"In jedem Minus steckt ein Plus. Vielleicht habe ich so etwas gesagt, aber man braucht das doch nicht allzu wörtlich zu nehmen."   R. Musil. Der Mann ohne Eigenschaften.

OUTLINE OF THE PROOF OF THE
GEOMETRIC LANGLANDS CONJECTURE FOR GL_2

DENNIS GAITSGORY

To Gérard Laumon

INTRODUCTION

0.1. **The goal of this paper.** The goal of this paper is to describe work-in-progress by D. Arinkin, V. Drinfeld and the author [1] towards the proof of the (categorical) geometric Langlands conjecture.

The contents of the paper can be summarized as follows: we reduce the geometric Langlands conjecture to a combination of two sets of statements.

The first set is what we call "quasi-theorems." These are plausible (and tractable) statements that involve Langlands duality, but either for proper Levi subgroups, or of local nature, or both. Hopefully, these quasi-theorems will soon turn into actual theorems.

The second set are two conjectures (namely, Conjectures 8.2.9 and 10.2.8), both of which are theorems for GL_n. However, these conjectures *do not* involve Langlands duality: Conjecture 8.2.9 only involves the geometric side of the correspondence, and and Conjecture 10.2.8 only the spectral side.

0.2. **Strategy of the proof.** In this subsection we will outline the general scheme of the argument. We will be working over an algebraically closed field k of characteristic 0. Let X be a smooth and complete curve over k, and G a reductive group. We let $\check{G}$ denote the Langlands dual group, also viewed as an algebraic group over k.

0.2.1. *Formulation of the conjecture.* The categorical geometric Langlands conjecture is supposed to compare two triangulated (or rather DG categories). One is the "geometric" (or "automorphic") side that has to do with D-modules on the stack Bun_G of G-bundles on X. The other is the "spectral" (or "Galois") side that has to do with quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}$ on $\check{G}$-local systems on X.

In our formulation of the conjecture, the geometric side is taken "as is." I.e., we consider the DG category D-mod(Bun_G) of D-modules on Bun_G. We refer the reader to [DrGa2] for the definition of this category and a discussion of its general properties (e.g., this category is *compactly generated* for non-tautological reasons).

A naive guess for the spectral side is the DG category QCoh($\mathrm{LocSys}_{\check{G}}$). However, this guess turns out to be slightly wrong, whenever G is not a torus. A quick way to see that it is wrong is via the compatibility of the conjectural geometric Langlands equivalence with the functor of

---

Date: November 12, 2014.

[1]The responsibility for any deficiency or undesired outcome of this paper lies with the author of this paper.

Eisenstein series, see Property Ei stated in Sect. 6.4.8. Namely, if P is a parabolic of G with Levi quotient M, we have the Eisenstein series functors

$$\mathrm{Eis}_P : \text{D-mod}(\mathrm{Bun}_M) \to \text{D-mod}(\mathrm{Bun}_G) \text{ and } \mathrm{Eis}_{\check{P},\mathrm{spec}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}),$$

that are supposed to match up under the geometric Langlands equivalence (up to a twist by some line bundles). However, this cannot be the case because the functor Eis_P preserves compactness (see [DrGa3]), whereas $\mathrm{Eis}_{\check{P},\mathrm{spec}}$ does not.

Our "fix" for the spectral side is designed to make the above problem with Eisenstein series go away in a minimal way (see Proposition 6.4.7). We observe that the non-preservation of compactness by the functor $\mathrm{Eis}_{\check{P},\mathrm{spec}}$ has to do with the fact that the stack $\mathrm{LocSys}_{\check{G}}$ is not smooth. Namely, it expresses itself in that some coherent complexes on $\mathrm{LocSys}_{\check{G}}$ are non-perfect.

Our modified version for the spectral side is the category that we denote

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}),$$

see Sect. 3.3.2. It is a certain enlargement of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, whose definition uses the fact that $\mathrm{LocSys}_{\check{G}}$ is a derived locally complete intersection, and the theory of singular support of coherent sheaves for such stacks developed in [AG].

0.2.2. *Idea of the proof.* The idea of the comparison between the categories $\text{D-mod}(\mathrm{Bun}_G)$ and $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ pursued in this paper is the following: we embed each side into a more tractable category and compare the essential images.

For the geometric side, the more tractable category in question is the category that we denote $\mathrm{Whit}^{\mathrm{ext}}(G, G)$, and refer to it as the *extended Whittaker category*; the nature of this category is explained in Sect. 0.2.3 below. The functor

$$\text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}^{\mathrm{ext}}(G, G)$$

(which, according to Conjecture 8.2.9, is supposed to be fully faithful) is that of *extended Whittaker coefficient*, denoted $\mathrm{coeff}_{G,G}^{\mathrm{ext}}$.

For the spectral side, the more tractable category is denoted $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$, and the functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Glue}(\check{G})_{\mathrm{spec}}$$

is denoted by $\mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}})$ (this functor is fully faithful by Theorem 9.3.8). The idea of the pair $(\mathrm{Glue}(\check{G})_{\mathrm{spec}}, \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}))$ is explained in Sect. 0.2.4.

We then claim (see Quasi-Theorems 9.4.2 and 9.4.5) that there exists a canonically defined fully faithful functor

$$\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} : \mathrm{Glue}(\check{G})_{\mathrm{spec}} \to \mathrm{Whit}^{\mathrm{ext}}(G, G).$$

Thus, we have the following diagram

$$
(0.1) \qquad
\begin{array}{ccc}
\mathrm{Glue}(\check{G})_{\mathrm{spec}} & \xrightarrow{\ \mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}}\ } & \mathrm{Whit}^{\mathrm{ext}}(G, G) \\[2mm]
{\scriptstyle \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}})}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{coeff}_{G,G}^{\mathrm{ext}}} \\[2mm]
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & & \text{D-mod}(\mathrm{Bun}_G),
\end{array}
$$

with all the arrows being fully faithful.

Assume that the essential images of the functors

$$(0.2) \qquad\qquad \mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \text{ and } \mathrm{coeff}_{G,G}^{\mathrm{ext}}$$

coincide. We then obtain that diagram (0.1) can be (uniquely) completed to a commutative diagram by means of a functor

$$\mathbb{L}_G : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G),$$

and, moreover, $\mathbb{L}_G$ is automatically an equivalence.

The required fact about the essential images of the functors (0.2) follows from Conjecture 10.2.8.

0.2.3. *The extended Whittaker category.* The extended Whittaker category $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ is defined as the DG category of D-modules on a certain space (prestack), by imposing a certain equivariance condition. It may be easiest to explain what $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ is via an analogy with the classical adelic picture.

Assume for simplicity that G has a connected center. Consider the adelic quotient $G(\mathbb{A})/G(\mathbb{O})$. Let $\mathfrak{ch}(K)$ denote the set of characters of the adèle group $N(\mathbb{A})$ (here N is the unipotent radical of the Borel group B) that are trivial on $N(K) \subset N(\mathbb{A})$ (here K denotes the global field corresponding to X). The set $\mathfrak{ch}(K)$ is naturally acted on by the Cartan group $T(K)$ by conjugation.

The space of functions which is the analog of the category $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ is the subspace of all functions on the set

$$G(\mathbb{A})/G(\mathbb{O}) \times \mathfrak{ch}(K)$$

that satisfy the following two conditions:

- $f(t \cdot g, \mathrm{Ad}_t(\chi)) = f(g,\chi), \quad t \in T(K), \quad g \in G(\mathbb{A})/G(\mathbb{O}), \quad \chi \in \mathfrak{ch}(K)$.
- $f(n \cdot g, \chi) = \chi(n) \cdot f(g,\chi), \quad g \in G(\mathbb{A})/G(\mathbb{O}), \quad \chi \in \mathfrak{ch}(K), \quad n \in N(\mathbb{A})$.

The analog of the functor $\mathrm{coeff}_{G,G}^{\mathrm{ext}}$ is the map from the space of functions on

$$G(K)\backslash G(\mathbb{A})/G(\mathbb{O})$$

that takes a function $\widetilde{f}$ to

$$f(g,\chi) := \int\limits_{N(K)\backslash N(\mathbb{A})} \widetilde{f}(n \cdot g) \cdot \chi^{-1}(n).$$

By construction, the category $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ is glued from the categories that we denote $\mathrm{Whit}(G,P)$ (here P is a parabolic in G) and call "degenerate Whittaker categories." In the function-theoretic analogy, for a parabolic P, the category $\mathrm{Whit}(G,P)$ corresponds to the subspace of functions supported on those characters $\chi \in \mathfrak{ch}(K)$ that satisfy:

- χ is non-zero on any simple root subgroup corresponding to roots inside M;
- χ is zero on any simple root *not* in M.

One can rewrite this subspace as the space of functions f on the set $G(\mathbb{A})/G(\mathbb{O})$ that satisfy

- f is invariant with respect to the subgroup $Z_M(K)$;
- f is invariant with respect to $N(P)(\mathbb{A})$, where $N(P)$ is the unipotent radical of P;

- f is equivariant with respect to $N(M)(\mathbb{A})$ against a fixed non-degenerate character, where $N(M) := N \cap M$ and M is the Levi subgroup of P.

In particular, the "open stratum" in $\mathrm{Whit}^{\mathrm{ext}}(G, G)$ is a version of the usual Whittaker category $\mathrm{Whit}(G, G)$ (with the imposed extra condition of equivariance with respect to the group of rational points of Z_G).

The other extreme is the "closed stratum", which is the principal series category denoted $\mathrm{I}(G, B)$. The latter is the analog of the space of functions on the double quotient

$$T(K) \cdot N(\mathbb{A})\backslash G(\mathbb{A})/G(\mathbb{O}).$$

The functor $\mathrm{coeff}^{\mathrm{ext}}_{G,G}$ can be thus thought of as taking for each parabolic the corresponding functor of *constant term*, and then taking the non-degenerate Whittaker coefficients for the Levi.

The category $\mathrm{Whit}^{\mathrm{ext}}(G, G)$ is more tractable than the original category $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ because it is comprised of the categories $\mathrm{Whit}(G, P)$, each of which is a combination of local information and that involving a proper Levi subgroup.

0.2.4. *The glued category on the spectral side.* The category $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$ is defined by explicitly gluing certain categories

$$\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})),$$

where $\check{P}$ runs through the poset of parabolic subgroups of $\check{P}$.

Each category $\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}))$ is defined as follows. We consider the map

$$\mathsf{p}_{\check{P},\mathrm{spec}} : \mathrm{LocSys}_{\check{P}} \to \mathrm{LocSys}_{\check{G}},$$

and $\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}))$ is the DG category of quasi-coherent sheaves on $\mathrm{LocSys}_{\check{P}}$ equipped with a connection along the fibers of the map $\mathsf{p}_{\check{P},\mathrm{spec}}$.

The gluing functors, and the functor $\mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}})$ are defined naturally via pull-back, see Sect. 9.3 for details.

To explain the reason why the category $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$ is more tractable than the original category $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$, let us consider the "open stratum", i.e., the category

$$\mathsf{F}_{\check{G}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})) = \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}).$$

We claim that this category embeds fully faithfully into the "open stratum" on the geometric side, i.e., the category $\mathrm{Whit}(G, G)$. This is shown by combining the following two results:

One is Proposition 4.3.4 that says that the category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ admits a fully faithful functor

$$\text{co-Loc}_{\check{G},\mathrm{spec}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)},$$

where $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ is a version of the category $\mathrm{Rep}(\check{G})$ *spread over the Ran space of X*. [2]

The second is a geometric version of the Casselman-Shalika formula, Quasi-Theorem 5.9.2, that says that $\mathrm{Whit}(G, G)$ is equivalent to a category obtained by slightly modifying $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$.

---

[2]The notation "co-Loc$_{\check{G},\mathrm{spec}}$" is not intended to suggest that this functor is a *co-localization* in the sense of category theory (i.e., admits a fully faithful left adjoint). Rather, it is the right adjoint to a functor $\mathrm{Loc}_{\check{G},\mathrm{spec}}$, which is a localization-type functor in the sense of [BB]. The latter happens to be a localization in the sense of category theory as its right adjoint, i.e., co-Loc$_{\check{G},\mathrm{spec}}$ is fully faithful.

0.2.5. *Comparing the essential images.* Finally, let us comment on the last step of the proof, namely, the comparison of the essential images in diagram (0.1).

The idea is to show that there exist two families of objects

$$\mathcal{F}_a \in \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \text{ and } \mathcal{M}_a \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G),$$

parameterized by the same set A, such that
- The objects $\mathcal{F}_a$ generate $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$;
- The objects $\mathcal{M}_a$ generate $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$;
- For each $a \in A$ we have an isomorphism

$$\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}})(\mathcal{F}_a) \simeq \mathrm{coeff}_{G,G}^{\mathrm{ext}}(\mathcal{M}_a).$$
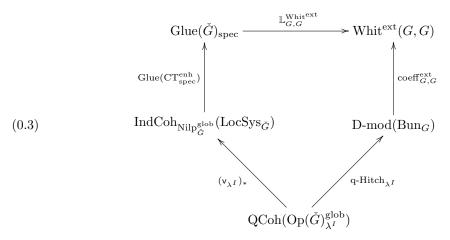
We construct the required families $\mathcal{F}_a$ and $\mathcal{M}_a$ as follows. By induction on the rank, we can assume that the geometric Langlands conjecture holds for proper Levi subgroups of G. Then Quasi-Theorem 6.7.2 implies that for a proper parabolic P with Levi quotient M, we have a diagram

$$
\begin{array}{ccc}
\mathrm{Glue}(\check{G})_{\mathrm{spec}} & \xrightarrow{\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}}} & \mathrm{Whit}^{\mathrm{ext}}(G,G) \\
\Big\uparrow {\scriptstyle \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}})} & & \Big\uparrow {\scriptstyle \mathrm{coeff}_{G,G}^{\mathrm{ext}}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\
\Big\uparrow {\scriptstyle \mathrm{Eis}_{\check{P},\mathrm{spec}}} & & \Big\uparrow {\scriptstyle \mathrm{Eis}_P} \\
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) & \xrightarrow{\mathbb{L}_M} & \mathrm{D\text{-}mod}(\mathrm{Bun}_M)
\end{array}
$$

that commutes up to a (specific) self-equivalence of $\mathrm{D\text{-}mod}(\mathrm{Bun}_M)$. Here $\mathrm{Eis}_{\check{P},\mathrm{spec}}$ and Eis_P are the Eisenstein series functors on the spectral and geometric sides, respectively.

However, the essential images of the functor $\mathrm{Eis}_{\check{P},\mathrm{spec}}$ (resp., Eis_P) for all proper parabolics P are not sufficient to generate the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ (resp., $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$). Namely, on the spectral side we are missing the entire locus of *irreducible* local systems, and on the geometric side the full subcategory $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$ corresponding to cuspidal objects.

Another family of objects is provided by the commutative diagram

$$(0.3)$$

$$
\begin{array}{ccc}
\mathrm{Glue}(\check{G})_{\mathrm{spec}} & \xrightarrow{\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}}} & \mathrm{Whit}^{\mathrm{ext}}(G,G) \\
\Big\uparrow {\scriptstyle \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}})} & & \Big\uparrow {\scriptstyle \mathrm{coeff}_{G,G}^{\mathrm{ext}}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\
& \nwarrow {\scriptstyle (\mathsf{v}_{\lambda I})_*} \qquad {\scriptstyle \mathrm{q\text{-}Hitch}_{\lambda I}} \nearrow & \\
& \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda I}^{\mathrm{glob}}) &
\end{array}
$$

Here $\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I})$ is the scheme of global *opers* on the curve X with specified singularities (encoded by the index λ^I), see Sect. 10.

The functor $(\mathsf{v}_{\lambda^I})_*$ is that of direct image with respect to the natural forgetful map

$$\mathsf{v}_{\lambda^I} : \mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I}) \to \mathrm{LocSys}_{\check{G}}.$$

The functor q-Hitch$_{\lambda^I}$ is obtained by generalizing the construction of [BD2] that attaches objects in D-mod(Bun$_G$) to quasi-coherent sheaves on the scheme of opers.

Now, the essential images of the functors Eis$_P$ (for all proper parabolics P) and those of the functors q-Hitch$_{\lambda^I}$ do generate D-mod(Bun$_G$) by Theorem 11.1.1.

The generation of $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$ by the essential images of the functors Eis$_{\check{P},\mathrm{spec}}$ and $(\mathsf{v}_{\lambda^I})_*$ follows from Conjecture 10.2.8.

0.2.6. *Summary.* One can summarize the idea of the proof as playing off against each other the operations of taking the (extended) Whittaker coefficient and the Beilinson-Drinfeld construction of D-modules on Bun$_G$ via opers, and tracing through the corresponding operations on the spectral side.

We should remark that the compatibility of the two operations on the geometric and spectral sides is the limiting case of the more general quantum Langlands phenomenon. This idea was present and explored in the papers [Fr] (specifically, Sect. 6.4) and [Sto]; these papers record part of the research in this direction, carried out by B. Feigin, E. Frenkel and A. Stoyanovsky in the early 90's.

0.3. **Other approaches to the construction of the functor.**

0.3.1. *The Drinfeld-Laumon approach: the case of an arbitrary reductive group.* Let G be still an arbitrary reductive group. Let

$$\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}} \overset{\jmath}{\hookrightarrow} \mathrm{LocSys}_{\check{G}}$$

be the embedding of the locus of irreducible local systems. By a slight abuse of notation we shall denote by $\jmath_*$ the functor [3]

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}).$$

The resulting functor

$$\mathbb{L}_G \circ \jmath_* : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

can be described as follows:

Starting from $\mathcal{F} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}})$, we regard it as an object of $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$ extended by zero from the "open stratum"

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \hookrightarrow \mathrm{Glue}(\check{G})_{\mathrm{spec}}.$$

Applying the functor

$$\mathbb{L}^{\mathrm{Whit}^{\mathrm{ext}}}_{G,G} : \mathrm{Glue}(\check{G})_{\mathrm{spec}} \to \mathrm{Whit}^{\mathrm{ext}}(G,G),$$

we obtain an object extended by zero from the "open stratum"

$$\mathrm{Whit}(G,G) \hookrightarrow \mathrm{Whit}^{\mathrm{ext}}(G,G).$$

---

[3]In fact, the difference between the two categories $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ and $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$ disappears once we restrict to $\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}$.

I.e., we do not need to worry about constant terms and gluing; our sought-for object of Bun_G will be cuspidal, and thus will only have non-degenerate Whittaker coefficients.

One can interpret the Drinfeld-Laumon approach (which takes its origin in the classical theory of automorphic functions) as attempting to prove directly that the above object

$$\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \jmath_*(\mathcal{F}) \in \mathrm{Whit}^{\mathrm{ext}}(G,G)$$

uniquely descends to an object

$$\mathbb{L}_G \circ \jmath_*(\mathcal{F}) \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

The difference between this approach and one in the present paper is that instead of proving the descent statement mentioned above for an arbitrary $\mathcal{F} \in \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}})$, we do it on the set of generators of that category. These are given as direct images

$$\mathsf{v}_{\lambda^I}(\mathcal{F}), \quad \mathcal{F} \in \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}}).$$

For such objects descent is proved by pinpointing the corresponding object of $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$. Namely, it is one given by the Beilinson-Drinfeld construction, i.e., $\mathrm{q\text{-}Hitch}_{\lambda^I}(\mathcal{F})$.

0.3.2. The Drinfeld-Laumon approach: the case of GL_n. In reality, the Drinfeld-Laumon approach as it appears in [Dr], [Lau1] and [Lau2], and developed further in [FGKV] and [FGV2], is specialized to the case of $G = GL_n$.

The main feature of this special case is that one can replace the space (prestack) on which we realize $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ by an actual algebraic stack [4], at the cost of losing fully-faithfulness of the functor

$$\mathrm{coeff}_{G,G}^{\mathrm{ext}} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{Whit}^{\mathrm{ext}}(G,G).$$

In our notations, the construction of [FGV2] can be interpreted as follows. One introduces a certain full subcategory

$$\mathrm{Whit}(G,G)^{\mathrm{non\text{-}polar;ext}} \subset \mathrm{Whit}^{\mathrm{ext}}(G,G),$$

(whose definition only involves usual algebraic stacks). The above inclusion admits a right adjoint, denoted

$$\Upsilon : \mathrm{Whit}^{\mathrm{ext}}(G,G) \to \mathrm{Whit}(G,G)^{\mathrm{non\text{-}polar;ext}},$$

and one considers the functor

$$\mathrm{coeff}_{G,G}^{\mathrm{non\text{-}polar;ext}} = \Upsilon \circ \mathrm{coeff}_{G,G}^{\mathrm{ext}}, \quad \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G,G)^{\mathrm{non\text{-}polar;ext}}.$$

The functor $\mathrm{coeff}_{G,G}^{\mathrm{non\text{-}polar;ext}}$ is no longer fully faithful. However, it has the property that it is fully faithful on the subcategory

$$\mathrm{D\text{-}mod}(\overset{\circ}{\mathrm{Bun}}_G)^{\heartsuit} \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G),$$

where $\overset{\circ}{\mathrm{Bun}}_G$ is the open substack corresponding to G-bundles (i.e., rank n vector bundles) with vanishing H^1, and where the superscript "$\heartsuit$" denotes the heart of the t-structure.

Starting from an irreducible n-dimensional local system σ on X, one wants to construct the corresponding object

$$\mathcal{M}_\sigma := \mathbb{L}_G(k_\sigma) \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G),$$

where k_σ is the sky-scraper at the point $\sigma \in \mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$.

---

[4]This was crucial at the time of writing of [FGV2], as it was not clear how to define or deal with D-modules on arbitrary prestacks.

To σ one explicitly associates an object of $\mathrm{Whit}(G, G)^{\mathrm{non\text{-}polar;ext}}$, which in our notations is

$$(0.4) \qquad \Upsilon \circ \mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \circ \jmath_*(k_\sigma) \in \mathrm{Whit}(G, G)^{\mathrm{non\text{-}polar;ext}},$$

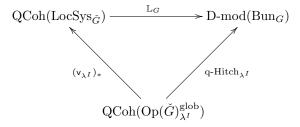or, by slightly abusing the notation and ignoring the contributions of proper parabolics,

$$\Upsilon \circ \mathbb{L}_{G,G}^{\mathrm{Whit}} \circ \text{co-Loc}_{\check{G},\mathrm{spec}}(k_\sigma).$$

One constructs the restriction of $\mathcal{M}_\sigma$ to Bun_G^d (the connected component of Bun_G corresponding to vector bundles of degree d) for $d \gg 0$ by showing that the direct summand of (0.4) living over Bun_G^d descends to (=canonically comes as the image under $\mathrm{coeff}_{G,G}^{\mathrm{non\text{-}polar;ext}}$ of) an object of $\mathrm{D\text{-}mod}(\mathrm{Bun}_G^d)$ by making heavy use of the t-structures on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ and $\mathrm{Whit}(G, G)^{\mathrm{non\text{-}polar;ext}}$.

0.3.3. *The Beilinson-Drinfeld approach via opers.* Extending the construction of [BD2], one may attempt to define a functor

$$\mathbb{L}_G|_{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

by requiring that the diagram

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\;\;\mathbb{L}_G\;\;} \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

$$(v_{\lambda^I})_* \qquad\qquad \text{q-Hitch}_{\lambda^I}$$

$$\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}})$$

be commutative for every parameter λ^I.

This would be possible if one knew Conjecture 10.5.10.

0.3.4. *Beilinson's spectral projector.* There exists yet one more approach to the construction of the functor

$$\mathbb{L}_G|_{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

It is based on the idea that the Hecke functors applied at all points of X comprise an action of the symmetric monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$. A precise statement along these lines is formulated as Theorem 4.5.2.

This does indeed define the restriction of the functor $\mathbb{L}_G$ to

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

by applying the above action to the object of $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ that is supposed to correspond under $\mathbb{L}_G$ to $\mathcal{O}_{\mathrm{LocSys}_{\check{G}}} \in \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$. This object is identified in Sect. 5.9.4.

0.4. **What is new in this paper?**

0.4.1. *Old and new ideas.* Many of the ideas present in this paper are not at all new.

The basic initial idea is the same one as in the Drinfeld-Laumon approach. It consists of accessing the category D-mod(Bun$_G$) through the Whittaker model, while the latter can be directly compared to the spectral side.

The next idea was already mentioned above: it consists of playing off the functors

$$\text{coeff}_{G,G} : \text{D-mod}(\text{Bun}_G) \to \text{Whit}(G,G) \text{ and } \text{Loc}_G : \text{KL}(G, \text{crit})_{\text{Ran}(X)} \to \text{D-mod}(\text{Bun}_G)$$

against each other, and comparing them to their counterparts on the spectral side.

I.e., we want to complete both the Drinfeld-Laumon approach and the Beilinson-Drinfeld approach to an equivalence of categories, by comparing them to each other. As was mentioned already, the fruitfulness of such a comparison was explored already in [Fr] and [Sto].

Among the new ideas one could mention the following ones: (a) the modification of the spectral side, given by $\text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}})$; (b) the idea that one can consider D-modules on arbitrary prestacks rather than algebraic stacks or ind-algebraic stacks; (c) categories living over the Ran space and "local-to-global" constructions they give rise to; (d) contractibility of the space of generically defined maps from X to a connected algebraic group.

All of these ideas became available as a result of bringing the machinery of derived algebraic geometry and higher category theory to the paradigm of Geometric Langlands. We learned about these subjects from J. Lurie.

0.4.2. *Ind-coherent sheaves and singular support.*

The definition of $\text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}})$ is based on the theory of singular support of coherent sheaves on a scheme (or algebraic stack) which is a *derived locally complete intersection.* This theory is developed in [AG] and reviewed in Sect. 2.

The idea of singular support is also an old one, and apparently goes back to D. Quillen. Given a triangulated category $\mathcal{C}$, an object $c \in \mathcal{C}$ and an evenly graded commutative algebra A mapping to the algebra

$$\underset{i}{\oplus} \text{Ext}_{\mathcal{C}}^{2i}(c, c),$$

it makes sense to say that c is supported over a given Zariski-closed subset of $\text{Spec}(A)$.

When $\mathcal{C}$ is the derived category of quasi-coherent sheaves on an affine DG scheme X, we take A to be the even part of $\text{HH}(X)$, the Hochschild cohomology algebra of X.

So, at the end of the day, singular support of a coherent sheaf $\mathcal{F}$ measures which cohomological operations $\mathcal{F} \to \mathcal{F}[2i]$ vanish when iterated a large number of times.

Further details are given in Sect. 2.

0.4.3. *D-modules on prestacks.* The idea of considering D-modules on prestacks is really an essential one for this paper. Here is a typical example of a prestack, considered in Sect. 5 and denoted $\text{Bun}_G^{B\text{-gen}}$, which is used in the definition of the category $\text{Whit}^{\text{ext}}(G,G)$.

The prestack $\text{Bun}_G^{B\text{-gen}}$ classifies G-bundles on X, equipped with a reduction to the Borel subgroup B, *defined generically on X.*

The idea to realize $\text{Whit}^{\text{ext}}(G,G)$ on $\text{Bun}_G^{B\text{-gen}}$ defined as above was suggested by J. Barlev.

Of course, for an arbitrary prestack $\mathcal{Y}$, the category D-mod($\mathcal{Y}$), although well-defined, will be pretty intractable. In the case of $\text{Bun}_G^{B\text{-gen}}$, the nice properties of D-mod($\text{Bun}_G^{B\text{-gen}}$) are ensured

by Proposition 5.1.3 that says that $\mathrm{Bun}_G^{B\text{-gen}}$ can be realized as a quotient of an algebraic stack by a schematic and proper equivalence relation.

One feature of D-mod($\mathrm{Bun}_G^{B\text{-gen}}$) is that it *does not have* a t-structure with the usual properties of a t-structure of the category of D-modules on a scheme or algebraic stack. For many people, including the author, this is one of the reasons why this category has not been considered earlier.

Another class of examples of prestacks has to do with the *Ran space* of X, denoted $\mathrm{Ran}(X)$, which classifies *non-empty finite subsets of X*.

0.4.4. *Local-to-global.* To give an example of a "local-to-global" principle employed in this paper, we consider the category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$. This is a "global" object, since the stack $\mathrm{LocSys}_{\check{G}}$ itself is of global nature, as it depends on the curve X.

The corresponding local category is $\mathrm{Rep}(\check{G})$ of algebraic representations of $\check{G}$. We spread it over the Ran space and obtain the category, denoted $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$, introduced in Sect. 4.2. The "local-to-global" principle for this case, stated in Proposition 4.3.4, says that there is a pair of adjoint functors

$$\mathrm{Loc}_{\check{G},\mathrm{spec}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \rightleftarrows \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) : \text{co-}\mathrm{Loc}_{\check{G},\mathrm{spec}}$$

with the right adjoint co-$\mathrm{Loc}_{\check{G},\mathrm{spec}}$ being fully faithful.

Hence, we obtain a fully faithful functor from a "global" category to a "local" one, which is what we mean by a "local-to-global" principle.

We will also encounter non-trivial generalizations of the above example in Quasi-Theorems 6.6.2 and 7.4.2. However, the corresponding "local-to-global" principle will not appear in the statement, but rather constitutes one of the steps in the proof, which is not discussed explicitly in the paper.

To explain its flavor, we consider the following example. Consider the natural map

$$\mathsf{p}_{\check{B},\mathrm{spec}} : \mathrm{LocSys}_{\check{B}} \to \mathrm{LocSys}_{\check{G}} .$$

We are interested in the category, denoted by $\mathsf{F}_{\check{B}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{B}}))$, mentioned in Sect. 0.2.4. It is equipped with a pair of adjoint functors:

$$\mathbf{ind}_{\mathsf{F}_{\check{B}}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{B}}) \rightleftarrows \mathsf{F}_{\check{B}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{B}})) : \mathbf{oblv}_{\mathsf{F}_{\check{B}}},$$

The composition $\mathbf{oblv}_{\mathsf{F}_{\check{B}}} \circ \mathbf{ind}_{\mathsf{F}_{\check{B}}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{B}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{B}})$ has thus a structure of monad (i.e., algebra object in the monoidal category $\mathrm{End}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{B}}))$). We would like to describe this monad in "local" terms.

The latter turns out to be possible. The answer is given in terms of the Ran version of the *spectral Hecke stack* (see Sect. 4.7.1) and is obtained by generalizing the construction of [Ro].

0.4.5. *Contractibility.* Finally, the contractibility result mentioned in Sect. 0.4.1 says that if H is a connected affine algebraic group, then the prestack $\mathbf{Maps}(X, H)^{\mathrm{gen}}$ that classifies maps from X to H, *defined generically on X*, is homologically contractible.

The latter means that the pull-back functor

$$\mathrm{Vect} = \mathrm{D\text{-}mod}(\mathrm{pt}) \to \mathrm{D\text{-}mod}(\mathbf{Maps}(X, H)^{\mathrm{gen}})$$

is fully faithful.

This result is the reason behind the validity of Theorem 8.2.10 (fully faithfulness of the functor $\mathrm{coeff}_{G,G}^{\mathrm{ext}}$) for GL_n.

We also note (although this is not used in this paper), that the above-mentioned contractibility provides a "local-to-global" principle on the geometric side. Namely, as is explained in [Ga2, Sect. 4.1], it implies that the pull-back functor

$$\text{D-mod}(\text{Bun}_G) \to \text{D-mod}(\text{Gr}_{G,\text{Ran}(X)})$$

is fully faithful, where $\text{Gr}_{G,\text{Ran}(X)}$ is the Ran version of the affine Grassmannian of the group G.

0.5. **Notations and conventions.**

0.5.1. *The theory of ∞-categories.* Even though the statement of the categorical geometric Langlands conjecture can be perceived as an equivalence of two triangulated categories (rather than DG categories), the language of ∞-categories is essential for this paper. The main reason they appear is the following:

Some of the crucial constructions in this paper use fact that we can define the (DG) category of D-modules (and quasi-coherent sheaves) on an arbitrary prestack. The latter category is, by definition, constructed as a limit *taken in the ∞-category of DG categories*, see Sect. 2.7.1.

So, essentially all we need is to have the notion of *diagram of DG categories, parameterized by some index category (which is typically an ordinary category)*, and to have the ability to take the limit of such a diagram. Now, to have such an ability (and to know some of its basic properties) amounts to including [Lu, Chapters 1-5] into our tool kit.

We will not attempt to review the theory of ∞-categories here. [5] An excellent review is provided by [Lu, Chapter 1]. So, our suggestion to the reader is to familiarize oneself with *loc.cit.* and start using the theory pretending having a full knowledge of it (knowing the proofs from the bulk of [Lu] will not really enhance one's ability to understand how the theory is applied in practice).

0.5.2. The conventions in this paper regarding ∞-categories and DG categories follow verbatim those adopted in [DrGa1]. The most essential ones are:

(i) When we say "category" by default we mean "$(\infty, 1)$"-category.

(ii) For a category $\mathbf{C}$ and objects $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$ we shall denote by $\text{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the ∞-groupoid of maps between them. We shall denote by $\text{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the *set* $\pi_0(\text{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2))$, i.e., Hom in the ordinary category $\text{Ho}(\mathbf{C})$.

(iii) All DG categories are assumed to be pretriangulated and, unless explicitly stated otherwise, cocomplete (that is, they contain arbitrary direct sums). All functors between DG categories are assumed to be exact and continuous (that is, commuting with arbitrary direct sums, or equivalently, with all colimits). In particular, all subcategories are by default assumed to be closed under arbitrary direct sums.

(iv) We let Vect denote the DG category of complexes of vector spaces; thus, the usual category of k-vector spaces is denoted by $\text{Vect}^{\heartsuit}$.

(v) The category of ∞-groupoids is denoted by ∞-Grpd.

0.5.3. Our conventions regarding DG schemes and prestacks follow verbatim those adopted in [DrGa1], Sect. 0.6.8-0.6.9.

---

[5]That said, the reader who is completely new to ∞-categories can pretend that the notion of ∞-category is an enhancement of that of ordinary category. The main point of difference is that morphisms between two objects no longer form a set, but rather an ∞-groupoid, i.e., a non-discrete homotopy type.

1. A ROADMAP TO THE CONTENTS

The structure of the main body of the paper may not make it obvious what role each section plays in the construction of the geometric Langlands equivalence, so we shall now proceed to describe the contents and main ideas of each section.

1.1. **Singular support and the statement of the geometric Langlands equivalence.**

1.1.1. In Sect. 2 we review the theory of singular support of coherent sheaves. This is needed in order to define the spectral side of geometric Langlands.

We first define the notion of quasi-smooth DG scheme (a.k.a. derived locally complete intersection). These are DG schemes for which the notion of singular support is defined. We then proceed to the definition of singular support itself via cohomological operations.

In the next step we review the theory of ind-coherent sheaves, and define the main player for the spectral side of geometric Langlands, the category of ind-coherent sheaves with specified singular support. We first do it for DG schemes, and then for algebraic stacks.

In the process we review the construction of QCoh on an arbitrary prestack, and of its renormalized version, denoted IndCoh, on an algebraic stack.

1.1.2. In Sect. 3 we state the geometric Langlands conjecture according to the point of view taken in this paper.

We first take a look at the stack $\mathrm{LocSys}_{\check{G}}$ and explain why it is quasi-smooth, and describe the corresponding stack $\mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$. Then we introduce the spectral side of geometric Langlands as the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$.

We state the geometric Langlands conjecture as the existence and uniqueness of an equivalence

$$\mathbb{L}_G : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

satisfying the property of compatibilty with the extended Whittaker model, denoted $\mathrm{Wh}^{\mathrm{ext}}$. This property itself will be stated in Sect. 9 after a good deal of preparations. As part of

the statement of the geometric Langlands conjecture we include the compatibility with Hecke functors, Eisenstein series, and Kac-Moody localization.

Finally, we introduce the full subcategory

$$\text{D-mod}(\text{Bun}_G)_{\text{temp}} \subset \text{D-mod}(\text{Bun}_G)$$

which under the (conjectural) equivalence $\mathbb{L}_G$ corresponds to

$$\text{QCoh}(\text{LocSys}_{\check{G}}) \subset \text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}}).$$

1.2. Hecke action.

1.2.1. Sect. 4 is devoted to the discussion of Hecke functors on both the geometric and spectral sides of geometric Langlands. This is needed in order to formulate the property of the functor $\mathbb{L}_G$ which has been traditionally perceived as the main property satisfied by Langlands correspondence, and also for one of the crucial steps in the proof of the existence of $\mathbb{L}_G$ (used in Sects. 11.2.5-11.2.6).

We begin by discussing what we call *the naive geometric Satake*. We consider the category $\text{Rep}(\check{G})$, and consider its version spread over the Ran space, denoted $\text{Rep}(\check{G})_{\text{Ran}(X)}$, and the pair of adjoint functors

$$\text{Loc}_{\check{G},\text{spec}} : \text{Rep}(\check{G})_{\text{Ran}(X)} \rightleftarrows \text{QCoh}(\text{LocSys}_{\check{G}}) : \text{co-Loc}_{\check{G},\text{spec}},$$

already mentioned in Sect. 0.4.4.

By Proposition 4.3.4, the functor $\text{Loc}_{\check{G},\text{spec}}$ realizes $\text{QCoh}(\text{LocSys}_{\check{G}})$ as a monoidal quotient category of $\text{Rep}(\check{G})_{\text{Ran}(X)}$.

We quote Proposition 4.4.4 which can be regarded as stating the existence of the *naive geometric Satake functor*, denoted

$$\text{Sat}(G)_{\text{Ran}(X)}^{\text{naive}} : \text{Rep}(\check{G})_{\text{Ran}(X)} \to \text{D-mod}(\text{Hecke}(G)_{\text{Ran}(X)}).$$

The functor $\text{Sat}(G)_{\text{Ran}(X)}^{\text{naive}}$ defines an action of the monoidal category $\text{Rep}(\check{G})_{\text{Ran}(X)}$ on $\text{D-mod}(\text{Bun}_G)$. We then proceed to Theorem 4.5.2, which says that the above action factors through an action of the monoidal category $\text{QCoh}(\text{LocSys}_{\check{G}})$ on $\text{D-mod}(\text{Bun}_G)$.

The property of compatibility of the geometric Langlands equivalence with the Hecke action says that $\mathbb{L}_G$ intertwines the natural action of $\text{QCoh}(\text{LocSys}_{\check{G}})$ on $\text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}})$ (by pointwise tensor product) with the above action of $\text{QCoh}(\text{LocSys}_{\check{G}})$ on $\text{D-mod}(\text{Bun}_G)$.

