \mathrm{Loc}_{G,\kappa}} \\ \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}_\subseteq} & \xleftarrow[\mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)}]{} & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq}. \end{array}$$

We claim:

Lemma 13.5.4. *The natural transformation in* (13.29) *becomes an isomorphism after precomposition with the functor*

$$\text{ins. unit.} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq}.$$

We will now use Lemma 13.5.4 to describe the composition

$$\mathbf{oblv}^l_\kappa \circ \mathrm{Loc}_{G,\kappa}$$

more explicitly in terms of the functor $\mathrm{Loc}_G^{\mathrm{QCoh}}$.

13.5.5. Note that the functor $\mathrm{Loc}_G^{\mathrm{QCoh}}$ essentially amounts to integration over Ran (parameterized by points of Bun_G).

Given a factorization algebra $\mathcal{A} \in \mathrm{Rep}(\mathfrak{L}^+(G))$ we can consider an analog of the functor of factorization homology:

$$\mathrm{C}^{\mathrm{fact}}(X; \mathcal{A}, -)^{\mathrm{Bun}_G}_{\mathrm{Ran}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G)))_{\mathrm{Ran}} \to \text{D-mod}(\mathrm{Ran}) \otimes \mathrm{QCoh}(\mathrm{Bun}_G),$$

so that analogs of Lemmas 13.4.8 and 13.4.10 hold.

13.5.6. Set

$$\mathcal{A}_{G,\kappa} := \mathbf{oblv}_{(\widehat{\mathfrak{g}}, \mathfrak{L}^+(G)) \to \mathfrak{L}^+(G)}(\mathrm{Vac}(G)_\kappa),$$

viewed as a factorization algebra in $\mathrm{Rep}(\mathfrak{L}^+(G))$. Note that the functor $\mathbf{oblv}_{(\widehat{\mathfrak{g}}, \mathfrak{L}^+(G)) \to \mathfrak{L}^+(G)}$ enhances to a functor

$$\mathbf{oblv}^{\mathrm{enh}}_{(\widehat{\mathfrak{g}}, \mathfrak{L}^+(G)) \to \mathfrak{L}^+(G)} : \mathrm{KL}(G)_\kappa \to \mathcal{A}_{G,\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G))).$$

Note that the composition

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq} \overset{\mathbf{oblv}_{(\widehat{\mathfrak{g}}, \mathfrak{L}^+(G)) \to \mathfrak{L}^+(G)}}{\longrightarrow} \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}_\subseteq}$$

can be identified with

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathbf{oblv}^{\mathrm{enh}}_{(\widehat{\mathfrak{g}}, \mathfrak{L}^+(G)) \to \mathfrak{L}^+(G)}}{\longrightarrow} \mathcal{A}_{G,\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G)))_{\mathrm{Ran}} \overset{\mathrm{ins.vac.}}{\longrightarrow}$$
$$\to \mathcal{A}_{G,\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G)))_{\mathrm{Ran}_\subset} \overset{\mathbf{oblv}_{\mathcal{A}_{G,\kappa}}}{\longrightarrow} \mathrm{Rep}(\mathfrak{L}^+(G))_{\mathrm{Ran}_\subseteq}.$$

Hence, we can rewrite the composition

$$\mathrm{Loc}_G^{\mathrm{QCoh}} \circ \mathbf{oblv}_{(\widehat{\mathfrak{g}}, \mathfrak{L}^+(G)) \to \mathfrak{L}^+(G)} \circ \text{ins. unit}$$

as

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathbf{oblv}^{\mathrm{enh}}_{(\widehat{\mathfrak{g}}, \mathfrak{L}^+(G)) \to \mathfrak{L}^+(G)}}{\longrightarrow} \tag{13.30}$$
$$\to \mathcal{A}_{G,\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(G)))_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}(X; \mathcal{A}_{G,\kappa}, -)^{\mathrm{Bun}_G}}{\longrightarrow} \mathrm{QCoh}(\mathrm{Bun}_G).$$

Hence, combining Lemma 13.5.4 with Lemma 13.4.8 we obtain:

Corollary 13.5.7. *The functor*

$$\mathbf{oblv}^l_\kappa \circ \mathrm{Loc}_{G,\kappa} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{Bun}_G)$$

identifies canonically with (13.30).

13.6. **Localization and restriction.**

13.6.1. Let $\phi : N' \to G$ be a group homomorphism. We restrict the level κ to N', and consider the corresponding Kazhdan-Lustig category

$$\mathrm{KL}(N')_\kappa := \widehat{\mathfrak{n}}'\text{-mod}^{\mathfrak{L}^+(N')}_\kappa$$

and the localization functor

$$\mathrm{Loc}_{N',\kappa} : \mathrm{KL}(N')_{\kappa,\mathrm{Ran}} \to \text{D-mod}_\kappa(\mathrm{Bun}_{N'}).$$

The triple $(\mathrm{KL}(N')_\kappa, \text{D-mod}_\kappa(\mathrm{Bun}_{N'}), \mathrm{Loc}_{N',\kappa})$ fits the paradigm of Sect. 13.3.1.

13.6.2. The map ϕ gives rise to (factorization) restriction functors

$$\mathrm{Rep}(\mathfrak{L}^+(G)) \xrightarrow{\mathrm{res}^\phi} \mathrm{Rep}(\mathfrak{L}^+(N')) \text{ and } \mathrm{KL}(G)_\kappa \xrightarrow{\mathrm{res}^\phi} \mathrm{KL}(N')_\kappa.$$

In addition, the map ϕ gives rise to a map

$$\phi^{\mathrm{glob}} : \mathrm{Bun}_{N'} \to \mathrm{Bun}_G,$$

which is compatible with the twistings, and thus gives rise to a functor

$$(\phi^{\mathrm{glob}})^!_\kappa : \text{D-mod}_\kappa(\mathrm{Bun}_G) \to \text{D-mod}_\kappa(\mathrm{Bun}_{N'}),$$

which makes the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Bun}_G) & \xrightarrow{(\phi^{\mathrm{glob}})^*} & \mathrm{QCoh}(\mathrm{Bun}_{N'}) \\ {\scriptstyle \mathbf{oblv}^l_\kappa}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}^l_\kappa} \\ \text{D-mod}_\kappa(\mathrm{Bun}_G) & \xrightarrow{(\phi^{\mathrm{glob}})^!_\kappa} & \text{D-mod}_\kappa(\mathrm{Bun}_{N'}) \end{array}$$

commute.

13.6.3. It follows from the definition of the localization functors that we have a natural transformation

(13.31)
$$\begin{array}{ccc} \text{D-mod}_\kappa(\mathrm{Bun}_G) & \xrightarrow{(\phi^{\mathrm{glob}})^!_\kappa} & \text{D-mod}_\kappa(\mathrm{Bun}_{N'}) \\ {\scriptstyle \mathrm{Loc}_{G,\kappa}}\uparrow & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_{N',\kappa}} \\ \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} & \xrightarrow[\mathrm{res}^\phi]{} & \mathrm{KL}(N')_{\kappa,\mathrm{Ran}}. \end{array}$$

13.6.4. The natural transformation in (13.31) is not an isomorphism (unless ϕ itself is). We will now draw another diagram, in which the natural transformation is an isomorphism, and which expresses the composition

$$(\phi^{\mathrm{glob}})^!_\kappa \circ \mathrm{Loc}_{G,\kappa}$$

via $\mathrm{Loc}_{N',\kappa}$.

13.6.5. The diagram (13.31) gives rise to a diagram:

(13.32)
$$\begin{array}{ccc} \text{D-mod}_\kappa(\mathrm{Bun}_G) & \xrightarrow{(\phi^{\mathrm{glob}})^!_\kappa} & \text{D-mod}_\kappa(\mathrm{Bun}_{N'}) \\ {\scriptstyle \mathrm{Loc}_{G,\kappa}}\uparrow & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_{N',\kappa}} \\ \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq} & \xrightarrow[\mathrm{res}^\phi]{} & \mathrm{KL}(N')_{\kappa,\mathrm{Ran}_\subseteq}. \end{array}$$

We will prove:

Proposition 13.6.6. *The natural transformation in* (13.32) *becomes an isomorphism after precomposing with*

(13.33)
$$\text{ins. unit} : \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \to \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq}.$$

Taking into account (13.16), from Proposition 13.6.6 we obtain:

Corollary 13.6.7. *We have a commutative diagram of functors*

$$\begin{array}{ccccc} \mathrm{D\text{-}mod}_\kappa(\mathrm{Bun}_G) & & \xrightarrow{(\phi^{\mathrm{glob}})^!_\kappa} & & \mathrm{D\text{-}mod}_\kappa(\mathrm{Bun}_{N'}) \\ {\scriptstyle \mathrm{Loc}_{G,\kappa}}\uparrow & & & & \uparrow{\scriptstyle \mathrm{Loc}_{N',\kappa}} \\ \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} & \xrightarrow{\mathrm{ins.unit}} & \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq} & \xrightarrow{\mathrm{res}^\phi} & \mathrm{KL}(N')_{\kappa,\mathrm{Ran}_\subseteq} \end{array}$$

Note that Corollary 13.6.7 can be reformulated as follows:

Corollary 13.6.8. *The natural transformation in* (13.31) *becomes an isomorphism after precomposing with*

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \xrightarrow{\mathrm{ins.unit}} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq} \xrightarrow{(\mathrm{pr}_{\mathrm{big}})_*} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}}.$$

13.6.9. The rest of this subsection is devoted to the proof of Proposition 13.6.6.

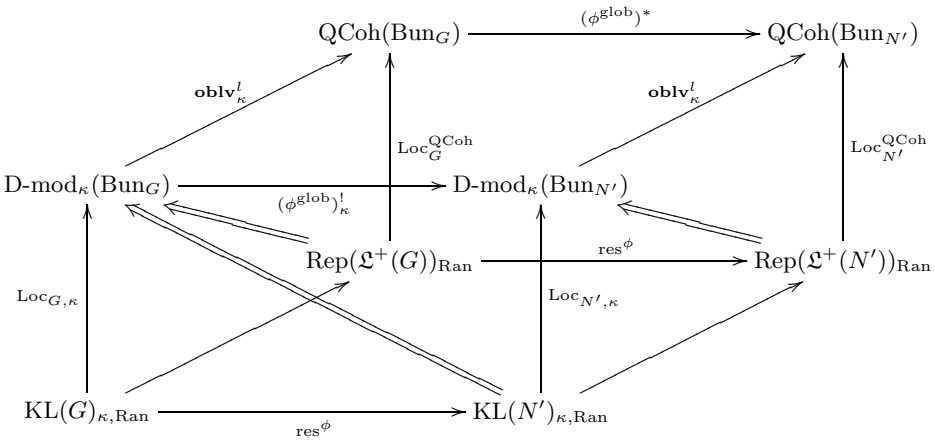
Since the functor

$$\mathbf{oblv}^l_\kappa : \mathrm{D\text{-}mod}_\kappa(\mathrm{Bun}_{N'}) \to \mathrm{QCoh}(\mathrm{Bun}_{N'}) \tag{13.34}$$

is conservative, it is sufficient to prove that the natural transformation in (13.31) becomes an isomorphism after precomposing with (13.33) and postcomposing with (13.34).

13.6.10. By the construction of the natural transformation in (13.31), it fits into the cube:

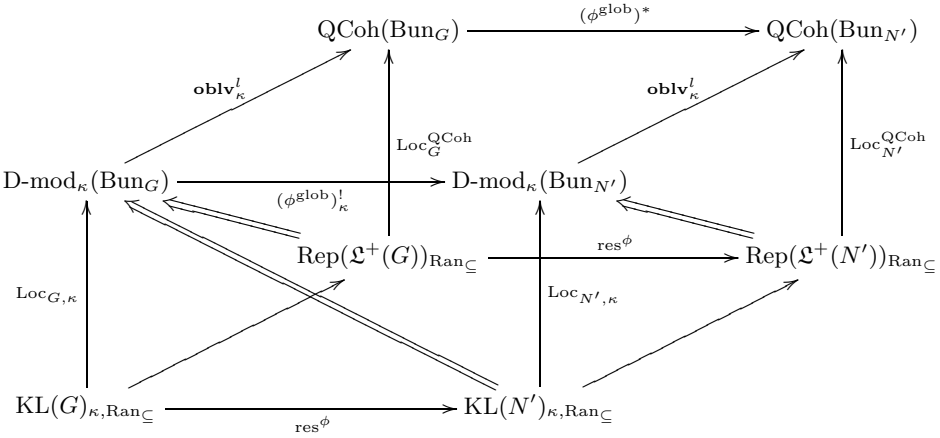
(13.35)



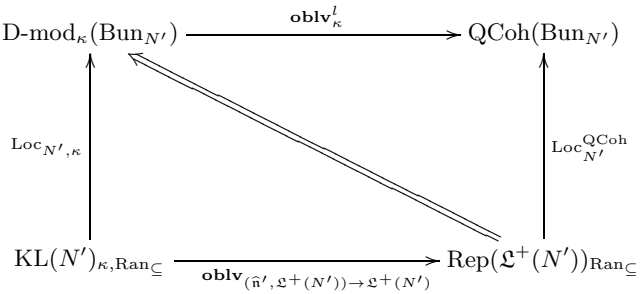
where:

- The side natural transformations are (13.28) for G and N'; respectiely;
- The front natural transformation is (13.31);
- All other faces naturally commute.

The cube (13.35) gives rise to the cube

(13.36)


Applying Lemma 13.5.4 for G, we obtain that it suffices to show that the natural transformation in

(13.37)


becomes an isomorphism after precomposing with

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\text{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq} \overset{\mathrm{res}^\phi}{\longrightarrow} \mathrm{KL}(N')_{\kappa,\mathrm{Ran}_\subseteq}. \tag{13.38}$$

13.6.11. Let $\mathcal{A}_{G,\kappa}$ (resp., $\mathcal{A}_{N',\kappa}$) be the factorization algebra in $\mathrm{Rep}(\mathfrak{L}^+(G))$ (resp., $\mathrm{Rep}(\mathfrak{L}^+(N'))$) as in Sect. 13.5.6. Let $\mathcal{A}_{G,N',\kappa}$ be the factorization algebra in $\mathrm{Rep}(\mathfrak{L}^+(N'))$ equal to

$$\mathrm{res}^\phi(\mathcal{A}_{G,\kappa}).$$

The functor

$$\mathrm{res}^\phi \circ \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)} : \mathrm{KL}(G)_\kappa \to \mathrm{Rep}(\mathfrak{L}^+(N'))$$

enhances to a functor

$$(\mathrm{res}^\phi \circ \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)})^{\mathrm{enh}} : \mathrm{KL}(G)_\kappa \to \mathcal{A}_{G,N',\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(N'))).$$

We can rewrite the composition of (13.38) with the counter-clockwise circuit in (13.37) as

$$\begin{aligned}\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} &\overset{(\mathrm{res}^\phi \circ \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)})^{\mathrm{enh}}}{\longrightarrow} \\ &\to \mathcal{A}_{G,N',\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\mathfrak{L}^+(N')))_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}(X;\mathcal{A}_{G,N',\kappa},-)^{\mathrm{Bun}_{N'}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{Bun}_{N'}).\end{aligned} \tag{13.39}$$

Using Corollary 13.5.7, we rewrite the composition of (13.38) with the clockwise circuit in (13.37) as

$$\begin{aligned}(13.40)\quad \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} &\overset{(\mathrm{res}^{\phi}\circ \mathbf{oblv}_{(\widehat{\mathfrak{g}},\mathfrak{L}^+(G))\to\mathfrak{L}^+(G)})^{\mathrm{enh}}}{\longrightarrow} \mathcal{A}_{G,N',\kappa}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{ins.vac.}}{\longrightarrow} \\ &\to \mathcal{A}_{G,N',\kappa}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_\subseteq} \overset{\mathbf{oblv}_{\mathcal{A}_{G,N',\kappa}\to\mathcal{A}_{N',\kappa}}}{\longrightarrow} \mathcal{A}_{N',\kappa}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_\subseteq} \overset{\mathrm{C}^{\mathrm{fact}}(X;\mathcal{A}_{N',\kappa},-)^{\mathrm{Bun}_{N'}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{Bun}_{N'}).\end{aligned}$$

Now, the expressions in (13.39) and (13.40) match by Lemma 13.4.10.

□[Proposition 13.6.6]

13.7. **Localization for unipotent group-schemes.**

13.7.1. Let N' be a unipotent group-scheme over X. Let $\delta_{N'}$ denote the integer $\dim(\mathrm{Bun}_{N'})$.

Note that the canonical line bundle of $\mathrm{Bun}_{N'}$, i.e.,

$$\det(T^*(\mathrm{Bun}_{N'})),$$

is canonically constant. Let $\mathfrak{l}_{N'}$ denote the corresponding (ungraded) line.

13.7.2. Consider the factorization category

$$\mathrm{KL}(N') := \mathfrak{L}(\mathfrak{n}')\text{-mod}^{\mathfrak{L}^+(N')}$$

and the localization functor

$$\mathrm{Loc}_{N'} : \mathrm{KL}(N')_{\mathrm{Ran}} \to \text{D-mod}(\mathrm{Bun}_{N'}).$$

The triple $(\mathrm{KL}(N'), \text{D-mod}(\mathrm{Bun}_{N'}), \mathrm{Loc}_{N'})$ fits the paradigm of Sect. 13.3.1.

13.7.3. Consider the (factorization) functor of semi-infinite cohomology with respect to $\mathfrak{L}(\mathfrak{n}')$:

$$\mathrm{BRST}_{\mathfrak{n}'} : \mathfrak{L}(\mathfrak{n}')\text{-mod} \to \mathrm{Vect}\,.$$

Its value on the factorization unit

$$\mathbf{1}_{\mathfrak{L}(\mathfrak{n}')\text{-mod}} = \mathrm{Vac}(N')$$

is the (commutative) factorization algebra

$$\Omega(\mathfrak{n}') := \mathrm{C}^{\cdot}(\mathfrak{L}^+(\mathfrak{n}')).$$

Thus, $\mathrm{BRST}_{\mathfrak{n}'}$ enhances to a functor

$$\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}'} : \mathfrak{L}(\mathfrak{n}')\text{-mod} \to \Omega(\mathfrak{n}')\text{-mod}^{\mathrm{fact}}.$$

By a slight abuse of notation, we will denote by the same symbols $\mathrm{BRST}_{\mathfrak{n}'}$ and $\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}'}$ the restrictions of the above functors along

$$\mathrm{KL}(N') \to \mathfrak{L}(\mathfrak{n}')\text{-mod}.$$

13.7.4. We now recall the following result of [CF2, Theorem 4.0.5(4)]:[20]

Theorem 13.7.5. *The composition*

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{Loc}_{N'}}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N'}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'},-)}{\longrightarrow} \mathrm{Vect}$$

identifies canonically with the functor

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}'}}{\longrightarrow} \Omega(\mathfrak{n}')\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}(X;\Omega(\mathfrak{n}'),-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N'}[\delta_{N'}]}{\longrightarrow} \mathrm{Vect}\,.$$

[20] Apply *loc.cit.* to the projection $N' \to \mathrm{pt}$ and the factorization unit $\mathrm{Vac}(N')$. Note that the Tate twist (Example 3.3.9 of *loc.cit.*) for N' is trivial, hence $\mathrm{KL}(N') = \mathrm{KL}(N')_{\mathrm{co}}$ as factorization categories.

Remark 13.7.6. Note that the composition

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}'}}{\longrightarrow} \Omega(\mathfrak{n}')\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\Omega(\mathfrak{n}'),-)}{\longrightarrow} \mathrm{Vect},$$

appearing in Theorem 13.7.5 can be also described as follows:

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(N')_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{BRST}_{\mathfrak{n}'}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}_{\subseteq}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_{\subseteq}),-)}{\to} \mathrm{Vect},$$

and also as

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(N')_{\mathrm{Ran}_{\subseteq}} \overset{(\mathrm{pr}_{\mathrm{big}})_*}{\longrightarrow} \mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{BRST}_{\mathfrak{n}'}}{\longrightarrow} \text{D-mod}(\mathrm{Ran} \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}),-)}{\to} \mathrm{Vect},$$

13.7.7. *Example.* Take the object

$$\mathbf{1}_{\mathrm{KL}(N'),\mathrm{Ran}} \in \mathrm{KL}(N')_{\mathrm{Ran}}.$$

Then

$$\mathrm{Loc}_{N'}(\mathbf{1}_{\mathrm{KL}(N'),\mathrm{Ran}}) \simeq \mathrm{Diff}(\mathrm{Bun}_{N'}),$$

viewed as a *left* D-module. The corresponding right D-module is

$$\mathbf{ind}^r(\omega_{\mathrm{Bun}_{N'}}).$$

Hence, on the one hand,

$$\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'}, \mathrm{Loc}_{N'}(\mathbf{1}_{\mathrm{KL}(N'),\mathrm{Ran}})) \simeq \Gamma(\mathrm{Bun}_{N'}, \omega_{\mathrm{Bun}_{N'}}) \simeq \Gamma(\mathrm{Bun}_{N'}, \mathcal{O}_{\mathrm{Bun}_{N'}}) \otimes \mathfrak{l}_{N'}[\delta_{N'}].$$

On the other hand,

$$\mathrm{BRST}_{\mathfrak{n}'}(\mathbf{1}_{\mathrm{KL}(N')}) = \Omega(\mathfrak{n}'),$$

while

$$\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \Omega(\mathfrak{n}')) \simeq \Gamma(\mathrm{Bun}_{N'}, \mathcal{O}_{\mathrm{Bun}_{N'}}),$$

as desired.

13.7.8. *A variant.* Let χ be a (factorization) character

$$\mathfrak{L}(\mathfrak{n}') \to \mathbb{G}_a,$$

assumed trivial in $\mathfrak{L}^+(\mathfrak{n}')$. The character χ gives rise to a map $\mathrm{Bun}_{N'} \to \mathbb{G}_m$, which we denote by the same symbol χ.

Let $\mathrm{BRST}_{\mathfrak{n}',\chi}$ be the χ-twisted version of the semi-infinite cohomology functor, i.e.

$$\mathrm{BRST}_{\mathfrak{n}',\chi}(-) = \mathrm{BRST}_{\mathfrak{n}'}(- \otimes \chi).$$

Note that

$$\mathrm{Vac}(N') \otimes \chi \simeq \mathrm{Vac}(N').$$

Hence,

$$\mathrm{BRST}_{\mathfrak{n}',\chi}(\mathrm{Vac}(N')) \simeq \mathrm{BRST}_{\mathfrak{n}'}(\mathrm{Vac}(N')) \simeq \Omega(\mathfrak{n}')$$

as factorization algebras.

Let $\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}',\chi}$ be the enhancement of $\mathrm{BRST}_{\mathfrak{n}',\chi}$

$$\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}',\chi} : \mathfrak{L}(\mathfrak{n}')\text{-mod} \to \Omega(\mathfrak{n}')\text{-mod}^{\mathrm{fact}}.$$

13.7.9. The next assertion results formally from Theorem 13.7.5:

Corollary 13.7.10. *The composition*

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{Loc}_{N'}}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N'}) \overset{-\overset{*}{\otimes}\chi^*(\exp)}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N'}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N'},-)}{\longrightarrow} \mathrm{Vect}$$

identifies canonically with the functor

$$\mathrm{KL}(N')_{\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}',\chi}}{\longrightarrow} \mathrm{C}^{\cdot}(\mathfrak{L}^+(\mathfrak{n}'))\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\Omega(\mathfrak{n}'),-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes\mathfrak{l}_{N'}[\delta_{N'}]}{\longrightarrow} \mathrm{Vect}\,.$$

13.8. **Integrating over (twists of)** Bun_N**.**

13.8.1. Let $\mathcal{P}_T$ be a T-bundle on X. Consider the unipotent group scheme $N_{\mathcal{P}_T}$. Denote the corresponding moduli stack $\mathrm{Bun}_{N_{\mathcal{P}_T}}$ identifies with $\mathrm{Bun}_{N,\mathcal{P}_T}$.

The resulting map $\mathrm{Bun}_{N_{\mathcal{P}_T}} \to \mathrm{Bun}_G$ can be thought of as

$$\mathrm{Bun}_{N_{\mathcal{P}_T}} \to \mathrm{Bun}_{G_{\mathcal{P}_T}} \overset{\alpha^{-1}_{\mathcal{P}_T,\mathrm{taut}}}{\longrightarrow} \mathrm{Bun}_G\,.$$

13.8.2. Recall that the pullback of $\det_{\mathrm{Bun}_G}$ to $\mathrm{Bun}_{N,\mathcal{P}_T}$ is constant. Hence the pullback of the de Rham twisting $\mathrm{dlog}(\det_{\mathrm{Bun}_G})$ to $\mathrm{Bun}_{N,\mathcal{P}_T}$ is canonically trivial.

Hence, for the de Rham twisting on Bun_G giving by an invariant form κ, we have a well-defined functor

$$\mathfrak{p}^!_\kappa : \text{D-mod}_\kappa(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}). \tag{13.41}$$

13.8.3. Note that the embedding

$$N_{\mathcal{P}_T} \to G_{\mathcal{P}_T}$$

gives rise to a map

$$\mathfrak{L}(\mathfrak{n}_{\mathcal{P}_T}) \to \widehat{\mathfrak{g}}_{\kappa,\mathcal{P}_T}.$$

In particular, we obtain a well-defined restriction functor

$$\mathrm{KL}(G)_\kappa \overset{\alpha_{\mathcal{P}_T},\mathrm{taut}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T} \to \mathrm{KL}(N_{\mathcal{P}_T}).$$

Denote

$$\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})_\kappa := \mathrm{BRST}_{\mathfrak{n}_{\mathcal{P}_T}}(\mathrm{Vac}(G)_{\kappa,\mathcal{P}_T}).$$

This is a factorization algebra, which receives a homomorphism from $\Omega(\mathfrak{n}_{\mathcal{P}_T})$.

Thus, the composition

$$\mathrm{KL}(G)_\kappa \overset{\alpha_{\mathcal{P}_T},\mathrm{taut}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T} \to \mathrm{KL}(N_{\mathcal{P}_T}) \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T})\text{-mod}^{\mathrm{fact}}$$

further enhances to a (factorization) functor

$$\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T}} : \mathrm{KL}(G)_\kappa \to \Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})_\kappa\text{-mod}^{\mathrm{fact}}.$$

13.8.4. From Theorem 13.7.5 we are going to deduce the following assertion:

Theorem 13.8.5. *The composition*

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\kappa}}{\longrightarrow} \text{D-mod}_\kappa(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_\kappa}{\to} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Bun}_{N,\mathcal{P}_T},-)}{\longrightarrow} \mathrm{Vect} \tag{13.42}$$

identifies with the functor

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})_\kappa,-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N_{\mathcal{P}_T}}[\delta_{N_{\mathcal{P}_T}}]}{\longrightarrow} \mathrm{Vect}, \tag{13.43}$$

where the notations $\delta_{N_{\mathcal{P}_T}}$ and $\mathfrak{l}_{N_{\mathcal{P}_T}}$ are as in Sect. 13.7.1.

Remark 13.8.6. Note that the composition

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})_\kappa,-)}{\longrightarrow} \mathrm{Vect}$$

appearing in Theorem 13.8.5 can be also described as follows:

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq} \to \mathrm{KL}(N_{\mathcal{P}_T})_{\mathrm{Ran}_\subseteq} \overset{\mathrm{BRST}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}_\subseteq) \overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq),-)}{\to} \mathrm{Vect},$$

and also as

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}_\subseteq} \overset{(\mathrm{pr}_{\mathrm{big}})_*}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{BRST}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}_\subseteq) \overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq),-)}{\to} \mathrm{Vect}\,.$$

13.8.7. *Proof of Theorem 13.8.5.* First, we rewrite the functor

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\kappa}}{\longrightarrow} \text{D-mod}_{\kappa}(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_{\kappa}}{\to} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T})$$

using (a $\mathcal{P}_T$-twisted version of) Corollary 13.6.7.

We obtain that it identifies with

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\alpha_{\mathcal{P}_T},\mathrm{taut}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T,\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T,\mathrm{Ran}_{\subseteq}} \to$$
$$\to \mathrm{KL}(N_{\mathcal{P}_T})_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{Loc}_{N_{\mathcal{P}_T},\kappa}}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}).$$

By Theorem 13.7.5, the functor

$$\mathrm{KL}(N_{\mathcal{P}_T})_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{Loc}_{N_{\mathcal{P}_T},\kappa}}{\longrightarrow} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N,\mathcal{P}_T},-)}{\longrightarrow} \mathrm{Vect}$$

identifies with

$$\mathrm{KL}(N_{\mathcal{P}_T})_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}_{\mathcal{F}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\Omega(\mathfrak{n}_{\mathcal{P}_T}),-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N_{\mathcal{P}_T}}[\delta_{N_{\mathcal{P}_T}}]}{\longrightarrow} \mathrm{Vect}\,.$$

Hence, the functor (13.42) identifies with the composition

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\alpha_{\mathcal{P}_T},\mathrm{taut}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T,\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\kappa,\mathcal{P}_T,\mathrm{Ran}_{\subseteq}} \to \mathrm{KL}(N_{\mathcal{P}_T})_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{BRST}^{\mathrm{enh}}_{\mathfrak{n}_{\mathcal{F}_T}}}{\longrightarrow}$$
$$\to \Omega(\mathfrak{n}_{\mathcal{P}_T})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\Omega(\mathfrak{n}_{\mathcal{P}_T}),-)}{\longrightarrow} \mathrm{Vect}\,. \tag{13.44}$$

We can rewrite (13.44) tautologically as

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{ins.vac}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} \overset{\mathbf{oblv}_{\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\to\Omega(\mathfrak{n}_{\mathcal{P}_T})}}{\longrightarrow}$$
$$\to \Omega(\mathfrak{n}_{\mathcal{P}_T})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\Omega(\mathfrak{n}_{\mathcal{P}_T}),-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N_{\mathcal{P}_T}}[\delta_{N_{\mathcal{P}_T}}]}{\longrightarrow} \mathrm{Vect}\,. \tag{13.45}$$

Thus, it suffices to show that the composition

$$\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{ins.vac}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} \overset{\mathbf{oblv}_{\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\to\Omega(\mathfrak{n}_{\mathcal{P}_T})}}{\longrightarrow}$$
$$\to \Omega(\mathfrak{n}_{\mathcal{P}_T})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\Omega(\mathfrak{n}_{\mathcal{P}_T}),-)}{\longrightarrow} \mathrm{Vect}$$

identifies with

$$\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\Omega(\mathfrak{n}_{\mathcal{P}_T},\mathfrak{g})_{\kappa},-)}{\longrightarrow} \mathrm{Vect}\,.$$

However, this follows from Lemma 13.4.10.

□[Theorem 13.8.5]

13.8.8. Note that the same proof applies in the situation twisted by a character. Namely, χ be a character of $\mathfrak{n}_{\mathcal{P}_T}$ as in Sect. 13.7.8.

Denote

$$\Omega(\mathfrak{n}_{\mathcal{P}_T},\chi,\mathfrak{g})_{\kappa} := \mathrm{BRST}_{\mathfrak{n}_{\mathcal{P}_T},\chi}(\mathrm{Vac}(G)_{\kappa,\mathcal{P}_T}).$$

Consider the corresponding functor

$$\mathrm{BRST}^{\mathfrak{g}\text{-enh}}_{\mathfrak{n}_{\mathcal{P}_T},\chi} : \mathrm{KL}(G)_{\kappa} \to \Omega(\mathfrak{n}_{\mathcal{P}_T},\chi,\mathfrak{g})_{\kappa}\text{-mod}^{\mathrm{fact}}.$$

Then:

Theorem 13.8.9. *The composition*

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\kappa}}{\longrightarrow} \text{D-mod}_{\kappa}(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_{\kappa}}{\to} \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \overset{\overset{*}{\otimes}\chi^*(\exp)}{\longrightarrow}$$
$$\to \text{D-mod}(\mathrm{Bun}_{N,\mathcal{P}_T}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N,\mathcal{P}_T},-)}{\longrightarrow} \mathrm{Vect}$$

identifies with the functor

$$\mathrm{KL}(G)_{\kappa,\mathrm{Ran}} \overset{\mathrm{BRST}^{\mathfrak{g}\text{-enh},\chi}_{\mathfrak{n}_{\mathcal{P}_T}}}{\longrightarrow} \Omega(\mathfrak{n}_{\mathcal{P}_T},\chi,\mathfrak{g})\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\Omega(\mathfrak{n}_{\mathcal{P}_T},\chi,\mathfrak{g})_{\kappa},-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N_{\mathcal{P}_T}}[\delta_{N_{\mathcal{P}_T}}]}{\longrightarrow} \mathrm{Vect}\,.$$

14. The composition of localization and coefficient functors

The goal of this section is to give an expression via chiral homology of the composition

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_G}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{coeff}}{\longrightarrow} \mathrm{Whit}^!(G)_{\mathrm{Ran}}. \tag{14.1}$$

This expression (combined with the $(\mathrm{FLE}_{G,\mathrm{crit}}, \mathrm{FLE}_{\check{G},\infty})$-compatibility expressed by Corollary 7.5.2) will be used in Part IV in order to show that the Langlands functor is compatible with the critical localization and the spectral Poincaré series functor via the critical FLE.

14.1. **The vacuum case.**

14.1.1. Let us specialize the setting of Theorem 13.8.9 to the case $\kappa = \mathrm{crit}$ and $\mathcal{P}_T = \rho(\omega_X)$. In this case, the integer that we denoted $\delta_{N_{\mathcal{P}_T}}$ is $\delta_{N_{\rho(\omega_X)}}$. Denote the corresponding line $\mathfrak{l}_{N_{\mathcal{P}_T}}$ by

$$\mathfrak{l}_{N_{\rho(\omega_X)}}. \tag{14.2}$$

14.1.2. We obtain:

Theorem 14.1.3. *The composition*

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_{G,\mathrm{crit}}}{\longrightarrow} \text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) \overset{\mathfrak{p}^!_{\mathrm{crit}}}{\to} \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{\overset{*}{\otimes}\chi^*(\exp)}{\longrightarrow}$$
$$\to \text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N,\rho(\omega_X)},-)}{\longrightarrow} \mathrm{Vect}$$

identifies with the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),\mathrm{Ran}} \overset{\mathrm{DS}^{\mathrm{enh,coarse}}}{\longrightarrow}$$
$$\to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})},-)}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{N_{\rho(\omega_X)}}[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}\,.$$

14.1.4. From (12.11) we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect} \\
{\scriptstyle \mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Bun}_{N,\rho(\omega_X)},-)\circ(-\overset{*}{\otimes}\chi^*(\exp))}\uparrow & & \uparrow{\scriptstyle -\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes\frac{1}{2}}} \\
\text{D-mod}(\mathrm{Bun}_{N,\rho(\omega_X)}) & & \mathrm{Vect} \\
{\scriptstyle \mathfrak{p}^!_{\mathrm{crit}}}\uparrow & & \uparrow{\scriptstyle \mathrm{coeff}^{\mathrm{Vac,glob}}_G} \\
\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G) & \xrightarrow{(12.2)} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \\
{\scriptstyle \mathrm{Loc}_{G,\mathrm{crit}}}\uparrow & & \uparrow{\scriptstyle \mathrm{Loc}_G} \\
\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{Id}} & \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}}.
\end{array}$$

Recall also that

$$\mathrm{coeff}^{\mathrm{Vac,glob}}_G \simeq \mathrm{coeff}^{\mathrm{Vac}}_G[2\delta_{N_{\rho(\omega_X)}}].$$

14.1.5. Hence, Theorem 14.1.3 can be restated as:

Theorem 14.1.6. *The composition*

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_G}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{coeff}_G^{\mathrm{Vac}}}{\longrightarrow} \mathrm{Vect} \overset{-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1}[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}$$

identifies with the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),\mathrm{Ran}} \overset{\mathrm{DS}^{\mathrm{enh,coarse}}}{\longrightarrow} \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})},-)}{\longrightarrow} \mathrm{Vect}\,.$$

14.2. **Composition of coefficient and localization functors: the general case.** We are now ready to state the general theorem, describing the composition of the functors

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_G}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1}[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

and

$$\mathrm{coeff}_G : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{Whit}^!(G)_{\mathrm{Ran}}.$$

14.2.1. Recall that the category $\mathrm{Whit}^!(G)_{\mathrm{Ran}}$ is the dual of $\mathrm{Whit}_*(G)_{\mathrm{Ran}}$. Hence, the description of the above composition is equivalent to describing the pairing

$$\begin{aligned}(14.3)\quad \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} &\overset{\mathrm{Loc}_G \otimes \mathrm{Id}}{\longrightarrow} \\ &\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{(-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1}[\delta_{N_{\rho(\omega_X)}}])\otimes \mathrm{Id}}{\longrightarrow} \\ &\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{coeff}_G \otimes \mathrm{Id}}{\longrightarrow} \mathrm{Whit}^!(G)_{\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \mathrm{Vect}\,.\end{aligned}$$

14.2.2. We will prove:

Theorem 14.2.3. *The functor* (14.3) *identifies canonically with*

$$\begin{aligned}(14.4)\quad \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)_{\mathrm{Ran}} &\overset{\mathrm{ins.unit} \otimes \mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_\subseteq} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)_{\mathrm{Ran}_\subseteq} \to \\ &\to (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G))_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \overset{\mathsf{P}^{\mathrm{loc,enh,coarse}}_G}{\longrightarrow} \\ &\to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \overset{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X;\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})},-)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq) \to \\ &\overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq,-)}{\longrightarrow} \mathrm{Vect},\end{aligned}$$

where the fiber product $\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq$ *is formed using the projections* $\mathrm{pr}_{\mathrm{big}} : \mathrm{Ran}_\subseteq \to \mathrm{Ran}$.

Remark 14.2.4. Note that the functor (14.4), appearing in Theorem 14.2.3 can also be rewritten as

$$\begin{aligned}\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)_{\mathrm{Ran}} &\overset{\mathrm{ins.unit} \otimes \mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_\subseteq} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)_{\mathrm{Ran}_\subseteq} \to \\ &\to (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G))_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \overset{(\mathrm{pr}_{\mathrm{big}} \times \mathrm{pr}_{\mathrm{big}})_*}{\longrightarrow} (\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G))_{\mathrm{Ran}} \to \\ &\overset{\mathsf{P}^{\mathrm{loc}}_G}{\longrightarrow} \text{D-mod}(\mathrm{Ran}) \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran},-)}{\longrightarrow} \mathrm{Vect}\,.\end{aligned}$$

14.2.5. The rest of this subsection is devoted to the proof of Theorem 14.2.3.

We rewrite the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{Loc}_G \otimes \mathrm{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{coeff}_G \otimes \mathrm{Id}}{\longrightarrow} \to \mathrm{Whit}^!(G)_{\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \mathrm{Vect}\,. \tag{14.5}$$

as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \mathrm{Vect} \tag{14.6}$$

14.2.6. Using Lemmas 13.5.4 and 13.4.10, we can rewrite the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \overset{\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}$$

as

$$\begin{aligned}\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} &\overset{\mathrm{ins.unit} \otimes \mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),\mathrm{Ran}_\subseteq} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)_{\mathrm{Ran}_\subseteq} \to \\ &\to \left(\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)\right)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \to \\ &\overset{(\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq}}{\longrightarrow} \to \text{D-mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq)) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \\ &\overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq, -) \otimes \mathrm{Id} \otimes \mathrm{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}.\end{aligned}$$

Hence, we can rewrite (14.6) as

$$\begin{aligned}\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} &\overset{\mathrm{ins.unit} \otimes \mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X),\mathrm{Ran}_\subseteq} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)_{\mathrm{Ran}_\subseteq} \to \\ &\to \left(\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)\right)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \to \\ &\overset{(\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq}}{\longrightarrow} \to \text{D-mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq)) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \\ &\overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq, -) \otimes \mathrm{Id} \otimes \mathrm{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \mathrm{Vect}\,.\end{aligned} \tag{14.7}$$

Hence, it is enough to identify the composition

$$\begin{aligned}&\left(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)\right)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \to \\ &\overset{(\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq}}{\longrightarrow} \to \text{D-mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq)) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \\ &\overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq, -) \otimes \mathrm{Id} \otimes \mathrm{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \\ &\to \mathrm{Vect} \overset{(-\otimes \mathrm{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathrm{l}_{N_{\rho(\omega_X)}}^{\otimes -1})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}\end{aligned} \tag{14.8}$$

with

$$\begin{aligned}&\left(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G)\right)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \overset{\mathsf{P}_G^{\mathrm{loc,enh,coarse}}}{\longrightarrow} \\ &\to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X; \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}, -)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq}}{\longrightarrow} \text{D-mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq) \to \\ &\overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq, -)}{\longrightarrow} \mathrm{Vect}\,.\end{aligned} \tag{14.9}$$

14.2.7. Note that both functors (14.8) and (14.9) factor naturally via

$$(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G))_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \to \left(\mathrm{KL}(G)_{\mathrm{crit}} \underset{\mathrm{Sph}_G}{\otimes} \mathrm{Whit}_*(\mathrm{Gr}_G)\right)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq},$$

and in particular via

$$(\mathrm{KL}(G)_{\mathrm{crit}} \otimes \mathrm{Whit}_*(\mathrm{Gr}_G))_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \to \left(\mathrm{KL}(G)_{\mathrm{crit}} \underset{\mathrm{Rep}(\check{G})}{\otimes} \mathrm{Whit}_*(\mathrm{Gr}_G)\right)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq},$$

where $\mathrm{Rep}(\check{G})$ maps to Sph_G via $\mathrm{Sat}_G^{-1,\mathrm{nv}}$.

Hence, using the fact that the action of $\mathrm{Rep}(\check{G})$ on $\mathbf{1}_{\mathrm{Whit}_*(G)}$ is an equivalence, we are reduced to showing to establishing an identification between

$$\begin{aligned}(14.10)\quad (\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} &\overset{(\mathrm{Loc}_G \otimes \mathrm{Poinc}_{G,*})_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \otimes \mathrm{Poinc}^{\mathrm{Vac}}_{G,*}}{\longrightarrow} \\ \to \mathrm{D\text{-}mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq)) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} &\overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq, -) \otimes \mathrm{Id} \otimes \mathrm{Id}}{\longrightarrow} \\ \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \mathrm{Vect} &\overset{(-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}\end{aligned}$$

and

$$\begin{aligned}(14.11)\quad (\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} &\overset{\alpha_{\rho(\omega_X),\mathrm{taut}}}{\simeq} \left(\mathrm{KL}(G)_{\mathrm{crit},\rho(\omega_X)}\right)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \overset{\mathrm{DS}^{\mathrm{enh,coarse}}}{\longrightarrow} \\ \to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} &\overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X; \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}, -)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq}}{\longrightarrow} \mathrm{D\text{-}mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq) \to \\ &\overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq, -)}{\longrightarrow} \mathrm{Vect}\,.\end{aligned}$$

14.2.8. We rewrite (14.10) as

$$\begin{aligned}(14.12)\quad (\mathrm{KL}(G)_{\mathrm{crit}})_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} &\overset{(\mathrm{Loc}_G)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq}}{\longrightarrow} \\ \to \mathrm{D\text{-}mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq)) \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) &\overset{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq, -) \otimes \mathrm{Id}}{\longrightarrow} \\ \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{coeff}^{\mathrm{Vac}}_G}{\longrightarrow} \mathrm{Vect} &\overset{(-\otimes \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes -1})[\delta_{N_{\rho(\omega_X)}}]}{\longrightarrow} \mathrm{Vect}\,.\end{aligned}$$

14.2.9. Now the isomorphism between (14.12) and (14.11) follows from Theorem 14.1.6.

□[Theorem 14.2.3]

14.3. **Composition of coefficient and localization functors: the twisted case.** We will now consider the variant of Theorem 14.2.3 in the situation twisted by a $Z_{\check{G}}$-torsor $\mathcal{P}_{Z^0_{\check{G}}}$.

14.3.1. Consider the functor

$$\mathrm{Loc}_{G,\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} : \mathrm{KL}(G)_{\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}}),\mathrm{Ran}} \to \mathrm{D\text{-}mod}_{\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}(\mathrm{Bun}_G),$$

see Sect. 13.2.3.

