MATH 221, PROBLEM SET 4, DUE: WED. 15.

1. Let $\phi : A \to B$ be a map of rings. Define the map $\Phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ by $\Phi(\dot{\mathfrak{q}}) = \dot{\mathfrak{p}}$, where $\mathfrak{p} := \phi^{-1}(\mathfrak{q})$.

(a) Show that this map is continuous.

(b) Show that $\Phi^{-1}(V(I)) = V(B \cdot \phi(I))$.

(c) Show that $\Phi^{-1}(U_f) = U_{\phi(f)}$.

(d) Let J be an ideal in B. Show that $V(\phi^{-1}(J)) = \overline{\Phi(V(J))}$.

(e) Deduce that Φ has a dense image if and only if $\ker(\phi) \subset rad(0)$.

2. Show that if $A \to B$ is surjective, then Φ homeomorphs $\mathrm{Spec}(B)$ onto $V(I)$, where $I = \ker(\phi)$.

3. Let A be a ring and $S \subset A$ a multiplicative set.

(a) Let $I \subset A$ be an ideal. Show that $I_S \subset A_S$ is trivial if and only if $S \cap I \neq \emptyset$, and that A_S/I_S is the localization of A/I with respect to the image of S in A/I.

(b) Show that if $\mathfrak{p} \subset A$ is prime, and $\mathfrak{p} \cap S = \emptyset$, then $\mathfrak{p}_S$ is also prime.

(c) For an ideal $J \subset A_S$ let $cl(J)$ denote its preimage in A. For an ideal $I \subset A$ let $res(I) := I_S \subset A_S$. Show that $res(cl(J)) = J$ and $I \subset cl(res(I))$; give an example where the latter inclusion is not equality. (Notation "cl" is for "closure", "res" is for "restriction", see below.)

(d) Let $\mathfrak{p} \subset A$ be a prime idea such that $\mathfrak{p} \cap S = \emptyset$. Show that in this case $\mathfrak{p} = cl(res(\mathfrak{p}))$.

4. Let A be a ring and $S \subset A$ a multiplicative set.

(a) Show that the homomorphism of rings $A \to A_S$ induces a map of topological spaces $\mathrm{Spec}(A_S) \to \mathrm{Spec}(A)$, which is a homeomorphism onto the subspace consisting of $\{\dot{\mathfrak{p}} \mid S \cap \mathfrak{p} = \emptyset\}$.

(b) Let $I \subset A$ be an ideal. Show that $V(I) \cap \mathrm{Spec}(A_S) = V(res(I))$.

(c) Let J be an ideal in $\mathrm{Spec}(A_S)$. Show that $V(cl(J)) = \overline{V(J)}$.

5. Let A be a ring.

(a) Show that open subsets U_f form a basis of the Zariski topology.

(b) Show that $U_f \cap U_g = U_{f \cdot g}$.

(c) Show that $f_1, ..., f_n$ cover $\mathrm{Spec}(A)$ if and only if $f_1, ..., f_n$ generate the unit ideal.

(d) Show that $A \to A_f$ induces a homeomorphism $\mathrm{Spec}(A_f) \to U_f$.

6. Recall that a topological space is called irreducible if and only the intersection of any two non-empty open subsets is non-empty.

(a) Show that $V(I) \subset \mathrm{Spec}(A)$ is irreducible if and only if $rad(I)$ is prime.

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(b) A closed subspace $Y \subset X$ is an "irreducible component" of X if Y is irreducible and Y is not contained in $\overline{X - Y}$. Show that if $V(\mathfrak{p}) \subset \mathrm{Spec}(A)$ is an irreducible component, then $\mathfrak{p}$ is a minimal prime.

(c) A topological space X is called Noetherian if any decreasing sequence $Y_1 \supset Y_2 \supset Y_3 \supset ...$ of closed subsets stabilizes. Show that $\mathrm{Spec}(A)$ is Noetherian if A is Noetherian. Show that the converse is not always true.

(d) Let X be a Noetherian topological space. Show that X can be uniquely written as $\underset{i=1,...,n}{\cup} Y_i$, where Y_i are (distinct) irreducible components. Show that every irreducible component appears in the above decomposition.

(e) Let A be Noetherian. Show that if a prime ideal $\mathfrak{p}$ is minimal, then $V(\mathfrak{p})$ is an irreducible component.

(f) Deduce that in a Noetherian ring A there are finitely many minimal prime ideals and that their intersection is the set of nilpotent elements in A.

7. Let A be a finitely generated k-algebra, where k is a field. We define $\mathrm{Specm}(A)$ as a subset of $\mathrm{Spec}(A)$ consisting of maximal ideals, with the induced topology.

(a) Show that a map of finitely generated algebras $\phi : A \to B$ induces a map $\Phi : \mathrm{Specm}(B) \to \mathrm{Specm}(A)$.

From now on assume that k is algebraically closed.

(b) Determine $\mathrm{Specm}(k[x])$, $\mathrm{Specm}(k[x]/x^2)$, $\mathrm{Specm}(k[x_1, ..., x_n])$, $\mathrm{Specm}(k[x, y]/xy)$, $\mathrm{Specm}(k[x, y]/x^2 - y)$.

(c) Let a map $\phi : k[x_1, ..., x_n] \to k[y_1, ..., y_m]$ be given by $x_i \mapsto f_i \in k[y_1, ..., y_m]$, $i = 1, ..., n$. Show that the map

$$\Phi : k^m \simeq \mathrm{Specm}(k[y_1, ..., y_m]) \to \mathrm{Specm}(k[x_1, ..., x_n]) \simeq k^n$$

is given by $(c_1, ..., c_m) \mapsto (f_1(c_1, ..., c_m), ..., f_n(c_1, ..., c_m))$.

(d) Show that the topological space $\mathrm{Specm}(A)$ is T1.

(e) Show that $\mathrm{Specm}(A)$ is Hausdorff if and only if $A/rad(0)$ is the direct sum of several copies of the field k. Hint: use Problem 6(d).