1.2.2. Next we indicate (but do not discuss in full detail) the extension of the *naive geometric Satake* to the *full geometric Satake*. The latter involves an analog of the Hecke stack on the spectral side, and says that the functor $\text{Sat}(G)_{\text{Ran}(X)}^{\text{naive}}$ can be extended to a monoidal functor

$$\text{Sat}(G)_{\text{Ran}(X)} : \text{IndCoh}(\text{Hecke}(\check{G}, \text{spec})_{\text{Ran}(X)}^{\text{loc}})) \to \text{D-mod}(\text{Hecke}(G)_{\text{Ran}(X)}).$$

The functor $\text{Sat}(G)_{\text{Ran}(X)}$ can be used to intrinsically characterise the full subcategory

$$\text{D-mod}(\text{Bun}_G)_{\text{temp}} \subset \text{D-mod}(\text{Bun}_G)$$

mentioned above.

1.3. **Whittaker and parabolic categories.** Sects. 5-9 contain the bulk of the geometric constructions in this paper. It is these sections that contain the "quasi-theorems" on which hinges the proof of the geometric Langlands conjecture.

These sections deal with the various versions (i.e., genuine, degenerate and extended) of the Whittaker category, and the parabolic category. In each case there is a quasi-theorem that describes the corresponding category in spectral terms. Some of the quasi-theorems rely on the validity of the geometric Langlands conjecture for proper Levi subgroups of G, and some do not.

1.3.1. *The genuine Whittaker category.* Sect. 5 is devoted to the discussion of two versions of the genuine Whittaker category, denoted $\mathrm{Whit}(G)$ and $\mathrm{Whit}(G,G)$, respectively.

In the function-theoretic analogy, the category $\mathrm{Whit}(G)$ corresponds to the space of functions on $G(\mathbb{A})/G(\mathbb{O}) \times \mathfrak{ch}(K)$ that satisfy

$$f(n \cdot g, \chi) = \chi(n) \cdot f(g, \chi), \quad g \in G(\mathbb{A})/G(\mathbb{O}), \quad n \in N(\mathbb{A}),$$

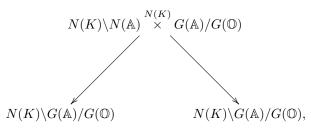where χ is a non-degenerate charcater on $N(\mathbb{A})$ trivial on $N(K)$.

The category $\mathrm{Whit}(G,G)$ is a full subcategory of $\mathrm{Whit}(G)$, where we impose an extra condition of invariance with respect to $Z_G(K)$.

In what follows, for simplicity, we will discuss $\mathrm{Whit}(G)$. We realize $\mathrm{Whit}(G)$ as the category of D-modules on a certain prestack satisfying an equivariance condition with respect to a certain groupoid against a canonically defined character.

The prestack in question, denoted $\mathcal{Q}_G$, is a version of the prestack $\mathrm{Bun}_G^{B\text{-gen}}$ mentioned above. The difference is that in addition to the data of a generic reduction of our G-bundle to B, we specify the data of (generic) identification of the induced T-bundle with one induced by the cocharacter $2\check{\rho}$ from the line bundle $\omega_X^{\frac{1}{2}}$, where ω_X is the canonical line bundle on X, and $\omega_X^{\frac{1}{2}}$ is its (chosen once and for all) square root. Up to generically trivializing $\omega_X^{\frac{1}{2}}$, the prestack $\mathcal{Q}_G$ identifies with $\mathrm{Bun}_G^{N\text{-gen}}$, and its set of k-points of $\mathcal{Q}_G$ identifies with the double quotient

$$N(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

1.3.2. The definition of the groupoid involved in the definition of $\mathrm{Whit}(G)$, denoted $\mathbf{N}$, is trickier. At the level of functions, one would like to consider the groupoid

$$N(K)\backslash N(\mathbb{A}) \overset{N(K)}{\times} G(\mathbb{A})/G(\mathbb{O})$$

$$N(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \qquad N(K)\backslash G(\mathbb{A})/G(\mathbb{O}),$$

where $- \overset{N(K)}{\times} -$ means the quotient by the diagonal action of $N(K)$. (In the above diagram the left arrow is the projection on the second factor, and the right arrow is given by the action of $N(\mathbb{A})$ on $G(\mathbb{A})/G(\mathbb{O})$.)

However, it is not clear how to implement this idea in algebraic geometry, i.e., how to realize such a groupoid as a prestack. Instead we use a certain surrogate, whose idea is explained in Sect. 5.3. Here we will just mention that it relies on the phenomenon of *strong approximation* for the group N.

1.3.3. *Whittaker category via the affine Grassmannian.* One could also approach the definition of $\mathrm{Whit}(G)$ slightly differently (and the same applies to the degenerate, extended and parabolic versions). Namely, instead of the prestack $\mathcal{Q}_G$, we can realize our category as a full subcategory of $\mathrm{D\text{-}mod}(\mathrm{Gr}_{G,\mathrm{Ran}(X)})$, where $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ is the Ran version of the affine Grassmannian. The point is that $\mathcal{Q}_G$ is isomorphic to the quotient of $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ by the action of the group-prestack $\mathbf{Maps}(X,N)^{\mathrm{gen}}$ of generically defined maps $X \to N$.

This way of realizing $\mathcal{Q}_G$ gives rise to a more straightforward way of imposing the equivariance condition needed for the definition of $\mathrm{Whit}(G)$. [6]

In any case, having this other approach to $\mathrm{Whit}(G)$ (and its degenerate, extended and parabolic versions) is necessary in order to prove its description in spectral terms.

1.3.4. *Spectral description of the Whittaker category.*

The spectral description of $\mathrm{Whit}(G)$, given by Quasi-Theorem 5.9.2, says that it is equivalent to the *unital* version of the category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$, denoted $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{unital}}$. This is the first of the quasi-theorems in this paper, and it is expected to follow rather easily from the already known results.

We note that Quasi-Theorem 5.9.2 is a geometric version of the Casselman-Shalika formula that describes the unramified Whittaker model in terms of Satake parameters.

1.4. **The parabolic category.** In Sect. 6 we discuss the parabolic category, denoted $\mathrm{I}(G,P)$, for a given parabolic subgroup $P \subset G$ with Levi quotient M.

1.4.1. *The idea of the parabolic category.* In terms of the function-theoretic analogy, the category $\mathrm{I}(G,P)$ corresponds to the space of functions on the double quotient

$$M(K) \cdot N(P)(\mathbb{A}) \backslash G(\mathbb{A}) / G(\mathbb{O}).$$

The actual defintion of $\mathrm{I}(G,P)$ uses the prestack $\mathrm{Bun}_G^{P\text{-}\mathrm{gen}}$ of G-bundles equipped with a generic reduction to P. We define $\mathrm{I}(G,P)$ to be the category of D-modules on $\mathrm{Bun}_G^{P\text{-}\mathrm{gen}}$ that are equivariant with respect to the appropriately defined groupoid. The idea of this groupoid, denoted $\mathbf{N}(\mathbf{P})$, is similar to that of $\mathbf{N}$, mentioned in Sect. 1.3.1. As in the case of $\mathrm{Whit}(G)$, we can alternatively define $\mathrm{I}(G,P)$ using the Ran version of the affine Grassmannian.

The forgetful functor

$$\mathrm{I}(G,P) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-}\mathrm{gen}})$$

is actually fully faithful due to a unipotence property of the groupoid $\mathbf{N}(\mathbf{P})$.

We note that in addition to the prestack $\mathrm{Bun}_G^{P\text{-}\mathrm{gen}}$, we have the usual algebraic stack Bun_P classifying P-bundles on X. There exists a naturally defined map

$$\iota_P : \mathrm{Bun}_P \to \mathrm{Bun}_G^{P\text{-}\mathrm{gen}},$$

which defines a bijection at the level of k-points. We can think of $\mathrm{Bun}_G^{P\text{-}\mathrm{gen}}$ as decomposed into locally closed sub-prestacks, with Bun_P being the disjoint union of the strata. Accordingly, we have a conservative restriction functor

$$\iota_P^{\dagger} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-}\mathrm{gen}}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_P),$$

---

[6]The price that one has to pay if one uses only this approach in the case of the parabolic category $\mathrm{I}(G,P)$ is that the definition of the functors of *enhanced Eisenstein series* and *constant term* becomes more cumbersome.

and we can think of D-mod($\mathrm{Bun}_G^{P\text{-gen}}$) as glued from D-mod($\mathrm{Bun}_P$) on the various connected components of Bun_P in a highly non-trivial way. For the above restriction functor we have a commutative diagram

(1.1)

$$
\begin{array}{ccc}
\mathrm{I}(G,P) & \longrightarrow & \text{D-mod}(\mathrm{Bun}_M) \\
\downarrow & & \downarrow \\
\text{D-mod}(\mathrm{Bun}_G^{P\text{-gen}}) & \xrightarrow{\ \iota_P^\dagger\ } & \text{D-mod}(\mathrm{Bun}_P),
\end{array}
$$

with fully faithful vertical arrows, which is moreover a *pull-back diagram*. Here the right vertical arrow is the functor of pull-back along the projection $\mathsf{q}_P : \mathrm{Bun}_P \to \mathrm{Bun}_M$; it is fully faithful, since the map q_P is smooth with contractible fibers.

1.4.2. *Eisenstein and constant term functors.* The category $\mathrm{I}(G,P)$ is related to the category D-mod(Bun_G) by a pair of adjoint functors

$$\mathrm{Eis}_P^{\mathrm{enh}} : \mathrm{I}(G,P) \rightleftarrows \text{D-mod}(\mathrm{Bun}_G) : \mathrm{CT}_P^{\mathrm{enh}},$$

that we refer to as *enhanced Eisenstein series and constant term* functors.

These functors are closely related (but carry significantly more information) than the corresponding "usual" Eisenstein and constant term functors
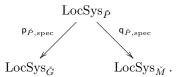
$$\mathrm{Eis}_P : \mathrm{I}(G,P) \rightleftarrows \text{D-mod}(\mathrm{Bun}_G) : \mathrm{CT}_P,$$

defined by pull-push along the diagram

$$
\begin{array}{ccc}
 & \mathrm{Bun}_P & \\
{\scriptstyle \mathsf{p}_P}\swarrow & & \searrow{\scriptstyle \mathsf{q}_P} \\
\mathrm{Bun}_G & & \mathrm{Bun}_M\,.
\end{array}
$$

For example, the functor CT_P is the composition of the functor $\mathrm{CT}_P^{\mathrm{enh}}$, followed by the restriction functor (the top horizontal arrow in the diagram (1.1)).

1.4.3. *Parabolic category on the spectral side.* We now discuss the spectral counterpart of the above picture. For the "usual" Eisenstein series functor, the picture is what one would naively expect. We consider the diagram

$$
\begin{array}{ccc}
 & \mathrm{LocSys}_{\check P} & \\
{\scriptstyle \mathsf{p}_{\check P,\mathrm{spec}}}\swarrow & & \searrow{\scriptstyle \mathsf{q}_{\check P,\mathrm{spec}}} \\
\mathrm{LocSys}_{\check G} & & \mathrm{LocSys}_{\check M}\,.
\end{array}
$$

and the corresponding pull-push functor

$$\mathrm{Eis}_{\check P,\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check M}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check M}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check G}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check G}).$$

The geometric Langlands equivalence is supposed to make the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check G}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check G}) & \xrightarrow{\ \mathbb{L}_G\ } & \text{D-mod}(\mathrm{Bun}_G) \\
{\scriptstyle \mathrm{Eis}_{\check P,\mathrm{spec}}}\uparrow & & \uparrow{\scriptstyle \mathrm{Eis}_P} \\
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check M}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check M}) & \xrightarrow{\ \mathbb{L}_M\ } & \text{D-mod}(\mathrm{Bun}_M),
\end{array}
$$

up to a twist by a (specific) line bundle on $\mathrm{LocSys}_{\check M}$.

The situation with enhanced Eisenstein series is more involved and more interesting. First, we need to give a spectral description of the category $\mathrm{I}(G,P)$. As is natural to expect, for the latter we need to assume the validity of the Langlands conjecture for the group M.

First, we consider the appropriate modification of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})$ given by the singular support condition. We denote the resulting category by $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})$. Next, we consider the map

$$\mathsf{p}_{\check{P},\mathrm{spec}} : \mathrm{LocSys}_{\check{P}} \to \mathrm{LocSys}_{\check{G}},$$

and we consider the category of objects of $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})$, endowed with a *right action of vector fields on* $\mathrm{LocSys}_{\check{P}}$ *along the (derived) fibers of the map* $\mathsf{p}_{\check{P},\mathrm{spec}}$. We denote this category by

$$\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}))$$

(the notation $\mathsf{F}_{\check{P}}$ stands for the monad induced by the action of the vector fields mentioned above). We have a naturally defined functor, denoted, and given by pushforward:

$$\mathrm{Eis}_{P,\mathrm{spec}}^{\mathrm{enh}} : \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$$

1.4.4. *Geometric Langlands equivalence for parabolic categories.*

One of the central quasi-theorems in this paper, namely Quasi-Theorem 6.6.2, says that we have a canonically defined equivalence:

$$\mathbb{L}_P : \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) \to \mathrm{I}(G,P).$$

It is supposed to be related to the geometric Langlands equivalence for G via the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\;\mathbb{L}_G\;} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\
{\scriptstyle \mathrm{Eis}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}} \big\uparrow & & \big\uparrow {\scriptstyle \mathrm{Eis}_P^{\mathrm{enh}}} \\
\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) & \xrightarrow{\;\mathbb{L}_P\;} & \mathrm{I}(G,P).
\end{array}
$$

1.5. Degenerate and extended Whittaker categories.

1.5.1. Sect. 7 deals with the degenerate Whittaker category, denoted $\mathrm{Whit}(G,P)$, and Sect. 8 with the extended Whittaker category, denoted $\mathrm{Whit}^{\mathrm{ext}}(G,G)$. The function-theoretic analogues of these categories were explained in Sect. 0.2.3.

The relevance of the categories $\mathrm{Whit}(G,P)$ (as P runs through the set of conjugacy classes of parabolics) is that they constitute buliding blocks of the category $\mathrm{Whit}^{\mathrm{ext}}(G,G)$. The category $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ plays a crucial role, being the recipient of the functor

$$\mathrm{coeff}(G,G)^{\mathrm{ext}} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{Whit}^{\mathrm{ext}}(G,G).$$

As was mentioned earlier, a crucial conjecture (which is a quasi-theorem for GL_n) says that the functor $\mathrm{coeff}(G,G)^{\mathrm{ext}}$ is fully faithful. This statement is at the heart of our approach to proving the geometric Langlands equivalence.

We omit a detailed discussion of the contents of these two sections as the ideas involved essentially combine those from Sects. 5 and 6, except for the following:

Recall the set $\mathfrak{ch}(K)$ mentioned in Sect. 0.2.3. The definition as given in *loc.cit.* is correct only when G has a connected center. In general, the definition needs to be modified, see

Sect. 8.1, and involves a certain canonically defined toric variety acted on by T, such that the stabilizer of each point is the *connected* center of the corresponding Levi subgroup.

1.5.2. The goal of Sect. 9 is to provide a spectral description of the category $\mathrm{Whit}^{\mathrm{ext}}(G, G)$. As was explained in Sect. 0.2, this is another crucial step in our approach to proving the geometric Langlands conjecture.

First, we recall the general pattern of gluing of DG categories, mimicking the procedure of describing the category of sheaves on a topological space from the knowledge of the corresponding categories on strata of a given stratification.

We observe that, more or less tautologically, the category $\mathrm{Whit}^{\mathrm{ext}}(G, G)$ is glued from the categories $\mathrm{Whit}(G, P)$.

Next, we explicitly construct the glued category on the spectral side, by taking as building blocks the categories

$$\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})).$$

We now arrive at a crucial assertion, Quasi-Theorem 9.4.2 that says that the glued category on the spectral side embeds fully faithfully into $\mathrm{Whit}^{\mathrm{ext}}(G, G)$.

So far, this Quasi-Theorem has been verified in a particular case (assuming Quasi-Theorem 6.6.2 for $P = B$), when we want to glue the open stratum (corresponding to $P = G$) to the closed stratum (corresponding to $P = B$); this case, however, suffices for the group $G = GL_2$. The proof of Quasi-Theorem 9.4.2 in the above case is a rather illuminating explicit calculation, which we unfortunately have to omit for reasons of length of this paper.

1.6. **Kac-Moody localization.**

1.6.1. Sect. 10 deals with a construction of objects of $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ of a nature totally different from one discussed in Sects. 5-9.

The previous sections approach D-modules on Bun_G geometrically, i.e., by considering various spaces that map to Bun_G and appying functors of direct and inverse image. In particular, these constructions make sense not just in the category of D-modules, but also in that of ℓ-adic sheaves (modulo the technical issue of the existence of the formalism of ℓ-adic sheaves as a functor of ∞-categories).

By contrast, in Sect. 10 we construct D-modules on Bun_G "by generators and relations." In particular, we (implicitly) use the forgetful functor $\mathrm{D}\text{-mod}(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{Bun}_G)$ (or, rather, its left adjoint). More precisely, the construction that we use is that of *localization* of modules over the Kac-Moody algebra (at a given level).

This construction is needed in order to create the commutative diagram (0.3), which is another crucial ingredient in the proof of the geometric Langands conjecture.

1.6.2. Historically, the pattern of localization originated form [BB]. In [BD2] it was extended to the following situation: if we have a group H acting on a scheme Y, and $H' \subset H$ is a subgroup, then we have a canonical functor of localization

$$(\mathfrak{h}, H')\text{-mod} \to \mathrm{D}\text{-mod}(H' \backslash Y),$$

where $(\mathfrak{h}, H')$-mod is the DG category of H'-equivariant objects in the DG category $\mathfrak{h}$-mod of $\mathfrak{h}$-modules (also known as the DG category of modules over the Harish-Chandra pair $(\mathfrak{h}, H')$).

If one looks at what this construction does in down-to-earth terms, it associates to a $(\mathfrak{h}, H')$-module a certain quotient of the free D-module, where relations are given by the action of vector fields in Y induced by the action of elements of $\mathfrak{h}$.

1.6.3. In [BD2], this construction was applied to H being the (critical central extension of the) loop group ind-scheme $\mathfrak{L}(G) = G((t))$, and H' being the group of arcs $\mathfrak{L}^+(G) := G[\![t]\!]$. The corresponding category of Harish-Chandra modules is denoted $\mathrm{KL}(G, \mathrm{crit})$. [7]

The scheme Y in question is $\mathrm{Bun}_{G,x}$, the moduli space of G-bundles on X with a full level structure at a point x. Here we think of $k[\![t]\!]$ as the completed local ring of X at x. We do not review this construction in this paper, but rather refer the reader to [BD2]; we should note, however, that modern technology allows to rewrite this construction in a more concise way.

In fact, we need an extension of the above construction to the situation, when instead of a fixed point $x \in X$ we have a finite number of points that are allowed to move along X. Ultimately, we obtain a functor

$$\mathrm{Loc}_{G,\mathrm{Ran}(X)} : \mathrm{KL}(G, \mathrm{crit})_{\mathrm{Ran}(X)} \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

A crucial property of the functor Loc_G is that "almost all D-modules on Bun_G lie in its essential image." The word *almost* is important here. We refer the reader to Proposition 10.1.6 for a precise formulation.

We also remark that the functor Loc_G should be thought of as a *quantum* case of the functor

$$\mathrm{Loc}_{G,\mathrm{spec}} : \mathrm{Rep}(G)_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_G),$$

mentioned earlier. In fact, the two are the special cases of a family whose intermediate values correspond to the situation of quantum geometric Langlands.

1.6.4. In the rest of Sect. 10 we review the connection between the category $\mathrm{KL}(G, \mathrm{crit})_{\mathrm{Ran}(X)}$ and the scheme of *local opers*. The key input is a generalization of the result of [BD2] that relates the functor Loc_G to the scheme of *global opers*. All of this is needed in order to form the diagram (0.3).

1.6.5. Finally, in Sect. 11, we assemble the ingredients developed in the previous sections in order to prove the geometric Langlands conjecture, modulo Conjectures 8.2.9 and 10.2.8, and the Quasi-Theorems.

The proof proceeds along the lines indicated in Sect. 0.2, modulo the fact that the last step of the proof, namely, one described in Sect. 0.2.5, is a bit of an oversimplification. For the actual proof, we break the category $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ into "cuspidal" and "Eisenstein" parts, and deal with each separately.

2. The theory of singular support

2.1. **Derived locally complete intersections.** The contents of this subsection are a brief review of [AG, Sect. 2]. We refer the reader to *loc.cit.* for the proofs.

We remind that throughout the paper we will be working with an algebraically closed field k of characteristic 0.

---

[7] "KL" stands for Kazhdan-Lusztig, who were the first to systematically study this category in the negative level case.

2.1.1. The theory of singular support for coherent sheaves makes substantial use of derived algebraic geometry. We cannot afford to make a thorough review here, but let us mention the following few facts, which is all we will need for this paper:

(1) Let A be a CDGA (commutative differential graded algebra) over k, which lives in cohomological degrees ≤ 0. To A one attaches the affine DG scheme $\mathrm{Spec}(A)$. If $A \to A'$ is a quasi-isomorphism, then the corresponding map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is, by definition, an isomorphism of DG schemes. The underlying topological space of $\mathrm{Spec}(A)$ is the same as that of the classical scheme $\mathrm{Spec}(H^0(A))$. The basic affine opens of $\mathrm{Spec}(A)$ are of the form $\mathrm{Spec}(A_f)$, where $f \in H^0(A)$ (more generally, it makes sense to take localizations of A with respect to multiplicative subsets of $H^0(A)$).

(1') Arbitrary DG schemes are glued from affines in the same sense as in classical algebraic geometry.

(2) There exists a fully faithful functor $\mathrm{Sch} \to \mathrm{DGSch}$ from classical schemes to derived schemes. This functor admits a right adjoint, which we will refer to as *taking the underlying classical scheme* and denote by $Y \mapsto {}^{cl}Y$. For affine DG schemes the latter functor corresponds to sending A to $H^0(A)$. In general, it is convenient to have the following analogy in mind "classical schemes to derived schemes are what reduced classical schemes are to all schemes."

(3) The DG category of quasi-coherent sheaves on a DG scheme is defined so that

$$\mathrm{QCoh}(\mathrm{Spec}(A)) = A\text{-mod},$$

the latter being the DG category of all A-modules (i.e., no finiteness assumptions).

(4) The category of DG schemes admits fiber products: for $\mathrm{Spec}(A_1) \to \mathrm{Spec}(A) \leftarrow \mathrm{Spec}(A_2)$, we have

$$\mathrm{Spec}(A_1) \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(A_2) = \mathrm{Spec}(A_1 \underset{A}{\otimes} A_2),$$

where the tensor product $A_1 \underset{A}{\otimes} A_2$ is understood in the derived sense (in particular, $A_1 \underset{A}{\otimes} A_2$ may be derived even if A, A_1 and A_2 are classical).

(4') A basic non-trivial example of a DG scheme is

$$\mathrm{pt} \underset{V}{\times} \mathrm{pt},$$

where V is a finite-dimensional vector space (considered as a scheme). The above DG scheme is by definition $\mathrm{Spec}(\mathrm{Sym}(V^*[1]))$. Here $\mathrm{pt} := \mathrm{Spec}(k)$.

(5) Let

$$
\begin{array}{ccc}
Y_1' & \xrightarrow{\ g_1\ } & Y_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y_2' & \xrightarrow{\ g_2\ } & Y_2
\end{array}
$$

be a Cartesian square of DG schemes with the vertical morphisms quasi-compact and quasi-separated. Then the base change natural transformation

$$g_2^* \circ f_* \to f_*' \circ g_1^*$$

is an isomorphism. (Note that the corresponding fact is *false* in classical algebraic geometry: i.e., even if Y_1, Y_2 and Y_2' are classical, we need to understand Y_1' is the derived sense.)

(6) One word of warning is necessary: the category DGSch is not an ordinary category, but an ∞-category, i.e., maps between objects no longer form sets, but rather ∞-groupoids (in

the various models of the theory of ∞-categories the latter can be realized as simplicial sets, topological spaces, etc.).

2.1.2. We shall now define what it means for a DG scheme Y to be a *derived locally complete intersection*, a.k.a. *quasi-smooth*.

The condition is Zariski-local, so we can assume that Y is affine.

Definition 2.1.3. *We shall say that Y is quasi-smooth if it can be realized as a derived fiber product*

(2.1)
$$
\begin{array}{ccc}
Y & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow f \\
\mathrm{pt} & \xrightarrow{\ v\ } & \mathcal{V},
\end{array}
$$

where $\mathcal{U}$ and $\mathcal{V}$ are smooth classical schemes.

More invariantly, one can phrase this definition as follows:

Definition 2.1.4. *A DG scheme Y is quasi-smooth if it is locally* almost of finite type[8] *and for each k-point $y \in Y$, the derived cotangent space $T_y^*(Y)$ has cohomologies only in degrees 0 and -1.*

In fact, for Y written as in (2.1), the derived cotangent space at $y \in Y$ is canonically isomorphic to the complex

$$
T_{f(y)}^*(\mathcal{V}) \to T_y^*(\mathcal{U}).
$$

2.1.5. It follows easily from the definitions that a classical scheme which is a locally complete intersection in the classical sense is such in the derived sense, i.e., quasi-smooth as a derived scheme.

2.2. The Sing space of a quasi-smooth scheme.

2.2.1. Let Y be a quasi-smooth derived scheme. We are going to attach to it a classical scheme $\mathrm{Sing}(Y)$ that measures the extent to which Y fails to be smooth.

Suppose that Y is locally written as a fiber product (2.1). Consider the vector bundles $T^*(\mathcal{U})|_{{}^{cl}Y}$ and $T^*(\mathcal{V})|_{{}^{cl}Y}$, considered as schemes over ${}^{cl}Y$.

The differential of f defines a map of classical schemes

(2.2)
$$
T^*(\mathcal{V})|_{{}^{cl}Y} \to T^*(\mathcal{U})|_{{}^{cl}Y}.
$$

We let $\mathrm{Sing}(Y)$ be the pre-image under the map (2.2) of the zero-section ${}^{cl}Y \to T^*(\mathcal{U})|_{{}^{cl}Y}$.

The scheme $\mathrm{Sing}(Y)$ carries a natural action of the group $\mathbb{G}_m$ inherited from one on $T^*(\mathcal{V})|_{{}^{cl}Y}$.

2.2.2. Explicitly, one can describe k-points if $\mathrm{Sing}(Y)$ as follows. These are pairs (y, ξ), where y is a k-point of Y, and ξ is an element in

$$
\ker\left(df : T_v^*(\mathcal{V}) \to T_y^*(\mathcal{U}) \right).
$$

In particular, f is smooth (which is equivalent to Y being a smooth classical scheme) if and only if the projection $\mathrm{Sing}(Y) \to Y$ is an isomorphism, i.e., if $\mathrm{Sing}(Y)$ consists of the zero-section.

---

[8]This means that the underlying classical scheme ${}^{cl}Y$ is locally of finite type over k, and the cohomology sheaves $H^i(\mathcal{O}_Y)$ are finitely generated over $H^0(\mathcal{O}_Y) = \mathcal{O}_{{}^{cl}Y}$.

2.2.3. More invariantly, one can think of ξ as an element in the vector space

$$H^{-1}(T_y^*(Y)).$$

This implies that $\mathrm{Sing}(Y)$ is well-defined in the sense that it is independent of the presentation of Y as a fiber product as in (2.1). In particular, we can define $\mathrm{Sing}(Y)$ for Y not necessarily affine.

2.3. Cohomological operations.

2.3.1. Let Y be a quasi-smooth DG scheme written as in (2.1). Let us denote by V the tangent space of $\mathcal{V}$ at the point v, and let V^* be its dual, i.e., the cotangent space.

We claim that for every $\mathcal{F} \in \mathrm{QCoh}(Y)$ there is a canonically define map of graded algebras

$$(2.3) \qquad\qquad \mathrm{Sym}(V) \to \underset{i}{\oplus}\, \mathrm{Hom}_{\mathrm{QCoh}(Y)}(\mathcal{F}, \mathcal{F}[i]),$$

where we set $\deg(V) = 2$.

We shall define (2.3) in the framework of the following geometric construction.

2.3.2. First, we consider the derived fiber product

$$\mathrm{pt} \underset{\mathcal{V}}{\times} \mathrm{pt}\,.$$

This is a groupoid over pt, i.e., a derived group-scheme (a group object in the category of derived schemes).

In particular, the category

$$\mathrm{QCoh}(\mathrm{pt} \underset{\mathcal{V}}{\times} \mathrm{pt})$$

acquires a monoidal structure given by convolution.

The unit in this category is k_{pt}, the direct image of $k \in \mathrm{QCoh}(\mathrm{pt})$ under the diagonal morphism

$$\mathrm{pt} \to \mathrm{pt} \underset{\mathcal{V}}{\times} \mathrm{pt}\,.$$

2.3.3. We claim that the derived group-scheme $\mathrm{pt} \underset{\mathcal{V}}{\times} \mathrm{pt}$ canonically acts on Y. This follows from the next diagram

in which both squares are Cartesian.

2.3.4. In particular, we obtain that the category $\mathrm{QCoh}(Y)$ acquires an action of the monoidal category $\mathrm{QCoh}(\mathrm{pt} \underset{\mathcal{V}}{\times} \mathrm{pt})$.

Hence, every $\mathcal{F} \in \mathrm{QCoh}(Y)$ acquires an action of the algebra of the endomorphisms of the unit object of $\mathrm{QCoh}(\mathrm{pt} \underset{\mathcal{V}}{\times} \mathrm{pt})$, i.e., we have a canonical map of graded algebras

$$\underset{i}{\oplus} \mathrm{Hom}_{\mathrm{QCoh}(\mathrm{pt} \underset{\mathcal{V}}{\times} \mathrm{pt})}(k_{\mathrm{pt}}, k_{\mathrm{pt}}[i]) \to \underset{i}{\oplus} \mathrm{Hom}(\mathcal{F}, \mathcal{F}[i]).$$

2.3.5. Finally, to construct the map (2.3) we notice we have a canonical isomorphism of graded algebras

$$\mathrm{Sym}(V) \to \underset{i}{\oplus} \mathrm{Hom}_{\mathrm{QCoh}(\mathrm{pt} \underset{\mathcal{V}}{\times} \mathrm{pt})}(k_{\mathrm{pt}}, k_{\mathrm{pt}}[i]).$$

2.4. **The singular support of a coherent sheaf.** The material of this subsection corresponds the approach to singular support in [AG, Sects. 5.3]. We refer the reader to *loc.cit.* for the proofs of the statements quoted here.

2.4.1. We continue to assume that Y is a quasi-smooth DG shceme written as (2.1). Note that by construction, $\mathrm{Sing}(Y)$ is a conical Zariski-closed ($=\mathbb{G}_m$-invariant) closed subset in $^{cl}Y \times V^*$.

From (2.3), we obtain that for $\mathcal{F} \in \mathrm{QCoh}(Y)$ we have a map of graded commutative algebras

$$(2.4) \qquad \Gamma(^{cl}Y \times V^*, \mathcal{O}_{^{cl}Y \times V^*}) \simeq \Gamma(Y, \mathcal{O}_{^{cl}Y}) \otimes \mathrm{Sym}(V) \to \underset{i}{\oplus} \mathrm{Hom}_{\mathrm{QCoh}(Y)}(\mathcal{F}, \mathcal{F}[i]).$$

We have the following assertion:

Lemma 2.4.2. *Let $f \in \Gamma(^{cl}Y \times V^*, \mathcal{O}_{^{cl}Y \times V^*})$ be a homogeneous element that vanishes when restricted to* $\mathrm{Sing}(Y)$. *Then some power of f belongs to the kernel of the map* (2.4).

The above lemma allows to define the notion of singular support of coherent sheaves.

2.4.3. Let $\mathrm{Coh}(Y) \subset \mathrm{QCoh}(Y)$ be the full subcategory that consists of *coherent* sheaves. I.e., these are objects that have only finitely many non-zero cohomologies, and such that each cohomology is finitely generated over $\mathcal{O}_{^{cl}Y}$.

Definition 2.4.4. *The singular support of $\mathcal{F} \in \mathrm{Coh}(Y)$ is the conical Zariski-closed subset*

$$\mathrm{sing.\,supp.}(\mathcal{F}) \subset {}^{cl}Y \times V^*,$$

corresponding to the ideal, given by the kernel of the map (2.4).

Note that by Lemma 2.4.2, we automatically have

$$\mathrm{sing.\,supp.}(\mathcal{F}) \subset \mathrm{Sing}(Y),$$

as Zariski-closed subsets of $^{cl}Y \times V^*$.

2.4.5. Dually, given a conical Zarsiki-closed subset $\mathcal{N} \subset \mathrm{Sing}(Y)$, we let

$$\mathrm{Coh}_{\mathcal{N}}(Y) \subset \mathrm{Coh}(Y)$$

be the full subcategory, consisting of objects whose singular support is contained in $\mathcal{N}$.

By definition, for $\mathcal{F} \in \mathrm{Coh}(Y)$, we have $\mathcal{F} \in \mathrm{Coh}_{\mathcal{N}}(Y)$ if and only if for every homogeneous element $f \in \Gamma(^{cl}Y \times V^*, \mathcal{O}_{^{cl}Y \times V^*})$ such that $f|_{\mathcal{N}} = 0$, some power of f lies in the kernel of (2.4).

2.4.6. We have the following assertion:

Proposition 2.4.7. *For* $\mathcal{F} \in \mathrm{Coh}(Y)$*, the subset* $\mathrm{sing.\,supp.}(\mathcal{F}) \subset \mathrm{Sing}(Y)$ *is independent of the choice of presentation of* Y *as in* (2.1)*.*

Thus, the notion of singular support of an object of $\mathrm{Coh}(Y)$ and the category $\mathrm{Coh}_{\mathcal{N}}(Y)$ make sense for any quasi-smooth DG scheme (not necessarily affine).

In addition, we have:

Proposition 2.4.8. *For* $\mathcal{F} \in \mathrm{Coh}(Y)$*, its singular support is the zero-section* $\{0\} \subset \mathrm{Sing}(Y)$ *if and only if* $\mathcal{F}$ *is* perfect.

2.5. **Ind-coherent sheaves.** The material in this subsection is a summary of [Ga3, Sect. 1].

2.5.1. Let Y be a quasi-compact DG scheme almost of finite type. We consider the DG category $\mathrm{IndCoh}(Y)$ to be the ind-completion of $\mathrm{Coh}(Y)$.

I.e., this is a cocomplete DG category, equipped with a functor $\mathrm{Coh}(Y) \to \mathrm{IndCoh}(Y)$, which is universal in the following sense: for a cocomplete DG category $\mathbf{C}$, a functor

$$\mathrm{Coh}(Y) \to \mathbf{C}$$

uniquely extends to a *continuous* functor $\mathrm{IndCoh}(Y) \to \mathbf{C}$.

One shows that the functor $\mathrm{Coh}(Y) \to \mathrm{IndCoh}(Y)$ is fully faithful and that its essential image compactly generates $\mathrm{IndCoh}(Y)$.

By the universal property of $\mathrm{IndCoh}(Y)$, the tautological embedding $\mathrm{Coh}(Y) \hookrightarrow \mathrm{QCoh}(Y)$ canonically extends to a continuous functor

(2.5) $$\Psi_Y : \mathrm{IndCoh}(Y) \to \mathrm{QCoh}(Y).$$

Note however, that the functor (2.5) is *no longer* fully faithful.

Another crucial piece of structure on $\mathrm{IndCoh}(Y)$ is that we have a canonical action of $\mathrm{QCoh}(Y)$, regarded as a monoidal category, on $\mathrm{IndCoh}(Y)$. It is obtained by ind-extending the action of $\mathrm{QCoh}(Y)^{\mathrm{perf}}$ on $\mathrm{Coh}(Y)$ be tensor products.

2.5.2. Suppose now that Y is eventually coconnective, which means that its structure sheaf has finitely many non-zero cohomologies. For example, any quasi-smooth DG scheme has this property.

In this case we have an inclusion $\mathrm{QCoh}(Y)^{\mathrm{perf}} \subset \mathrm{Coh}(Y)$ as full subcategories of $\mathrm{QCoh}(Y)$.

By the functoriality of the construction of forming the ind-completion, we have a naturally defined functor

(2.6) $$\mathrm{Ind}(\mathrm{QCoh}(Y)^{\mathrm{perf}}) \to \mathrm{IndCoh}(Y).$$

Note, however, that by the Thomason-Trobaugh theorem (see, e.g., [Ne]), the natural functor

$$\mathrm{Ind}(\mathrm{QCoh}(Y)^{\mathrm{perf}}) \to \mathrm{QCoh}(Y)$$

is an equivalence.

Hence, from (2.6) we obtain a functor

(2.7) $$\Xi_Y : \mathrm{QCoh}(Y) \to \mathrm{IndCoh}(Y).$$

It follows from the construction, that the functor (2.7) is fully faithful and provides a left adjoint of the functor (2.5).

Thus, we obtain that $\mathrm{QCoh}(Y)$ is a *co-localization* of $\mathrm{IndCoh}(Y)$. I.e., $\mathrm{IndCoh}(Y)$ is a "refinenement" of $\mathrm{QCoh}(Y)$.

Of course, if Y is a smooth classical scheme, there is no difference between $\mathrm{QCoh}(Y)^{\mathrm{perf}}$ and $\mathrm{Coh}(Y)$, and the functors (2.5) and (2.7) are mutually inverse equivalences.

2.6. Ind-coherent sheaves with prescribed support. The material of this subsection corresponds to [AG, Sect. 4.1-4.3].

2.6.1. Assume now that Y is quasi-smooth. In a similar way to the definition of $\mathrm{IndCoh}(Y)$, starting from $\mathrm{Coh}_{\mathcal{N}}(Y)$, we construct the category $\mathrm{IndCoh}_{\mathcal{N}}(Y)$.

As in the case of $\mathrm{IndCoh}(Y)$, we have a canonical monoidal action of $\mathrm{QCoh}(Y)$ on $\mathrm{IndCoh}_{\mathcal{N}}(Y)$.

We recover all of $\mathrm{IndCoh}(Y)$ by setting $\mathcal{N} = \mathrm{Sing}(Y)$.

2.6.2. Note that by Proposition 2.4.8, for $\mathcal{N}$ being the zero-section $\{0\} \subset Y$ we have

$$\mathrm{Coh}_{\{0\}}(Y) = \mathrm{QCoh}(Y)^{\mathrm{perf}},$$

so

$$\mathrm{IndCoh}_{\{0\}}(Y) \simeq \mathrm{Ind}(\mathrm{QCoh}(Y)^{\mathrm{perf}}) \simeq \mathrm{QCoh}(Y),$$

and we have tautologically defined fully faithful functors

$$\mathrm{QCoh}(Y) \simeq \mathrm{IndCoh}_{\{0\}}(Y) \hookrightarrow \mathrm{IndCoh}_{\mathcal{N}}(Y) \hookrightarrow \mathrm{IndCoh}(Y),$$

whose composition is the functor (2.7).