Since the de Rham twisting $\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})$ corresponds to a line bundle on Bun_G, we have a canonical equivalence

$$\mathrm{D\text{-}mod}_{\mathrm{crit}-\mathrm{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}(\mathrm{Bun}_G) \simeq \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G).$$

Composing further with (12.2), we obtain an equivalence

$$\text{D-mod}_{\text{crit}-\text{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}(\text{Bun}_G) \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G). \tag{14.13}$$

Denote by Loc_G the composite functor

$$\text{KL}(G)_{\text{crit}-\text{dlog}(\mathcal{P}_{Z^0_{\check{G}}}),\text{Ran}} \overset{\text{Loc}_{G,\text{crit}-\text{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}}{\longrightarrow} \text{D-mod}_{\text{crit}-\text{dlog}(\mathcal{P}_{Z^0_{\check{G}}})}(\text{Bun}_G) \overset{(14.13)}{\simeq} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G).$$

14.3.2. Consider the composition

$$\begin{aligned}\text{KL}(G)_{\text{crit}-\text{dlog}(\mathcal{P}_{Z^0_{\check{G}}}),\text{Ran}} \otimes \text{Whit}_*(G)_{\text{Ran}} &\overset{\text{Loc}_G \otimes \text{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \otimes \text{Whit}_*(G)_{\text{Ran}} \to \\ &\overset{(-\otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[\delta_{N_{\rho(\omega_X)}}]) \otimes \text{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \otimes \text{Whit}_*(G)_{\text{Ran}} \overset{\text{coeff}_G \otimes \text{Id}}{\longrightarrow} \\ &\to \text{Whit}^!(G)_{\text{Ran}} \otimes \text{Whit}_*(G)_{\text{Ran}} \to \text{Vect}\,.\end{aligned} \tag{14.14}$$

Consider now the functor

$$\begin{aligned}\text{KL}(G)_{\text{crit}-\text{dlog}(\mathcal{P}_{Z^0_{\check{G}}}),\text{Ran}} \otimes \text{Whit}_*(G)_{\text{Ran}} &\overset{\text{ins.unit} \otimes \text{ins.unit}}{\longrightarrow} \\ &\to \text{KL}(G)_{\text{crit}-\text{dlog}(\mathcal{P}_{Z^0_{\check{G}}}),\text{Ran}_\subseteq} \otimes \text{Whit}_*(\text{Gr}_G)_{\text{Ran}_\subseteq} \to \\ &\to \left(\text{KL}(G)_{\text{crit}-\text{dlog}(\mathcal{P}_{Z^0_{\check{G}}})} \otimes \text{Whit}_*(\text{Gr}_G)\right)_{\text{Ran}_\subseteq \underset{\text{Ran}}{\times} \text{Ran}_\subseteq} \overset{\mathsf{P}^{\text{loc,enh,coarse}}_G}{\longrightarrow} \\ &\to \mathcal{O}_{\text{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D})}\text{-mod}^{\text{fact}}_{\text{Ran}_\subseteq \underset{\text{Ran}}{\times} \text{Ran}_\subseteq} \overset{\text{C}^{\text{fact}}_\cdot(X;\mathcal{O}_{\text{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D})},-)_{\text{Ran}_\subseteq \underset{\text{Ran}}{\times} \text{Ran}_\subseteq}}{\longrightarrow} \text{D-mod}(\text{Ran}_\subseteq \underset{\text{Ran}}{\times} \text{Ran}_\subseteq) \to \\ &\overset{\text{C}^\cdot_{\text{dR}}(\text{Ran}_\subseteq \underset{\text{Ran}}{\times} \text{Ran}_\subseteq,-)}{\longrightarrow} \text{Vect},\end{aligned} \tag{14.15}$$

where $\mathsf{P}^{\text{loc,enh,coarse}}_G$ is the composition of the pairing $\mathsf{P}^{\text{loc,enh}}_G$ of (7.30) with the forgetful functor

$$\Gamma(\text{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}),-)^{\text{enh}} : \text{IndCoh}^*(\text{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times)) \to \mathcal{O}_{\text{Op}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D})}\text{-mod}^{\text{fact}}.$$

14.3.3. Parallel to Theorem 14.2.3, we have:

Theorem 14.3.4. *The functors* (14.14) *and* (14.15) *are canonically isomorphic.*

15. Localization and constant term functors

In this section we define the constant term functor

$$\text{CT}^-_* : \text{D-mod}(\text{Bun}_G) \to \text{D-mod}(\text{Bun}_M),$$

along with its twisted versions, when we introduce a level, and a shifted version, when we apply a translation on Bun_M using a Z_M-bundle.

Our main goal is to describe the composite functor

$$\text{KL}(G)_{\text{crit},\text{Ran}} \overset{\text{Loc}_G}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \overset{\text{CT}^-_{*,\rho_P(\omega_X)}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)$$

as a composition of the BRST functor

$$\text{BRST}^-_{\rho_P(\omega_X)} : \text{KL}(G)_{\text{crit},\text{Ran}} \to \text{KL}(M)_{\text{crit}-\check{p}_P(\omega_X),\text{Ran}}$$

and the localization functor

$$\text{KL}(M)_{\text{crit}-\check{p}_P(\omega_X),\text{Ran}} \overset{\text{Loc}_M}{\to} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M),$$

with the *insertion of vacuum* in the middle.

Such a description will be a consequence of the results of the previous section.

Here is one thing to watch out for: there are two "ρ-shifts" around: one is the translation by $\rho_P(\omega_X)$, which is artificially introduced in order to match the spectral side. And there is the twist by the line bundle on Bun_M, corresponding to $\check{\rho}_P(\omega_X) \in \mathrm{Bun}_{Z^0_M}$. It is intrinsic to the functor of BRST reduction, and it will correspond to the Miura shift on the spectral side.

15.1. **The (untwisted) functor of constant term.**

15.1.1. Let P^- be a standard negative parabolic. The usual constant term functor

$$\text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_M)$$

is defined to be

$$\mathrm{CT}^-_* := (\mathfrak{q}^-)_* \circ (\mathfrak{p}^-)^! [-\dim.\mathrm{rel}(\mathrm{Bun}_{P^-} / \mathrm{Bun}_M)], \tag{15.1}$$

for the morphisms in the diagram

$$\mathrm{Bun}_G \overset{\mathfrak{p}^-}{\leftarrow} \mathrm{Bun}_{P^-} \overset{\mathfrak{q}^-}{\to} \mathrm{Bun}_M, \tag{15.2}$$

and where the amount of the shift $\dim.\mathrm{rel}(\mathrm{Bun}_{P^-} / \mathrm{Bun}_M)$ depends on the connected component of Bun_M.

15.1.2. *The ρ-translated constant term functors.* The functor that actually plays a role in Langlands theory is the following translated version of the functor CT^-_*:

We will denote by

$$\mathrm{CT}^-_{*,\rho_P(\omega_X)} : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_M) \tag{15.3}$$

the functor, given by composing the following three functors:

- The functor $(\mathfrak{q}^-)_* \circ (\mathfrak{p}^-)^!$, i.e., !-pull *-push along the diagram (15.2);
- The functor $\mathrm{transl}^*_{\rho_P(\omega_X)} : \text{D-mod}(\mathrm{Bun}_M) \to \text{D-mod}(\mathrm{Bun}_M)$;
- Over the connected component of Bun_M of degree λ, the cohomological shift to the right by the amount
$$\delta_{N(P^-)_{\rho_P(\omega_X)}} + \langle \lambda, 2\check{\rho}_P \rangle,$$
where
$$\delta_{N(P^-)_{\rho_P(\omega_X)}} := \dim(\mathrm{Bun}_{N(P^-)_{\rho_P(\omega_X)}}). \tag{15.4}$$

15.1.3. Note that the composition of the first two functors comprising $\mathrm{CT}^-_{*,\rho_P(\omega_X)}$ is just pull-push along

$$\mathrm{Bun}_G \overset{\mathfrak{p}^-}{\leftarrow} \mathrm{Bun}_{P^-,\rho_P(\omega_X)} \overset{\mathfrak{q}^-}{\to} \mathrm{Bun}_M,$$

where

$$\mathrm{Bun}_{P^-,\rho_P(\omega_X)} := \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_M, \mathrm{transl}_{\rho_P(\omega_X)}}{\times} \mathrm{Bun}_M .$$

Note also that the relative dimension of the morphism

$$\mathrm{Bun}_{P^-,\rho_P(\omega_X)} \to \mathrm{Bun}_M$$

over the connected component of Bun_M of degree λ equals

$$\delta_{N(P^-)_{\rho_P(\omega_X)}} + \langle \lambda, 2\check{\rho}_P \rangle.$$

So, the third functor comprising $\mathrm{CT}^-_{*,\rho_P(\omega_X)}$ can be thought of as the cohomological shift to the right by

$$\dim.\mathrm{rel}(\mathrm{Bun}_{P^-,\rho_P(\omega_X)} / \mathrm{Bun}_M).$$

15.2. **Constant term functor on gerbe-twisted D-modules.** For our purposes, we will need to consider the functor $\mathrm{CT}^-_{*,\rho_P(\omega_X)}$ not on $\text{D-mod}(\mathrm{Bun}_G)$ but rather on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

15.2.1. Let us denote by $\mathfrak{l}_{G,P^-,M,\rho_P(\omega_X)}$ the (non-graded) line

$$\det_{\mathrm{Bun}_G}|_{\rho_P(\omega_X)} \otimes \det{}^{\otimes -1}_{\mathrm{Bun}_M}|_{\rho_P(\omega_X)} \otimes \mathfrak{l}^{\otimes -2}_{N(P^-)_{\rho_P(\omega_X)}},$$

where the notation $\mathfrak{l}_{N(P^-)_{\rho_P(\omega_X)}}$ is as in Sect. 13.7.1.[21]

We record the following (elementary) observation:

Lemma 15.2.2. *There exists a canonical isomorphism of lines*

$$\mathfrak{l}_{G,P^-,M,\rho_P(\omega_X)} \simeq \left(\mathfrak{l}^{\otimes -\frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{M,N(M)_{\rho_M(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N(M)_{\rho_M(\omega_X)}}\right)^{\otimes 2} \otimes$$
$$\otimes \det(\Gamma(X,\mathcal{O}_X)\otimes\mathfrak{g})^{\otimes -2} \otimes \det(\Gamma(X,\mathcal{O}_X)\otimes\mathfrak{m})^{\otimes 2},$$

where

$$\mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \text{ and } \mathfrak{l}_{N_{\rho(\omega_X)}}$$

are the lines introduced in (12.9) *and* (14.2)*, respectively, and* $\mathfrak{l}^{\otimes \frac{1}{2}}_{M,N(M)_{\rho_M(\omega_X)}}$ *and* $\mathfrak{l}_{N(M)_{\rho_M(\omega_X)}}$ *are the corresponding lines for* M*.*

Corollary 15.2.3. *The line* $\mathfrak{l}_{G,P^-,M,\rho_P(\omega_X)}$ *admits a canonical square root.*

In what follows we will denote by

$$\mathfrak{l}^{\otimes \frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)} \tag{15.5}$$

the square root of $\mathfrak{l}_{G,P^-,M,\rho_P(\omega_X)}$, given by Corollary 15.2.3.

15.2.4. Let $\mathcal{L}_{\check{\rho}_P(\omega_X)}$ be the line bundle on Bun_M corresponding to the Z^0_M-torsor $\check{\rho}_P(\omega_X)$, see Sect. 13.2.3.

We claim:

Proposition 15.2.5. *There is a canonical isomorphism between the following two line bundles on* $\mathrm{Bun}_{P^-,\rho_P(\omega_X)}$*:*

- *The pullback of* $\det_{\mathrm{Bun}_G}$ *along* $\mathrm{Bun}_{P^-,\rho_P(\omega_X)} \to \mathrm{Bun}_G$*;*
- *The tensor product of:*
  - *The pullback of* $\det_{\mathrm{Bun}_M}$ *along* $\mathrm{Bun}_{P^-,\rho_P(\omega_X)} \to \mathrm{Bun}_M$*;*
  - $\det(T^*(\mathrm{Bun}_{P^-,\rho_P(\omega_X)} / \mathrm{Bun}_M))^{\otimes 2}$*;*
  - *The pullback of* $\mathcal{L}^{\otimes -2}_{\check{\rho}_P(\omega_X)}$ *along* $\mathrm{Bun}_{P^-,\rho_P(\omega_X)} \to \mathrm{Bun}_M$*;*
  - *The line* $\mathfrak{l}_{G,P^-,M,\rho_P(\omega_X)}$*.*

Proof. We first identify the two lines bundles up to a constant line.

Fix an M-bundle $\mathcal{P}_M$. Comparing the two sides, we need to establish an isomorphism

$$\det(\Gamma(X,\mathfrak{n}(P^-)_{\rho_P(\omega_X)\cdot\mathcal{P}_M}\otimes\omega_X)) \simeq \det(\Gamma(X,(\mathfrak{g}/\mathfrak{n}(P^-))_{\rho_P(\omega_X)\cdot\mathcal{P}_M}\otimes\omega_X)) \otimes \mathrm{Weil}(-2\check{\rho}(\mathcal{P}_M),\omega_X),$$

up to a constant line.

For a root $\check{\alpha}\in\mathfrak{n}(P)$, let $\mathcal{E}_\alpha$ denote the line bundle $\check{\alpha}(\mathcal{P}_M)$. It is enough to show that for every α we have

$$\det(\Gamma(X,\mathcal{E}_\alpha^{\otimes -1}\otimes\omega_X^{\langle -\check{\alpha},\rho\rangle+1})) \simeq \det(\Gamma(X,\mathcal{E}_\alpha\otimes\omega_X^{\langle\check{\alpha},\rho\rangle+1})) \otimes \mathrm{Weil}(\mathcal{E}_\alpha,\omega_X)^{\otimes -1},$$

up to a constant line.

By Serre dualty,

$$\det(\Gamma(X,\mathcal{E}_\alpha^{\otimes -1}\otimes\omega_X^{\langle -\check{\alpha},\rho\rangle+1})) \simeq \det(\Gamma(X,\mathcal{E}_\alpha\otimes\omega_X^{\langle\check{\alpha},\rho\rangle})).$$

Hence, we have to show

$$\det(\Gamma(X,\mathcal{E}_\alpha\otimes\omega_X^{\langle\check{\alpha},\rho\rangle})) \simeq \det(\Gamma(X,\mathcal{E}_\alpha\otimes\omega_X^{\langle\check{\alpha},\rho\rangle+1})) \otimes \langle\mathcal{E}_\alpha,\omega_X\rangle^{\otimes -1},$$

[21]Note, however, that since ρ_P is central in M, the line $\det_{\mathrm{Bun}_M}|_{\rho_P(\omega_X)}$ is canonically trivial.

up to a constant line. However, this follows from (12.6).

In order to determine the constant line, we take the fibers of both sides at the trivial M-bundle, hence the result.

□

Remark 15.2.6. An assertion parallel to Proposition 15.2.5 holds for the untranslated Bun_{P^-}. The only difference is that instead of the line $\mathfrak{l}_{G,P^-,M,\rho_P(\omega_X)}$ we will have $\mathfrak{l}_{N(P^-)}^{\otimes -2}$.

15.2.7. Combining Proposition 15.2.5 and Corollary 15.2.3, we obtain that the line bundles

$$\det_{\mathrm{Bun}_G}|_{\mathrm{Bun}_{P^-},\rho_P(\omega_X)} \text{ and } \det_{\mathrm{Bun}_M}|_{\mathrm{Bun}_{P^-},\rho_P(\omega_X)}$$

differ by a line bundle that admits a canonical square root.

Hence, the $\mathbb{Z}/2\mathbb{Z}$-gerbes

$$\det_{\mathrm{Bun}_G}^{\frac{1}{2}}|_{\mathrm{Bun}_{P^-},\rho_P(\omega_X)} \text{ and } \det_{\mathrm{Bun}_M}^{\frac{1}{2}}|_{\mathrm{Bun}_{P^-},\rho_P(\omega_X)}$$

are canonically identified.

This allows to define the functor

$$\mathrm{CT}^-_{*,\rho_P(\omega_X)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M). \tag{15.6}$$

15.2.8. Similarly, parallel to Remark 15.2.6, we have the untranslated functor

$$\mathrm{CT}^-_* : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M). \tag{15.7}$$

15.3. **Constant term and localization: a general level.**

15.3.1. Let κ be a general level. Note that the pullbacks to Bun_{P^-} of the de Rham twistings corresponding to κ on Bun_G and Bun_P are canonically isomorphic.

Hence, we have a well-defined functor

$$\mathrm{CT}^-_* : \text{D-mod}_\kappa(\mathrm{Bun}_G) \to \text{D-mod}_\kappa(\mathrm{Bun}_M).$$

15.3.2. Note also that as in Proposition 15.2.5 (see Remark 15.2.6), we have a canonical isomorphism of the de Rham twistings

$$\begin{aligned} \frac{1}{2}\cdot \mathrm{dlog}(\det_{\mathrm{Bun}_G})|_{\mathrm{Bun}_{P^-}} \simeq \\ \simeq \frac{1}{2}\cdot \mathrm{dlog}(\det_{\mathrm{Bun}_M})|_{\mathrm{Bun}_{P^-}} + \mathrm{dlog}(\det(T^*(\mathrm{Bun}_{P^-}/\mathrm{Bun}_M))) - \mathrm{dlog}(\mathcal{L}_{\check\rho_P(\omega_X)}). \end{aligned} \tag{15.8}$$

This allows to consider the functor

$$\mathrm{CT}^-_{*,\mathrm{crit}} : \text{D-mod}_{\mathrm{crit}+\kappa}(\mathrm{Bun}_G) \to \text{D-mod}_{\mathrm{crit}+\kappa}(\mathrm{Bun}_M)$$

equal to

$$\begin{aligned} &\text{D-mod}_{\mathrm{crit}+\kappa}(\mathrm{Bun}_G) =: \text{D-mod}_{\mathrm{crit}_G+\kappa}(\mathrm{Bun}_G) \xrightarrow{\mathrm{CT}^-_*} \text{D-mod}_{\mathrm{crit}_{G|_{\mathfrak{m}}}+\kappa}(\mathrm{Bun}_M) \overset{(15.8)}{\simeq} \\ &\simeq \text{D-mod}_{\mathrm{crit}_M+\kappa+\mathrm{dlog}(\det(T^*(\mathrm{Bun}_{P^-}/\mathrm{Bun}_M)))-\mathrm{dlog}(\mathcal{L}_{\check\rho_P(\omega_X)})}(\mathrm{Bun}_M) \xrightarrow{-\otimes\det(T^*(\mathrm{Bun}_{P^-}/\mathrm{Bun}_M))^{\otimes -1}} \\ &\simeq \text{D-mod}_{\mathrm{crit}_M+\kappa-\mathrm{dlog}(\mathcal{L}_{\check\rho_P(\omega_X)})}(\mathrm{Bun}_M) := \text{D-mod}_{\mathrm{crit}+\kappa-\check\rho_P}(\mathrm{Bun}_M) \xrightarrow{[-\dim.\mathrm{rel}(\mathrm{Bun}_{P^-}/\mathrm{Bun}_M)]} \\ &\to \text{D-mod}_{\mathrm{crit}+\kappa-\check\rho_P}(\mathrm{Bun}_M). \end{aligned}$$

15.3.3. We have the following general statement (see [CF2, Theorem 4.0.5]:[22])

Theorem 15.3.4. *The following diagram of functors commutes:*

$$\begin{array}{ccccc} \mathrm{D\text{-}mod}_{\mathrm{crit}+\kappa}(\mathrm{Bun}_G) & & \xrightarrow{\mathrm{CT}^-_{*,\mathrm{crit}}} & & \mathrm{D\text{-}mod}_{\mathrm{crit}+\kappa-\check{\rho}_P}(\mathrm{Bun}_M) \\ {\scriptstyle \mathrm{Loc}_{G,\mathrm{crit}+\kappa}}\uparrow & & & & \uparrow{\scriptstyle \mathrm{Loc}_{M,\mathrm{crit}+\kappa-\check{\rho}_P}} \\ \mathrm{KL}(G)_{\mathrm{crit}+\kappa,\mathrm{Ran}} & \xrightarrow{\mathrm{ins.unit}} & \mathrm{KL}(G)_{\mathrm{crit}+\kappa,\mathrm{Ran}_\subseteq} & \xrightarrow{\mathrm{BRST}^-} & \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P,\mathrm{Ran}_\subseteq}. \end{array}$$

Remark 15.3.5. Not that at the level of fibers at points of Bun_M, the assertion of Theorem 15.3.4 exactly reproduces the assertion of Theorem 13.8.5 (with N replaced by $N^-(P)$ and the torus T replaced by $Z^0(M)$).

Remark 15.3.6. Note that the counter-clockwise composition in Theorem 15.3.4 can be also rewritten as

$$\begin{aligned} \mathrm{KL}(G)_{\mathrm{crit}+\kappa,\mathrm{Ran}} \xrightarrow{\mathrm{ins.unit}} \mathrm{KL}(G)_{\mathrm{crit}+\kappa,\mathrm{Ran}_\subseteq} \xrightarrow{(\mathrm{pr}_{\mathrm{big}})^*} \mathrm{KL}(G)_{\mathrm{crit}+\kappa,\mathrm{Ran}} \xrightarrow{\mathrm{BRST}^-} \\ \to \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P,\mathrm{Ran}} \xrightarrow{\mathrm{Loc}_{M,\mathrm{crit}+\kappa-\check{\rho}_P}} \mathrm{D\text{-}mod}_{\mathrm{crit}+\kappa-\check{\rho}_P}(\mathrm{Bun}_M). \end{aligned}$$

15.3.7. Let $\mathcal{P}_{Z_M}$ be a Z_M-torsor. Denote by $\mathrm{CT}^-_{*,\mathrm{crit},\mathcal{P}_{Z_M}}$ for the functor

$$\mathrm{D\text{-}mod}_{\mathrm{crit}+\kappa}(\mathrm{Bun}_G) \xrightarrow{\mathrm{CT}^-_{*,\mathrm{crit}}} \mathrm{D\text{-}mod}_{\mathrm{crit}+\kappa-\check{\rho}_P}(\mathrm{Bun}_M) \xrightarrow{\mathrm{transl}^*_{\mathcal{P}_{Z_M}}} \mathrm{D\text{-}mod}_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}(\mathrm{Bun}_M),$$

see Sect. 13.2.8 for the notation.

Concatenating Theorem 15.3.4 with (13.8), we obtain:

Corollary 15.3.8. *The following diagram of functors commutes:*

$$\begin{array}{ccccc} \mathrm{D\text{-}mod}_{\mathrm{crit}+\kappa}(\mathrm{Bun}_G) & & \xrightarrow{\mathrm{CT}^-_{*,\mathrm{crit},\mathcal{P}_{Z_M}}} & & \mathrm{D\text{-}mod}_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}(\mathrm{Bun}_M) \\ {\scriptstyle \mathrm{Loc}_{G,\mathrm{crit}+\kappa}}\uparrow & & & & \uparrow{\scriptstyle \mathrm{Loc}_{M,\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-)}} \\ \mathrm{KL}(G)_{\mathrm{crit}+\kappa,\mathrm{Ran}} & \xrightarrow{\mathrm{ins.unit}} & \mathrm{KL}(G)_{\mathrm{crit}+\kappa,\mathrm{Ran}_\subseteq} & \xrightarrow{\mathrm{BRST}^-_{\mathcal{P}_{Z_M}}} & \mathrm{KL}(M)_{\mathrm{crit}+\kappa-\check{\rho}_P-\kappa(\mathrm{dlog}(\mathcal{P}_{Z_M}),-),\mathrm{Ran}_\subseteq}, \end{array}$$

where $\mathrm{BRST}^-_{\mathcal{P}_{Z_M}}$ *is the functor of* (4.17).

15.3.9. In particular, for $\kappa = 0$, Corollary 15.3.8 specializes to the commutative diagram

$$\begin{array}{ccccc} \mathrm{D\text{-}mod}_{\mathrm{crit}}(\mathrm{Bun}_G) & & \xrightarrow{\mathrm{CT}^-_{*,\mathrm{crit},\mathcal{P}_{Z_M}}} & & \mathrm{D\text{-}mod}_{\mathrm{crit}-\check{\rho}_P}(\mathrm{Bun}_M) \\ {\scriptstyle \mathrm{Loc}_{G,\mathrm{crit}}}\uparrow & & & & \uparrow{\scriptstyle \mathrm{Loc}_{M,\mathrm{crit}-\check{\rho}_P}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{ins.unit}} & \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_\subseteq} & \xrightarrow{\mathrm{BRST}^-_{\mathcal{P}_{Z_M}}} & \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P,\mathrm{Ran}_\subseteq}. \end{array} \tag{15.9}$$

[22] The commutative diagram below is the combination of two commutative diagrams: one is obtained by applying Theorem 4.0.5(3) of *loc.cit.* to $P \to G$, the other by applying Theorem 4.0.5(4) to $P \to M$. These two diagrams are combined using Theorem 4.0.5(2), which produces the $\check{\rho}_P$-shift. Also, see [CF2, Sect. 2.4] for a similar diagram but only for $X = \mathbb{P}^1$ and $P = B$.

15.3.10. We will mostly use Corollary 15.3.8 and (15.9) when $\mathcal{P}_{Z_M} = \rho_P(\omega_X)$, in which case they read as

(15.10)

$$\begin{array}{ccc} \text{D-mod}_{\text{crit}+\kappa}(\text{Bun}_G) & \xrightarrow{\text{CT}^-_{*,\text{crit},\rho_P(\omega_X)}} & \text{D-mod}_{\text{crit}+\kappa-\check{\rho}_P-\kappa(\rho_P,-)}(\text{Bun}_M) \\ {\scriptstyle \text{Loc}_{G,\text{crit}+\kappa}}\uparrow & & \uparrow{\scriptstyle \text{Loc}_{M,\text{crit}+\kappa-\check{\rho}_P-\kappa(\rho_P,-)}} \\ \text{KL}(G)_{\text{crit}+\kappa,\text{Ran}} & \xrightarrow{\text{ins.unit}} \text{KL}(G)_{\text{crit}+\kappa,\text{Ran}_\subseteq} & \xrightarrow{\text{BRST}^-_{\rho_P(\omega_X)}} \text{KL}(M)_{\text{crit}+\kappa-\check{\rho}_P-\kappa(\rho_P,-),\text{Ran}_\subseteq}, \end{array}$$

and

(15.11)

$$\begin{array}{ccc} \text{D-mod}_{\text{crit}}(\text{Bun}_G) & \xrightarrow{\text{CT}^-_{*,\text{crit},\rho_P(\omega_X)}} & \text{D-mod}_{\text{crit}-\check{\rho}_P}(\text{Bun}_M) \\ {\scriptstyle \text{Loc}_{G,\text{crit}}}\uparrow & & \uparrow{\scriptstyle \text{Loc}_{M,\text{crit}-\check{\rho}_P}} \\ \text{KL}(G)_{\text{crit},\text{Ran}} & \xrightarrow{\text{ins.unit}} \text{KL}(G)_{\text{crit},\text{Ran}_\subseteq} & \xrightarrow{\text{BRST}^-_{\rho_P(\omega_X)}} \text{KL}(M)_{\text{crit}-\check{\rho}_P,\text{Ran}_\subseteq}, \end{array}$$

respectively.

Note that the functors $\text{CT}^-_{*,\text{crit},\rho_P(\omega_X)}$ and $\text{BRST}^-_{\rho_P(\omega_X)}$ are different from their non-translated counterparts, even though the target categories are the same.

15.4. **Constant term and localization: the half-twisted case.**

15.4.1. We are finally ready to state the main result of this section:

Theorem 15.4.2. *The following diagram of functors commutes*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\text{CT}^-_{*,\rho_P(\omega_X)}} & \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \\ {\scriptstyle \text{Loc}_G \otimes \mathfrak{l}^{\otimes\frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle \text{Loc}_M} \\ \text{KL}(G)_{\text{crit},\text{Ran}} & \xrightarrow{\text{ins.unit}} \text{KL}(G)_{\text{crit},\text{Ran}_\subseteq} & \xrightarrow{\text{BRST}^-_{\rho_P(\omega_X)}} \text{KL}(M)_{\text{crit}-\check{\rho}_P,\text{Ran}_\subseteq}, \end{array}$$

where $\delta_{N(P^-)_{\rho_P(\omega_X)}}$ *is the integer* (15.4) *and* $\mathfrak{l}^{\otimes\frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)}$ *is the line* (15.5).

15.4.3. *Proof of Theorem 15.4.2.* Given (15.11), we only have to show that the following diagram commutes

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\text{CT}^-_{*,\rho_P(\omega_X)}} & \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \\ {\scriptstyle (12.2)\otimes \mathfrak{l}^{\otimes\frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle (12.2)} \\ \text{D-mod}_{\text{crit}}(\text{Bun}_G) & \xrightarrow{\text{CT}^-_{*,\text{crit},\rho_P(\omega_X)}} & \text{D-mod}_{\text{crit}-\check{\rho}_P}(\text{Bun}_M). \end{array}$$

However, the latter is a straightforward consequence of the constructions.

□[Theorem 15.4.2]

15.4.4. In a completely similar way, we have the following compatibility assertion for the untranslated half-twisted constant term functor:

Theorem 15.4.5. *The following diagram of functors commutes*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\text{CT}^-_*} & \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \\ {\scriptstyle \text{Loc}_G \otimes \mathfrak{l}^{\otimes -1}_{N(P^-)}}\uparrow & & \uparrow{\scriptstyle \text{Loc}_M} \\ \text{KL}(G)_{\text{crit},\text{Ran}} & \xrightarrow{\text{ins.unit}} \text{KL}(G)_{\text{crit},\text{Ran}_\subseteq} & \xrightarrow{\text{BRST}^-} \text{KL}(M)_{\text{crit}-\check{\rho}_P,\text{Ran}_\subseteq}. \end{array}$$

16. Localization and constant term functors: the enhanced version

This section can be skipped on the first pass, and returned to, when necessary. Here We introduce global enhancements of the objects we studied in the previous section.

We first introduce the global enhanced category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}}$, essentially by tensoring $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)$ with the Ran version of the local semi-infinite category $\text{I}(G,P^-)^{\text{loc}}$ over Sph_M. This category is related to the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)$ by a pair of adjoint functors

$$\mathbf{ind}_{\text{enh}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}} : \mathbf{oblv}_{\text{enh}}.$$

We introduce the *enhanced* constant term functor

$$\text{CT}_*^{-,\text{enh}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}},$$

so that $\text{CT}_*^- \simeq \mathbf{oblv}_{\text{enh}} \circ \text{CT}_*^{-,\text{enh}}$. This is done by a geometric procedure that involves modifying the G-bundle at points "along the P^--direction."

We introduce the enhanced version of the localization functor

$$\text{Loc}_M^{-,\text{enh}} : \text{KL}(M)^{-,\text{enh}}_{\text{crit},\text{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}}.$$

The main result of this section is a generalization of Theorem 15.4.2, given by Theorem 16.4.2, which expresses the composition

$$\text{CT}_*^{-,\text{enh}} \circ \text{Loc}_G$$

as a pre-composition of $\text{Loc}_M^{-,\text{enh}}$ with the *enhanced* functor of BRST reduction at the local level.

16.1. The enhanced recipient category.

16.1.1. Recall the (factorization) category $\text{I}(G,P^-)^{\text{loc}}$. It is equipped with an action of Sph_M (see Sect. 2.2.2).

Consider $\text{I}(G,P^-)^{\text{loc}}_{\text{Ran}}$ and $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \otimes \text{D-mod}(\text{Ran})$ as categories over Ran. Consider the tensor product

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\text{Ran}}} := \text{I}(G,P^-)^{\text{loc}}_{\text{Ran}} \underset{\text{Sph}_{M,\text{Ran}}}{\otimes} \left(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \otimes \text{D-mod}(\text{Ran})\right).$$

The (monadic) pair of adjoint functors

$$\mathbf{ind}_{\text{Sph}\to\frac{\infty}{2}} : \text{Sph}_M \rightleftarrows \text{I}(G,P^-)^{\text{loc}} : \mathbf{oblv}_{\frac{\infty}{2}\to\text{Sph}} \tag{16.1}$$

gives rise to a monadic adjunction

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \otimes \text{D-mod}(\text{Ran}) \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\text{Ran}}}. \tag{16.2}$$

16.1.2. More generally, let $\mathcal{Z}$ be a space mapping to Ran. Then we can form the category

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\mathcal{Z}}} := \text{I}(G,P^-)^{\text{loc}}_{\mathcal{Z}} \underset{\text{Sph}_{M,\mathcal{Z}}}{\otimes} \left(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \otimes \text{D-mod}(\mathcal{Z})\right),$$

equipped with a monadic adjunction

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \otimes \text{D-mod}(\mathcal{Z}) \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\mathcal{Z}}}.$$

In particular, for $\underline{x} \in \text{Ran}$ we have the category

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\underline{x}}}$$

and a monadic adjunction

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\underline{x}}}.$$

16.1.3. Define $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}}$ to be the fiber product

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\text{Ran}}} \underset{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)\otimes\text{D-mod}(\text{Ran})}{\times} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M),$$

where:

- The functor
$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\text{Ran}}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)\otimes\text{D-mod}(\text{Ran})$$
is the right adjoint (forgetful) functor from (16.2);
- The functor $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)\otimes\text{D-mod}(\text{Ran})$ is
$$-\boxtimes\omega_{\text{Ran}}.$$

Note that, due to the contractibility of the Ran space, the functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}} \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}_{\text{Ran}}}$$

is fully faithful.

16.1.4. The adjunction (16.2) gives rise to a monadic adjunction

$$\mathbf{ind}_{\text{enh}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}} : \mathbf{oblv}_{\text{enh}}. \tag{16.3}$$

16.1.5. Here is another way to think about the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}}$. Recall the associative factorization algebra $\widetilde{\Omega} \in \text{Sph}_M$, see Sect. 2.2.5.

The monoidal operation on Ran endows $\text{Sph}_{M,\text{Ran}}$ with a structure of mononoidal category, and $\widetilde{\Omega}_{\text{Ran}}$ with a structure of an associative algebra in it.

We have a monoidal action of $\text{Sph}_{M,\text{Ran}}$ on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)$, and a tautological equivalence

$$\widetilde{\Omega}_{\text{Ran}}\text{-mod}(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)) \simeq \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}},$$

so that the adjunction (16.3) becomes

$$\mathbf{ind}_{\widetilde{\Omega}_{\text{Ran}}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \rightleftarrows \widetilde{\Omega}_{\text{Ran}}\text{-mod}(\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)) : \mathbf{oblv}_{\widetilde{\Omega}_{\text{Ran}}}.$$

Remark 16.1.6. One can show that the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}}$ introduced above is equivalent to the category $\text{I}(G,P^-)^{\text{glob}}$ from [Gai1, Sect. 6.1], defined as follows.

For a group H, let $\text{Bun}_H^{\text{gen}}$ denote the prestack of H-bundles defined generically on X. There exists a tautological map

$$\text{Bun}_H \to \text{Bun}_H^{\text{gen}}.$$

Consider the prestack

$$\text{Bun}_{P^-}^{\text{gen}} \underset{\text{Bun}_G^{\text{gen}}}{\times} \text{Bun}_G.$$

The category $\text{I}(G,P^-)^{\text{glob}}$ is by definition the fiber product

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{P^-}^{\text{gen}} \underset{\text{Bun}_G^{\text{gen}}}{\times} \text{Bun}_G) \underset{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{P^-})}{\times} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M),$$

where the functor

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{P^-})$$

is $(\mathsf{q}^-)^*$.

In other words, $\text{I}(G,P^-)^{\text{glob}}$ is the full subcategory of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{P^-}^{\text{gen}} \underset{\text{Bun}_G^{\text{gen}}}{\times} \text{Bun}_G)$, consisting of objects, whose pullback to Bun_{P^-} lies in the essential image of the (fully faithful) functor $(\mathsf{q}^-)^*$.

Under the equivalence

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M)^{-,\text{enh}} \simeq \text{I}(G,P^-)^{\text{glob}},$$

the functor $\mathbf{oblv}_{\mathrm{enh}}$ corresponds to the tautological functor

$$\mathrm{I}(G,P^-)^{\mathrm{glob}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M).$$

16.2. **The enhanced constant term functor.** Our current goal is to define the enhanced constant term functor

$$\mathrm{CT}_*^{-,\mathrm{enh}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}$$

so that

$$\mathbf{oblv}_{\mathrm{enh}} \circ \mathrm{CT}_*^{-,\mathrm{enh}} \simeq \mathrm{CT}_*^-,$$

where CT_*^- is the functor from Sect. 15.2.8.

16.2.1. To simplify the notation, we will fix a point $\underline{x} \in \mathrm{Ran}$, and describe the corresponding functor

$$\mathrm{CT}_*^{-,\mathrm{enh}_{\underline{x}}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\underline{x}} \underset{\mathrm{Sph}_{M,\underline{x}}}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) =: \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\underline{x}}}. \tag{16.4}$$

By duality, the datum of a functor (16.4) is equivalent to that of a $\mathrm{Sph}_{M,\underline{x}}$-linear functor

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\underline{x}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M), \tag{16.5}$$

where

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\underline{x}} := \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(M)_{\underline{x}}}_{\mathfrak{L}(N(P^-))_{\underline{x}}},$$

see Sect. 3.1.2.

16.2.2. Consider the Hecke global stack

$$\mathrm{Bun}_G \overset{\overleftarrow{h}_G}{\leftarrow} \mathrm{Hecke}^{\mathrm{glob}}_{G,\underline{x}} \overset{\overrightarrow{h}_G}{\to} \mathrm{Bun}_G\,.$$

Denote by s the projection

$$\mathrm{Hecke}^{\mathrm{glob}}_{G,\underline{x}} \to \mathrm{Hecke}^{\mathrm{loc}}_{G,\underline{x}} := \mathfrak{L}^+(G)_{\underline{x}} \backslash \mathrm{Gr}_{G,\underline{x}}.$$

Note that we have a canonical identification of line bundles on $\mathrm{Hecke}^{\mathrm{glob}}_{G,\underline{x}}$

$$\overleftarrow{h}{}^*_G(\det_{\mathrm{Bun}_G}) \simeq \overrightarrow{h}{}^*_G(\det_{\mathrm{Bun}_G}) \otimes \mathsf{s}^*(\det_{\mathrm{Gr}_{G,\underline{x}}}).$$

In particular, we have a canonical identification of $\mathbb{Z}/2\mathbb{Z}$-gerbes

$$\overleftarrow{h}{}^*_G(\det^{\frac{1}{2}}_{\mathrm{Bun}_G}) \simeq \overrightarrow{h}{}^*_G(\det^{\frac{1}{2}}_{\mathrm{Bun}_G}) \otimes \mathsf{s}^*(\det^{\frac{1}{2}}_{\mathrm{Gr}_{G,\underline{x}}}). \tag{16.6}$$

16.2.3. Consider the fiber product

$$\mathrm{Hecke}^{\mathrm{glob}}_{G,P^-,\underline{x}} := \mathrm{Hecke}^{\mathrm{glob}}_{G,\underline{x}} \underset{\overrightarrow{h}_{G,\mathrm{Bun}_G}}{\times} \mathrm{Bun}_{P^-}\,. \tag{16.7}$$

Denote the resulting maps by

$$\mathrm{Bun}_G \overset{\overleftarrow{h}_{G,P^-}}{\leftarrow} \mathrm{Hecke}^{\mathrm{glob}}_{G,\underline{x}} \overset{\overrightarrow{h}_{G,P^-}}{\to} \mathrm{Bun}_{P^-}\,.$$

We have a natural map

$$\mathrm{Hecke}^{\mathrm{glob}}_{G,\underline{x}} \to \mathfrak{L}^+(P^-)_{\underline{x}} \backslash \mathrm{Gr}_{G,\underline{x}},$$

which we denote by the same symbol s.

Due to Remark 15.2.6, the identification (16.6) gives rise to an identification of $\mathbb{Z}/2\mathbb{Z}$-gerbes

$$\overleftarrow{h}{}^*_{G,P^-}(\det^{\frac{1}{2}}_{\mathrm{Bun}_G}) \simeq (\overrightarrow{h}_{G,P^-})^* \circ (\mathfrak{q}^-)^*(\det^{\frac{1}{2}}_{\mathrm{Bun}_M}) \otimes \mathsf{s}^*(\det^{\frac{1}{2}}_{\mathrm{Gr}_{G,\underline{x}}}).$$

16.2.4. Thus, we have a well-defined functor

$$(\overrightarrow{h}_{G,P^-})_* \left(\mathsf{s}^!(-) \overset{!}{\otimes} (\overleftarrow{h}_{G,P^-})^!(-) \right), \tag{16.8}$$

which maps

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{(\mathfrak{q}^-)^*(\det^{\frac{1}{2}}_{\mathrm{Bun}_M})}(\mathrm{Bun}_{P^-}).$$

16.2.5. Consider the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \tag{16.9}$$

equal to the composition of (16.8) with:

- The functor

$$\mathfrak{q}^-_* : \text{D-mod}_{(\mathfrak{q}^-)^*(\det^{\frac{1}{2}}_{\mathrm{Bun}_M})}(\mathrm{Bun}_{P^-}) \to \text{D-mod}_{\det^{\frac{1}{2}}_{\mathrm{Bun}_M}}(\mathrm{Bun}_{P^-}) =: \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M);$$

- The functor of cohomological shift to the right by the amount

$$\dim. \mathrm{rel}(\mathrm{Bun}_{P^-} \,/\, \mathrm{Bun}_M)$$

over a given connected component of Bun_M.

It is easy to see that that functor (16.9) factors via the quotient

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(M)_{\underline{x}}}_{\mathfrak{L}^+(N(P^-))_{\underline{x}}} \twoheadrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(M)_{\underline{x}}}_{\mathfrak{L}(N(P^-))_{\underline{x}}}$$

along the first factor.

The resulting functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(M)_{\underline{x}}}_{\mathfrak{L}(N(P^-))_{\underline{x}}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$$

is the sought-for functor (16.5).

16.2.6. By construction, the functor (16.5) is compatible with the actions of $\mathrm{Sph}_{G,\underline{x}}$.

16.3. **The enhanced localization functor.**

16.3.1. *Notational convention.* In order to avoid overburdening the notation, for the duration of Sects. 16.3-16.5 we will denote by

$$\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P}$$

the (untwisted) category

$$\mathrm{I}(G,P^-)^{\mathrm{loc}} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}$$

(cf. (4.22)).

We will restore the original meaning of $\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P}$, i.e.,

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_M}{\otimes} \mathrm{KL}(M)_{\mathrm{crit}-\check{\rho}_P}$$

in Sect. 16.6.

16.3.2. Fix a point $\underline{x} \in \mathrm{Ran}$, and recall that the corresponding localization functor

$$\mathrm{Loc}_{M,\underline{x}} : \mathrm{KL}(M)_{\mathrm{crit},\underline{x}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$$

respects the actions of $\mathrm{Sph}_{M,\underline{x}}$.

Hence, it gives rise to a functor, to be denoted $\mathrm{Loc}_M^{-,\mathrm{enh}_{\underline{x}}}$

$$\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\underline{x}} := \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\underline{x}} \underset{\mathrm{Sph}_{M,\underline{x}}}{\otimes} \mathrm{KL}(M)_{\mathrm{crit},\underline{x}} \overset{\mathrm{Id}\otimes \mathrm{Loc}_{M,\underline{x}}}{\longrightarrow} \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\underline{x}} \underset{\mathrm{Sph}_{M,\underline{x}}}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) =: \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\underline{x}}}, \tag{16.10}$$

compatible with the adjunctions

$$\mathrm{KL}(M)_{\mathrm{crit},\underline{x}} \rightleftarrows \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\underline{x}}$$

and

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\underline{x}}}.$$

16.3.3. Making the point $\underline{x}$ vary along Ran, from (16.10) we obtain a functor

$$\mathrm{Loc}_M^{-,\mathrm{enh}_{\mathrm{Ran}}} : \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\mathrm{Ran}}}. \tag{16.11}$$

More generally, for $\mathcal{Z} \to \mathrm{Ran}$, we obtain a functor

$$\mathrm{Loc}_M^{-,\mathrm{enh}_{\mathcal{Z}}} : \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathcal{Z}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\mathcal{Z}}}. \tag{16.12}$$

16.3.4. Consider the space $\mathrm{Ran}_{\subseteq}$, thought of as mapping to Ran using $\mathrm{pr}_{\mathrm{big}}$. Consider the functor

$$\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{ins.unit}}{\longrightarrow} \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathrm{Ran}_{\subseteq}} \overset{\mathrm{Loc}_M^{-,\mathrm{enh}_{\mathrm{Ran}_{\subseteq}}}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\mathrm{Ran}_{\subseteq}}} \overset{\mathrm{Id}\otimes(\mathrm{pr}_{\mathrm{big}})_*}{\longrightarrow} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\mathrm{Ran}}}. \tag{16.13}$$

The following assertion results from the isomorphism (13.17):

Lemma 16.3.5. *The functor* (16.13) *takes values in*

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\mathrm{Ran}}} \underset{\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)\otimes\text{D-mod}(\mathrm{Ran})}{\times} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) =: \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}.$$

16.3.6. Thanks to Lemma 16.3.5 we obtain a well-defined functor, to be denoted

$$\mathrm{Loc}_M^{-,\mathrm{enh}} : \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}. \tag{16.14}$$

By a similar token, we define the corresponding functor in the twisted setting:

$$\mathrm{Loc}_M^{-,\mathrm{enh}} : \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P,\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}. \tag{16.15}$$

The above functors possess the unitality property from Sect. 13.3. Following the conventions of Sect. 13.3.3, for a space $\mathcal{Z}$ mapping to Ran, we will denote by the same symbol $\mathrm{Loc}_M^{-,\mathrm{enh}}$ the resulting functor

$$\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathcal{Z}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}},$$

i.e.,

$$\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathcal{Z}} \overset{\mathrm{Loc}_{M,\mathcal{Z}}^{-,\mathrm{enh}}}{\longrightarrow} \text{D-mod}(Z) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \overset{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathcal{Z},-)\otimes\mathrm{Id}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}, \tag{16.16}$$

and similarly in the $\check{\rho}_P$-shifted case.

Remark 16.3.7. Note that the functor (16.16) can also be written as

$$\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathcal{Z}} \to \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{Loc}_M^{-,\mathrm{enh}}}{\to} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}},$$

where the first arrow is (de Rham) direct image along $\mathcal{Z} \to \mathrm{Ran}$.

16.4. **Compatibility of localization with constant terms–enhanced version.**

16.4.1. The following assertion is an analog of Theorem 15.4.5 for the enhanced constant term functor:

Theorem 16.4.2. *The following diagram of functors commutes:*

$$\begin{array}{ccccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \xrightarrow{\mathrm{CT}_*^{-,\mathrm{enh}}} & & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}_{N(P^-)}^{\otimes -1}}\uparrow & & & & \uparrow{\scriptstyle \mathrm{Loc}_M^{-,\mathrm{enh}}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{ins.unit}} & \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_\subseteq} & \xrightarrow{\mathrm{BRST}^{-,\mathrm{enh}}} & \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P,\mathrm{Ran}_\subseteq}, \end{array}$$

where $\mathrm{BRST}^{-,\mathrm{enh}}$ *is the factorization functor of* (4.20)*, and* $\mathrm{Loc}_M^{-,\mathrm{enh}}$ *is as in* (16.15).

