MATH 221, PROBLEM SET 13, DUE: MON., DEC. 22.

1. Let A be a Noetherian ring and $\mathfrak{a} \subset A$ an ideal. Let M be a f.g. A-module endowed with an $\mathfrak{a}$-good filtration. Show that $\mathrm{gr}(M)$ is f.g. an $\mathrm{gr}(A)$-module. (For a partial converse see Problem 2.)

2. Let A be a commutative ring; $\mathfrak{a} \subset A$ an ideal. Let M be an A-module, endowed with an $\mathfrak{a}$-compatible filtration, such that M is complete in the resulting topology. Assume that $\mathrm{gr}(M)$ is f.g. as an $\mathrm{gr}(A)$-module. Show that the filtration on M is $\mathfrak{a}$-good.

3. Let A be Noetherian commutative ring, complete in the topology defined by an ideal $\mathfrak{a} \subset A$. Let M be a f.g. A-module.

(a) Show that M is complete in the topology defined by any $\mathfrak{a}$-good filtration.

(b*) Let $M' \subset M$ be an A-submodule. Show that M' is closed. Hint: use point (a) and the Artin-Rees lemma.

4. Take $A = k[t]$ with the t-adic filtration. Let $M = k[t, t^{-1}]/k[t]$. Show that the t-adic filtration on M is non-Hausdorff. In fact, show that $\hat{M} = 0$. Show, however, that $M \underset{k[t]}{\otimes} k[[t]] \neq 0$.

5.(a) Let $K_2 \supset K_1$ be a field extension of *tr.deg.* equal n and $K_3 \supset K_2$ be a field extension of *tr.deg.* equal m. Show that $K_3 \supset K_1$ is a field extension of *tr.deg.* equal $n + m$.

NB: In what follows for an algebra A, finitely generated over a field k, we shall denote by $\dim_{Alg}(A)$ its dimension defined e.g. via Hilbert polynomials. This should not be confused with $\dim_k(A)$, when A is regarded as a mere vector space over k when it happens to be finite-dimensional. BTW, if A is f.d. over k, then $\dim_{Alg}(A) = 0$.

(b) Let $A \to B$ be a map of algebras f.g. over a field k. Assume that A is a domain and let K denote its field of fractions. Assume that for every irreducible component $Y \subset \mathrm{Spec}(B)$, the map $Y \to \mathrm{Spec}(A)$ is dominant (i.e., has a dense image). Show that $\dim_{Alg}(B) - \dim_{Alg}(A)$ equals $\dim_{Alg}(B \underset{A}{\otimes} K)$, where the latter is regarded as a f.g. K-algebra.

6. Let $\phi : A \to B$ be a map of algebras finitely generated over a field k. Assume that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are irreducible. Let $\mathfrak{p} \subset A$ be a prime ideal and $k_{\mathfrak{p}}$ the corresponding residue field, and assume that $B \underset{A}{\otimes} k_{\mathfrak{p}}$ is non-zero and regard it as a f.g. algebra over $k_{\mathfrak{p}}$.

(a) Show that $\dim_{Alg}(B \underset{A}{\otimes} k_{\mathfrak{p}})$ (regarded as a f.g. algebra over $k_{\mathfrak{p}}$) is $\geq \dim_{Alg}(B) - \dim_{Alg}(A)$.

Date: December 14, 2008.

Remark: In Problem 9 we are going to show that the inequality in question is an equality as long as $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is dominant for $\dot{\mathfrak{p}}$ belonging to some non-empty Zariski open subset of $\operatorname{Spec}(A)$.