We shall denote the above functor $\mathrm{QCoh}(Y) \to \mathrm{IndCoh}_{\mathcal{N}}(Y)$ by $\Xi_{Y,\mathcal{N}}$, and its right adjoint (tautologically given by $\Psi_Y|_{\mathrm{IndCoh}_{\mathcal{N}}(Y)}$) by $\Psi_{Y,\mathcal{N}}$.

2.6.3. The category $\mathrm{IndCoh}_{\mathcal{N}}(Y)$ will be the principal actor on the spectral side of the geometric Langlands conjecture.

2.7. QCoh and IndCoh on stacks. This subsection makes a brief review of the material of [GL:QCoh, Sects. 1.1, 1.2, 2.1 and 5.1] and [Ga3, Sects. 11] relevant for this paper.

2.7.1. For later use we give the following definition. Let $\mathcal{Y}$ be a prestack, i.e., an arbitrary functor

$$(\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd} .$$

We define the category $\mathrm{QCoh}(\mathcal{Y})$ as

$$\varprojlim_{S \to \mathcal{Y}} \mathrm{QCoh}(S),$$

where the inverse limit is taken over the category of affine DG schemes over $\mathcal{Y}$.

I.e., informally, an object $\mathcal{F} \in \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ is an assignment for every map $S \to \mathcal{Y}$ of an object

$$\mathcal{F}_S \in \mathrm{QCoh}(S),$$

and for map $f : S_1 \to S_2$ over $\mathcal{Y}$ of an isomorphism

$$f^*(\mathcal{F}_{S_2}) \simeq \mathcal{F}_{S_1},$$

where these isomorphisms must be equipped with a data of homotopy-coherence for higher order compositions.

For a map of prestacks $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ we have a tautologically defined functor

$$f^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1).$$

If f is schematic quasi-compact and quasi-separated (i.e., its base change by a DG scheme yields a quasi-compact and quasi-separated DG scheme), the functor f^* admits a continuous right adjoint, denoted f_*.

2.7.2. A prestack $\mathcal{Y}$ is said to be *classical* if in the category $\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}}$ of affine DG schemes mapping to $\mathcal{Y}$ the full subcategory $\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$ is cofinal. I.e., if any map $S \to \mathcal{Y}$, where $S \in \mathrm{DGSch}^{\mathrm{aff}}$ can be factored as

$$S \to S' \to \mathcal{Y},$$

where S' is classical, and the category of such factorizations is *contractible*.

If $\mathcal{Y}$ is classical, then the category $\mathrm{QCoh}(\mathcal{Y})$ can be recovered just from the knowledge of $\mathrm{QCoh}(S)$ for classical schemes S over $\mathcal{Y}$. Precisely, the restriction functor

$$\mathrm{QCoh}(\mathcal{Y}) := \varprojlim_{S \in (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S) \to \varprojlim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S)$$

is an equivalence.

2.7.3. Let now $\mathcal{Y}$ be a (derived) algebraic stack (see [DrGa1, Sect. 1.1] for our conventions regarding algebraic stacks). In this case, one can rewrite the definition of $\mathrm{QCoh}(\mathcal{Y})$ as follows:

Instead of taking the limit over the category of all affine DG schemes over $\mathcal{Y}$, we can replace it by a full subcategory

$$(\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y},\mathrm{smooth}},$$

where we restrict objects to those $y : S \to \mathcal{Y}$ for which the map y is smooth, and morphisms to those maps $f : S_1 \to S_2$ over $\mathcal{Y}$ for which f is smooth.

2.7.4. Suppose that $\mathcal{Y}$ is a (derived) algebraic stack *locally almost of finite type* (i.e., it admits a smooth atlas consisting of DG schemes that are almost of finite type). In this case one can define $\mathrm{IndCoh}(\mathcal{Y})$ as

$$\varprojlim_{S \in ((\mathrm{DGSch}^{\mathrm{aff}})_{/\mathcal{Y},\mathrm{smooth}})^{\mathrm{op}}} \mathrm{IndCoh}(\mathcal{Y}).$$

Informally, an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{Y})$ is an assignment for every smooth map $S \to \mathcal{Y}$ of an object

$$\mathcal{F}_S \in \mathrm{IndCoh}(S),$$

and for every smooth map $f : S_1 \to S_2$ over $\mathcal{Y}$ of an isomorphism

$$f^*(\mathcal{F}_{S_2}) \simeq \mathcal{F}_{S_1},$$

where these isomorphisms must be equipped with a data of homotopy-coherence for higher order compositions.

2.7.5. If $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a schematic quasi-compact map of algebraic DG stacks (both assumed locally almost of finite type), we have a naturally defined continuous pushforward functor

$$f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(\mathcal{Y}_1) \to \mathrm{IndCoh}(\mathcal{Y}_2).$$

In addition, if f is an *arbitrary* map between algebraic DG stacks, there exists a well-defined functor

$$f^! : \mathrm{IndCoh}(\mathcal{Y}_2) \to \mathrm{IndCoh}(\mathcal{Y}_1).$$

The functor $f^!$ is the *right adjoint* of f^{IndCoh}_* if f is schematic and proper, and is the *left adjoint* of f^{IndCoh}_* if f is an open embedding.

The functors of pushforward and !-pull-back satisfy a base change property: for a Cartesian square of algebraic DG stacks almost of finite type

$$
\begin{array}{ccc}
\mathcal{Y}_1' & \xrightarrow{g_1} & \mathcal{Y}_1 \\
f' \downarrow & & \downarrow f \\
\mathcal{Y}_2' & \xrightarrow{g_2} & \mathcal{Y}_2,
\end{array}
$$

with the vertical maps being schematic and quasi-compact, there is a *canonically defined* isomorphism of functors

(2.8) $$ g_2^! \circ f_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ g_1^!. $$

Note, however, that unless f is proper or open, there is a priori no map in either direction in (2.8).

Finally, if f is locally of finite Tor-dimension, we also have a functor

$$ f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Y}_2) \to \mathrm{IndCoh}(\mathcal{Y}_1). $$

If f is schematic and quasi-compact then $f^{\mathrm{IndCoh},*}$ is the left-adjoint of f_*^{IndCoh}. If f is smooth (or more generally, Gorenstein), then the functors $f^!$ and $f^{\mathrm{IndCoh},*}$ differ by a twist by the relative dualizing line bundle.

2.8. Singular support on algebraic stacks.
The material of this subsection corresponds to [AG, Sect. 8].

2.8.1. Let $\mathcal{Y}$ be a (derived) algebraic stack. We shall say that $\mathcal{Y}$ is quasi-smooth if for any DG scheme and a smooth map $Y \to \mathcal{Y}$, the DG scheme Y is quasi-smooth.

Equivalently, $\mathcal{Y}$ is quasi-smooth if it admits a smooth atlas consisting of quasi-smooth DG schemes.

One can also express this in terms of the cotangent complex of $\mathcal{Y}$. Namely, $\mathcal{Y}$ is quasi-smooth if and only if it is locally almost of finite type, and for any k-point $y \in Y$, the derived cotangent space $T_y^*(\mathcal{Y})$ lives in degrees $[-1, 1]$. (The cohomology in degree 1 is responsible for the Lie algebra of the algebraic group of automorphisms of y.)

2.8.2. For a quasi-smooth derived algebraic stack $\mathcal{Y}$, one defines the classical algebraic stack

$$ \mathrm{Sing}(\mathcal{Y}) \to \mathcal{Y} $$

using descent:

For a smooth map $Y \to \mathcal{Y}$, where Y is a DG scheme, we have

$$ Y \underset{\mathcal{Y}}{\times} \mathrm{Sing}(\mathcal{Y}) \simeq \mathrm{Sing}(Y). $$

The fact that this is well-defined relies in the following lemma:

Lemma 2.8.3. *For a smooth map of quasi-smooth DG schemes $Y_1 \to Y_2$, the natural map*

$$ Y_1 \underset{Y_2}{\times} \mathrm{Sing}(Y_2) \to \mathrm{Sing}(Y_1) $$

is an isomorphism.

More invariantly, $\mathrm{Sing}(Y)$ consists of pairs (y, ξ), where $y \in \mathcal{Y}$, and $\xi \in H^{-1}(T_x^*(\mathcal{Y}))$.

2.8.4. Let $\mathcal{N} \subset \operatorname{Sing}(\mathcal{Y})$ be a conical Zariski-closed subset. We define the category $\operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ to be the full subcategory of $\operatorname{IndCoh}(\mathcal{Y})$ introduced in Sect. 2.7.4 defined by the following condition:

An object $\mathcal{F} \in \operatorname{IndCoh}(\mathcal{Y})$ belongs to $\operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ if for every $Y \in \operatorname{DGSch}^{\operatorname{aff}}$ equipped with a smooth map $Y \to \mathcal{Y}$ (equivalently, for some atlas of such Y's), the corresponding object $\mathcal{F}_Y \in \operatorname{IndCoh}(Y)$ belongs to

$$\operatorname{IndCoh}_{Y \underset{\mathcal{Y}}{\times} \mathcal{N}}(Y) \subset \operatorname{IndCoh}(Y).$$

2.8.5. By construction, we have a canonically defined action of the monoidal category $\operatorname{QCoh}(\mathcal{Y})$ on $\operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y})$.

By Sect. 2.6.2 we have an adjoint pair of continuous functors

$$\Xi_{\mathcal{Y},\mathcal{N}} : \operatorname{QCoh}(\mathcal{Y}) \rightleftarrows \operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y}) : \Psi_{\mathcal{Y},\mathcal{N}}$$

with $\Xi_{\mathcal{Y},\mathcal{N}}$ fully faithful.

3. Statement of the categorical geometric Langlands conjecture

For the rest of the paper, we fix X to be a smooth and complete curve over k.

3.1. The de Rham functor.

3.1.1. The following general construction will be useful in the sequel. Ley $\mathcal{Y}$ be an arbitrary prestack, see Sect. 2.7.1.

We define a new prestack $\mathcal{Y}_{\operatorname{dr}}$ by

$$\operatorname{Maps}(S, \mathcal{Y}_{\operatorname{dr}}) = \operatorname{Maps}(({}^{cl}S)_{red}, \mathcal{Y}), \quad S \in \operatorname{DGSch}^{\operatorname{aff}}.$$

In the above formula $({}^{cl}S)_{red}$ denotes the reduced classical scheme underlying S.

3.1.2. For what follows we define the DG category D-mod$(\mathcal{Y})$ of D-modules on $\mathcal{Y}$ by

$$\text{D-mod}(\mathcal{Y}) := \operatorname{QCoh}(\mathcal{Y}_{\operatorname{dr}}).$$

We refer the reader to [GR], where this point of view on the theory of D-modules is developed.

If $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a map of prestacks, we shall denote by $f^{\dagger}$ the resulting pull-back functor

$$f^{\dagger} : \text{D-mod}(\mathcal{Y}_2) \to \text{D-mod}(\mathcal{Y}_1).$$

I.e., $f^{\dagger} := (f_{\operatorname{dr}})^*$, where $f_{\operatorname{dr}} : (\mathcal{Y}_1)_{\operatorname{dr}} \to (\mathcal{Y}_2)_{\operatorname{dr}}$.

3.1.3. The following observation makes life somewhat easier:

Let $\mathcal{Y}$ be a prestack, which is *locally almost of finite type* (see [GL:Stacks, Sect. 1.3.9] for the definition[9]). In this case we have (see [GR, Proposition 1.3.3]):

Lemma 3.1.4. *The prestack $\mathcal{Y}_{\operatorname{dr}}$ is classical (see Sect. 2.7.2 for what this means) and locally of finite type.*

The upshot of this lemma is that in order to "know" the category D-mod$(\mathcal{Y}) := \operatorname{QCoh}(\mathcal{Y}_{\operatorname{dr}})$, it is sufficient to consider maps $({}^{cl}S)_{red} \to \mathcal{Y}$, where S is classical and of finite type. In particular, *we do not need derived algebraic geometry* when we study D-modules.

3.2. The stack of local systems. The contents of this subsection are a brief digest of [AG, Sect. 10]. We refer the reader to *loc.cit.* for the proofs of the statements that we quote.

---

[9]This is a techincal condition satisfied for the prestacks of interest to us.

3.2.1. Let G be an algebraic group. We let $\mathrm{pt}\,/G$ be the algebraic stack that classifies G-torsors. We define the prestacks $\mathrm{Bun}_G(X)$ and $\mathrm{LocSys}_G(X)$ by

$$\mathrm{Maps}(S, \mathrm{Bun}_G(X)) = \mathrm{Maps}(S \times X, \mathrm{pt}\,/G)$$

and

$$\mathrm{Maps}(S, \mathrm{LocSys}_G(X)) = \mathrm{Maps}(S \times X_{\mathrm{dr}}, \mathrm{pt}\,/G).$$

Note that we have a natural forgetful map $\mathrm{LocSys}_G(X) \to \mathrm{Bun}_G(X)$ corresponding to the tautological map $X \to X_{\mathrm{dr}}$.

One shows that $\mathrm{Bun}_G(X)$ is in fact a smooth classical algebraic stack, and that $\mathrm{LocSys}_G(X)$ is a derived algebraic stack.

As X is fixed, we will simply write Bun_G and LocSys_G, omitting X from the notation.

3.2.2. We claim that LocSys_G is in fact quasi-smooth. Indeed, the cotangent space at a point $\sigma \in \mathrm{LocSys}_G$ is canonically isomorphic to

$$\Gamma_{\mathrm{dr}}(X, \mathfrak{g}_\sigma^*)[1],$$

where $\mathfrak{g}_\sigma^*$ is the local system of vector spaces corresponding to σ and the co-adjoint representation of G.

In particular, the complex $\Gamma_{\mathrm{dr}}(X, \mathfrak{g}_\sigma^*)[1]$ has cohomologies in degrees $[-1, 1]$, as required.

3.2.3. The same computation provides a description of the stack $\mathrm{Sing}(\mathrm{LocSys}_G)$:

Corollary 3.2.4. *The (classical) stack* $\mathrm{Sing}(\mathrm{LocSys}_G)$ *is the moduli space of pairs* (σ, A) *where* $\sigma \in \mathrm{LocSys}_G$, *and* A *is a horizontal section of the local system* $\mathfrak{g}_\sigma^*$, *associated with the co-adjoint representation of* G.

3.2.5. The following property of LocSys_G is shared by any quasi-smooth algebraic stack which can be globally written as a complete intersection, see [AG, Corollary 9.2.7 and Sect. 10.6]:

Lemma 3.2.6. *For any conical Zariski-closed subset* $\mathcal{N} \subset \mathrm{Sing}(\mathrm{LocSys}_G)$, *the category* $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_G)$ *is compactly generated by its subcategory* $\mathrm{Coh}_{\mathcal{N}}(\mathrm{LocSys}_G)$.

3.3. **The spectral side of geometric Langlands.** From now on we will assume that G is a connected reductive group. We let $\check{G}$ denote the Langlands dual of G.

3.3.1. Consider the stack $\mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$. We will also denote it by $\mathrm{Arth}_{\check{G}}$. This is the stack of Arthur parameters.

Let $\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}$ be the conical Zariski-lcosed subset of $\mathrm{Arth}_{\check{G}}$ corresponding to those pairs (σ, A) (see Corollary 3.2.4) for which A is *nilpotent*, i.e., its value at any (equivalently, some) point of X lies in the cone of nilpotent elements of $\check{\mathfrak{g}}^*$.

3.3.2. According to Sect. 2.8.4, we have a well-defined DG category

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$$

This is the category that we propose as the spectral (i.e., Galois) side of the categorical geometric Langlands conjecture.

3.3.3. By Sect. 2.8.5, we have an adjoint pair of functors

(3.1) $\Xi_{\check{G}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) : \Psi_{\check{G}}$

with $\Xi_{\check{G}}$ fully faithful (we use the subscript "$\check{G}$" as a shorthand for "$\mathrm{LocSys}_{\check{G}}, \mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}$").

In other words, the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ is a modification of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ that has to do with the fact that the derived algebraic stack $\mathrm{LocSys}_{\check{G}}$ is not smooth, but only quasi-smooth.

In particular, the functor $\Xi_{\check{G}}$ becomes an equivalence once we restrict to the open substack of $\mathrm{LocSys}_{\check{G}}$ that consists of irreducible local systems (i.e., ones that do not admit a reduction to a proper parabolic). In fact, the equivalence takes place over a larger open substack; namely, one corresponding to those local systems that do not admit a unipotent subgroup of automorphisms.

3.3.4. Finally, note that if G (and hence $\check{G}$) is a torus, then $\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}$ is the zero-section of $\mathrm{Arth}_{\check{G}}$. So, for tori, the spectral side of geometric Langlands is the usual category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$.

3.4. The geometric side.

3.4.1. We consider the algebraic stack Bun_G and the corresponding category $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ as defined in Sect. 3.1.2.

The categorical geometric Langlands conjecture says:

Conjecture 3.4.2.
(a) *There exists a uniquely defined equivalence of categories*

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\mathbb{L}_G} \mathrm{D\text{-}mod}(\mathrm{Bun}_G),$$

satisfying Property $\mathrm{Wh}^{\mathrm{ext}}$ *stated in Sect. 9.4.6.*
(b) *The functor* $\mathbb{L}_G$ *satisfies Properties* $\mathrm{He}^{\mathrm{naive}}$, $\mathrm{Ei}^{\mathrm{enh}}$ *and* $\mathrm{Km}^{\mathrm{prel}}$, *stated in Sects. 4.4.5, 6.6.4, and 10.3.5, respectively.*

3.4.3. In the rest of the paper we will show that Conjecture 3.4.2 can be deduced, modulo a number of more tractable results that we call "quasi-theorems", from two more conjectures, namely Conjectures 8.2.9 and 10.2.8, the former pertaining exclusively to $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$, and the latter exclusively to $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$.

The "quasi-theorems" referred to above are very close to being theorems for $G = GL_2$ (and we hope will be soon turned into ones for general G). In addition, Conjectures 8.2.9 and 10.2.8 are also are theorems for $G = GL_n$. So, we obtain that Conjecture 3.4.2 is very close to be a theorem for GL_2, and is within reach for GL_n.

The case of an arbitrary G remains wide open.

3.5. **The tempered subcategory.** In this subsection we will assume the validity of Conjecture 3.4.2.

3.5.1. Recall the fully faithful embedding $\Xi_{\check{G}}$ of (3.1). We obtain that the DG category $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ contains a full subcategory that under the equivalence of Conjecture 3.4.2 corresponds to

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \overset{\Xi_{\check{G}}}{\hookrightarrow} \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$$

We denote this subcategory $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{temp}}$. We regard it as a geometric analog of the subspace of automorphic functions corresponding to tempered ones.

3.5.2. It is a natural question to ask whether one can define the subcategory

$$\text{D-mod}(\text{Bun}_G)_{\text{temp}} \subset \text{D-mod}(\text{Bun}_G)$$

intrinsically, i.e., without appealing to the spectral side of Langlands correspondence.

This is indeed possible, using the *derived Satake equivalence*, see [AG, Sect. 12.8] for a precise statement (see also Sect. 4.6.7 below).

3.5.3. The equivalence

$$\text{QCoh}(\text{LocSys}_{\check{G}}) \simeq \text{D-mod}(\text{Bun}_G)_{\text{temp}}$$

implies, in particular, that to every k-point $\sigma \in \text{LocSys}_G$ one can attach an object $\mathcal{M}_\sigma \in$ D-mod$(\text{Bun}_G)_{\text{temp}}$; moreover $\mathcal{M}_\sigma$ is acted on by the group of automorphisms of σ.

3.5.4. However, it is not clear (and perhaps not true) that the assignment

$$\sigma \rightsquigarrow \mathcal{M}_\sigma$$

can be extended to points of $\text{Nilp}_{\check{G}}^{\text{glob}}$. Indeed, there is no obvious way to assign to points of $\text{Nilp}_{\check{G}}^{\text{glob}}$ objects of $\text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}$.

I.e., at the moment we see no reason that there should be a way of assigning objects of D-mod(Bun_G) to Arthur parameters. Rather, what we have is that for an object $\mathcal{M} \in$ D-mod(Bun_G), there is a well-defined support, which is a closed subset of $\text{Nilp}_{\check{G}}^{\text{glob}}$.

4. THE HECKE ACTION

4.1. The Ran space.

4.1.1. We define the Ran space $\text{Ran}(X)$ of X to be the following prestack:

For $S \in \text{DGSch}^{\text{aff}}$, the ∞-groupoid $\text{Maps}(S, \text{Ran}(X))$ is the *set* (i.e., a discrete ∞-groupoid) of *non-empty finite subsets* of the set

$$\text{Maps}(S, X_{\text{dr}}) = \text{Maps}((^{cl}S)_{red}, X).$$

Note that by construction, the map $\text{Ran}(X) \to \text{Ran}(X)_{\text{dr}}$ is an isomorpism.

4.1.2. One can right down $\text{Ran}(X)$ explicitly as a colimit in PreStk:

$$\text{Ran}(X) \simeq \underset{I}{colim}\, (X^I)_{\text{dr}},$$

where the colimit is taken over the category $(\text{fSet}_{\text{surj}})^{\text{op}}$ opposite to that of non-empty finite sets and surjective maps [10]. (Here for a surjection of finite sets $I_2 \twoheadrightarrow I_1$, the map $X^{I_1} \to X^{I_2}$ is the corresponding diagonal embedding.)

---

[10]The definition of the Ran space as a colimit was in fact the original definition in [BD1]. The definition from Sect. 4.1.1 was suggested in [Bar].

4.1.3. We shall symbolically denote points of $\mathrm{Ran}(X)$ by $\underline{x}$. For each $\underline{x} \in \mathrm{Maps}(S, \mathrm{Ran}(X))$ we let $\Gamma_{\underline{x}}$ be the Zariski-closed *subset* of $S \times X$ equal to the union of the graphs of the maps $(^{cl}S)_{red} \to X$ that comprise $\underline{x}$.

In particular, we obtain an open subset

$$S \times X - \{\underline{x}\} := S \times X - \Gamma_{\underline{x}} \subset S \times X.$$

In addition, we have a well-defined formal scheme $\mathcal{D}_{\underline{x}}$ obtained as the formal completion of $S \times X$ along $\Gamma_{\underline{x}}$. This formal scheme should be thought of as the S-family of formal disc in X around the points that comprise $\underline{x}$.

4.1.4. A crucial piece of structure that exists on $\mathrm{Ran}(X)$ is that of commutative semi-group object in the category of prestacks. The corresponding operation on $\mathrm{Maps}(S, \mathrm{Ran}(X))$ is that of union of finite sets. We denote the resulting map

$$\mathrm{Ran}(X) \times \mathrm{Ran}(X) \to \mathrm{Ran}(X)$$

by $\cup$.

4.1.5. Another fundamental fact about the Ran space is its *contractibility*. We will use it its weaker form, namely *homological contractibility* (see [Ga2, Sect. 6] for the proof):

Proposition 4.1.6. *The functor*

$$\mathrm{Vect} = \mathrm{QCoh}(\mathrm{pt}) \xrightarrow{p^*} \mathrm{QCoh}(\mathrm{Ran}(X))$$

is fully faithful, where p denotes the projection $\mathrm{Ran}(X) \to \mathrm{pt}$.

(Note also that the fact that the map $\mathrm{Ran}(X) \to \mathrm{Ran}(X)_{\mathrm{dr}}$ is an isomorphism implies that the natural forgetful functor $\mathrm{D\text{-}mod}(\mathrm{Ran}(X)) \to \mathrm{QCoh}(\mathrm{Ran}(X))$ is an equivalence.)

4.2. **Representations spread over the Ran space.**

4.2.1. We shall now define the Ran version of the category of representations of $\check{G}$ (here $\check{G}$ may be any algebraic group). In fact we are going to start with an arbitrary prestack $\mathcal{Y}$ (in our case $\mathcal{Y} = \mathrm{pt}/\check{G}$) and attach to it a new prestack, denoted $\mathcal{Y}_{\mathrm{Ran}(X)}$, equipped with a map to $\mathrm{Ran}(X)$.

Namely, we let an S-point of $\mathcal{Y}_{\mathrm{Ran}(X)}$ to be the data of a pair $(\underline{x}, y)$, where $\underline{x}$ is an S-point of $\mathrm{Ran}(X)$, and y is a datum of a map

$$(\mathcal{D}_{\underline{x}})_{\mathrm{dr}} \underset{S_{\mathrm{dr}}}{\times} S \to \mathcal{Y}.$$

4.2.2. In order to decipher this definition, let us describe explicitly the fiber of $\mathcal{Y}_{\mathrm{Ran}(X)}$ over a given k-point $\underline{x}$ of $\mathrm{Ran}(X)$.

Let $\underline{x}$ correspond to the finite collection of distinct points $x_1, ..., x_n$ of X. We claim that the fiber product

$$\mathcal{Y}_{\mathrm{Ran}(X)} \underset{\mathrm{Ran}(X)}{\times} \mathrm{pt}$$

identifies with the product of copies of $\mathcal{Y}$, one for each index i.

This follows from the fact that $\mathcal{D}_{\underline{x}}$ is the disjoint union of the formal discs $\mathcal{D}_{x_i}$. Hence, the prestack $(\mathcal{D}_{\underline{x}})_{\mathrm{dr}}$ identifies with the disjoint union of copies of pt, one for each x_i.

4.2.3. We set
$$\mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}(X)} := \mathrm{QCoh}(\mathcal{Y}_{\mathrm{Ran}(X)}).$$

We claim that the DG category $\mathrm{QCoh}(\mathcal{Y})_{\mathrm{Ran}(X)}$ has a naturally defined structure of (non-unital) symmetric monoidal category.

Namely, consider the fiber product
$$\mathcal{Y}_{\mathrm{Ran}(X)} \underset{\mathrm{Ran}(X)}{\times} (\mathrm{Ran}(X) \times \mathrm{Ran}(X)),$$

where the map $\mathrm{Ran}(X) \times \mathrm{Ran}(X) \to \mathrm{Ran}(X)$ is $\cup$.

We have a diagram
$$\mathcal{Y}_{\mathrm{Ran}(X)} \underset{\mathrm{Ran}(X)}{\times} (\mathrm{Ran}(X) \times \mathrm{Ran}(X)) \xrightarrow{\ \mathsf{res}\ } \mathcal{Y}_{\mathrm{Ran}(X)} \times \mathcal{Y}_{\mathrm{Ran}(X)}$$
$$\mathsf{id} \times \cup \downarrow$$
$$\mathcal{Y}_{\mathrm{Ran}(X)},$$

where the map res corresponds to restricting maps to $\mathcal{Y}$ along
$$\mathcal{D}_{\underline{x}'} \to \mathcal{D}_{\underline{x}' \cup \underline{x}''} \leftarrow \mathcal{D}_{\underline{x}''}.$$

We define the functor
$$\mathrm{QCoh}(\mathcal{Y}_{\mathrm{Ran}(X)}) \otimes \mathrm{QCoh}(\mathcal{Y}_{\mathrm{Ran}(X)}) \to \mathrm{QCoh}(\mathcal{Y}_{\mathrm{Ran}(X)})$$

to be the composition
$$(\mathsf{id} \times \cup)_! \circ (\mathsf{res})^*,$$

where $(\mathsf{id} \times \cup)_!$ is the *left* adjoint [11] of the functor $(\mathsf{id} \times \cup)^!$.

4.2.4. Thus, we set
$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} := \mathrm{QCoh}(\mathrm{pt}\,/\check{G})_{\mathrm{Ran}(X)} := \mathrm{QCoh}((\mathrm{pt}\,/\check{G})_{\mathrm{Ran}(X)}).$$

We view it as a (non-unital) symmetric monoidal category.

4.3. **Relation to the stack of local systems.**

4.3.1. Note that by construction we have the following diagram of prestacks

(4.1)
$$\mathrm{LocSys}_{\check{G}} \times \mathrm{Ran}(X) \xrightarrow{\ \mathrm{ev}\ } (\mathrm{pt}\,/\check{G})_{\mathrm{Ran}(X)}$$
$$\mathsf{id} \times p \downarrow$$
$$\mathrm{LocSys}_{\check{G}},$$

where the map ev corresponds to restriction of a map to the target $\mathrm{pt}\,/\check{G}$ along $(\mathcal{D}_{\underline{x}})_{\mathrm{dr}} \to X_{\mathrm{dr}}$.

We have a pair of mutually adjoint functors

(4.2) $\quad (\mathsf{id} \times p)_! \circ \mathrm{ev}^* : \mathrm{QCoh}((\mathrm{pt}\,/\check{G})_{\mathrm{Ran}(X)}) \rightleftarrows \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) : \mathrm{ev}_* \circ (\mathsf{id} \times p)^*.$

The left adjoint functor (i.e., $(\mathsf{id} \times p)_! \circ \mathrm{ev}^*$) has a natural symmetric monoidal structure, where the symmetric monoidal structure on $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ is the usual tensor product.

---

[11]The fact that this left adjoint exists requires a proof; in our case this essentially follows from the fact that map $\cup$ is proper.

Remark 4.3.2. We note that the diagram (4.1) and the functors (4.2) makes sense more generally, when pt $/\check{G}$ is replaced by an arbitrary prestack $\mathcal{Y}$. In this case instead of $\mathrm{LocSys}_{\check{G}}$ we have the prestack $\mathbf{Maps}(X_{\mathrm{dr}}, \mathcal{Y})$, defined so that

$$\mathrm{Maps}(S, \mathbf{Maps}(X_{\mathrm{dr}}, \mathcal{Y})) = \mathrm{Maps}(S \times X_{\mathrm{dr}}, \mathcal{Y}).$$

4.3.3. We denote

$$\mathrm{Loc}_{\check{G},\mathrm{spec}} := (\mathrm{id} \times p)_! \circ \mathrm{ev}^* \text{ and } \mathrm{co\text{-}Loc}_{\check{G},\mathrm{spec}} := \mathrm{ev}_* \circ (\mathrm{id} \times p)^*.$$

We have the following result:

Proposition 4.3.4 (joint with J. Lurie, unpublished)**.** *The functor*

$$\mathrm{co\text{-}Loc}_{\check{G}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{QCoh}((\mathrm{pt}\,/\check{G})_{\mathrm{Ran}(X)}) = \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$$

is fully faithful.

Thus, Proposition 4.3.4 realizes a "local-to-global" principle for $\mathrm{LocSys}_{\check{G}}$, namely, it embeds the "global" category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ into a "local" one, namely, $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$.

Remark 4.3.5. The assertion of Proposition 4.3.4 is valid more generally, when the stack pt $/\check{G}$ is replaced by an arbitrary quasi-compact derived algebraic stack $\mathcal{Y}$ locally almost of finite type with an affine diagonal.

4.4. **Hecke action.**

4.4.1. We define the Ran version of the Hecke stack $\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$ as follows: its S-points are quadruples $(\underline{x}, \mathcal{P}_G^1, \mathcal{P}_G^2, \beta)$, where $\underline{x}$ is an S-point of $\mathrm{Ran}(X)$, $\mathcal{P}_G^1$ and $\mathcal{P}_G^2$ are two S-points of Bun_G, and β is the isomorphism of G-bundles

$$\mathcal{P}_G^1|_{S \times X - \underline{x}} \simeq \mathcal{P}_G^2|_{S \times X - \underline{x}}.$$

We let $\overset{\leftarrow}{h}$ and $\vec{h}$ denote the two forgetful maps $\mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \to \mathrm{Bun}_G$.

4.4.2. We claim that the category $\mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_{\mathrm{Ran}(X)})$ has a naturally defined (non-unital) monoidal structure, and that the resulting monoidal category acts on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$.

These two pieces of structure are constructed by pull-push as in Sect. 4.2.3 using the diagrams

$$\mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \underset{\vec{h},\mathrm{Bun}_G,\overset{\leftarrow}{h}}{\times} \mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \longrightarrow \mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \times \mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$$

$$\downarrow$$

$$\mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \underset{\mathrm{Ran}(X)}{\times} (\mathrm{Ran}(X) \times \mathrm{Ran}(X))$$

$$\mathrm{id} \times \cup \downarrow$$

$$\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$$

and

$$\mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \xrightarrow{\mathrm{id} \times \vec{h}} \mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \times \mathrm{Bun}_G$$

$$\overset{\leftarrow}{h} \downarrow$$

$$\mathrm{Bun}_G$$

respectively.

4.4.3. We have the following input from the geometric Satake equivalence:

Proposition 4.4.4. *There exists a canonically defined monoidal functor*

$$(4.3) \qquad \operatorname{Sat}(G)^{\mathrm{naive}}_{\operatorname{Ran}(X)} : \operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)} \to \operatorname{D-mod}(\operatorname{Hecke}(G)_{\operatorname{Ran}(X)}).$$

The functor $\operatorname{Sat}(G)^{\mathrm{naive}}_{\operatorname{Ran}(X)}$ follows from the *naive* (or "usual") geometric Satake equivalence, and is essentialy constructed in [MV].

4.4.5. *Compatibility with the Hecke action.* We are now able to formulate Property He$^{\mathrm{naive}}$ ("He" stands for "Hecke") of the geometric Langlands functor $\mathbb{L}_G$ in Conjecture 3.4.2:

Property He$^{\mathrm{naive}}$: *We shall say that the functor $\mathbb{L}_G$ satisfies* Property He$^{\mathrm{naive}}$ *if it intertwines the monoidal actions of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)}$ on the categories $\operatorname{IndCoh}_{\operatorname{Nilp}^{\mathrm{glob}}_{\check{G}}}(\operatorname{LocSys}_{\check{G}})$ and $\operatorname{D-mod}(\operatorname{Bun}_G)$, where:*

- *The action of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)}$ on $\operatorname{IndCoh}_{\operatorname{Nilp}^{\mathrm{glob}}_{\check{G}}}(\operatorname{LocSys}_{\check{G}})$ is obtained via the the monoidal functor*

  $$\operatorname{Loc}_{\check{G},\mathrm{spec}} : \operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)} = \operatorname{QCoh}((\operatorname{pt}/\check{G})_{\operatorname{Ran}(X)}) \to \operatorname{QCoh}(\operatorname{LocSys}_{\check{G}})$$

  and the action of $\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}})$ on $\operatorname{IndCoh}_{\operatorname{Nilp}^{\mathrm{glob}}_{\check{G}}}(\operatorname{LocSys}_{\check{G}})$ (see Sect. 2.8.4);
- *The action of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)}$ on $\operatorname{D-mod}(\operatorname{Bun}_G)$ is obtained via the monoidal functor $\operatorname{Sat}(G)^{\mathrm{naive}}_{\operatorname{Ran}(X)}$ and the action of $\operatorname{D-mod}(\operatorname{Hecke}(G)_{\operatorname{Ran}(X)})$ on $\operatorname{D-mod}(\operatorname{Bun}_G)$ (see Sect. 4.4.2).*

4.5. The vanishing theorem.

4.5.1. Consider again the action of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)}$ on $\operatorname{D-mod}(\operatorname{Bun}_G)$, described above. We claim:

Theorem 4.5.2. *The action of the monoidal ideal*

$$\ker\left(\operatorname{Loc}_{\check{G},\mathrm{spec}} : \operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)} \to \operatorname{QCoh}(\operatorname{LocSys}_{\check{G}})\right)$$

on $\operatorname{D-mod}(\operatorname{Bun}_G)$ is zero.

The proof of this theorem will be sketched in Sect. 11.1. It uses the same basic ingredients as the proof of Conjecture 3.4.2, but is much simpler. A more detailed exposition can be found in [GL:GenVan].

Remark 4.5.3. We note that Theorem 4.5.2 is a generalization of a vanishing theorem proved in [Ga1] that concerned the case of $G = GL_n$ and a particular object of the category $\operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)}$ lying in the kernel of $\operatorname{Loc}_{\check{G},\mathrm{spec}}$.

4.5.4. Combining Proposition 4.3.4 and Theorem 4.5.2, we obtain:

Corollary 4.5.5. *The monoidal action of $\operatorname{Rep}(\check{G})_{\operatorname{Ran}(X)}$ on $\operatorname{D-mod}(\operatorname{Bun}_G)$ uniquely factors through a monoidal action of $\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}})$ on $\operatorname{D-mod}(\operatorname{Bun}_G)$.*

4.6. Derived Satake.
The material of this subsection is not crucial for the understanding of the outline of the proof of Conjecture 3.4.2 presented in the rest of the paper.
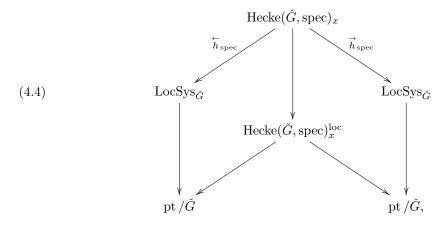
4.6.1. Let us fix a k-point $x \in X$. We let $\mathrm{Hecke}(\check{G}, \mathrm{spec})_x$ denote the DG algebraic stack whose S-points are triples

$$((\mathcal{P}^1_{\check{G}}, \nabla^1), (\mathcal{P}^2_{\check{G}}, \nabla^2), \beta),$$

where $(\mathcal{P}^i_{\check{G}}, \nabla^i)$ are objects of $\mathrm{Maps}(S, \mathrm{LocSys}_{\check{G}})$ and β is an isomorphism of the resulting two maps $S \times (X - x)_{\mathrm{dr}} \to \mathrm{pt}\,/\check{G}$ obtained from $(\mathcal{P}^i_{\check{G}}, \nabla^i)$ by restriction along

$$S \times (X - x)_{\mathrm{dr}} \hookrightarrow S \times X_{\mathrm{dr}}.$$

The two projections $\overleftarrow{h}_{\mathrm{spec}}, \overrightarrow{h}_{\mathrm{spec}} : \mathrm{Hecke}(\check{G}, \mathrm{spec})_x \to \mathrm{LocSys}_{\check{G}}$ define on $\mathrm{Hecke}(\check{G}, \mathrm{spec})_x$ a structure of groupoid acting on $\mathrm{LocSys}_{\check{G}}$. In fact, we have a canonicaly defined commutative diagram, in which both sides are Cartesian

(4.4)

$$
\begin{array}{ccc}
 & \mathrm{Hecke}(\check{G}, \mathrm{spec})_x & \\
 \overleftarrow{h}_{\mathrm{spec}} \swarrow & \downarrow & \searrow \overrightarrow{h}_{\mathrm{spec}} \\
 \mathrm{LocSys}_{\check{G}} & & \mathrm{LocSys}_{\check{G}} \\
 & \mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}} & \\
 \downarrow & \swarrow \quad \searrow & \downarrow \\
 \mathrm{pt}\,/\check{G} & & \mathrm{pt}\,/\check{G},
\end{array}
$$

where

$$\mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}} := (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt})/\check{G},$$

see [AG, Sect. 12.7].

4.6.2. The structure of groupoid on $\mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}}$ defines on $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}})$ a structure of monoidal category, where we use the $(\mathrm{IndCoh}, *)$-pushforward and !-pull-back as our pull-push functors.