Remark 16.4.3. Note that the counter-clockwise composition in Theorem 16.4.2 can be rewritten as

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \xrightarrow{\mathrm{ins.unit}} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_\subseteq} \xrightarrow{(\mathrm{pr}_{\mathrm{big}})_*} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \xrightarrow{\mathrm{BRST}^{-,\mathrm{enh}}} \\ \to \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P,\mathrm{Ran}} \xrightarrow{\mathrm{Loc}_M^{-,\mathrm{enh}}} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}.$$

16.4.4. The proof of this theorem will occupy the rest of this subsection and the next one.

Recall that we have the natural transformations

(16.17)
$$\begin{array}{ccccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{(\mathfrak{p}^-)^!} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_{P^-}) & \longrightarrow & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}_{N(P^-)}^{\otimes -1}}\uparrow & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_{P^-} \otimes \mathfrak{l}_{N(P^-)}^{\otimes -1}} & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_M} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow[\mathrm{res}]{} & \mathrm{KL}(P^-)_{\mathrm{crit}_G|_{P^-},\mathrm{Ran}} & \xrightarrow[\mathrm{BRST}^-]{} & \mathrm{KL}(M)_{\mathrm{crit}_M-\check{\rho}_P,\mathrm{Ran}}, \end{array}$$

where:

- $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_{P^-}) := \mathrm{D\text{-}mod}_{\det_{\mathrm{Bun}_G}^{\frac{1}{2}}|_{\mathrm{Bun}_{P^-}}}(\mathrm{Bun}_{P^-})$;
- The functor $\mathrm{Loc}_{P^-} : \mathrm{KL}(P^-)_{\mathrm{crit}_G|_{P^-},\mathrm{Ran}} \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_{P^-})$ is defined in a way parallel to (13.3);
- The functor $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_{P^-}) \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$ is

$$(\mathfrak{q}^-)_* \circ (- \otimes \det(T^*(\mathrm{Bun}_{P^-} / \mathrm{Bun}_M))^{\otimes -1})[-\dim.\mathrm{rel}(\mathrm{Bun}_{P^-} / \mathrm{Bun}_M)].$$

Moreover, the resulting natural transformation

(16.18)
$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathrm{CT}_*^-} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}_{N(P^-)}^{\otimes -1}}\uparrow & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_M} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow[\mathrm{BRST}^-]{} & \mathrm{KL}(M)_{\mathrm{crit}_M-\check{\rho}_P,\mathrm{Ran}}, \end{array}$$

becomes an isomorphism after precomposing with

(16.19)
$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \xrightarrow{\mathrm{ins.unit}} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_\subseteq} \xrightarrow{(\mathrm{pr}_{\mathrm{big}})_*} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}}.$$

We will show that the diagram (16.18) can be enhanced to

(16.20)
$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathrm{CT}_*^{-,\mathrm{enh}}} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \\ \uparrow{\scriptstyle \mathrm{Loc}_G \otimes \mathrm{I}^{\otimes -1}_{N(P^-)}} & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_M^{-,\mathrm{enh}}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow[\mathrm{BRST}^{-,\mathrm{enh}}]{} & \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}_{M-\check{\rho}_P},\mathrm{Ran}}. \end{array}$$

This will imply Theorem 16.4.2.

16.4.5. In order to unburden the notation, instead of constructing (16.20), we will be working over a fixed point $\underline{x} \in \mathrm{Ran}$.

Thus, we want to construct the diagram

(16.21)
$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathrm{CT}_*^{-,\mathrm{enh}_{\underline{x}}}} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\underline{x}}} \\ \uparrow{\scriptstyle \mathrm{Loc}_{G,\underline{x}} \otimes \mathrm{I}^{\otimes -1}_{N(P^-)}} & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_M^{-,\mathrm{enh}_{\underline{x}}}} \\ \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} & \xrightarrow[\mathrm{BRST}^{-,\mathrm{enh}}]{} & \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}_{M-\check{\rho}_P},\underline{x}}. \end{array}$$

16.4.6. By duality, the datum of (16.21) is equivalent to that of

$$\begin{array}{ccc} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\underline{x}} \otimes \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{(16.5)} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \\ \uparrow{\scriptstyle \mathrm{Id} \otimes \left(\mathrm{Loc}_{G,\underline{x}} \otimes \mathrm{I}^{\otimes -1}_{N(P^-)}\right)} & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_{M,\underline{x}}} \\ \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\underline{x}} \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} & \longrightarrow & \mathrm{KL}(M)_{\mathrm{crit}_{M-\check{\rho}_P},\underline{x}}, \end{array}$$

where the bottom horizontal arrow is the composition

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\underline{x}} \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \to \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\underline{x}} \underset{\mathrm{Sph}_{G,\underline{x}}}{\otimes} \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \xrightarrow{(4.19)} \mathrm{KL}(M)_{\mathrm{crit}_{M-\check{\rho}_P},\underline{x}}.$$

Equivalently, we need to construct a diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{(16.9)} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \\ \uparrow{\scriptstyle \mathrm{Id}\otimes\left(\mathrm{Loc}_{G,\underline{x}}\otimes \mathrm{I}^{\otimes -1}_{N(P^-)}\right)} & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_{M,\underline{x}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} & \longrightarrow & \mathrm{KL}(M)_{\mathrm{crit}_M - \check{\rho}_P,\underline{x}}, \end{array} \tag{16.22}$$

where the bottom horizontal arrow is the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} \overset{\star}{\to} \widehat{\mathfrak{g}}\text{-mod}_{\mathrm{crit}}^{\mathfrak{L}^+(P^-)_{\underline{x}}} \overset{\mathrm{BRST}}{\to} \mathrm{KL}(M)_{\mathrm{crit}_M - \check{\rho}_P,\underline{x}}.$$

We will construct the desired diagram (16.22) in Sect. 16.5.6.

16.5. **Localization in the presence of level structure.**

16.5.1. Fix a point $\underline{x} \in \mathrm{Ran}$, and the scheme

$$\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}.$$

For a level κ, we will consider the corresponding category of twisted D-modules

$$\text{D-mod}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}).$$

It is acted on by the group $\mathfrak{L}(G)_{\underline{x}}$ at level κ.

16.5.2. Parallel to the functor

$$\mathrm{Loc}_{G,\kappa,\underline{x}} : \mathrm{KL}(G)_{\kappa,\underline{x}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G),$$

there exists a functor

$$\mathrm{Loc}_{G,\kappa,\underline{x}} : \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}} \to \text{D-mod}_\kappa(\mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}),$$

compatible with the actions of $\mathfrak{L}(G)_{\underline{x}}$.

In particular, for a subgroup $H \subset \mathfrak{L}^+(G)_{\underline{x}}$, we have a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_\kappa(\mathrm{Gr}_{G,\underline{x}})^H \otimes \text{D-mod}_\kappa(\mathrm{Bun}_G) & \xrightarrow{\star} & \text{D-mod}_\kappa(H\backslash \mathrm{Bun}_G^{\mathrm{level}_{\underline{x}}}) \\ \uparrow{\scriptstyle \mathrm{Id}\otimes \mathrm{Loc}_{G,\kappa,\underline{x}}} & & \uparrow{\scriptstyle \mathrm{Loc}_{G,\kappa,\underline{x}}} \\ \text{D-mod}_\kappa(\mathrm{Gr}_{G,\underline{x}})^H \otimes \mathrm{KL}(G)_{\kappa,\underline{x}} & \xrightarrow{\star} & \widehat{\mathfrak{g}}\text{-mod}^H_{\kappa,\underline{x}} \end{array} \tag{16.23}$$

16.5.3. Consider the particular case of (16.23) when $H = \mathfrak{L}^+(P^-)_{\underline{x}}$. We obtain a commutative diagram:

$$\begin{array}{ccc} \text{D-mod}_\kappa(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \text{D-mod}_\kappa(\mathrm{Bun}_G) & \xrightarrow{\star} & \text{D-mod}_\kappa(\mathrm{Bun}_G \underset{\mathrm{pt}/\mathfrak{L}^+(G)_{\underline{x}}}{\times} \mathrm{pt}/\mathfrak{L}^+(P^-)_{\underline{x}}) \\ \uparrow{\scriptstyle \mathrm{Id}\otimes \mathrm{Loc}_{G,\kappa,\underline{x}}} & & \uparrow{\scriptstyle \mathrm{Loc}_{G,\kappa,\underline{x}}} \\ \text{D-mod}_\kappa(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \mathrm{KL}(G)_{\kappa,\underline{x}} & \xrightarrow{\star} & \widehat{\mathfrak{g}}\text{-mod}_{\kappa,\underline{x}}^{\mathfrak{L}^+(P^-)_{\underline{x}}}. \end{array} \tag{16.24}$$

16.5.4. Note now that the map

$$\mathfrak{p}^- : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G$$

naturally factors via a map

$$'\mathfrak{p}^- : \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G \underset{\mathrm{pt}/\mathfrak{L}^+(G)_{\underline{x}}}{\times} \mathrm{pt}/\mathfrak{L}^+(P^-)_{\underline{x}}.$$

As in (13.31), we obtain a diagram

$$\begin{array}{ccc} \text{D-mod}_\kappa(\mathrm{Bun}_G \underset{\mathrm{pt}/\mathfrak{L}^+(G)_{\underline{x}}}{\times} \mathrm{pt}/\mathfrak{L}^+(P^-)_{\underline{x}}) & \xrightarrow{('\mathfrak{p}^-)^!_\kappa} & \text{D-mod}_\kappa(\mathrm{Bun}_{P^-}) \\ \uparrow{\scriptstyle \mathrm{Loc}_{G,\kappa,\underline{x}}} & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_{P^-,\kappa,\underline{x}}} \\ \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(P^-)_{\underline{x}}}_{\mathrm{crit},\underline{x}} & \longrightarrow & \mathrm{KL}(P^-)_{\mathrm{crit}_G|_{P^-},\mathrm{Ran}}, \end{array} \tag{16.25}$$

where the bottom horizontal arrow is the natural restriction functor.

16.5.5. Specializing to the critical level, (16.24) is equivalent to

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\star} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G \underset{\mathrm{pt}/\mathfrak{L}^+(G)_{\underline{x}}}{\times} \mathrm{pt}/\mathfrak{L}^+(P^-)_{\underline{x}} \\ {\scriptstyle \mathrm{Id}\otimes \mathrm{Loc}_{G,\underline{x}}}\uparrow & & \uparrow{\scriptstyle \mathrm{Loc}_{G,\underline{x}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(P^-)_{\underline{x}}} \otimes \mathrm{KL}(G)_{\mathrm{crit},\underline{x}} & \xrightarrow{\star} & \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(P^-)_{\underline{x}}}_{\mathrm{crit},\underline{x}}, \end{array} \tag{16.26}$$

and (16.25) is equivalent to

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G \underset{\mathrm{pt}/\mathfrak{L}^+(G)_{\underline{x}}}{\times} \mathrm{pt}/\mathfrak{L}^+(P^-)_{\underline{x}}) & \xrightarrow{('\mathfrak{p}^-)^!} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{P^-}) \\ \uparrow{\scriptstyle \mathrm{Loc}_{G,\underline{x}}} & \swarrow & \uparrow{\scriptstyle \mathrm{Loc}_{P^-,\underline{x}}} \\ \widehat{\mathfrak{g}}\text{-mod}^{\mathfrak{L}^+(P^-)_{\underline{x}}}_{\mathrm{crit},\underline{x}} & \longrightarrow & \mathrm{KL}(P^-)_{\mathrm{crit}_G|_{P^-},\mathrm{Ran}}. \end{array} \tag{16.27}$$

16.5.6. *End of proof of Theorem 16.4.2.* We are now ready to construct (16.22) and thereby prove Theorem 16.4.2.

Namely, (16.22) is obtained by horizontally concatenating the diagram (16.26) with (16.27) and the right square in (16.17).

□

16.6. **Enhanced categories–twisted versions.**

16.6.1. We can repeat the contents of the preceding subsections when we replace the category $\mathrm{I}(G,P^-)^{\mathrm{loc}}$ by its twisted version $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)}$.

We consider $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)}$ as acted on by Sph_M according to the convention in Sect. 2.3.5.

Recall also that we have an equivalence

$$\mathrm{I}(G,P^-)^{\mathrm{loc}} \xrightarrow{\alpha_{\rho_P(\omega_X),\mathrm{taut}}} \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)}, \tag{16.28}$$

which is compatible with the actions of Sph_G, and is compatible with the actions of Sph_M via the automorphism

$$\mathrm{Sph}_M \overset{\mathrm{transl}^*_{\rho_P(\omega_X)}}{\simeq} \mathrm{Sph}_M\,. \tag{16.29}$$

16.6.2. Denote the resulting global enhanced category by

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)}.$$

We denote by the same symbols

$$\mathbf{ind}_{\mathrm{enh}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} : \mathbf{oblv}_{\mathrm{enh}} \tag{16.30}$$

the resulting pair of adjoint functors.

16.6.3. Here is another way to think about $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}$:

Let $\widetilde{\Omega}_{\rho_P(\omega_X)}$ be as in Sect. 2.3.2. Let

$$\widetilde{\Omega}_{\rho_P(\omega_X),\mathrm{Ran}} \in \mathrm{Sph}_{M,\mathrm{Ran}}$$

be the corresponding associative algebra object.

Then the adjunction (16.30) identifies with

$$\mathbf{ind}_{\widetilde{\Omega}_{\rho_P(\omega_X),\mathrm{Ran}}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \rightleftarrows \widetilde{\Omega}_{\rho_P(\omega_X),\mathrm{Ran}}\text{-mod}(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)) : \mathbf{oblv}_{\widetilde{\Omega}_{\rho_P(\omega_X),\mathrm{Ran}}}.$$

16.6.4. Note that the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \overset{\mathrm{transl}^*_{\rho_P(\omega_X)}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$$

is also compatible with the Sph_M-actions via (16.29).

Hence, tensoring $\mathrm{transl}^*_{\rho_P(\omega_X)}$ with (16.28), we obtain an equivalence

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)}, \tag{16.31}$$

to be denoted

$$(\mathrm{transl}^*_{\rho_P(\omega_X)})^{-,\mathrm{enh}},$$

which makes the diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathrm{transl}^*_{\rho_P(\omega_X)}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \\ {\scriptstyle \mathbf{oblv}_{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}_{\mathrm{enh}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xrightarrow{(\mathrm{transl}^*_{\rho_P(\omega_X)})^{-,\mathrm{enh}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \end{array}$$

commute.

16.6.5. A twisted version of the functor $\mathrm{CT}^{-,\mathrm{enh}}_*$ gives rise to a functor

$$\mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)},$$

so that the diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xrightarrow{(\mathrm{transl}^*_{\rho_P(\omega_X)})^{-,\mathrm{enh}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \\ {\scriptstyle \mathrm{CT}^{-,\mathrm{enh}}_*}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathrm{Id}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \end{array}$$

commutes.

16.6.6. Let $\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P}$ be as in (4.22) (see Sect. 16.3.1 for our notational conventions). We have the corresponding localization functor

$$\mathrm{Loc}^{-,\mathrm{enh}}_{M,\rho_P(\omega_X)} : \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}. \tag{16.32}$$

16.6.7. The following assertion follows formally from Theorem 16.4.2 by applying the functor $\mathrm{transl}^*_{\rho_P(\omega_X)}$:

Theorem 16.6.8. *The following diagram of functors commutes:*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}_{G,P^-,M,\rho_P(\omega_X)}}\uparrow & & \uparrow{\scriptstyle \mathrm{Loc}^{-,\mathrm{enh}}_{M,\rho_P(\omega_X)}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{ins.unit}} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}_\subseteq} \xrightarrow{\mathrm{BRST}^{-,\mathrm{enh}}_{\rho_P(\omega_X)}} & \mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\check{\rho}_P,\mathrm{Ran}_\subseteq}, \end{array}$$

where $\mathrm{BRST}^{-,\mathrm{enh}}_{\rho_P(\omega_X)}$ *is the factorization functor of* (4.23)*, and* $\mathrm{Loc}^{-,\mathrm{enh}}_{M,\rho_P(\omega_X)}$ *is as in* (16.32)*.*

17. Spectral Poincaré and global section functors

In this section we start dealing with the local-to-global constructions on the spectral side, i.e., when the recipient category is $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X))$.

We introduce two versions of the spectral Poincaré functor:

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$$

and

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

However, we show (Theorem 17.2.4) that they are intertwined by the "self-duality" functor

$$\Theta_{\mathrm{Op}(\check{G})} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)),$$

up to tensoring by a graded line.

Next we recall the definition of the *spectral localization and global sections functors*

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) : \Gamma^{\mathrm{spec}}_{\check{G}}.$$

Finally, we give the expression for the composition

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \overset{\Gamma^{\mathrm{spec}}_{\check{G}}}{\longrightarrow} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}$$

via chiral homology, which exactly matches the composition (14.1) under $\mathrm{FLE}_{G,\mathrm{crit}}$ and $\mathrm{FLE}_{\check{G},\infty}$.

17.1. The spectral Poincaré functor.

17.1.1. The spectral Poincaré functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X))$$

is comprised of the functors

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)), \quad \underline{x} \in \mathrm{Ran},$$

where

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}} := (\pi_{\underline{x}})_* \circ (s_{\underline{x}})^!$$

for the morphisms

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}) \overset{s_{\underline{x}}}{\leftarrow} \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X - \underline{x}) \overset{\pi_{\underline{x}}}{\to} \mathrm{LS}_{\check{G}}(X)$$

and

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X - \underline{x}) := \mathrm{Op}_{\check{G}}(X - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(X-\underline{x})}{\times} \mathrm{LS}_{\check{G}}(X).$$

17.1.2. The functor $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}$ has the unitality property spelled out in Sect. 13.3. Concretely, this means the following (cf. Sect. 12.3.8):

Fix a point $\underline{x}, \underline{x}' \in \mathrm{Ran}$ be two points with $\underline{x} \subset \underline{x}'$. Then the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) & \longrightarrow & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'})) \\ \downarrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}}} & & \downarrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}'}} \\ \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)) & \xrightarrow{=} & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)), \end{array} \tag{17.1}$$

commutes, where the top horizontal arrow is given by pull-push along

$$\begin{array}{ccccc} & & \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}) & & \\ & \swarrow & & \searrow & \\ \mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}) & & & & \mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}) \\ \| & & & & \| \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}) & & & & \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'}), \end{array} \tag{17.2}$$

in which the slanted arrows are given by restriction along the inclusions

$$\mathcal{D}^\times_{\underline{x}} \to (\mathcal{D}_{\underline{x}'} - \underline{x}) \leftarrow \mathcal{D}^\times_{\underline{x}'},$$

respectively.

Indeed, the commutativity of (17.2) follows from the fact that the diagram

$$\begin{array}{ccc} \mathrm{Op}_{\check{G}}(X - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(X - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(X) & \longrightarrow & \mathrm{Op}_{\check{G}}(X - \underline{x}') \underset{\mathrm{LS}_{\check{G}}(X - \underline{x}')}{\times} \mathrm{LS}_{\check{G}}(X) \\ \downarrow & & \downarrow \\ \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}) & \longrightarrow & \mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}) \end{array}$$

is Cartesian.

17.1.3. A basic property of the spectral Poincaré functor is the following:

Proposition 17.1.4. *The essential image of the functor* $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}$ *lies in*

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \subset \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)).$$

Proof. Fix a point $\underline{x} \in \mathrm{Ran}$ and x_0, disjoint from $\underline{x}$, and denote $\underline{x}' = \underline{x} \cup x_0$. Let us interpret the functor $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}}$ via the corresponding diagram (17.1).

Note that all terms in the diagram

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x} \cup x_0}) \overset{s_{\underline{x} \cup x_0}}{\leftarrow} \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X - (\underline{x} \cup x_0)) \overset{\pi_{\underline{x} \cup x_0}}{\to} \mathrm{LS}_{\check{G}}(X)$$

are acted on by the spectral Hecke groupoid $\mathrm{IndCoh}(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},x_0})$, and hence, the the functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},\underline{x} \cup x_0} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \otimes \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{x_0})) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X))$$

is manifestly equivariant with respect to this action.

Hence, it is sufficient to show that the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x_0}$ on $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{x_0}))$ factors via

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},x_0} \to \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp},x_0}$$

(see Sect. 6.4.1) for the notation.

However, this follows from the fact that the compact generators of $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{x_0}))$ can be obtained as !-pullbacks of objects in $\mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{x_0}))$.

□

Remark 17.1.5. Note that the above fact that the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{G}}$ on $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))$ factors through $\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{temp}}$ is a spectral counterpart of the fact that the action of Sph_G on $\mathrm{KL}(G)_{\mathrm{crit}}$ factors through $\mathrm{Sph}_{G,\mathrm{temp}}$, see Remark 7.3.6.

17.2. **Another version of the spectral Poincaré functor.**

17.2.1. We now consider the category $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}}$, and a functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)),$$

comprised of the functors

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x}} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)), \quad \underline{x} \in \mathrm{Ran},$$

where

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x}} = (\pi_{\underline{x}})_* \circ (s_{\underline{x}})^*,$$

for the morphisms $\pi_{\underline{x}}$ and $s_{\underline{x}}$ as in Sect. 17.1.1.

17.2.2. Denote by $\mathfrak{l}_{\mathrm{Kost}(\check{G})}$ the (non-graded) line

$$(\det \Gamma(X, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}))^{\otimes -1},$$

where $\mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}$ is as in Sect. 5.3.3.

17.2.3. Recall now that we have a canonical equivalence

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \overset{\Theta_{\mathrm{Op}(\check{G})}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}}, \tag{17.3}$$

see Sect. 5.3.

The main result of the present section reads:

Theorem 17.2.4. *There is an isomorphism of functors*

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!} \simeq \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*} \circ \Theta_{\mathrm{Op}(\check{G})} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[\delta_G].$$

In the above theorem and elsewhere

$$\delta_G := \dim(\mathrm{Bun}_G) = (g-1) \cdot \dim(G).$$

17.2.5. The rest of this subsection is devoted to the proof of this theorem.

To simplify the notation, we will establish the required isomorphism at a fixed point $\underline{x} \in \mathrm{Ran}$, i.e.,

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}} \simeq \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x}} \circ \Theta_{\mathrm{Op}(\check{G})} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[\delta_G], \tag{17.4}$$

with respect to the equivalence

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \overset{\Theta_{\mathrm{Op}(\check{G})}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \tag{17.5}$$

of Sect. 5.3.

Note that by the construction of $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}}$ and $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x}}$, it is enough to establish the commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) & \xleftarrow{(17.5)} & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \\ \downarrow{\scriptstyle s^*_{\underline{x}}} & & \downarrow{\scriptstyle s^!_{\underline{x}}} \\ \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})) & \xleftarrow{\mathrm{Id}\otimes \mathfrak{l}^{\otimes -1}_{\mathrm{Kost}(\check{G})}[-\delta_G]} & \mathrm{IndCoh}(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})). \end{array} \tag{17.6}$$

17.2.6. Recall that the equivalence (17.5) is such that the equivalence (17.5) is such that the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) & \xrightarrow{\Theta_{\mathrm{Op}(\check{G})}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \\ \uparrow & & \uparrow \\ \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) & \xrightarrow{\Theta_{\mathrm{Op}(\check{G})}} & \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \end{array}$$

commutes, where the bottom horizontal arrow is the equivalence of Lemma 5.3.5, and the vertical arrows are given by !-pullback.

Hence, by base change, the commutativity of (17.6) follows from the commutativity of the next diagram:

$$\begin{array}{ccc} \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) & \xleftarrow{\Theta_{\mathrm{Op}(\check{G})}} & \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \\ \downarrow{\scriptstyle s^*_{\underline{x}}} & & \downarrow{\scriptstyle s^!_{\underline{x}}} \\ \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(X-\underline{x})) & \xleftarrow{\mathrm{Id}\otimes \mathfrak{l}^{\otimes -1}_{\mathrm{Kost}(\check{G})}[-\delta_G]} & \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(X-\underline{x})), \end{array} \tag{17.7}$$

where by a slight abuse of notation we denote by the same symbol $s_{\underline{x}}$ the map

$$\mathrm{Op}_{\check{G}}(X-\underline{x}) \to \mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}),$$

so that the two instances of this functor are obtained from one another by base change

$$\mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})}{\times} -.$$

17.2.7. Let

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})} \in \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \tag{17.8}$$

be the image of the dualizing sheaf

$$\omega^!_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})} \in \mathrm{IndCoh}^!(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})),$$

see (5.12).

Since $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})$ is ind-pro-smooth, the commutativity of (17.7) is equivalent to the existence of a canonical isomorphism

$$s^*_{\underline{x}}(\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})}) \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[\delta_G] \simeq \omega_{\mathrm{Op}_{\check{G}}(X-\underline{x})}. \tag{17.9}$$

17.2.8. Let us recall the explicit shape of the object (17.8). Recall that the indscheme $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})$ is an affine space with respect to the Tate vector space

$$\mathbf{V} := \Gamma(\mathcal{D}^\times_{\underline{x}}, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}).$$

Denote by $\mathbf{L}_0 \subset \mathbf{V}$ the standard lattice, i.e.,

$$\mathbf{L}_0 := \Gamma(\mathcal{D}_{\underline{x}}, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})$$

so that

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}) \simeq \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}}) \overset{\mathbf{L}_0}{\times} \mathbf{V}.$$

Write

$$\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}) = \operatorname*{colim}_{\mathbf{L}} \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}}) \overset{\mathbf{L}_0}{\times} \mathbf{L},$$

where $\mathbf{L}$ runs over the (filtered) poset of lattices containing $\mathbf{L}_0$. Denote

$$\mathrm{Op}^{\mathbf{L}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}) := \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}}) \overset{\mathbf{L}_0}{\times} \mathbf{L}.$$

Then

$$\omega^{*,\mathrm{fake}}_{\mathrm{Op}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}})} \simeq \operatorname*{colim}_{\mathbf{L}} \mathcal{O}_{\mathrm{Op}^{\mathbf{L}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}})} \otimes \det(\mathbf{L}/\mathbf{L}_0)^{\otimes -1}[\dim(\mathbf{L}/\mathbf{L}_0)].$$

17.2.9. Thus, (17.9) is equivalent to a compatible family of isomorphisms

$$\mathcal{O}_{\mathrm{Op}^{\mathbf{L}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}) \cap \mathrm{Op}_{\check{G}}(X-\underline{x})} \otimes \det(\mathbf{L}/\mathbf{L}_0)^{\otimes -1} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[\delta_G + \dim(\mathbf{L}/\mathbf{L}_0)] \simeq \tag{17.10}$$
$$\simeq \omega_{\mathrm{Op}^{\mathbf{L}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}) \cap \mathrm{Op}_{\check{G}}(X-\underline{x})}.$$

Denote by Γ the co-lattice

$$\Gamma(X - \underline{x}, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}),$$

so that $\mathrm{Op}_{\check{G}}(X - \underline{x})$ is an affine space with respect to Γ.

Then $\mathrm{Op}^{\mathbf{L}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}) \cap \mathrm{Op}_{\check{G}}(X - \underline{x})$ is an affine space with respect to $\mathbf{L} \cap \Gamma$, and we have

$$\omega_{\mathrm{Op}^{\mathbf{L}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}) \cap \mathrm{Op}_{\check{G}}(X-\underline{x})} \simeq \mathcal{O}_{\mathrm{Op}^{\mathbf{L}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}) \cap \mathrm{Op}_{\check{G}}(X-\underline{x})} \otimes \det(\mathbf{L} \cap \Gamma)^{\otimes -1}[\dim(\mathbf{L} \cap \Gamma)].$$

17.2.10. Thus, (17.10) reduces to an identification of graded lines:

$$\det(\mathbf{L}/\mathbf{L}_0)^{\otimes -1} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[\delta_G + \dim(\mathbf{L}/\mathbf{L}_0)] \simeq \det(\mathbf{L} \cap \Gamma)^{\otimes -1}[\dim(\mathbf{L} \cap \Gamma)],$$

which in turn reduces to the existence of an isomorphism

$$\det(\mathbf{L}_0 \cap \Gamma)[-\dim(\mathbf{L}_0 \cap \Gamma)] = \mathfrak{l}^{\otimes -1}_{\mathrm{Kost}(\check{G})}[-\delta_G].$$

However, the latter isomorphism is just the fact that

$$\mathbf{L}_0 \cap \Gamma \simeq \Gamma(X, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X}),$$

combined with the fact that

$$\dim(\Gamma(X, \mathfrak{a}(\check{\mathfrak{g}})_{\omega_X})) = \dim(\mathrm{Bun}_G).$$

□[Theorem 17.2.4]

17.3. **The spectral localization and global sections functors.**

17.3.1. The spectral localization functor

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$$

is defined as pull-push along the diagram

$$\mathrm{LS}_{\check{G}}(\mathcal{D})_{\mathrm{Ran}} \leftarrow \mathrm{LS}_{\check{G}}(X) \times \mathrm{Ran} \to \mathrm{LS}_{\check{G}}(X), \tag{17.11}$$

where:

- We identify $\mathrm{Rep}(\check{G})$ and $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathcal{D}))$ as factorization categories;
- The map $\mathrm{LS}_{\check{G}}(X) \times \mathrm{Ran} \to \mathrm{LS}_{\check{G}}(\mathcal{D})_{\mathrm{Ran}}$ is comprised of the maps $\mathrm{LS}_{\check{G}}(X) \times \underline{x} \to \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}})$, given by restriction.

The functor $\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$ possesses the unitality property spelled out in Sect. 13.3.

17.3.2. The functor $\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}$ admits a right adjoint, denoted

$$\Gamma^{\mathrm{spec}}_{\check{G}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}},$$

obtained by applying pull-push along (17.11) in the opposite direction.

Explicitly, for a given $\underline{x} \in \mathrm{Ran}$, the corresponding functor

$$\Gamma^{\mathrm{spec}}_{\check{G},\underline{x}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{Rep}(\check{G})_{\underline{x}}$$

is given by *-direct image along

$$\mathrm{LS}_{\check{G}}(X) \to \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}).$$

17.3.3. Note also that the categories $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ and $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ are both canonically self-dual, and with respect to these dualities, we have

$$(\mathrm{Loc}^{\mathrm{spec}}_{\check{G}})^\vee \simeq \Gamma^{\mathrm{spec}}_{\check{G}}.$$

17.4. **Composing spectral Poincaré and global sections functors.**

17.4.1. Our current goal is to study the composite functor

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \xrightarrow{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \xrightarrow{\Gamma^{\mathrm{spec}}_{\check{G}}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}. \tag{17.12}$$

Applying the canonical self-duality of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, the datum of the functor (17.12) is equivalent to the datum of the pairing

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect},$$

given by

$$\begin{aligned}\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} &\xrightarrow{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}\otimes \mathrm{Id}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \\ &\xrightarrow{\Gamma^{\mathrm{spec}}_{\check{G}}\otimes \mathrm{Id}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}\,.\end{aligned} \tag{17.13}$$

We will prove (cf. Theorem 14.2.3):

Theorem 17.4.2. *The functor* (17.13) *identifies canonically with*

$$\begin{aligned}\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} &\xrightarrow{\mathrm{ins.unit}\otimes \mathrm{ins.unit}} \\ &\to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}_\subseteq} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}_\subseteq} \to \\ &\to \Big(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \otimes \mathrm{Rep}(\check{G})\Big)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \xrightarrow{\mathsf{P}^{\mathrm{loc,enh,coarse}}_{\check{G}}} \\ &\to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \xrightarrow{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})},-)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq}} \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq) \to \\ &\xrightarrow{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq,-)} \mathrm{Vect},\end{aligned} \tag{17.14}$$

where $\mathsf{P}^{\mathrm{loc,enh,coarse}}_{\check{G}}$ *is the functor introduced in Sect. 7.5.5.*

Remark 17.4.3. Note that the functor (17.14), appearing in Theorem 17.4.2 can also be rewritten as

$$\begin{aligned}\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} &\xrightarrow{\mathrm{ins.unit}\otimes \mathrm{ins.unit}} \\ &\to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}_\subseteq} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}_\subseteq} \to \\ &\to \Big(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \otimes \mathrm{Rep}(\check{G})\Big)_{\mathrm{Ran}_\subseteq \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_\subseteq} \xrightarrow{(\mathrm{pr}_{\mathrm{big}} \times \mathrm{pr}_{\mathrm{big}})_*} \\ &\to \Big(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \otimes \mathrm{Rep}(\check{G})\Big)_{\mathrm{Ran}} \xrightarrow{\mathsf{P}^{\mathrm{loc}}_{\check{G}}} \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Ran}) \xrightarrow{\mathrm{C}^\cdot_{\mathrm{dR}}(\mathrm{Ran},-)} \mathrm{Vect}\,.\end{aligned}$$

17.4.4. The rest of this subsection is devoted to the proof of Theorem 17.4.2.

First, using the (non-derived) Satake action, as in the proof of Theorem 14.2.3, we obtain that the assertion of the theorem is equivalent to that of the following:

Theorem 17.4.5. *The functor*

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))_{\mathrm{Ran}} \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \overset{\Gamma(\mathrm{LS}_{\check{G}}(X),-)}{\longrightarrow} \mathrm{Vect}$$

identifies canonically with

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times))_{\mathrm{Ran}} \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \overset{\Gamma^{\mathrm{enh}}(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times),-)}{\longrightarrow}$$
$$\to \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}} \overset{\mathrm{C}^{\mathrm{fact}}_\cdot(X;\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})},-)}{\longrightarrow} \mathrm{Vect}\,.$$

17.4.6. In order to simply the exposition we will replace the situation over Ran by one with a fixed $\underline{x} \in \mathrm{Ran}$. So, we want to show that the composition

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times)_{\underline{x}}) \overset{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*,\underline{x}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \overset{\Gamma(\mathrm{LS}_{\check{G}}(X),-)}{\longrightarrow} \mathrm{Vect} \tag{17.15}$$

identifies canonically with

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times_{\underline{x}})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \simeq \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \overset{\mathrm{C}^{\mathrm{ch}}_\cdot(X,\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})},-)_{\underline{x}}}{\longrightarrow} \mathrm{Vect}\,.$$

17.4.7. The functor (17.15) can be tautologically rewritten as the composition

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times_{\underline{x}})) \overset{s^*_{\underline{x}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X-\underline{x})) \overset{\Gamma(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(X-\underline{x}),-)}{\longrightarrow} \mathrm{Vect},$$

and further, by base change along

$$\begin{array}{ccc} \mathrm{Op}_{\check{G}}^{\text{mon-free}}(X-\underline{x}) & \overset{s_{\underline{x}}}{\longrightarrow} & \mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times_{\underline{x}}) \\ \downarrow & & \downarrow \\ \mathrm{Op}_{\check{G}}(X-\underline{x}) & \overset{s_{\underline{x}}}{\longrightarrow} & \mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}) \end{array}$$

as

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}^{\text{mon-free}}(\mathcal{D}^\times_{\underline{x}})) \to \mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \overset{s^*_{\underline{x}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(X-\underline{x})) \overset{\Gamma(\mathrm{Op}_{\check{G}}(X-\underline{x}),-)}{\longrightarrow} \mathrm{Vect}\,.$$

17.4.8. Thus, we need to establish a canonical isomorphism between the functors

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \overset{s^*_{\underline{x}}}{\longrightarrow} \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(X-\underline{x})) \overset{\Gamma(\mathrm{Op}_{\check{G}}(X-\underline{x}),-)}{\longrightarrow} \mathrm{Vect}$$

and

$$\mathrm{IndCoh}^*(\mathrm{Op}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \simeq \mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})}\text{-mod}^{\mathrm{fact}}_{\underline{x}} \overset{\mathrm{C}^{\mathrm{ch}}_\cdot(X,\mathcal{O}_{\mathrm{Op}_{\check{G}}(\mathcal{D})},-)_{\underline{x}}}{\longrightarrow} \mathrm{Vect}\,.$$

The latter is, however, a general feature of affine D-schemes, as is explained in the next subsection.

17.4.9. Let $\mathcal{A}$ be a commutative factorization algebra on X, and let $\mathcal{Y} := \mathrm{Spec}_X(\mathcal{A})$ be the corresponding affine D-scheme. Assume that $\mathcal{A}$ is locally D-free, i.e., that it is non-canonically isomorphic to

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathrm{Diff}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M}),$$

where $\mathcal{M}$ is a vector bundle on X.

For a point $\underline{x} \in \mathrm{Ran}$, consider the indschemes

$$\mathrm{Sect}(\mathcal{D}^\times_{\underline{x}}, \mathcal{Y}) \text{ and } \mathrm{Sect}(X-\underline{x}, \mathcal{Y})$$

of horizontal sections of $\mathcal{Y}$ over $\mathcal{D}^\times_{\underline{x}}$ and $X-\underline{x}$, respectively. Let $s_{\underline{x}}$ denote the closed embedding

$$\mathrm{Sect}(X-\underline{x}, \mathcal{Y}) \hookrightarrow \mathrm{Sect}(\mathcal{D}^\times_{\underline{x}}, \mathcal{Y}).$$

Note that the functor of global sections

$$\Gamma(\mathrm{Sect}(\mathcal{D}^{\times}_{\underline{x}}, \mathcal{Y}), -) : \mathrm{IndCoh}^*(\mathrm{Sect}(\mathcal{D}^{\times}_{\underline{x}}, \mathcal{Y})) \to \mathrm{Vect}$$

enhances naturally to a functor

$$\Gamma^{\mathrm{enh}}(\mathrm{Sect}(\mathcal{D}^{\times}_{\underline{x}}, \mathcal{Y}), -) : \mathrm{IndCoh}^*(\mathrm{Sect}(\mathcal{D}^{\times}_{\underline{x}}, \mathcal{Y})) \to \mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\underline{x}}.$$

We have:

Lemma 17.4.10. *The functor*

$$\mathrm{IndCoh}^*(\mathrm{Sect}(\mathcal{D}^{\times}_{\underline{x}}, \mathcal{Y})) \xrightarrow{s^*_{\underline{x}}} \mathrm{IndCoh}^*(\mathrm{Sect}(X - \underline{x}, \mathcal{Y})) \xrightarrow{\Gamma(\mathrm{Sect}(X-\underline{x}, \mathcal{Y}), -)} \mathrm{Vect}$$

is canonically isomorphic to

$$\mathrm{IndCoh}^*(\mathrm{Sect}(\mathcal{D}^{\times}_{\underline{x}}, \mathcal{Y})) \xrightarrow{\Gamma^{\mathrm{enh}}(\mathrm{Sect}(\mathcal{D}^{\times}_{\underline{x}}, \mathcal{Y}), -)} \mathcal{A}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\underline{x}} \xrightarrow{\mathrm{C}^{\mathrm{ch}}_{\cdot}(X, \mathcal{A}, -)_{\underline{x}}} \mathrm{Vect}\,.$$

□[Theorem 17.4.2]

17.5. **The twisted case.**

17.5.1. Fix a $Z^0_{\check{G}}$-torsor $\mathcal{P}_{Z^0_{\check{G}}}$ on X, and consider the corresponding D-scheme $\mathrm{Op}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}$. Mimicking Sects. 17.1.1 and 17.2.1 we define the functors

$$\mathrm{Poinc}_{\check{G},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^{\times}))_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$$

and

$$\mathrm{Poinc}_{\check{G},*} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^{\times}))_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)).$$

The assertion of Theorem 17.2.4 translates verbatim to the present context.

17.5.2. We have the following counterpart of Theorem 17.4.2:

Theorem 17.5.3. *The functor*

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^{\times}))_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*} \otimes \mathrm{Id}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to$$

$$\xrightarrow{\Gamma^{\mathrm{spec}}_{\check{G}} \otimes \mathrm{Id}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}\,.$$

identifies canonically with

$$\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^{\times}))_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\mathrm{ins.unit} \otimes \mathrm{ins.unit}}$$

$$\to \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^{\times}))_{\mathrm{Ran}_{\subseteq}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}_{\subseteq}} \to$$

$$\to \left(\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^{\times})) \otimes \mathrm{Rep}(\check{G})\right)_{\mathrm{Ran}_{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_{\subseteq}} \xrightarrow{\mathsf{P}^{\mathrm{loc,enh,coarse}}_{\check{G}}}$$

$$\to \mathcal{O}_{\mathrm{Op}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D})}\text{-}\mathrm{mod}^{\mathrm{fact}}_{\mathrm{Ran}_{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_{\subseteq}} \xrightarrow{\mathrm{C}^{\mathrm{fact}}_{\cdot}(X; \mathcal{O}_{\mathrm{Op}_{\check{G}, \mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D})}, -)_{\mathrm{Ran}_{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_{\subseteq}}} \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Ran}_{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_{\subseteq}) \to$$

$$\xrightarrow{\mathrm{C}^{\cdot}_{\mathrm{dR}}(\mathrm{Ran}_{\subseteq} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_{\subseteq}, -)} \mathrm{Vect},$$

where $\mathsf{P}^{\mathrm{loc,enh,coarse}}_{\check{G}}$ *is the corresponding coarsened version of the functor* (7.31).

18. Spectral Poincaré and constant terms functors

In this section we introduce the spectral constant term functor

$$\mathrm{CT}^{-,\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$$

and establish its compatibility with the spectral Poincaré functors

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$$

and

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{M},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))_{\mathrm{Ran}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

This is the spectral counterpart of the compatibility of localization and constant term functors, studied in Sect. 15. The two pictures will be intertwined by the Langlands functor, see diagram (21.2).

18.1. **The spectral constant term functor.**

18.1.1. Consider the maps

$$\mathrm{LS}_{\check{G}}(X) \overset{\mathsf{p}^{\mathrm{glob}}}{\leftarrow} \mathrm{LS}_{\check{P}^-}(X) \overset{\mathsf{q}^{\mathrm{glob}}}{\to} \mathrm{LS}_{\check{M}}(X).$$

We define the spectral Eisenstein functor

$$\mathrm{Eis}^{-,\mathrm{spec}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X))$$

by

$$\mathrm{Eis}^{-,\mathrm{spec}} := (\mathsf{p}^{\mathrm{glob}})_* \circ (\mathsf{q}^{\mathrm{glob}})^*.$$

Here, $(\mathsf{q}^{\mathrm{glob}})^*$ is well-defined as a functor

$$\mathrm{IndCoh}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X)),$$

since the morphism $\mathsf{q}^{\mathrm{glob}}$ has a finite Tor-amplitude (in fact, it is quasi-smooth).

18.1.2. We define the spectral constant term functor

$$\mathrm{CT}^{-,\mathrm{spec}} : \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{M}}(X))$$

as the right adjoint of the functor $\mathrm{Eis}^{-,\mathrm{spec}}$, i.e.,

$$\mathrm{CT}^{-,\mathrm{spec}} := (\mathsf{q}^{\mathrm{glob}})_* \circ (\mathsf{p}^{\mathrm{glob}})^!.$$

18.1.3. It is shown in [AG, Proposition 13.2.6] that the functor $\mathrm{Eis}^{-,\mathrm{spec}}$ sends

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

Similarly, it follows from [AG, Theorem 7.1.3] that the functor $\mathrm{CT}^{-,\mathrm{spec}}$ also sends

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).$$

Thus, the functors $(\mathrm{Eis}^{-,\mathrm{spec}}, \mathrm{CT}^{-,\mathrm{spec}})$ form an adjoint pair

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

18.2. **The spectral Poincaré vs constant term compatibility.**

18.2.1. The goal of the rest of this section is to prove the following result (cf. Theorem 15.4.2):

Theorem 18.2.2. *The following diagram of functors commutes:*

$$\begin{array}{ccc} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) & \xrightarrow{\mathrm{CT}^{-,\mathrm{spec}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}}\uparrow & & \\ \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{M},!}} \\ {\scriptstyle \mathrm{ins.unit}}\downarrow & & \\ \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}\subseteq} & \xrightarrow{J^{-,\mathrm{spec},!}} & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))_{\mathrm{Ran}\subseteq} \end{array}$$

Remark 18.2.3. Note that the counter-clockwise circuit in Theorem 18.2.2 can be rewritten as

$$\begin{aligned} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} &\xrightarrow{\mathrm{ins.unit}} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}\subseteq} \xrightarrow{(\mathrm{pr}_{\mathrm{big}})_*} \\ \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} &\xrightarrow{J^{-,\mathrm{spec},!}} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))_{\mathrm{Ran}} \xrightarrow{\mathrm{Poinc}^{\mathrm{spec}}_{\check{M},!}} \\ &\to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)). \end{aligned}$$

18.2.4. To simplify the notation, we will fix a point $\underline{x} \in \mathrm{Ran}$ and replace the source category by $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}))$. Thus, we need to establish the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) & \xrightarrow{\mathrm{CT}^{-,\mathrm{spec}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!,\underline{x}}}\uparrow & & \\ \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{M},!}} \\ {\scriptstyle \mathrm{ins.unit}}\downarrow & & \\ \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}_{\underline{x}\subseteq}\subseteq} & \xrightarrow{J^{-,\mathrm{spec},!}} & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))_{\mathrm{Ran}_{\underline{x}\subseteq}}, \end{array} \tag{18.1}$$

where the functor ins. unit is comprised of the functors

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})) \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'})), \quad \underline{x} \subset \underline{x}',$$

given by pull-push along the diagram (17.2).

18.2.5. Fur future reference for $\underline{x} \subseteq \underline{x}'$, denote

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x};\underline{x}'}) := \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'} - \underline{x})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}'}) \simeq \mathrm{Op}_{\check{G}}(\mathcal{D}_{\underline{x}'} - \underline{x}) \underset{\mathrm{LS}_{\check{G}}(\mathcal{D}^\times_{\underline{x}})}{\times} \mathrm{LS}_{\check{G}}(\mathcal{D}_{\underline{x}}). \tag{18.2}$$

Let

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}})$$

denote the relative indscheme over $\mathrm{Ran}_{\underline{x}\subseteq}$, whose fiber over $\underline{x}' \in \mathrm{Ran}_{\underline{x}\subseteq}$ is $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x};\underline{x}'})$.

18.3. **The clockwise circuit.**

18.3.1. We first rewrite the clockwise circuit in (18.1). By definition, it is given by !-pull-*-push along the diagram

$$\begin{array}{ccccc} & & \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{LS}_{\check{M}}(X) \\ & & \downarrow & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}) & \longleftarrow & \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X - \underline{x}) & \longrightarrow & \mathrm{LS}_{\check{G}}(X) \end{array}$$

which by base-change can be rewritten as !-pull-*-push along

(18.3)
$$\begin{array}{ccccc} \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{LS}_{\check{M}}(X) \\ \downarrow & & & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}) \longleftarrow & & \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}). & & \end{array}$$

18.3.2. Recall that $\mathrm{MOp}_{\check{G},\check{P}^-}$ denotes the D-scheme of P^--Miura opers (see Sect. 5.4.4). For a point $\underline{y} \in \mathrm{Ran}$, consider the indschemes

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{y}} := \mathrm{MOp}_{\check{G},\check{P}^-}(X-\underline{y}) \underset{\mathrm{LS}_{\check{P}^-}(X-\underline{y})}{\times} \mathrm{LS}_{\check{P}^-}(X)$$

and

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{y}} := \mathrm{MOp}_{\check{G},\check{P}^-}(\mathcal{D}^\times_{\underline{y}}) \underset{\mathrm{LS}_{\check{P}^-}(\mathcal{D}^\times_{\underline{y}})}{\times} \mathrm{LS}_{\check{P}^-}(\mathcal{D}_{\underline{y}}),$$

along with their Ran versions

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\mathrm{Ran}} \text{ and } (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\mathrm{Ran}},$$

and

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\mathrm{Ran}_{\underline{x}\subseteq}} \text{ and } (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\mathrm{Ran}_{\underline{x}\subseteq}},$$

respectively.

