MATH 221, PROBLEM SET 3, DUE: OCT. 6.

Problems marked with (*) are more difficult, but still mandatory.

1. Let $A \to B$ be a homomorphism of rings.
(a) Show that the preimage of a prime ideal is always prime.
(b) Give an example that a preimage of a maximal ideal is not maximal.
(c) Show that if A and B are k-algebras (where k is an arbitrary field) with B finitely generated, then the preimage of any maximal ideal is again maximal. Hint: you'll need to use EEKS.

2.(a) Let A be a commutative ring. Show that the intersection of all prime ideals coincides with the set of nilpotent elements in A. Hint: localize!
(b) Let A be f.g. over a field. Show that the intersection of all maximal ideals coincides with the set of nilpotent elements in A.

3. Let A be a ring and $S \subset A$ a multiplicative subset; consider the natural isomorphism $\phi_{can} : A \to A_S$. Show that ϕ_{can} is injective if and only if for $a \in A$ and $s \in S$, $a \cdot s = 0 \Rightarrow a = 0$.

4. Let A be an algebra and $S_1, S_2 \subset A$ two multiplicative subsets. Show that the A-algebra $A_{S_1} \underset{A}{\otimes} A_{S_2}$ is either 0, or is a localization of A with respect to a certain multiplicative subset (which one?).

5. Show that for any ring A and a multiplicative subset S, the A-algebra A_S is flat as an A-module.

6. Let $\phi : A \to B$ be an algebra homomorphism, and $S \subset A$ a multiplicative subset such that $0 \notin \phi(S)$. Show that $B_{\phi(S)}$ is naturally an A_S-algebra, which as an A_S-module is isomorphic to B_S.

7. Let A be a ring, let M be an A-module, and $S \subset A$ a multiplicative subset. Let $\phi_{can} : M \to M_S$ be the canonical map. We define a map

$$\{A-\text{submodules of } M\} \to \{A_S-\text{submodules of } M_S\}$$

by $M' \mapsto M'_S$. We define a map

$$sat : \{A_S-\text{submodules of } M_S\} \to \{A-\text{submodules of } M\}$$

by $N \mapsto \phi_{can}^{-1}(N)$.
(a) Show that $N = (sat(N))_S$.
(b) Show that $M' \subset sat(M'_S)$, and give an example that the inclusion is proper.
(c) Deduce that if A is Noetherian, then so is A_S.

8. Let A be a ring and $f \in A$ a non-nilpotent element. Show that for an A-module M,

$$M_f \simeq \underset{\to}{lim}\,(M \xrightarrow{f} M \xrightarrow{f} M...).$$

Date: October 1, 2008.