Moreover, the diagram (4.4) defines an action of $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}})$ on the category $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ that preserves the subcategory

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}).$$

There is a naturally defined monoidal functor

$$\mathrm{Rep}(\check{G}) = \mathrm{QCoh}(\mathrm{pt}\,/\check{G}) \to \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}})$$

corresponding to the diagonal map

$$\mathrm{pt}\,/\check{G} \to (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt})/\check{G} =: \mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}}.$$

4.6.3. Let $\mathrm{Hecke}(G)_x$ be the fiber of $\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$ over the point $\{x\} \in \mathrm{Ran}(X)$. The restrictions of the projections of $\overleftarrow{h}$ and $\overrightarrow{h}$ to $\mathrm{Hecke}(G)_x$ define on it a structure of groupoid acting on Bun_G. Hence, the category $\mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x)$ acquires a monoidal structure. Direct image (i.e., the functor *left adjoint* to restriction) defines a monoidal functor

$$\mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x) \to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}).$$

Similarly, we have a naturally defined monoidal functor

$$\mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)},$$

left adjoint to the restriction functor. Part of the construction of the functor $\mathrm{Sat}(G)^{\mathrm{naive}}_{\mathrm{Ran}(X)}$ is that we have a naturally defined monoidal functor

$$\mathrm{Sat}(G)^{\mathrm{naive}}_x : \mathrm{Rep}(\check{G}) \to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x)$$

that makes the diagram

$$
\begin{array}{ccc}
\mathrm{Rep}(\check{G}) & \xrightarrow{\mathrm{Sat}(G)^{\mathrm{naive}}_x} & \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x) \\
\downarrow & & \downarrow \\
\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} & \xrightarrow{\mathrm{Sat}(G)^{\mathrm{naive}}_{\mathrm{Ran}(X)}} & \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_{\mathrm{Ran}(X)})
\end{array}
$$

commute.

We now claim:

Proposition 4.6.4. *There exists a canonically defined monoidal functor*

$$\mathrm{Sat}(G)_x : \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x)$$

that makes the diagram

$$
\begin{array}{ccc}
\mathrm{Rep}(\check{G}) & \xrightarrow{\mathrm{Sat}(G)^{\mathrm{naive}}_x} & \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x) \\
\downarrow & & \downarrow{\scriptstyle\mathrm{id}} \\
\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) & \xrightarrow{\mathrm{Sat}(G)_x} & \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x)
\end{array}
$$

commute.

Proposition 4.6.4 follows from the local *full* (or "derived") geometric Satake equivalence, see [AG, Theorem 12.5.3] (which in turn follows from [BF, Theorem 5]).

Remark 4.6.5. Note that the stack

$$\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x \simeq (\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt})/\check{G}$$

is quasi-smooth, and the corresponding classical stack $\mathrm{Sing}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$ identifies canonically with the classical stack $\check{\mathfrak{g}}^*/\check{G}$. Let $\mathrm{Nilp}^{\mathrm{loc}}_{\check{G}} \subset \mathrm{Sing}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$ be the nilpotent locus, and consider the corresponding category $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{loc}}_{\check{G}}}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$.

One can show that the functor

$$\mathrm{Sat}(G)_x : \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x)$$

canonically factors as a composition of monoidal functors

$$(4.5) \quad \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \to \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{loc}}_{\check{G}}}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \to$$
$$\to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)^{\mathrm{loc}}_x) \to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_x),$$

where

- $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \to \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{loc}}_{\check{G}}}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$ is the co-localization functor, left adjoint to the tautological embedding

$$\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{loc}}_{\check{G}}}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \hookrightarrow \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x).$$

- $\mathrm{Hecke}(G)^{\mathrm{loc}}_x$ is the local version of the Hecke stack, i.e., $G(\widehat{\mathcal{O}}_x)\backslash G(\widehat{\mathcal{K}}_x)/G(\widehat{\mathcal{O}}_x)$, where $\widehat{\mathcal{O}}_x$ and $\widehat{\mathcal{K}}_x$ are the completed local ring and field at the point $x \in X$, respectively.

A salient feature of this situation is that the middle functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{loc}}_{\check{G}}}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)^{\mathrm{loc}}_x)$$

in (4.5) is an *equivalence* (unlike the version with $\mathrm{Sat}(G)^{\mathrm{naive}}_x$).

4.6.6. We can now formulate the following variant of Property $\mathrm{He}^{\mathrm{naive}}$ of the functor $\mathbb{L}_G$:

Property He^x: *We shall say that the functor $\mathbb{L}_G$ satisfies* Property He^x *if it intertwines the monoidal actions of* $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$ *on the categories* $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$ *and* $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$*, where:*

- *The action of* $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$ *on* $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$ *is one from Sect. 4.6.2;*
- *The action of* $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$ *on* $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ *is obtained via the monoidal functor* $\mathrm{Sat}(G)_x$ *and the action of* $\mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_{\mathrm{Ran}(X)})$ *on* $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ *(see Sect. 4.4.2).*

It will follow from the constructions carried out in the rest of the paper that, in the same circumstances under which we can prove Conjecture 3.4.2, the resulting functor $\mathbb{L}_G$ will also satisfy Property He^x for any $x \in X$.

4.6.7. The intrinsic characterization of the subcategory

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{temp}} \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

mentioned in Sect. 3.5.2 is formulated in terms of the above action of $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$ on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$:

An object $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ belongs to $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{temp}}$ if and only if the functor

$$\mathcal{F} \mapsto \mathcal{F} \star \mathcal{M}, \quad \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

factors through the quotient

$$\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x) \overset{\Psi_{\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x}}{\twoheadrightarrow} \mathrm{QCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$$

(for any chosen point x). In the above formula $-\star-$ denotes the monoidal action of $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x)$ on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$, and we remind that $\Psi_{\mat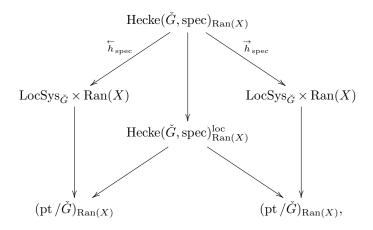hrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x}$ denotes the functor introduced in Sect. 2.8.5, for the stack $\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_x$.

4.7. **The Ran version of derived Satake.** The material of this subsection will not be used elsewhere in the paper. The reason we include it is to mention another important piece of structure present in the geometric Langlands picture, and one which is crucial for the proofs.[12]

4.7.1. Along with the stack $\mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}}$, one can consider its Ran version, denoted $\mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}}$ that fits into the Cartesian diagram

$$
\begin{array}{ccc}
 & \mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)} & \\
 \overset{\leftarrow}{h}_{\mathrm{spec}} \swarrow & \downarrow & \searrow \vec{h}_{\mathrm{spec}} \\
 \mathrm{LocSys}_{\check{G}} \times \mathrm{Ran}(X) & & \mathrm{LocSys}_{\check{G}} \times \mathrm{Ran}(X) \\
 & \mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}} & \\
 \downarrow \swarrow & & \searrow \downarrow \\
 (\mathrm{pt}\,/\check{G})_{\mathrm{Ran}(X)} & & (\mathrm{pt}\,/\check{G})_{\mathrm{Ran}(X)},
\end{array}
$$

The reason we do not formally give the definition of $\mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}}$ is that it involves the notion of $\check{G}$-local system on the *parameterized formal punctured disc* (as opposed to the parameterized formal non-punctured disc $\mathcal{D}_{\underline{x}}$), the discussion of which would be too lengthy for the intended scope of this paper.

Let us, nonetheless, indicate the formal structure of this piece of the picture:

4.7.2. Although the prestack $\mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}}$ is not a DG algebraic stack, the category $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}})$ is well-defined, carries a monoidal structure, and as such acts on $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ preserving $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$.

We have naturally defined monoidal functors

$$(4.6) \qquad \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \to \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}}) \leftarrow \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})_x^{\mathrm{loc}}),$$

and a monoidal functor

$$\mathrm{Sat}(G)_{\mathrm{Ran}(X)} : \mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}}) \to \mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_{\mathrm{Ran}(X)})$$

that restricts to the functors $\mathrm{Sat}(G)_x$ and $\mathrm{Sat}(G)_{\mathrm{Ran}(X)}^{\mathrm{naive}}$, respectively.

Remark 4.7.3. As in Remark 4.6.5, the functor $\mathrm{Sat}(G)_{\mathrm{Ran}(X)}$ factors through an equivalence from a co-localization $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{loc}}}(\mathrm{Hecke}(\check{G}, \mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}})$ to the appropriately defined category $\mathrm{D\text{-}mod}(\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}^{\mathrm{loc}})$.

---

[12]As of now, the material in this subsection does not have a reference in the existing literature.

4.7.4. The full Hecke comptatibility property reads:

Property He: *We shall say that the functor* $\mathbb{L}_G$ *satisfies* Property He *if it intertwines the monoidal actions of* $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_{\mathrm{Ran}(X)})$ *on the categories* $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$ *and* D-mod(Bun_G), *where:*

- *The action of* $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_{\mathrm{Ran}(X)})$ *on* $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$ *is as in Sect. 4.7.2*
- *The action of* $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G}, \mathrm{spec})^{\mathrm{loc}}_{\mathrm{Ran}(X)})$ *on* D-mod(Bun_G) *is obtained via the functor* $\mathrm{Sat}(G)_{\mathrm{Ran}(X)}$ *and the action of* D-mod($\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$) *on* D-mod($\mathrm{Bun}_G$) *(see Sect. 4.4.2).*

Tautologically, Property He implies both Properties He$^{\mathrm{naive}}$ and He^x.

As with Property He^x, it will follow from the constructions carried out in the rest of the paper that, in the same circumstances under which we can prove Conjecture 3.4.2, the resulting functor $\mathbb{L}_G$ will satisfy Property He.

5. THE WHITTAKER MODEL

5.1. **The space of generic reductions to the Borel.** In this subsection we are going to introduce a space (=prestack) $\mathrm{Bun}_G^{B\text{-gen}}$ that will figure prominently in this paper. This is the space that classifies pairs consisting of a G-bundle and its reduction to the Borel subgroup defined *generically* on X. The approach to $\mathrm{Bun}_G^{B\text{-gen}}$ described below was developed by J. Barlev in [Bar].[13]

In this subsection, as well as in Sects. 5.2-5.7, we will be exclusively dealing with D-modules, so derived algebraic geometry will play no role (see Sect. 3.1.3).

5.1.1. First, we consider the prestack that attaches to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the groupoid of triples

$$(\mathcal{P}_G, U, \alpha),$$

where

- $\mathcal{P}_G$ is a G-bundle on $S \times X$;
- U is a Zariski-open subset of $S \times X$, such that for each k-point of S, the corresponding open subset

$$\mathrm{pt} \underset{S}{\times} U \subset \mathrm{pt} \underset{S}{\times} (S \times X) \simeq X$$

  is non-empty (equivalently, dense in X);
- α is a datum of a reduction of $\mathcal{P}_G|_U$ to the Borel subgroup B.

In what follows we shall denote by $\mathcal{P}_{B,U}$ the B-bundle on U corresponding to α. We shall denote by $\mathcal{P}_{T,U}$ the induced T-bundle.

We define $\mathrm{Bun}_G^{B\text{-gen}}$ to be the prestack that attaches to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the quotient of the above groupoid of triples $(\mathcal{P}_G, U, \alpha)$ by the equivalence relation that identifies $(\mathcal{P}_G^1, U^1, \alpha^1)$ and $(\mathcal{P}_G^2, U^2, \alpha^2)$ if

$$\mathcal{P}_G^1 \simeq \mathcal{P}_G^2$$

and for this identification, the data of α^1 and α^2 coincide over $U^1 \cap U^2$.

---

[13]A much more cumbersome treatment, but one which only uses algebraic stacks or ind-algebraic stacks can be found in [GL:ExtWhit, Sects. 5 and 6].

5.1.2. We have a natural forgetful map $\mathsf{p}_B^{\mathrm{enh}} : \mathrm{Bun}_G^{B\text{-gen}} \to \mathrm{Bun}_G$. However, the fibers of this map are neither indschemes nor algebraic stacks.

Nonetheless, we have the following assertion established in [Bar, Proposition 3.3.2]:

Proposition 5.1.3. *There exists an algebraic stack* $\mathcal{Y}_0$, *equipped with a proper schematic map to* Bun_G, *and a proper schematic equivalence relation*

$$\mathcal{Y}_1 \rightrightarrows \mathcal{Y}_0$$

such that $\mathrm{Bun}_G^{B\text{-gen}}$ *identifies with the quotient of* $\mathcal{Y}_0$ *by* $\mathcal{Y}_1$, *up to sheafification in the Zariski topology.*

Remark 5.1.4. The pair $\mathcal{Y}_1 \rightrightarrows \mathcal{Y}_0$ is in fact very explicit. Namely, $\mathcal{Y}_0$ is the algebraic stack $\overline{\mathrm{Bun}}_B$ (the Drinfeld compactification), and $\mathcal{Y}_1$ is defined as

$$\overline{\mathrm{Bun}}_B \underset{\mathrm{Bun}_G^{B\text{-gen}}}{\times} \overline{\mathrm{Bun}}_B.$$

One shows that $\mathcal{Y}_1$ is indeed an algebraic stack, and the two projections from $\mathcal{Y}_1$ to $\mathcal{Y}_0$ are schematic and proper.

5.1.5. Consider now the usual stack Bun_B that classifies B-bundles on X. We have a tautological map

$$\imath_B : \mathrm{Bun}_B \to \mathrm{Bun}_G^{B\text{-gen}}.$$

The valuative criterion of properness implies that the map $\imath_B$ induces an isomorphism of groupoids of field-valued points. In particular, the groupoid of k-points of $\mathrm{Bun}_G^{B\text{-gen}}$ identifies canonically with the double quotient

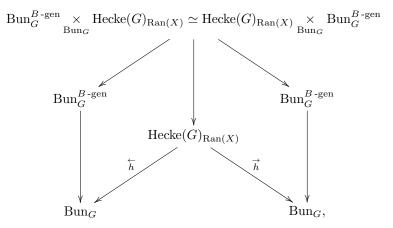$$B(K)\backslash G(\mathbb{A})/G(\mathbb{O}),$$

where K is the field of rational functions on X, $\mathbb{A}$ denotes the ring of adèles, and $\mathbb{O}$ is the ring of integral adèles.

However, the map $\imath_B$ itself is, of course, not an isomorphism. For example, one can show that connected components of $\mathrm{Bun}_G^{B\text{-gen}}$ are in bijection with those of Bun_G, i.e., $\pi_1(G)$, whereas connected components of Bun_B are in bijection with the coweight lattice $\check{\Lambda}$ of G.

One can view $\mathrm{Bun}_G^{B\text{-gen}}$ as equipped with a stratification, while the map $\imath_B$ is the map from the disjoint of the strata.

5.1.6. Recall the Hecke stack $\mathrm{Hecke}(G)_{\mathrm{Ran}(X)}$. We claim it naturally lifts to $\mathrm{Bun}_G^{B\text{-gen}}$ in the sense that we have a commutative diagram

$$\mathrm{Bun}_G^{B\text{-gen}} \underset{\mathrm{Bun}_G}{\times} \mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \simeq \mathrm{Hecke}(G)_{\mathrm{Ran}(X)} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_G^{B\text{-gen}}$$

In particular, we obtain a natural action of the monoidal category D-mod$(\mathrm{Hecke}(G)_{\mathrm{Ran}(X)})$ on D-mod$(\mathrm{Bun}_G^{B\text{-gen}})$.

Using the functor $\mathrm{Sat}(G)_{\mathrm{Ran}(X)}^{\mathrm{naive}}$ we thus obtain an action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on D-mod$(\mathrm{Bun}_G^{B\text{-gen}})$.

5.2. Replacing B by its unipotent radical.
In what follows we shall need a few variants of the space $\mathrm{Bun}_G^{B\text{-gen}}$.

5.2.1. First, we have the prestack $\mathrm{Bun}_G^{N\text{-gen}}$, defined in the same way as $\mathrm{Bun}_G^{B\text{-gen}}$, with B replaced by N. By construction, we have a natural projection
$$\mathrm{Bun}_G^{N\text{-gen}} \to \mathrm{Bun}_G^{B\text{-gen}}.$$

An analog of Proposition 5.1.3 holds with no modifications. The groupoid on k-points of $\mathrm{Bun}_G^{N\text{-gen}}$ is canonically isomorphic to the double quotient
$$N(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

5.2.2. For any target scheme (or even prestack) Y, we define the prestack $\mathbf{Maps}(X,Y)^{\mathrm{gen}}$ in a way analogous to the definition of $\mathrm{Bun}_G^{B\text{-gen}}$.

Namely, the groupoid of S-points of Y is the quotient of the set of pairs
$$(U \subset S \times X; y : U \to Y)$$
by the equivalence relation that identifies (U^1, y^1) with (U^2, y^2) if $y_1|_{U_1 \cap U_2} = y_2|_{U_1 \cap U_2}$.

5.2.3. Consider in particular the group-object in PreStk given by $\mathbf{Maps}(X,T)^{\mathrm{gen}}$.

We have a natural action of $\mathbf{Maps}(X,T)^{\mathrm{gen}}$ on $\mathrm{Bun}_G^{N\text{-gen}}$, and the quotient is easily seen to identify with $\mathrm{Bun}_G^{B\text{-gen}}$.

5.2.4. We can rewrite the definition of $\mathrm{Bun}_G^{N\text{-gen}}$ as follows. We consider the prestack that assigns to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the groupoid of quadruples
$$(\mathcal{P}_G, U, \alpha, \gamma),$$
where $(\mathcal{P}_G, U, \alpha)$ are as in the definition of $\mathrm{Bun}_G^{B\text{-gen}}$, and γ is a datum of a trivialization of the T-bundle $\mathcal{P}_{T,U}$, see Sect. 5.1.1 for the notation.

The prestack $\mathrm{Bun}_G^{N\text{-gen}}$ is obtained from the above prestack of quadruples by quotienting it by the equivalence relation that identifies $(\mathcal{P}_G^1, U^1, \alpha^1, \gamma^1)$ and $(\mathcal{P}_G^2, U^2, \alpha^2, \gamma^2)$ if the corresponding points $(\mathcal{P}_G^1, U^1, \alpha^1)$ and $(\mathcal{P}_G^2, U^2, \alpha^2)$ are identified, and the resulting isomorphism between $\mathcal{P}_{T,U}^1$ and $\mathcal{P}_{T,U}^2$ over $U_1 \cap U_2$ maps γ^1 to γ^2.

5.2.5. From now on in the paper we are going to fix a square root $\omega_X^{\frac{1}{2}}$ of the canonical line bundle on X. In particular, we obtain a well-defined T-bundle
$$\check{\rho}(\omega_X) := 2\check{\rho}(\omega_X^{\frac{1}{2}}).$$

We define the prestack $\mathcal{Q}_G := \mathrm{Bun}_G^{N^\omega\text{-gen}}$ to be a twist of $\mathrm{Bun}_G^{N\text{-gen}}$. Namely, in the data $(\mathcal{P}_G, U, \alpha, \gamma)$ we change the meaning of γ:

Instead of being a trivialization of $\mathcal{P}_{T,U}$ we now let γ be the datum of an isomorphism with $\check{\rho}(\omega_X)|_U$.

A choice of a generic trivialization of $\omega_X^{\frac{1}{2}}$ identifies $\mathcal{Q}_G$ with $\mathrm{Bun}_G^{N\text{-gen}}$, and in particular, the groupoid of its k-points with the double quotient

$$N(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

5.2.6. Yet another space that we will need is the quotient of $\mathrm{Bun}_G^{N\text{-gen}}$ (or, rather, $\mathrm{Bun}_G^{N^\omega\text{-gen}}$) by the action of

$$\mathbf{Maps}(X, Z_G^0)^{\mathrm{gen}},$$

where Z_G^0 is the connected component of the center of G.

We denote the resulting prestack by $\mathcal{Q}_{G,G}$ (its variant $\mathcal{Q}_{G,P}$, where $P \subset G$ is a parabolic and M is the Levi quotient of P, will be introduced in Sect. 7).

A choice of a generic trivialization of $\omega_X^{\frac{1}{2}}$ identifies the groupoid on k-points of $\mathcal{Q}_{G,G}$ with the double quotient

$$Z_G^0(K) \cdot N(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

5.2.7. The prestack $\mathcal{Q}_{G,G}$ can be explicitly described as follows.

We consider the prestack that assigns to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the groupoid of quadruples

$$(\mathcal{P}_G, U, \alpha, \gamma),$$

where $(\mathcal{P}_G, U, \alpha)$ as above, and γ is a datum of isomorphism over U between the bundles with respect to T/Z_G^0, one being induced from $\mathcal{P}_{T,U}$, and the other from $\check{\rho}(\omega_X)|_U$.

(Note that when G has a connected center, the data of γ amounts to an isomorphism of line bundles $\alpha_i(\mathcal{P}_{T,U}) \simeq \omega_X|_U$ for each simple root α_i of G. In particular, it is independent of the choice of $\omega_X^{\frac{1}{2}}$.)

The prestack $\mathcal{Q}_{G,G}$ is obtained from the above prestack of quadruples by quotienting by the equivalence relation, defined in the same way as in the case of $\mathrm{Bun}_G^{N^\omega\text{-gen}}$.

5.2.8. We claim:

Proposition 5.2.9. *The pull-back functors*

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G^{B\text{-gen}}) \to \mathrm{D\text{-}mod}(\mathcal{Q}_{G,G}) \to \mathrm{D\text{-}mod}(\mathcal{Q}_G)$$

are fully faithful.

Proof. Follows from the homological contractibility of the prestacks $\mathbf{Maps}(X,T)^{\mathrm{gen}}$ and $\mathbf{Maps}(X, Z_G^0)^{\mathrm{gen}}$, see [Ga2]. $\qquad\square$

5.2.10. As in Sect. 5.1.6, we have a canonical action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on both $\mathrm{D\text{-}mod}(\mathcal{Q}_G)$ and $\mathrm{D\text{-}mod}(\mathcal{Q}_{G,G})$.

5.3. **The groupoid: function-theoretic analogy.** In order to introduce the Whittaker category, as well as several other categories of primary interest for this paper, we will need to define a certain groupoid, denoted $\mathbf{N}$ that acts on $\mathrm{Bun}_G^{B\text{-gen}}$ and related geometric objects.

5.3.1. We will now explain the idea of the definition of this groupoid through a function-theoretic analogy.

As was mentioned above, the category D-mod($\text{Bun}_G^{N\text{-gen}}$) is the geometric analog of the space of functions on $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})$. What we want to achieve is to enforce the condition that our function, when considered as a function on $G(\mathbb{A})/G(\mathbb{O})$, be invariant with respect to all of $N(\mathbb{A})$ (resp., equivariant against a fixed character of $N(\mathbb{A})$, which is trivial on $N(K)$ and $N(\mathbb{O})$). However, we want to do this without actually lifting our function on $G(\mathbb{A})/G(\mathbb{O})$.

Here is how we will do this. The trick explained below stands behind the definition of the corresponding verions of the Whittaker category in [FGV1] and [Ga1].

5.3.2. Let $\underline{x}$ be a finite collection of points on X, and let $\mathbb{A}_{\underline{x}}$ denote the corresponding product of local fields. Let us say that we want to enforce invariance/equivariance with respect to the corresponding subgroup $N(\mathbb{A}_{\underline{x}}) \subset N(\mathbb{A})$.

Let $\underline{y}$ be another finite collection of points of X, which is *non-empty* and *disjoint* from $\overline{x}$. Let

$$N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}} \subset N(K)\backslash G(\mathbb{A})/G(\mathbb{O})$$

be the subset equal to

$$N(K)\backslash \left(G(\mathbb{A}^{\underline{y}})/G(\mathbb{O}^{\underline{y}}) \times N(\mathbb{A}_{\underline{y}})/N(\mathbb{O}_{\underline{y}}) \right),$$

where[14] $\mathbb{A}^{\underline{y}} := \prod_{z \notin \underline{y}} \widehat{\mathcal{K}}_z$, $\mathbb{O}^{\underline{y}} := \prod_{z \notin \underline{y}} \widehat{\mathbb{O}}_z$.

Clearly, the preimage of the subset $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}}$ in $G(\mathbb{A})/G(\mathbb{O})$ is invariant with respect to $N(\mathbb{A}_{\underline{x}})$. Moreover, Iwasawa decomposition implies that all of $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})$ can be covered by subsets of this form for various choices of $\underline{y}$.

Hence, it is sufficient to specify the invariance/equivariance condition for a function on $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}}$.

5.3.3. Set

$$^{\sim}N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}} := N(K)\backslash \left(G(\mathbb{A}^{\underline{y}})/G(\mathbb{O}^{\underline{y}}) \times N(\mathbb{A}_{\underline{y}}) \right).$$

This set is acted on (by right multiplication) by $N(\mathbb{A}_{\underline{y}})$, and the resulting action of the subgroup $N(\mathbb{O}_{\underline{y}}) \subset N(\mathbb{A}_{\underline{y}})$ makes the projection

$$^{\sim}N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}} \to N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}}$$

into a $N(\mathbb{O}_{\underline{y}})$-torsor.

5.3.4. We are now finally ready to explain how we will enforce the sought-for invariance/condition with respect to $N(\mathbb{A}_{\underline{x}})$ for a function on $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}}$. In fact, we will enforce equivariance with respect to all of $N(\mathbb{A}^{\underline{y}})$.

Namely, we require that the lift of our function to $^{\sim}N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}}$ be $N(\mathbb{A}_{\underline{y}})$-invariant/equivariant.

The fact that this is the right thing to do follows from the *strong approximation* for the group N, i.e., from the fact that the image of the map

$$N(K) \to N(\mathbb{A}^{\underline{y}}),$$

---

[14]I.e., $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})_{\text{goot at } \underline{y}}$ is different from all of $N(K)\backslash G(\mathbb{A})/G(\mathbb{O})$ is that for $z \in \underline{y}$ we take $N(\widehat{\mathcal{K}}_z)/N(\widehat{\mathbb{O}}_z)$ instead of $G(\widehat{\mathcal{K}}_z)/G(\widehat{\mathbb{O}}_z)$

given by Taylor expansion, is dense.

5.4. **The groupoid: algebro-geometric definition.** The actual algebro-geometric definition of the groupoid $\mathbf{N}$, given below, was suggested by J. Barlev.

5.4.1. We define the groupoid $\mathbf{N}$ is as follows. First, the space (=prestack) that it will act on is not $\mathrm{Bun}_G^{B\text{-gen}}$, but rather a certain open substack

$$\left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}$$

of $\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)$.

Namely, $\left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}$ corresponds those quadruples $(\mathcal{P}_G, U, \alpha, \gamma, \underline{y})$ for which U can be chosen to contain $\underline{y}$.

We consider the prestack that assigns to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the groupoid of the following data:

$$((\mathcal{P}_G^1, U, \alpha^1), (\mathcal{P}_G^2, U, \alpha^2), \underline{y}, \beta),$$

where $(\mathcal{P}_G^1, U, \alpha^1)$ and $(\mathcal{P}_G^2, U, \alpha^2)$ are as in Sect. 5.1.1, $\underline{y} \in U$, and β is a datum is isomorphism of B-bundles

$$\mathcal{P}_{B,U}^1|_{U-\underline{y}} \simeq \mathcal{P}_{B,U}^2|_{U-\underline{y}},$$

such that the induced isomorphism of T-bundles

$$\mathcal{P}_{T,U}^1|_{U-\underline{y}} \simeq \mathcal{P}_{T,U}^2|_{U-\underline{y}}$$

extends to all of U, and such that the induced isomorphism of the G-bundles

$$\mathcal{P}_G^1|_{U-\underline{y}} \simeq \mathcal{P}_G^2|_{U-\underline{y}}$$

extends to all of $S \times X - \underline{y}$.

We let $\mathrm{Maps}(S, \mathbf{N})$ be the quotient of the above prestack by the equivalence relation defined in a way similar to the case of $\mathrm{Bun}_G^{B\text{-gen}}$.

We have the following assertion:

Proposition 5.4.2. *The fibers of the groupoid $\mathbf{N}$ are homologically contractible, i.e., the functors*

$$p_1^\dagger, p_2^\dagger : \mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right) \to \mathrm{D\text{-}mod}(\mathbf{N})$$

are fully faithful, where p_1 and p_2 are the two projections $\mathbf{N} \to \left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}$.

This proposition essentially follows from the fact that the group N is homologically contractible.

5.4.3. The groupoid $\mathbf{N}_{\mathcal{Q}_G}$ (resp., $\mathbf{N}_{\mathcal{Q}_{G,G}}$) acting on $\mathcal{Q}_G$ (resp., $\mathcal{Q}_{G,G}$) is defined similarly.

Note that we have a Cartesian diagram

$$
\begin{array}{ccccc}
(\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}} & \xleftarrow{\ p_1\ } & \mathbf{N}_{\mathcal{Q}_G} & \xrightarrow{\ p_2\ } & (\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{Q}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}} & \xleftarrow{\ p_1\ } & \mathbf{N}_{\mathcal{Q}_{G,G}} & \xrightarrow{\ p_2\ } & (\mathcal{Q}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}} \\
\downarrow & & \downarrow & & \downarrow \\
\left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}} & \xleftarrow{\ p_1\ } & \mathbf{N} & \xrightarrow{\ p_2\ } & \left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}.
\end{array}
$$

From Proposition 5.4.2 we obtain the corresponding assertion for $\mathbf{N}_{\mathcal{Q}_G}$ (resp., $\mathbf{N}_{\mathcal{Q}_{G,G}}$).

5.5. The character.

5.5.1. We now consider the groupoid $\mathbf{N}_{\mathcal{Q}_{G,G}}$ and we claim that it admits a canonically defined homomorphism χ to $\mathbb{G}_a$.

In fact, there are homomorphisms χ_i, one for each simple root α_i of G, and we will let χ be their sum.

5.5.2. For a simple root α_i, let $B_i \simeq \mathbb{G}_m \ltimes \mathbb{G}_a$ be the quotient group of B by $N(P_i)$ (the unipotent radical of the sub-minimal parabolic P_i) and Z_{M_i} (the center of the Levi M_i of P_i). In particular, the map $T \to \mathbb{G}_m$ is given by the simple root α_i.

For a point $(\mathcal{P}_G, U, \alpha, \gamma)$ of $\mathcal{Q}_{G,G}$ we let $\mathcal{P}_{B_i,U}$ denote the induced B_i-bundle defined over U. Note that the data of γ identifies the line bundle corresponding to $B_i \to \mathbb{G}_m$ with $\omega_X|_U$. Hence, we can think of $\mathcal{P}_{B_i,U}$ as a short exact sequence of vector bundles

$$0 \to \omega_U \to \mathcal{F}_i \to \mathcal{O}_U \to 0.$$

5.5.3. A point

$$((\mathcal{P}_G^1, U, \alpha^1, \gamma^1), (\mathcal{P}_G^2, U, \alpha^2, \gamma^2), \underline{y}, \beta)$$

of $\mathbf{N}_{\mathcal{Q}_{G,G}}$ defines an isomorphism of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_X|_{U-\underline{u}} & \longrightarrow & \mathcal{F}_i^1|_{U-\underline{y}} & \longrightarrow & \mathcal{O}_{U-\underline{y}} & \longrightarrow & 0 \\
& & \text{id} \downarrow & & \beta_i \downarrow & & \downarrow \text{id} & & \\
0 & \longrightarrow & \omega_X|_{U-\underline{y}} & \longrightarrow & \mathcal{F}_i^2|_{U-\underline{y}} & \longrightarrow & \mathcal{O}_{U-\underline{y}} & \longrightarrow & 0,
\end{array}
$$

and hence a section of the quasi-coherent sheaf

$$(5.1) \qquad \omega_X|_{U-\underline{y}}/\omega_X|_U \simeq \omega_X|_{S\times X-\underline{y}}/\omega_X|_{S\times X} \simeq (\mathcal{O}_S \boxtimes \omega_X)(\infty \cdot \underline{y})/(\mathcal{O}_S \boxtimes \omega_X),$$

where we think of $\underline{y}$ as a relative Cartier divisor $D \subset S \times X$ over S.

Now, the residue map assigns to sections of (5.1) a section of $\mathcal{O}_S$, i.e., a map $S \to \mathbb{G}_a$.

5.5.4. By composing, the above character χ on $\mathbf{N}_{\mathcal{Q}_{G,G}}$ gives rise to one on $\mathbf{N}_{\mathcal{Q}_G}$. We will not distinguish the two notationally.

5.6. The Whittaker category. We are finally able to define the main actor for this section, the Whittaker category for G.

5.6.1. First, we consider the equivariant category

$$\text{D-mod}\left((\mathcal{Q}_G \times \text{Ran}(X))_{\text{good}}\right)^{\mathbf{N}_{\mathcal{Q}_G},\chi}$$

of $\text{D-mod}\left((\mathcal{Q}_G \times \text{Ran}(X))_{\text{good}}\right)$ with respect to the groupoid $\mathbf{N}_{\mathcal{Q}_G}$ against the character χ.

In other words, we consider the simplicial object $\mathbf{N}_{\mathcal{Q}_G}^{\mathbf{\Delta}}$ of PreStk corresponding to the groupoid $\mathbf{N}_{\mathcal{Q}_G}$. We consider the co-simplicial category $\text{D-mod}(\mathbf{N}_{\mathcal{Q}_G}^{\mathbf{\Delta}})$, and its twist, denoted

$$\text{D-mod}(\mathbf{N}_{\mathcal{Q}_G}^{\mathbf{\Delta}})^{\chi},$$

corresponding to the pull-back by means of χ of the exponential D-module on $\mathbb{G}_a$. By definition,

$$\text{D-mod}\left((\mathcal{Q}_G \times \text{Ran}(X))_{\text{good}}\right)^{\mathbf{N}_{\mathcal{Q}_G},\chi} := \text{Tot}(\text{D-mod}(\mathbf{N}_{\mathcal{Q}_G}^{\mathbf{\Delta}})^{\chi}).$$

The following results from Proposition 5.4.2:

Proposition 5.6.2. *The forgetful functor*

$$\text{D-mod}\left((\mathfrak{Q}_G \times \text{Ran}(X))_{\text{good}}\right)^{\mathbf{N}_{\mathfrak{Q}_G},\chi} \to \text{D-mod}\left((\mathfrak{Q}_G \times \text{Ran}(X))_{\text{good}}\right)$$

is fully faithful.

5.6.3. We define the Whittaker category $\text{Whit}(G)$ to be the full subcategory of $\text{D-mod}(\mathfrak{Q}_G)$ equal to the preimage of

$$\text{D-mod}\left((\mathfrak{Q}_G \times \text{Ran}(X))_{\text{good}}\right)^{\mathbf{N}_{\mathfrak{Q}_G},\chi} \subset \text{D-mod}\left((\mathfrak{Q}_G \times \text{Ran}(X))_{\text{good}}\right)$$

under the pull-back functor

$$\text{D-mod}(\mathfrak{Q}_G) \to \text{D-mod}\left((\mathfrak{Q}_G \times \text{Ran}(X))_{\text{good}}\right).$$

In other words,

$$\text{Whit}(G) := \text{D-mod}(\mathfrak{Q}_G) \underset{\text{D-mod}((\mathfrak{Q}_G \times \text{Ran}(X))_{\text{good}})}{\times} \text{D-mod}\left((\mathfrak{Q}_G \times \text{Ran}(X))_{\text{good}}\right)^{\mathbf{N}_{\mathfrak{Q}_G},\chi}.$$

5.6.4. Consider the fully faithful embedding

$$\text{Whit}(G) \hookrightarrow \text{D-mod}(\mathfrak{Q}_G).$$

One shows that it admits a right adjoint; we will denote it by $\text{Av}^{\mathbf{N},\chi}$.

In addition, one shows:

Proposition 5.6.5. *The action of the monoidal category* $\text{Rep}(\check{G})_{\text{Ran}(X)}$ *on* $\text{D-mod}(\mathfrak{Q}_G)$ *preserves the full subcategory*

$$\text{Whit}(G) \subset \text{D-mod}(\mathfrak{Q}_G)$$

and commutes with the functor $\text{Av}^{\mathbf{N},\chi}$.

5.6.6. The category $\text{Whit}(G)$ contains a distinguished object that we shall denote by $\mathcal{W}_{\text{vac}}$:

Analogously to the map $\imath_B : \text{Bun}_B \to \text{Bun}_G^{B\text{-gen}}$, we have a canonically defined map

$$\imath_{N^\omega} : \text{Bun}_{N^\omega} \to \text{Bun}_G^{N\text{-gen}}.$$

Note that, analogously to Sect. 5.5, there exists a canonically defined map

$$\text{Bun}_{N^\omega} \to \mathbb{G}_a.$$

Let $\overset{\circ}{\mathcal{W}}_{\text{vac}} \in \text{D-mod}(\text{Bun}_{N^\omega})$ denote the pullback of the exponential D-module on $\mathbb{G}_a$ under this map.

The object $\mathcal{W}_{\text{vac}} \in \text{D-mod}(\text{Bun}_G^{N\text{-gen}})$ is defined by

$$\mathcal{W}_{\text{vac}} := (\imath_{N^\omega})_{\dagger}(\overset{\circ}{\mathcal{W}}_{\text{vac}}),$$

where for a morphism f we denote by $f_\dagger$ the (partially defined) left adjoint of $f^\dagger$; one shows that the (partially defined) functor $(\imath_{N^\omega})_\dagger$ *is* defined on the object $\overset{\circ}{\mathcal{W}}_{\text{vac}}$ due to the holonomicity property of the latter.

5.6.7. *A variant.* We define the category $\mathrm{Whit}(G, G)$ in a similar way to $\mathrm{Whit}(G)$, using the prestack $\mathcal{Q}_{G,G}$ instead of $\mathcal{Q}_G$.

As in Sect. 5.6.4, the fully faithful embedding

$$\mathrm{Whit}(G) \hookrightarrow \text{D-mod}(\mathcal{Q}_G).$$

it admits a right adjoint, and the analog of Proposition 5.6.5 holds.

Let us remind that the categories $\mathrm{Whit}(G)$ and $\mathrm{Whit}(G, G)$ are geometric counterparts of the spaces of functions on $G(\mathbb{A})/G(\mathbb{O})$ and $Z_G^0(K)\backslash G(\mathbb{A})/G(\mathbb{O})$, respectively, that are equivariant with respect to $N(\mathbb{A})$ against the character χ. In particular, we have a naturally defined pullback functor

$$\mathrm{Whit}(G, G) \to \mathrm{Whit}(G).$$

Now, it follows from Proposition 5.2.9 that this functor is fully faithful.

5.7. **The functor of Whittaker coefficient and Poincaré series.** In this subsection we will relate the Whittaker categories $\mathrm{Whit}(G)$ and $\mathrm{Whit}(G, G)$ to the main object on the geometric side, the category $\text{D-mod}(\mathrm{Bun}_G)$.