18.3.3. For $\underline{x}' \in \mathrm{Ran}_{\underline{x}\subseteq}$ denote

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x};\underline{x}'} := (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x}'} \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}')}{\times} \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x})$$

and

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{x};\underline{x}'} := (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{x}'} \underset{\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}'})}{\times} \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x};\underline{x}'}),$$

where $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x};\underline{x}'})$ is as in (18.2).

Let

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} \text{ and } (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}}$$

denote the corresponding Ran versions.

18.3.4. We have a naturally defined map

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} \to \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{P}^-}(X). \tag{18.4}$$

The key step in establishing the commutativity of (18.1) is the following assertion:

Lemma 18.3.5. *The functor given by !-pull-*-push along* (18.3) *is canonically isomorphic to the functor given by !-pull-*-push along*

(18.5)
$$\begin{array}{ccccc} (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} & \longrightarrow & \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{LS}_{\check{M}}(X) \\ \downarrow & & & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times_{\underline{x}}) \longleftarrow & & \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}). & & \end{array}$$

The proof of the lemma will be given in Sect. 18.5.

18.4. **Morphing into the anti-clockwise circuit.** The rest of the proof will be essentially a diagram chase.

18.4.1. We rewrite the functor pull-push along (18.5) as pull-push and along

$$\begin{array}{ccccccc} (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} & \longleftarrow & (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} & \longrightarrow & \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{LS}_{\check{M}}(X) \\ \downarrow & & & & & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}), & & & & & & \end{array}$$

which we expand as

$$\begin{array}{ccccccc} (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} & \longrightarrow & (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\mathrm{Ran}_{\underline{x}\subseteq}} & \longrightarrow & \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{LS}_{\check{M}}(X) \\ \downarrow & & & & & & \\ (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} & & & & & & \\ \downarrow & & & & & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}}) & & & & & & \\ \downarrow & & & & & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}). & & & & & & \end{array} \tag{18.6}$$

18.4.2. Since the square

$$\begin{array}{ccc} (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} & \longrightarrow & (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\mathrm{Ran}_{\underline{x}\subseteq}} \\ \downarrow & & \downarrow \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}}) & \longrightarrow & \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\mathrm{Ran}_{\underline{x}\subseteq}}) \end{array}$$

is Cartesian, by base change, we rewrite the pull-push along (18.6) as the pull-push alonh

$$\begin{array}{ccccccc} & & (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\mathrm{Ran}_{\underline{x}\subseteq}} & \longrightarrow & \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{LS}_{\check{M}}(X) \\ & & \downarrow & & & & \\ & & (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\mathrm{Ran}_{\underline{x}\subseteq}} & & & & \\ & & \downarrow & & & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}}) & \longrightarrow & \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\mathrm{Ran}_{\underline{x}\subseteq}}) & & & & \\ \downarrow & & & & & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}}). & & & & & & \end{array} \tag{18.7}$$

18.4.3. Note that the pull-push along the lower left corner of (18.7) effects the functor of the lower left vertical arrow in (18.1). Hence, it is enough to compare the resulting two functors with source $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\mathrm{Ran}_{\underline{x}\subseteq}}))$.

To simplify the notation we now fix a point $\underline{x}' \in \mathrm{Ran}_{\underline{x}\subseteq}$. Thus, we need to show that the pull-push along

$$\begin{array}{ccccccc} (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x}'} & \longrightarrow & \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{LS}_{\check{M}}(X) \\ \downarrow & & & & \\ (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{x}'} & & & & \\ \downarrow & & & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}'}) & & & & \end{array} \tag{18.8}$$

is canonically isomorphic to the pull-push along

$$\begin{array}{ccc} & & \mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(X-\underline{x}') \longrightarrow \mathrm{LS}_{\check{M}}(X) \\ & & \downarrow \\ (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{x}'} \longrightarrow & \mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^{\times}_{\underline{x}'}) & \\ \downarrow & & \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^{\times}_{\underline{x}'}) & & \end{array} \tag{18.9}$$

18.4.4. We rewrite the pushforward along the upper row in (18.8) as pushforward along

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x}'} \to \mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(X-\underline{x}') \to \mathrm{LS}_{\check{M}}(X).$$

The desired isomorphism follows now by base change from the fact that the square

$$\begin{array}{ccc} (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x}'} & \longrightarrow & \mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(X-\underline{x}') \\ \downarrow & & \downarrow \\ (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{loc}}_{\underline{x}'} & \longrightarrow & \mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^{\times}_{\underline{x}'}) \end{array}$$

is Cartesian.

18.5. **Proof of Lemma 18.3.5.**

18.5.1. By the projection formula, it is enough to show that the direct image of the dualizing sheaf along (18.4) is isomorphic to the dualizing sheaf on $\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{P}^-}(X)$.

We will show that the map (18.4) is proper with $\mathcal{O}$-contractible fibers.

18.5.2. Consider the fiber product

$$\mathrm{Bun}_{\check{B}}(X-\underline{x}) \underset{\mathrm{Bun}_{\check{G}}(X-\underline{x})}{\times} \mathrm{Bun}_{\check{P}^-}(X)$$

and its open subspace

$$\mathrm{Zast} := \left(\mathrm{Bun}_{\check{B}}(X-\underline{x}) \underset{\mathrm{Bun}_{\check{G}}(X-\underline{x})}{\times} \mathrm{Bun}_{\check{P}^-}(X)\right)^{\text{gen.trans}} \tag{18.10}$$

corresponding to the condition that the B-reduction and the P^--reduction are transversal at the generic point of the curve.

We have the obvious forgetful map

$$\mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{P}^-}(X) \to \mathrm{Bun}_{\check{B}}(X-\underline{x}) \underset{\mathrm{Bun}_{\check{G}}(X-\underline{x})}{\times} \mathrm{Bun}_{\check{P}^-}(X)$$

whose image lands in Zast.

18.5.3. For $\underline{x} \subset \underline{x}'$, denote

$$\mathrm{Zast}^{\circ}_{\underline{x}'} :=$$

$$\mathrm{Bun}_{\check{B}}(X-\underline{x}) \underset{\mathrm{Bun}_{\check{B}}(X-\underline{x}')}{\times} \left(\mathrm{Bun}_{\check{B}}(X-\underline{x}') \underset{\mathrm{Bun}_{\check{G}}(X-\underline{x}')}{\times} \mathrm{Bun}_{\check{P}^-}(X-\underline{x}')\right)^{\text{trans}} \underset{\mathrm{Bun}_{\check{P}^-}(X-\underline{x}')}{\times} \mathrm{Bun}_{\check{P}^-}(X),$$

where the suprescript "trans" indicates transversality on all of $X-\underline{x}'$.

Let $\mathrm{Zast}^{\circ}_{\underline{x}\subseteq}$ be the space over $\mathrm{Ran}_{\underline{x}\subseteq}$, whose fiber over $\underline{x}'$ is $\mathrm{Zast}^{\circ}_{\underline{x}'}$.

We have a tautological map

$$(\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\mathrm{glob}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} \to \mathrm{Zast}^{\circ}_{\underline{x}\subseteq}\,.$$

18.5.4. Restriction defines a map

$$\mathrm{Zast}^{\circ}_{\underline{x}\subseteq} \to \mathrm{Zast} \tag{18.11}$$

that fits into a Cartesian square

$$\begin{array}{ccc} (\mathrm{MOp}^{\text{mon-free}}_{\check{G},\check{P}^-})^{\text{glob}}_{\underline{x};\mathrm{Ran}_{\underline{x}\subseteq}} & \longrightarrow & \mathrm{Zast}^{\circ}_{\underline{x}\subseteq} \\ \downarrow & & \downarrow \\ \mathrm{Op}^{\text{mon-free}}_{\check{G}}(X-\underline{x}) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{P}^-}(X) & \longrightarrow & \mathrm{Zast}\,. \end{array}$$

Hence, it is enough to show that the map (18.11) is proper with $\mathcal{O}$-contractible fibers.

18.5.5. The map (18.11) factors as

$$\mathrm{Zast}^{\circ}_{\underline{x}} \to \mathrm{Zast} \times \mathrm{Ran}_{\underline{x}\subseteq} \to \mathrm{Zast},$$

and we claim that the first arrow is a closed embedding.

Indeed, we can describe $\mathrm{Zast}^{\circ}_{\underline{x}}$ as the subspace of $\mathrm{Zast} \times \mathrm{Ran}_{\underline{x}\subseteq}$, consisting of quadruples

$$\{\mathcal{P}_{\check{B}}, \mathcal{P}_{\check{P}^-}, \check{G} \overset{\check{B}}{\times} \mathcal{P}_{\check{B}} \overset{\beta}{\simeq} \check{G} \overset{\check{P}^-}{\times} \mathcal{P}_{\check{P}^-}|_{X-\underline{x}}, \underline{x}'\}$$

corresponding to the *closed condition* that the isomorphism β is transversal over $X - \underline{x}'$.

This implies that the map (18.11) is proper.

18.5.6. Let us now show that the fibers of (18.11) are $\mathcal{O}$-contractible. For a given point

$$\{\mathcal{P}_{\check{B}}, \mathcal{P}_{\check{P}^-}, \check{G} \overset{\check{B}}{\times} \mathcal{P}_{\check{B}} \overset{\beta}{\simeq} \check{G} \overset{\check{P}^-}{\times} \mathcal{P}_{\check{P}^-}|_{X-\underline{x}}\}$$

of Zast, let $U \subset X - \underline{x}$ be the locus where the isomorphism β is transversal.

Write $U = X - \underline{y}$. Then the fiber of (18.11) over the above point identifies with $\mathrm{Ran}_{\underline{y}\subseteq}$. The $\mathcal{O}$-contractibility assertion follows now from the contractibility of the relative Ran space.

□[Lemma 18.3.5]

19. The enhanced spectral constant term functor

This section is a spectral counterpart of Sect. 16, and it can also be skipped on the first pass, and returned to when necessary.

We introduce the *enhanced recipient category* on the spectral side, denoted

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}},$$

which is essentially obtained by tensoring $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ with $\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{Ran}}$ over $\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$. It is related to $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ by a pair of adjoint functors

$$\mathbf{ind}_{\mathrm{enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} : \mathbf{oblv}_{\mathrm{enh}}.$$

We introduce the enhanced spectral constant term functor

$$\mathrm{CT}^{-,\mathrm{spec,enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$$

so that

$$\mathrm{CT}^{-,\mathrm{spec}} \simeq \mathbf{oblv}_{\mathrm{enh}} \circ \mathrm{CT}^{-,\mathrm{spec,enh}}.$$

We introduce an enhanced spectral Poincaré series functor

$$\mathrm{Poinc}^{-,\mathrm{spec,enh}}_{\check{M},!} : \mathrm{IndCoh}^{!}(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^{\times}))^{-,\mathrm{enh}}_{\mathrm{Ran}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}},$$

and establish an enhanced version of the spectral Poincaré-vs-constant term compatibility from the previous section.

We also introduce *partially enhanced* versions of the above constructions. These have a much more transparent meaning. For example, the corresponding category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}$ is the subcategory

$$\mathrm{IndCoh}_{\check{M}^-\,\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \subseteq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)),$$

singled out by a natural singular support condition. The partially enhanced versions will be needed for the proof of the main result, Theorem 24.1.2.

19.1. **The enhanced recipient category on the spectral side.**

19.1.1. Recall that factorization category $\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}$, equipped with an action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{M},\mathrm{Ran}}$.

Parallel to Sect. 16.1.1, define

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}}} := \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{Ran}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M},\mathrm{Ran}}}{\otimes} (\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \text{D-mod}(\mathrm{Ran})).$$

The (monadic) adjunction

$$\mathbf{ind}_{\mathrm{Sph}\to\frac{\infty}{2}} : \mathrm{Sph}^{\mathrm{spec}}_{\check{M},\mathrm{Ran}} \rightleftarrows \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} : \mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}}$$

gives rise to a (monadic) adjunction

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \text{D-mod}(\mathrm{Ran}) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}}}. \tag{19.1}$$

19.1.2. Parallel to Sect. 16.1.2, for $\mathcal{Z} \to \mathrm{Ran}$, define

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathcal{Z}}} := \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathcal{Z}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M},\mathcal{Z}}}{\otimes} (\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \text{D-mod}(\mathcal{Z})).$$

We have a monadic adjunction

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \text{D-mod}(\mathcal{Z}) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathcal{Z}}}.$$

In particular, for a point $\underline{x} \in \mathrm{Ran}$, we have the category

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\underline{x}}},$$

and a monadic adjunction

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\underline{x}}}.$$

19.1.3. Parallel to Sect. 16.1.3, we define the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$ to be the fiber product

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}}} \underset{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))\otimes\text{D-mod}(\mathrm{Ran})}{\times} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).$$

The (monadic) adjunction (19.1) gives rise to a monadic adjunction

$$\mathbf{ind}_{\mathrm{enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} : \mathbf{oblv}_{\mathrm{enh}}. \tag{19.2}$$

19.1.4. Recall the associative (factorization) algebra $\widetilde{\Omega}^{\mathrm{spec}} \in \mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$, see Sect. 2.4.4. Consider the corresponding associative algebra object

$$\widetilde{\Omega}^{\mathrm{spec}}_{\mathrm{Ran}} \in \mathrm{Sph}^{\mathrm{spec}}_{\check{M},\mathrm{Ran}}.$$

Parallel to Sect. 16.1.5, we can identify

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \simeq \widetilde{\Omega}^{\mathrm{spec}}_{\mathrm{Ran}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))), \tag{19.3}$$

so that the adjunction (19.2) becomes

$$\mathbf{ind}_{\widetilde{\Omega}^{\mathrm{spec}}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightleftarrows \widetilde{\Omega}^{\mathrm{spec}}_{\mathrm{Ran}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))) : \mathbf{oblv}_{\widetilde{\Omega}^{\mathrm{spec}}}.$$

19.1.5. Parallel to Remark 16.1.6, we have the following alternative description of the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$:

Consider the prestack

$$\mathrm{LS}_{\check{G}}(X) \underset{(\mathrm{LS}_{\check{G}}(X))_{\mathrm{dR}}}{\times} \mathrm{LS}_{\check{P}^-}(X))_{\mathrm{dR}},$$

and the category

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,glob}} :=$$

$$= \mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \underset{\mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X))}{\times} \left(\mathrm{IndCoh}\Big(\mathrm{LS}_{\check{G}}(X) \underset{(\mathrm{LS}_{\check{G}}(X))_{\mathrm{dR}}}{\times} (\mathrm{LS}_{\check{P}^-}(X))_{\mathrm{dR}}\Big)\right),$$

where:

- $$\mathrm{IndCoh}\Big(\mathrm{LS}_{\check{G}}(X) \underset{(\mathrm{LS}_{\check{G}}(X))_{\mathrm{dR}}}{\times} (\mathrm{LS}_{\check{P}^-}(X))_{\mathrm{dR}}\Big) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X))$$
  is the functor of pullback along
  $$\mathrm{LS}_{\check{P}^-}(X) \to \mathrm{LS}_{\check{G}}(X) \underset{(\mathrm{LS}_{\check{G}}(X))_{\mathrm{dR}}}{\times} (\mathrm{LS}_{\check{P}^-}(X))_{\mathrm{dR}};$$
- (19.4) $$\mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \subset \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X))$$
  is the full subcategory generated by the essential image of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ along the pullback functor
  $$(\mathfrak{q}^{\mathrm{glob}})^* : \mathrm{IndCoh}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X)).$$

We have the following result, which is a particular case of [Roz, Theorem 4.6.6]:

Theorem 19.1.6. *There exists a canonical equivalence*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \simeq \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,glob}}. \tag{19.5}$$

Under this equivalence, the forgetful functor

$$\mathbf{oblv}_{\mathrm{enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$$

corresponds to the functor

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,glob}} \to \mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \xrightarrow{\mathfrak{q}^{\mathrm{glob}}_*} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)). \tag{19.6}$$

Remark 19.1.7. Since the morphism $\mathfrak{q}^{\mathrm{glob}} : \mathrm{LS}_{\check{P}^-}(X) \to \mathrm{LS}_{\check{M}}(X)$ is co-affine, one can characterize the above subcategory (19.4) by a singular support condition. Namely, it consists of objects whose singular support belongs to

$$\mathrm{Sing}(\mathrm{LS}_{\check{M}}(X)) \underset{\mathrm{LS}_{\check{M}}(X)}{\times} \mathrm{LS}_{\check{P}^-}(X) \subset \mathrm{Sing}(\mathrm{LS}_{\check{P}^-}(X)).$$

19.2. **Partial enhancement.** Consider the functor (19.6). We will now describe the corresponding factorization of the functor $\mathbf{oblv}_{\mathrm{enh}}$ on the other side of the equivalence (19.5).

19.2.1. Set

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}_{\mathrm{Ran}}} := \left(\mathrm{Rep}(\check{P}^-)_{\mathrm{Ran}} \underset{\mathrm{Rep}(\check{M})_{\mathrm{Ran}}}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \text{D-mod}(\mathrm{Ran})\right)$$

and

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} := \tag{19.7}$$

$$= \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}_{\mathrm{Ran}}} \underset{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))\otimes\text{D-mod}(\mathrm{Ran})}{\times} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).$$

Let

$$\mathbf{ind}_{\mathrm{part.enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} : \mathbf{oblv}_{\mathrm{part.enh}} \tag{19.8}$$

denote the corresponding (monadic) adjunction.

19.2.2. The following is an elementary particular case of Theorem 19.1.6:

Proposition 19.2.3. *There exists a canonical equivalence*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \simeq \mathrm{IndCoh}_{\check{M}-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)).$$

Under this equivalence, the forgetful functor $\mathbf{oblv}_{\mathrm{part.enh}}$ *corresponds to the functor* $\mathsf{q}^{\mathrm{glob}}_*$.

19.2.4. *Proof of Proposition 19.2.3.* Recall the commutative (factorization) algebra $\Omega^{\mathrm{spec}} \in \mathrm{Rep}(\check{M})$, see Sect. 2.5.2. Let

$$\Omega^{\mathrm{spec}}_{\mathrm{Ran}} \in \mathrm{Rep}(\check{M})_{\mathrm{Ran}}$$

be the corresponding (commutative) algebra object.

As in Sect. 19.1.4, we can identify

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \simeq \Omega^{\mathrm{spec}}_{\mathrm{Ran}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))), \tag{19.9}$$

where $\mathrm{Rep}(\check{M})_{\mathrm{Ran}}$ acts on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ via

$$\mathrm{nv} : \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \to \mathrm{Sph}^{\mathrm{spec}}_{\check{M},\mathrm{Ran}}.$$

Under the equivalence (19.9), the adjunction (19.8) corresponds to

$$\mathbf{ind}_{\Omega^{\mathrm{spec}}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightleftarrows \Omega^{\mathrm{spec}}_{\mathrm{Ran}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))) : \mathbf{oblv}_{\Omega^{\mathrm{spec}}}.$$

A version of Lemma 17.4.10 for *co-affine* morphisms (the morphism in question is $\mathsf{q} : \mathrm{LS}_{\check{P}^-} \to \mathrm{LS}_{\check{M}}$) shows that the monad on the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$, defined by the action of $\Omega^{\mathrm{spec}}_{\mathrm{Ran}}$ identifies with the one given by the action of

$$\mathsf{q}_*(\mathcal{O}_{\mathrm{LS}_{\check{P}^-}(X)}) \in \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)).$$

This makes the assertion of Proposition 19.2.3 manifest.

□[Proposition 19.2.3]

19.2.5. Note that the forgetful functor

$$\mathbf{oblv}_{\frac{\infty}{2}\to\mathrm{Sph}} : \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \to \mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$$

factors as

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \to \mathrm{Rep}(\check{P}^-) \underset{\mathrm{Rep}(\check{M})}{\otimes} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \xrightarrow{\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-),-)\otimes\mathrm{Id}} \mathrm{Rep}(\check{M}) \underset{\mathrm{Rep}(\check{M})}{\otimes} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} = \mathrm{Sph}^{\mathrm{spec}}_{\check{M}}.$$

Indeed, this follows from interpreting $\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}$ as $\widetilde{\Omega}^{\mathrm{spec}}$-mod$(\mathrm{Sph}_{\check{M}})$, using the homomorphism

$$\mathrm{nv}(\Omega^{\mathrm{spec}}) \to \widetilde{\Omega}^{\mathrm{spec}}$$

and the identification

$$\Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Rep}(\check{M})) \simeq \mathrm{Rep}(\check{P}^-).$$

The above factorization allows us to factor the functor $\mathbf{oblv}_{\mathrm{enh}}$ as

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \xrightarrow{\mathrm{full}\to\mathrm{part}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \xrightarrow{\mathbf{oblv}_{\mathrm{part.enh}}} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)). \tag{19.10}$$

19.2.6. One can view the factorization (19.10) in terms of the identifications (19.3) and (19.9) as

$$\widetilde{\Omega}^{\mathrm{spec}}_{\mathrm{Ran}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))) \overset{\mathbf{oblv}_{\widetilde{\Omega}^{\mathrm{spec}}_{\mathrm{Ran}} \to \Omega^{\mathrm{spec}}_{\mathrm{Ran}}}}{\longrightarrow}$$
$$\to \Omega^{\mathrm{spec}}_{\mathrm{Ran}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))) \overset{\mathbf{oblv}_{\Omega^{\mathrm{spec}}_{\mathrm{Ran}}}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).$$

Remark 19.2.7. One can show that the factorization (19.10) indeed corresponds under the equivalence (19.5) to the given factorization of the functor (19.6).

19.3. **The enhanced spectral constant term functor.** In this subsection we will upgrade the spectral constant term functor

$$\mathrm{CT}^{-,\mathrm{spec}} := \mathsf{q}^{\mathrm{glob}}_* \circ (\mathsf{p}^{\mathrm{glob}})^!, \quad \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$$

to a functor

$$\mathrm{CT}^{-,\mathrm{spec,enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$$

so that

$$\mathrm{CT}^{-,\mathrm{spec}} \simeq \mathbf{oblv}_{\mathrm{enh}} \circ \mathrm{CT}^{-,\mathrm{spec,enh}}.$$

19.3.1. To simplify the notation, we will fix a point $\underline{x} \in \mathrm{Ran}$ and describe the corresponding functor

$$\mathrm{CT}^{-,\mathrm{spec,enh}_x} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\underline{x}}}. \tag{19.11}$$

By duality, the datum of a functor (19.11) is equivalent to that of a $\mathrm{Sph}^{\mathrm{spec}}_{\check{M},\underline{x}}$-linear functor

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{co},\underline{x}} \otimes \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)), \tag{19.12}$$

where

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{co}} := \mathrm{IndCoh}^!(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-}),$$

see (2.8).

19.3.2. Set

$$\mathrm{Hecke}^{\mathrm{spec,glob}}_{\check{G},\check{P}^-,\underline{x}} := \mathrm{LS}_{\check{G}}(X) \underset{\mathrm{LS}_{\check{G}}(X-\underline{x})}{\times} \mathrm{LS}_{\check{P}^-}(X-\underline{x}) \underset{\mathrm{LS}_{\check{M}}(X-\underline{x})}{\times} \mathrm{LS}_{\check{M}}(X).$$

It is equipped with the maps

$$\mathrm{LS}_{\check{G}}(X) \overset{\overleftarrow{h}^{\mathrm{spec}}_{\check{G},\check{P}^-}}{\leftarrow} \mathrm{Hecke}^{\mathrm{spec,glob}}_{\check{G},\check{P}^-,\underline{x}} \overset{\overrightarrow{h}^{\mathrm{spec}}_{\check{G},\check{P}^-}}{\to} \mathrm{LS}_{\check{M}}(X),$$

and also with a map

$$\mathfrak{s}^{\mathrm{spec}} : \mathrm{Hecke}^{\mathrm{spec,glob}}_{\check{G},\check{P}^-,\underline{x}} \to \mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G},\check{P}^-,\underline{x}},$$

given by restriction along $\mathcal{D}_{\underline{x}} \to X$.

The functor

$$(\overrightarrow{h}^{\mathrm{spec}}_{\check{G},\check{P}^-})_* \left((\mathfrak{s}^{\mathrm{spec}})^!(-) \overset{!}{\otimes} (\overleftarrow{h}^{\mathrm{spec}}_{\check{G},\check{P}^-})^!(-) \right)$$

defines the sought-for functor (19.12).

19.3.3. Denote

$$\mathrm{CT}^{-,\mathrm{spec,part.enh}} := (\mathrm{full} \to \mathrm{part}) \circ \mathrm{CT}^{-,\mathrm{spec,enh}},$$

which is a functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}.$$

Unwinding the constructions we obtain:

Lemma 19.3.4. *The functor* $\mathrm{CT}^{-,\mathrm{spec},\mathrm{part.enh}}$ *corresponds under the identification of Proposition 19.2.3 to the composition*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \hookrightarrow \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)) \overset{(\mathsf{p}^{\mathrm{glob}})^!}{\longrightarrow} \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X)) \twoheadrightarrow \mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)),$$

where the last arrow is the right adjoint to the tautological embedding

$$\mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \hookrightarrow \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X)).$$

Remark 19.3.5. The functor $\mathrm{CT}^{-,\mathrm{spec},\mathrm{enh}}$ has a natural description on the other side of the equivalence of (19.5). Namely, the corresponding functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec},\mathrm{glob}}$$

is the composition of:

- The embedding $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \hookrightarrow \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X))$;
- The pullback functor along $\mathrm{LS}_{\check{G}}(X) \underset{(\mathrm{LS}_{\check{G}}(X))_{\mathrm{dR}}}{\times} \mathrm{LS}_{\check{P}^-}(X))_{\mathrm{dR}} \to \mathrm{LS}_{\check{G}}(X)$;
- The right adjoint of the tautological embedding

$$\mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \underset{\mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X))}{\times} \left(\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X) \underset{(\mathrm{LS}_{\check{G}}(X))_{\mathrm{dR}}}{\times} (\mathrm{LS}_{\check{P}^-}(X))_{\mathrm{dR}})\right) \hookrightarrow$$
$$\hookrightarrow \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X) \underset{(\mathrm{LS}_{\check{G}}(X))_{\mathrm{dR}}}{\times} (\mathrm{LS}_{\check{P}^-}(X))_{\mathrm{dR}}).$$

19.4. **The enhanced spectral Poincaré series functor.** The material in this subsection is parallel to that of Sect. 16.3.

19.4.1. Fix a point $\underline{x} \in \mathrm{Ran}$. The spectral Poincaré series functor

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{M},!,\underline{x}} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times_{\underline{x}})) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$$

is compatible with the action of $\mathrm{Sph}^{\mathrm{spec}}_{\check{M},\underline{x}}$.

Hence, it induces a functor

$$\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}_{\underline{x}}}_{\check{M},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times_{\underline{x}}))^{-,\mathrm{enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\underline{x}}},$$

where $\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times_{\underline{x}}))^{-,\mathrm{enh}}$ is as in Sect. 5.7.7.

The functors $\mathrm{Poinc}^{\mathrm{spec}}_{\check{M},!,\underline{x}}$ and $\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}_{\underline{x}}}_{\check{M},!}$ are compatible with the adjunctions

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times_{\underline{x}})) \rightleftarrows \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times_{\underline{x}}))^{-,\mathrm{enh}}$$

and

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\underline{x}}}.$$

19.4.2. Making the point $\underline{x}$ vary along Ran, we obtain a functor

$$\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}_{\mathrm{Ran}}}_{\check{M},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times_{\underline{x}}))^{-,\mathrm{enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}}}.$$

More generally, for $\mathcal{Z} \to \mathrm{Ran}$, we obtain a functor

$$\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}_{\mathcal{Z}}}_{\check{M},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times_{\underline{x}}))^{-,\mathrm{enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathcal{Z}}}.$$

19.4.3. Consider the space $\mathrm{Ran}_{\subseteq}$, viewed as mapping to Ran by means of $\mathrm{pr}_{\mathrm{big}}$. Consider the functor

$$\begin{aligned}(19.13)\quad \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathrm{Ran}} &\xrightarrow{\text{ins.unit}} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathrm{Ran}_\subseteq} \xrightarrow{\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},!}} \\ &\to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}_\subseteq}} \xrightarrow{\mathrm{Id}\otimes(\mathrm{pr}_{\mathrm{big}})_*} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}}}.\end{aligned}$$

The following assertion results from the isomorphism (13.17):

Lemma 19.4.4. *The functor* (19.13) *takes values in*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}}}.$$

19.4.5. Thanks to Lemma 19.4.4 we obtain a well-defined functor, to be denoted

$$(19.14)\qquad \mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},!} : \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathrm{Ran}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}.$$

For a space $\mathcal{Z}$ mapping to Ran, we will denote by the sam symbol $\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},!}$ the resulting functor

$$\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathcal{Z}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}},$$

cf. Sect. 16.3.6.

19.5. **The enhanced spectral Poincaré vs constant term compatibility.**

19.5.1. The following assertion is an enhanced version of Theorem 18.2.2:

Theorem 19.5.2. *The following diagram of functors commutes:*

$$\begin{array}{ccc}
\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) & \xrightarrow{\mathrm{CT}^{-,\mathrm{spec},\mathrm{enh}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\
{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},!}}\uparrow & & \\
\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} & & \uparrow{\scriptstyle \mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},!}} \\
{\scriptstyle \text{ins.unit}}\downarrow & & \\
\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}_\subseteq} & \xrightarrow{J^{-,\mathrm{spec},!,\mathrm{enh}}} & \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathrm{Ran}_\subseteq}.
\end{array}$$

We omit the proof, as it is obtained by tracing the same sequence of diagrams as that of Theorem 18.2.2.

Remark 19.5.3. Note that the counter-clockwise circuit in Theorem 19.5.2 can we also written as

$$\begin{aligned}\mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} &\xrightarrow{\text{ins.unit}} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}_\subseteq} \xrightarrow{(\mathrm{pr}_{\mathrm{big}})_*} \\ \to \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} &\xrightarrow{J^{-,\mathrm{spec},!,\mathrm{enh}}} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathrm{Ran}} \xrightarrow{\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},!}} \\ &\to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}.\end{aligned}$$

Part IV. Applications to the Langlands functor

In this Part we will assemble the ingredients developed in Parts I-III and study the global Langlands functor

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)).$$

The geometric Langlands conjecture (Conjecture 20.3.8) says that $\mathbb{L}_G$ is an equivalence.

In this part:

- We wecall the construction of $\mathbb{L}_G$ along with its defining property, which is the compatibility with the functors

$$\text{coeff}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{Whit}^!(G)_{\text{Ran}} \text{ and } \Gamma^{\text{spec}}_{\check{G}} : \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)) \to \text{Rep}(\check{G})_{\text{Ran}}.$$

- We recall also the compatibility[23] of $\mathbb{L}_G$ with the *Eisenstein series* functors

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \xrightarrow{\text{Eis}^-_!} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \text{ and } \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{M}}(X)) \xrightarrow{\text{Eis}^{-,\text{spec}}} \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)).$$

- We prove that $\mathbb{L}_G$ satisfies another local-to-global compatibility, namely, with the functors

$$\text{KL}(G)_{\text{crit,Ran}} \xrightarrow{\text{Loc}_G} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \text{ and } \text{IndCoh}(\text{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times)) \xrightarrow{\text{Poinc}^{\text{spec}}_{\check{G},*}} \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)).$$

- We state and prove the central result of this Part, Theorem 21.2.2 (along with its enhanced version, Theorem 22.2.4), which establish the compatibility of the Langlands functor with the geometric and spectral constant term functors.

As an application, we show that $\mathbb{L}_G$ admits a left adjoint, and we relate this left adjoint to the functor *dual* to $\mathbb{L}_G$.

We show that the composition $\mathbb{L}_G \circ \mathbb{L}^L_G$, viewed as an endofunctor of $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))$, is given by tensoring by an (associative algebra object)

$$\mathcal{A}_G \in \text{QCoh}(\text{LS}_{\check{G}}(X)).$$

We show that the geometric Langlands conjecture reduces to the assertion that the unit map

$$\mathcal{O}_{\text{LS}_{\check{G}}(X)} \to \mathcal{A}_G$$

is an isomorphism.

20. The Langlands functor

In this section we recall the construction of the Langlands functor in carry out the first three bullet points described above. Here is what is used in each of them:

- The construction of $\mathbb{L}_G$ uses the geometric Casselman-Shalika formula (Theorem 1.4.2), the *spectral action*, i.e., the assertion that the Hecke action of $\text{Rep}(\check{G})_{\text{Ran}}$ on $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ factors through an action of $\text{QCoh}(\text{LS}_{\check{G}}(X))$, and a cohomological estimate from [GR1].
- The compatibility with the Eisenstein functors uses the compatibility of *local Jacquet functors* with the equivalence $\text{FLE}_{\check{G},\infty}$, given by (2.31);
- The compatibility with Loc_G and $\text{Poinc}^{\text{spec}}_{\check{G},*}$ uses Theorems 14.2.3 and 17.4.2, as well as the compatibility between $\text{FLE}_{G,\text{crit}}$ and $\text{FLE}_{\check{G},\infty}$, expressed by Corollary 7.5.8.

20.1. **Recollections on the Langlands functor–the coarse version.** In this and the next subsections we recall the construction of the Langlands functor

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)). \tag{20.1}$$

[23] This compatibility is actually an ingredient in showing that $\mathbb{L}_G$ is well-defined.

20.1.1. Recall that the functor

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \tag{20.2}$$

is naturally (symmetric) monoidal, and is a *localization* (i.e., it admits a fully faithful right adjoint).

The following is a key feature of the Hecke action (see [Gai1, Corollary 4.5.5]):

Theorem 20.1.2. *The action of* $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ *on* $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ *obtained via*

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \overset{\mathrm{Sat}^{-1,\mathrm{nv}}_G}{\longrightarrow} \mathrm{Sph}_{G,\mathrm{Ran}}$$

factors through (20.2).

20.1.3. Thanks to Theorem 20.1.2, we have a canonically defined action of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ on $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$. In particular, a choice of an object in $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ defines a $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$-linear functor $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$.

We define the functor

$$\mathbb{L}^L_{G,\mathrm{temp}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G). \tag{20.3}$$

to correspond to the object

$$\mathrm{Poinc}^{\mathrm{Vac,glob}}_{G,!} \in \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G), \tag{20.4}$$

see Sect. 12.5.4.

20.1.4. By construction, the functor $\mathbb{L}^L_{G,\mathrm{temp}}$ makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Whit}^!(\mathrm{Gr}_{G,\mathrm{Ran}}) & \xleftarrow{\mathrm{CS}^{-1}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ \Big\downarrow{\scriptstyle \mathrm{Poinc}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]} & & \Big\downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}^L_{G,\mathrm{temp}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{20.5}$$

20.1.5. Note that the counter-clockwise circuit in (20.5) commutes with the actions.

$$\mathrm{Sph}_{G,\mathrm{Ran}} \overset{\mathrm{Sat}_G}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{G},\mathrm{Ran}}. \tag{20.6}$$

Since the right vertical arrow in (20.5) also has this property and is a localization, we obtain that the functor $\mathbb{L}^L_{G,\mathrm{temp}}$ also commutes with the actions of (20.6).

This implies, in particular, that the essential image of $\mathbb{L}^L_{G,\mathrm{temp}}$ lands in the full subcategory

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \tag{20.7}$$

(see [FR2] for the definition of this subcategory).

20.1.6. Since the object (20.4) is compact, the functor $\mathbb{L}^L_{G,\mathrm{temp}}$ preserves compactness. Hence, it admits a $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$-linear (automatically continuous) right adjoint, to be denoted

$$\mathbb{L}_{G,\mathrm{coarse}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)). \tag{20.8}$$

It follows from Sect. 20.1.5 that the functor $\mathbb{L}_{G,\mathrm{coarse}}$ factors as

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \twoheadrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \overset{\mathbb{L}_{G,\mathrm{temp}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)), \tag{20.9}$$

where the first arrow is the right adjoint to the embedding (20.7).

20.1.7. It follows by rigidity that the functor $\mathbb{L}_{G,\mathrm{coarse}}$ also respects the actions of (20.6).

20.1.8. Passing to the right adjoints in (20.5), we obtain that the functor $\mathbb{L}_{G,\mathrm{coarse}}$ makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Whit}^!(\mathrm{Gr}_{G,\mathrm{Ran}}) & \xrightarrow{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{20.10}$$

20.2. **The case $G = T$.**

20.2.1. Let $G = T$ be a torus. Consider the Fourier-Mukai equivalence

$$\mathrm{FM} : \mathrm{QCoh}(\mathrm{Bun}_T) \to \mathrm{QCoh}(\mathrm{Bun}_{\check{T}}),$$

given by the Poincaré line bundle

$$\mathcal{L}_{\mathrm{Poinc}} \in \mathrm{QCoh}(\mathrm{Bun}_T \times \mathrm{Bun}_{\check{T}}),$$

as a kernel, where $\mathcal{L}_{\mathrm{Poinc}}$, viewed as a map $\mathrm{Bun}_T \times \mathrm{Bun}_{\check{T}} \to B\mathbb{G}_m$ is given by the Weil pairing.

20.2.2. It is known that FM can be enhanced to an equivalence

$$\mathrm{FM}^{\mathrm{enh}} : \mathrm{D\text{-}mod}(\mathrm{Bun}_T) \to \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X)),$$

that makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{D\text{-}mod}(\mathrm{Bun}_T) & \xrightarrow{\mathrm{FM}^{\mathrm{enh}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X)) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathrm{Bun}_T) & \xrightarrow{\mathrm{FM}} & \mathrm{QCoh}(\mathrm{Bun}_{\check{T}}), \end{array}$$

where:

- The functor $\mathrm{D\text{-}mod}(\mathrm{Bun}_T) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_T)$ is $\mathbf{oblv}^r$, the forgetful functor for "right" D-modules;
- The functor $\mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X)) \to \mathrm{QCoh}(\mathrm{Bun}_{\check{T}})$ is direct image along the projection

$$\mathrm{LS}_{\check{T}}(X) \to \mathrm{Bun}_{\check{T}}\,.$$

20.2.3. Unwinding the definitions, we obtain that the functor $\mathbb{L}_T := \mathbb{L}_{T,\mathrm{coarse}}$ identifies with

$$\mathrm{FM}^{\mathrm{enh}} \circ \tau_T,$$

where τ_T is the Cartan involution, i.e., the inversion automorphism, of T.

20.2.4. Let $\mathcal{P}_T$ be a point of Bun_T, and let $\mathcal{L}_{\mathcal{P}_T}$ be the line bundle on $\mathrm{Bun}_{\check{T}}$, obtained by Weil pairing with $\mathcal{P}_T$. By a slight abuse of notation, we will denote by the same character $\mathcal{L}_{\mathcal{P}_T}$ its pullback to $\mathrm{LS}_{\check{T}}(X)$.

We obtain that the following diagram commutes

$$\begin{array}{ccc} \mathrm{D\text{-}mod}(\mathrm{Bun}_T) & \xrightarrow{\mathbb{L}_T} & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X)) \\ {\scriptstyle (\mathrm{transl}_{\mathcal{P}_T})_*}\uparrow & & \uparrow{\scriptstyle -\otimes \mathcal{L}_{\mathcal{P}_T}^{\otimes -1}} \\ \mathrm{D\text{-}mod}(\mathrm{Bun}_T) & \xrightarrow{\mathbb{L}_T} & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X)). \end{array}$$

20.2.5. Similarly, let σ be a $\check{T}$-local system. Let $\mathcal{F}_\sigma$ be the corresponding character sheaf on Bun_T. Then the diagram

$$\begin{array}{ccc} \mathrm{D\text{-}mod}(\mathrm{Bun}_T) & \xrightarrow{\mathbb{L}_T} & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X)) \\ {\scriptstyle -\otimes \mathcal{F}_\sigma}\uparrow & & \uparrow{\scriptstyle \mathrm{transl}_\sigma} \\ \mathrm{D\text{-}mod}(\mathrm{Bun}_T) & \xrightarrow{\mathbb{L}_T} & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X)). \end{array}$$

commutes.

20.3. **The actual Langlands functor.**

20.3.1. We now quote the following result established in [GR1]:

Theorem 20.3.2. *The functor $\mathbb{L}_{G,\mathrm{coarse}}$ sends compact objects of $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ to objects of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))^{>-\infty}$ (i.e., objects cohomologically bounded below).*

20.3.3. Consider the tautological embedding

$$\Xi_{0,\mathrm{Nilp}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \hookrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)),$$

and its right adjoint

$$(\Xi_{0,\mathrm{Nilp}})^R : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)).$$

According to [AG, Proposition 4.4.5], the functor $(\Xi_{0,\mathrm{Nilp}})^R$ is t-exact and induces an equivalence

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{>-\infty} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))^{>-\infty}. \tag{20.11}$$

Hence, using Theorem 20.3.2, we obtain that the functor

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \xrightarrow{\mathbb{L}_{G,\mathrm{coarse}}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))^{>-\infty}$$

can be uniquely lifted to a functor, to be denoted

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^c \xrightarrow{\mathbb{L}_G} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{>-\infty}. \tag{20.12}$$

20.3.4. Finally, we define the sought-for functor (20.1) to be the (unique) extension of (20.12) to a continuous functor

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

20.3.5. Since the compact generators of (20.6) act on $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ by cohomologically bounded functors, from Sect. 20.1.7 we obtain that the functor $\mathbb{L}_G$ is compatible with the actions of (20.6) on the two sides.

Combining with Theorem 20.1.2, we obtain that the functor $\mathbb{L}_G$ is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$-linear.

20.3.6. By construction, the composition

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \xrightarrow{\mathbb{L}_G} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \xrightarrow{(\Xi_{0,\mathrm{Nilp}})^R} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$$

is the functor $\mathbb{L}_{G,\mathrm{coarse}}$.

Hence, the functor $\mathbb{L}_G$ makes the diagram

$$\begin{array}{ccc} \mathrm{Whit}^!(\mathrm{Gr}_{G,\mathrm{Ran}}) & \xrightarrow{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \end{array} \tag{20.13}$$

commute, where by a slight abuse of notation, we denote by the same symbol $\Gamma^{\mathrm{spec}}_{\check{G}}$ the composition

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \xrightarrow{(\Xi_{0,\mathrm{Nilp}})^R} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \xrightarrow{\Gamma^{\mathrm{spec}}_{\check{G}}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

20.3.7. The following is the statement of the categorical (global, unramified) geometric Langlands conjecture in the de Rham context:

Conjecture 20.3.8. *The functor*

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))$$

is an equivalence.

The goal of the present sequence of papers is to prove this conjecture.

Remark 20.3.9. From diagram (20.10), we obtain a diagram

(20.14)
$$\begin{array}{ccc} \text{Whit}^!(\text{Gr}_{G,\text{Ran}}) & \xrightarrow{\text{CS}_G} & \text{Rep}(\check{G})_{\text{Ran}} \\ \downarrow{\scriptstyle \text{Poinc}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]} & \swarrow & \downarrow{\scriptstyle \text{Loc}^{\text{spec}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\text{coarse}}} & \text{QCoh}(\text{LS}_{\check{G}}(X)). \end{array}$$

One can show that the assertion that $\mathbb{L}_G$ is an equivalence is logically equivalent to the assertion that the natural transformation in (20.14) is an isomorphism.

20.4. **The Langlands functor and Eisenstein series.**

20.4.1. Let $\text{Eis}^-_! : \text{D-mod}(\text{Bun}_M) \to \text{D-mod}(\text{Bun}_G)$ be the functor left adjoint to CT^-_*. I.e., it s given by *-pull and !-push along the diagram (15.2), combined with the cohomological shift *to the left* by $\dim.\,\text{rel}(\text{Bun}_{P^-} \,/\, \text{Bun}_M)$ over a given connected component of Bun_M.

The functor $\text{Eis}^-_!$ is rigged so that it is the left adjoint of the functor CT^-_* of (15.1).

20.4.2. Let

$$\text{Eis}^-_{!,\rho_P(\omega_X)} : \text{D-mod}(\text{Bun}_M) \to \text{D-mod}\,\text{Bun}_G)$$

be the functor equal to the composition of

- Over a connected component of Bun_M of degree λ, the cohomological shift to the left by the amount
$$\delta_{N(P^-)_{\rho_P(\omega_X)}} + \langle \lambda, 2\check{\rho}_P \rangle,$$
- Pushforward along the translation map
$$(\rho_P(\omega_X) \cdot -) : \text{D-mod}(\text{Bun}_M) \to \text{D-mod}(\text{Bun}_M);$$
- The functor of !-pullback along $\text{Bun}_{P^-} \to \text{Bun}_M$;
- The functor of !-pushforward along $\text{Bun}_{P^-} \to \text{Bun}_G$.

The functor $\text{Eis}^-_{!,\rho_P(\omega_X)}$ is rigged to be the left adjoint of the functor $\text{CT}^-_{*,\rho_P(\omega_X)}$ of (15.3).

20.4.3. Using the identifications of the corresponding $\mathbb{Z}/2\mathbb{Z}$-gerbes in Sect. 15.2.7, we define the functors

$$\text{Eis}^-_{!,\rho_P(\omega_X)} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

and

$$\text{Eis}^-_! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G),$$

which are the left adjoints of the functors (15.6) and (15.7), respectively.

20.4.4. We claim:

Theorem 20.4.5. *There exists a canonical datum of commutativity for the diagram*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{20.15}$$

The rest of this subsection, and the next one, are devoted to the proof of Theorem 20.4.5.

20.4.6. It is enough to show that the two circuits in Theorem 20.4.5 are isomorphic as functors out of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^c$.