Suggested strategy: Assume by induction that the assertion holds for all A with $\dim(A) < n$. Reduce to the case when both A and B are domains. Consider two cases: $\mathfrak{p} = 0$ and $\mathfrak{p} \neq 0$; in the former case the assertion should follow from Problem 5 (see also Problem 8 on PS 9). Hence, it remains to consider the case $\mathfrak{p} \neq 0$. Let f be a non-zero element of $\mathfrak{p}$; consider the quotient algebras $A' = A/fA$, B'/fB. Let $Y_i = \operatorname{Spec}(B'_i)$ be an irreducible component of $\operatorname{Spec}(B')$ such that $\exists \dot{\mathfrak{q}} \in Y_i$ that maps to $\dot{\mathfrak{p}}$; let X_i be an irreducible component of $\operatorname{Spec}(A')$ that contains the image of Y_i. Apply Krull's Hauptidealsatz [1] and the induction hypothesis to deduce the assertion.

(b) Let $A = k[x, y]$, $B = k[z, w]$ and ϕ be such that $\phi(x) = z$ and $\phi(y) = z \cdot w$. Find the unique prime ideal in A for which in inequality in (a) is strict.

7. Let A, B be f.g. algebras over a field$k$. Assume that both A and B are *equidimensional.* [2] Let $\phi : A \to B$ be a map of k-algebras such that B is flat as an A-module. Show that for a prime ideal $\mathfrak{p} \subset A$, whenever $B \underset{A}{\otimes} k_{\mathfrak{p}}$ is non-zero, we have $\dim_{Alg}(B \underset{A}{\otimes} k_{\mathfrak{p}}) = \dim_{Alg}(B) - \dim_{Alg}(A)$.

Suggested strategy: Assume by induction that the assertion holds for all A with $\dim(A) < n$. Reduce to the case when A is a domain. Show that every irreducible component of $\operatorname{Spec}(B)$ maps dominantly to $\operatorname{Spec}(A)$. As in Problem 6, distinguish two cases $\mathfrak{p} = 0$ and $\mathfrak{p} \neq 0$. The former case should follow from Problem 5(b). In the latter case, choose a non-zero element $f \in \mathfrak{p}$, and consider the quotient algebras $A' = A/fA$; $B' = B/fB$. Use the flatness assumption to show that the image of f in B is a non-zero divisor. Apply Krull's Hauptidealsatz and the induction hypothesis to deduce the assertion.

8. Let $\phi : A \to B$ be a map of domains finitely generated over the ground field k. Show that there exist elements $g \in B$, $f \in A$ such that ϕ is well-defined as a map $A_f \to B_g$ and such that B_g is flat as an A_f-module.

Suggested strategy: show that if the assertion is true for maps $A \to B$ and $B \to C$, then it's true for $A \to C$. Argue by induction that one can assume that B is generated as an A-algebra by a single element f. Distinguish two cases: either $A[t] \to B, t \mapsto f$ is an isomorphism, and then we are done. Or, there exist coefficients $a_0, a_1, ..., a_n$ such that $a_n \cdot f^n + ... + a_1 \cdot f + a_0 = 0$. Show that in this case B_{a_0} is flat over A_{a_0}.

9. Let A, B, ϕ be as in Problem 6. Show that there exists a non-empty Zariski open $U \subset \operatorname{Spec}(A)$ such that for $\dot{\mathfrak{p}} \in U$ the inequality of Problem 6(a) becomes an equality.

Suggested strategy: Argue by induction on both $\dim(A)$ and $\dim(B)$. Reduce to the case when both A and B are integral. Use Problems 7 and 8.

---

[1] Krull's Hauptidealsatz, we remind, is the statement that for a domain C f.g. over a field and a non-zero divisor $f \in C$, for every irreducible component $\operatorname{Spec}(C'_k) = Z \subset \operatorname{Spec}(C/fC)$, we have $\dim_{Alg}(C'_k) = \dim_{Alg}(C) - 1$.

[2] An algebra C is called equidemensional if for every irreducible component $\operatorname{Spec}(C_k) = Z \subset \operatorname{Spec}(C)$, we have $\dim_{Alg}(C_k) = \dim_{Alg}(C)$.