5.7.1. Let $\mathfrak{r}_G$ (resp., $\mathfrak{r}_{G,G}$) denote the forgetful map $\mathcal{Q}_G \to \mathrm{Bun}_G$ (resp., $\mathcal{Q}_{G,G} \to \mathrm{Bun}_G$). In particular, we obtain the functors

$$(\mathfrak{r}_G)^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathcal{Q}_G) \text{ and } (\mathfrak{r}_{G,G})^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathcal{Q}_{G,G}).$$

5.7.2. We denote the composed functors

$$\mathrm{Av}^{\mathbf{N},\chi} \circ (\mathfrak{r}_G)^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G)$$

and

$$\mathrm{Av}^{\mathbf{N},\chi} \circ (\mathfrak{r}_{G,G})^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G, G)$$

by coeff_G and $\mathrm{coeff}_{G,G}$, respectively.

These are the two closely related versions of the *functor of Whittaker coefficient.*

5.7.3. By Proposition 5.6.5, the functor coeff_G (resp., $\mathrm{coeff}_{G,G}$) intertwines the actions of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on $\text{D-mod}(\mathrm{Bun}_G)$ and $\mathrm{Whit}(G)$ (resp., $\mathrm{Whit}(G, G)$).

5.7.4. The functor $(\mathfrak{r}_G)^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathcal{Q}_G)$ does not in general admit a left adjoint. However, one shows that the (partially defined) left adjoint $(\mathfrak{r}_G)_\dagger$ *is defined* on the full subcategory $\mathrm{Whit}(G) \subset \text{D-mod}(\mathcal{Q}_G)$.

We denote the resulting functor $\mathrm{Whit}(G) \to \text{D-mod}(\mathrm{Bun}_G)$ by Poinc_G, and refer to it as the functor of Poincaré series. By construction, this functor is the left adjoint of the functor coeff_G.

In particular, we obatin a canonically defined object

$$\mathrm{Poinc}_G(\mathcal{W}_{\mathrm{vac}}) \in \text{D-mod}(\mathrm{Bun}_G),$$

where $\mathcal{W}_{\mathrm{vac}}$ is as in Sect. 5.6.6.

5.8. **Digression: "unital" categories over the Ran space.** Our next goal is to give a description of the categories $\mathrm{Whit}(G)$ and $\mathrm{Whit}(G, G)$ in spectral terms. This subsection contains some preliminaries needed in order to describe the spectral side.

These preliminaries have to do with the fact that the symmetric monoidal categories $\mathrm{D\text{-}mod}(\mathrm{Ran}(X))$ and $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ are non-unital, and in this subsection we will show how to modify them to make them unital. [15]

5.8.1. We note that a group homomorphism $G_1 \to G_2$ gives rise to a symmetric monoidal functor
$$\mathrm{Rep}(G_2)_{\mathrm{Ran}(X)} \to \mathrm{Rep}(G_1)_{\mathrm{Ran}(X)}.$$
In particular, taking $\check{G}_1 = \check{G}$ and $\check{G}_2 = \{1\}$, we obtain a symmetric monoidal functor
$$\mathrm{D\text{-}mod}(\mathrm{Ran}(X)) \simeq \mathrm{QCoh}(\mathrm{Ran}(X)) \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}.$$

By taking $G_1 = \check{G}$ and $G_2 = \check{G}/[\check{G}, \check{G}]$ we obtain a symmetric monoidal functor
$$\mathrm{Rep}(\check{G}/[\check{G}, \check{G}])_{\mathrm{Ran}(X)} \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}.$$

5.8.2. Consider the symmetric monoidal functor
$$\mathrm{D\text{-}mod}(\mathrm{Ran}(X)) \xrightarrow{p_\dagger} \mathrm{D\text{-}mod}(\mathrm{pt}) = \mathrm{Vect}$$
(the left adjoint to the pull-back functor $p^\dagger$). It can also be viewed as the functor $\mathrm{Loc}_{\{1\},\mathrm{spec}}$, where $\{1\}$ is the trivial group. Consider the category
$$\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)} := \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \underset{\mathrm{D\text{-}mod}(\mathrm{Ran}(X))}{\otimes} \mathrm{Vect}.$$

One can show that the functor
$$\mathrm{Vect} \simeq \mathrm{D\text{-}mod}(\mathrm{Ran}(X)) \underset{\mathrm{D\text{-}mod}(\mathrm{Ran}(X))}{\otimes} \mathrm{Vect} \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \underset{\mathrm{D\text{-}mod}(\mathrm{Ran}(X))}{\otimes} \mathrm{Vect} =: \mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}$$

defines a *unit* for the symmetric monoidal structure on $\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}$; we shall denote the corresponding unit object by
$$\mathbf{1}_{\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}} \in \mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}.$$

I.e., unlike $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$, the symmetric monodical category $\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}$ is unital.

5.8.3. It follows from the definition that the symmetric monoidal functor
$$\mathrm{Loc}_{\check{G},\mathrm{spec}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$
canonically factors as

(5.2) $$\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \to \mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}).$$

We denote the resulting functor
$$\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$
by $\mathrm{Loc}^{\mathrm{unital}}_{\check{G},\mathrm{spec}}$.

Passing to right adjoints in (5.4), we obtain the functors
$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)} \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)},$$
all of which are fully faithful by Proposition 4.3.4.

---

[15]The reader who is afraid of being overwhelmed by the notation can skip this subsection and return to it when necessary.

We shall denote the resulting (fully faithful) functor

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}$$

by co-$\mathrm{Loc}^{\mathrm{unital}}_{\check{G},\mathrm{spec}}$.

5.8.4. *Variant.* Consider now the symmetric monoidal functor

$$\mathrm{Loc}_{\check{G}/[\check{G},\check{G}],\mathrm{spec}} : \mathrm{Rep}(\check{G}/[\check{G},\check{G}])_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}/[\check{G},\check{G}]}).$$

Consider also the category

$$\mathrm{Rep}(\check{G})^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}_{\mathrm{Ran}(X)} := \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \underset{\mathrm{Rep}(\check{G}/[\check{G},\check{G}])_{\mathrm{Ran}(X)}}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}/[\check{G},\check{G}]}),$$

and the functor

(5.3) $\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)} \simeq$

$$\simeq \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \underset{\mathrm{Rep}(\check{G}/[\check{G},\check{G}])_{\mathrm{Ran}(X)}}{\otimes} \mathrm{Rep}(\check{G}/[\check{G},\check{G}])_{\mathrm{Ran}(X)} \underset{\mathrm{D\text{-}mod}(\mathrm{Ran}(X))}{\otimes} \mathrm{Vect} \xrightarrow{\mathrm{Id} \otimes \mathrm{Loc}^{\mathrm{unital}}_{\check{G}/[\check{G},\check{G}],\mathrm{spec}}}$$

$$\to \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \underset{\mathrm{Rep}(\check{G}/[\check{G},\check{G}])_{\mathrm{Ran}(X)}}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}/[\check{G},\check{G}]}) =: \mathrm{Rep}(\check{G})^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}_{\mathrm{Ran}(X)}.$$

It follows from the construction that the functor

$$\mathrm{Loc}^{\mathrm{unital}}_{\check{G},\mathrm{spec}} : \mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

introduced above canonically factors as

(5.4) $\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)} \xrightarrow{(5.3)} \mathrm{Rep}(\check{G})^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}),$

We denote the resulting functor

$$\mathrm{Rep}(\check{G})^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

by $\mathrm{Loc}^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}_{\check{G},\mathrm{spec}}$.

Passing to right adjoints in (5.4), we obtain functors

(5.5) $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Rep}(\check{G})^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}_{\mathrm{Ran}(X)} \to \mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}$

all of which are fully faithful by Proposition 4.3.4.

We denote the resulting (fully faithful) functor

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Rep}(\check{G})^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}_{\mathrm{Ran}(X)}$$

by co-$\mathrm{Loc}^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}_{\check{G},\mathrm{spec}}$.

5.9. **Description of the Whittaker category in spectral terms.** A key feature of the Whittaker categories $\mathrm{Whit}(G)$ and $\mathrm{Whit}(G,G)$, and the reason for why the figure so prominently in geometric Langlands, is that these categories can be directly described in terms of the spectral side of the correspondence.

5.9.1. The following assertion is a geometric version of the Casselman-Shalika formula. It expresses the categories $\mathrm{Whit}(G)$ and $\mathrm{Whit}(G, G)$, respectively, in terms of the Langlands dual group.

Quasi-Theorem 5.9.2.

(a) *There exists a canonical equivalence*

$$\mathbb{L}_G^{\mathrm{Whit}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{unital}} \to \mathrm{Whit}(G),$$

compatible with the action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$.

(b) *There is a canonical equivalance*

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}} \to \mathrm{Whit}(G, G),$$

compatible with the action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$.

(c) *We have a commutative diagram*

$$
(5.6) \qquad
\begin{array}{ccc}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{unital}} & \xrightarrow{\;\mathbb{L}_G^{\mathrm{Whit}}\;} & \mathrm{Whit}(G) \\
\uparrow & & \uparrow \\
\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}} & \longrightarrow & \mathrm{Whit}(G, G),
\end{array}
$$

where the left vertical arrow is the right adjoint of (5.3).

This quasi-theorem is very close to being a theorem and is being worked out by D. Beraldo.

We shall denote the composed functor

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\mathrm{co\text{-}Loc}_{\check{G},\mathrm{spec}}^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}}} \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{unital}_{\check{G}/[\check{G},\check{G}]}} \xrightarrow{\sim} \mathrm{Whit}(G, G)$$

by $\mathbb{L}_{G,G}^{\mathrm{Whit}}$. By the above, $\mathbb{L}_{G,G}^{\mathrm{Whit}}$ is fully faithful.

5.9.3. We will now formulate Property Wh ("Wh" stands for Whittaker) of the geometric Langlands functor $\mathbb{L}_G$. It is a particular case of Property $\mathrm{Wh}^{\mathrm{ext}}$, formulated in Sect. 9.4.6:

Property Wh: *We shall say that the functor* $\mathbb{L}_G$ *satisfies* Property Wh *if the following diagram is commutative:*

$$
(5.7) \qquad
\begin{array}{ccc}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{unital}} & \xrightarrow{\;\mathbb{L}_G^{\mathrm{Whit}}\;} & \mathrm{Whit}(G) \\
{\scriptstyle \mathrm{co\text{-}Loc}_{\check{G},\mathrm{spec}}^{\mathrm{unital}} \circ \Psi_{\check{G}}} \uparrow & & \uparrow {\scriptstyle \mathrm{coeff}_G} \\
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\;\mathbb{L}_G\;} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G).
\end{array}
$$

We remind that the functor $\Psi_{\check{G}}$ appearing in the left vertical arrow in (5.7) is the right adjoint of the fully faithful embedding

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\Xi_{\check{G}}} \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$$

5.9.4. By passing to left adjoints in the diagram (5.7), from Property Wh we obtain a commutative diagram

$$(5.8) \qquad \begin{array}{ccc}
\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)} & \xrightarrow{\ \mathbb{L}^{\mathrm{Whit}}_G\ } & \mathrm{Whit}(G) \\
{\scriptstyle \Xi_{\check{G}} \circ \mathrm{Loc}^{\mathrm{unital}}_{\check{G},\mathrm{spec}}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Poinc}_G} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\ \mathbb{L}_G\ } & \text{D-mod}(\mathrm{Bun}_G).
\end{array}$$

As part of the construction of the equivalence of Quasi-Theorem 5.9.2, we have that the object $\mathcal{W}_{\mathrm{vac}} \in \mathrm{Whit}(G)$ identifies with

$$\mathbb{L}^{\mathrm{Whit}}_G(\mathbf{1}_{\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}}),$$

where we remind that $\mathbf{1}_{\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}}$ is the unit object in the monoidal category $\mathrm{Rep}(\check{G})^{\mathrm{unital}}_{\mathrm{Ran}(X)}$, see Sect. 5.8.2.

5.9.5. In particular, from (5.8), we obtain:

$$(5.9) \qquad \qquad \mathbb{L}_G(\Xi_{\check{G}}(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}})) \simeq \mathrm{Poinc}_G(\mathcal{W}_{\mathrm{vac}}).$$

So, the object on the geometric side that corresponds to

$$\Xi_{\check{G}}(\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}) \in \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$$

is

$$\mathrm{Poinc}_G(\mathcal{W}_{\mathrm{vac}}) \in \text{D-mod}(\mathrm{Bun}_G).$$

5.9.6. Note that since the vertical arrows in the diagram (5.6) are fully faithful, we can reformulate Property Wh as the commutativity of the diagram

$$(5.10) \qquad \begin{array}{ccc}
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\ \mathbb{L}^{\mathrm{Whit}}_{G,G}\ } & \mathrm{Whit}(G,G) \\
{\scriptstyle \Psi_{\check{G}}} \Big\uparrow & & \Big\uparrow {\scriptstyle \mathrm{coeff}_{G,G}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\ \mathbb{L}_G\ } & \text{D-mod}(\mathrm{Bun}_G).
\end{array}$$

Remark 5.9.7. Note that if we believe in Conjecture 3.4.2, we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\ \mathbb{L}^{\mathrm{Whit}}_{G,G}\ } & \mathrm{Whit}(G,G) \\
{\scriptstyle \Psi_{\check{G}}} \Big\uparrow & & \Big\uparrow {\scriptstyle \mathrm{coeff}_{G,G}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\ \mathbb{L}_G\ } & \text{D-mod}(\mathrm{Bun}_G) \\
{\scriptstyle \Xi_{\check{G}}} \Big\uparrow & & \Big\uparrow \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & \longrightarrow & \text{D-mod}(\mathrm{Bun}_G)_{\mathrm{temp}},
\end{array}$$

where the composed left vertical arrow is the identity functor. Hence, the composed functor

$$\text{D-mod}(\mathrm{Bun}_G)_{\mathrm{temp}} \hookrightarrow \text{D-mod}(\mathrm{Bun}_G) \xrightarrow{\mathrm{coeff}_{G,G}} \mathrm{Whit}(G,G)$$

is fully faithful. I.e., the tempered category is Whittaker non-degenerate in the strong sense that not only does the functor coeff_G not annihilate anything, but it is actually fully faithful.

6. Parabolic induction

In this subsection we study how the automorphic category D-mod(Bun_G) can be related to the corresponding categories for proper Levi subgroups of G, and a similar phenomenon on the spectral side of Langlands correspondence.

6.1. **The space of generic parabolic reductions.** In this subsection we will introduce the "parabolically induced" category, denoted I(G, P).

6.1.1. Let $P \subset G$ be a parabolic. Let M denote its Levi quotient.

We define the prestack $\mathrm{Bun}_G^{P\text{-gen}}$ in the same way as we defined $\mathrm{Bun}_G^{B\text{-gen}}$, substituting P for B.

We let $\mathsf{p}_P^{\mathrm{enh}}$ denote the natural forgetful map $\mathrm{Bun}_G^{P\text{-gen}} \to \mathrm{Bun}_G$, and by ι_P the map

$$\mathrm{Bun}_P \to \mathrm{Bun}_G^{P\text{-gen}}.$$

As in the case of $P = B$, the map ι_P defines an isomorphism at the level of groupoids of field-valued points. In particular, the groupoid of k-points of $\mathrm{Bun}_G^{P\text{-gen}}$ identifies canonically with the double quotient

$$P(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

From here one deduces:

Lemma 6.1.2. *The forgetful functor* $(\iota_P)^\dagger : \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_P)$ *is conservative.*

6.1.3. Let $N(P)$ denote the unipotent radical of P. To it we associate a groupoid that we denote by $\mathbf{N(P)}$ acting on $\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}$ in the same way as we defined the groupoid $\mathbf{N}$ acting on $\left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}$.

We consider the $\mathbf{N(P)}$-equivariant category of $\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}$, i.e.,

$$\mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right)^{\mathbf{N(P)}} := \mathrm{Tot}\left(\mathrm{D\text{-}mod}(\mathbf{N(P)}^{\boldsymbol{\Delta}})\right),$$

where $\mathbf{N(P)}^{\boldsymbol{\Delta}}$ is the simplicial object of PreStk corresponding to the groupoid $\mathbf{N(P)}$.

As in Proposition 5.6.2, we have:

Proposition 6.1.4. *The forgetful functor*

$$\mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right)^{\mathbf{N(P)}} \to \mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right)$$

is fully faithful.

6.1.5. We define I(G, P) as the full subcategory of D-mod($\mathrm{Bun}_G^{P\text{-gen}}$) equal to the preimage of

$$\mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right)^{\mathbf{N(P)}} \subset \mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right)$$

under the pull-back functor

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}) \to \mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right).$$

I.e.,

$$\mathrm{I}(G,P) :=$$

$$= \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}) \underset{\mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right)}{\times} \mathrm{D\text{-}mod}\left(\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}\right)^{\mathbf{N(P)}}.$$

Remark 6.1.6. The category $\mathrm{I}(G,P)$ is the geometric counterpart of the space of functions on the double quotient

$$M(K) \cdot N(P)(\mathbb{A}) \backslash G(\mathbb{A}) / G(\mathbb{O}).$$

6.1.7. As in the case of $\mathrm{Whit}(G)$, one shows that the fully faithful embedding

$$\mathrm{I}(G,P) \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}})$$

admits a right adjoint, that we denote by $\mathrm{Av}^{\mathbf{N(P)}}$.

As in Sect. 5.1.6, we have a canonically defined action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}})$, and as in Proposition 5.6.5, this action preserves the full subcategory

$$\mathrm{I}(G,P) \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}})$$

and commutes with the functor $\mathrm{Av}^{\mathbf{N(P)}}$.

6.2. **A strata-wise description of the parabolic category.** One can describe the full subcategory $\mathrm{I}(G,P) \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}})$ explicitly via the morphism

$$\iota_P : \mathrm{Bun}_P \to \mathrm{Bun}_G^{P\text{-gen}}.$$

This is the subject of the present subsection.

6.2.1. Let p_P and q_P denote the natural forgetful maps from Bun_P to Bun_G and Bun_M, respectively. For instance, we have:

$$\mathsf{p}_P = \mathsf{p}_P^{\mathrm{enh}} \circ \iota_P.$$

Note that the map q_P is smooth. Hence, the functor

$$(\mathsf{q}_P)^\bullet : \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_P)$$

(the Verdier conjugate of $(\mathsf{q}_P)^\dagger$) is well-defined [16]. Note that the fibers of q_P are contractible, so the functor $(\mathsf{q}_P)^\bullet$ is fully faithful.

6.2.2. We have:

Lemma 6.2.3. *The category* $\mathrm{I}(G,P)$ *fits into a pull-back square:*

(6.1)
$$
\begin{array}{ccc}
\mathrm{I}(G,P) & \longrightarrow & \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}) \\
\downarrow & & \downarrow{\scriptstyle (\iota_P)^\dagger} \\
\mathrm{D\text{-}mod}(\mathrm{Bun}_M) & \xrightarrow{\ (\mathsf{q}_P)^\bullet\ } & \mathrm{D\text{-}mod}(\mathrm{Bun}_P).
\end{array}
$$

In other words, the above lemma says that an object $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}})$ belongs to $\mathrm{I}(G,P)$ if and only if $(\iota_P)^\dagger(\mathcal{M}) \in \mathrm{D\text{-}mod}(\mathrm{Bun}_P)$ belongs to the essential image of the functor $(\mathsf{q}_P)^\bullet$.

---

[16]Since q_P is smooth, the functors $(\mathsf{q}_P)^\bullet$ and $(\mathsf{q}_P)^\dagger$ are in fact isomorphic up to a cohomological shift, which depends on the connected component of Bun_M.

6.2.4. We denote the resulting (conservative) functor $\mathrm{I}(G,P) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_M)$ by $(\imath_M)^\dagger$.

One shows that the square obtained by passing to right adjoints along the horizontal arrows in (6.1) is also commutative:

$$
(6.2) \quad
\begin{array}{ccc}
\mathrm{I}(G,P) & \xleftarrow{\mathrm{Av}^{\mathbf{N(P)}}} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}) \\
{\scriptstyle(\imath_M)^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle(\imath_P)^\dagger} \\
\mathrm{D\text{-}mod}(\mathrm{Bun}_M) & \xleftarrow{(\mathsf{q}_P)_\bullet} & \mathrm{D\text{-}mod}(\mathrm{Bun}_P).
\end{array}
$$

6.2.5. In addition, one shows that the partially defined left adjoint $(\imath_P)_\dagger$ of $(\imath_P)^\dagger$ is defined on the essential image of $(\mathsf{q}_P)^\bullet$. We denote the resulting functor $\mathrm{D\text{-}mod}(\mathrm{Bun}_M) \to \mathrm{I}(G,P)$ by $(\imath_M)_\dagger$.

By passing to left adjoints in (6.2), we obtain a commutative diagram

$$
(6.3) \quad
\begin{array}{ccc}
\mathrm{I}(G,P) & \xrightarrow{} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}) \\
{\scriptstyle(\imath_M)_\dagger}\big\uparrow & & \big\uparrow{\scriptstyle(\imath_P)_\dagger} \\
\mathrm{D\text{-}mod}(\mathrm{Bun}_M) & \xrightarrow{(\mathsf{q}_P)^\bullet} & \mathrm{D\text{-}mod}(\mathrm{Bun}_P).
\end{array}
$$

6.3. **The "enhanced" constant term and Eisenstein functors.** As in the classical theory of automorphic functions, the parabolic category $\mathrm{I}(G,P)$ is related to the automorphic category $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ by a pair of functors, called "constant term" and "Eisenstein series."

6.3.1. We define the functor of *enhanced constant term*

$$
\mathrm{CT}_P^{\mathrm{enh}} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{I}(G,P)
$$

as the composition

$$
\mathrm{CT}_P^{\mathrm{enh}} = \mathrm{Av}^{\mathbf{N(P)}} \circ (\mathsf{p}_P^{\mathrm{enh}})^\dagger.
$$

6.3.2. We claim that the functor $\mathrm{CT}_P^{\mathrm{enh}}$ admits a left adjoint. This follows from the next lemma:

Lemma 6.3.3. *The partially defined left adjoint $(\mathsf{p}_P^{\mathrm{enh}})_\dagger$ of $(\mathsf{p}_P^{\mathrm{enh}})^\dagger$ is defined on the full subcategory* $\mathrm{I}(G,P) \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}})$.

Thus, the functor

$$
\mathrm{Eis}_P^{\mathrm{enh}} := (\mathsf{p}_P^{\mathrm{enh}})_\dagger|_{\mathrm{I}(G,P)}, \quad \mathrm{I}(G,P) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)
$$

is well-defined and provides a left adjoint to $\mathrm{CT}_P^{\mathrm{enh}}$.

We will refer to $\mathrm{Eis}_P^{\mathrm{enh}}$ as the functor of *enhanced Eisenstein series*.

6.3.4. Consider now the diagram

$$
\begin{array}{ccc}
 & \mathrm{Bun}_P & \\
{\scriptstyle\mathsf{p}_P}\swarrow & & \searrow{\scriptstyle\mathsf{q}_P} \\
\mathrm{Bun}_G & & \mathrm{Bun}_M\,.
\end{array}
$$

We define the *usual* constant term and Eisenstein functors as follows:

$$
\mathrm{CT}_P = (\mathsf{q}_P)_\bullet \circ (\mathsf{p}_P)^\dagger,
$$

where $(\mathsf{q}_P)_\bullet$ is the right adjoint of the functor $(\mathsf{q}_P)^\bullet$ (i.e., $(\mathsf{q}_P)_\bullet$ is the functor of usual direct image for D-modules).

6.3.5. The functor Eis_P (called the usual functor of Eisenstein series), left adjoint to CT_P, is described as

$$(\mathsf{p}_P)_\dagger \circ (\mathsf{q}_P)^\bullet.$$

The functor $(\mathsf{p}_P)_\dagger$ is the partially defined left adjoint to $(\mathsf{p}_P)^\dagger$, and as in Lemma 6.3.3 one shows that it is defined on the essential image of $(\mathsf{q}_P)^\bullet$.

6.3.6. From (6.2) we obtain that the functor CT_P can be expressed through $\mathrm{CT}_P^{\mathrm{enh}}$ as
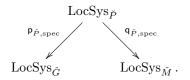
$$\mathrm{CT}_P \simeq (\imath_M)^\dagger \circ \mathrm{CT}_P^{\mathrm{enh}}.$$

Similarly, from (6.3), we obtain that the functor Eis_P can be expressed through $\mathrm{Eis}_P^{\mathrm{enh}}$ as

$$\mathrm{Eis}_P \simeq \mathrm{Eis}_P^{\mathrm{enh}} \circ (\imath_M)_\dagger.$$

6.4. **Spectral Eisenstein series.** The functors of constant term and Eisenstein series on the geometric side have their respective counterparts on the spectral side. In this subsection we will introduce the spectral counterparts of the naive functors Eis_P and CT_P; their enhanced versions will be introduced in Sect. 6.5.

6.4.1. Consider the derived stack $\mathrm{LocSys}_{\check{P}}$ and the diagram

$$
\begin{array}{ccc}
 & \mathrm{LocSys}_{\check{P}} & \\
\mathsf{p}_{\check{P},\mathrm{spec}} \swarrow & & \searrow \mathsf{q}_{\check{P},\mathrm{spec}} \\
\mathrm{LocSys}_{\check{G}} & & \mathrm{LocSys}_{\check{M}}.
\end{array}
$$

We note that the morphism $\mathsf{q}_{\check{P},\mathrm{spec}}$ is quasi-smooth (i.e., its geometric fibers are quasi-smooth), and in particular of finite Tor dimension. Hence, the functor

$$\mathsf{q}_{\check{P},\mathrm{spec}}^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}),$$

is well-defined, see Sect. 2.7.5.

We also note that the morphism $\mathsf{p}_{\check{P},\mathrm{spec}}$ is schematic and proper. Hence, the functor

$$\mathsf{p}_{\check{P},\mathrm{spec}}^{!} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}),$$

right adjoint to $(\mathsf{p}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}}$, is well-defined and is continuous, see again Sect. 2.7.5.

6.4.2. We let $\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}$ be the conical Zariski-closed subset of $\mathrm{Sing}(\mathrm{LocSys}_{\check{P}})$ that corresponds to pairs (σ, A), where σ is a $\check{P}$-local system, and A is a horizontal section of $\check{\mathfrak{p}}_\sigma^*$ that belongs to $\check{\mathfrak{m}}_\sigma^* \subset \check{\mathfrak{p}}_\sigma^*$, and is nilpotent as a section of $\check{\mathfrak{m}}_\sigma^*$.

We consider the corresponding category

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}) \subset \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}).$$

The following is shown in [AG, Propositions 13.2.6]:

Lemma 6.4.3.

(a) *The functor* $\mathsf{q}_{\check{P},\mathrm{spec}}^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$ *sends the subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}})$ *to the subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}).$

(b) *The functor* $(\mathsf{p}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}),$ *sends the subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})$ *to the subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$

6.4.4. Hence, we obtain well-defined functors

$$\mathsf{q}_{\check{P},\mathrm{spec}}^{\mathrm{IndCoh},*} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})$$

and

$$(\mathsf{p}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}),$$

that admit (continuous) right adjoints

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \leftarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}) : (\mathsf{q}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}}$$

and

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}) \leftarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) : \mathsf{p}_{\check{P},\mathrm{spec}}^!,$$

respectively.

6.4.5. We define the spectral Eisenstein series functor

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

as

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} := (\mathsf{p}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}} \circ \mathsf{q}_{\check{P},\mathrm{spec}}^{\mathrm{IndCoh},*}.$$

We introduce the spectral constant term functor

$$\mathrm{CT}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}})$$

as

$$\mathrm{CT}_{\check{P},\mathrm{spec}} := (\mathsf{q}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}} \circ \mathsf{p}_{\check{P},\mathrm{spec}}^!.$$

By construction, $\mathrm{CT}_{\check{P},\mathrm{spec}}$ is the right adjoint of $\mathrm{Eis}_{\check{P},\mathrm{spec}}$.

6.4.6. In addition to the adjoint pair

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) : \mathrm{CT}_{\check{P},\mathrm{spec}}$$

we shall also consider the corresponding adjoint pair

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} \circ \Xi_{\check{M}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) : \Psi_{\check{M}} \circ \mathrm{CT}_{\check{P},\mathrm{spec}}.$$

In a certain sense the above two adjoint pairs end up retaining the same information. More precisely, we have the following result of [AG, Corollary 13.3.10 and Theorem 13.3.6]:

Proposition 6.4.7.

(a) *The essential images of the functors*

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} \circ \Xi_{\check{M}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

for all parabolics P (including $P = G$) generate $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$

(b) *The essential images of the functors*

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} \circ \Xi_{\check{M}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

for all proper *parabolics generate the full subcategory equal to the kernel of the restriction functor*

$$\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}).$$

6.4.8. We can now formulate Property Ei ("Ei" stands for Eisenstein) of the compatibility of the geometric Langlands functor for the group G and its Levi subgroups. It is a particular case of Porperty Ei$^{\mathrm{enh}}$ formulated in Sect. 6.6.4.

Property Ei: *We shall say that the functor* $\mathbb{L}_G$ *satisfies* Porperty Ei *if the following diagram of functors commutes:*

$$(6.4) \qquad
\begin{array}{ccc}
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{M}}}(\mathrm{LocSys}_{\check{M}}) & \xrightarrow{\;\mathbb{L}_M\;} & \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \\
{\scriptstyle -\otimes \check{\mathfrak{l}}_{M,G}}\Big\downarrow & & \Big\downarrow{\scriptstyle -\otimes \mathfrak{l}_{M,G}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{M}}}(\mathrm{LocSys}_{\check{M}}) & & \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \\
{\scriptstyle \mathrm{Eis}_{\check{P},\mathrm{spec}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Eis}_P} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\;\mathbb{L}_G\;} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G)
\end{array}$$

where $-\otimes \check{\mathfrak{l}}_{M,G}$ *and* $-\otimes \mathfrak{l}_{M,G}$ *are auto-equivalences defined below.*

The functor $-\otimes \check{\mathfrak{l}}_{M,G}$ appearing in the statement of Property Ei is that given by tensor product by a line bundle $\check{\mathfrak{l}}_{M,G}$ on $\mathrm{LocSys}_{\check{M}}$ equal to the pull-back under

$$\mathrm{LocSys}_{\check{M}} \overset{2\check{\rho}_P}{\to} \mathrm{LocSys}_{\mathbb{G}_m} \to \mathrm{Bun}_{\mathbb{G}_m} = \mathrm{Pic}$$

of the line bundle on Pic corresponding to the canonical line bundle $\omega_X^{\frac{1}{2}}$ on X, and where $2\check{\rho}_P$ is the character $\check{M} \to \mathbb{G}_m$ correspinding to the determinant of the adjoint action on $\check{\mathfrak{p}}$.
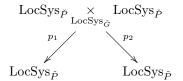
The functor $-\otimes \mathfrak{l}_{M,G}$ is given by tensor product by the (constant) D-module on Bun_M, which on the connected component $\mathrm{Bun}_M^{\check{\mu}}$ corresponding to $\check{\mu} : M \to \mathbb{G}_m$ is given by the cohomological shift by $\langle 2\rho_P, 2(g-1)\check{\rho}_P - \check{\mu}\rangle$.

6.4.9. By adjunction, Property Ei implies that the following diagram of functors commutes as well:

$$(6.5) \qquad
\begin{array}{ccc}
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{M}}}(\mathrm{LocSys}_{\check{M}}) & \xrightarrow{\;\mathbb{L}_M\;} & \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \\
{\scriptstyle -\otimes \check{\mathfrak{l}}^{-1}_{M,G}}\Big\uparrow & & \Big\uparrow{\scriptstyle -\otimes \mathfrak{l}^{-1}_{M,G}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{M}}}(\mathrm{LocSys}_{\check{M}}) & & \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \\
{\scriptstyle \mathrm{CT}_{\check{P},\mathrm{spec}}}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{CT}_P} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\;\mathbb{L}_G\;} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G)
\end{array}$$

6.5. **The spectral parabolic category.** The goal of this subsection is to define a spectral counterpart of the category $\mathrm{I}(G,P)$ and the functors $\mathrm{Eis}_P^{\mathrm{enh}}$ and $\mathrm{CT}_P^{\mathrm{enh}}$.

6.5.1. Consider the groupoid

$$\mathrm{LocSys}_{\check{P}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{P}}$$

$$p_1 \swarrow \qquad \searrow p_2$$

$$\mathrm{LocSys}_{\check{P}} \qquad\qquad \mathrm{LocSys}_{\check{P}}$$

over $\mathrm{LocSys}_{\check{P}}$.

Since the map p_i $(i = 1, 2)$ is schematic and proper, we have an adjoint pair of (continuous) functors

$$(p_i)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{P}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}) : p_i^!.$$

6.5.2. We let

$$(6.6) \qquad \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{P}})_\Delta \hookrightarrow \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{P}})$$

denote the full subcategory consisting of objects that are set-theoretically supported on the image of the diagonal embedding

$$\mathrm{LocSys}_{\check{P}} \to \mathrm{LocSys}_{\check{P}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{P}}.$$

We let $(p_{i,\Delta})_*^{\mathrm{IndCoh}}$ denote the restriction of $(p_i)_*^{\mathrm{IndCoh}}$ to the subcategory (6.6). We let $p_{i,\Delta}^!$ denote the right adjoint of $(p_{i,\Delta})_*^{\mathrm{IndCoh}}$, which is isomorphic to the composition of $p_i^!$ and the right adjoint to the embedding (6.6).

6.5.3. The structure of groupoid on $\mathrm{LocSys}_{\check{P}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{LocSys}_{\check{P}}$ endows the endo-functor

$$(p_{2,\Delta})_*^{\mathrm{IndCoh}} \circ p_{1,\Delta}^!$$

of $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$ with a structure of monad. We shall denote this monad by $\mathsf{F}_{\check{P}}$.

We have:

Lemma 6.5.4. *Let $\mathcal{N} \subset \mathrm{Sing}(\mathrm{LocSys}_{\check{P}})$ be any conical Zariski-closed subset. Then the functor $\mathsf{F}_{\check{P}}$ sends the full subcategory*

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_{\check{P}}) \subset \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$$

to itself.

(The lemma holds more generally when $\mathrm{LocSys}_{\check{G}}$ and $\mathrm{LocSys}_{\check{P}}$ are replaced by arbitrary quasi-smooth algebraic stacks.)

6.5.5. By construction, the action of the monad $\mathsf{F}_{\check{P}}$ on the category $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$ commutes with the action of the (symmetric) monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, where the latter acts on $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$ via the (symmetric) monoidal functor

$$\mathsf{p}_{\check{P},\mathrm{spec}}^* : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})$$

and the canonical action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})$ on $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$.

6.5.6. We consider the category

$$\mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}(\text{LocSys}_{\check{P}}))$$

of $\mathsf{F}_{\check{P}}$-modules in $\text{IndCoh}(\text{LocSys}_{\check{P}})$. We let

$$\mathbf{ind}_{\mathsf{F}_{\check{P}}} : \text{IndCoh}(\text{LocSys}_{\check{P}}) \rightleftarrows \mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}(\text{LocSys}_{\check{P}})) : \mathbf{oblv}_{\mathsf{F}_{\check{P}}}$$

be the corresponding adjoint pair of forgetful and induction functors.

By Lemma 6.5.4 we also have well-defined full subcategories

$$\mathsf{F}_{\check{P}}\text{-mod}(\text{QCoh}(\text{LocSys}_{\check{P}})) \subset \mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}})) \subset$$

$$\subset \mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}(\text{LocSys}_{\check{P}}))$$

and the functors

$$\mathbf{ind}_{\mathsf{F}_{\check{P}}} : \text{QCoh}(\text{LocSys}_{\check{P}}) \rightleftarrows \mathsf{F}_{\check{P}}\text{-mod}(\text{QCoh}(\text{LocSys}_{\check{P}})) : \mathbf{oblv}_{\mathsf{F}_{\check{P}}}$$

and

$$\mathbf{ind}_{\mathsf{F}_{\check{P}}} : \text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}}) \rightleftarrows \mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}})) : \mathbf{oblv}_{\mathsf{F}_{\check{P}}}$$

that commute with the corresponding fully faithful embeddings and their right adjoints, denoted $(\Xi_{\check{P}}, \Psi_{\check{P}})$, respectively.

In particular, we have the following commutative daigarms

$$
\begin{array}{ccc}
\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}}) & \xrightarrow{\mathbf{ind}_{\mathsf{F}_{\check{P}}}} & \mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}})) \\
\Xi_{\check{P}} \uparrow & & \uparrow \Xi_{\check{P}} \\
\text{QCoh}(\text{LocSys}_{\check{P}}) & \xrightarrow{\mathbf{ind}_{\mathsf{F}_{\check{P}}}} & \mathsf{F}_{\check{P}}\text{-mod}(\text{QCoh}(\text{LocSys}_{\check{P}}))
\end{array}
$$

and

$$
\begin{array}{ccc}
\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}}) & \xrightarrow{\mathbf{ind}_{\mathsf{F}_{\check{P}}}} & \mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}})) \\
\Psi_{\check{P}} \downarrow & & \downarrow \Psi_{\check{P}} \\
\text{QCoh}(\text{LocSys}_{\check{P}}) & \xrightarrow{\mathbf{ind}_{\mathsf{F}_{\check{P}}}} & \mathsf{F}_{\check{P}}\text{-mod}(\text{QCoh}(\text{LocSys}_{\check{P}})).
\end{array}
$$

Finally, it follows from Sect. 6.5.5 that the category $\mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}}))$ is naturally acted on by the monoidal category $\text{QCoh}(\text{LocSys}_{\check{G}})$, and the functors $\mathbf{ind}_{\mathsf{F}_{\check{P}}}$ and $\mathbf{oblv}_{\mathsf{F}_{\check{P}}}$ commute with this action.

6.5.7. Consider again the functors

$$(\mathsf{p}_{\check{P},\text{spec}})_*^{\text{IndCoh}} : \text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}}) \rightleftarrows \text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}}) : \mathsf{p}_{\check{P},\text{spec}}^{!}.$$

It follows from the definitions that the functor $\mathsf{p}_{\check{P},\text{spec}}^{!}$ canonically factors as a composition

$$\text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}}) \to \mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}})) \xrightarrow{\mathbf{oblv}_{\mathsf{F}_{\check{P}}}}$$

$$\to \text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}}).$$

We denote the resulting functor

$$\text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}}) \to \mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}}))$$

by $\mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}$.

6.5.8. It follows formally from the Barr-Beck-Lurie theorem (see [GL:DG, Proposition 3.1.1]) that there exists a canonically defined functor, to be denoted $\mathrm{Eis}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}$,

$$\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

equipped with an isomorphism

$$\mathrm{Eis}_{\check{P},\mathrm{spec}}^{\mathrm{enh}} \circ \mathbf{ind}_{\mathsf{F}_{\check{P}}} \simeq (\mathsf{p}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}}.$$

Furthermore, the functor $\mathrm{Eis}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}$ is the *left* adjoint of $\mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}$.