First, we claim that both functors in question send $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^c$ to bounded below objects in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$.

For the counter-clockwise circuit, this follows from the fact that (a) the functor $\mathrm{Eis}^-_{!,\rho_P(\omega_X)}$ preserves compactness (being a left adjoint), and (b) Theorem 20.3.2 (for G).

For the clockwise circuit this follows from (a') Theorem 20.3.2 (for M), and (b') the fact that the functor $\mathrm{Eis}^{-,\mathrm{spec}}$ is of bounded cohomological amplitude.

20.4.7. Hence, using the equivalence (20.11), it suffices to establish the commutativity of the following diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_{M,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)), \end{array} \tag{20.16}$$

where

$$\mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}} := \mathsf{p}^{\mathrm{glob}}_* \circ (\mathsf{q}^{\mathrm{glob}})^*, \quad \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)).$$

20.4.8. Combining with (20.10), since the functor $\Gamma^{\mathrm{spec}}_{\check{G}}$ is fully faithful, we obtain that it suffices to establish the commutativity of the next diagram

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xrightarrow{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_{M,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)). \end{array} \tag{20.17}$$

20.4.9. By duality, the commutativity of (20.17) is equivalent to the commutativity of the following diagram

(20.18)

$$\begin{array}{ccc}
\mathrm{Whit}_*(G)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{(\mathrm{FLE}_{\check{G},\infty})^{-1}\otimes \mathbb{L}_{M,\mathrm{coarse}}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \\
{\scriptstyle \mathrm{Poinc}_{G,*}\otimes \mathrm{Eis}^-_{!,\rho_P(\omega_X)}}\downarrow{\scriptstyle [\delta_{N(P^-)_{\rho_P(\omega_X)}}+2\delta_{N_{\rho(\omega_X)}}]} & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}\otimes \mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \\
{\scriptstyle \Gamma_{\mathrm{dR}}(\mathrm{Bun}_G,-\overset{!}{\otimes}-)}\downarrow & & \downarrow{\scriptstyle \Gamma(\mathrm{LS}_{\check{G}}(X),-\overset{*}{\otimes}-)} \\
\mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect}\,.
\end{array}$$

We will give a local-to-global expression to the composite vertical arrows in (20.18), and will show that they match under the functor $(\mathrm{FLE}_{\check{G},\infty})^{-1}\otimes \mathbb{L}^{-,\mathrm{enh}}_M$.

20.5. **The commutativity of** (20.18)**.**

20.5.1. First, we will show that the right vertical arrow in (20.18) identifies with the composition

$$\begin{aligned}(20.19)\quad \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) &\overset{\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-),-)\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \overset{\mathrm{Loc}^{\mathrm{spec}}_{\check{M}}\otimes \mathrm{Id}}{\longrightarrow} \\ &\to \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \overset{\Gamma(\mathrm{LS}_{\check{M}}(X),-\overset{*}{\otimes}-)}{\longrightarrow} \mathrm{Vect}\,.\end{aligned}$$

Note that since the functor

$$\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-),-) : \mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{M})$$

is unital as a functor between factorization categories, we can rewrite (20.19) as

$$\begin{aligned}(20.20)\quad \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) &\overset{\mathrm{ins.unit}\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Rep}(\check{G})_{\mathrm{Ran}\subseteq} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \overset{\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-),-)\otimes \mathrm{Id}}{\longrightarrow} \\ &\to \mathrm{Rep}(\check{M})_{\mathrm{Ran}\subseteq} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \overset{\mathrm{Loc}^{\mathrm{spec}}_{\check{M}}\otimes \mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \to \\ &\overset{\Gamma(\mathrm{LS}_{\check{M}}(X),-\overset{*}{\otimes}-)}{\longrightarrow} \mathrm{Vect}\,.\end{aligned}$$

20.5.2. Next we note that the functor $\mathrm{Eis}^{-,\mathrm{spec}}_{\mathrm{coarse}}$ is the dual of the functor[24]

$$\mathrm{CT}^{-,\mathrm{spec}}_{\mathrm{coarse}} := \mathsf{q}^{\mathrm{glob}}_* \circ (\mathsf{p}^{\mathrm{glob}})^*, \quad \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)).$$

Hence, the right vertical composition in (20.18) can be rewritten as

$$\begin{aligned}(20.21)\quad \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) &\overset{\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}\otimes \mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \overset{\mathrm{CT}^{-,\mathrm{spec}}_{\mathrm{coarse}}\otimes \mathrm{Id}}{\longrightarrow} \\ &\to \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \overset{\Gamma(\mathrm{LS}_{\check{G}}(X),-\overset{*}{\otimes}-)}{\longrightarrow} \mathrm{Vect}\,.\end{aligned}$$

20.5.3. Hence, we obtain that it suffices to establish the isomorphism of the following diagram

$$\begin{array}{ccccc}
\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) & & \xrightarrow{\mathrm{CT}^{-,\mathrm{spec}}_{\mathrm{coarse}}} & & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \\
{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}}\uparrow & & & & \uparrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{M}}} \\
\mathrm{Rep}(\check{G})_{\mathrm{Ran}} & \xrightarrow{\mathrm{ins.unit}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}\subseteq} & \xrightarrow{\mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-),-)} & \mathrm{Rep}(\check{M})_{\mathrm{Ran}\subseteq}.
\end{array}$$

However, this follows from (the parametrized version of) Lemma 17.4.10, applied to the co-affine morphism

$$\mathrm{LS}_{\check{P}^-}(X) \to \mathrm{LS}_{\check{M}}(X).$$

[24]Warning: the functor $\mathrm{CT}^{-,\mathrm{spec}}_{\mathrm{coarse}}$ is *not* simply the coarsened version of $\mathrm{CT}^{-,\mathrm{spec}}$; two differ by a tensor product by a graded line bundle, see (24.16).

20.5.4. The next assertion follows from the standard Zastava space calculation:

Lemma 20.5.5. *The composite left vertical arrow in* (20.18) *identifies with*

$$\begin{aligned}(20.22)\quad &\mathrm{Whit}_*(G)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \overset{J^{-,*}\otimes \mathrm{Id}}{\longrightarrow} \mathrm{Whit}_*(M)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \\ &\overset{\mathrm{Poinc}_{M,*}\otimes \mathrm{Id}[2\delta_{N(M)}\rho_M(\omega_X)]}{\longrightarrow} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \overset{\Gamma_{\mathrm{dR}}(\mathrm{Bun}_M, -\overset{!}{\otimes}-)}{\longrightarrow} \mathrm{Vect},\end{aligned}$$

where $J^{-,*}$ is the factorization functor from (2.30).

20.5.6. Thus, we obtain that in order to establish the commutativity of (20.18), we need to establish the commutativity of

$$\begin{array}{ccc}
\mathrm{Whit}_*(G)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{(\mathrm{FLE}_{\check{G},\infty})^{-1}\otimes \mathbb{L}_{M,\mathrm{coarse}}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \\
\downarrow{\scriptstyle J^{-,*}\otimes \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{C}^\cdot(\mathfrak{n}(\check{P}^-),-)\otimes \mathrm{Id}} \\
\mathrm{Whit}_*(M)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & & \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \\
\downarrow{\scriptstyle \mathrm{Poinc}_{G,*}\otimes \mathrm{Id}[2\delta_{N(M)}\rho_M(\omega_X)]} & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{M}}\otimes \mathrm{Id}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \\
\downarrow{\scriptstyle \Gamma_{\mathrm{dR}}(\mathrm{Bun}_M, -\overset{!}{\otimes}-)} & & \downarrow{\scriptstyle \Gamma(\mathrm{LS}_{\check{G}}(X), -\overset{*}{\otimes}-)} \\
\mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect}\,.
\end{array}$$

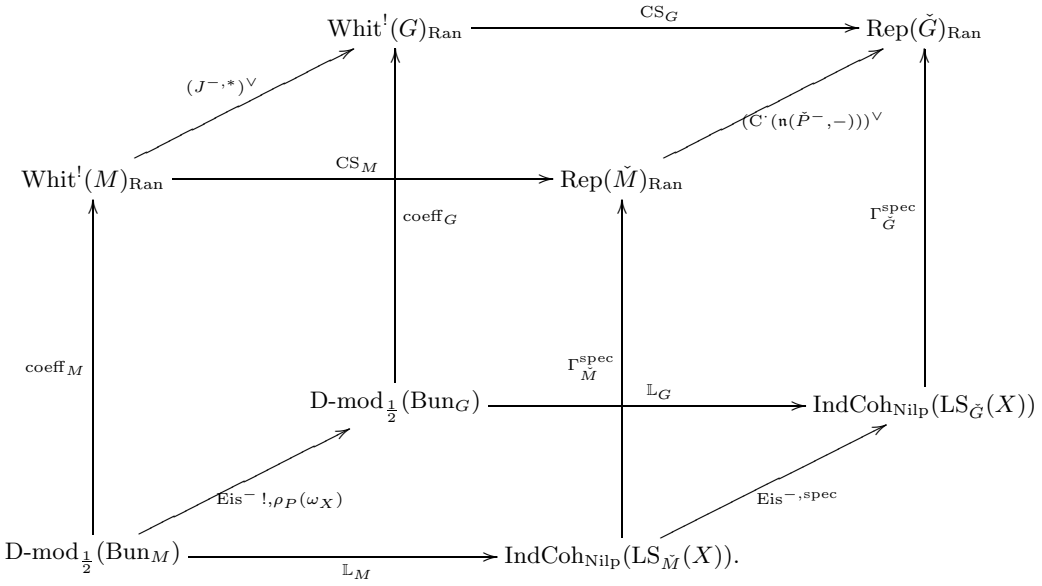
However, this follows from (2.31) and the commutative diagram

$$\begin{array}{ccc}
\mathrm{Whit}_*(M)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{(\mathrm{FLE}_{\check{M},\infty})^{-1}\otimes \mathbb{L}_{M,\mathrm{coarse}}} & \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \\
\downarrow{\scriptstyle \mathrm{Poinc}_{G,*}\otimes \mathrm{Id}[2\delta_{N(M)}\rho_M(\omega_X)]} & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{M}}\otimes \mathrm{Id}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & & \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \\
\downarrow{\scriptstyle \Gamma_{\mathrm{dR}}(\mathrm{Bun}_M, -\overset{!}{\otimes}-)} & & \downarrow{\scriptstyle \Gamma(\mathrm{LS}_{\check{G}}(X), -\overset{*}{\otimes}-)} \\
\mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect},
\end{array}$$

which is equivalent to the M-version of the commutative diagram (20.10).

Remark 20.5.7. The proof of Theorem 20.4.5 can be summarized by the following cube (in which the arrows are marked up to cohomological shifts and tensor products by constant lines):

(20.23)



The assertion of Theorem 20.4.5 is that the bottom lid of this cube commutes, We have deduced this by showing that the remaining five faces of (20.23) commute.

20.6. **Compatibility of the the Langlands functor with critical localization.**

20.6.1. The following theorem expresses the compatibility of the Langlands functor with critical localization:

Theorem 20.6.2. *The diagram*

$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \\ \mathrm{KL}(G)_{\mathrm{crit,Ran}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \end{array}$$

commutes, where the lines $\mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}}$ *and* $\mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}$ *are as in* (12.9) *and* (14.2)*, respectively.*

In a completely similar fashion, we have the following twisted version of Theorem 20.6.2:

Theorem 20.6.3. *Let* $\mathcal{P}_{Z^0_{\check{G}}}$ *be a* $Z^0_{\check{G}}$*-torsor on* X*. Then The diagram*

$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \\ \mathrm{KL}(G)_{\mathrm{crit-dlog}(\mathcal{P}_{Z^0_{\check{G}}}),\mathrm{Ran}} & \xrightarrow[\mathrm{FLE}_{G,\mathrm{crit-dlog}(\mathcal{P}_{Z^0_{\check{G}}})}]{} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G},\mathcal{P}_{Z^0_{\check{G}}}}(\mathcal{D}^\times))_{\mathrm{Ran}} \end{array}$$

commutes.

The rest of the subsection is devoted to the proof of Theorem 20.6.2.

20.6.4. First, we observe that since the action of $\mathrm{Sph}_{G,\mathrm{Ran}}$ on $\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}}$ factors through the tempered quotient, the essential image of the functor Loc_G lands in

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

Hence, taking into account Proposition 17.1.4 and Theorem 17.2.4, we obtain that the commutativity of the diagram in (20.6.2) is equivalent to the commutativity of the following one:

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} & \xrightarrow{\mathbb{L}_{G,\mathrm{temp}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \\ \uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}}, \end{array}$$

and further equivalent to

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \\ \uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}}. \end{array} \tag{20.24}$$

20.6.5. Since the right vertical arrow in (20.10) is fully fully faithful, it suffices to show that the two circuits in (20.24) become isomorphic after composing with the functor $\Gamma^{\mathrm{spec}}_{\check{G}}$.

Since the diagram (20.10) is commutative, we obtain that it suffices to establish the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xrightarrow{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes\frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}}, \end{array}$$

or which is the same

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xrightarrow{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{coeff}_G}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec}}_{\check{G}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \\ {\scriptstyle \mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes\frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}}\uparrow{\scriptstyle [\delta_{N_{\rho(\omega_X)}}]} & & \uparrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}} \\ \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} & \xrightarrow{\mathrm{FLE}_{G,\mathrm{crit}}} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}}, \end{array} \tag{20.25}$$

20.6.6. Applying duality, we obtain that it suffices to show that the pairing

$$\begin{aligned} \mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} &\xrightarrow{\mathrm{Loc}_G \otimes \mathrm{Id}} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \xrightarrow{\mathrm{coeff}_G \otimes \mathrm{Id}} \\ &\to \mathrm{Whit}^!(G)_{\mathrm{Ran}} \otimes \mathrm{Whit}_*(G)_{\mathrm{Ran}} \to \mathrm{Vect} \xrightarrow{(-\otimes \mathfrak{l}^{\otimes\frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}})[\delta_{N_{\rho(\omega_X)}}]} \mathrm{Vect}\,. \end{aligned} \tag{20.26}$$

agrees under the FLE equivalences

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \overset{\mathrm{FLE}_{G,\mathrm{crit}}}{\simeq} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \text{ and } \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \overset{\mathrm{FLE}_{\check{G},\infty}}{\simeq} \mathrm{Whit}_*(\mathrm{Gr}_G)$$

with

$$\begin{aligned}(20.27)\quad \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} &\xrightarrow{\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}\otimes \mathrm{Id}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \\ &\xrightarrow{\Gamma^{\mathrm{spec}}_{\check{G}}\otimes \mathrm{Id}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{Vect}\,.\end{aligned}$$

20.6.7. By Theorem 14.2.3, the functor (20.26) identifies canonically with (14.4). By Theorem 17.4.2, the functor (20.27) identifies canonically with (17.14).

The desired assertion follows now from Corollary 7.5.8.

□[Theorem 20.6.2]

21. Compatibility of the Langlands functor with constant terms

In this section we will establish one of the main results of this paper, which says that the Langlands functor admits *a* compatibility isomorphism with the constant term functors. I.e., we will establish the commutativity of the diagram

$$\begin{aligned}(21.1)\quad &\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{CT}^-_{*,\rho_P(\omega_X)^{[-\delta_{N(P^-)}\rho_P(\omega_X)]}}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \end{array}\end{aligned}$$

The proof is based on the local-to-global approach, namely, we will deduce the global compatibility from the local one, given by Theorem 9.1.3, i.e., the compatibility of the critical FLE with local Jacquet functors.

A caveat in the proof is that the functor

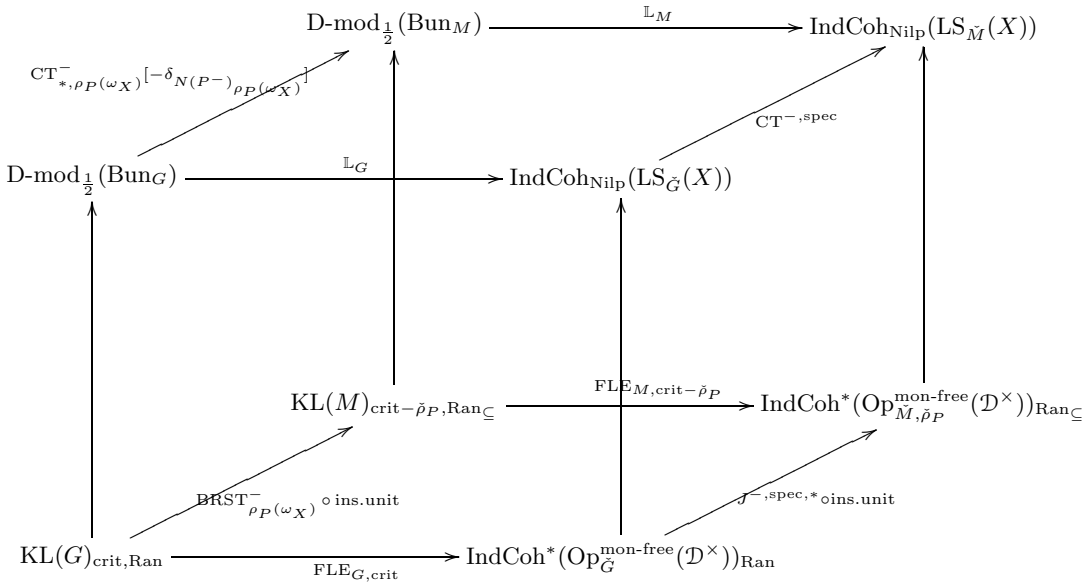
$$\mathrm{Loc}_G : \mathrm{KL}(G)_{\mathrm{crit,Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

is not a quotient (it is not even essentially surjective). However, we will show that $\mathrm{KL}(G)_{\mathrm{crit,Ran}}$ "dominates" $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ enough, so that we can draw the global compatibility from the local one.

21.1. **The cube.**

21.1.1. Consider the 1-skeleton of the cube (cf. (20.23)):

(21.2)



where the vertical arrows are as follows:

- The functor KL$(G)_{\mathrm{crit},\mathrm{Ran}} \to$ D-mod$_{\frac{1}{2}}$(Bun$_G$) is

$$\mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes\frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}];$$

- The functor IndCoh*(Op$^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \to$ IndCoh$_{\mathrm{Nilp}}$(LS$_{\check{G}}(X)$) is Poinc$^{\mathrm{spec}}_{\check{G},*}$;

- The functor KL$(M)_{\mathrm{crit}-\check{\rho}_P,\mathrm{Ran}_\subseteq} \to$ D-mod$_{\frac{1}{2}}$(Bun$_M$) is

$$\mathrm{Loc}_M \otimes \mathfrak{l}^{\otimes\frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -\frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)}[-\delta_{N_{\rho(\omega_X)}} - \delta_{N(P^-)_{\rho_P(\omega_X)}}],$$

  where $\mathfrak{l}^{\otimes\frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)}$ is as in (15.5);

- The functor IndCoh*(Op$^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))_{\mathrm{Ran}_\subseteq} \to$ IndCoh$_{\mathrm{Nilp}}$(LS$_{\check{M}}(X)$) is

$$\mathrm{Poinc}^{\mathrm{spec}}_{\check{M},*} \otimes \mathfrak{l}^{\otimes\frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -\frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)} \otimes$$
$$\otimes\, \mathfrak{l}^{\otimes -\frac{1}{2}}_{M,N(M)_{\rho_M(\omega_X)}} \otimes \mathfrak{l}_{N(M)_{\rho_M(\omega_X)}}[\delta_{N(M)_{\rho_M(\omega_X)}} - \delta_{N_{\rho(\omega_X)}} - \delta_{N(P^-)_{\rho_P(\omega_X)}}].$$

21.1.2. Note that the front and the back faces of the above cube commute, thanks to Theorems 20.6.2 and 20.6.3.

21.1.3. The left face of (21.2) commutes commutes thanks to Theorem 15.4.2.

21.1.4. The bottom face of cube (21.2) commutes thanks Theorem 9.1.3.

21.1.5. We now claim that the right face of (21.2) commutes as well. Using Theorems 17.2.4 and 18.2.2, this boils down to the numerical identity

$$\delta_G - \delta_M = \delta_{N(P^-)_{\rho_P(\omega_X)}} + \delta_{N_{\rho(\omega_X)}} - \delta_{N(M)_{\rho_M(\omega_X)}}$$

and following identification of (ungraded) lines:

$$\mathfrak{l}_{\mathrm{Kost}(\check{G})} \otimes \mathfrak{l}_{\mathrm{Kost}(M)}^{\otimes -1} \simeq \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes -\frac{1}{2}} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}} \otimes \mathfrak{l}_{M,N(M)_{\rho_M(\omega_X)}}^{\otimes \frac{1}{2}} \otimes \mathfrak{l}_{N(M)_{\rho_M(\omega_X)}}^{\otimes -1} \otimes \mathfrak{l}_{G,P^-,M,\rho_P(\omega_X)}^{\otimes \frac{1}{2}} \tag{21.3}$$

where $\mathfrak{l}_{\mathrm{Kost}(\check{G})}$ is as in Sect. 17.2.2, and $\mathfrak{l}_{\mathrm{Kost}(M)}$ is the corresponding line for M.

Taking into account Lemma 15.2.2, the required identification of lines follows from the next assertion:

Proposition 21.1.6. *There is a canonical isomorphism*

$$\mathfrak{l}_{\mathrm{Kost}(\check{G})} \simeq \mathfrak{l}_{G,N_{\rho(\omega_X)}}^{\otimes -1} \otimes \mathfrak{l}_{N_{\rho(\omega_X)}}^{\otimes 2} \otimes \det(\Gamma(X, \mathcal{O}_X) \otimes \mathfrak{g})^{\otimes -1}.$$

Proof. First, using the Killing form, we identify $\mathfrak{a}(\check{\mathfrak{g}})$ with $\mathfrak{a}(\mathfrak{g})$.

We have:

$$\mathfrak{l}_{G,N_{\rho(\omega_X)}} \otimes \det(\Gamma(X, \mathcal{O}_X) \otimes \mathfrak{g}) = \det(\Gamma(X, \mathfrak{g}_{\rho(\omega_X)}))$$

and

$$\mathfrak{l}_{N_{\rho(\omega_X)}} = \det(\Gamma(X, \mathfrak{n}_{\rho(\omega_X)})).$$

Thus, we need to establish an isomorphism

$$\det(\Gamma(X, \mathfrak{a}(\mathfrak{g})_{\omega_X})) \otimes \det(\Gamma(X, \mathfrak{n}_{\rho(\omega_X)})) \simeq \det(\Gamma(X, (\mathfrak{g}/\mathfrak{n})_{\rho(\omega_X)})).$$

It is easy to reduce to the case when $\mathfrak{g}$ is adjoint, so we will make this assumption for the duration of the proof.

Decompose

$$\mathfrak{g} \simeq \underset{e>0}{\oplus} V^e, \quad V^e = \underset{n}{\oplus} V^e(n)$$

as in Sect. 12.2.5.

Note that we can identify

$$\mathfrak{a}(\mathfrak{g}) \simeq \underset{e}{\oplus} V^e(e),$$

however, that canonical $\mathbb{G}_m$-action on $\mathfrak{a}$ is shifted by 1 relative to the action of $\mathbb{G}_m \hookrightarrow SL_2$ on $V^e(e)$, so that

$$\det(\Gamma(X, \mathfrak{a}(\mathfrak{g})_{\omega_X})) \simeq \underset{e}{\otimes} \det(\Gamma(X, V^e(e)_{\rho(\omega_X)} \otimes \omega_X)).$$

We will show that for every e,

$$\det(\Gamma(X, V^e(e)_{\rho(\omega_X)} \otimes \omega_X)) \otimes \det(\Gamma(X, \underset{n>0}{\oplus} V^e(n)_{\rho(\omega_X)})) \simeq \det(\Gamma(X, \underset{n\leq 0}{\oplus} V^e(n)_{\rho(\omega_X)})). \tag{21.4}$$

More precisely, we will show that

$$\det(\Gamma(X, V^e(e)_{\rho(\omega_X)} \otimes \omega_X)) \simeq \det(\Gamma(X, V^e(-e)_{\rho(\omega_X)})) \tag{21.5}$$

and for every $n > 0$

$$\det(\Gamma(X, V^e(n)_{\rho(\omega_X)})) \simeq \det(\Gamma(X, V^e(-n+2)_{\rho(\omega_X)})). \tag{21.6}$$

Indeed, the Killing form identifies $V^e(e)$ with the dual of $V^e(-e)$, so that by Serre duality

$$\Gamma(X, V^e(e)_{\rho(\omega_X)} \otimes \omega_X)^\vee \simeq \Gamma(X, V^e(-e)_{\rho(\omega_X)})[1].$$

This implies (21.5).

Similarly, for $n > 0$, the Killing form and the action of the positive generator of $\mathfrak{sl}_2$ identifies $V^e(n)$ with the dual vector space of $V^e(-n+2)$, and hence by Serre duality

$$\Gamma(X, V^e(n)_{\rho(\omega_X)}))^\vee \simeq \Gamma(X, V^e(-n+2)_{\rho(\omega_X)})[1].$$

This implies (21.6).

□

21.2. **The statement of the (unenhanced) compatibility theorem.**

21.2.1. We can now state the (unenhanced version) of the theorem that expresses the compatibility of the Langlands functor with constant terms:

Main Theorem 21.2.2. *Assume that the geometric Langlands conjecture (i.e., Conjecture 20.3.8) holds for M. Then there exists a unique datum of commutativity for the diagram* (21.1)*, such that along with the datum of commutativity of the other five of the faces of the cube* (21.2)*, the entire cube commutes.*

Remark 21.2.3. Recall that by Theorem 20.4.5, the diagram

(21.7)
$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{N(P^-)}\rho_P(\omega_X)]}\searrow & & \swarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}_G]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \end{array}$$

is equipped with a datum of commutativity. This equips the diagram (21.1) with an *a priori non-invertible* 2-morphism

(21.8)
$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{CT}^-_{*,\rho_P(\omega_X)}[-\delta_{N(P^-)}\rho_P(\omega_X)]}\nearrow & \Vert & \nearrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}_G]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array}$$

We do not know whether the natural transformation in (21.8) equals (i.e., is homotopic to) the isomorphism of Theorem 21.2.2.

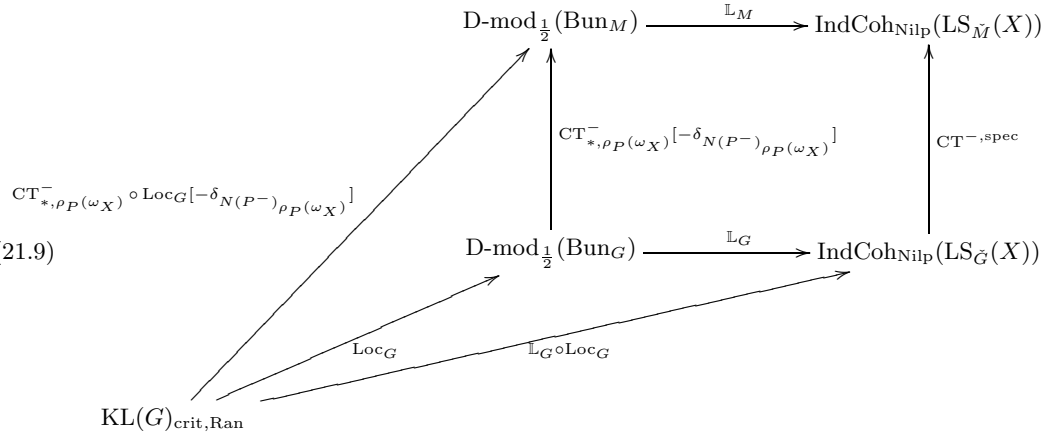
However, we will eventually show that the natural transformation in (21.8) is an isomorphism, see Corollary 24.1.4 (a posteriori, this follows immediately from GLC).

One can show that the two natural transformations (one from Theorem 21.2.2 and another from (21.8)) differ by a scalar. However, we do not know (and cannot confidently conjecture) that this scalar equals 1.

21.2.4. We now commence the proof of Theorem 21.2.2.

The commutativity of the five of the faces of the cube (21.2) established above implies that the *outer* diagram in

(21.9)



is endowed with a commutativity datum.

The statement of Theorem 21.2.2 is equivalent to the fact that this datum comes from a uniquely defined commutativity datum of the inner square in (21.9).

For expositional purposes, we first consider the case when $P = B$.

21.3. **Proof of Theorem 21.2.2 for $P = B$.**

21.3.1. The category $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_T) \simeq \mathrm{D\text{-}mod}(\mathrm{Bun}_T)$ splits as a direct sum according to the connected components of Bun_T, which are indexed by the coweight lattice of T. For each coweight μ, let $\mathrm{CT}^{-,\mu}_{*,\rho_P(\omega_X)}$ denote the corresponding direct summand of $\mathrm{CT}^-_{*,\rho_P(\omega_X)}$.

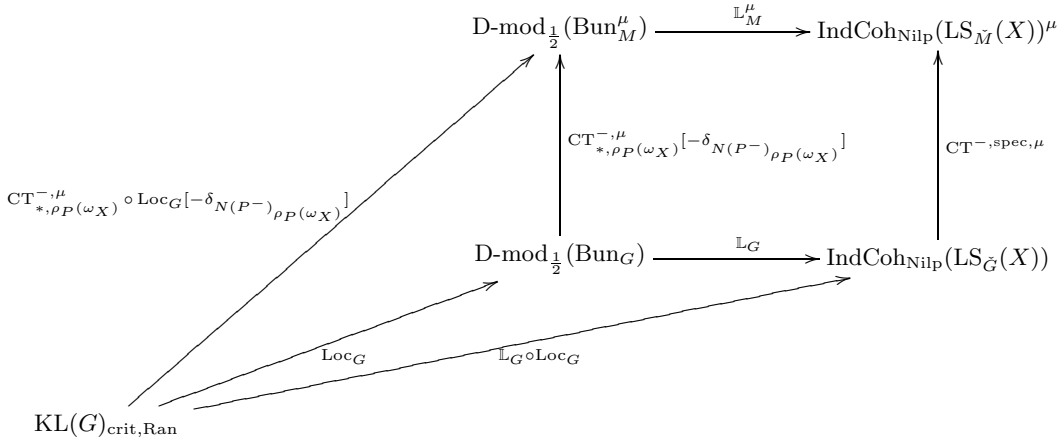
Let $\mathrm{CT}^{-,\mathrm{spec},\mu}$ denote the corresponding direct summand of $\mathrm{CT}^{-,\mathrm{spec}}$. It corresponds to the direct summand $\mathrm{QCoh}(\mathrm{LS}_{\check{T}})^\mu$ of

$$\mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X)) = \mathrm{IndCoh}_{\{0\}}(\mathrm{LS}_{\check{T}}(X)) = \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{T}}(X))$$

consisting of objects, on which the action of $\check{T}$ by 1-automorphisms of $\mathrm{LS}_{\check{T}}(X)$ has character $-\mu$ (here we regard μ as a weight of $\check{T}$), see Sect. 20.2.

Thus, in proving Theorem 21.2.2, instead of the diagram (21.9), we can consider

(21.10)



for a fixed μ.

21.3.2. For a fixed μ, let $\lambda \in \Lambda_G^{+,\mathbb{Q}}$ be large enough so that the image of the map

$$\mathrm{Bun}_B^\mu \to \mathrm{Bun}_G$$

is contained in the open Harder-Narasimhan stratum $\mathrm{Bun}_G^{(<\lambda)}$ (see [DG, Sect. 7.4] regarding our conventions regarding the parameterization of the Harder-Narasimhan strata).

By construction, we have:

Lemma 21.3.3. *The functor* $\mathrm{CT}^{-,\mu}_{*,\rho_P(\omega_X)}$ *factors as*

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \twoheadrightarrow \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G^{(<\lambda)}) \overset{(\mathrm{CT}^{-,\mu}_{*,\rho_P(\omega_X)})^{(<\lambda)}}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_T).$$

21.3.4. Consider the closed substack

$$\mathrm{Bun}_G^{(\geq\lambda)} \subset \mathrm{Bun}_G,$$

so that

$$\mathrm{Bun}_G^{(<\lambda)} \subset \mathrm{Bun}^{(/\geq\lambda)} := \mathrm{Bun}_G - \mathrm{Bun}_G^{(\geq\lambda)}.$$

Note that the open $\mathrm{Bun}_G^{(/\geq\lambda)}$ has the property that its intersection with every connected component of Bun_G is quasi-compact.

It follows from Lemma 21.3.3 that the functor $\mathrm{CT}^{-,\mu}_{*,\rho_P(\omega_X)}$ also factors as

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \twoheadrightarrow \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G^{(/\geq\lambda)}) \overset{(\mathrm{CT}^{-,\mu}_{*,\rho_P(\omega_X)})^{(/\geq\lambda)}}{\longrightarrow} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_T).$$

21.3.5. Let P' be a standard parabolic in G with Levi quotient M'. Recall that $\Lambda^+_{G,P'}$ denotes the quotient of Λ by the root lattice of M', i.e.,

$$\Lambda^+_{G,P'} \simeq \pi_{1,\mathrm{alg}}(M') \simeq \pi_0(\mathrm{Bun}_{M'}).$$

Recall (see [DG, Sects. 7.1.3-7.1.5]) that we can view $\Lambda^+_{G,P'}$ as a subset of $\Lambda_G^{+,\mathbb{Q}}$. Denote

$$\Lambda^+_{G,P'} := \Lambda_{G,P'} \cap \Lambda_G^{+,\mathbb{Q}}.$$

Let

$$\mathrm{Bun}_{M'}^{\overset{\geq\lambda}{G}} \subset \mathrm{Bun}_{M'}$$

be the union of connected components, indexed by coweights $\lambda' \in \Lambda^+_{G,P'}$ with

$$\lambda' \underset{G}{\geq} \lambda.$$

With the above notations, we can identify $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G^{(/\geq\lambda)})$ with the quotient of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$ by the full subcategory generated by the essential images of $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_{M'}^{\underset{G}{\geq}\lambda})$ along the Eisenstein functors

$$\text{Eis}_! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_{M'}) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

for all standard proper parabolics P'.

21.3.6. For $\lambda' \in \Lambda_{G,P'}$, let $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{M}'}(X))^{\lambda'}$ be the direct summand of $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{M}'}(X))$ consisting of objects on which the action of $Z_{\check{M}'}$ by 1-automorphisms of $\text{LS}_{\check{M}'}(X)$ has character $-\lambda'$.

Let $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))^{(/\geq\lambda)}$ denote the quotient of $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))$ by the full subcategory generated by the essential images of[25]

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{M'})^{\lambda'}, \quad \lambda' \underset{G}{\geq} \lambda - 2(g-1)\cdot\rho_{P'}.$$

We will prove:

Proposition 21.3.7. *For a fixed μ, and λ large enough in the order relation $\underset{G}{\geq}$, for every standard parabolic P' and $\lambda' \in \Lambda_{G,P'}$ satisfying $\lambda' \underset{G}{\geq} \lambda$, the functor*

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{M}'}(X))^{\lambda'} \overset{\text{Eis}^{\text{spec}}}{\longrightarrow} \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)) \overset{\text{CT}^{-,\text{spec},\mu}}{\longrightarrow} \text{QCoh}(\text{LS}_{\check{T}}(X))^{\mu}$$

vanishes.

Assuming Proposition 21.3.7 for a moment, we obtain that for λ sufficiently large, the functor $\text{CT}^{-,\text{spec},\mu}$ factors as

$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)) \twoheadrightarrow \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))^{(/\geq\lambda)} \overset{(\text{CT}^{-,\text{spec},\mu})^{(/\geq\lambda)}}{\longrightarrow} \text{QCoh}(\text{LS}_{\check{T}}(X))^{\mu}.$$

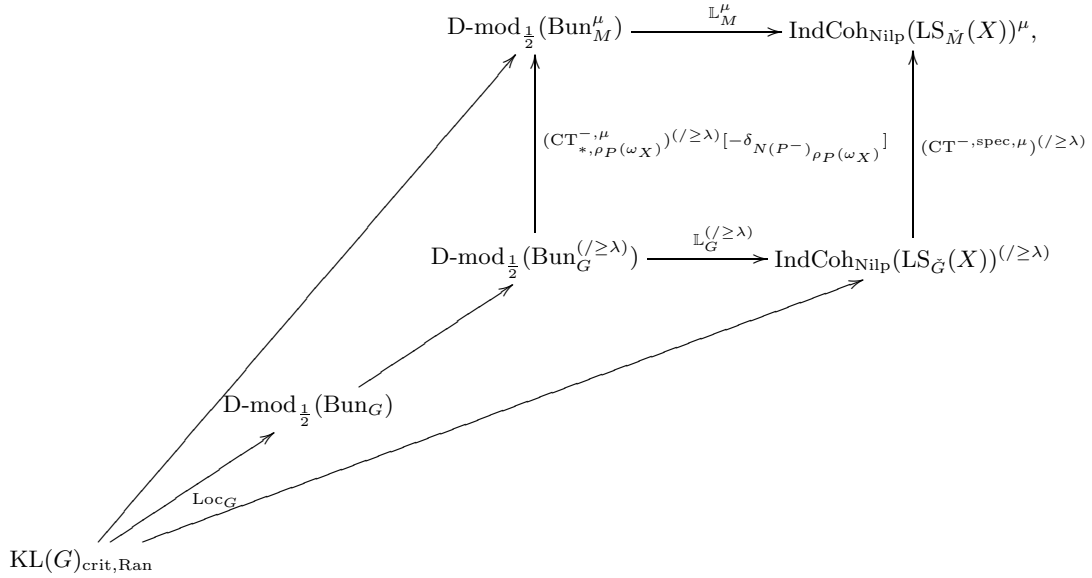
21.3.8. The compatibility of the Langlands functor with the Eisenstein functors given by Theorem 20.4.5 implies that the functor $\mathbb{L}_G$ descends to a well-defined functor

$$\mathbb{L}_G^{(/\geq\lambda)} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G^{(/\geq\lambda)}) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))^{(/\geq\lambda)}.$$

We obtain that the commutativity datum for (21.10) is equivalent to that of the commutativity datum for the inner square in

[25] The shift by $2(g-1)\cdot\rho_{P'}$ in the formula below is due to the fact that on the geometric side in Theorem 21.2.2, we are dealing with the functor $\text{Eis}^-_{!,\rho_P(\omega_X)}$ rather than just $\text{Eis}^-_!$.

(21.11)



compatible with the existing commutativity datum for the outer diagram.

However, the resulting assertion follows now from the next observation:

Lemma 21.3.9. *The functor*

$$\mathrm{KL}(G)_{\mathrm{crit,Ran}} \overset{\mathrm{Loc}_G}{\to} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \twoheadrightarrow \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G^{(/\geq\lambda)})$$

is a Verdier quotient.

Proof. Indeed, this assertion holds (at any level) for any open substack of Bun_G whose intersection with every connected component is quasi-compact.

□

21.4. **Proof of Proposition 21.3.7.** For the duration of the proof, we will change the notation from P' to P.

21.4.1. By base change, the functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \overset{\mathrm{Eis}^{\mathrm{spec}}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \overset{\mathrm{CT}^{-,\mathrm{spec}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(X))$$

can be rewritten as the composition of:

- *-pull along $\mathrm{LS}_{\check{P}}(X) \to \mathrm{LS}_{\check{M}}(X)$;
- !-pull along $\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-} \to \mathrm{LS}_{\check{P}}(X)$;
- *-push along $\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X) \to \mathrm{LS}_{\check{B}^-}(X) \to \mathrm{LS}_{\check{T}}(X)$.

However, since the morphism $\mathrm{LS}_{\check{P}}(X) \to \mathrm{LS}_{\check{M}}(X)$ is quasi-smooth, up to shifting the degree , we can replace the *-pull by !-pull. So the functor in question becomes !-pull followed by *-push along the

diagram

$$\begin{array}{ccc} & \mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X). & \\ \swarrow & & \searrow \\ \mathrm{LS}_{\check{M}}(X) & & \mathrm{LS}_{\check{T}}(X) \end{array}$$

21.4.2. We decompose $\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X)$ according to relative positions of the two reductions, which are indexed by the elements of $W_M \backslash W$.

For each $w \in W_M \backslash W$, let

$$(\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-})_w \subset \mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X)$$

denote the corresponding locally closed substack, and let

$$(\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X))^{\wedge}_w$$

denote its formal completion inside $\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X)$.

We will show that for every w and λ large enough, the pull-push functor along

$$\begin{array}{ccc} & (\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X))^{\wedge}_w & \\ \swarrow & & \searrow \\ \mathrm{LS}_{\check{M}}(X) & & \mathrm{LS}_{\check{T}}(X) \end{array} \tag{21.12}$$

has the property that its (λ', μ) component vanishes for

$$\lambda' \underset{G}{\geq} \lambda, \quad \lambda' \in \Lambda^+_{G,P}. \tag{21.13}$$

21.4.3. We will first establish the corresponding fact for the diagram

$$\begin{array}{ccc} & (\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X))_w & \\ \swarrow & & \searrow \\ \mathrm{LS}_{\check{M}}(X) & & \mathrm{LS}_{\check{T}}(X). \end{array} \tag{21.14}$$

Note that the diagram (21.14) can be factored as

$$\begin{array}{ccc} & (\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X))_w & \\ & \downarrow{\scriptstyle h} & \\ & \mathrm{LS}_{B^-(\check{M})}(X) & \\ \swarrow & & \searrow \\ \mathrm{LS}_{\check{M}}(X) & & \mathrm{LS}_{\check{T}}(X). \end{array} \tag{21.15}$$

Pull-push along (21.14) identifies with

$$(\mathsf{q}_M)_* \left(\mathsf{p}_M^!(-) \overset{!}{\otimes} h_*(\omega_h) \right), \tag{21.16}$$

where ω_h is the relative dualizing sheaf of the map h. Up to shift by a fixed weight, we can replace (21.16) by

$$(\mathsf{q}_M)_* \left(\mathsf{p}_M^!(-) \otimes h_*(\mathcal{O}_{(\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X))_w}) \right). \tag{21.17}$$

21.4.4. Fix a regular dominant cocharacter $\check{\theta} : \mathbb{G}_m \to Z_{\check{M}}$, and let us consider the resulting action of $\mathbb{G}_m$ by 1-automorphisms on the stacks in

$$\begin{array}{ccc} & \mathrm{LS}_{B^-(\check{M})}(X) & \\ \swarrow & & \searrow \\ \mathrm{LS}_{\check{M}}(X) & & \mathrm{LS}_{\check{T}}(X). \end{array}$$

The weights of this action that appear in $h_*(\mathcal{O}_{(\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X))_w})$ are of the form

$$\langle \alpha, \check{\theta} \rangle, \quad \alpha \text{ is a root in } \check{\mathfrak{n}}^- \cap w(\check{\mathfrak{n}}(\check{P})).$$

In particular, all such weights are negative.

Hence, for $\mathcal{F} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{\lambda'}$, the weights on its pull-push along (21.14) are of the form

$$-\langle \lambda', \check{\theta} \rangle + \mathbb{Z}^{\leq 0}.$$

For the μ-direct summand of the above object we must thus have

$$-\langle \lambda', \check{\theta} \rangle + \mathbb{Z}^{\leq 0} = -\langle \mu, \check{\theta} \rangle,$$

which impossible once (21.13) is satisfied with λ large enough.

21.4.5. We now prove the assertion for the pull-push along (21.12). This functor admits a filtration with subquotients of the form

$$(\mathsf{q}_M)_* \left(\mathsf{p}_M^!(-) \overset{!}{\otimes} h_*(\omega_h) \otimes \mathrm{Sym}(\mathrm{Norm}_w) \right),$$

where Norm_w is the normal bundle to $(\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X))_w$ inside $\mathrm{LS}_{\check{P}}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{B}^-}(X)$.

The proof follows the same logic, using the fact that the weights of $\mathbb{G}_m$ on Norm_w are of the form

$$\langle \alpha, \check{\theta} \rangle, \quad \alpha \text{ is a root in } \check{\mathfrak{n}}^-/\check{\mathfrak{n}}^- \cap w(\check{\mathfrak{n}}(\check{P})),$$

and all such weights are also negative.

21.5. **Proof of Theorem 21.2.2 for a general Levi.**

21.5.1. As was stated in Theorem 21.2.2, its proof relies on the validity of the geometric Langlands conjecture for Levi quotients of all proper parabolics of G.

Remark 21.5.2. What we really need to assume for the proof to go through is a certain property of the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$, see Sect. 21.5.7. This property takes place purely on the spectral side, and it follows from GLC.

Let us formulate this property for $\check{G}$. For $\lambda \in \Lambda_G^{+,\mathbb{Q}}$, consider the Verdier quotient category

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \twoheadrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{(\leq\lambda)}, \quad \lambda \in \Lambda_G^{+,\mathbb{Q}},$$

where we kill the subcategory generated by the essential images of the functors

$$\mathrm{Eis}^{\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}'}(X))^{\lambda'} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)), \quad \lambda' \in \Lambda^+_{G,P'},\ \lambda' \underset{G}{\not\leq} \lambda - 2(g-1)\cdot\rho_{P'}.$$

for all standard parabolics P' of G.

What we need is that the functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \lim_{\lambda,\leq_G} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{(<\lambda)}$$

is an equivalence.

With this property at hand, we could imitate the argument Sect. 21.2 essentially word-by-word.

Remark 21.5.3. For $\lambda \in \Lambda_G^{+,\mathbb{Q}}$, denote by $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))^{(<\lambda)}$ the corresponding quotient of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ so that we have a commutative diagram.

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) & \xleftarrow{\Xi^R_{0,\mathrm{Nilp}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))^{(<\lambda)} & \xleftarrow{\Xi^R_{0,\mathrm{Nilp}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{(<\lambda)}, \end{array} \tag{21.18}$$

Assuming GLC for G, it follows from the localization argument given below that the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{(\leq\lambda)}$ is generated by the essential image of

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \overset{\Xi_{0,\mathrm{Nilp}}}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{(<\lambda)}.$$

This implies that the bottom horizontal arrow in (21.18) is actually an equivalence.

