MATH 221, PROBLEM SET 5, DUE: MON., OCT. 20.

1. Let A be a commutative ring. Show that the property of an A-module to be flat is local.

2.(a) Let A be a commutaive ring. For A-modules M and N consider $\mathrm{Hom}_A(M,N)$ as an A-module. Let $S \subset A$ be a multiplicative set. Construct a map $(\mathrm{Hom}_A(M,N))_S \to \mathrm{Hom}_{A_S}(M_S, N_S)$.

(b) Show that the map in (a) is injective if M is finitely generated.

(c) Deduce that the map in (a) is an isomorphism, provided that M is finitely generated *and* A is Noetherian. Hint: consider a short exact sequence $0 \to M' \to A^n \to M \to 0$ and use the diagram chase.

(d) Show that over a Noetherian ring the property of being finitely generated and projective is local.

3.(a) Give an example of an infinitely generated module over $\mathbb{Z}$, whose support is not closed.

(b) Show that if $M_1 \subset M$, then $supp(M_1) \subset supp(M)$.

(c) Deduce that $supp(M) = \underset{M' \subset M}{\cup} supp(M')$, where the union is taken over the set of all finitely-generated submodules of M.

(d) Deduce that $supp(M)$ contains a closed point.

4.(a) Show that $M_{\mathfrak{p}} = 0$ for every prime $\mathfrak{p}$ implies $M = 0$.

(b) Show that if $M_{\mathfrak{p}}$ is flat for every prime $\mathfrak{p}$, then M is flat.

(c) Give an example of a module M over $\mathbb{C}[x]$ with $M_{\mathfrak{p}}$ being finitely-generated for every prime $\mathfrak{p}$, but M itself not finitely generated.

5.(a) Show that $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ implies $M = 0$. Hint: use 3(d).

(b) Show that $M_{\mathfrak{m}}$ is flat for every maximal ideal $\mathfrak{m}$, then M is flat.

6. Let A be a ring such that $\mathrm{Spec}(A)$ is disconnected, i.e., is a union of two closed subsets V_1 and V_2. Show that there exists a unique decomposition $A = A_1 \oplus A_2$ such that $V_1 = V(0 \oplus A_2) \simeq \mathrm{Spec}(A_1)$ and $V_2 = V(A_1 \oplus 0) \simeq \mathrm{Spec}(A_2)$.

7.(optional) Show that if A is a local Artinian ring and $\mathfrak{m}$ is its maximal ideal, then $\mathfrak{m}^n = 0$ for some n. Deduce that every Artinian ring is Noetherian.

8.(a) Let V be a finite-dimensional space over an algebraically closed field k and $T : V \to V$ a linear map. Deduce the decomposition of V into generalized eigenspaces from the structure theorem for Artinian rings.

(b) Let $A = \mathbb{C}[x_1, ..., x_n]$. Describe all reduced Artinian quotient rings of A.

(c) Let V be a finite-dimensional vector spaces and $T_1, ..., T_n$ pairwise commuting endomorphisms. Formulate and prove the decomposition of V into simultaneous generalized eigenspaces for $T_1, ..., T_n$.

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(d) Let A be a finite abelian group. Deduce the decomposition of A as a direct sum of p-groups from the structure theorem for Artinian rings.