6.5.9. By construction, the functors $\mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}$ and $\mathrm{Eis}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}$ intertwine the monoidal actions of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ on $\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}))$ and $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$, respectively.

6.5.10. We proclaim the category $\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}))$, equipped with the adjoint functors

$$\mathrm{Eis}_{\check{P},\mathrm{spec}}^{\mathrm{enh}} : \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) : \mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}$$

to be the spectral counterpart of the category $\mathrm{I}(G,P)$ equipped with the adjoint functors

$$\mathrm{Eis}^{\mathrm{enh}} : \mathrm{I}(G,P) \rightleftarrows \mathrm{D\text{-}mod}(\mathrm{Bun}_G) : \mathrm{CT}_P^{\mathrm{enh}}.$$

6.6. **Compatibility of Langlands correspondence with parabolic induction.** For the duration of this subsection we will assume the validity of Conjecture 3.4.2 for the reductive group M. In particular, this is unconditional for $P = B$, in which case M is a torus, and Conjecture 3.4.2 amounts to Fourier-Mukai transform.

The key observation is that although the categories

$$\mathrm{I}(G,P) \text{ and } \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}))$$

cannot be recovered purely in terms of the reductive group M (i.e., we need to know how it is realized as a Levi of G), this additional G-information is manageable, and so we can relate these categories by just knowing Langlands correspondence for M.

6.6.1. We have:

Quasi-Theorem 6.6.2.
(a) *There exists a canonically defined equivalence of categories*

$$\mathbb{L}_P : \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) \to \mathrm{I}(G,P)$$

that makes the following diagram commute:

$$(6.7) \quad \begin{array}{ccc} \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) & \xrightarrow{\mathbb{L}_M} & \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \\ {\scriptstyle -\otimes\check{\mathfrak{l}}_{M,G}}\downarrow & & \downarrow{\scriptstyle -\otimes\mathfrak{l}_{M,G}} \\ \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) & & \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \\ {\scriptstyle \mathbf{ind}_{\mathsf{F}_{\check{P}}}\circ(\mathsf{q}_{\check{P},\mathrm{spec}})^*}\downarrow & & \downarrow{\scriptstyle (\iota_M)_\dagger} \\ \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) & \xrightarrow{\mathbb{L}_P} & \mathrm{I}(G,P), \end{array}$$

where $- \otimes \check{\mathfrak{l}}_{M,G}$ *and* $- \otimes \mathfrak{l}_{M,G}$ *are the auto-equivalences defined in Sect. 6.4.8.*

(b) *The equivalence* $\mathbb{L}_P$ *is compatible with the action of the category* $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$, *where*

- $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ *acts on* $\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}))$ *via the symmetric monoidal functor*
$$\mathrm{Loc}_{\check{G},\mathrm{spec}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}});$$

- $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ *acts on* $\mathrm{I}(G,P)$ *as in Sect. 6.1.7.*

6.6.3. In the case of $P = B$, Quasi-Theorem 6.6.2 is work-in-progress by S. Raskin. The idea of the proof, applicable to any P, is the following:

The composite functors

(6.8) $$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \to \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}))$$

and

(6.9) $$\mathrm{D\text{-}mod}(\mathrm{Bun}_M) \to \mathrm{I}(G,P)$$

appearing in (6.7) admit continuous and consertavative right adjoints, which, up to twists by line bundles, are given by

$$(\mathsf{q}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}} \circ \mathbf{oblv}_{\mathsf{F}(P)} \text{ and } \iota_M^\dagger,$$

respectively. Hence, by the Barr-Beck-Lurie theorem, the statement of Quasi-Theorem 6.6.2 amounts to comparing the monads corresponding to the composition of the functors in (6.8) and (6.9), and their respective right adjoints.

One shows that the monad on the geometric side, i.e., $\mathrm{D\text{-}mod}(\mathrm{Bun}_M)$, is given by the action of an algebra object in the monoidal category $\mathrm{D\text{-}mod}(\mathrm{Hecke}(M)_{\mathrm{Ran}(X)})$ that comes via the functor $\mathrm{Sat}(G)_{\mathrm{Ran}(X)}$ from a canonically defined algebra object of the monoidal category $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G},\mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}})$, see Sect. 4.7.2. [17]

One then uses Bezrukavnikov's theory of [Bez] that describes various categories of D-modules on the affine Grassmannian in terms of the Langlands dual group to match the resulting monad with one appearing on the spectral side.

6.6.4. We can now state Property Ei$^{\mathrm{enh}}$ of the geometric Langlands functor $\mathbb{L}_G$ in Conjecture 3.4.2.

Property Ei$^{\mathrm{enh}}$: *We shall say that the functor* $\mathbb{L}_G$ *satisfies* Property Ei$^{\mathrm{enh}}$ *if the following diagram of functors commutes:*

(6.10)
$$\begin{array}{ccc}
\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) & \xrightarrow{\mathbb{L}_P} & \mathrm{I}(G,P) \\
{\scriptstyle \mathrm{Eis}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}} \downarrow & & \downarrow {\scriptstyle \mathrm{Eis}_P^{\mathrm{enh}}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\mathbb{L}_G} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G).
\end{array}$$

---

[17]We emphasize that the above algebra object of $\mathrm{IndCoh}(\mathrm{Hecke}(\check{G},\mathrm{spec})_{\mathrm{Ran}(X)}^{\mathrm{loc}})$ *does not* come from an algebra object of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ via the functor $\to$ in (4.6), so here one really needs to use the full derived Satake equivalence for the group M.

6.6.5. Note that by passing to right adjoint functors in (6.10) we obtain the following commutative diagram

$$(6.11) \qquad \begin{array}{ccc} \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) & \xrightarrow{\mathbb{L}_P} & \mathrm{I}(G,P) \\ {\scriptstyle\mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle\mathrm{CT}_P^{\mathrm{enh}}} \\ \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\mathbb{L}_G} & \text{D-mod}(\mathrm{Bun}_G). \end{array}$$

6.6.6. Finally, we note that Property Ei stated in Sect. 6.4.8 is a formal consequence of Property Ei$^{\mathrm{enh}}$: the commutative diagram (6.4) is obtained by concatenating (6.10) and (6.7).

6.7. Eisenstein and constant term compatibility.
Let now P and P' be two parabolic subgroups, and let us assume the validity of Conjecture 3.4.2 for the Levi quotient M' as well.

6.7.1. By concatenating diagrams (6.10) (for P') and (6.11) (for P) we obtain the following commutative diagram:

$$(6.12) \qquad \begin{array}{ccc} \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) & \xrightarrow{\mathbb{L}_P} & \mathrm{I}(G,P) \\ {\scriptstyle\mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle\mathrm{CT}_P^{\mathrm{enh}}} \\ \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & & \text{D-mod}(\mathrm{Bun}_G) \\ {\scriptstyle\mathrm{Eis}_{\check{P}',\mathrm{spec}}^{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle\mathrm{Eis}_{P'}^{\mathrm{enh}}} \\ \mathsf{F}_{\check{P}'}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob},\check{P}'}}(\mathrm{LocSys}_{\check{P}'})) & \xrightarrow{\mathbb{L}_{P'}} & \mathrm{I}(G,P') \end{array}$$

We have (again, assuming the validity of Conjecture 3.4.2 for M and M'):

Quasi-Theorem 6.7.2. *The diagram* (6.12) *commutes unconditionally (i.e., without assuming the validity of Conjecture 3.4.2 for G).*

Remark 6.7.3. One shows that both functors in (6.12) corresponding to the vertical arrows admit natural filtrations indexed by the poset

$$W_M \backslash W / W_{M'},$$

where W is the Weyl group of G, and W_M and $W_{M'}$ are the Weyl groups of M and M', respectively.

In order to prove Quasi-Theorem 6.7.2, one needs to identify the corresponding subquotients on both sides (via the equivalence of Quasi-Theorem 6.6.2 for M and M'), and then show that these subquotients glue in the same way on both sides.

For $G = GL_2$ (and when $M = M' = T$) the first step follows easily from Quasi-Theorem 6.6.2, and the second step is an explicit calculation of a class in an appropriate Ext^1 group.

7. The degenerate Whittaker model

This section develops a variant of the category $\mathrm{Whit}(G,G)$, denoted $\mathrm{Whit}(G,P)$ (for a fixed parabolic P), where we impose an equivariance condition with respect to a character of N, which is no longer non-degenerate, but is trivial on $N(P)$ and non-degenerate on $N(M)$, where $N(M) = N \cap M$ is the unipotent radical of the Borel subgroup of M.

The reason that $\mathrm{Whit}(G,P)$ is necessary to consider is that these categories, when P runs through the poset of standard parabolics, comprise the extended Whittaker category introduced in the next section, and which will be of central importance for the proof of Conjecture 3.4.2.

That said, we should remark that the present section does not contain any substantially new ideas. Furthermore, the material discussed here is relevant only for groups of semi-simple rank > 1, because the case $P = G$ is covered by Sect. 5, and in the case $P = B$ we have $\mathrm{Whit}(G,P) = \mathrm{I}(G,B)$. So the reader may prefer to skip this section on the first pass.

7.1. **Degenerate Whittaker categories.** The degenerate Whittaker category $\mathrm{Whit}(G,P)$ defined in this subsection is the geometric counterpart of the space of functions on the double quotient

$$Z_M^0(K)\backslash G(\mathbb{A})/G(\mathbb{O})$$

that are equivariant with respect to $N(\mathbb{A})$ against a character that factors via the surjection $N(\mathbb{A}) \to N(M)(\mathbb{A})$ and a non-degenerate character of $N(M)(\mathbb{A})$, trivial on $N(M)(K)$.

7.1.1. We define the prestack $\mathcal{Q}_{G,P}$ in a way similar to $\mathcal{Q}_{G,G}$. It classifies the data of

$$(\mathcal{P}_G, U, \alpha, \gamma),$$

where $(\mathcal{P}_G, U, \alpha)$ has the same meaning as for $\mathcal{Q}_{G,G}$ (i.e., it defines a point of $\mathrm{Bun}_G^{B\text{-gen}}$), but γ is now an identification of bundles with respect to the torus T/Z_M^0, one bundle being induced from $\mathcal{P}_{T,U}$, and the other from $\check{\rho}(\omega_X)$.

Equivalently, $\mathcal{Q}_{G,P}$ is the quotient of $\mathcal{Q}_Q$ by the action of $\mathbf{Maps}(X, Z_M^0)^{\mathrm{gen}}$.

(Note that when G has a connected center, the data of γ amounts to an isomorphism $\alpha_i(\mathcal{P}_{T,U}) \simeq \omega_X$ for every simple root α_i of M.)

A choice of a generic trivialization of $\omega_X^{\frac{1}{2}}$ identifies the groupoid on k-points of $\mathcal{Q}_{G,P}$ with the double quotient

$$Z_M^0(K) \cdot N(K)\backslash G(\mathbb{A})/G(\mathbb{O}).$$

By construction, if $P = G$, we have $\mathcal{Q}_{G,P} = \mathcal{Q}_{G,G}$ (so the notation is consistent). When $P = B$, we have $\mathcal{Q}_{G,P} = \mathrm{Bun}_G^{B\text{-gen}}$.

We let $\mathfrak{r}_{G,P}$ denote the forgetful map $\mathcal{Q}_{G,P} \to \mathrm{Bun}_G$.

7.1.2. The groupoid $\mathbf{N}$ acting $\left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}$ gives rise to a groupoid that we denote $\mathbf{N}_{\mathcal{Q}_{G,P}}$ over $(\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}$ so that the diagram

$$
\begin{array}{ccccc}
(\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}} & \xleftarrow{\ p_1\ } & \mathbf{N}_{\mathcal{Q}_{G,P}} & \xrightarrow{\ p_2\ } & (\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}} \\
\downarrow & & \downarrow & & \downarrow \\
\left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}} & \xleftarrow{\ p_1\ } & \mathbf{N} & \xrightarrow{\ p_2\ } & \left(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}}
\end{array}
$$

is Cartesian.

The groupoid $\mathbf{N}_{\mathcal{Q}_{G,P}}$ is endowed with a canonically defined character that we denote χ_P. The definition of χ_P is similar to that of χ with the difference that we only use the simple roots that lie in M.

7.1.3. We consider the twisted $\mathbf{N}_{\mathcal{Q}_{G,P}}$-equivariant category of D-mod $((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}})$, denoted D-mod $((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}_{G,P}}, \chi_P}$.

As in Proposition 5.6.2, the forgetful functor

$$\mathrm{D\text{-}mod}\left((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}\right)^{\mathbf{N}_{\mathcal{Q}_{G,P}}, \chi_P} \to \mathrm{D\text{-}mod}\left((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}\right)$$

is fully faithful.

7.1.4. We define the *degenerate Whittaker category* $\mathrm{Whit}(G, P)$ to be the full subcategory of D-mod$(\mathcal{Q}_{G,P})$ equal to the preimage of

$$\mathrm{D\text{-}mod}\left((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}\right)^{\mathbf{N}_{\mathcal{Q}_{G,P}}, \chi_P} \subset \mathrm{D\text{-}mod}\left((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}\right)$$

under the pull-back functor

$$\mathrm{D\text{-}mod}(\mathcal{Q}_{G,P}) \to \mathrm{D\text{-}mod}\left((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}\right).$$

In other words,

$$\mathrm{Whit}(G, P) :=$$
$$= \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P}) \underset{\mathrm{D\text{-}mod}((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}})}{\times} \mathrm{D\text{-}mod}\left((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}\right)^{\mathbf{N}_{\mathcal{Q}_{G,P}}, \chi_P}.$$

Note that for $P = G$ we recover the category $\mathrm{Whit}(G, G)$; for $P = B$, we recover the category $\mathrm{I}(G, B)$.

7.1.5. As in the case of $\mathrm{Whit}(G, G)$, the (fully faithful) forgetful functor

$$\mathrm{Whit}(G, P) \to \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})$$

admits a right adjoint that we denote $\mathrm{Av}^{\mathbf{N}, \chi_P}$.

As in the case of $\mathrm{Whit}(G, G)$, we have a canonical action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on D-mod$(\mathcal{Q}_{G,P})$, and this action preserves the full subcategory

$$\mathrm{Whit}(G, P) \subset \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P}).$$

Furthermore, the functor $\mathrm{Av}^{\mathbf{N}, \chi_P}$ commutes with the $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$-action.

7.1.6. We define the functor of *degenerate Whittaker coefficient*

$$\mathrm{coeff}_{G,P} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G, P)$$

by

$$\mathrm{coeff}_{G,P} := \mathrm{Av}^{\mathbf{N}, \chi_P} \circ (\mathfrak{r}_{G,P})^{\dagger}.$$

7.2. **Relation between constant term and degenerate Whittaker coefficient functors.** In this subsection we will show how to express the functor $\mathrm{coeff}_{G,P}$, introduced above, via the functor of enhanced constant term $\mathrm{CT}_P^{\mathrm{enh}}$, introduced in the previous section.

7.2.1. Note that we have a naturally defined forgetful map

$$\mathfrak{r}_{P,M} : \mathcal{Q}_{G,P} \to \mathrm{Bun}_G^{P\text{-gen}},$$

so that

$$\mathsf{p}_P^{\mathrm{enh}} \circ \mathfrak{r}_{P,M} = \mathfrak{r}_{G,P}.$$

In addition to the groupoid $\mathbf{N}_{\mathcal{Q}_{G,P}}$ over $(\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}$, there exists a canonically defined groupoid $\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}$ that fits into a Cartesian diagram

$$
\begin{array}{ccccc}
(\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}} & \xleftarrow{\ p_1\ } & \mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}} & \xrightarrow{\ p_2\ } & (\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}} \\
{\scriptstyle \mathfrak{r}_{G,P}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathfrak{r}_{G,P}} \\
\left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}} & \xleftarrow{\ p_1\ } & \mathbf{N}(\mathbf{P}) & \xrightarrow{\ p_2\ } & \left(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Ran}(X)\right)_{\mathrm{good}},
\end{array}
$$

7.2.2. By a slight abuse of notation, let us denote by

$$\mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}} \subset \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})$$

the full subcategory, defined in the same way as $\mathrm{Whit}(G,P)$, when instead of the groupoid $\mathbf{N}_{\mathcal{Q}_{G,P}}$, we use $\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}$. We shall denote by the

$$\mathrm{Av}^{\mathbf{N}(\mathbf{P})} : \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P}) \to \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}}$$

the right adjoint to the embedding.

Remark 7.2.3. The category $\mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}}$ is the geometric counterpart of the space of functions on

$$Z_M^0(K) \cdot N(P)(\mathbb{A}) \backslash G(\mathbb{A}) / G(\mathbb{O}).$$

7.2.4. We have a commutative diagram of functors

$$
\text{(7.1)} \qquad
\begin{array}{ccc}
\mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}} & \longrightarrow & \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P}) \\
\uparrow & & {\scriptstyle (\mathfrak{r}_{G,P})^{\dagger}}\uparrow \\
\mathrm{I}(G,P) & \longrightarrow & \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}).
\end{array}
$$

By a slight abuse of notation, we shall denote the resulting functor

$$\mathrm{I}(G,P) \to \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}}$$

by $(\mathfrak{r}_{G,P})^{\dagger}$.

Lemma 7.2.5. *The functor* $(\mathfrak{r}_{G,P})^{\dagger} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}) \to \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})$ *is fully faithful.*

Proof. Follows from the homological contractibility of $\mathbf{Maps}(X, M/Z_M^0)^{\mathrm{gen}}$. $\qquad\square$

Hence, we obtain that the above functor

$$(\mathfrak{r}_{G,P})^{\dagger} : \mathrm{I}(G,P) \to \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}}$$

is also fully faithful.

Finally, we note that the diagram

$$
\text{(7.2)} \qquad
\begin{array}{ccc}
\mathrm{D\text{-}mod}(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}} & \xleftarrow{\ \mathrm{Av}^{\mathbf{N}(\mathbf{P})}\ } & \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P}) \\
\uparrow & & {\scriptstyle (\mathfrak{r}_{G,P})^{\dagger}}\uparrow \\
\mathrm{I}(G,P) & \xleftarrow{\ \mathrm{Av}^{\mathbf{N}(\mathbf{P})}\ } & \mathrm{D\text{-}mod}(\mathrm{Bun}_G^{P\text{-gen}}),
\end{array}
$$

obtained from (7.1) by passing to right adjoints along the horizontal arrows, is also commutative.

7.2.6. There is a canonical map of groupoids $\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}} \to \mathbf{N}_{\mathcal{Q}_{G,P}}$, and the restriction of the character χ_P under this map is trivial. Hence, we obtain an inclusion of full subcategories of D-mod$(\mathcal{Q}_{G,P})$:

$$\mathrm{Whit}(G,P) \hookrightarrow \text{D-mod}(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}}.$$

This inclusion admits a right adjoint obtained by restricting the functor $\mathrm{Av}^{\mathbf{N},\chi_P}$ to D-mod$(\mathcal{Q}_{G,P})^{\mathbf{N}(\mathbf{P})_{\mathcal{Q}_{G,P}}}$.

7.2.7. Hence, we obtain a functor

$$\mathrm{coeff}_{P,M} : \mathrm{I}(G,P) \to \mathrm{Whit}(G,P),$$

defined as

$$\mathrm{coeff}_{P,M} := \mathrm{Av}^{\mathbf{N},\chi_P} \circ (\mathfrak{r}_{G,P})^\dagger.$$

By construction, the functor $\mathrm{coeff}_{P,M}$ respects the action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$.

Remark 7.2.8. The functor $\mathrm{coeff}_{P,M}$ is *not* fully faithful. However, it follows from Quasi-Theorem 7.4.2 formulated below that its retsriction to the full subcategory

$$\mathrm{I}(G,P)_{\mathrm{temp}} \subset \mathrm{I}(G,P)$$

is fully faithful, where $\mathrm{I}(G,P)_{\mathrm{temp}}$ is defined via the pull-back square

$$
\begin{array}{ccc}
\mathrm{I}(G,P)_{\mathrm{temp}} & \longrightarrow & \mathrm{I}(G,P) \\
\downarrow & & \downarrow {\scriptstyle (\iota_M)^\dagger} \\
\text{D-mod}(\mathrm{Bun}_M)_{\mathrm{temp}} & \longrightarrow & \text{D-mod}(\mathrm{Bun}_M)
\end{array}
$$

Remark 7.2.9. The analog of the functor $\mathrm{coeff}_{P,M}$ at the level of functions takes a function on

$$M(K) \cdot N(P)(\mathbb{A}) \backslash G(\mathbb{A}) / G(\mathbb{O})$$

and averages it on the left with respect to $N(M)(\mathbb{A})/N(M)(K)$ against the character χ.

7.2.10. From (7.2) we obtain that there exists a canonical isomorphism of functors

$$\text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G,P),$$

namely

(7.3) $$\mathrm{coeff}_{G,P} \simeq \mathrm{coeff}_{P,M} \circ \mathrm{CT}_P^{\mathrm{enh}}.$$

An intuitive picture behind the functor $\mathrm{coeff}_{G,P}$ will be suggested in Remark 7.3.6.

7.3. **A strata-wise description.**

7.3.1. Set

$$\mathcal{Q}_{P,M} := \mathcal{Q}_{G,P} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_P,$$

where the map $\mathrm{Bun}_P \to \mathrm{Bun}_G^{P\text{-gen}}$ is $\imath_P$. Denote the resulting map

$$\mathcal{Q}_{P,M} \to \mathrm{Bun}_P$$

by $'\mathfrak{r}_{P,M}$, and the map $\mathcal{Q}_{P,M} \to \mathcal{Q}_{G,P}$ by $'\imath_P$. I.e., we have a Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{Q}_{G,P} & \xleftarrow{\ '\imath_P\ } & \mathcal{Q}_{P,M} \\
{\scriptstyle \mathfrak{r}_{P,M}}\Big\downarrow & & \Big\downarrow{\scriptstyle '\mathfrak{r}_{P,M}} \\
\mathrm{Bun}_G^{P\text{-gen}} & \xleftarrow{\ \imath_P\ } & \mathrm{Bun}_P .
\end{array}
$$

We have:

Lemma 7.3.2. *There exists a canonically defined Cartesian square:*

$$
\begin{array}{ccc}
\mathcal{Q}_{P,M} & \xrightarrow{\ '\mathsf{q}_M\ } & \mathcal{Q}_{M,M} \\
{\scriptstyle '\mathfrak{r}_{P,M}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{r}_{M,M}} \\
\mathrm{Bun}_P & \xrightarrow{\ \mathsf{q}_M\ } & \mathrm{Bun}_M
\end{array}
$$

7.3.3. Consider the stack Bun_P. The groupoid $\mathbf{N}(\mathbf{P})$ gives rise to a groupoid $\mathbf{N}(\mathbf{P})_{\mathrm{Bun}_P}$ acting on $(\mathrm{Bun}_P \times \mathrm{Ran}(X))_{\mathrm{good}}$. Consider the corresponding full subcategory

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_P)^{\mathbf{N}(\mathbf{P})_{\mathrm{Bun}_P}} \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_P).$$

The groupoid $\mathbf{N}$ gives rise to a groupoid $\mathbf{N}_{\mathcal{Q}_{P,M}}$ acting on $(\mathcal{Q}_{P,M} \times \mathrm{Ran}(X))_{\mathrm{good}}$. We let

$$\mathrm{D\text{-}mod}(\mathcal{Q}_{P,M})^{\mathbf{N}_{\mathcal{Q}_{P,M}},\chi_P} \subset \mathrm{D\text{-}mod}(\mathcal{Q}_{P,M})$$

denote the resulting full subcategory.

Consider again the map

$$'\mathsf{q}_M : \mathcal{Q}_{P,M} \to \mathcal{Q}_{M,M}.$$

This map is smooth and has contractible fibers, and we consider the corresponding fully faithful functor

$$('\mathsf{q}_M)^\bullet : \mathrm{D\text{-}mod}(\mathcal{Q}_{M,M}) \to \mathrm{D\text{-}mod}(\mathcal{Q}_{P,M}).$$

We have:

Lemma 7.3.4.

(a) *The functor* $(\mathsf{q}_M)^\bullet$ *defines an equivalence*

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_M) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_P)^{\mathbf{N}(\mathbf{P})_{\mathrm{Bun}_P}}.$$

(b) *The functor* $('\mathsf{q}_M)^\bullet$ *defines an equivalence*

$$\mathrm{Whit}(M,M) \to \mathrm{D\text{-}mod}(\mathcal{Q}_{P,M})^{\mathbf{N}_{\mathcal{Q}_{P,M}},\chi_P}.$$

7.3.5. The functor
$$('\imath_P)^\dagger : \text{D-mod}(\mathcal{Q}_{G,P}) \to \text{D-mod}(\mathcal{Q}_{P,M})$$
gives rise to a (conservative) functor
$$\text{Whit}(G,P) \to \text{D-mod}(\mathcal{Q}_{P,M})^{\mathbf{N}_{\mathcal{Q}_{P,M}},\chi_P}.$$

We denote the resulting functor
$$\text{Whit}(G,P) \to \text{D-mod}(\mathcal{Q}_{P,M})^{\mathbf{N}_{\mathcal{Q}_{P,M}},\chi_P} \simeq \text{Whit}(M,M)$$
by $('\imath_M)^\dagger$.

We have a canonical isomorphism of functors
$$(7.4) \qquad\qquad ('\imath_M)^\dagger \circ \text{coeff}_{G,P} \simeq \text{coeff}_{M,M} \circ \text{CT}_P .$$

Remark 7.3.6. From (7.4) we obtain the following way of thinking about the functor $\text{coeff}_{G,P}$:

In the same way as the functor CT_P^{enh} captures more information than the usual functor CT_P, the functor $\text{coeff}_{G,P}$ captures more information than the composition
$$\text{coeff}_{M,M} \circ \text{CT}_P : \text{D-mod}(\text{Bun}_G) \to \text{Whit}(M,M).$$

7.4. **Spectral description of the degenerate Whittaker category.**

7.4.1. We have the following assertion:

Quasi-Theorem 7.4.2. *There exists a canonically defined fully faithful functor*
$$\mathbb{L}_{G,P}^{\text{Whit}} : \mathsf{F}_{\check{P}}\text{-mod}(\text{QCoh}(\text{LocSys}_{\check{P}})) \to \text{Whit}(G,P),$$
compatible with the actions of the monoidal category $\text{Rep}(\check{G})_{\text{Ran}(X)}$.

We note that Quasi-Theorem 7.4.2 *does not* assume Conjecture 3.4.2 for M; in particular it includes the case of $P = M = G$.

Note that for $P = G$, the corresponding functor $\mathbb{L}_{G,P}^{\text{Whit}}$ is the functor that we had earlier denoted $\mathbb{L}_{G,G}^{\text{Whit}}$, and it is fully faithful by Quasi-Theorem 5.9.2(b).

Note also that in the other extreme case, namely when $P = B$, the assertion of Quasi-Theorem 7.4.2 coincides with that of Quasi-Theorem 6.6.2.

7.4.3. The proof of Quasi-Theorem 7.4.2 is parallel but simpler than that of Quasi-Theorem 6.6.2.

Namely, we embed both sides into the corresponding local categories (i.e., ones living over $\text{Ran}(X)$) and use Bezrukavnikov's theory to relate the resulting category of D-modules on the affine Grassmannian to the Langlands dual group.

7.4.4. From now on, until the end of this subsection, we will assume that Conjecture 3.4.2 holds for M, and will relate Quasi-Theorem 7.4.2 to Quasi-Theorem 6.6.2.

The following assertion comes along with the proof:

Proposition 7.4.5. *We have a commutative diagram of functors:*

$$
\begin{array}{ccc}
\mathsf{F}_{\check{P}}\text{-mod}(\text{QCoh}(\text{LocSys}_{\check{P}})) & \xrightarrow{\;\mathbb{L}_{G,P}^{\text{Whit}}\;} & \text{Whit}(G,P) \\
{\scriptstyle\Psi_{\check{P}}}\big\uparrow & & \big\uparrow{\scriptstyle\text{coeff}_{P,M}} \\
\mathsf{F}_{\check{P}}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\check{P}}^{\text{glob}}}(\text{LocSys}_{\check{P}})) & \xrightarrow{\;\mathbb{L}_P\;} & \text{I}(G,P).
\end{array}
$$

7.4.6. We can now formulate the following property of the geometric Langlands functor $\mathbb{L}_G$ that contains Property Wh as a particular case for $P = G$:

Property Wh$^{\mathrm{deg}}$: *We shall say that the functor $\mathbb{L}_G$ satisfies* Property Wh$^{\mathrm{deg}}$ *for the parabolic P if the following diagram is commutative:*

$$(7.5)$$

$$
\begin{array}{ccc}
\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})) & \xrightarrow{\;\mathbb{L}_{G,P}^{\mathrm{Whit}}\;} & \mathrm{Whit}(G,P) \\[6pt]
\Psi_{\check{P}}\Big\uparrow & & \\[6pt]
\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) & & \Big\uparrow{\mathrm{coeff}_{G,P}} \\[6pt]
\mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}\Big\uparrow & & \\[6pt]
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\;\mathbb{L}_G\;} & \text{D-mod}(\mathrm{Bun}_G).
\end{array}
$$

Note, however, that Property Wh$^{\mathrm{deg}}$ is a formal consequence of Property Ei$^{\mathrm{enh}}$ and Proposition 7.4.5.

7.5. (Degenerate) Whittaker coefficients and Eisenstein series. Let $P' \subset G$ be another parabolic. In this subsection we will assume that Conjecture 3.4.2 holds for its Levi quotient M'. However, we will *not* be assuming that Conjecture 3.4.2 holds for M.

7.5.1. By concatenating the commutative diagrams (7.5) and (6.11) we obtain a commutative diagram

$$(7.6)$$

$$
\begin{array}{ccc}
\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})) & \xrightarrow{\;\mathbb{L}_{G,P}^{\mathrm{Whit}}\;} & \mathrm{Whit}(G,P) \\[6pt]
\Psi_{\check{P}}\Big\uparrow & & \\[6pt]
\mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) & & \Big\uparrow{\mathrm{coeff}_{G,P}} \\[6pt]
\mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}\Big\uparrow & & \\[6pt]
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & & \text{D-mod}(\mathrm{Bun}_G) \\[6pt]
\mathrm{Eis}_{\check{P}',\mathrm{spec}}^{\mathrm{enh}}\Big\uparrow & & \Big\uparrow{\mathrm{Eis}_{P'}^{\mathrm{enh}}} \\[6pt]
\mathsf{F}_{\check{P}'}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob},\check{P}'}}(\mathrm{LocSys}_{\check{P}'})) & \xrightarrow{\;\mathbb{L}_{P'}\;} & \mathrm{I}(G,P')
\end{array}
$$

7.5.2. We have:

Quasi-Theorem 7.5.3. *The diagram* (7.6) *commutes unconditionally (i.e., without assuming the validity of Conjecture 3.4.2 for G).*

Remark 7.5.4. Note that if we do assume that Conjecture 3.4.2 holds for M, then in this case the assertion of Quasi-Theorem 7.5.3 follows from Quasi-Theorem 6.7.2 and Proposition 7.4.5.

7.5.5. Finally, we remark that for $P = G$, the assertion of Quasi-Theorem 7.5.3 is built in the proof of Quasi-Theorem 7.4.2 (for P').

8. The extended Whittaker model

In this section we will introduce a crucial player for our approach to the geometric Langlands conjecture, the extended Whittaker category, denoted $\text{Whit}^{\text{ext}}(G, G)$. The idea is that, on the one hand, according to Conjecture 8.2.9, discussed below, the category $\text{Whit}^{\text{ext}}(G, G)$ receives a fully faithful functor from the automorphic category D-mod(Bun_G), and on the other hand, it can be related to the spectral side. How the latter is done will be the subject of Sect. 9.

8.1. The variety of characters.
When defining degenerate Whittaker categories, we had to consider characters of the group N, whose degeneracies varied with the parabolic. In this subsection we will combine all these categories into one family.

8.1.1. Let $\mathfrak{t}_{\text{adj}}$ denote the Lie algebra of the torus T/Z_G^0. In this subsection we will introduce a certain toric variety $\mathfrak{ch}(G)$ endowed with a finite map

$$(8.1) \qquad \mathfrak{ch}(G) \to \mathfrak{t}_{\text{adj}}.$$

The map (8.1) will be an isomorphism when G has a connected center.

8.1.2. Let Λ denote the weight lattice of G; let

$$\Lambda^{\text{pos}} \subset \Lambda \text{ and } \Lambda^{\text{pos},\mathbb{Q}} \subset \Lambda^{\mathbb{Q}} := \Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

be the sub-monoids of weights that can be expressed as integral (resp., rational) non-negative combinations of simple roots. Let $\Lambda^{\text{pos},\text{sat}_G}$ be the saturation of Λ^{pos}, i.e.,

$$\Lambda^{\text{pos},\text{sat}_G} := \Lambda \cap \Lambda^{\text{pos},\mathbb{Q}}.$$

Note that the inclusion $\Lambda^{\text{pos}} \hookrightarrow \Lambda^{\text{pos},\text{sat}_G}$ is an equality if G has a connected center.

8.1.3. We define

$$\mathfrak{ch}(G) := \text{Spec}(k[\Lambda^{\text{pos},\text{sat}_G}]).$$

I.e., $\mathfrak{ch}(G)$ classifies maps of monoids $\Lambda^{\text{pos},\text{sat}_G} \to \mathbb{A}^1$, where $\mathbb{A}^1$ is a monoid with respect to the operation of multiplication.

The group T/Z_G^0, which can be thought of that classifying maps of monoids $\Lambda^{\text{pos},\text{sat}_G} \to \mathbb{G}_m$, acts on $\mathfrak{ch}(G)$.

Let $\overset{\circ}{\mathfrak{ch}}(G) \subset \mathfrak{ch}(G)$ be the open subscheme corresponding to maps $\Lambda^{\text{pos},\text{sat}_G} \to (\mathbb{A}^1 - 0) = \mathbb{G}_m$. It is clear that the action of T/Z_G^0 on $\overset{\circ}{\mathfrak{ch}}(G)$ is simply transitive.

8.1.4. Let $P \subset G$ be a parabolic, with Levi quotient M. Consider the closed subscheme of $\mathfrak{ch}(G)$ that corresponds to maps $\Lambda^{\text{pos},\text{sat}_G} \to \mathbb{A}^1$ that vanish on any element μ with

$$\mu \in \Lambda^{\text{pos},\text{sat}_G} - \Lambda^{\text{pos},\text{sat}_M}.$$

It is easy to see that this subscheme identifies with the corresponding scheme $\mathfrak{ch}(M)$, in a way compatible with the actions of

$$T/Z_G^0 \twoheadrightarrow T/Z_M^0.$$

Furthermore, it is clear that $\mathfrak{ch}(G)$ decomposes as a union of locally closed subschemes

$$\mathfrak{ch}(G) \simeq \underset{P}{\sqcup} \overset{\circ}{\mathfrak{ch}}(M).$$

8.2. The extended Whittaker category.
In this subsection we will finally define the extended Whittaker category $\text{Whit}^{\text{ext}}(G, G)$. The definition will follow the same pattern as in the case of $\text{Whit}(G)$, $\text{I}(G, P)$ and $\text{Whit}(G, P)$.

8.2.1. We define the prestack $\mathfrak{Q}_{G,G}^{\mathrm{ext}}$ as follows. The definition repeats that of $\mathfrak{Q}_{G,G}$ with the following difference: when considering quadruples $(\mathcal{P}_G, U, \alpha, \gamma)$, we let γ be a section over U of the scheme $\mathfrak{ch}(G)_{\check{\rho}(\omega_X)|_U \otimes \mathcal{P}_T^{-1}}$.

In other words, the datum of γ assigns to every $\mu \in \Lambda^{\mathrm{pos, sat}_G}$ a map of line bundles over U:

$$(8.2) \qquad\qquad \gamma(\mu) : \mu(\mathcal{P}_T) \to (\omega_X^{\frac{1}{2}})^{\otimes \langle \mu, 2\check{\rho} \rangle}|_U.$$

(Note that when G has a connected center, the datum of γ amounts to a map $\alpha_i(\mathcal{P}_T) \to \omega_X|_U$ for every simple root α_i of G.)

We let $\mathfrak{r}_{G,G}^{\mathrm{ext}}$ denote the forgetful map $\mathfrak{Q}_{G,G}^{\mathrm{ext}} \to \mathrm{Bun}_G$.

8.2.2. The groupoid of k-points of $\mathfrak{Q}_{G,G}^{\mathrm{ext}}$ identifies with the quotient

$$T(K) \backslash \Big(N(K) \backslash G(\mathbb{A}) / G(\mathbb{O}) \times \mathfrak{ch}(G)(K) \Big),$$

where T acts on $\mathfrak{ch}$ via the projection $T \twoheadrightarrow T/Z_G^0$.

8.2.3. We let $\mathbf{N}_{\mathfrak{Q}_{G,G}^{\mathrm{ext}}}$ the groupoid on $\big(\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}}$ obtained by lifting the groupoid $\mathbf{N}$ on $\big(\mathrm{Bun}_G^{B\text{-gen}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}}$.

As in the case of $\mathfrak{Q}_{G,G}$, the groupoid $\mathbf{N}_{\mathfrak{Q}_{G,G}^{\mathrm{ext}}}$ is endowed with a canonical character χ^{ext} with values in $\mathbb{G}_a$.

8.2.4. We consider the twisted $\mathbf{N}_{\mathfrak{Q}_{G,G}^{\mathrm{ext}}}$-equivariant category of D-mod $\Big(\big(\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}} \Big)$, and as in Proposition 5.6.2, the forgetful functor

$$\text{D-mod} \Big(\big(\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}} \Big)^{\mathbf{N}_{\mathfrak{Q}_{G,G}^{\mathrm{ext}}}, \chi^{\mathrm{ext}}} \to \text{D-mod} \Big(\big(\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}} \Big)$$

is fully faithful.

8.2.5. We define the *extended Whittaker category* $\mathrm{Whit}^{\mathrm{ext}}(G, G)$ as the preimage of

$$\text{D-mod} \Big(\big(\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}} \Big)^{\mathbf{N}_{\mathfrak{Q}_{G,G}^{\mathrm{ext}}}, \chi^{\mathrm{ext}}} \subset \text{D-mod} \Big(\big(\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}} \Big)$$

under the pull-back functor

$$\text{D-mod}(\mathfrak{Q}_{G,G}^{\mathrm{ext}}) \to \text{D-mod} \Big(\big(\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}} \Big).$$

I.e.,

$\mathrm{Whit}^{\mathrm{ext}}(G, G) :=$

$$= \text{D-mod}(\mathfrak{Q}_{G,G}^{\mathrm{ext}}) \underset{\text{D-mod}\big((\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X))_{\mathrm{good}} \big)}{\times} \text{D-mod} \Big(\big(\mathfrak{Q}_{G,G}^{\mathrm{ext}} \times \mathrm{Ran}(X) \big)_{\mathrm{good}} \Big)^{\mathbf{N}_{\mathfrak{Q}_{G,G}^{\mathrm{ext}}}, \chi^{\mathrm{ext}}}.$$

8.2.6. As in the case of $\mathrm{Whit}(G,G)$, the (fully faithful) forgetful functor

$$\mathrm{Whit}^{\mathrm{ext}}(G,G) \to \mathrm{D\text{-}mod}(\mathcal{Q}^{\mathrm{ext}})$$

admits a right adjoint, that we denote by $\mathrm{Av}^{\mathbf{N},\chi^{\mathrm{ext}}}$.