In particular, we obtain that the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ can be recovered from the usual $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ also as

$$\lim_{\lambda,\leq_G} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))^{(<\lambda)}.$$

21.5.4. Fix $\mu \in \Lambda^+_M$, and let

$$\mathrm{Bun}_M^{(\leq\mu)} \subset \mathrm{Bun}_M$$

be the quasi-compact open equal to the union of Harder-Narasimhan strata $\mathrm{Bun}_M^{(\mu')}$ with

$$\mu' \underset{M}{\leq} \mu.$$

We consider $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M^{(\leq\mu)})$ as a quotient of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$. Assuming that μ is dominant enough (as a coweight of M), the kernel of the projection

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M^{(\leq\mu)})$$

is generated by the essential images of

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{M'}^{\mu'}), \quad \mu' \in \Lambda^+_{G,P'}, \quad \mu' \underset{M}{\not\leq} \mu - 2(g-1)\cdot\rho_{P'}$$

along the functors

$$\mathrm{Eis}_{!,-\rho_{P'}(\omega_X)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_{M'}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M),$$

where M' is the Levi of a standard parabolic P' of M.

21.5.5. Let $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(<\mu)}$ denote the quotient of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ by the full subcategory generated by the essential images of

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}'})^{\mu'}, \quad \mu' \in \Lambda^+_{G,P'}, \quad \mu' \underset{M}{\not\leq} \mu - 2(g-1)\cdot\rho_{P'}$$

along the functors

$$\mathrm{Eis}^{\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}'}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).$$

21.5.6. The compatibility of the Langlands functor $\mathbb{L}_M$ with Eisenstein series implies that there exists a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M^{(\leq\mu)}) & \xrightarrow[\sim]{\mathbb{L}_M^{(\leq\mu)}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq\mu)} \\ \uparrow & & \uparrow \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow[\sim]{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)). \end{array}$$

21.5.7. Since

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \lim_{\mu} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M^{(\leq\mu)})$$

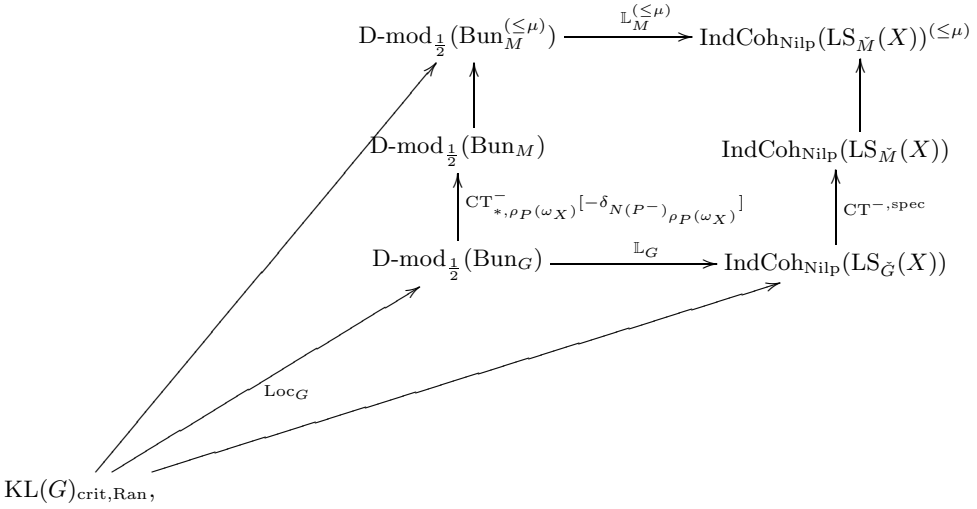
is an equivalence, we obtain that

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \lim_{\mu} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq\mu)}$$

is also an equivalence.

Hence, in order to construct a datum of commutativity for (21.9), it enough to construct a compatible data of commutativity for the diagrams (for varying μ)

(21.19)



compatiblle with the given data of commutativity for the outer diagrams.

21.5.8. Let $\mathrm{Bun}_G^{(<\lambda)} \subset \mathrm{Bun}_G$ be a quasi-compact open union of Harder-Narasimhan strata, such that the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{CT}^-_{*,\rho_P(\omega_X)}}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M^{(\leq\mu)})$$

factors via the quotient

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \twoheadrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G^{(<\lambda)}).$$

As in Sect. 21.3.4, we consider the larger open

$$\mathrm{Bun}_G^{(<\lambda)} \subset \mathrm{Bun}_G^{(/\leq\lambda)},$$

and the corresponding functor

$$(\mathrm{CT}^-_{*,\rho_P(\omega_X)})^{(\leq\mu),(/\geq\lambda)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G^{(/\leq\lambda)}) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M^{(\leq\mu)}).$$

21.5.9. Let

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \twoheadrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{(/\geq\lambda)}$$

denote the corresponding quotient, see Sect. 21.3.6.

The following is a generalization of Proposition 21.3.7:

Proposition 21.5.10. *For a fixed μ and λ large enough in the order relation $\underset{G}{\geq}$, for every standard parabolic P' and $\lambda' \in \Lambda_{G,P'}$ satisfying $\lambda' \underset{G}{\geq} \lambda$, the functor*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}'}(X))^{\lambda'} \overset{\mathrm{Eis}^{\mathrm{spec}}}{\longrightarrow} \\ \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \overset{\mathrm{CT}^{-,\mathrm{spec}}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq\mu)}$$

vanishes.

Let us assume this proposition for a moment and finish the proof of Theorem 21.2.2.

Corollary 21.5.11. *For λ large enough in the order relation $\underset{G}{\geq}$, the composite functor*

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathbb{L}_G}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \overset{\mathrm{CT}^{-,\mathrm{spec}}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq\mu)}$$

also factors via the quotient

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \twoheadrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G^{(/\geq\lambda)}).$$

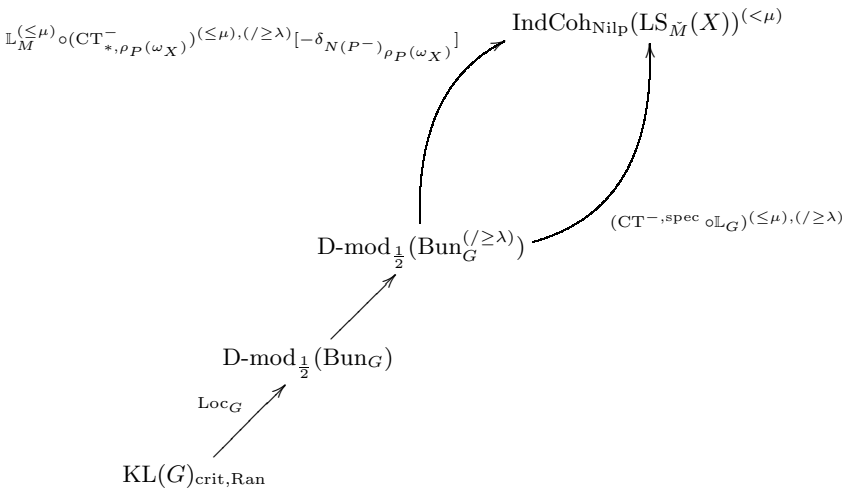
21.5.12. Denote the resulting functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G^{(/\geq\lambda)}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq\mu)}$$

by

$$(\mathrm{CT}^{-,\mathrm{spec}} \circ \mathbb{L}_G)^{(\leq\mu),(/\geq\lambda)}.$$

We obtain that a datum of commutativity for (21.19) is equivalent to that for the diagram



However, this follows again from Lemma 21.3.9.

□[Theorem 21.2.2]

21.5.13. *Proof of Proposition 21.5.10.* The proof proceeds along the same lines as that of Proposition 21.3.7, using the following generalization of the diagram (21.15):

Let P_1 and P_2 be a pair of standard parabolics of G with Levi quotients M_1 and M_2, respectively. For an element

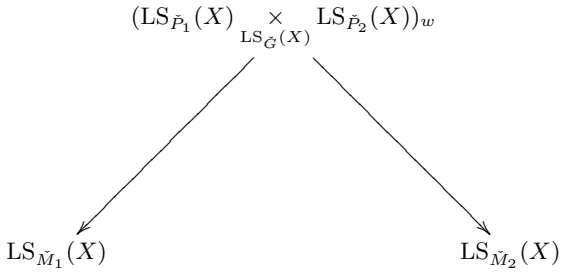
$$w \in W_1 \backslash W / W_2,$$

let

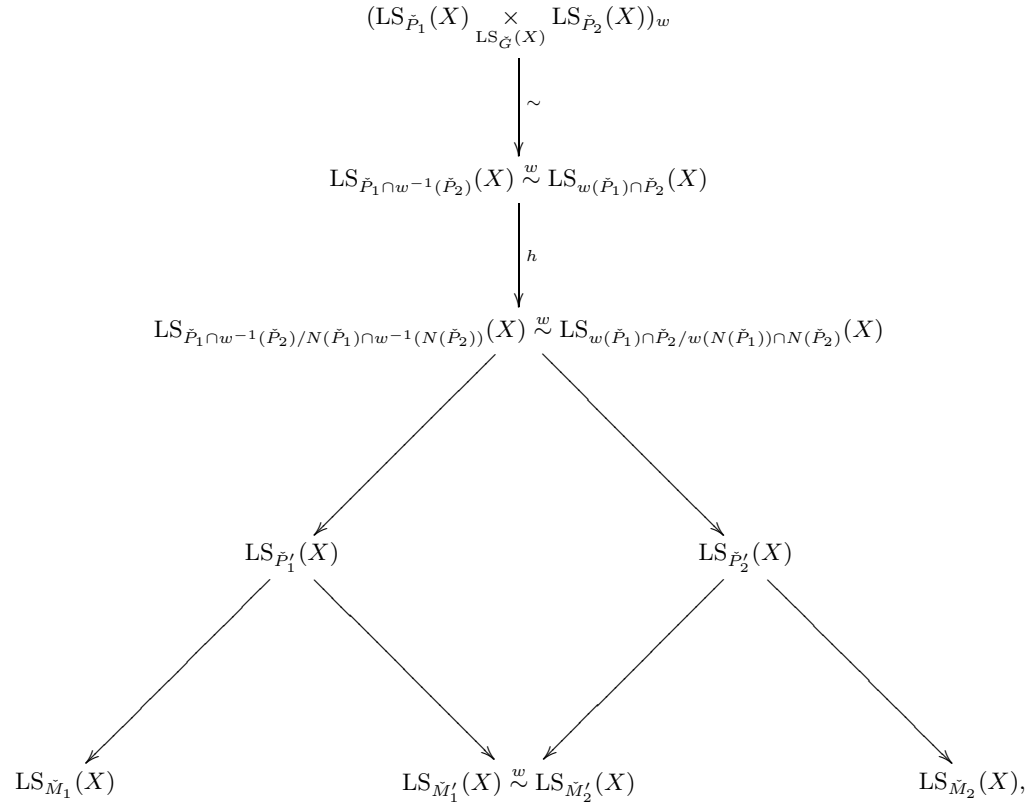
$$(\mathrm{LS}_{\check{P}_1}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{P}_2}(X))_w \subset \mathrm{LS}_{\check{P}_1}(X) \underset{\mathrm{LS}_{\check{G}}(X)}{\times} \mathrm{LS}_{\check{P}_2}(X)$$

be the corresponding locally closed substack.

Then the diagram



can be factored as



in which the middle diamond is Cartesian, where

$$\check{P}'_1 := \check{P}_1 \cap w^{-1}(\check{P}_2)/N(\check{P}_1) \cap w^{-1}(\check{P}_2) \text{ and } \check{P}'_2 := w(\check{P}_1) \cap \check{P}_2/w(N(\check{P}_1)) \cap N(\check{P}_2)$$

are standard parabolics in $\check{M}_1$ and $\check{M}_2$ with Levi quotients $\check{M}'_1$ and $\check{M}'_2$, respectively.

22. Compatibility of the Langlands functor with constant terms: enhanced version

In this section we will prove an enhanced version of Theorem 21.2.2, in which the target category is $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{\mathrm{enh},-}$ instead of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$.

This enhanced version will be used for the proof of the main result of this paper, Theorem 24.1.2.

22.1. **Some enhanced functors.**

22.1.1. Consider the Langlands functor for the group M

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \overset{\mathbb{L}_M}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).$$

It is compatible with the actions of

$$\mathrm{Sph}_M \overset{\mathrm{Sat}_M}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}}.$$

Since the functor $\mathrm{Sat}^{-,\frac{\infty}{2}}$ is compatible with the actions of

$$\mathrm{Sph}_M \overset{\mathrm{Sat}_{M,\tau}}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{M}},$$

it gives rise to a functor

$$\mathrm{Sat}^{-,\frac{\infty}{2}} \otimes \mathbb{L}_M : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\mathrm{Ran}}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}}}$$

(see (1.9)).

22.1.2. It is easy to see that the above functor $\mathrm{Sat}^{-,\frac{\infty}{2}} \otimes \mathbb{L}_M$ sends the full subcategory

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \subset \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\mathrm{Ran}}}$$

to the full subcategory

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}_{\mathrm{Ran}}}.$$

Denote the resulting functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$$

by $\mathbb{L}_M^{-,\mathrm{enh}}$.

22.1.3. Note also that by mimicking the construction of the functor $\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},!}$ (see (19.14)) we can produce a functor

$$\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},*} : \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathrm{Ran}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}. \tag{22.1}$$

It follows formally from Theorem 17.2.4 that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} & \xrightarrow{\mathrm{Id}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\ {\scriptstyle \mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},!}}\uparrow & & \uparrow{\scriptstyle \mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},*} \otimes \mathfrak{l}_{\mathrm{Kost}(\check{G})}[\delta_G]} \\ \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathrm{Ran}} & \xrightarrow[\Theta^{-,\mathrm{enh}}_{\mathrm{Op}(\check{M})}]{} & \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times))^{-,\mathrm{enh}}_{\mathrm{Ran}}, \end{array} \tag{22.2}$$

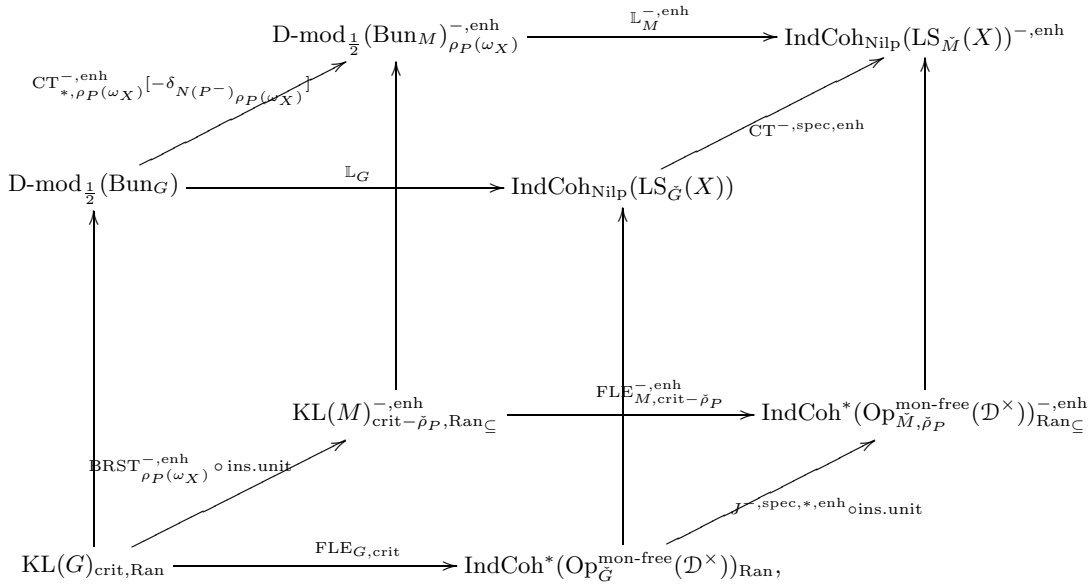
where $\Theta^{-,\mathrm{enh}}_{\mathrm{Op}(\check{M})} := \mathrm{Id} \otimes \Theta_{\mathrm{Op}(\check{M})}$ as the functor

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^!(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)) \to \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\check{\rho}_P}(\mathcal{D}^\times)).$$

22.2. **Enhanced version of the cube.**

22.2.1. Consider the 1-skeleton of the cube:

(22.3)



in which the vertical arrows are as follows:

- The functor $\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ is
$$\mathrm{Loc}_G \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}}[-\delta_{N_{\rho(\omega_X)}}];$$
- The functor $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{G}}(\mathcal{D}^\times))_{\mathrm{Ran}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ is $\mathrm{Poinc}^{\mathrm{spec}}_{\check{G},*}$;
- The functor $\mathrm{KL}(M)^{-,\mathrm{enh}}_{\mathrm{crit}-\tilde{\rho}_P,\mathrm{Ran}_\subseteq} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)}$ is
$$\mathrm{Loc}^{-,\mathrm{enh}}_{M,\rho_P(\omega_X)} \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -\frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)}[-\delta_{N_{\rho(\omega_X)}} - \delta_{N(P^-)_{\rho_P(\omega_X)}}].$$
where $\mathfrak{l}^{\otimes \frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)}$ is as in (15.5);
- The functor $\mathrm{IndCoh}^*(\mathrm{Op}^{\text{mon-free}}_{\check{M},\tilde{\rho}_P}(\mathcal{D}^\times))_{\mathrm{Ran}_\subseteq} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$ is

$$\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},*} \otimes \mathfrak{l}^{\otimes \frac{1}{2}}_{G,N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -1}_{N_{\rho(\omega_X)}} \otimes \mathfrak{l}^{\otimes -\frac{1}{2}}_{G,P^-,M,\rho_P(\omega_X)} \otimes$$
$$\otimes \mathfrak{l}^{\otimes -\frac{1}{2}}_{M,N(M)_{\rho_M(\omega_X)}} \otimes \mathfrak{l}_{N(M)_{\rho_M(\omega_X)}}[\delta_{N(M)_{\rho_M(\omega_X)}} - \delta_{N_{\rho(\omega_X)}} - \delta_{N(P^-)_{\rho_P(\omega_X)}}],$$
where $\mathrm{Poinc}^{-,\mathrm{spec},\mathrm{enh}}_{\check{M},*}$ is as in (22.1).

22.2.2. We claim now that all but the top face of this cube are endowed with a datum of commutativity.

The front face is identical to that of the cube (21.2), and hence commutes. The back face is obtained formally by the enhancement procedure from the back face of the cube (21.2), and hence also commutes.

The commutation of the left face is the content of Theorem 16.6.8. The commutation of the right face is the content of Theorem 19.5.2.

Finally, the commutation of the bottom face in (22.3) is the content of Theorem 9.1.7.

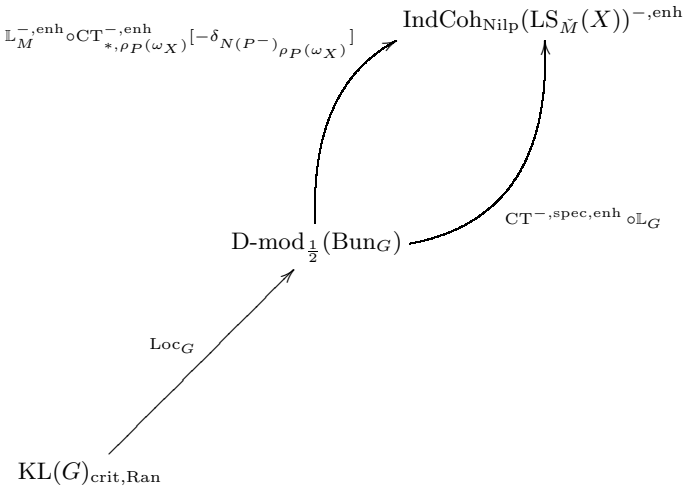
22.2.3. We are now ready to state the enhanced version of Theorem 21.2.2:

Main Theorem 22.2.4. *Assume that the geometric Langlands conjecture holds for M. Then there exists a unique datum of commutativity for the top face in* (22.3)*, such that along with the datum of commutativity of the other five of the faces, the entire cube* (22.3) *commutes.*

22.3. **Proof of Theorem 22.2.4.**

22.3.1. The commutativity of all but the top face in (22.3) implies that the two curved arrows in the diagram

(22.4)



become isomorphic after precomposing with the functor Loc_G. The statement of the theorem is equivalent to the fact that this isomorphism comes from a uniquely defined isomorphism between the curved arrows themselves.

22.3.2. Recall that the forgetful functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$$

is monadic, and that the corresponding monad is given by the action of the associative algebra object

$$\widetilde{\Omega}^{\mathrm{spec}}_{\mathrm{Ran}} \in \mathrm{Sph}^{\mathrm{spec}}_{\check{M},\mathrm{Ran}}.$$

The endofunctor F of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ underlying the above monad has the following property:

It is naturally filtered by the poset $\Lambda^{\mathrm{pos}}_{G,P}$, so that

$$\mathsf{F} \simeq \operatorname*{colim}_{\lambda \in \Lambda^{\mathrm{pos}}_{G,P}} \mathsf{F}_\lambda.$$

22.3.3. Let $\mathsf{F}_{\underline{\lambda}}$ be a composite of a finite collection of the functors F_λ. Let

$$\mathsf{F}_{\underline{\lambda}}\text{-mod}$$

denote the category of

$$\{x \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)), \alpha : \mathsf{F}_{\underline{\lambda}}(x) \to x\}.$$

The category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$ can be identified with a limit of categories of the form $\mathsf{F}_{\underline{\lambda}}$.

22.3.4. We will show that for each $\underline{\lambda}$ *separately*, there exists a *unique* isomorphism between the post-composition of the two curved arrows in (22.4) with the projection

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \to \mathsf{F}_{\underline{\lambda}}\text{-mod},$$

so that after pre-composing with Loc_G we obtain the already existing isomorphism.

The uniqueness assertion implies that these isomorphisms give rise to a uniquely defined isomorphism for the diagram (22.4) itself.

22.3.5. For a finite collection $\underline{\mu}$ of (sufficiently large) elements of Λ^+_M, let

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu})}$$

denote the quotient of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ by the full subcategory generated by the essential images of

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}'})^{\mu'}, \quad \mu' \in \Lambda^+_{G,P'}, \quad \mu' \underset{M}{\not\leq} \mu - 2(g-1)\cdot\rho_{P'},\ \mu \in \underline{\mu}$$

along the functors

$$\mathrm{Eis}^{\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}'}) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).$$

The collections $\underline{\mu}$ naturally form a (filtered) poset, and as in Sect. 21.5.7, the assumption that GLC holds for M implies that the functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \lim_{\underline{\mu}} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu})}$$

is an equivalence.

22.3.6. The functors F_λ (and, hence, their compositions $\mathsf{F}_{\underline{\lambda}}$) have the following property:

For every $\underline{\mu}$, the composite

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \overset{\mathsf{F}_{\underline{\lambda}}}{\to} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu})}$$

factors as

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \twoheadrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu}')} \overset{\mathsf{F}^{\underline{\mu}',\underline{\mu}}_{\underline{\lambda}}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu})}$$

for some sufficiently large $\underline{\mu}'$.

22.3.7. For $\underline{\lambda}$ and $\underline{\mu} \leq \underline{\mu}'$ as above, let

$$\mathsf{F}_{\underline{\lambda}}^{\underline{\mu}',\underline{\mu}}\text{-mod}$$

denote the category of

$$\{x_{\underline{\mu}'} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu}')}, \alpha : \mathsf{F}_{\underline{\lambda}}^{\underline{\mu}',\underline{\mu}}(x_{\underline{\mu}'}) \to x_{\underline{\mu}}\},$$

where $x_{\underline{\mu}}$ denotes the image of $x_{\underline{\mu}'}$ along the projection

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu}')} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu})}.$$

We have:

$$\mathsf{F}_{\underline{\lambda}}\text{-mod} \simeq \lim_{\underline{\mu} \leq \underline{\mu}'} \mathsf{F}_{\underline{\lambda}}^{\underline{\mu}',\underline{\mu}}\text{-mod}.$$

Hence, it is enough to show that for each $\underline{\mu} \leq \underline{\mu}'$, there exists a *unique* isomorphism between the post-composition of the two curved arrows in (22.4) with the projection

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \to \mathsf{F}_{\underline{\lambda}}^{\underline{\mu}',\underline{\mu}}\text{-mod},$$

so that after pre-composing with Loc_G we obtain the already existing isomorphism.

22.3.8. We now note that the for a fixed $\underline{\mu} \leq \underline{\mu}'$, the resulting two arrows

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \rightrightarrows \mathsf{F}_{\underline{\lambda}}^{\underline{\mu}',\underline{\mu}}\text{-mod}$$

both factor as

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^{/\leq \nu} \rightrightarrows \mathsf{F}_{\underline{\lambda}}^{\underline{\mu}',\underline{\mu}}\text{-mod}$$

for some ν.

Indeed, since the functor

$$\mathsf{F}_{\underline{\lambda}}^{\underline{\mu}',\underline{\mu}}\text{-mod} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu}')}$$

is conservative, this follows from the corresponding property of the two arrows

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \rightrightarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{(\leq \underline{\mu}')},$$

established in the course of the proof of Theorem 21.2.2.

22.3.9. Now, the required assertion follows the fact that the functor

$$\mathrm{KL}(G)_{\mathrm{crit},\mathrm{Ran}} \xrightarrow{\mathrm{Loc}_G} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^{/\leq \nu}$$

is a Verdier quotient.

□[Theorem 22.2.4]

23. The left adjoint of the Langlands functor

In this section we start reaping the benefits from the work done until this point.

- We show that the functor $\mathbb{L}_G$ admits a left adjoint (to be denoted $\mathbb{L}_G^L$), which is also compatible with the geometric and spectral Eisenstein series functors;
- We show that, up to a cohomological shift, the functor $\mathbb{L}_G^L$ identifies with the composition

$$\begin{aligned}(23.1)\quad \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) &\overset{\mathrm{Serre}}{\simeq} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{\vee} \xrightarrow{\mathbb{L}_G^{\vee}} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)^{\vee} \overset{\mathrm{Verdier}}{\simeq} \\ &\simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \xrightarrow{\tau_G} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),\end{aligned}$$

  where $\mathrm{Mir}_{\mathrm{Bun}_G}$ is the *Miraculous functor*, and τ_G is the Cartan involution;
- We show that the composition $\mathbb{L}_G \circ \mathbb{L}_G^L$, which is an endofunctor of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ is given by tensor product by an associative algebra object $\mathcal{A}_G \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$.

23.1. **The existence of the left adjoint.**

23.1.1. The goal of this subsection is to prove the following statement:

Theorem 23.1.2. *The functor $\mathbb{L}_G$ admits a left adjoint. Moreover, for every standard parabolic P, we have a commutative diagram*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xleftarrow{(\mathbb{L}_M)^L} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)[\delta_{N(P^-)_{\rho_P(\omega_X)}}]}}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{(\mathbb{L}_G^L)} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{23.2}$$

The rest of the subsection is devoted to the proof of this result.

23.1.3. As was shown in [AG, Theorem 13.3.6], the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ is generated by the essential images of $\mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ along the functors

$$\mathrm{Eis}^{-,\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

In Theorem 21.2.2, we have constructed a commutative square

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\ {\scriptstyle \mathrm{CT}^-_{*,\rho_P(\omega_X)[-\delta_{N(P^-)_{\rho_P(\omega_X)}}]}}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)), \end{array}$$

in which the vertical arrows are the right adjoints to the ones in (23.2).

It follows formally that in order to prove that $(\mathbb{L}_G)^L$ exists and makes (23.2) commute, it suffices to show that for every M, the (a priori partially defined) left adjoint $(\mathbb{L}_M)^L$, is actually defined on $\mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X)) \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$.

23.1.4. Up to changing the notation, we can assume that $M = G$. However, then the existence of $(\mathbb{L}_G)^L|_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))}$ was built in to the construction of $\mathbb{L}_G$: this is the functor $\mathbb{L}^L_{G,\mathrm{temp}}$ of (20.3).

□[Theorem 23.1.2]

23.2. **The left adjoint as a dual.**

23.2.1. Recall that $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}$ denotes the dual of the category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$. Consider the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \xleftarrow{\mathbb{L}^\vee_G} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)),$$

dual to $\mathbb{L}^\vee_G$, where we identify $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ with its own dual via Serre duality.

We now recall the *Miraculous Functor*.

$$\mathrm{Mir}_{\mathrm{Bun}_G} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

see [Gai2, Sect. 2.1.1].

Remark 23.2.2. In [Gai2], the functor $\mathrm{Mir}_{\mathrm{Bun}_G}$ was defined in the untwisted setting, i.e., for $\text{D-mod}(\mathrm{Bun}_G)$ rather than for $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$. However, the same procedure defines is also for $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$, and the resulting functor has the same properties: the two setting are actually equivalent, see Remark 12.1.6.

23.2.3. Define the functor

$$\Phi_G : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \xrightarrow{\mathbb{L}^\vee_G} \text{D-mod}(\mathrm{Bun}_G)$$

as

$$\Phi_G := \tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathbb{L}^\vee_G[4\delta_G - 2\delta_{N_{\rho(\omega_X)}}].$$

23.2.4. We are going to prove:

Theorem 23.2.5. *There exists a canonical isomorphism*

$$\Phi_G \simeq (\mathbb{L}_G)^L.$$

23.3. **Proof of Theorem 23.2.5.**

23.3.1. Both functors appearing in Theorem 23.2.5 are (automatically) compatible with the derived Hecke action via

$$\mathrm{Sph}_G \overset{\mathrm{Sat}_G}{\simeq} \mathrm{Sph}^{\mathrm{spec}}_{\check{G}},$$

see Sect. 1.8.8.

Hence, they both define functors

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \underset{\Phi_{G,\mathrm{temp}}}{\overset{\mathbb{L}^L_{G,\mathrm{temp}}}{\leftrightarrows}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$$

that make both diagrams

$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \underset{\Phi_G}{\overset{\mathbb{L}^L_G}{\longleftarrow}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\ \uparrow & & \uparrow{\scriptstyle \Xi_{0,\mathrm{Nilp}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} & \underset{\Phi_{G,\mathrm{temp}}}{\overset{\mathbb{L}^L_{G,\mathrm{temp}}}{\longleftarrow}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \underset{\Phi_G}{\overset{\mathbb{L}_G}{\longleftarrow}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\ \downarrow & & \downarrow{\scriptstyle (\Xi_{0,\mathrm{Nilp}})^R} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} & \underset{\Phi_{G,\mathrm{temp}}}{\overset{\mathbb{L}^L_{G,\mathrm{temp}}}{\longleftarrow}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \end{array} \tag{23.3}$$

commute.

23.3.2. We will first show that the functors $\mathbb{L}^L_{G,\mathrm{temp}}$ and $\Phi_{G,\mathrm{temp}}$ are (canonically) isomorphic.

Since the functor

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$$

is a Verdier quotient, it suffices to establish an equivalence

$$\mathbb{L}^L_{G,\mathrm{temp}} \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G}} \simeq \Phi_{G,\mathrm{temp}} \circ \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}. \tag{23.4}$$

By construction, the functor $\mathbb{L}^L_{G,\mathrm{temp}}$ makes the diagram

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \underset{}{\overset{(\mathrm{CS}_G)^{-1}}{\longleftarrow}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{Poinc}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} & \underset{}{\overset{\mathbb{L}^L_{G,\mathrm{temp}}}{\longleftarrow}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \end{array}$$

commute.

Hence, it suffices to check that the diagram

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \overset{(\mathrm{CS}_G)^{-1}}{\longleftarrow} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{Poinc}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} & \overset{\tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathbb{L}^\vee_G|_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))}[4\delta_G - 2\delta_{N_{\rho(\omega_X)}}]}{\longleftarrow} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \end{array}$$

commutes as well.

23.3.3. Recall (see Lemma 1.4.11) that

$$\tau_G \circ (\mathrm{CS}_G)^{-1} \simeq \Theta_{\mathrm{Whit}(G)} \circ \mathrm{FLE}_{\check{G},\infty}.$$

Thus, we need to establish the commutativity of

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xleftarrow{\Theta_{\mathrm{Whit}(G)} \circ \mathrm{FLE}_{\check{G},\infty}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{Poinc}_{G,!}}\downarrow & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} & \xleftarrow{\mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathbb{L}^{\vee}_G|_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))}[4\delta_G]} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \end{array} \tag{23.5}$$

23.3.4. Recall that the functors

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \leftrightarrows \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) : \Gamma^{\mathrm{spec}}_{\check{G}}$$

are mutually dual, when we identify $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ with its own dual via the usual duality (i.e., usual dualization on perfect comlexes).

Hence, when we use the identification

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))^{\vee} \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)),$$

induced by the Serre duality on $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X))$, the dual of the functor $\Gamma^{\mathrm{spec}}_{\check{G}}$ becomes identified with

$$\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}[\dim(\mathrm{LS}_{\check{G}}(X))] = \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}[2\delta_G].$$

Here we are using the fact that $\mathrm{LS}_{\check{G}}$ is quasi-smooth and *derived-symplectic*, so that

$$\omega_{\mathrm{LS}_{\check{G}}} \simeq \mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}[\dim(\mathrm{LS}_{\check{G}}(X))].$$

Hence, by taking the duals in the commutative diagram

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xrightarrow{\mathrm{CS}_G = (\mathrm{FLE}_{\check{G},\infty})^{\vee}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec}}_{\check{G}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} & \xrightarrow{\mathbb{L}_{G,\mathrm{temp}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)), \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Whit}_*(\mathrm{Gr}_G)_{\mathrm{Ran}} & \xleftarrow{\mathrm{FLE}_{\check{G},\infty}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{Poinc}_{G,*}[2\delta_{N_{\rho(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}[2\delta_G]} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co,temp}} & \xleftarrow{\mathbb{L}^{\vee}_G|_{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{23.6}$$

23.3.5. We now recall the following result, established in [Lin, Theorem 1.1.6]:

Theorem 23.3.6. *The diagram*

$$\begin{array}{ccc} \mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xleftarrow{\Theta_{\mathrm{Whit}(G)}} & \mathrm{Whit}_*(\mathrm{Gr}_G)_{\mathrm{Ran}} \\ {\scriptstyle \mathrm{Poinc}_{G,!}}\downarrow & & \downarrow{\scriptstyle \mathrm{Poinc}_{G,*}[2\delta_{N_{\rho(\omega_X)}}]} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathrm{Mir}_{\mathrm{Bun}_G}[2\delta_G]} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}. \end{array} \tag{23.7}$$

23.3.7. Now, concatenating the diagrams (23.6) and (23.7). we obtain the desired commutative diagram (23.5). Thus, we have established the isomorphism

$$\mathbb{L}^L_{G,\mathrm{temp}} \simeq \Phi_{G,\mathrm{temp}}.$$

23.3.8. We will now deduce

$$\mathbb{L}^L_{G,\mathrm{temp}} \simeq \Phi_{G,\mathrm{temp}} \Rightarrow (\mathbb{L}_G)^L \simeq \Phi_G.$$

The functor $(\mathbb{L}_G)^L$ being a left adjoint, preserves compactness. We will prove:

Lemma 23.3.9. *The functor Φ_G also preserves compactness.*

Lemma 23.3.10. *The functor*

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}},$$

right adjoint to the tautological embedding, is fully faithful on compact objects.

Assuming these two lemmas for a moment, we obtain the assertion of Theorem 23.2.5 from the commutative diagram (23.3).

23.4. **Proof of Lemma 23.3.9.**

23.4.1. It is enough to show that Φ_G sends objects of the form $\mathrm{Eis}^{-,\mathrm{spec}}(\mathcal{F})$, for $\mathcal{F} \in \mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X))^c$ to compacts.

23.4.2. Note the functors

$$\mathrm{Eis}^{-,\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \leftrightarrows \mathrm{CT}^{-,\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$$

are mutually dual, up to tensoring by a line bundle on $\mathrm{LS}_{\check{M}}$.

Hence

$$\mathbb{L}^\vee_G \circ \mathrm{Eis}^{-,\mathrm{spec}}$$

is isomorphic to

$$(\mathrm{CT}^{-,\mathrm{spec}} \circ \mathbb{L}_G)^\vee,$$

up to tensoring by a line bundle.

Hence, combining with Theorem 21.2.2, we obtain that $\mathbb{L}^\vee_G \circ \mathrm{Eis}^{-,\mathrm{spec}}$ is isomorphic to

$$(\mathbb{L}_M \circ \mathrm{CT}^-_*)^\vee \simeq \mathrm{Eis}^-_* \circ \mathbb{L}^\vee_M,$$

again up to tensoring by a line bundle.

23.4.3. Now, by [Gai2, Theorem 4.1.2],

$$\tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}^-_* \simeq \mathrm{Eis}^-_! \circ \mathrm{Mir}_{\mathrm{Bun}_M} \circ \tau_M.$$

Combining, we obtain that

$$\Phi_G \circ \mathrm{Eis}^{-,\mathrm{spec}} \simeq \mathrm{Eis}^-_! \circ \Phi_M,$$

again up to tensoring by a line bundle.

Now, the assertion follows from the fact that the functors Φ_M send compacts in $\mathrm{QCoh}(\mathrm{LS}_{\check{M}}(X))$ to compacts in $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$, which follows from the isomorphism $\mathbb{L}^L_{M,\mathrm{temp}} \simeq \Phi_{M,\mathrm{temp}}$.

□[Lemma 23.3.9]

23.5. **Proof of Lemma 23.3.10.** For this proof we can (and will) identify $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$ with $\text{D-mod}(\mathrm{Bun}_G)$.

23.5.1. For an object $\mathcal{F} \in \text{D-mod}(\mathrm{Bun}_G)$, let

$$\mathcal{F}_{\mathrm{temp}} \to \mathcal{F} \to \mathcal{F}_{\text{anti-temp}}$$

be the fiber sequence associated with

$$\text{D-mod}(\mathrm{Bun}_G)_{\mathrm{temp}} \hookrightarrow \text{D-mod}(\mathrm{Bun}_G).$$

We have to show that for a pair of compact object $\mathcal{F}_1, \mathcal{F}_2 \in \text{D-mod}(\mathrm{Bun}_G)$, the map

$$\mathrm{Hom}_{\text{D-mod}(\mathrm{Bun}_G)}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\text{D-mod}(\mathrm{Bun}_G)}(\mathcal{F}_{1,\mathrm{temp}}, \mathcal{F}_{2,\mathrm{temp}})$$

is an isomorphism.

23.5.2. As we shall see, just the assumption that $\mathcal{F}_2$ be compact will suffice. Thus, we have to show that if $\mathcal{F}_1$ is *anti-tempered*, i.e., if

$$\mathcal{F}_1 \to \mathcal{F}_{1,\text{anti-temp}}$$

is an isomorphism, and $\mathcal{F}_2$ is compact, then

$$\mathrm{Hom}_{\text{D-mod}(\mathrm{Bun}_G)}(\mathcal{F}_1, \mathcal{F}_{2,\text{temp}}) = 0.$$

Recall (see [Gai2, Theorem 3.1.5]) that the functor $\mathrm{Mir}_{\mathrm{Bun}_G}$ is an equivalence. Hence, it is enough to show that

$$\mathrm{Hom}_{\text{D-mod}(\mathrm{Bun}_G)_{\mathrm{co}}}(\mathrm{Mir}^{-1}_{\mathrm{Bun}_G}(\mathcal{F}_1), \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}(\mathcal{F}_2)) = 0. \tag{23.8}$$

23.5.3. Recall that compact objects in D-mod(Bun_G) are of the form

$$j_U(\mathcal{F}_U),$$

where

$$U \overset{j_U}{\hookrightarrow} \mathrm{Bun}_G$$

is the embedding of a quasi-compact open, and $\mathcal{F}_U \in \text{D-mod}(U)^c$.

With no restriction of generality, we can assume that U is *co-truncative* (see [DG, Sect. 3.1] for what this means). In this case we have

$$\mathrm{Mir}^{-1}_{\mathrm{Bun}_G} \circ (j_U)_! \simeq (j_U)_{*,co},$$

where

$$(j_U)_{*,co} : \text{D-mod}(U) \to \text{D-mod}(\mathrm{Bun}_G)_{\mathrm{co}}$$

is the tautological functor.

Hence, in order the prove (23.8), it suffices to show that if $\mathcal{F} \in \text{D-mod}(\mathrm{Bun}_G)$ is anti-tempered, then

$$(j_U)^*_{\mathrm{co}} \circ \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}(\mathcal{F}) = 0,$$

where $(j_U)^*_{\mathrm{co}}$ is the left adjoint of $(j_U)_{*,co}$.

23.5.4. Let

$$\mathrm{Id}^{\mathrm{nv}} : \text{D-mod}(\mathrm{Bun}_G)_{\mathrm{co}} \to \text{D-mod}(\mathrm{Bun}_G)$$

be the naive functor (see [Gai2, Sect. 2.1]). Recall that

$$(j_U)^*_{\mathrm{co}} \simeq (j_U)^* \circ \mathrm{Id}^{\mathrm{nv}}.$$

Hence, it suffices to show that the compisite functor

$$\mathrm{Id}^{\mathrm{nv}} \circ \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}$$

annihilates the anti-tempered subcategory.

23.5.5. Note that both functors $\mathrm{Id}^{\mathrm{nv}}$ and $\mathrm{Mir}_{\mathrm{Bun}_G}$ commute with the Hecke action. Hence, it suffices to side for any $\mathcal{F} \in \text{D-mod}(\mathrm{Bun}_G)$, we have

$$(\mathrm{Id}^{\mathrm{nv}} \circ \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}(\mathcal{F}))_{\text{anti-temp}} = 0.$$

However, this follows follows from the next result of [Ber2]:

Theorem 23.5.6. *The functor* $\mathrm{Id}^{\mathrm{nv}} \circ \mathrm{Mir}^{-1}_{\mathrm{Bun}_G}$ *sends* D-mod(Bun_G) *to* D-mod(Bun_G)$_{\mathrm{temp}}$.

□[Lemma 23.3.10]

23.6. **The composition $\mathbb{L}_G \circ (\mathbb{L}_G)^L$.** Recall that the geometric Langlands conjecture says that the functor $\mathbb{L}_G$ is an equivalence. We can now reformulate this as saying that the unit of the adjunction

$$\mathrm{Id} \to \mathbb{L}_G \circ (\mathbb{L}_G)^L$$

is an isomorphism as endofunctors of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$, combined with the fact that the functor $\mathbb{L}_G$ is conservative (the latter will be proved in Part V of this paper, see Sect. 24.3.2).

In this subsection we commence the study of the composition $\mathbb{L}_G \circ (\mathbb{L}_G)^L$.

23.6.1. Note that $\mathbb{L}_G \circ (\mathbb{L}_G)^L$, viewed as an endofunctor of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$, is $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$-linear. Hence, it it is a priori given by an object in

$$\mathcal{A}_G \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

The goal of this subsection is to prove the following assertion:

Theorem 23.6.2. *The object $\mathcal{A}_G$ belongs to*

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \subset$$
$$\subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

In other words, this theorem implies that we have an isomorphism

$$\mathbb{L}_G \circ (\mathbb{L}_G)^L \simeq \mathcal{A}_G \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} -, \quad \mathcal{A}_G \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)).$$

Remark 23.6.3. Once Theorem 23.6.2 is proved, and given the fact that the functor $\mathbb{L}_G$ is conservative, we will have interpreted Conjecture 20.3.8 as the statement that the unit of the adjunction

$$\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)} \to \mathcal{A}_G \tag{23.9}$$

is an isomorphism in $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$.

In Part V of this paper, we will show that the map (23.9) becomes an isomorphism when restricted to the *reducible* locus of $\mathrm{LS}_{\check{G}}(X)$. This will reduce Conjecture 20.3.8 to the study of the restriction of $\mathcal{A}_G$ to the *irreducible* locus $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X)(X)$.

In Paper 3 of this series, we will show that

$$\mathcal{A}_{G,\mathrm{irred}} := \mathcal{A}_G|_{\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X)}$$

is a *classical* vector bundle equipped with a flat connection.

In Paper 4 of the series, we will deduce from this that (23.9) is an isomorphism also over $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X)$, thereby proving Conjecture 20.3.8.

23.7. **Proof of Theorem 23.6.2.**

23.7.1. Let $j : \mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X) \subset \mathrm{LS}_{\check{G}}(X)$ denote the embedding of the locus of irreducible local systems. Let $i : \mathrm{LS}^{\mathrm{red}}_{\check{G}}(X) \to \mathrm{LS}_{\check{G}}(X)$ be the embedding of its complement (with any scheme-theoretic structure).

Let

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \tag{23.10}$$

be the full subcategory of objects set-theoretically supported on $\mathrm{LS}^{\mathrm{red}}_{\check{G}}$.

In other words,

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} = \mathrm{ket}(j^* : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X))).$$

Denote by $\widehat{i}_!$ the tautological embedding,

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \hookrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$$

and by $\widehat{i}^!$ ist right adjoint.

$$\widehat{i}_! : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) : \widehat{i}^!.$$

23.7.2. In order to prove that $\mathcal{A}_G$ belongs to $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$, it suffices to show that

$$(j^* \otimes j^*)(\mathcal{A}_G) \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X))$$

and

$$(\widehat{i}^! \otimes \widehat{i}^!)(\mathcal{A}_G) \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)). \tag{23.11}$$

The former is automatic, since the spectral nilpotent cone Nilp restricted to $\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X)$ consists only of the zero section, so the embedding

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X)) \subseteq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X))$$

is an equality, see also [AG, Proposition 13.3.3].

We now proceed to proving (23.11).

23.7.3. Since both functors $\mathbb{L}_G$ and $\mathbb{L}^L_G$ are compatible with the inclusions and projections

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \leftrightarrows \mathrm{QCoh}(\mathrm{LS}_G) \text{ and } \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \leftrightarrows \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}},$$

so is the composition $\mathbb{L}_G \circ \mathbb{L}^L_G$.

Denote the induced endofunctor of $\mathrm{QCoh}(\mathrm{LS}_G)$ by $\mathcal{A}_{G,\mathrm{temp}}$. In other words,

$$\mathbb{L}_{G,\mathrm{temp}} \circ (\mathbb{L}^L_G)_{\mathrm{temp}} := \mathcal{A}_{G,\mathrm{temp}} \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} -, \quad \mathcal{A}_{G,\mathrm{temp}} \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)).$$

In order to prove (23.11), it suffices to show that

$$(\mathbb{L}_G \circ \mathbb{L}^L_G)|_{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}} \simeq \mathcal{A}_{G,\mathrm{temp}} \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} -$$

as endofunctors of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}$.

For that end, it suffices to establish a functorial isomorphism

$$\mathcal{A}_G(\mathcal{M}) \simeq \mathcal{A}_{G,\mathrm{temp}} \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} \mathcal{M}, \quad \mathcal{M} \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^c_{\mathrm{red}}. \tag{23.12}$$

23.7.4. For $\mathcal{M}$ as above, let

$$\mathcal{M}_{\mathrm{temp}} \text{ and } \mathcal{A}_G(\mathcal{M})_{\mathrm{temp}}$$

denote the projections of $\mathcal{M}$ and $\mathcal{A}_G(\mathcal{M})$, respectively, along

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}, \tag{23.13}$$

where

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} := \ker\left(j^* : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X))\right). \tag{23.14}$$

By construction

$$\mathcal{A}_G(\mathcal{M})_{\mathrm{temp}} \simeq \mathcal{A}_{G,\mathrm{temp}} \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} \mathcal{M}_{\mathrm{temp}}. \tag{23.15}$$

Thus, we wish to show that (23.15) implies (23.12).

23.7.5. We claim:

Proposition 23.7.6. *The functor* $(\mathbb{L}_G \circ \mathbb{L}^L_G)|_{\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}}$ *preserves compactness.*

Let us assume this proposition temporarily and finish the proof of Theorem 23.6.2.

23.7.7. We claim that Proposition 23.7.6 implies the following:

Corollary 23.7.8. *The restriction of* $\mathcal{A}_{G,\mathrm{temp}}$ *to the formal completion* $(\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)}$ *is perfect as an object of* $\mathrm{QCoh}((\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)})$.

Proof of Corollary 23.7.8. The assertion of Corollary 23.7.8 can be reformulated as saying that

$$\mathcal{A}_{G,\mathrm{temp}}\big|_{(\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)}}$$

is dualizable as an object of $\mathrm{QCoh}((\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)})$, and equivalently, as saying that the functor

$$\mathcal{A}_{G,\mathrm{temp}} \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} - : \mathrm{QCoh}((\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)}) \to \mathrm{QCoh}((\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)})$$

admits a right adjoint.

Note that restriction along

$$(\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)} \hookrightarrow \mathrm{LS}_{\check{G}}(X)$$

defines an equivalence

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \to \mathrm{QCoh}((\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}(X)}),$$

where

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \subset \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$$

is as in (23.14).

According to [AG, Corollary 9.2.7], the category $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}$ is compactly generated by

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))^c \cap \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \subset \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}.$$

Hence, it suffices to show that the functor

$$\mathcal{A}_{G,\mathrm{temp}} \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} - : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}$$

preserves compactness. However, this follows from Proposition 23.7.6 via the commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} & \xrightarrow{\mathbb{L}_G \circ \mathbb{L}^L_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} & \xrightarrow{\mathcal{A}_{G,\mathrm{temp}} \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} -} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}. \end{array}$$

Indeed, an object of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))$ is compact if and only if its image in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ is compact.

□

23.7.9. Let $\mathcal{M}$ be as in (23.12). By Proposition 23.7.6,

$$\mathcal{A}_G(\mathcal{M}) \in \mathrm{Coh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} := \mathrm{Coh}(\mathrm{LS}_{\check{G}}(X)) \cap \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)),$$

and by Corollary 23.7.8, the object $\mathcal{A}_{G,\mathrm{temp}} \underset{\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}}{\otimes} \mathcal{M}$ also belongs to this subcategory.

The projections of these objects along (23.13) are identified by (23.15). The isomorphism (23.13) now follows, since the restriction of (23.13) to

$$\mathrm{Coh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \subset \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}$$

is fully faithful.

□[Theorem 23.6.2]

23.8. **Proof of Proposition 23.7.6.**

23.8.1. Let

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

denote the full subcategory generated by the essential images of the functors

$$\text{Eis}_! : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)$$

for Levi quotients M of proper parabolics $P \subset G$.

23.8.2. We claim that the functors

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \leftrightarrows \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)) : \mathbb{L}_G^L$$

send the full subcategories

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \text{ and } \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}} \subset \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))$$

to one another, and the resulting functors

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \leftrightarrows \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}} : \mathbb{L}_G^L$$

preserve compactness.

This would imply that $(\mathbb{L}_G \circ \mathbb{L}_G^L)|_{\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}}}$ also preserves compactness.

23.8.3. For the functor $\mathbb{L}_G$, this follows from its compatibility with the Eisenstein procedure, expressed by Theorem 20.4.5.

23.8.4. To prove the assertion for $\mathbb{L}_G^L$, we note that the subcategory (23.10) is generated by the essential images of the functors

$$\text{Eis}^{\text{spec}} : \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{M}}(X)) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))$$

for proper parabolics (indeed, the collection of their right adjoints, i.e., the functors CT^{spec} is conservative on $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)_{\text{red}})$.

Now, the assertion concerning $\mathbb{L}_G^L$ follows from the commutative diagram (23.2).

□[Proposition 23.7.6]

Part V. The Langlands functor is an equivalence on Eisenstein subcategories

In this Part we prove the main result of this paper, Theorem 24.1.2, which says that the Langlands functor $\mathbb{L}_G$ and its left adjoint $\mathbb{L}_G^L$ define mutually inverse equivalences between the following full subcategories on the geometric and spectral sides, respectively:

- On the geometric side, this is the subcategory
$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G),$$
generated by the essential images of the Eisenstein functors $\text{Eis}_!$ for all *proper* parabolics;
- On the spectral side, this is the full subcategory
$$\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}} \subset \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X)),$$
consisting of objects, set-theoretically supported on the locus $\text{LS}_{\check{G}}^{\text{red}}(X) \subset \text{LS}_{\check{G}}(X)$, consisting of *reducible* local systems.

As an almost immediate corollary of this result, we will obtain a proof of the geometric Langlands conjecture (Conjecture 20.3.8) when the group G is GL_n, see Sect. 24.2.

We will reduce the assertion of Theorem 24.1.2 to the following one (Theorem 24.4.2): the functors
$$\text{CT}^{-,\text{spec},\text{part.enh}} \text{ and } \text{CT}^{-,\text{spec},\text{part.enh}} \circ \mathbb{L}_G \circ \mathbb{L}_G^L,$$
are canonically isomorphic, where $\text{CT}^{-,\text{spec},\text{part.enh}}$ is the *partially enhanced* spectral constant term functor from Sect. 19.3.3.

In its turn, the proof of Theorem 24.4.2 will use the following ingredients:

- The expression for $\mathbb{L}_G^L$ via $\mathbb{L}_G^\vee$ (see formula (23.1));
- The self-duality of $\text{I}(G, P^-)^{\text{loc}}$ (see Theorem 3.2.2);
- The relation of the above self-duality to the Miraculous functor $\text{Mir}_{\text{Bun}_G}$ (Theorem 25.2.3);
- The relation of the above self-duality to the partial enhancement, expressed by (the innocuous-looking) Lemma 3.4.2.

24. Statement of the equivalence

In this section we state our main result, Theorem 24.1.2 and commence it proof, by reducing it to Theorem 24.4.2, and further, to Theorem 24.5.7.

We will first prove the unenhanced version of Theorem 24.5.7, given by Theorem 24.6.2. This proof of Theorem 24.5.7 will follow the same pattern, once we decorate it with appropriate manipulations on the local semi-infinite categories, i.e.,
$$\text{I}(G, P^-)^{\text{loc}} \text{ and } \text{I}(\check{G}, \check{P}^-)^{\text{loc},\text{spec}}.$$

As a first application of Theorem 24.1.2, we give a proof of the geometric Langlands conjecture for the group GL_n.

24.1. Statement of the result.

24.1.1. In Sect. 23.8.1, we have considered the pair of (mutually adjoint) functors

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \leftrightarrows \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}} : \mathbb{L}_G^L. \tag{24.1}$$

In this section we will formulate and begin the proof the main result of this paper:

Main Theorem 24.1.2. *Let us assume that GLC is valid for Levi subgroups of all proper parabolics of G. Then the adjoint functors in* (24.1) *are mutually inverse equivalences.*

24.1.3. From Theorem 24.1.2 we obtain:

Corollary 24.1.4. *The natural transformation in* (21.8) *is an isomorphism.*

Proof. By passing to the right adjoints along the vertical arrows in

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\text{Bun}_M) & \xrightarrow[\sim]{\mathbb{L}_M} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{M}}(X)) \\ \Big\downarrow{\scriptstyle \text{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(\omega_X)}}]} & & \Big\downarrow{\scriptstyle \text{Eis}^{-,\text{spec}}} \\ \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} & \xrightarrow[\sim]{\mathbb{L}_G} & \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}}, \end{array}$$

we obtain that the natural transformation in question is an equivalence, once both circuits on the diagram are restricted to

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}} \subset \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G).$$

Hence, in order to prove the corollary, it suffices to show that both circuits vanish when restricted to

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} := (\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}})^{\perp}.$$

The vanishing is tautological for the clockwise circuit. For the vanishing of the anti-clockwise circuit, it suffices to show that there exists *some* isomorphism

$$\text{CT}^{-,\text{spec}} \circ \mathbb{L}_G \simeq \mathbb{L}_M \circ \text{CT}^-_{*,\rho_P(\omega_X)}[-\delta_{N(P^-)_{\rho_P(\omega_X)}}].$$

However, the latter is given by Theorem 21.2.2.

□

Remark 24.1.5. As was already mentioned, we do not know whether the isomorphism

$$\text{CT}^{-,\text{spec}} \circ \mathbb{L}_G \simeq \mathbb{L}_M \circ \text{CT}^-_{*,\rho_P(\omega_X)}[-\delta_{N(P^-)_{\rho_P(\omega_X)}}],$$

constructed in Corollary 24.1.4 above equals the one given by Theorem 21.2.2.

Remark 24.1.6. Note that Theorem 24.1.2 implies that the GLC is equivalent to the statement that the functors $(\mathbb{L}^L_G, \mathbb{L}_G)$ define mutually inverse equivalences

$$\mathbb{L}_{G,\text{irred}} : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \leftrightarrows \text{IndCoh}_{\text{Nilp}}(\text{LS}^{\text{irred}}_{\check{G}}(X)) \simeq \text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}(X)) : \mathbb{L}^L_{G,\text{irred}}. \tag{24.2}$$

As was mentioned in Remark 23.6.3, this is equivalent to showing that the map

$$\mathcal{O}_{\text{LS}^{\text{irred}}_{\check{G}}(X)} \to \mathcal{A}_{G,\text{irred}}, \tag{24.3}$$

induced by (23.9), is an isomorphism.

24.2. **Proof of Conjecture 20.3.8 for $G = GL_n$.** Assuming Theorem 24.1.2, in this subsection we will give a proof of Conjecture 20.3.8 in the case when $G = GL_n$.

24.2.1. First off, by induction on n, we can assume that Conjecture 20.3.8 holds for all proper Levi subgroups of G. Hence, the conditions of Theorem 24.1.2 are satisfied.

Hence, by Remark 24.1.6 it remains to show that the (mutually adjoint) functors in (24.2) are (mutually inverse) equivalences of categories.

We need to show that the map (24.3) is an isomorphism. It suffices to show that it induces an isomorphism at the level of fibers at all geometric points of $\text{LS}^{\text{irred}}_{\check{G}}(X)$.

24.2.2. We now recall the following assertion, which follows from the main result [Ber1]:

Theorem 24.2.3. *Suppose that $G = GL_n$. Then the composite functor*

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \hookrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \overset{\mathrm{coeff}_G}{\to} \mathrm{Whit}^!(G)_{\mathrm{Ran}} \tag{24.4}$$

is fully faithful.

Remark 24.2.4. Note that once Conjecture 20.3.8 is proved, we would know that the assertion of Theorem 24.2.3 holds for any G.

Note also that, thanks to [FR1], we already know that the above functor is *conservative* for any G.

24.2.5. Passing left to adjoint functors in (24.4), we obtain that the functor

$$\mathrm{Whit}^!(G)_{\mathrm{Ran}} \overset{\mathrm{Poinc}^!_G}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \twoheadrightarrow \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}}$$

is a *localization*.

Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Whit}^!(\mathrm{Gr}_{G,\mathrm{Ran}}) & \xleftarrow[\sim]{\mathrm{CS}_G^{-1}} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
{\scriptstyle \mathrm{Poinc}_{G,!}[-2\delta_{N_{\rho(\omega_X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xleftarrow{\mathbb{L}^L_{G,\mathrm{temp}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \\
\downarrow & & \downarrow \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} & \xleftarrow{\mathbb{L}^L_{G,\mathrm{irred}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)),
\end{array} \tag{24.5}$$

in which the upper portion is (20.5).

As we have just seen, the left composite vertical arrow in (24.5) is a localization. Note that the composite right vertical arrow in (24.5) is also a localization: indeed, each of the right vertical arrows has this property.

Since the upper horizontal arrow is an equivalence, we obtain that the functor $\mathbb{L}^L_{G,\mathrm{irred}}$ is a localization.

24.2.6. Let

$$\sigma : \mathrm{Spec}(K) \to \mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X)$$

be a geometric point. Applying base change

$$- \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X))}{\otimes} \mathrm{Vect}_K$$

to (24.2), we obtain an adjunction

$$\mathbb{L}_{G,\sigma} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{cusp}} \underset{\mathrm{QCoh}(\mathrm{LS}^{\mathrm{irred}}_{\check{G}}(X))}{\otimes} \mathrm{Vect}_K \leftrightarrows \mathrm{Vect}_K : \mathbb{L}^L_{G,\sigma} \tag{24.6}$$

and the corresponding morphism of K-algebras

$$K \to \mathcal{A}_{G,\sigma}, \tag{24.7}$$

where $\mathcal{A}_{G,\sigma}$ is the fiber of $\mathcal{A}_G$ at σ.

The fact that $\mathbb{L}^L_{G,\mathrm{irred}}$ is a localization implies that $\mathbb{L}^L_{G,\sigma}$ is also a localization. This means that either the map (24.7) is an isomorphism (which is what we want to show), or $\mathcal{A}_{G,\sigma} = 0$.

We claim, however, that the latter is impossible (note that in the argument given below we will use yet another additional piece of knowledge about GL_n).

24.2.7. Note that if $\mathcal{A}_{G,\sigma}$ were 0, this would mean that the category

$$\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma := \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}} \underset{\text{QCoh}(\text{LS}^{\text{irred}}_{\check{G}}(X))}{\otimes} \text{Vect}_K$$

is 0.

Performing base change $k \rightsquigarrow K$, we identify $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma$ with the category of *Hecke eigensheaves* with respect to σ.

However, it was shown in [FGV] that for an irreducible σ, the category $\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_\sigma$ contains a non-zero object.

□[Conjecture 20.3.8 for GL_n]

24.3. **Proof of Theorem 24.1.2: initial observations.**

24.3.1. In the course of the proof of Proposition 23.7.6, we have seen that the essential image of each of the functors in (24.1) generates the target category.

In particular, we obtain that the functor $\mathbb{L}_G|_{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}}}$ is conservative.

Hence, in order to prove Theorem 24.1.2, it suffices to show that the functor $\mathbb{L}^L_G|_{\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}}}$ is fully faithful.

I.e., we need to show that the unit of the adjunction

$$\text{Id}_{\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}}} \to (\mathbb{L}_G \circ \mathbb{L}^L_G)|_{\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))_{\text{red}}} \tag{24.8}$$

is an isomorphism.

24.3.2. As an aside, let us show that the functor

$$\mathbb{L}_G : \text{D-mod}_{\frac{1}{2}}(\text{Bun}_G) \to \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))$$

is itself conservative.

Indeed, given the conservativity of $\mathbb{L}_G|_{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{Eis}}}$, it suffices to show that $\mathbb{L}_G|_{\text{D-mod}_{\frac{1}{2}}(\text{Bun}_G)_{\text{cusp}}}$ is conservative. However, the latter was established in [FR1].

24.3.3. By Theorem 23.6.2, the endofunctor $\mathbb{L}_G \circ \mathbb{L}^L_G$ is given by tensor product with an object

$$\mathcal{A}_G \in \text{QCoh}(\text{LS}_{\check{G}}(X)).$$

Moreover, the structure of monad on $\mathbb{L}_G \circ \mathbb{L}^L_G$ corresponds to a structure of associative algebra on $\mathcal{A}_G$ as an object of the (symmetric) monoidal category $\text{QCoh}(\text{LS}_{\check{G}}(X))$.

Under this identification, the unit of the adjunction

$$\text{Id}_{\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}(X))} \to \mathbb{L}_G \circ \mathbb{L}^L_G$$

corresponds to the map of associative algebras

$$\mathcal{O}_{\text{LS}_{\check{G}}(X)} \to \mathcal{A}_G. \tag{24.9}$$

24.3.4. In order to show that (24.8) is an isomorphism, it suffices to show that the *-restriction of the map (24.9) to the formal completion

$$(\text{LS}_{\check{G}}(X))^\wedge_{\text{LS}^{\text{red}}_{\check{G}}} \subset \text{LS}_{\check{G}}$$

is an isomorphism.

24.3.5. Note that the composition

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}} \hookrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \xrightarrow{\widehat{i}^*} \mathrm{QCoh}((\mathrm{LS}_{\check{G}}(X))^{\wedge}_{\mathrm{LS}^{\mathrm{red}}_{\check{G}}})$$

is an equivalence.

Note also that the collection of functors

$$(\mathsf{p}^{\mathrm{glob}})^* : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \to \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(X)),$$

for proper standard parabolics of G, is conservative on $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X))_{\mathrm{red}}$.

Hence, it suffices to show that for each proper parabolic as above, the induced map

$$\mathcal{O}_{\mathrm{LS}_{\check{P}^-}(X)} \to (\mathsf{p}^{\mathrm{glob}})^*(\mathcal{A}_G) \tag{24.10}$$

is an isomorphism.

24.3.6. We will deduce this from the following statement:

Main Lemma 24.3.7. *For every proper parabolic, the object*

$$(\mathsf{p}^{\mathrm{glob}})^*(\mathcal{A}_G) \in \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(X))$$

is a line bundle.

Indeed, the fact that Main Lemma 24.3.7 implies that (24.10) is an isomorphism follows from the next observation:

Proposition 24.3.8. *Let $\mathcal{Y}$ be any prestack and let $\mathcal{A}_{\mathcal{Y}}$ be a unital associative algebra object in* $\mathrm{QCoh}(\mathcal{Y})$. *Assume that $\mathcal{A}_{\mathcal{Y}}$ is a line bundle. Then the unit map $u : \mathcal{O}_{\mathcal{Y}} \to \mathcal{A}_{\mathcal{Y}}$ is an isomorphism.*

Proof. Briefly, the cone of the unit map is a perfect complex. In general, one can verify the vanishing of a perfect complex by checking the vanishing of its fibers at field-valued points. This reduces us to the case where $\mathcal{Y}$ is the spectrum of a field, where the assertion is obvious.

We also present an alternative argument that adapts more generally when $\mathcal{A}_{\mathcal{Y}}$ is a unital associative algebra in a symmetric monoidal stable ∞-category whose underlying object $\mathcal{A}_{\mathcal{Y}}$ is invertible.

After tensoring $m : \mathcal{A}_{\mathcal{Y}} \otimes \mathcal{A}_{\mathcal{Y}} \to \mathcal{A}_{\mathcal{Y}}$ on the left with the inverse line bundle $\mathcal{A}_{\mathcal{Y}}^{\otimes -1}$, we obtain a map $v : \mathcal{A}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}$. It is straightforward to check that v is the inverse to u.

Indeed, first one sees that the composition $v \circ u$ is the identity for $\mathcal{O}_{\mathcal{Y}}$ by tensoring with the line bundle $\mathcal{A}_{\mathcal{Y}}$ and using the definition of v. This means $\mathcal{A}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{Y}} \oplus \mathcal{K}$ for a summand $\mathcal{K}$ and compatibly with u. It suffices to see that the map $\alpha : \mathcal{K} \to \mathcal{A}_{\mathcal{Y}}$ is nullhomotopic to deduce that $\mathcal{K}$ is zero. The composition

$$\mathcal{K} \xrightarrow{u \otimes \mathrm{id}} \mathcal{A} \otimes \mathcal{K} \xrightarrow{\mathrm{id} \otimes \alpha} \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$$

clearly equals α, but as $m = \mathrm{id} \otimes v$ by definition of v and as $v \circ \alpha = 0$ by definition of $\mathcal{K}$, we obtain the desired nullhomotopy.

□

□[Theorem 24.1.2]

24.4. **Proof of Main Lemma 24.3.7.**

24.4.1. The key ingredient in the proof of Main Lemma 24.3.7 is the following assertion:

Theorem 24.4.2. *Assume that Conjecture 20.3.8 holds for M. Then the functors*

$$\mathrm{CT}^{-,\mathrm{spec,enh}} \circ \mathbb{L}_G \circ \mathbb{L}_G^L \text{ and } \mathrm{CT}^{-,\mathrm{spec,enh}}$$

become isomorphic after composing with the forgetful functor

$$\mathrm{full} \to \mathrm{part} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{-,\mathrm{enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{-,\mathrm{part.enh}}.$$

The proof of Theorem 24.4.2 will occupy the rest of the paper.

Remark 24.4.3. Once Theorem 24.1.2 is proved, it will follow formally that the functors

$$\mathrm{CT}^{-,\mathrm{spec,enh}} \circ \mathbb{L}_G \circ \mathbb{L}_G^L \text{ and } \mathrm{CT}^{-,\mathrm{spec,enh}}$$

themselves are isomorphic.

24.4.4. Let us show how Theorem 24.4.2 implies the statement of Main Lemma 24.3.7.

Since $\mathsf{p}^{\mathrm{glob}}$ is a morphism between quasi-smooth stacks, it suffices to show that

$$(\mathsf{p}^{\mathrm{glob}})^!(\mathcal{A}_G) \simeq (\mathsf{p}^{\mathrm{glob}})^!(\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}(X)), \tag{24.11}$$

when view both $\mathcal{A}_G$ and $\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}(X)$ as objects of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$ via

$$\Xi_{0,\mathrm{Nilp}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \hookrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}}(\check{G}).$$

24.4.5. According to Lemma 19.3.4, the composition

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \overset{\mathrm{CT}^{-,\mathrm{spec,enh}}}{\longrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{-,\mathrm{enh}} \overset{\mathrm{full}\to\mathrm{part}}{\longrightarrow} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))^{-,\mathrm{part.enh}} \tag{24.12}$$

identifies with

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \hookrightarrow \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)) \overset{(\mathsf{p}^{\mathrm{glob}})^!}{\longrightarrow} \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X)) \twoheadrightarrow \mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)),$$

where the last arrow is the right adjoint to the tautological embedding

$$\mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \hookrightarrow \mathrm{IndCoh}(\mathrm{LS}_{\check{P}^-}(X)).$$

In particular, the restriction of (24.12) to

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$$

identifies with

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \overset{(\mathsf{p}^{\mathrm{glob}})^!}{\longrightarrow} \mathrm{QCoh}(\mathrm{LS}_{\check{P}^-}(X)) \hookrightarrow \mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)).$$

24.4.6. Thus, in order to prove (24.11), it suffices to show that

$$(\mathrm{full} \to \mathrm{part}) \circ \mathrm{CT}^{-,\mathrm{spec,enh}}(\mathcal{A}_G) \simeq (\mathrm{full} \to \mathrm{part}) \circ \mathrm{CT}^{-,\mathrm{spec,enh}}(\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}).$$

However, by Theorem 23.6.2,

$$\mathcal{A}_G \simeq \mathbb{L}_G \circ \mathbb{L}_G^L(\mathcal{O}_{\mathrm{LS}_{\check{G}}(X)}) \tag{24.13}$$

as objects of $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$.

Hence, (24.13) follows from Theorem 24.4.2.

□[Main Lemma 24.3.7]

Remark 24.4.7. The above proof produces *an* isomorphism

$$(\mathsf{p}^{\mathrm{glob}})^*(\mathcal{A}_G) \simeq \mathcal{O}_{\mathrm{LS}_{\check{P}^-}(X)}.$$

However, we do not claim that this isomorphism equals the unit morphism (24.10).

24.5. **The partially enhanced geometric constant term functor.**

24.5.1. Recall the factorization algebra

$$\Omega \in \mathrm{Sph}_M,$$

see Sect. 3.3.

Consider the category

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}},$$

constructed in a way parallel to $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}$, where at the local level we perform the operation

$$\Omega\text{-mod}(\mathrm{Sph}_M) \underset{\mathrm{Sph}_M}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \underset{\mathrm{Sph}_M}{\simeq} \mathrm{Rep}(\check{P}^-) \underset{\mathrm{Rep}(\check{M})}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$$

instead of

$$\mathrm{I}(G,B)^{\mathrm{loc}}_{\rho_P(\omega_X)} \underset{\mathrm{Sph}_M}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) := \widetilde{\Omega}_{\rho_P(\omega_X)}\text{-mod}(\mathrm{Sph}_M) \underset{\mathrm{Sph}_M}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \simeq$$
$$\simeq \widetilde{\Omega}^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}) \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}}{\otimes} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M).$$

24.5.2. The homomorphism (3.13) gives rise to a forgetful functor

$$\widetilde{\Omega}_{\rho_P(\omega_X)}\text{-mod}(\mathrm{Sph}_M) \to \Omega\text{-mod}(\mathrm{Sph}_M),$$

and hence to a functor

$$\mathrm{full} \to \mathrm{part} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}.$$

The functor

$$\mathrm{oblv}_\Omega : \Omega\text{-mod}(\mathrm{Sph}_M) \to \mathrm{Sph}_M$$

gives rise to a functor

$$\mathbf{oblv}_{\mathrm{part.enh}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M),$$

so that

$$\mathbf{oblv}_{\mathrm{enh}} \simeq \mathbf{oblv}_{\mathrm{part.enh}} \circ (\mathrm{full} \to \mathrm{part}).$$

24.5.3. Denote

$$\mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)} := (\mathrm{full} \to \mathrm{part}) \circ \mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}.$$

We have

$$\mathbf{oblv}_{\mathrm{part.enh}} \circ \mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)} \simeq \mathrm{CT}^{-}_{*,\rho_P(\omega_X)}.$$

24.5.4. As in Sect. 22.1.1, the functor $\mathbb{L}_M$ upgrades to a functor

$$\mathbb{L}^{-,\mathrm{part.enh}}_M : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}$$

so that we have a commutative diagram

$$\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\
\downarrow{\scriptstyle \mathrm{full}\to\mathrm{part}} & & \downarrow{\scriptstyle \mathrm{full}\to\mathrm{part}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{part.enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\
\downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{part.enh}}} & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{part.enh}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).
\end{array}$$

24.5.5. In a parallel fashion, the functor Φ_M gives rise to functors

$$\Phi_M^{-,\mathrm{part.enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}$$

and

$$\Phi_M^{-,\mathrm{enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)}$$

so that the diagram

$$\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} & \xleftarrow{\Phi_M^{-,\mathrm{enh}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\
\downarrow{\scriptstyle \text{full}\to\text{part}} & & \downarrow{\scriptstyle \text{full}\to\text{part}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xleftarrow{\Phi_M^{-,\mathrm{part.enh}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\
\downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{part.enh}}} & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{part.enh}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xleftarrow{\Phi_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)).
\end{array}$$

commutes.

24.5.6. It follows formally from Theorem 22.2.4 that we have an isomorphism of functors

$$\mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)} \circ \mathbb{L}_G[-\delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \mathbb{L}_M^{-,\mathrm{part.enh}} \circ \mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}.$$

Note that the assumption that $\mathbb{L}_M$ is an equivalence, formally implies that so is $\mathbb{L}_M^{-,\mathrm{part.enh}}$, with inverse given by $\Phi_M^{-,\mathrm{part.enh}}$.

Hence, using Theorem 23.2.5, in order to prove Theorem 24.4.2, it suffices to prove the following:

Theorem 24.5.7. *There exists a canonical isomorphism*

$$\mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)} \circ \Phi_G[-\delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \Phi_M^{-,\mathrm{part.enh}} \circ \mathrm{CT}^{-,\mathrm{spec,part.enh}}.$$

The rest of the paper will be devoted to the proof of Theorem 24.5.7.

Remark 24.5.8. As in Remark 24.4.3, once Theorem 24.1.2 is proved, it would formally follow that we actually have an isomorphism of functors

$$\mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)} \circ \Phi_G[-\delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \Phi_M^{-,\mathrm{enh}} \circ \mathrm{CT}^{-,\mathrm{spec,enh}}. \tag{24.14}$$

The reason we cannot prove (24.14) directly is that we do not know a certain (expected) self-duality statement for the local category

$$\mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,loc}},$$

see Sect. 25.4.5.

24.6. **The unenhanced version of Theorem 24.5.7.**

24.6.1. For expositional purposes, we will first prove an unenhanced version of Theorem 24.5.7:

Theorem 24.6.2. *There exists a canonical isomorphism*

$$\mathrm{CT}^{-}_{*,\rho_P(\omega_X)} \circ \Phi_G[-\delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \Phi_M \circ \mathrm{CT}^{-,\mathrm{spec}}.$$

The rest of this section will be devoted to the proof of Theorem 24.6.2.

24.6.3. Dualizing and applying the definition of the functors Φ_G and Φ_M, we obtain that the required isomorphism is equivalent to

$$\mathbb{L}_G \circ \tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}^-_{*,\rho_P(\omega_X)}[4\delta_G - 2\delta_{N_{\rho(\omega_X)}} - \delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \tag{24.15}$$
$$\simeq (\mathrm{CT}^{-,\mathrm{spec}})^\vee \circ \mathbb{L}_M \circ \tau_M \circ \mathrm{Mir}_{\mathrm{Bun}_M}[4\delta_M - 2\delta_{N(M)_{\rho_M(\omega_X)}}],$$

where

$$\mathrm{Eis}^-_{*,\rho_P(\omega_X)} := (\mathrm{CT}^-_{*,\rho_P(\omega_X)})^\vee.$$

In other words, $\mathrm{Eis}^-_{*,\rho_P(\omega_X)}$ is the composition of the following functors:

- Over the connected component of Bun_M of degree λ, the cohomological shift to the right by the amount
$$\langle \lambda, 2\check{\rho}_P \rangle + \delta_{N(P^-)_{\rho_P(\omega_X)}};$$
- Pushforward along the translation map
$$\mathrm{transl}_{\rho_P(\omega_X)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M);$$
- The functor $(\mathfrak{p}^-)_* \circ (\mathfrak{q}^-)^!$, i.e., !-pull and *-push along the diagram (15.2).

24.6.4. Let $\mathcal{L}_{\rho_P(\omega_X)}$ be the line bundle on $\mathrm{LS}_{\check{M}}(X)$, given by Weil pairing with $\rho_P(\omega_X) \in \mathrm{Bun}_{Z_M}$ (see Sect. 20.2.4).

Proposition 24.6.5. *There exists a canonical identification of (non-graded) line bundles on* $\mathrm{LS}_{\check{P}^-}(X)$*:*

$$\mathfrak{q}^*(\mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)}) \simeq \det(T^*(\mathrm{LS}_{\check{P}^-}(X)/\mathrm{LS}_{\check{M}}(X))).$$

Let us accept Proposition 24.6.5 temporarily and proceed with the proof of (24.15).

We obtain a canonical isomorphism

$$(\mathrm{CT}^{-,\mathrm{spec}})^\vee \simeq \mathrm{Eis}^{-,\mathrm{spec}} \circ (- \otimes \mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)})[\dim(\mathrm{LS}_{\check{P}^-}(X)) - \dim(\mathrm{LS}_{\check{M}}(X))], \tag{24.16}$$

where we also note that

$$\dim(\mathrm{LS}_{\check{P}^-}(X)) - \dim(\mathrm{LS}_{\check{M}}(X)) = \delta_G - \delta_M.$$

24.6.6. Note also that under $\mathbb{L}_M$, the endofunctor

$$- \otimes \mathcal{L}_{\rho_P(\omega_X)}$$

of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ corresponds to $(\mathrm{transl}_{-\rho_P(\omega_X)})_*$ as an endofunctor of $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$, cf. Sect. 20.2.4.

Hence, we can rewrite the desired isomorphism (24.15) as

$$\mathbb{L}_G \circ \tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}^-_{*,\rho_P(\omega_X)}[3\delta_G - 2\delta_{N_{\rho(\omega_X)}} - \delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \tag{24.17}$$
$$\simeq \mathrm{Eis}^{-,\mathrm{spec}} \circ \mathbb{L}_M \circ (\mathrm{transl}_{-2\rho_P(\omega_X)})_* \circ \tau_M \circ \mathrm{Mir}_{\mathrm{Bun}_M}[3\delta_M - 2\delta_{N(M)_{\rho_M(\omega_X)}}].$$

24.6.7. We now apply the isomorphism of Theorem 20.4.5:

$$\mathbb{L}_G \circ \mathrm{Eis}^-_{!,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \mathrm{Eis}^{-,\mathrm{spec}} \circ \mathbb{L}_M. \tag{24.18}$$

This allows to rewrite the desired isomorphism (24.17) as

$$\mathbb{L}_G \circ \tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}^-_{*,\rho_P(\omega_X)}[3\delta_G - 2\delta_{N_{\rho(\omega_X)}} - 2\delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \tag{24.19}$$
$$\simeq \mathbb{L}_G \circ \mathrm{Eis}^-_{!,\rho_P(\omega_X)} \circ (\mathrm{transl}_{-2\rho_P(\omega_X)})_* \circ \tau_M \circ \mathrm{Mir}_{\mathrm{Bun}_M}[3\delta_M - 2\delta_{N(M)_{\rho_M(\omega_X)}}].$$

I.e., it is sufficient to establish the isomorphism

$$\tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}^-_{*,\rho_P(\omega_X)}[3\delta_G - 2\delta_{N_{\rho(\omega_X)}} - 2\delta_{N(P^-)_{\rho_P(\omega_X)}}] \simeq \tag{24.20}$$
$$\simeq \mathrm{Eis}^-_{!,\rho_P(\omega_X)} \circ (\mathrm{transl}_{-2\rho_P(\omega_X)})_* \circ \tau_M \circ \mathrm{Mir}_{\mathrm{Bun}_M}[3\delta_M - 2\delta_{N(M)_{\rho_M(\omega_X)}}]$$

purely on the geometric side.

24.6.8. Let

$$\mathrm{Eis}_!^{\text{un-ren}}, \mathrm{Eis}_!^{-,\text{un-ren}}, \mathrm{Eis}_*^{-,\text{un-ren}}$$

be the un-renormalized Eisenstein functors: i.e., we perform the pull-push along (15.2) without applying cohomological shifts.

Then over the connected component of Bun_M corresponding to $\lambda \in \Lambda_{G,P}$, the left-hand side of (24.20) is

$$\tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}_*^{-,\text{un-ren}} \circ (\mathrm{transl}_{\rho_P(\omega_X)})_* [3\delta_G - 2\delta_{N_{\rho(\omega_X)}} - 3\delta_{N(P^-)_{\rho_P(\omega_X)}} - \langle \lambda, 2\check{\rho}_P \rangle] \tag{24.21}$$

and the right-hand side is

$$\mathrm{Eis}_!^{-,\text{un-ren}} \circ (\mathrm{transl}_{\rho_P(\omega_X)})_* \circ (\mathrm{transl}_{-2\rho_P(\omega_X)})_* \circ \tau_M \circ \mathrm{Mir}_{\mathrm{Bun}_M} \tag{24.22}$$
$$[3\delta_M - 2\delta_{N(M)_{\rho_M(\omega_X)}} + \delta_{N(P^-)_{\rho_P(\omega_X)}} - \langle \lambda + 2\rho_P(2g-2), 2\check{\rho}_P \rangle].$$

Thus, we have to establish the isomorphism of functors

$$\begin{aligned} \tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}_*^{-,\text{un-ren}} \circ (\mathrm{transl}_{\rho_P(\omega_X)})_* \simeq \\ \simeq \mathrm{Eis}_!^{-,\text{un-ren}} \circ (\mathrm{transl}_{\rho_P(\omega_X)})_* \circ (\mathrm{transl}_{-2\rho_P(\omega_X)})_* \circ \tau_M \circ \mathrm{Mir}_{\mathrm{Bun}_M} = \\ = \mathrm{Eis}_!^{-,\text{un-ren}} \circ (\mathrm{transl}_{-\rho_P(\omega_X)})_* \circ \tau_M \circ \mathrm{Mir}_{\mathrm{Bun}_M} \end{aligned} \tag{24.23}$$

and the numerical identity

$$3\delta_G - 2\delta_{N_{\rho(\omega_X)}} - 3\delta_{N(P^-)_{\rho_P(\omega_X)}} = 3\delta_M - 2\delta_{N(M)_{\rho_M(\omega_X)}} + \delta_{N(P^-)_{\rho_P(\omega_X)}} - \langle 2\rho_P(2g-2), 2\check{\rho}_P \rangle \tag{24.24}$$

24.6.9. Recall that, according to [Gai2], we have a canonical isomorphism of functors

$$\mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}_*^{-,\text{un-ren}} \simeq \mathrm{Eis}_!^{\text{un-ren}} \circ \mathrm{Mir}_{\mathrm{Bun}_M}. \tag{24.25}$$

This implies (24.23) using the fact that

$$\tau_G \circ \mathrm{Eis}_!^{\text{un-ren}} \circ \tau_M \simeq \mathrm{Eis}_!^{-,\text{un-ren}}$$

and that

$$\tau_M \circ (\mathrm{transl}_{\rho_P(\omega_X)})_* \simeq (\mathrm{transl}_{-\rho_P(\omega_X)})_* \circ \tau_M.$$

The identity (24.24) is a straightforward verification using Riemann-Roch.

□[Theorem 24.6.2]

Remark 24.6.10. Thus, modulo the overall cohomological shifts, the proof of (24.15) amounts to the following diagram

$$\begin{array}{ccccccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_M}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\tau_M} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\
 & & & & \downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})_*} & & \downarrow{\scriptstyle -\otimes \mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)}} \\
\downarrow{\scriptstyle \mathrm{Eis}^-_{*,\rho_P(\omega_X)}} & & \downarrow{\scriptstyle \mathrm{Eis}_{!,\rho_P(\omega_X)}} & & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\mathbb{L}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \\
 & & & & \downarrow{\scriptstyle \mathrm{Eis}^-_{!,\rho_P(\omega_X)}} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow[\mathrm{Mir}_{\mathrm{Bun}_G}]{} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\tau_G]{} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}_G]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)),
\end{array}$$

in which the two left squares commute up to cohomological shifts, and the composite right vertical arrow identifies with $(\mathrm{CT}^{-,\mathrm{spec}})^\vee$, again up to a cohomological shift.

24.7. **Proof of Proposition 24.6.5.**

24.7.1. The proof is based on the following general observation:

Lemma 24.7.2. *Let E be a local system on X and let $\mathcal{E}$ be the underlying vector bundle. Then there exists a canonical isomorphism*

$$\det(\Gamma_{\mathrm{dR}}(X,E)[1]) \simeq \mathrm{Weil}(\det(\mathcal{E}),\omega_X).$$

Proof. We calculate $\Gamma_{\mathrm{dR}}(X,E)$ using the de Rham complex

$$\Gamma(X,\mathcal{E}) \to \Gamma(X,\mathcal{E}\otimes\omega_X).$$

Hence,

$$\det(\Gamma_{\mathrm{dR}}(X,E)[1]) \simeq \det(\Gamma(X,\mathcal{E}\otimes\omega_X))\otimes\det(\Gamma(X,\mathcal{E}))^{\otimes -1}.$$

Using formula (12.6), we have

$$\det(\Gamma(X,\mathcal{E}\otimes\omega_X))\otimes\det(\Gamma(X,\mathcal{E}))^{\otimes -1} \simeq \mathrm{Weil}(\det(\mathcal{E}),\omega_X)\otimes\det(\Gamma(X,\omega_X))^{\otimes \mathrm{rk}(\mathcal{E})}\otimes\det(\Gamma(X,\mathcal{O}_X))^{\otimes -\mathrm{rk}(\mathcal{E})}.$$

However,

$$\det(\Gamma(X,\omega_X)) \simeq \det(\Gamma(X,\mathcal{O}_X)),$$

whence the result.

□

24.7.3. Let $\sigma_{\check{M}}$ be a $\check{M}$-local system, and let $\mathcal{P}_{\check{M}}$ be the underlying $\check{M}$-bundle. The fiber of $T^*(\mathrm{LS}_{\check{P}^-}(X)/\mathrm{LS}_{\check{M}}(X))$ over $\sigma_{\check{M}}$ is

$$(\Gamma_{\mathrm{dR}}(X,\mathfrak{n}(\check{P}^-)_{\sigma_{\check{M}}})[1])^*,$$

which using Verdier duality can be rewritten as

$$\Gamma_{\mathrm{dR}}(X,(\mathfrak{n}(\check{P}^-)^*)_{\sigma_{\check{M}}})[1].$$

Hence, using Lemma 24.7.2, we can rewrite the fiber of $\det(T^*(\mathrm{LS}_{\check{P}^-}(X)/\mathrm{LS}_{\check{M}}(X)))$ at $\sigma_{\check{M}}$ as

$$\mathrm{Weil}(\det((\mathfrak{n}(\check{P}^-)^*)_{\mathcal{P}_{\check{M}}}),\omega_X).$$

Since the Langlands dual Lie algebra is equipped with a pinning, the line bundle $\det((\mathfrak{n}(\check{P}^-)^*)_{\mathcal{P}_{\check{M}}}$ identifies with $2\rho_P(\mathcal{P}_{\check{M}})$. This implies the assertion of the proposition.

□[Proposition 24.6.5]

25. A digression: enhanced Eisenstein series functors

In this section, we prepare the ground for the proof of Theorem 24.5.7. In fact, we will make an attempt to prove a fully (as opposed to partially) enhanced version of Theorem 24.5.7, given by (24.14), but we will encounter an obstruction, when trying to carry out the duality step on the spectral side (see Sect. 25.4).

Concretely, we will:

- Introduce enhanced Eisenstein series on the geometric side;
- Study their relation to the Miraculous functor $\mathrm{Mir}_{\mathrm{Bun}_G}$;
- Introduce enhanced spectral Eisenstein series;
- Prove a generalization of Theorem 20.4.5, given by Theorem 25.3.5, which establishes the compatibility of the Langlands functor with the *enhanced* Eisenstein functors.

25.1. **Enhanced Eisenstein series functors on the geometric side.**

25.1.1. For a fixed $\underline{x} \in \mathrm{Ran}$, consider the prestack (16.7). The operation

$$(\overleftarrow{h}_{G,P^-,\underline{x}})_! \left(\mathsf{s}^*(-) \overset{*}{\otimes} (\overrightarrow{h}_{G,P^-,\underline{x}})^* \circ (\mathfrak{q}^-)^*(-) \right) [\dim. \mathrm{rel}(\mathrm{Bun}_{P^-} / \mathrm{Bun}_M)]$$

defines a functor

$$\mathrm{I}(G,P)^{\mathrm{loc}}_{\underline{x}} \underset{\mathrm{Sph}_{M,\underline{x}}}{\otimes} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) =$$
$$= \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}(N(P^-))_{\underline{x}} \cdot \mathfrak{L}^+(M)_{\underline{x}}} \underset{\mathrm{Sph}_{M,\underline{x}}}{\otimes} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

Varying $\underline{x}$, we obtain a functor, denoted

$$\mathrm{Eis}^{-,\mathrm{enh}}_! : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

25.1.2. Unwinding the constructions, one can see that the functor $\mathrm{Eis}^{-,\mathrm{enh}}_!$ is the left adjoint of $\mathrm{CT}^{-,\mathrm{enh}}_*$.

25.1.3. For a fixed $\underline{x} \in \mathrm{Ran}$, define the category

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}_{\underline{x}}}_{\mathrm{co}} := \mathrm{I}(G,P)^{\mathrm{loc}}_{\mathrm{co},\underline{x}} \underset{\mathrm{Sph}_{M,\underline{x}}}{\otimes} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}}.$$

Varying $\underline{x}$, as in Sect. 16.1, we produce the category denoted

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}}.$$

The adjunction (3.3) gives rise to a (monadic) adjunction

$$\mathbf{ind}_{\mathrm{enh,co}} : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} \rightleftarrows \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}} : \mathbf{oblv}_{\mathrm{enh,co}}.$$

25.1.4. The operation

$$(\overleftarrow{h}_{G,P^-,\underline{x}})_* \left(\mathsf{s}^!(-) \overset{!}{\otimes} (\overrightarrow{h}_{G,P^-,\underline{x}})^! \circ (\mathfrak{q}^-)^!(-) \right) [-\dim. \mathrm{rel}(\mathrm{Bun}_{P^-} / \mathrm{Bun}_M)]$$

defines a functor

$$\mathrm{I}(G,P)^{\mathrm{loc}}_{\mathrm{co},\underline{x}} \underset{\mathrm{Sph}_{M,\underline{x}}}{\otimes} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) =$$
$$= \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Gr}_{G,\underline{x}})^{\mathfrak{L}^+(M)_{\underline{x}}}_{\mathfrak{L}(N(P^-))_{\underline{x}}} \underset{\mathrm{Sph}_{M,\underline{x}}}{\otimes} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}.$$

Varying $\underline{x}$, we obtain a functor, denoted

$$\mathrm{Eis}^{-,\mathrm{enh}}_* : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}} \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

We have

$$\mathrm{Eis}^{-,\mathrm{enh}}_* \circ \mathbf{ind}_{\mathrm{enh,co}} \simeq \mathrm{Eis}^-_*,$$

where

$$\mathrm{Eis}^-_* = (\mathrm{CT}^-_*)^\vee,$$

i.e., Eis^-_* is the version of $\mathrm{Eis}^-_{*,\rho_P(\omega_X)}$, but without the $\rho_P(\omega_X)$-translation.