We observe that as in Proposition 5.6.5, we have a canonical action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on $\mathrm{D\text{-}mod}(\mathcal{Q}^{\mathrm{ext}})$ that preserves the full subcategory

$$\mathrm{Whit}^{\mathrm{ext}}(G,G) \subset \mathrm{D\text{-}mod}(\mathcal{Q}^{\mathrm{ext}})$$

and commutes with the functor $\mathrm{Av}^{\mathbf{N},\chi^{\mathrm{ext}}}$.

8.2.7. We introduce the functor of extended Whittaker coefficient

$$\mathrm{coeff}^{\mathrm{ext}}_{G,G} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{Whit}^{\mathrm{ext}}(G,G)$$

to be

$$\mathrm{coeff}^{\mathrm{ext}}_{G,G} := \mathrm{Av}^{\mathbf{N},\chi^{\mathrm{ext}}} \circ (\mathfrak{r}^{\mathrm{ext}}_{G,G})^{\dagger}.$$

By construction, the functor $\mathrm{coeff}^{\mathrm{ext}}_{G,G}$ is compatible with the action of the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$.

8.2.8. We propose the following crucial conjecture:

Conjecture 8.2.9. *The functor* $\mathrm{coeff}^{\mathrm{ext}}_{G,G}$ *is fully faithful.*

We have:

Theorem 8.2.10. *Conjecture 8.2.9 holds for* $G = GL_n$.

This theorem has been recently established by D. Beraldo. The proof uses the mirabolic subgroup and the classical strategy of expressing the functor $\mathrm{coeff}^{\mathrm{ext}}_{G,G}$ as a composition of $n-1$ Fourier transform functors.

8.3. Extended vs. degenerate Whittaker models.

8.3.1. Let P be a parabolic in G with Levi quotient M. Note that we have a canonically defined locally closed embedding of prestacks:

$$\mathbf{i}_P : \mathcal{Q}_{G,P} \to \mathcal{Q}^{\mathrm{ext}}_{G,G}.$$

Namely, it corresponds to the locally closed subscheme $\overset{\circ}{\mathfrak{ch}}(M) \subset \mathfrak{ch}(G)$. In other words, $\mathrm{Maps}(S, \mathcal{Q}_{G,P})$ is a subgroupoid of $\mathrm{Maps}(S, \mathcal{Q}^{\mathrm{ext}}_{G,G})$, corresponding to those $(\mathcal{P}_G, U, \alpha, \gamma)$, for which the maps $\gamma(\mu)$ of (8.2) satisfy:

- For $\mu \notin \Lambda^{\mathrm{pos,sat}_M}$, we have $\gamma(\mu) = 0$.
- For $\mu \in \Lambda^{\mathrm{pos,sat}_M}$, the map $\gamma(\mu)$ is an isomorphism (possibly, after shrinking the open subset U).

8.3.2. For $P = G$ we will sometimes use the notation $\mathbf{j}$ instead of $\mathbf{i}_G$, to emphasize that we are dealing with an open embedding.

For the same reason, we will use the notation $\mathbf{j}^{\bullet}$ instead of $\mathbf{j}^{\dagger}$. The functor $\mathbf{j}^{\bullet}$ admits a *right* adjoint, denoted $\mathbf{j}_{\bullet}$, given by the D-module direct image.

8.3.3. The restriction of the groupoid $\mathbf{N}_{\mathcal{Q}_{G,G}^{\mathrm{ext}}}$ to

$$(\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}}$$

identifies with $\mathbf{N}_{\mathcal{Q}_{G,P}}$, and the character χ^{ext} restricts to χ_P.

Hence, the functor $(\mathbf{i}_P)^{\dagger}$ gives rise to a functor

$$(\mathbf{i}_P)^{\dagger} : \mathrm{Whit}^{\mathrm{ext}}(G, G) \to \mathrm{Whit}(G, P).$$

One shows that the partially defined left adjoint $(\mathbf{i}_P)_{\dagger}$ to $(\mathbf{i}_P)^{\dagger}$ is well-defined on the full subcategory

$$\mathrm{Whit}(G, P) \subset \mathrm{D\text{-}mod}(\mathcal{Q}_{G,P}).$$

Hence, we obtain a functor

$$(\mathbf{i}_P)_{\dagger} : \mathrm{Whit}(G, P) \to \mathrm{Whit}^{\mathrm{ext}}(G, G),$$

which is fully faithful, since $\mathbf{i}_P$ is a locally closed embedding.

8.3.4. In particular, the functor $\mathrm{coeff}_{G,G}^{\mathrm{ext}}$ contains the information of all the functors $\mathrm{coeff}_{G,P}$:

$$(8.3) \qquad\qquad\qquad\qquad \mathrm{coeff}_{G,P} \simeq (\mathbf{i}_P)^{\dagger} \circ \mathrm{coeff}_{G,G}^{\mathrm{ext}}.$$

Note that we have the following consequence of Conjecture 8.2.9:

Corollary-of-Conjecture 8.3.5. *Let* $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ *be such that* $\mathrm{coeff}_{G,P}(\mathcal{M}) = 0$ *for all parabolics* P *(including* $P = G$*). Then* $\mathcal{M} = 0$.

8.4. Cuspidality.

8.4.1. We shall call an object $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ *cuspidal* if it is annihilated by the functors CT_P for all *proper* parabolics P of G. We let

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{cusp}} \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

the full subcategory spanned by cuspidal objects.

8.4.2. Note that since for a given parabolic P, the functor $i_P^{\dagger}$ is conservative, an object $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ is annihilated by CT_P if and only if it is annihilated by $\mathrm{CT}_P^{\mathrm{enh}}$.

From (7.3) we obtain that if $\mathcal{M}$ is cuspidal then all $\mathrm{coeff}_{G,P}(\mathcal{M})$ (for P being a proper parabolic) are zero. In particular, we have:

Corollary 8.4.3. *Let* $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ *be cuspidal. Then the canonical map*

$$\mathrm{coeff}_{G,G}^{\mathrm{ext}}(\mathcal{M}) \to \mathbf{j}_{\bullet} \circ \mathbf{j}^{\bullet}(\mathrm{coeff}_{G,G}^{\mathrm{ext}}(\mathcal{M}))$$

is an isomorphism.

8.4.4. Note, however, that from (7.4) and Corollary 8.3.5 (applied to proper Levi subgroups of G), we obtain:

Corollary-of-Conjecture 8.4.5. *If* $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ *is such that* $\mathrm{coeff}_{G,P}(\mathcal{M}) = 0$ *for all proper parabolics* P*, then* $\mathcal{M}$ *is cuspidal.*

And, hence:

Corollary-of-Conjecture 8.4.6. *Let* $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ *be such that the map*

$$\mathrm{coeff}_{G,G}^{\mathrm{ext}}(\mathcal{M}) \to \mathbf{j}_{\bullet} \circ \mathbf{j}^{\bullet}(\mathrm{coeff}_{G,G}^{\mathrm{ext}}(\mathcal{M}))$$

is an isomorphism. Then $\mathcal{M}$ *is cuspidal.*

9. THE GLUING PROCEDURE

In this section we will match the category $\mathrm{Whit}^{\mathrm{ext}}(G, G)$ with a category that can be described purely in spectral terms.

9.1. **Gluing of DG categories, a digression.** In subsection we will describe the general paradigm in which one can define the procedure of gluing of DG categories.

9.1.1. Let A be an index category, and let $\mathbf{C}$

$$(a \in A) \mapsto \mathbf{C}_a, \quad (a_1 \xrightarrow{\phi} a_2) \mapsto (\mathbf{C}_{a_1} \xrightarrow{\mathbf{C}_\phi} \mathbf{C}_{a_2})$$

be a *lax* diagram of DG categories, parameterized by A.

Informally, this means that for a pair of composable arrows

$$a_1 \xrightarrow{\phi} a_2 \xrightarrow{\psi} a_3$$

we have a *natural transformation* (but not necessarily an isomorphism)

(9.1) $$\mathbf{C}_\psi \circ \mathbf{C}_\phi \to \mathbf{C}_{\psi \circ \phi},$$

equipped with a homotopy-coherent system of compatibilities for higher-order compositions.

In the ∞-categorical language, we should think of $\mathbf{C}$ as a category $\mathbf{C}_A$, equipped with a functor to A, which is a *locally co-Cartesian fibration*.

9.1.2. To $\mathbf{C}$ as above we assign its *lax limit*

$$\mathrm{Glue}(\mathbf{C}) \in \mathrm{DGCat}_{\mathrm{cont}}.$$

In the ∞-categorical language, $\mathrm{Glue}(\mathbf{C})$ is the category of sections of the functor $\mathbf{C}_A \to A$.

One can characterize $\mathrm{Glue}(\mathbf{C})$ by the following universal property. For $\mathbf{D} \in \mathrm{DGCat}_{\mathrm{cont}}$, the datum of a continuous functor

$$F : \mathbf{D} \to \mathrm{Glue}(\mathbf{C})$$

is equivalent to that of a collection of continuous functors

$$F_a : \mathbf{D} \to \mathbf{C}_a, \quad a \in A,$$

equipped with a compatible system of natural transformations

$$\mathbf{C}_\phi \circ F_{a_1} \xrightarrow{F_\phi} F_{a_2} \text{ for } a_1 \xrightarrow{\phi} a_2.$$

Note, however, that we *do not* require that the natural transformations F_ϕ be isomorphisms.

Taking $\mathbf{D}$ to be Vect, we obtain a description of the ∞-groupoid of objects of $\mathrm{Glue}(\mathbf{C})$. These are assignments

$$(a \in A) \mapsto \mathbf{c}_a \in \mathbf{C}_a, \quad (a_1 \xrightarrow{\phi} a_2) \mapsto (\mathbf{C}_\phi(\mathbf{c}_{a_1}) \xrightarrow{\mathbf{c}_\phi} \mathbf{c}_{a_2}),$$

equipped with a homotopy-coherent system of compatibilities for higher-order compositions.

Remark 9.1.3. The category $\mathrm{Glue}(\mathbf{C})$ contains a full subcategory, denoted $\mathrm{Glue}(\mathbf{C})^{\mathrm{strict}}$, that consists of those assigments for which the maps $\mathbf{c}_\phi$ above are isomorphisms.

If $\mathbf{C}$ was itself a strict functor $A \to \mathrm{DGCat}_{\mathrm{cont}}$ (i.e., if the natural transformations (9.1) were isomorphisms, or equivalently $\mathbf{C}_A \to A$ was a *co-Cartesian fibration*), then $\mathrm{Glue}(\mathbf{C})^{\mathrm{strict}}$ identifies with the limit of $\mathbf{C}$,

$$\lim_{a \in A} \mathbf{C}_a \in \mathrm{DGCat}_{\mathrm{cont}}.$$

9.1.4. We have the natural evaluation functors

$$\mathrm{ev}_a : \mathrm{Glue}(\mathbf{C}) \to \mathbf{C}_a, \quad a \in A.$$

These functors admit left adjoints, denoted ins_a[18].

Explicitly, the composition

$$\mathrm{ev}_{a_2} \circ \mathrm{ins}_{a_1} : \mathbf{C}_{a_1} \to \mathbf{C}_{a_2}$$

is calculated as the colimit in $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_{a_1}, \mathbf{C}_{a_2})$ over the ∞-groupoid $\mathrm{Maps}_A(a_1, a_2)$ of the functor

$$(\phi \in \mathrm{Maps}_A(a_1, a_2)) \mapsto (\mathbf{C}_\phi \in \mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}_{a_1}, \mathbf{C}_{a_2})).$$

In particular, we have:

Lemma 9.1.5. *Suppose that $a \in A$ is such that $\mathrm{Maps}_A(a, a)$ contractible. Then the functor ins_a is fully faithful.*

9.1.6. Here are is a typical example of the above situation. Let $\mathcal{Y}$ be a topological space and let

$$\mathcal{Y} = \bigcup_{a \in A} \mathcal{Y}_a$$

be its decomposition into locally closed subsets, indexed by a poset A, so that

$$\mathcal{Y}_{a_1} \cap \overline{\mathcal{Y}}_{a_2} \neq \emptyset \Rightarrow a_1 \geq a_2.$$

For each index a let $\mathbf{i}_a$ denote the corresponding locally closed embedding, and let

$$(\mathbf{i}_a)_\dagger : \mathrm{Shv}(\mathcal{Y}_a) \rightleftarrows \mathrm{Shv}(\mathcal{Y}) : (\mathbf{i}_a)^\dagger$$

be the corresponding adjoint pair.

We define the diagram $\mathbf{C}$ by sending $a \mapsto \mathrm{Shv}(\mathcal{Y}_a)$ and $(a_1 \leq a_2)$ to the functor

$$(\mathbf{i}_{a_2})^\dagger \circ (\mathbf{i}_{a_1})_\dagger : \mathrm{Shv}(\mathcal{Y}_{a_1}) \to \mathrm{Shv}(\mathcal{Y}_{a_2}).$$

Consider the resulting category $\mathrm{Glue}(\mathbf{C})$. We have a naturally defined functor

(9.2) $$\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Glue}(\mathbf{C}),$$

given by sending $a \mapsto (\mathbf{i}_a)^\dagger$ and $(a_1 \leq a_2)$ to the natural transformation

$$(\mathbf{i}_{a_2})^\dagger \circ (\mathbf{i}_{a_1})_\dagger \circ (\mathbf{i}_{a_1})^\dagger \to (\mathbf{i}_{a_2})^\dagger.$$

It is well known that the functor (9.2) is an equivalence. This is the source of the name "gluing" for the construction of Sect. 9.1.2.

9.1.7. Let now $F : \mathbf{C}' \to \mathbf{C}''$ be a *lax* natural transformation. Informally, this means having a collection of functors

$$F_a : \mathbf{C}'_a \to \mathbf{C}''_a, \quad a \in A,$$

equipped with natural transformations

(9.3) $$\mathbf{C}''_\phi \circ F_{a_1} \to F_{a_2} \circ \mathbf{C}'_\phi, \quad a_1 \xrightarrow{\phi} a_2,$$

and a homotopy-coherent system of compatibilities for higher-order compositions.

In the ∞-categorical language, the datum of F amounts to that of a functor $F_A : \mathbf{C}'_A \to \mathbf{C}''_A$, compatible with the projections to A.

We shall say that F is *strict* if the natural transformations (9.3) are isomorphisms. In the ∞-categorical language, this can be formulated as saying that F_A takes co-Cartesian arrows to co-Cartesian arrows.

---

[18]The notation "ins" is for "insert".

Given F as above, we have a naturally defined functor

$$\mathrm{Glue}(F) : \mathrm{Glue}(\mathbf{C}') \to \mathrm{Glue}(\mathbf{C}'').$$

We have:

Lemma 9.1.8. *Assume that each of the functors* $F_a : \mathbf{C}'_a \to \mathbf{C}''_a$ *is fully faithful and that* F *is strict. Then* $\mathrm{Glue}(F)$ *is fully faithful.*

9.2. **The extended Whittaker model as a glued category.** In this subsection we will see that the category $\mathrm{Whit}^{\mathrm{ext}}(G, G)$, introduced in Sect. 8, can be naturally obtained by a gluing procedure from the categories $\mathrm{Whit}(G, P)$,

9.2.1. We let A be the category $\mathrm{Par}(G)$ opposite to the poset of standard parabolics of G. For each parabolic we consider the category

$$\mathrm{Whit}(G, P).$$

We extend the assignment $P \mapsto \mathrm{Whit}(G, P)$ to a lax diagram of DG categories, parameterized by $\mathrm{Par}(G)$, by sending an inclusion $P_1 \subset P_2$ to the functor

$$(\mathbf{i}_{P_1})^\dagger \circ (\mathbf{i}_{P_2})_\dagger.$$

Let $\mathrm{Glue}(G)_{\mathrm{geom}}$ denote the resulting lax limit category.

9.2.2. We have a naturally defined functor

$$\mathrm{Whit}^{\mathrm{ext}}(G, G) \to \mathrm{Glue}(G)_{\mathrm{geom}}$$

corresponding to the collection of functors $(\mathbf{i}_{P_1})^\dagger$.

As in Sect. 9.1.6, we have:

Lemma 9.2.3. *The above functor* $\mathrm{Whit}^{\mathrm{ext}}(G, G) \to \mathrm{Glue}(G)_{\mathrm{geom}}$ *is an equivalence.*

By definition, the resulting adjoint pair of functors

$$\mathrm{ins}_P : \mathrm{Whit}(G, P) \rightleftarrows \mathrm{Glue}(G)_{\mathrm{geom}} : \mathrm{ev}_P$$

identifies with the adjoint pair $(\mathbf{i}_P)_\dagger, (\mathbf{i}_P)^\dagger$.

9.3. **The glued category on the spectral side.** In this subsection we will perform another construction, crucial for our approach to geometric Langlands.

We will show that the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$, appearing on the spectral side of the correspondence can be embedded into a category, obtained by a gluing procedure from the QCoh-categories for the parabolics of G.

This gives a precise expression to the idea that the difference between the categories $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ and $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ is captured by the proper parabolics of G.

9.3.1. Consider again the category $\mathrm{Par}(G)$. For each parabolic we consider the category

$$\mathsf{F}_{\check{P}}\text{-}\mathrm{mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})).$$

We are now going to upgrade the assignment

$$P \mapsto \mathsf{F}_{\check{P}}\text{-}\mathrm{mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}))$$

to a lax diagram of DG categories, parameterized by $\mathrm{Par}(G)$.

9.3.2. For $P_1 \subset P_2$, let $\mathsf{p}_{P_1/P_2,\mathrm{spec}}$ denote the corresponding map

$$\mathrm{LocSys}_{\check{P}_1} \to \mathrm{LocSys}_{\check{P}_2}.$$

As in Sect. 6.5, we have an adjoint pair of functors

$$(\mathsf{p}_{P_1/P_2,\mathrm{spec}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}_1}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}_1}) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}_2}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}_2}) : \mathsf{p}_{P_1/P_2,\mathrm{spec}}^!,$$

and the same-named pair of functors

$$(\mathsf{p}_{P_1/P_2,\mathrm{spec}})_*^{\mathrm{IndCoh}} : \mathsf{F}_{P_1}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}_1}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}_1})) \rightleftarrows$$

$$\mathsf{F}_{P_2}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}_2}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}_2})) : \mathsf{p}_{P_1/P_2,\mathrm{spec}}^!$$

that commute with the forgetful functors

$$\mathbf{oblv}_{\mathsf{F}_{P_i}} : \mathsf{F}_{P_i}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}_i}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}_i})) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}_i}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}_i}).$$

9.3.3. Recall also the functors

$$\Xi_{\check{P}_i} : \mathsf{F}_{P_i}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}_i})) \rightleftarrows \mathsf{F}_{P_i}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}_i}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}_i})) : \Psi_{\check{P}_i}.$$

We define the functor

$$\mathsf{F}_{P_2}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}_2})) \to \mathsf{F}_{P_1}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}_1}))$$

to be the composition

$$(9.4) \qquad\qquad\qquad \Psi_{\check{P}_1} \circ \mathsf{p}_{P_1/P_2,\mathrm{spec}}^! \circ \Xi_{\check{P}_2}.$$

9.3.4. We denote the resulting lax limit category by

$$\mathrm{Glue}(\check{G})_{\mathrm{spec}}.$$

For a parabolic P, we let $\mathrm{ev}_{\check{P},\mathrm{spec}}$ denote the corresponding evaluation functor

$$\mathrm{Glue}(\check{G})_{\mathrm{spec}} \to \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})),$$

and by $\mathrm{ins}_{\check{P},\mathrm{spec}}$ its left adjoint.

By Lemma 9.1.5, the functors $\mathrm{ins}_{\check{P},\mathrm{spec}}$ are fully faithful.

Note that since the functors (9.4) are compatible with the action of the monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, the category $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$ also acquires a $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$-action.

9.3.5. We now claim that there exists a canonically defined functor

$$\mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Glue}(\check{G})_{\mathrm{spec}}.$$

Namely, it is given by the collection of functors for each parabolic P:

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \overset{\mathrm{CT}_{\check{P},\mathrm{spec}}^{\mathrm{enh}}}{\longrightarrow}$$

$$\to \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) \overset{\Psi_{\check{P}}}{\longrightarrow} \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})).$$

By construction, the functor $\mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}})$ respects the action of the monoidal category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$.

9.3.6. We propose:

Conjecture 9.3.7. *The functor* $\mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}})$ *is fully faithful.*

The following has been recently proved by D. Arinkin and the author:

Theorem 9.3.8. *Conjecture 9.3.7 holds for all reductive groups G.*

9.4. **Extended Whittaker compatibility.** Having expressed $\mathrm{Whit}^{\mathrm{ext}}(G, G)$ as a glued category, we can now relate it to the spectral side. This will be done in the present subsection.

9.4.1. We now make the following crucial statement:

Quasi-Theorem 9.4.2.
(a) *The assignment that sends a parabolic P to the functor*

$$\mathbb{L}^{\mathrm{Whit}}_{G,P} : \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})) \to \mathrm{Whit}(G, P)$$

extends to a strict *natural transformation of the corresponding* lax *diagrams.*
(b) *The resulting functor* $\mathbb{L}^{\mathrm{Whit}^{\mathrm{ext}}}_{G,G}$

$$\mathrm{Glue}(\check{G})_{\mathrm{spec}} \to \mathrm{Glue}(G)_{\mathrm{geom}} \simeq \mathrm{Whit}^{\mathrm{ext}}(G, G)$$

is compatible with the actions of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$.

Remark 9.4.3. In fact, Quasi-Theorem 9.4.2 is a theorem modulo Quasi-Theorem 6.6.2. By definition, its statement amounts to a compatible family of commutative diagrams

$$
\begin{array}{ccc}
\mathsf{F}_{P_2}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}_2})) & \xrightarrow{\mathbb{L}^{\mathrm{Whit}}_{G,P_2}} & \mathrm{Whit}(G, P_2) \\
{\scriptstyle \Psi_{\check{P}_1} \circ \mathsf{p}^!_{P_1/P_2,\mathrm{spec}} \circ \Xi_{\check{P}_2}} \Big\downarrow & & \Big\downarrow {\scriptstyle (\mathbf{i}_{P_1})^{\dagger} \circ (\mathbf{i}_{P_2})_{\dagger}} \\
\mathsf{F}_{P_1}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}_1})) & \xrightarrow{\mathbb{L}^{\mathrm{Whit}}_{G,P_1}} & \mathrm{Whit}(G, P_1)
\end{array}
$$

for $P_1 \subset P_2$. Thus, the proof of Quasi-Theorem 9.4.2 amounts an explicit understanding of the gluing functors

$$(\mathbf{i}_{P_1})^{\dagger} \circ (\mathbf{i}_{P_2})_{\dagger} : \mathrm{Whit}(G, P_2) \to \mathrm{Whit}(G, P_1).$$

9.4.4. Combined with Lemmas 9.2.3 and 9.1.8, Quasi-Theorem 9.4.2 implies:

Quasi-Theorem 9.4.5. *The functor*

$$\mathbb{L}^{\mathrm{Whit}^{\mathrm{ext}}}_{G,G} : \mathrm{Glue}(\check{G})_{\mathrm{spec}} \to \mathrm{Whit}^{\mathrm{ext}}(G, G)$$

is fully faithful.

9.4.6. We are now ready to state Property $\mathrm{Wh}^{\mathrm{ext}}$ of the geometric Langlands functor $\mathbb{L}_G$ in Conjecture 3.4.2:

Property $\mathrm{Wh}^{\mathrm{ext}}$: *We shall say that the functor $\mathbb{L}_G$ satisfies* Property $\mathrm{Wh}^{\mathrm{ext}}$ *if the following diagram is commutative:*

$$
(9.5) \qquad
\begin{array}{ccc}
\mathrm{Glue}(\check{G})_{\mathrm{spec}} & \xrightarrow{\mathbb{L}^{\mathrm{Whit}^{\mathrm{ext}}}_{G,G}} & \mathrm{Whit}^{\mathrm{ext}}(G, G) \\
{\scriptstyle \mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}})} \Big\uparrow & & \Big\uparrow {\scriptstyle \mathrm{coeff}^{\mathrm{ext}}_{G,G}} \\
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\mathbb{L}_G} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G).
\end{array}
$$

Note that Property Wh^{ext} contains as a particular case Property Wh^{deg}, by concatenating (9.5) with the commutative diagram

$$
\begin{array}{ccc}
\mathsf{F}_{\check{P}}\text{-mod}(\text{QCoh}(\text{LocSys}_{\check{P}})) & \xrightarrow{\;\mathbb{L}_{G,P}^{\text{Whit}}\;} & \text{Whit}(G,P) \\
{\scriptstyle \text{ev}_P}\big\uparrow & & \big\uparrow{\scriptstyle (\mathbf{i}_P)^{\dagger}} \\
\text{Glue}(\check{G})_{\text{spec}} & \xrightarrow{\;\mathbb{L}_{G,G}^{\text{Whit}^{\text{ext}}}\;} & \text{Whit}^{\text{ext}}(G,G)
\end{array}
$$

9.4.7. Note that by combining Conjecture 8.2.9, Theorem 9.3.8 and Quasi-Theorem 9.4.5, we obtain:

Corollary-of-Conjecture 9.4.8.

(a) *Property* Wh^{ext} *determines the equivalence* $\mathbb{L}_G$ *uniquely, and if the latter exists, it satisfies property* He^{naive}.

(b) *The equivalence* $\mathbb{L}_G$ *exists if and only if the essential images of*

$$\text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}}) \ \text{ and } \ \text{D-mod}(\text{Bun}_G)$$

in $\text{Whit}^{\text{ext}}(G,G)$ *under the functors*

$$\mathbb{L}_{G,G}^{\text{Whit}^{\text{ext}}} \circ \text{Glue}(\text{CT}_{\text{spec}}^{\text{enh}}) \ \text{ and } \ \text{coeff}_{G,G}^{\text{ext}},$$

respectively, coincide.

In Sect. 11.3 we will show (assuming Quasi-Theorem 9.4.2 and Quasi-Theorem 9.5.3 below) that the condition of Corollary 9.4.8(b) is satisfied for $G = GL_2$, thereby proving Conjecture 3.4.2 in this case.

9.4.9. Let us for a moment assume the validity of Conjecture 3.4.2. We obtain the following geometric characterization of the full subcategory

$$\text{D-mod}(\text{Bun}_G)_{\text{temp}} \subset \text{D-mod}(\text{Bun}_G).$$

Corollary-of-Conjecture 9.4.10. *An object* $\mathcal{M} \in \text{D-mod}(\text{Bun}_G)$ *belongs to the subcategory* $\text{D-mod}(\text{Bun}_G)_{\text{temp}}$ *if and only if the canonical map*

$$\mathbf{j}_{\dagger}(\text{coeff}_{G,G}(\mathcal{M})) \to \text{coeff}_{G,G}^{\text{ext}}(\mathcal{M})$$

is an isomorphism.

9.5. **Extended Whittaker coefficients and Eisenstein series compatibility.** Let $P' \subset G$ be another parabolic. In this subsection we will assume that Conjecture 3.4.2 holds for its Levi quotient M'.

9.5.1. By concatenating the commutative diagrams (9.5) and (6.11) we obtain the following commutative diagram

(9.6)
$$
\begin{array}{ccc}
\text{Glue}(\check{G})_{\text{spec}} & \xrightarrow{\;\mathbb{L}_{G,G}^{\text{Whit}^{\text{ext}}}\;} & \text{Whit}^{\text{ext}}(G,G) \\
{\scriptstyle \text{Glue}(\text{CT}_{\text{spec}}^{\text{enh}})}\big\uparrow & & \big\uparrow{\scriptstyle \text{coeff}_{G,G}^{\text{ext}}} \\
\text{IndCoh}_{\text{Nilp}_{\check{G}}^{\text{glob}}}(\text{LocSys}_{\check{G}}) & & \text{D-mod}(\text{Bun}_G) \\
{\scriptstyle \text{Eis}_{\check{P}',\text{spec}}^{\text{enh}}}\big\uparrow & & \big\uparrow{\scriptstyle \text{Eis}_{P'}^{\text{enh}}} \\
\mathsf{F}_{\check{P}'}\text{-mod}(\text{IndCoh}_{\text{Nilp}_{\text{glob},\check{P}'}}(\text{LocSys}_{\check{P}'})) & \xrightarrow{\;\mathbb{L}_{P'}\;} & \text{I}(G,P').
\end{array}
$$

9.5.2. We have:

Quasi-Theorem 9.5.3. *The diagram* (9.6) *commutes unconditionally (i.e., without assuming the validity of Conjecture 3.4.2 for G).*

10. Compatibility with Kac-Moody localization and opers

We will now change gears and discuss a very different approach to the construction of objects of D-mod(Bun$_G$). This construction has to do with localization of modules over the Kac-Moody algebra, first explored by [BD2].

As was explained in the introduction, we need this other construction for our approach to the proof of geometric Langlands: some of the objects of D-mod(Bun$_G$) obtained in this way will provide generators of this category, on which the functor coeff$_{G,G}^{\text{ext}}$ can be calculated explicitly.

The spectral counterpart of the Kac-Moody localization construction has to do with the scheme of $\check{G}$-opers, also studied in this section.

10.1. The category of Kac-Moody modules.
In this subsection we will define what we mean by the category of Kac-Moody modules.

Many of the objects discussed in this subsection do not, unfortunately, admit adequate references in the existing literature. Hopefully, these gaps will be filled soon.

10.1.1. Let κ be a *level*, i.e., a G-invariant quadratic form on $\mathfrak{g}$. We consider the corresponding affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}_\kappa$, which is a central extension

$$0 \to k \to \mathfrak{L}(\mathfrak{g}, \kappa) \to \mathfrak{g}((t)) \to 0,$$

and the category $\mathfrak{L}(\mathfrak{g}, \kappa)$-mod as defined in [FG2, Sect. 23.1].[19]

We consider the group-scheme $\mathfrak{L}^+(G) := G[\![t]\!]$, and our primary interest is the category KL(G, κ) of $\mathfrak{L}^+(G)$-equivariant objects in $\mathfrak{L}(\mathfrak{g}, \kappa)$-mod.

Remark 10.1.2. The eventually coconnective part of KL(G, κ) (the subcategory of objects that are $> -\infty$ with respect to the natural t-structure) can be defined by the procedure of [FG3, Sect. 20.8]. The entire KL(G, κ) is defined so that it is compactly generated by the Weyl modules.

10.1.3. For this paper we will need the following generalization of the category KL(G, κ):

For a finite set I, we consider the variety X^I. Over it there exists a group ind-scheme $\mathfrak{L}(G)_{X^I}$, equipped with a connection; and a group subscheme $\mathfrak{L}^+(G)_{X^I}$. Let $\mathfrak{L}(\mathfrak{g})_{X^I}$ denote the corresponding sheaf of topological Lie algebras over $(X^I)_{\text{dr}}$.

We let $\mathfrak{L}(\mathfrak{g}, \kappa)_{X^I}$ be the central extension of $\mathfrak{L}(\mathfrak{g})_{X^I}$ corresponding to κ, and we consider the corresponding category

$$\text{KL}(G, \kappa)_{X^I} := \mathfrak{L}(\mathfrak{g})_{X^I}\text{-mod}^{\mathfrak{L}^+(G)_{X^I}}.$$

---

[19]In *loc.cit.* it was referred to as the *renormalized* category of Kac-Moody modules.

10.1.4. For a surjective map of finite sets $I_2 \twoheadrightarrow I_2$ there is a naturally defined functor

(10.1) $$\mathrm{KL}(G,\kappa)_{X^{I_1}} \to \mathrm{KL}(G,\kappa)_{X^{I_2}}.$$

The assignment $I \mapsto \mathrm{KL}(G,\kappa)_{X^I}$ extends to a functor

$$(\mathrm{fSet}_{\mathrm{surj}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

(see Sect. 4.1.2 for the notation) and we set

$$\mathrm{KL}(G,\kappa)_{\mathrm{Ran}(X)} := \underset{I \in (\mathrm{fSet}_{\mathrm{surj}})^{\mathrm{op}}}{\underrightarrow{colim}} \mathrm{KL}(G,\kappa)_{X^I}.$$

10.1.5. Assume now that κ is integral, which means by definition, that the central exetension of Lie algebras

$$0 \to k \to \mathfrak{L}(\mathfrak{g},\kappa) \to \mathfrak{L}(\mathfrak{g}) \to 0$$

comes from a central extension of group ind-schemes

$$1 \to \mathbb{G}_m \to \mathfrak{L}(G,\kappa)_{X^I} \to \mathfrak{L}(G)_{X^I} \to 1,$$

functorial in $I \in \mathrm{fSet}_{\mathrm{surj}}$.

In this case, for every finite set I, there exists a canonically localization functor

$$\mathrm{Loc}_{G,X^I} : \mathrm{KL}(G,\kappa)_{X^I} \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

These functors are compatible with the functors (10.1), and hence we obtain a functor

$$\mathrm{Loc}_G : \mathrm{KL}(G,\kappa)_{\mathrm{Ran}(X)} \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

We have the following assertion:

Proposition 10.1.6. *Let* $\mathsf{U} \subset \mathrm{Bun}_G$ *be an open substack such that its intersection with every connected component of* Bun_G *is quasi-compact. Then the composed functor*

$$\mathrm{KL}(G,\kappa)_{\mathrm{Ran}(X)} \overset{\mathrm{Loc}_G}{\longrightarrow} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \overset{restriction}{\longrightarrow} \mathrm{D\text{-}mod}(\mathsf{U})$$

is a localization, i.e., admits a fully faithful right adjoint.

Note that from Proposition 10.1.6 we obtain:

Corollary 10.1.7. *Let* $\mathsf{U} \subset \mathrm{Bun}_G$ *be an open substack such that its intersection with every connected component of* Bun_G *is quasi-compact. Then the essential images of the functors*

$$\mathrm{KL}(G,\kappa)_{X^I} \overset{\mathrm{Loc}_{G,X^I}}{\longrightarrow} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{D\text{-}mod}(\mathsf{U}),$$

as I runs over fSet, *generate* $\mathrm{D\text{-}mod}(\mathsf{U})$.

Remark 10.1.8. For any κ, we have a functor Loc_G from $\mathrm{KL}(G,\kappa)_{\mathrm{Ran}(X)}$ to the corresponding category of κ-twisted D-modules on Bun_G, and the analog of Proposition 10.1.6 holds. The proof amounts to a calculation of chiral homology of the chiral algebra of differential operators on G, introduced in [ArkhG].

10.1.9. From now on we will fix κ to be the critical level, i.e., $-\frac{\kappa_{\mathrm{Kil}}}{2}$, where κ_{Kil} is the Killing form.

Remark 10.1.10. Using an intrinsic characterization of the subcategory

$$\text{D-mod}(\mathrm{Bun}_G)_{\mathrm{temp}} \subset \text{D-mod}(\mathrm{Bun}_G)$$

described in Sect. 4.6.7, and using the properties of the category $\mathrm{KL}(G, \mathrm{crit})$ with respect to the Hecke action (essentially, given by [FG3, Theorem 8.22]), one shows that the essential image of the functor

$$\mathrm{Loc}_G : \mathrm{KL}(G, \mathrm{crit})_{\mathrm{Ran}(X)} \to \text{D-mod}(\mathrm{Bun}_G)$$

lands inside $\text{D-mod}(\mathrm{Bun}_G)_{\mathrm{temp}}$. For the latter it is crucial that the value of κ is critical (as opposed to arbitrary integral).

10.2. **The spaces of local and global opers.** In this subsection we will introduce the scheme of $\check{G}$-opers. Quasi-coherent sheaves on the scheme of opers will be the spectral counterpart of Kac-Moody representations.

10.2.1. Let I be again a finite non-empty set, and let λ^I be a map from I to the set Λ^+ of dominant weights of G, which are the same as dominant co-weights of $\check{G}$.

Local λ^I-opers for the group $\check{G}$ form a DG scheme mapping to X^I, denoted $\mathrm{Op}(\check{G})^{\mathrm{loc}}_{\lambda^I}$, defined as follows.

For $S \in \mathrm{DGSch}^{\mathrm{aff}}$, an S-point of $\mathrm{Op}(\check{G})^{\mathrm{loc}}_{\lambda^I}$ is the data of

$$(\underline{x}, \mathcal{P}_{\check{G}}, \mathcal{P}_{\check{B}}, \alpha, \nabla),$$

where:

- $\underline{x}$ is an S-point of X^I; we let $\mathcal{D}_{\underline{x}}$ be the corresponding parameterized family of formal discs over S.

- $\mathcal{P}_{\check{G}}$ is a $\check{G}$-bundle over $\mathcal{D}_{\underline{x}}$.

- α is a datum of reduction of $\mathcal{P}_{\check{G}}$ to a $\check{B}$-bundle $\mathcal{P}_{\check{B}}$, whose induced $\check{T}$-bundle $\mathcal{P}_{\check{T}}$ is identified with $\rho(\omega_X) \otimes (-\lambda^I \cdot \underline{x})|_{\mathcal{D}_{\underline{x}}}$, where we regard $\lambda^I \cdot \underline{x}$ as a Λ-valued Cartier divisor on $S \times X$.

- ∇ is a datum of "vertical" connection on $\mathcal{P}_{\check{G}}$ along the fibers of the map $\mathcal{D}_{\underline{x}} \to S$, i.e., a datum of lift of $\mathcal{P}_{\check{G}}$ from a $\check{G}$-bundle on $(\mathcal{D}_{\underline{x}})_{\mathrm{dr}} \underset{S_{\mathrm{dr}}}{\times} S$.

Note that the discrepancy between α and ∇ is given by a section of

$$\nabla \bmod \check{B} \in (\check{\mathfrak{g}}/\check{\mathfrak{b}})_{\mathcal{P}_{\check{B}}} \otimes \omega_X|_{\mathcal{D}_{\underline{x}}}.$$

We require that the following compatibility condition be satisfied:

- $\nabla \bmod \check{B}$ belongs to the sub-bundle $(\check{\mathfrak{g}}^{-1}/\check{\mathfrak{b}})_{\mathcal{P}_{\check{B}}} \otimes \omega_X|_{\mathcal{D}_{\underline{x}}}$; where $(\check{\mathfrak{g}}^{-1}/\check{\mathfrak{b}}) \subset (\check{\mathfrak{g}}/\check{\mathfrak{b}})$ is the $\check{B}$-subrepresentation spanned by negative simple roots.

- For each vertex of the Dynkin diagram i, the resulting section of

$$(\check{\mathfrak{g}}^{-1,\check{\alpha}_i}/\check{\mathfrak{b}})_{\mathcal{P}_{\check{B}}} \otimes \omega_X|_{\mathcal{D}_{\underline{x}}} \simeq -\check{\alpha}_i(\mathcal{P}_{\check{T}}) \otimes \omega_X|_{\mathcal{D}_{\underline{x}}}$$

is the canonical map

$$\mathcal{O}_{\mathcal{D}_{\underline{x}}} \to \mathcal{O}(\langle \lambda^I, \check{\alpha}_i \rangle \cdot \underline{x})|_{\mathcal{D}_{\underline{x}}} \simeq -\check{\alpha}_i(\mathcal{P}_{\check{T}}) \otimes \omega_X|_{\mathcal{D}_{\underline{x}}}.$$

Remark 10.2.2. As S is a derived scheme, some care is needed to makes sense of the expression $\nabla \bmod \check{B}$ (see, e.g., [AG, Sect. 10.5] for how to do this). However, a posteriori, one can show that the DG scheme $\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{loc}}$ is classical, so we could restrict our attention to those $S \in \mathrm{DGSch}^{\mathrm{aff}}$ that are themselves classical.

10.2.3. We define the DG scheme $\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}}$ similarly, with the only difference that instead of the parameterized formal disc $\mathcal{D}_{\underline{x}}$ we consider the entire scheme $S \times X$.

By construction, we have the forgetful maps

$$\mathrm{LocSys}_{\check{G}}$$
$$\mathsf{v}_{\lambda^I} \uparrow$$
$$\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}} \xrightarrow{\ \mathsf{u}_{\lambda^I}\ } \mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{loc}}.$$

Remark 10.2.4. We note that, unlike, $\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{loc}}$, the DG scheme $\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}}$ is typically *not* classical.

10.2.5. Let $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}} \subset \mathrm{LocSys}_{\check{G}}$ be the open substack corresponding to irreducible local systems. Let

$$\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}} \subset \mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}}$$

be the preimage of $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$ under the map v_{λ^I}.

We have:

Lemma 10.2.6. *The map*

$$\mathsf{v}_{\lambda^I} : \mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}} \to \mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}$$

is proper.

Consider the functor

$$(10.2) \qquad\qquad (\mathsf{v}_{\lambda^I})^! : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}) \to \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}})$$

right adjoint to

$$(10.3) \qquad\qquad (\mathsf{v}_{\lambda^I})_* : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}).$$

10.2.7. The next conjecture, along with Conjecture 8.2.9, is the second element in the proof of geometric Langlands that still remains mysterious in the case of an arbitrary group G:

Conjecture 10.2.8. *Let* $\mathcal{F} \in \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}})$ *be such that* $(\mathsf{v}_{\lambda^I})^!(\mathcal{F}) = 0$ *for all finite sets* I *and* $\lambda^I : I \to \Lambda^+$. *Then* $\mathcal{F} = 0$.

However, we have:

Theorem 10.2.9. *Conjecture 10.2.8 holds for* $G = GL_n$.

We also note that recent progress made by D. Kazhdan and T. Schlank implies that Conjecture 10.2.8 holds also for $G = Sp(2n)$.

10.2.10. We can reformulate Conjecture 10.2.8 as follows:

Corollary-of-Conjecture 10.2.11. *The union of the essential images of the functors*

$$(\mathsf{v}_{\lambda^I})_* : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}})$$

over all finite sets I *and* $\lambda^I : I \to \Lambda^+$, *generates* $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}})$.

10.3. **Compatibility between opers and Kac-Moody localization.** In this subsection we will match the local category on the geometric side, i.e., $\mathrm{KL}(G, \mathrm{crit})_{X^I}$, with the local category on the spectral side, i.e., $\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{loc}}_{\lambda^I})$.

10.3.1. The following is an extension of [FG1, Proposition 3.5]:

Proposition 10.3.2.

(a) *For a finite set I and $\lambda^I : I \to \Lambda^+$ there exists a canonically defined functor*

$$\mathbb{L}_G^{\mathrm{Op}_{\lambda^I}} : \mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{loc}}_{\lambda^I}) \to \mathrm{KL}(G, \mathrm{crit})_{X^I}.$$

(b) *For a fixed finite set I, the union of essential images of the functors $\mathbb{L}_G^{\mathrm{Op}_{\lambda^I}}$ over $\lambda^I : I \to \Lambda^+$ generates $\mathrm{KL}(G, \mathrm{crit})_{X^I}$.*

10.3.3. The next theorem is a moving-point version of [BD2, Theorem 5.2.9]:

Theorem 10.3.4.

(a) *The composed functor*

$$\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{loc}}_{\lambda^I}) \xrightarrow{\mathbb{L}_G^{\mathrm{Op}_{\lambda^I}}} \mathrm{KL}(G, \mathrm{crit})_{X^I} \xrightarrow{\mathrm{Loc}_{G, X^I}} \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

canonically factors as

$$\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{loc}}_{\lambda^I}) \xrightarrow{(\mathsf{u}_{\lambda^I})^*} \mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I}) \xrightarrow{\mathrm{q\text{-}Hitch}_{\lambda^I}} \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

(b) *The resulting functor* [20]

$$\mathrm{q\text{-}Hitch}_{\lambda^I} : \mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

respects the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$, where the latter acts on $\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I})$ via the composition $(\mathsf{v}_{\lambda^I})^* \circ \mathrm{Loc}_{\check{G}, \mathrm{spec}}$ and on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ via the functor $\mathrm{Sat}(G)^{\mathrm{naive}}_{\mathrm{Ran}(X)}$.

10.3.5. We are finally ready to state Property $\mathrm{Km}^{\mathrm{prel}}$ of the geometric Langlands functor $\mathbb{L}_G$ in Conjecture 3.4.2 ("Km" stands for Kac-Moody):

Property $\mathrm{Km}^{\mathrm{prel}}$: *We shall say that the functor $\mathbb{L}_G$ satisfies* Property $\mathrm{Km}^{\mathrm{prel}}$ *if for every finite set I and $\lambda^I : I \to \Lambda^+$, the following diagram is commutative:*

(10.4)

$$
\begin{array}{ccc}
\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\mathbb{L}_G} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\
\Xi_{\check{G}} \uparrow & & \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & & \uparrow \mathrm{q\text{-}Hitch}_{\lambda^I} \\
(\mathsf{v}_{\lambda^I})_* \uparrow & & \\
\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I}) & \xrightarrow{\mathrm{Id}} & \mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I}).
\end{array}
$$

10.4. **The oper vs. Whittaker compatibility.** The reason the localization procedure is useful is that one can explicitly control the Whittaker coefficients of D-modules obtained in this way. How this is done will be explained in the present subsection.

---

[20]The notation "q-Hitch" stands for "quantized Hitchin map".

10.4.1. For a finite set I and a map $\Lambda^I : I \to \Lambda^+$ let us concatenate the diagrams (10.4) and (5.7).

Using the fact that $\Psi_{\check{G}} \circ \Xi_{\check{G}} \simeq \mathrm{Id}$, we obtain a commutative diagram

(10.5)
$$
\begin{array}{ccc}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \underset{\mathrm{D\text{-}mod}(\mathrm{Ran}(X))}{\otimes} \mathrm{Vect} & \xrightarrow{\ \mathbb{L}_G^{\mathrm{Whit}}\ } & \mathrm{Whit}(G) \\[2mm]
\Big\uparrow {\scriptstyle \mathrm{co\text{-}Loc}_{\check{G},\mathrm{spec}}^{\mathrm{unital}}} & & \Big\uparrow {\scriptstyle \mathrm{coeff}_G} \\[2mm]
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\[2mm]
\Big\uparrow {\scriptstyle (\mathsf{v}_{\lambda^I})_*} & & \Big\uparrow {\scriptstyle \mathrm{q\text{-}Hitch}_{\lambda^I}} \\[2mm]
\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}}) & \xrightarrow{\ \mathrm{Id}\ } & \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}}).
\end{array}
$$

We claim:

Theorem 10.4.2. *The diagram* (10.5) *commutes unconditionally, i.e., without assuming the validity of Conjecture 3.4.2.*

Remark 10.4.3. The proof of Theorem 10.4.2 amounts to a computation of chiral homology of the chiral algebra responsible for the scheme of opers, and the Feigin-Frenkel isomorphism that identifies it with the Whittaker BRST reduction of the chiral algebra corresponding to $\mathfrak{L}(\mathfrak{g}, \mathrm{crit})$.

10.4.4. As a corollary of Theorem 10.4.2 we obtain:

Corollary 10.4.5. *The following diagram is commutative*

$$
\begin{array}{ccc}
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\ \mathbb{L}_{G,G}^{\mathrm{Whit}}\ } & \mathrm{Whit}(G,G) \\[2mm]
\Big\uparrow {\scriptstyle \Psi_{\check{G}}} & & \Big\uparrow {\scriptstyle \mathrm{coeff}_{G,G}} \\[2mm]
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\[2mm]
\Big\uparrow {\scriptstyle \Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*} & & \Big\uparrow {\scriptstyle \mathrm{q\text{-}Hitch}_{\lambda^I}} \\[2mm]
\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}}) & \xrightarrow{\ \mathrm{Id}\ } & \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}}).
\end{array}
$$

10.5. **Full compatibility with opers.** The material in this subsection will not be used elsewhere in the paper. We will discuss a stronger version of Property $\mathrm{Km}^{\mathrm{prel}}$ of the geometric Langlands functor $\mathbb{L}_G$ that we call Property Km. As adequate references are not available, we will only indicate the formal structure of the theory once the appropriate definitions are given.

10.5.1. For every finite set I one can introduce prestacks $\mathrm{Op}(\check{G})_{X^I}^{\mathrm{loc}}$ and $\mathrm{Op}(\check{G})_{X^I}^{\mathrm{glob}}$, by considering "opers with singularities but without monodromy", instead of λ^I-opers for a specified $\lambda^I : I \to \Lambda^+$. For $I = \{1\}$ the local version is defined in [FG1, Sect. 2.2].

We have a diagram

$$
\begin{array}{c}
\mathrm{LocSys}_{\check{G}} \\[2mm]
\Big\uparrow {\scriptstyle \mathsf{v}_{X^I}} \\[2mm]
\mathrm{Op}(\check{G})_{X^I}^{\mathrm{glob}} \xrightarrow{\ \mathsf{u}_{X^I}\ } \mathrm{Op}(\check{G})_{X^I}^{\mathrm{loc}}.
\end{array}
$$

10.5.2. We have:

Conjecture 10.5.3. *There exists a canonically defined equivalence*

$$\mathbb{L}_G^{\mathrm{Op}_{X^I}} : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{X^I}^{\mathrm{loc}}) \to \mathrm{KL}(G, \mathrm{crit})_{X^I},$$

extending the functors $\mathbb{L}_G^{\mathrm{Op}_{\lambda^I}}$ *of Proposition 10.3.2(a).*

10.5.4. Passing to the limit over $I \in (\mathrm{fSet}_{\mathrm{surj}})^{\mathrm{op}}$, we obtain a diagram

$$\mathrm{LocSys}_{\check{G}}$$

(10.6)
$$\mathsf{v}_{\mathrm{Ran}(X)} \Big\uparrow$$

$$\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{glob}} \xrightarrow{\;\mathsf{u}_{\mathrm{Ran}(X)}\;} \mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{loc}}$$

and an equivalence

$$\mathbb{L}_G^{\mathrm{Op}_{\mathrm{Ran}(X)}} : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{loc}}) \to \mathrm{KL}(G, \mathrm{crit})_{\mathrm{Ran}(X)}.$$

Push-pull along (10.6) defines a functor

$$\mathrm{Poinc}_{\check{G}, \mathrm{spec}} : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{loc}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}).$$

10.5.5. The full Property Km of the geometric Langlands functor $\mathbb{L}_G$ reads:

Property Km: *We shall say that the functor* $\mathbb{L}_G$ *satisfies* Property Km *if the diagram is commutative:*

$$
\begin{array}{ccc}
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\;\mathbb{L}_G\;} & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\
\Xi_{\check{G}} \Big\uparrow & & \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) & & \Big\uparrow \mathrm{Loc}_G \\
\mathrm{Poinc}_{\check{G}, \mathrm{spec}} \Big\uparrow & & \\
\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{loc}}) & \xrightarrow{\;\mathbb{L}_G^{\mathrm{Op}_{\mathrm{Ran}(X)}}\;} & \mathrm{KL}(G, \mathrm{crit})_{\mathrm{Ran}(X)}.
\end{array}
$$

10.5.6. Finally, we propose the following two closely related conjectures:

Conjecture 10.5.7. *The fibers of the map* $\mathsf{v}_{\mathrm{Ran}(X)}$ *are* $\mathcal{O}$*-contractible, i.e., the functor*

$$\mathsf{v}_{\mathrm{Ran}(X)}^* : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{glob}})$$

is fully faithful.

Remark 10.5.8. Conjecture 10.5.7 can be reformulated as saying that the functor

$$(\mathsf{v}_{\mathrm{Ran}(X)})_* : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{glob}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

is a *co-localization*, i.e., it identifies the homotopy category of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ with a Verdier quotient of $\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}^{\mathrm{glob}})$. Note that this equivalent to $(\mathsf{v}_{\mathrm{Ran}(X)})_*$ being a *localization*, i.e., that its (not necessarily continuous) *right* adjoint is fully faithful.

Remark 10.5.9. Note that Conjecture 10.5.7 is a strengthening of Conjecture 10.2.8, and it should be within reach for $G = GL_n$. The recent work of D. Kazhdan and T. Schlank indicates that it also holds for $G = Sp(2n)$.

Conjecture 10.5.10. *The functor* $\mathrm{Poinc}_{\check{G},\mathrm{spec}}$ *is a* localization, *i.e., it identifies the homotopy category of* $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ *with a Verdier quotient of* $\mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{loc}}_{\mathrm{Ran}(X)})$.

11. The proof modulo the conjectures

In this section we will assemble the ingredients developed in the previous sections to prove Conjecture 3.4.2, assuming Conjectures 8.2.9 and 10.2.8 and all the quasi-theorems.

11.1. **Proof of the vanishing theorem.** One of the steps in the proof of Conjecture 3.4.2 is Theorem 4.5.2. The proofs of both Theorem 4.5.2 of Conjecture 3.4.2 rely on the following description of generators of the category $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$:

Theorem 11.1.1. *The union of the essential images of the functors*

$$\mathrm{Loc}_{G,X^I} : \mathrm{KL}(G,\kappa)_{X^I} \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

and

$$\mathrm{Eis}_P : \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

for proper *parabolics* $P \subset G$, *generates* $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$.

Sketch of proof. Let $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ be right-orthogonal to both the essential image of Eis_P for all proper parabolics $P \subset G$ and Loc_{G,X^I}. We need to show that $\mathcal{M} = 0$.

Reduction theory (see [DrGa3, Proposition 1.4.6]) implies that there exists an open substack

$$\mathsf{U} \xrightarrow{f} \mathrm{Bun}_G$$

such that its intersection with every connected component of Bun_G is quasi-compact, and which has the following property:

For every object $\mathcal{M}' \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$, the canonical arrow

$$(11.1) \qquad\qquad\qquad \mathcal{M}' \to f_\bullet \circ f^\bullet(\mathcal{M}')$$

is an isomorphism.

On the one hand, the assumption that $\mathcal{M}$ is right-orthogonal to the essential image of Eis_P for all proper parabolics $P \subset G$, implies that $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$. Now the fact that $\mathcal{M} = 0$ follows from (11.1) and Corollary 10.1.7.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

11.1.2. We are now ready to prove Theorem 4.5.2:

Proof. Let $\mathcal{F} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ be an object such that $\mathrm{Loc}_{\check{G},\mathrm{spec}}(\mathcal{F}) = 0$. We need to show that the action of $\mathcal{F}$ on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ is also zero. For that it is sufficient to show that $\mathcal{F}$ acts by zero on a subcategory of $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ that generates it.

By Theorem 10.3.4(b), the action of $\mathcal{F}$ on objects in the essential image of the functor q-$\mathrm{Hitch}_{\lambda^I}$ (for any finite set I and $\lambda^I : I \to \Lambda^+$) is zero. Combined with Proposition 10.3.2, this implies that $\mathcal{F}$ acts by zero on the essential image of Loc_{G,X^I} for any finite set I.

By Theorem 11.1.1, it remains to show that $\mathcal{F}$ acts by zero on the essential image of Eis_P for all proper parabolics $P \subset G$.

This follows from the next assertion, which is itself a particular case of Quasi-Theorem 6.6.2, but can be proved in a more elementary way by generalizing the argument of [BG, Theorem 1.11]:

Proposition 11.1.3. *Assume that Theorem 4.5.2 holds for the Levi quotient M; in particular we have an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})$ on $\mathrm{D\text{-}mod}(\mathrm{Bun}_M)$.*

(a) *The functor*

$$\mathrm{Eis}_P : \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

canonically factors as

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_M) \xrightarrow{\mathsf{q}_{\check{P},\mathrm{spec}}^* \otimes \mathrm{Id}} \mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \xrightarrow{\mathrm{Eis}_{\check{P}}^{\mathrm{int}}}$$

$$\to \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

(b) *For $\mathcal{F} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ and $\mathcal{M} \in \mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_M)$, we have a canonical isomorphism*

$$\mathcal{F} \star \mathrm{Eis}_P^{\mathrm{int}}(\mathcal{M}) \simeq \mathrm{Eis}_P^{\mathrm{int}}\left(\mathsf{p}_{\check{P},\mathrm{spec}}^*(\mathrm{Loc}_{\check{G},\mathrm{spec}}(\mathcal{F})) \otimes \mathcal{M}\right),$$

where $- \star -$ denotes the monoidal action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$.

$\square$

Remark 11.1.4. The functor $\mathrm{Eis}_P^{\mathrm{int}}$ can also be interpreted as a composition of $\mathrm{Eis}_P^{\mathrm{enh}}$ with a canonically defined functor

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{I}(G, P),$$

which in terms of the equivalence $\mathbb{L}_P$ corresponds to

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{M}})}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \xrightarrow{\mathsf{q}_{\check{P},\mathrm{spec}}^*}$$

$$\to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}}) \xrightarrow{\mathbf{ind}_{\mathsf{F}_{\check{P}}}} \mathbf{ind}_{\mathsf{F}_{\check{P}}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})),$$

up to an auto-equivalence of $\mathrm{D\text{-}mod}(\mathrm{Bun}_M)$.

11.2. Construction of the functor. From now on, we shall assume the validity of Conjectures 8.2.9 and 10.2.8 (which are theorems for GL_n) and all the Quasi-Theorems, and deduce Conjecture 3.4.2.

By induction on the rank, we can assume the validity of Conjecture 3.4.2 for proper Levi subgroups of G.

11.2.1. By Conjecture 8.2.9, the existence of the functor $\mathbb{L}_G$ amounts to showing that the essential image of the functor

$$\mathrm{Glue}(\check{G})_{\mathrm{spec}} \xrightarrow{\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}}} \mathrm{Whit}^{\mathrm{ext}}(G, G)$$

$$\mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \uparrow$$

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

is contained in the essential image of the functor

$$\mathrm{Whit}^{\mathrm{ext}}(G, G)$$

$$\uparrow \mathrm{coeff}_{G,G}^{\mathrm{ext}}$$

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

The latter is enough to check on the generators of $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$.

By Conjecture 10.2.11 and Proposition 6.4.7(b), the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ is generated by the union of the essential images of the following functors:

- $\mathrm{Eis}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ for all *proper* parabolics $P \subset G$.

- $\Xi_{\check{G}} \circ \jmath_* \circ (\mathsf{v}_{\lambda^I})_* : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ for all finite sets I and $\lambda^I : I \to \Lambda^+$, where $\jmath$ denotes the open embedding $\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}} \hookrightarrow \mathrm{LocSys}_{\check{G}}$.

11.2.2. The containment of Sect. 11.2.1 for the functors

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

is equivalent to that for the functors

$$\mathrm{Eis}_{\check{P},\mathrm{spec}}^{\mathrm{enh}} : \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{P}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{P}})) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$$

For the latter, it follows from Quasi-Theorem 9.5.3.

11.2.3. It remains to show the containment of Sect. 11.2.1 for the functor

$$\Xi_{\check{G}} \circ \jmath_* \circ (\mathsf{v}_{\lambda^I})_* : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

for a fixed finite set I and $\lambda^I : I \to \Lambda^+$.

Let $\mathcal{F}$ be an object of $\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob,irred}})$. We claim that

$$(11.2) \qquad \mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \circ \Xi_{\check{G}} \circ \jmath_* \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F}) \simeq \mathrm{coeff}_{G,G}^{\mathrm{ext}} \circ \text{q-Hitch}_{\lambda^I}(\mathcal{F}).$$

Clearly, (11.2) would imply the required assertion.

11.2.4. First, we note that the isomorphsim

$$\mathbf{j}^\bullet \left(\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \circ \Xi_{\check{G}} \circ \jmath_* \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F}) \right) \simeq \mathbf{j}^\bullet \left(\mathrm{coeff}_{G,G}^{\mathrm{ext}} \circ \text{q-Hitch}_{\lambda^I}(\mathcal{F}) \right)$$

follows from Corollary 10.4.5.

Second, we note that, by definition,

$$\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \circ \Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F}) \simeq \mathbf{j}_\dagger \circ \mathbf{j}^\bullet \left(\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \circ \Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F}) \right).$$

Hence, it is enough to show that the canonical map

$$(11.3) \qquad \mathbf{j}_\dagger \circ \mathbf{j}^\bullet \left(\mathrm{coeff}_{G,G}^{\mathrm{ext}} \circ \text{q-Hitch}_{\lambda^I}(\mathcal{F}) \right) \to \mathrm{coeff}_{G,G}^{\mathrm{ext}} \circ \text{q-Hitch}_{\lambda^I}(\mathcal{F})$$

is an isomorphism.

11.2.5. We consider the category $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ as acted on by the monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$, and recall that the functor $\mathrm{coeff}^{\mathrm{ext}}_{G,G}$ respects this action.

Consider the cone of the map (11.3); denote it by $\mathcal{M}'$. On the one hand, by Theorem 10.3.4(b), the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on $\mathcal{M}'$ factors through

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \overset{\mathrm{Loc}_{\check{G},\mathrm{spec}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \overset{\jmath^*}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}).$$

On the other hand, we claim that for any $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$, the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on

$$\mathcal{M}' := \mathrm{Cone}\left(\mathbf{j}_\dagger \circ \mathbf{j}^\bullet(\mathrm{coeff}^{\mathrm{ext}}_{G,G}(\mathcal{M})) \to \mathrm{coeff}^{\mathrm{ext}}_{G,G}(\mathcal{M})\right)$$

factors through

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \overset{\mathrm{Loc}_{\check{G},\mathrm{spec}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})_{\mathrm{LocSys}^{\mathrm{red}}_{\check{G}}},$$

where $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})_{\mathrm{LocSys}^{\mathrm{red}}_{\check{G}}}$ is quotient of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ by the monoidal ideal given by the embedding $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{irred}}_{\check{G}}) \overset{\jmath_*}{\hookrightarrow} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$.

This would imply that $\mathcal{M}' = 0$.

11.2.6. The object $\mathcal{M}'$ admits a filtration with subquotients of the form

$$(\mathbf{i}_P)_\dagger \circ (\mathbf{i}_P)^\dagger(\mathrm{coeff}^{\mathrm{ext}}_{G,G}(\mathcal{M})) \simeq (\mathbf{i}_P)_\dagger \circ \mathrm{coeff}_{G,P}(\mathcal{M})$$

for the proper parabolics P of G.

Hence, it suffices to check that for any proper parabolic $P \subset G$, the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ on

$$\mathrm{coeff}_{G,P}(\mathcal{M}) \in \mathrm{Whit}(G,P)$$

factors through

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \overset{\mathrm{Loc}_{\check{G},\mathrm{spec}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \overset{\mathsf{p}^*_{\check{P},\mathrm{spec}}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}).$$

By (7.3), it suffices to establish the said factorization for the object

$$\mathrm{CT}^{\mathrm{enh}}_P(\mathcal{M}) \in \mathrm{I}(G,P).$$

Now the required assertion follows from Quasi-Theorem 6.6.2(b).

11.3. **Proof of the equivalence.** We will now show that the functor $\mathbb{L}_G$, whose existence was proved above, is an equivalence.

11.3.1. First, we claim that $\mathbb{L}_G$ is fully faithful. This follows from Theorem 9.3.8 and Conjecture 8.2.9.

To prove that $\mathbb{L}_G$ is essentially surjective, it is enough to show that the generators of $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ belong to the essential image of $\mathbb{L}_G$.

By Theorem 11.1.1 and Proposition 10.3.2(b), the category $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ is generated by the union of the essential images of the following functors:

- $\mathrm{Eis}_P : \mathrm{D\text{-}mod}(\mathrm{Bun}_M) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ for all proper parabolics $P \subset M$.
- $\mathrm{q\text{-}Hitch}_{\lambda^I} : \mathrm{QCoh}(\mathrm{Op}(\check{G})^{\mathrm{glob}}_{\lambda^I}) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ for all finite sets I and $\lambda^I : I \to \Lambda^+$.

11.3.2. First, we claim that the essential image of Eis_P is contained in the essential image of $\mathbb{L}_G$. This is equivalent for the corresponding assertion for the functor $\mathrm{Eis}_P^{\mathrm{enh}}$.

For the latter it suffices to show that the essential image of the functor

$$
\begin{array}{c}
\mathrm{Whit}^{\mathrm{ext}}(G,G) \\
\uparrow{\scriptstyle \mathrm{coeff}_{G,G}^{\mathrm{ext}}} \\
\mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\
{\scriptstyle \mathrm{Eis}_P^{\mathrm{enh}}}\uparrow \\
\mathrm{I}(G,P)
\end{array}
$$

is contained in the essential image of the functor

$$
\begin{array}{c}
\mathrm{Glue}(\check{G})_{\mathrm{spec}} \xrightarrow{\ \mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}}\ } \mathrm{Whit}^{\mathrm{ext}}(G,G) \\
{\scriptstyle \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}})}\uparrow \\
\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).
\end{array}
$$

However, this follows from from Quasi-Theorem 9.5.3.

11.3.3. *A digression.* Let $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{Eis}}$ denote the full subcategory of $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ generated by the essential images of the functors Eis_P, for all proper parabolics $P \subset G$. By the above, the subcategory $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{Eis}}$ is contained in the essential image of the functor $\mathbb{L}_G$.

This, every $\mathcal{M} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ canonically fits in an exact triangle

$$\mathcal{M}_{\mathrm{Eis}} \to \mathcal{M} \to \mathcal{M}_{\mathrm{cusp}},$$

where $\mathcal{M}_{\mathrm{Eis}} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{Eis}}$ and $\mathcal{M}_{\mathrm{cusp}} \in \mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$.

11.3.4. It remains to show that for a finite set I, $\lambda^I : I \to \Lambda^+$, and $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\lambda^I}^{\mathrm{glob}})$, the object $\mathrm{q\text{-}Hitch}_{\lambda^I}(\mathcal{F})$ belongs to the essential image of $\mathbb{L}_G$.

By the above, it is sufficient to show that the object

$$\left(\mathrm{q\text{-}Hitch}_{\lambda^I}(\mathcal{F})\right)_{\mathrm{cusp}}$$

belongs to the essential image of $\mathbb{L}_G$.

We will construct an isomorphism

$$\left(\mathbb{L}_G(\Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F}))\right)_{\mathrm{cusp}} \simeq \left(\mathrm{q\text{-}Hitch}_{\lambda^I}(\mathcal{F})\right)_{\mathrm{cusp}}.$$

Remark 11.3.5. It will follow *a posteriori* that we actually have an isomorphism

$$\mathbb{L}_G(\Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F})) \simeq \mathrm{q\text{-}Hitch}_{\lambda^I}(\mathcal{F}),$$

which amounts to Property $\mathrm{Km}^{\mathrm{prel}}$ in Conjecture 3.4.2.

11.3.6. Let us construct a map

$$(11.4) \qquad \mathbb{L}_G(\Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F})) \to \text{q-Hitch}_{\lambda^I}(\mathcal{F}).$$

By Conjecture 8.2.9, this amounts to a map

$$(11.5) \qquad \text{coeff}^{\text{ext}}_{G,G} \circ \mathbb{L}_G(\Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F})) \to \text{coeff}^{\text{ext}}_{G,G}(\text{q-Hitch}_{\lambda^I}(\mathcal{F})).$$

Note that by Corollary 10.4.5, we have an isomorphism

$$\text{coeff}_{G,G} \circ \mathbb{L}_G(\Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F})) \xrightarrow{\sim} \text{coeff}_{G,G}(\text{q-Hitch}_{\lambda^I}(\mathcal{F})).$$

Furthermore, by construction, the map

$$\mathbf{j}_\dagger \circ \mathbf{j}^\bullet \left(\text{coeff}^{\text{ext}}_{G,G} \circ \mathbb{L}_G(\Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F})) \right) \to \text{coeff}^{\text{ext}}_{G,G} \circ \mathbb{L}_G(\Xi_{\check{G}} \circ (\mathsf{v}_{\lambda^I})_*(\mathcal{F}))$$

is an isomorphism.

This gives rise to the desired map in (11.5).

11.3.7. Let $\mathcal{M}$ denote the cone of the map (11.4). By construction,

$$\mathbf{j}^\bullet \circ \text{coeff}^{\text{ext}}_{G,G}(\mathcal{M}) = \text{coeff}_{G,G}(\mathcal{M}) = 0.$$

We wish to show that $\mathcal{M}_{\text{cusp}} = 0$, which is equivalent to showing that the canonical map

$$\mathcal{M} \to \mathcal{M}_{\text{cusp}}$$

vanishes.

By Conjecture 8.2.9, it suffices to show that the map

$$\text{coeff}^{\text{ext}}_{G,G}(\mathcal{M}) \to \text{coeff}^{\text{ext}}_{G,G}(\mathcal{M}_{\text{cusp}})$$

vanishes.

However, since $\mathcal{M}_{\text{cusp}} \in \text{D-mod}(\text{Bun}_G)_{\text{cusp}}$, the canonical map

$$\text{coeff}^{\text{ext}}_{G,G}(\mathcal{M}_{\text{cusp}}) \to \mathbf{j}_\bullet \circ \mathbf{j}^\bullet(\text{coeff}^{\text{ext}}_{G,G}(\mathcal{M}_{\text{cusp}}))$$

is an isomorphism (see Corollary 8.4.3).

Hence, the required vanishing holds by the $(\mathbf{j}^\bullet, \mathbf{j}_\bullet)$-adjunction.

11.4. **Proof of the properties and further remarks.**

11.4.1. Thus, the equivalence $\mathbb{L}_G$, satisfying Property Wh^{ext}, claimed in Conjecture 3.4.2(a) has been constructed. Let us now prove the properties claimed in Conjecture 3.4.2(b).

Property He^{naive} follows from Corollary 9.4.8.

Property Ei^{enh} follows from Property Wh^{ext} and Quasi-Theorem 9.6.

Property Km^{prel} follows from Property Wh^{ext} and Theorem 10.4.5, combined with the fact that the essential image of the functor Loc_G belongs to $\text{D-mod}(\text{Bun}_G)_{\text{temp}}$, using the intrinsic characterization of the latter given in Sect. 4.6.7 (see Remark 10.1.10).

11.4.2. *Interdependence of the conjectures.* Recall, however, that the above proof of Conjecture 3.4.2 was conditional on the validity of Conjectures 8.2.9 and 10.2.8.

Let us now assume Conjecture 3.4.2 and comment on the above supporting conjectures.

First, we note that Conjecture 10.2.8 follows formally from Theorem 11.1.1 modulo Conjecture 3.4.2.

Second, we note that Conjecture 8.2.9 is equivalent to Theorem 9.3.8 modulo Conjecture 3.4.2.

I.e., we obtain that Conjectures 8.2.9 and 10.2.8 are forced by Conjecture 3.4.2.

11.4.3. *Implications for the cuspidal category.* Let

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{cusp}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

be the full subcategory equal to the right orthogonal of the essential images of the functors

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{M}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$$

for proper parabolics $P \subset G$.

The following results, e.g., from Proposition 6.4.7:

Corollary 11.4.4. *The subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{cusp}}$ *equals the image of*

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}) \xrightarrow{\jmath_*} \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \overset{\Xi_{\check{G}}}{\rightrightarrows} \mathrm{IndCoh}_{\mathrm{Nilp}_{\check{G}}^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})_{\mathrm{cusp}}.$$

Hence, assuming Conjecture 3.4.2, we obtain:

Corollary-of-Conjecture 11.4.5.

(a) *We have an inclusion* $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{cusp}} \subset \mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{temp}}$.

(b) *We have:*

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{cusp}} = \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{irred}}) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

as subcategories of $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$.

11.4.6. *Generation by Kac-Moody representations.* Finally, we note that if we accept Conjecture 10.5.10, by combining with Conjecture 3.4.2, we obtain:

Corollary-of-Conjecture 11.4.7. *The functor*

$$\mathrm{Loc}_G : \mathrm{KL}(G,\kappa)_{\mathrm{Ran}(X)} \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{temp}}$$

is a localization, *i.e., identifies the homotopy category of the target with a Verdier quotient of the source.*

REFERENCES

[AG] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, arXiv: 1201.6343.

[ArkhG] S. Arkhipov and D. Gaitsgory, *Differential operators on the loop group via chiral algebras*, Internat. Math. Res. Notices, **2002-4**, 165–210 (2002).

[Bar] J. Barlev, *D-modules on spaces of rational maps and on other generic data*, arXiv: 1204.3469.

[BB] A. Beilinson and J. Bernstein, *Localisation de g-modules*, C.R. Acad. Sci. Paris Ser. I Math. **292**, 15–18 (1981).

[BD1] A. Beilinson and V. Drinfeld, *Chiral algebras*, AMS Colloquium Publications **51**, AMS (2004).

[BD2] A. Beilinson and V. Drinfeld, *Quantization of Hitchins integrable system and Hecke eigensheaves*, available at http://math.uchicago.edu/ mitya/langlands.html.

[Bez] R. Bezrukavnikov, *On two geometric realizations of the affine Hecke algebra*, arXiv:1209.0403.

[BF] R. Bezrukavnikov and M. Finkelberg, *Equivariant Satake category and Kostant-Whittaker reduction*, Mosc. Math. J. **8**, 39–72 (2008).

[BG] A. Braverman, D Gaitsgory, *Deformations of local systems and Eisenstein series*, Geometric and functional analysis, **17**, 1788–1850 (2008).

[Dr] V. Drinfeld, *Two-dimensional l-adic representations of the fundamental group of a curve over a finite field and automorphic forms on GL(2)*, Amer. Jour. of Math. **105**, 85–114 (1983)

[DrGa1] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, GAFA **23** (2013), 149–294.

[DrGa2] V. Drinfeld and D. Gaitsgory, *Compact generation of the category of D-modules on the stack of G-bundles on a curve*, arXiv: 1112.2402.

[DrGa3] V. Drinfeld and D. Gaitsgory, *Geometric constant term functor(s)*, arXiv: 1311.2071.

[Fr] E. Frenkel, *Affine Algebras, Langlands Duality and Bethe Ansatz*, in Proceedings of the International Congress of Mathematical Physics, Paris, 1994, ed. D. Iagolnitzer, 606–642, International Press (1995); arXiv:q-alg/9506003.

[FG1] E. Frenkel and D. Gaitsgory, *Local geometric Langlands correspondence: the spherical case*, in Algebraic analysis and around: In honor of Professor Masaki Kashiwara's 60th Birthday, Advanced Studies in Pure Mathematics **54**, 167–186 (2009); arXiv:0711.1132.

[FG2] E. Frenkel and D. Gaitsgory, *D-modules on the affine flag variety and representations of affine Kac-Moody algebras*, J. Represent. Theory **13**, 470–608 (2009).

[FG3] E. Frenkel and D. Gaitsgory, *Local geometric Langlands correspondence and affine Kac-Moody algebras*, in Algebraic Geometry and Number Theory, Progr. Math. **253**, 69–260 (2006).

[FGKV] E. Frenkel, D. Gaitsgory, D. Kazhdan and K. Vilonen, *Geometric realization of Whittaker functions and the Langlands conjecture*, J. Amer. Math. Soc. **11**, 151–484 (1998).

[FGV1] E. Frenkel, D. Gaitsgory and K. Vilonen, *Whittaker patterns in the geometry of moduli spaces of bundles on curves*, Annals of Math. **153**, 699–748 (2001).

[FGV2] E. Frenkel, D. Gaitsgory and K. Vilonen *On the geometric Langlands conjecture*, J. Amer. Math. Soc. **15**, 367–417 (2002).

[Ga1] D. Gaitsgory, *On a vanishing conjecture appearing in the geometric Langlands correspondence*, Ann. of Math. (2) **160**, no. 2, 617–682 (2004).

[Ga2] D. Gaitsgory, *Contractibility of the space of rational maps*, Invent. Math. **191** (2013), 91–196.

[Ga3] D. Gaitsgory, *Ind-coherent sheaves*, Moscow Mathematical J. **13** (2013), 399–528.

[GR] D. Gaitsgory and N. Rozenblyum, *Crystals and D-modules*, arXiv: 1111.2087.

[GL:DG] Notes on Geometric Langlands, *DG categories*,
available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:Stacks] Notes on Geometric Langlands, *Stacks*,
available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:QCoh] Notes on Geometric Langlands, *Quasi-coherent sheaves on stacks*,
available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:GenVan] Notes on Geometric Langlands, *A Generalized vanishing conjecture*,
available at http://www.math.harvard.edu/ gaitsgde/GL/.

[GL:ExtWhit] Notes on Geometric Langlands, *The extended Whittaker category*,
available at http://www.math.harvard.edu/ gaitsgde/GL/.

[Lau1] G. Laumon, *Correspondence de Langlands géométrique pour les corps de fonctions*, Duke Math. Jour. **54**, 309–359 (1987)

[Lau2] G. Laumon, *Faisceaux automorphes pour GLn: la première construction de Drinfeld*, arXiv:alg-geom/9511004.

[Lu] J. Lurie, *Higher Topos Theoery*, Princeton Univ. Press (2009).

[MV] I. Mirkovic and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. **166**, 95–143 (2007).

[Ne] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9**, 205–236 (1996).

[Ro] N. Rozenblyum, *Connections on Conformal Blocks*, PhD Thesis, MIT (2011).

[Sto] A. Stoyanovsky, *Quantum Langlands duality and conformal field theory*, arXiv:math/0610974