25.1.5. The identification

$$(\mathrm{I}(G,P)^{\mathrm{loc}})^{\vee} \simeq \mathrm{I}(G,P)^{\mathrm{loc}}_{\mathrm{co}}$$

as factorization categories and

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\vee} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}}$$

gives rise to an identification

$$\left(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}\right)^{\vee} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}}. \tag{25.1}$$

With respect to this identification, we have

$$\mathbf{ind}^{\vee}_{\mathrm{enh}} \simeq \mathbf{oblv}_{\mathrm{enh,co}} \text{ and } \mathbf{oblv}^{\vee}_{\mathrm{enh}} \simeq \mathbf{ind}_{\mathrm{enh,co}}.$$

25.1.6. Unwinding the definitions, we obtain that under the identification (25.1), the functor $\mathrm{Eis}^{-,\mathrm{enh}}_{*}$ identifies with the *dual* of the functor $\mathrm{CT}^{-,\mathrm{enh}}_{*}$.

25.1.7. In a similar fashion we can consider the functor

$$\mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

which is the left adjoint of $\mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}$.

Recall, however (see (16.31)) that we have a canonical equivalence

$$(\mathrm{transl}^{*}_{\rho_P(\omega_X)})^{-,\mathrm{enh}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)}.$$

With respect to this identification the diagram

$$\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xrightarrow{(\mathrm{transl}^{*}_{\rho_P(\omega_X)})^{-,\mathrm{enh}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \\
\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!}} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathrm{Id}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)
\end{array}$$

commutes.

25.1.8. Proceeding as in Sect. 16.6, we introduce a translated version

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)}$$

of the category

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}},$$

along with the functor

$$\mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

We still have an identification

$$\left(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)}\right)^{\vee} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)},$$

with respect to which we have

$$\mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)} \simeq (\mathrm{CT}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)})^{\vee}.$$

25.1.9. As in (16.31), we have an equivalence

$$(\mathrm{transl}^*_{\rho_P(\omega_X)})^{-,\mathrm{enh}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}} \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)}, \tag{25.2}$$

and the following diagrams commute:

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}} & \xrightarrow{(25.2)} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} \\ \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_*} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathrm{Id}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \end{array}$$

and

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} & \xrightarrow{\mathrm{transl}^*_{\rho_P(\omega_X)}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} \\ \uparrow{\scriptstyle \mathbf{oblv}_{\mathrm{enh,co}}} & & \uparrow{\scriptstyle \mathbf{oblv}_{\mathrm{enh,co}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}} & \xrightarrow{(\mathrm{transl}^*_{\rho_P(\omega_X)})^{-,\mathrm{enh}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)}. \end{array}$$

25.1.10. Finally, the above constructions make sense when instead of the parabolic P^- we use its opposite, and/or instead of the translation by $\rho_P(\omega_X)$ we use any other Z_M-torsor, in particular, $-\rho_P(\omega_X)$.

25.2. **Miraculous functor and enhanced Eisenstein series.**

25.2.1. Recall the equivalence of factorization categories

$$\Upsilon^{\mathrm{loc}} : \mathrm{I}(G, P^-)^{\mathrm{loc}}_{\mathrm{co}} \to \mathrm{I}(G, P)^{\mathrm{loc}},$$

see Sect. 3.2.4.

Tensoring Υ^{loc} with the functor

$$\mathrm{Mir}_{\mathrm{Bun}_M} : \text{D-mod}(\mathrm{Bun}_M)_{\mathrm{co}} \to \text{D-mod}(\mathrm{Bun}_M),$$

we obtain a functor to be denoted

$$\Upsilon^{\mathrm{glob}} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}.$$

25.2.2. We now quote the following assertion (see [Che1, Theorem 5.3.5(b)]), which generalizes (24.25):

Theorem 25.2.3. *We have the following commutative diagram of functors*[26]

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}} \\ \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_*[\delta_{N(P^-)}]} & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_!{}[-\delta_{N(P)}]} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G). \end{array}$$

[26]The presence of the cohomological shifts in the diagram below is due to the fact that they were artificially built into the functors $\mathrm{Eis}^{-,\mathrm{enh}}_*$ and $\mathrm{Eis}^{-,\mathrm{enh}}_*$, respectively.

25.2.4. Note that the Cartan involution on G defines an equivalence

$$\tau_G : \mathrm{I}(G,P)^{\mathrm{loc}} \to \mathrm{I}(G,P^-)^{\mathrm{loc}},$$

and when combined with the equivalence

$$\tau_M : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M),$$

we obtain an equivalence

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}},$$

to be denoted τ_P.

The following diagrams commute by construction:

$$\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}} & \xrightarrow{\tau_P} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \\
{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_!}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_!} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G), \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}} & \xrightarrow{\tau_P} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \\
{\scriptstyle \mathrm{CT}^{\mathrm{enh}}_*}\uparrow & & \uparrow{\scriptstyle \mathrm{CT}^{-,\mathrm{enh}}_*} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G),
\end{array}$$

and

$$\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) & \xrightarrow{\tau_M} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \\
{\scriptstyle \mathbf{oblv}_{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}_{\mathrm{enh}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}} & \xrightarrow{\tau_P} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}.
\end{array}$$

25.2.5. Concatenating with Theorem 25.2.3 we thus obtain the following commutative diagram

$$\begin{array}{ccccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co}} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}} & \xrightarrow{\tau_P} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \\
{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_*[\delta_{N(P^-)}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_![-\delta_{N(P)}]} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_![-\delta_{N(P^-)}]} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).
\end{array} \tag{25.3}$$

25.2.6. We now consider the $\rho_P(\omega_X)$-translated of the above constructions. We have an equivalence

$$\Upsilon^{\mathrm{loc}} : \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\rho_P(\omega_X)} \to \mathrm{I}(G,P)^{\mathrm{loc}}_{\rho_P(\omega_X)},$$

and a commutative diagram

$$\begin{array}{ccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} \\
{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_{!,\rho_P(\omega_X)}[-\delta_{N(P)_{\rho_P(X)}}]} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).
\end{array} \tag{25.4}$$

The Cartan involution τ_P is now an equivalence

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} \xrightarrow{\tau_P} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)}.$$

We obtain the following variant of (25.3):

$$\begin{array}{ccccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\tau_P} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} \\
{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(X)}}]}\downarrow & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_{!,\rho_P(\omega_X)}[-\delta_{N(P)_{\rho_P(X)}}]} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,-\rho_P(\omega_X)}[-\delta_{N(P^-)_{-\rho_P(X)}}]} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).
\end{array}$$

Remark 25.2.7. We can expand the latter diagram as

$$\begin{array}{ccccc}
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\tau_P} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} \\
 & & & & \downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{enh}}_*} \\
\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}} & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_{!,\rho_P(\omega_X)}} & & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \\
 & & & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G),
\end{array}$$

in which the squares *commute up to overall cohomologcal shifts*, and where $(\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{enh}}_*$ is the functor

$$\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{-\rho_P(\omega_X)} \xrightarrow{(((\mathrm{transl}_{-\rho_P(\omega_X)})^*)^{-,\mathrm{enh}})^{-1}} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} \to$$
$$\xrightarrow{((\mathrm{transl}_{\rho_P(\omega_X)})^*)^{-,\mathrm{enh}}} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)}.$$

Thus, we have constructed an enhancement of the left portion of the diagram from Remark 24.6.10.

25.3. **Enhanced spectral Eisenstein series.**

25.3.1. For a fixed $\underline{x} \in \mathrm{Ran}$, consider the paradigm of Sect. 19.3.2. The operation

$$(\overleftarrow{h}^{\mathrm{spec}}_{\check{G},\check{P}^-})_* \left((\mathfrak{s}^{\mathrm{spec}})^*(-) \overset{!}{\otimes} (\overrightarrow{h}^{\mathrm{spec}}_{\check{G},\check{P}^-})^*(-) \right)$$

defines a functor

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\underline{x}} \underset{\mathrm{Sph}^{\mathrm{spec}}_{\check{M},\underline{x}}}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

Letting the point $\underline{x}$ vary along Ran, we obtain a functor, denoted

$$\mathrm{Eis}^{-,\mathrm{spec,enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).$$

25.3.2. Unwinding the construction, we obtain that the functor $\mathrm{Eis}^{-,\mathrm{spec,enh}}$ is the left adjoint to the functor $\mathrm{CT}^{-,\mathrm{spec,enh}}$, as defined in Sect. 19.3.

25.3.3. Recall the adjoint pair

$$\mathbf{ind}_{\mathrm{enh}} : \mathrm{Sph}^{\mathrm{spec}}_{\check{M}} \rightleftarrows \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} : \mathbf{oblv}_{\mathrm{enh}}.$$

Unwinding the construction, we obtain an identification of functors

$$\mathrm{Eis}^{-,\mathrm{spec,enh}} \circ \mathbf{ind}_{\mathrm{enh}} \simeq \mathrm{Eis}^{-,\mathrm{spec,enh}}.$$

25.3.4. The following assertion is an enhancement of Theorem 20.4.5:

Theorem 25.3.5. *There exists a canonical datum of commutativity for the diagram*

$$\begin{array}{ccc}
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\
\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(\omega_X)}}]} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec,enh}}} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).
\end{array} \tag{25.5}$$

The proof will be given in Sect. 25.5.

Remark 25.3.6. Assuming Theorem 25.3.5 we can further expand the diagram in Remark 25.2.7, by concatenating it on the right the diagram

$$\begin{array}{ccc}
 & & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} \\
 & & \Big\downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{enh}}_*} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\
\Big\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}} & & \Big\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec,enh}}} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}_G]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)),
\end{array}$$

which also commutes *commute up to an overall cohomological shift*. I.e., we obtain the diagram

$$\begin{array}{ccccc}
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\tau_P} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} \\
 & & & & \Big\downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{enh}}_*} \\
\Big\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}} & & \Big\downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_{!,\rho_P(\omega_X)}} & & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_M} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\
 & & & & \Big\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}} \qquad \Big\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec,enh}}} \\
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \xrightarrow[\mathbb{L}_G]{} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)).
\end{array}$$

25.4. **What is missing for a direct proof of** (24.14)**?**

25.4.1. Dualizing (24.14), we obtain that it is equivalent to the existence of an isomorphism of functors

$$\mathbb{L}_G \circ \tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ \mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)} \simeq (\mathrm{CT}^{-,\mathrm{spec,enh}})^\vee \circ (\Phi^{-,\mathrm{enh}}_M)^\vee, \tag{25.6}$$

up to a cohomological shift.

Given the diagram in Remark 25.3.6, we can break this into a combination of the following three statements:

(1) There exists a (canonical) identification

$$\Big(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}\Big)^\vee \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}; \tag{25.7}$$

(2) Under the identification (25.7), we have

$$(\mathrm{CT}^{-,\mathrm{spec,enh}})^\vee \simeq \mathrm{Eis}^{-,\mathrm{spec,enh}};$$

(3) Under the identification (25.7), the functor

$$(\Phi^{-,\mathrm{enh}}_M)^\vee : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$$

identifies, up to a cohomological shift, with the composition

$$\begin{array}{l}
\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} \xrightarrow{\Upsilon^{\mathrm{glob}}} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} \xrightarrow{\tau_P} \\
\quad \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},-\rho_P(\omega_X)} \xrightarrow{(\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{enh}}_*} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_M} \\
\qquad\qquad \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}.
\end{array}$$

Remark 25.4.2. Recall the identification

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \simeq \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,glob}},$$

given by Theorem 19.1.6.

The category $\mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,glob}}$ is naturally endowed with a self-duality datum

$$\Big(\mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,glob}}\Big)^\vee \simeq \mathrm{I}(\check{G}, \check{P}^-)^{\mathrm{spec,glob}},$$

and under this identification we have

$$(\mathrm{CT}^{-,\mathrm{spec,enh}})^\vee \simeq \mathrm{Eis}^{-,\mathrm{spec,enh}}.$$

Thus, the pathway towards establishing (25.6) boils down to verifying point (3) in Sect. 25.4.1.

25.4.3. Here is, however, how we envisage a direct local-to-global approach to verifying properties (1)-(3) in Sect. 25.4.1.

Let

$$(\mathrm{transl}_{-2\rho_P(\omega_X)})^{*,\mathrm{spec}}$$

be the (factorization) monoidal automorphism of $\mathrm{Sph}_{\check{M}}^{\mathrm{spec}}$ that corresponds under Sat_M to the (factorization) monoidal automorphism $(\mathrm{transl}_{-2\rho_P(\omega_X)})^*$ of Sph_M.

Note that the functor

$$(-\otimes \mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)}) : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$$

intertwines the actions of $\mathrm{Sph}^{\mathrm{spec}}_{\check{M},\mathrm{Ran}}$ up to $(\mathrm{transl}_{-2\rho_P(\omega_X)})^{*,\mathrm{spec}}$.

25.4.4. Let $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}_{\mathrm{co}}$ denote the variant of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}$, where instead of $\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}$ we use $\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{co}}$.

Note that the Serre duality on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$ defines a natural identification

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}_{\mathrm{co}} \simeq \left(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}\right)^\vee. \tag{25.8}$$

25.4.5. We propose:

Conjecture 25.4.6. *There exists an equivalence of factorization categories*

$$\mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{co}} \overset{\Theta_{\mathrm{spec},\frac{\infty}{2}}}{\simeq} \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}, \tag{25.9}$$

with the following properties:

(1) *The equivalence* $\Theta_{\mathrm{spec},\frac{\infty}{2}}$ *intertwines the* $\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}$*-actions on the two sides up to the automorphism* $(\mathrm{transl}_{-2\rho_P(\omega_X)})^{*,\mathrm{spec}}$.

(2) *The identification* (25.7) *obtained via* (25.8) *by tensoring*

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \xrightarrow{(-\otimes \mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)})[\delta_G-\delta_M]} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$$

with $\Theta_{\mathrm{spec},\frac{\infty}{2}}$ *satisfies point (2) in Sect. 25.4.1.*

(3) *The diagram*

$$\begin{array}{ccc}
\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\rho_P(\omega_X)} & \xleftarrow{(\mathrm{Sat}^{-,\frac{\infty}{2}})^\vee} & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{co}} \\
\Upsilon^{\mathrm{loc}}\downarrow\sim & & \sim\downarrow\Theta_{\mathrm{spec},\frac{\infty}{2}} \\
\mathrm{I}(G,P)^{\mathrm{loc}}_{\rho_P(\omega_X)} & & \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}} \\
\tau_G\downarrow\sim & & \sim\uparrow\mathrm{Sat}^{-,\frac{\infty}{2}} \\
\mathrm{I}(G,P^-)^{\mathrm{loc}}_{-\rho_P(\omega_X)} & \xrightarrow{\alpha_{\rho_P(\omega_X),\mathrm{taut}}\circ\alpha^{-1}_{-\rho_P(\omega_X),\mathrm{taut}}} & \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)}
\end{array} \tag{25.10}$$

commutes.

Note that point (3) in Conjecture 25.4.6 implies point (3) in Sect. 25.4.1.

Remark 25.4.7. Conjecture 25.4.6 allows us to expand the diagram from Remark 25.2.7, by concatenating it on the right with

$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_{M,\mathrm{co}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}_{\mathrm{co}} \\ \downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{enh}}_*} & & \downarrow{\scriptstyle (-\otimes \mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)})\otimes \Theta_{\mathrm{spec},\frac{\infty}{2}}[\delta_G-\delta_M]} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_{M}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\ \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec,enh}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}_G]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)), \end{array} \tag{25.11}$$

where $\mathbb{L}^{-,\mathrm{enh}}_{M,\mathrm{co}}$ is obtained by tensoring

$$\mathbb{L}_M : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))$$

with the (factorization) equivalence

$$\mathrm{I}(G,P^-)^{\mathrm{loc}}_{-\rho_P(\omega_X)} \to \mathrm{I}(\check{G},\check{P}^-)^{\mathrm{spec,loc}}_{\mathrm{co}}$$

resulting from (25.10).

Thus, we obtain an enhanced version of the diagram from Remark 24.6.10:

$$\begin{array}{ccccccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\tau_P} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_{M,\mathrm{co}}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}_{\mathrm{co}} \\ & & & & \downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{enh}}_*} & & \downarrow{\scriptstyle (-\otimes \mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)})\otimes \Theta_{\mathrm{spec},\frac{\infty}{2}}} \\ \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)}} & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_{!,\rho_P(\omega_X)}} & & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_{M}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\ & & & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec,enh}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}_G]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \end{array}$$

(in which we have ignored all cohomological shifts), and the proof of (25.6) follows the same logic as that of (24.15):

Indeed, according to point (2) in Conjecture 25.4.6, the right vertical arrow in (25.11) identifies with $(\mathrm{CT}^{-,\mathrm{spec,enh}})^\vee$, while the composite arrow

$$\begin{aligned} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} &\xrightarrow{\Upsilon^{\mathrm{glob}}} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} \xrightarrow{\tau_P} \\ &\to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_{M,\mathrm{co}}} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}_{\mathrm{co}} \end{aligned}$$

identifies with $(\Phi^{-,\mathrm{enh}}_M)^\vee$ (the latter, due to the commutation of (25.10)).

25.5. **Proof of Theorem 25.3.5.** The proof follows closely that of Theorem 20.4.5.

25.5.1. Since both circuits in (20.15) send compacts to compacts, the same is true for (25.5).

Hence, as in the proof of Theorem 20.4.5, it suffices to establish the commutativity of

$$\begin{array}{ccc} \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_{M}} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\ \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(\omega_X)}}]} & & \downarrow{\scriptstyle (\Xi_{0,\mathrm{Nilp}})^R\circ \mathrm{Eis}^{-,\mathrm{spec,enh}}} \\ \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_{G,\mathrm{coarse}}} & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)). \end{array}$$

And further, it suffices to establish the commutativity of:

$$\begin{array}{ccc}
\mathrm{Whit}^!(G)_{\mathrm{Ran}} & \xrightarrow{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
{\scriptstyle \mathrm{coeff}_G[2\delta_{N_{\rho(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{spec}}_{\check{G}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\
{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X),\mathrm{shft}}[\delta_{N(P^-)_{\rho_P(\omega_X)}}]}\uparrow & & \uparrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec},\mathrm{enh}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}}.
\end{array}$$

25.5.2. By duality, it suffices to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Whit}_*(G)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}} & \xrightarrow{(\mathrm{FLE}_{\check{G},\infty})^{-1}\otimes \mathbb{L}^{-,\mathrm{enh}}_M} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \\
{\scriptstyle \mathrm{Poinc}_{G,*}\otimes \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)}}\downarrow{\scriptstyle [\delta_{N(P^-)_{\rho_P(\omega_X)}}+2\delta_{N_{\rho(\omega_X)}}]} & & \downarrow{\scriptstyle \mathrm{Loc}^{\mathrm{spec}}_{\check{G}}\otimes \mathrm{Eis}^{-,\mathrm{spec},\mathrm{enh}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \otimes \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\
 & & \downarrow{\scriptstyle \overset{*}{\otimes}} \\
{\scriptstyle \Gamma_{\mathrm{dR}}(\mathrm{Bun}_G,-\overset{!}{\otimes}-)}\downarrow & & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \\
 & & \downarrow{\scriptstyle \Gamma(\mathrm{LS}_{\check{G}}(X),-)} \\
\mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect}\,.
\end{array}$$

25.5.3. In fact we will show that the compositions

$$\begin{aligned}
(25.12)\quad & \mathrm{Whit}_*(G)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \otimes \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X),\mathrm{Ran}} \to \\
& \xrightarrow{\mathrm{Poinc}_{G,*}\otimes \mathrm{Eis}^{-,\mathrm{enh}_{\mathrm{Ran}}}_{!,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(\omega_X)}}+2\delta_{N_{\rho(\omega_X)}}]} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \to \\
& \xrightarrow{\Gamma_{\mathrm{dR}}(\mathrm{Bun}_G,-\overset{!}{\otimes}-)} \mathrm{Vect}
\end{aligned}$$

and

$$\begin{aligned}
(25.13)\quad & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{I}(\check{G},\check{P}^-)^{-,\mathrm{spec},\mathrm{loc}}_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}^{\mathrm{spec}}_{\check{G}}\otimes \mathrm{Eis}^{-,\mathrm{spec},\mathrm{enh}_{\mathrm{Ran}}}} \\
& \to \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(X)) \otimes \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \xrightarrow{\overset{*}{\otimes}} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)) \xrightarrow{\Gamma(\mathrm{LS}_{\check{G}}(X),-)} \mathrm{Vect}
\end{aligned}$$

match under $(\mathrm{FLE}_{\check{G},\infty})^{-1} \otimes \mathbb{L}_M \otimes \mathrm{Sat}^{-,\frac{\infty}{2}}$.

25.5.4. As in Sect. 20.5, up to inserting ins. vac, the functor (25.13) identifies with

$$\begin{aligned}
(25.14)\quad & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \otimes \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{I}(\check{G},\check{P}^-)^{-,\mathrm{spec},\mathrm{loc}}_{\mathrm{Ran}} \to \\
& \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \Big(\mathrm{Rep}(\check{G}) \otimes \mathrm{I}(\check{G},\check{P}^-)^{-,\mathrm{spec},\mathrm{loc}}\Big)_{\mathrm{Ran}} \xrightarrow{\mathrm{Id}\otimes(2,22)} \\
& \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{Rep}(\check{M})_{\mathrm{Ran}} \xrightarrow{\mathrm{Id}\otimes \mathrm{Loc}^{\mathrm{spec}}_{\check{M}}} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \otimes \mathrm{QCoh}(\check{M}) \to \\
& \xrightarrow{\overset{*}{\otimes}} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \xrightarrow{\Gamma(\mathrm{LS}_{\check{M}}(X),-)} \mathrm{Vect}\,.
\end{aligned}$$

Again, a standard Zastava space calculation shows that, up to inserting ins. vac, the functor (25.12) identifies with

$$\begin{aligned}(25.15)\quad &\mathrm{Whit}_*(G)_{\mathrm{Ran}} \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \otimes \mathrm{I}(G,B)^{\mathrm{loc}}_{\rho_P(\omega_X),\mathrm{Ran}} \to \\ &\to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \otimes \left(\mathrm{Whit}_*(G) \otimes \mathrm{I}(G,B)^{\mathrm{loc}}_{\rho_P(\omega_X)}\right)_{\mathrm{Ran}} \overset{\mathrm{Id}\otimes(2.21)}{\longrightarrow} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \otimes \mathrm{Whit}_*(M) &\overset{\mathrm{Id}\otimes \mathrm{Poinc}_{M,*}[2\delta_{N(M)}\rho_M(\omega_X)]}{\longrightarrow} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \otimes \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} \to \\ &\overset{\Gamma_{\mathrm{dR}}(\mathrm{Bun}_M,-\overset{!}{\otimes}-)}{\longrightarrow} \mathrm{Vect}\,.\end{aligned}$$

Now the desired assertion follows from the commutation of the diagrams (2.23) and (20.10) (for $\check{M}$). □[Theorem 25.3.5]

26. Proof of Theorem 24.4.2

In this section we will finally prove Theorem 24.5.7.

We will follow the pattern of the (failed) proof in Sect. 25.4, but at the *partially enhanced level*. The difference now is that the required self-duality assertion on the spectral side is easy to obtain: it corresponds to the Serre duality on $\mathrm{LS}_{\check{P}^-}(X)$, up to a twist by a line bundle.

26.1. **Partially enhanced Eisenstein functors.**

26.1.1. Recall the category $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part,enh}}$, see Sect. 24.5. It comes equipped with a pair of adjoint functors

$$\mathrm{part} \to \mathrm{full} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \rightleftarrows \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} : \mathrm{full} \to \mathrm{part}\,.$$

Define the functor

$$\mathrm{Eis}^{-,\mathrm{part.enh}}_{!,\rho_P(\omega_X)} := \mathrm{Eis}^{-,\mathrm{enh}}_{!,\rho_P(\omega_X)} \circ (\mathrm{part} \to \mathrm{full}), \quad \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

By construction, the functor is the left adjoint of the functor $\mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}$.

26.1.2. Denote

$$\Omega^- := \tau_M(\Omega).$$

By a similar token, using the algebra Ω^- instead of Ω, we can consider the partial enhancement $\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part,enh}}$, and the functors

$$\mathrm{Eis}^{\mathrm{part.enh}}_{!,\rho_P(\omega_X)}, \mathrm{Eis}^{\mathrm{part.enh}}_{!,-\rho_P(\omega_X)}, \quad \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part,enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G).$$

26.1.3. Recall that due to (3.16), we have a canonical identification

$$(\Omega\text{-mod}(\mathrm{Sph}_M))^\vee \simeq \Omega^-\text{-mod}(\mathrm{Sph}_M). \tag{26.1}$$

Passing to the duals in the adjunction

$$\Omega\text{-mod}(\mathrm{Sph}_M) \rightleftarrows \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\rho_P(\omega_X)}$$

we obtain an adjunction

$$\Omega^-\text{-mod}(\mathrm{Sph}_M) \rightleftarrows \mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\rho_P(\omega_X)}. \tag{26.2}$$

26.1.4. Recall the category $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)}$ and the functor

$$\mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)} : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G),$$

see Sect. 25.1.8.

By a similar token to Sect. 24.5, using the category Ω^--mod(Sph_M) instead of $\mathrm{I}(G,P^-)^{\mathrm{loc}}_{\mathrm{co},\rho_P(\omega_X)}$, we define the category $\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co}}$.

The adjunction (26.2) gives rise to an adjunction

$$(\mathrm{part} \to \mathrm{full})_{\mathrm{co}} : \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co}} \rightleftarrows \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} : (\mathrm{full} \to \mathrm{part})_{\mathrm{co}}.$$

Define the functor

$$\mathrm{Eis}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)} := \mathrm{Eis}^{-,\mathrm{enh}}_{*,\rho_P(\omega_X)} \circ (\mathrm{part} \to \mathrm{full})_{\mathrm{co}}, \quad \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co},\rho_P(\omega_X)} \to \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}}.$$

26.1.5. Note that (26.1) gives rise to an identification

$$\left(\mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part,enh}}\right)^\vee \simeq \mathrm{D\text{-}mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co}}. \tag{26.3}$$

Under this identification, we have

$$(\mathrm{part} \to \mathrm{full})^\vee \simeq (\mathrm{full} \to \mathrm{part})_{\mathrm{co}} \text{ and } (\mathrm{full} \to \mathrm{part})^\vee \simeq (\mathrm{part} \to \mathrm{full})_{\mathrm{co}}.$$

Furthermore, we have

$$\left(\mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}\right)^\vee \simeq \mathrm{Eis}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}.$$

26.1.6. Recall the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}$, equipped with an adjoint pair

$$\mathrm{part} \to \mathrm{full} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \rightleftarrows \mathrm{IndCoh}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} : \mathrm{full} \to \mathrm{part}.$$

Define

$$\mathrm{Eis}^{-,\mathrm{spec,part.enh}} := \mathrm{Eis}^{-,\mathrm{spec,enh}} \circ (\mathrm{part} \to \mathrm{full}).$$

This functor is the left adjoint of $\mathrm{CT}^{-,\mathrm{spec,part.enh}}$. Under the identification of Proposition 19.2.3, the functor $\mathrm{Eis}^{-,\mathrm{spec,part.enh}}$ corresponds to

$$\mathrm{IndCoh}_{\check{M}-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \xrightarrow{\mathsf{p}_*} \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X)).$$

26.1.7. Note that the identification

$$\sigma^{\mathrm{spec}}(\Omega^{\mathrm{spec}}) \simeq \Omega^{\mathrm{spec}}$$

implies that we have a canonical identification

$$(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}})^\vee \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}, \tag{26.4}$$

so that under the Serre duality identification

$$(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)))^\vee \simeq (\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)),$$

we have

$$(\mathbf{ind}_{\mathrm{part.enh}})^\vee \simeq \mathbf{oblv}_{\mathrm{part.enh}} \text{ and } (\mathbf{oblv}_{\mathrm{part.enh}})^\vee \simeq \mathbf{ind}_{\mathrm{part.enh}}.$$

Note that we have a commutative diagram

$$\begin{array}{ccc}
(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}})^\vee & \xrightarrow{\text{Proposition 19.2.3}} & (\mathrm{IndCoh}_{\check{M}-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)))^\vee \\
{\scriptstyle (26.4)}\downarrow & & \downarrow{\scriptstyle \mathrm{Serre}_{\mathrm{LS}_{\check{P}^-}(X)} \otimes \mathsf{q}^*(\mathcal{L}^{\otimes -2}_{\rho_P(\omega_X)})\,[-\dim.\mathrm{rel}(\mathrm{LS}_{\check{P}^-}(X)/\,\mathrm{LS}_{\check{M}}(X))]} \\
\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} & \xrightarrow{\text{Proposition 19.2.3}} & \mathrm{IndCoh}_{\check{M}-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X))
\end{array}$$

In particular, since the map p is proper, and hence p_* is the dual of $\mathsf{p}^!$ *with respect to Serre duality*, we obtain that with respect to the identification (26.4), we have

$$(\mathrm{CT}^{-,\mathrm{spec,part.enh}})^\vee \simeq \mathrm{Eis}^{-,\mathrm{spec,part.enh}} \circ (- \otimes \mathsf{q}^*(\mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)}))[\dim.\mathrm{rel}(\mathrm{LS}_{\check{P}^-}(X)/\,\mathrm{LS}_{\check{M}}(X))]. \tag{26.5}$$

26.2. **Partial enhancement and the miraculous functor.**

26.2.1. Let $\mathrm{Mir}^{\mathrm{part.enh}}_{\mathrm{Bun}_M}$ denote the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co},\rho_P(\omega_X)} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part.enh}}_{\rho_P(\omega_X)}$$

obtained by tensoring

$$\mathrm{Mir}_{\mathrm{Bun}_M} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co},\rho_P(\omega_X)} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\rho_P(\omega_X)}$$

with the identity functor on Ω^--mod(Sph_M).

The following assertion results from Lemma 3.4.2:

Corollary 26.2.2. *The following diagram commutes*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co},\rho_P(\omega_X)} & \xrightarrow{\mathrm{Mir}^{\mathrm{part.enh}}_{\mathrm{Bun}_M}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part.enh}}_{\rho_P(\omega_X)} \\ \downarrow{\scriptstyle (\mathrm{part}\to\mathrm{full})_{\mathrm{co}}} & & \downarrow{\scriptstyle (\mathrm{part}\to\mathrm{full})} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\mathrm{co},\rho_P(\omega_X)} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)}. \end{array} \tag{26.6}$$

26.2.3. Concatenating with Theorem 25.2.3, we obtain:

Corollary 26.2.4. *The following diagram commutes*

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}^{\mathrm{part.enh}}_{\mathrm{Bun}_M}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part.enh}} \\ \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}[\delta_{N(P^-)_{\rho_P(X)}}]} & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{part.enh}}_{!,\rho_P(\omega_X)}[-\delta_{N(P)_{\rho_P(X)}}]} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow{\Upsilon^{\mathrm{glob}}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) \end{array} \tag{26.7}$$

26.2.5. Let $\tau^{\mathrm{part.enh}}_M$ denote the functor

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part.enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}$$

obtained by tensoring

$$\tau_M : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M) \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)$$

with

$$\tau_M : \Omega^-\text{-mod}(\mathrm{Sph}_M) \to \Omega\text{-mod}(\mathrm{Sph}_M).$$

The following diagram commutes tautologically:

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part.enh}} & \xrightarrow{\tau^{\mathrm{part.enh}}_M} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \\ \downarrow{\scriptstyle (\mathrm{part}\to\mathrm{full})} & & \downarrow{\scriptstyle (\mathrm{part}\to\mathrm{full})} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\tau_P} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)}. \end{array}$$

Concatenating with

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\tau_P} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} \\ \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{enh}}_{!,\rho_P(\omega_X)}[-\delta_{N(P)_{\rho_P(X)}}]} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{enh}}_{!,-\rho_P(\omega_X)}[-\delta_{N(P^-)_{-\rho_P(X)}}]} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G), \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part.enh}} & \xrightarrow{\tau^{\mathrm{part.enh}}_M} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \\ \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{part.enh}}_{!,\rho_P(\omega_X)}[-\delta_{N(P)_{\rho_P(X)}}]} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{part.enh}}_{!,-\rho_P(\omega_X)}[-\delta_{N(P^-)_{-\rho_P(X)}}]} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\tau_G} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G), \end{array} \tag{26.8}$$

26.2.6. Note that thanks to (3.14), we have a well-defined endo-functor

$$(\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{part.enh}}_* : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}},$$

which makes the diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{(\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{part.enh}}_*} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \\ \downarrow{\scriptstyle (\mathrm{part}\to\mathrm{full})} & & \downarrow{\scriptstyle (\mathrm{part}\to\mathrm{full})} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{-\rho_P(\omega_X)} & \xrightarrow[(\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{enh}}_*]{} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} \end{array}$$

commute.

Hence, we can rewrite the functor

$$\mathrm{Eis}^{-,\mathrm{part.enh}}_{!,-\rho_P(\omega_X)} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \to \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)$$

as

$$\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \xrightarrow{(\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{part.enh}}_*} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} \xrightarrow{\mathrm{Eis}^{-,\mathrm{part.enh}}_{!,\rho_P(\omega_X)}} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G). \tag{26.9}$$

26.2.7. Concatenating Theorem 25.3.5 with the commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{part.enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\ \downarrow{\scriptstyle (\mathrm{part}\to\mathrm{full})} & & \downarrow{\scriptstyle (\mathrm{part}\to\mathrm{full})} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{enh}}_{\rho_P(\omega_X)} & \xrightarrow{\mathbb{L}^{-,\mathrm{enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{enh}} \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{part.enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\ \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{part.enh}}_{!,\rho_P(\omega_X)}[\delta_{N(P^-)\rho_P(\omega_X)}]} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec,part.enh}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)). \end{array} \tag{26.10}$$

26.2.8. Finally, we note that we have a well-defined functor

$$(-\otimes \mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)})^{-,\mathrm{part.enh}} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}},$$

which in terms of the equivalence of Proposition 19.2.3 corresponds to

$$-\otimes \mathsf{q}^*(\mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)}) : \mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)) \to \mathrm{IndCoh}_{\check{M}^-\mathrm{Nilp}}(\mathrm{LS}_{\check{P}^-}(X)),$$

and which makes diagram

$$\begin{array}{ccc} \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{part.enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\ \downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{part.enh}}_*} & & \downarrow{\scriptstyle (-\otimes\mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)})^{-,\mathrm{part.enh}}} \\ \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{part.enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \end{array}$$

commute.

26.2.9. To summarize, we obtain the following *partial* enhancement of the diagram in Remark 24.6.10:

$$\begin{array}{ccccccc}
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co}} & \xrightarrow{\mathrm{Mir}^{\mathrm{part.enh}}_{\mathrm{Bun}_M}} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{\mathrm{part.enh}} & \xrightarrow{\tau^{\mathrm{part.enh}}_M} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{part.enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\
 & & & & \downarrow{\scriptstyle (\mathrm{transl}_{-2\rho_P(\omega_X)})^{-,\mathrm{part.enh}}_*} & & \downarrow{\scriptstyle (-\otimes\mathcal{L}^{\otimes 2}_{\rho_P(\omega_X)})^{-,\mathrm{part.enh}}} \\
\downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}} & & \downarrow{\scriptstyle \mathrm{Eis}^{\mathrm{part.enh}}_{!,\rho_P(\omega_X)}} & & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}} & \xrightarrow{\mathbb{L}^{-,\mathrm{part.enh}}_M} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}} \\
 & & & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{part.enh}}_{!,\rho_P(\omega_X)}} & & \downarrow{\scriptstyle \mathrm{Eis}^{-,\mathrm{spec,part.enh}}} \\
\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{co}} & \xrightarrow[\mathrm{Mir}_{\mathrm{Bun}_G}]{} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\tau_G]{} & \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow[\mathbb{L}_G]{} & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)),
\end{array}$$

which commutes up to overall cohomological shifts.

26.3. **Proof of Theorem 24.4.2.**

26.3.1. Dualizing, it suffices to establish an isomorphism between the functors

$$\mathbb{L}_G \circ \tau_G \circ \mathrm{Mir}_{\mathrm{Bun}_G} \circ (\mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)})^\vee \tag{26.11}$$

and

$$(\mathrm{CT}^{-,\mathrm{spec,part.enh}})^\vee \circ (\Phi^{-,\mathrm{part.enh}}_M)^\vee, \tag{26.12}$$

as functors

$$(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}})^\vee \to (\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)))^\vee,$$

up to a cohomological shift (the shift automatically works out thanks to Theorem 24.6.2, which has already been proved).

26.3.2. We identify

$$(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X)))^\vee \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}(X))$$

via Serre duality on $\mathrm{LS}_{\check{G}}(X)$.

We identify

$$(\text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}})^\vee \simeq \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)^{-,\mathrm{part.enh}}_{\mathrm{co}}$$

as in (26.3).

Under the latter identification, we have

$$(\mathrm{CT}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)})^\vee \simeq \mathrm{Eis}^{-,\mathrm{part.enh}}_{*,\rho_P(\omega_X)}.$$

Hence, the functor (26.11) identifies with the counter-clockwise circuit in the diagram in Sect. 26.2.9.

Thus, in order to prove Theorem 24.4.2, it remains to identify (26.12) with the clockwise circuit in the diagram in Sect. 26.2.9, up to an overall cohomological shift.

26.3.3. We identify

$$\Big(\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}\Big)^\vee \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X))^{-,\mathrm{part.enh}}$$

as in (26.4).

According to (26.5), the functor $(\mathrm{CT}^{-,\mathrm{spec,part.enh}})^\vee$ identifies with the right vertical composition in the diagram in Sect. 26.2.9, up to an overall cohomological shift.

Hence, it remains to identify the functor $(\Phi^{-,\mathrm{part.enh}}_M)^\vee$ with the top horizontal composition in the diagram in Sect. 26.2.9.

26.3.4. With respect to the identifications (26.3) and (26.4), the functor $(\Phi^{-,\mathrm{part.enh}}_M)^\vee$ is obtained by tensoring

$$\mathbb{L}_M \circ \tau_M \circ \mathrm{Mir}_{\mathrm{Bun}_M} : \text{D-mod}_{\frac{1}{2}}(\mathrm{Bun}_M)_{\mathrm{co}} \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{M}}(X)) \tag{26.13}$$

with the composition

$$\Omega^-\text{-mod}(\mathrm{Sph}_M) \simeq (\Omega\text{-mod}(\mathrm{Sph}_M))^\vee \simeq (\Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}))^\vee \simeq \Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}), \tag{26.14}$$

where

- The first equivalence is induced by (26.1);
- The second equivalence is obtained by duality from the identification

$$\Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}) \simeq \Omega\text{-mod}(\mathrm{Sph}_M), \tag{26.15}$$

  induced by Sat_M;
- The third equivalence is induced by the identification

$$\sigma^{\mathrm{spec}}(\Omega^{\mathrm{spec}})^o \simeq \Omega^{\mathrm{spec}}.$$

26.3.5. Unwinding, we obtain that (26.14) is the same as

$$\Omega^-\text{-mod}(\mathrm{Sph}_M) \overset{\mathrm{Id}}{\simeq} \Omega^-\text{-mod}(\mathrm{Sph}_M) \overset{\tau_M}{\simeq} \Omega\text{-mod}(\mathrm{Sph}_M) \overset{\mathrm{Sat}_M}{\longrightarrow} \Omega^{\mathrm{spec}}\text{-mod}(\mathrm{Sph}^{\mathrm{spec}}_{\check{M}}). \tag{26.16}$$

Matching the terms of the factorization (26.16) with those of (26.13), we obtain that $(\Phi^{-,\mathrm{part.enh}}_M)^\vee$ indeed identifies with the top horizontal composition in the diagram in Sect. 26.2.9.

Appendix A. IndCoh* and IndCoh$^!$

A.0.1. In Sect. 0.8, we sketched a definition, which we refer as the *pre-renormalized* one, of IndCoh$^*(Z)$ and IndCoh$^!(Z)$ on *any* prestack Z. But in fact, that definition is *not* (at least obviously) equivalent to the genuine definition, which we refer as the *renormalized* one, used in the main text. For example, it is not clear the *factorizable* geometric Satake equivalence (Theorem 1.7.2) still holds if we use the pre-renormalized definition. In fact, we don't even know IndCoh$^*(\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}})$ is a factorization category without renormalization.

A.0.2. In this appendix, instead of giving the detailed definition, we shall provide axioms that characterize these renormalized categories. Details and proofs will be provided in the next version of this paper, as well as in the incoming paper [CF1]. The readers can also find similar discussions in [Ras4].

A.0.3. First off, there is a full subcategory $\mathrm{PreStk}_{\mathrm{ren}} \subset \mathrm{PreStk}$ of *renormalizable prestacks*, which contains all those prestacks over which ind-coherent sheaves are considered in this paper. For example,

$$\mathrm{Hecke}^{\mathrm{spec,loc}}_{\check{G}},\ \mathrm{Op}_{\check{G}}(\mathcal{D}^{\times})$$

are renormalizable.

A.0.4. For any $Y \in \mathrm{PreStk}_{\mathrm{ren}}$, there are two *compactly generated* categories

$$\mathrm{IndCoh}^*_{\mathrm{ren}}(Y),\ \mathrm{IndCoh}^!_{\mathrm{ren}}(Y)$$

canonically dual to each other. If Y is laft, then both categories are canonically identified with IndCoh(Y) defined in [GR2].

A.0.5. Let

$$\mathrm{Coh}(Y) \subset \mathrm{IndCoh}^*_{\mathrm{ren}}(Y)$$

be the full subcategory of compact objects, which are called *coherent sheaves on* Y.

A.0.6. For any $Y \in \mathrm{PreStk}_{\mathrm{ren}}$, there is a canonical t-structure on $\mathrm{IndCoh}^*_{\mathrm{ren}}(Y)$ that is right complete and compatible with filtered colimit.[27] The subcategory Coh(Y) is closed under (cohomological truncations) and its objects are bounded.

A.0.7. If Y is a renormalizable qcqs scheme, then Zariski locally its coordinate (DG) ring is coherent (see [Lur1, Definition 7.2.4.16]), and Coh(Y) can be identified with the full subcategory of QCoh(Y) of quasi-coherent sheaves that are cohomologically bounded with locally finitely presented cohomologies.

A.0.8. If Y is a renormalizable *indscheme*, then

$$\mathrm{IndCoh}^*_{\mathrm{ren}}(Y),\ \mathrm{IndCoh}^!_{\mathrm{ren}}(Y)$$

can be identified with the pre-renormalized categories in Sect. 0.8.

A.0.9. The assignment

$$Y \mapsto \mathrm{IndCoh}^*_{\mathrm{ren}}(Y),\ Y \mapsto \mathrm{IndCoh}^!_{\mathrm{ren}}(Y)$$

is naturally covariant (resp. contravariant) in Y. In particular, for any morphism $f : Y \to Z$, there is a *-pushforward functor

$$f_* : \mathrm{IndCoh}^*_{\mathrm{ren}}(Y) \to \mathrm{IndCoh}^*_{\mathrm{ren}}(Z)$$

and a !-pullback functor

$$f^! : \mathrm{IndCoh}^!_{\mathrm{ren}}(Z) \to \mathrm{IndCoh}^!_{\mathrm{ren}}(Y).$$

[27] However, this t-structure is generally not left complete. In fact, it is (left) *anti-complete* in the sense of [Lur2, Sect. C.5.5]. Also, in general, $\mathrm{IndCoh}^!_{\mathrm{ren}}(Y)$ does not have a well-behaved t-structure.

A.0.10. If $f : Y \to Z$ is qcqs schematic and of bounded Tor dimension, or more generally if Y admits a fpqc cover Y' such that $Y' \to Z$ is so, then f_* admits a left adjoint

$$f^* : \mathrm{IndCoh}^*_{\mathrm{ren}}(Z) \to \mathrm{IndCoh}^*_{\mathrm{ren}}(Y)$$

and dually, $f^!$ admits a continuous right adjoint

$$f_? : \mathrm{IndCoh}^!_{\mathrm{ren}}(Y) \to \mathrm{IndCoh}^!_{\mathrm{ren}}(Z).$$

We have base-change equivalences between *-push and *-pull (and dually between ?-push and !-pull).

A.0.11. The bounded below part $\mathrm{IndCoh}^*_{\mathrm{ren}}(-)^+$ as well as $\mathrm{Coh}(-)$ satisfies fpqc descent with respect to the *-pullback functor.[28]

A.0.12. If $f : Y \to Z$ is ind-proper, then f_* admits a continuous right adjoint

$$f^? : \mathrm{IndCoh}^*_{\mathrm{ren}}(Z) \to \mathrm{IndCoh}^*_{\mathrm{ren}}(Y)$$

and dually, $f^!$ admits a left adjoint

$$f_! : \mathrm{IndCoh}^!_{\mathrm{ren}}(Y) \to \mathrm{IndCoh}^!_{\mathrm{ren}}(Z).$$

We have base-change equivalences between *-push and ?-pull (and dually between !-push and !-pull). Also, ?-pull and *-pull (and dually !-push and ?-push) commutes for Cartesian squares in $\mathrm{PreStk}_{\mathrm{ren}}$.

A.0.13. For $Y, Z \in \mathrm{PreStk}_{\mathrm{ren}}$, we have canonical product formulae:

$$\mathrm{IndCoh}^*_{\mathrm{ren}}(Y) \otimes \mathrm{IndCoh}^*_{\mathrm{ren}}(Z) \simeq \mathrm{IndCoh}^*_{\mathrm{ren}}(Y \times Z)$$

and similarly for $\mathrm{IndCoh}^!_{\mathrm{ren}}(-)$.

A.0.14. In particular, $\mathrm{IndCoh}_{\mathrm{ren}}(Y)$ has a natural symmetric monoidal structure. Moreover, there is a canonical symmetric monoidal functor

$$\Upsilon : \mathrm{QCoh}(Y) \to \mathrm{IndCoh}_{\mathrm{ren}}(Y),$$

which allows us to view

$$\mathrm{IndCoh}^*_{\mathrm{ren}}(Y),\ \mathrm{IndCoh}^!_{\mathrm{ren}}(Y)$$

as $\mathrm{QCoh}(Y)$-module categories.

[28] However, $\mathrm{IndCoh}^*_{\mathrm{ren}}(-)$ does not satisfy fpqc descent. This is the main difference between the renormalized category and the pre-renormalized one.

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