MATH 221, PROBLEM SET 12, DUE: MON., DEC. 8.

1. Let $A \to B \to C$ be homomorphisms of commutative rings.

(a) Construct a natural map of B-modules $\Omega_A(B) \to \Omega_A(C)$ and a natural map of C-modules $\Omega_A(C) \to \Omega_B(C)$.

(b) Prove that we have an exact sequence

$$C \underset{B}{\otimes} \Omega_A(B) \to \Omega_A(C) \to \Omega_B(C) \to 0.$$

(c) Deduce that if B is f.g. as an A-algebra, then $\Omega_A(B)$ is f.g. as a B-module.

(d) Take $A = k$ (a field), $B = k[y,z]/y^3 = z^2$, $C = k[x]$, $B \to C$ given by $y \mapsto x^2, z \mapsto x^3$. Show that in this case $C \underset{B}{\otimes} \Omega_A(B) \to \Omega_A(C)$ is not injective. Hint: Problem 2 gives an algorithm to calculate $\Omega_A(B)$.

2. Let $B_1 \twoheadrightarrow B_2$ be a surjection of commutative A-algebras, and let $I \subset B_1$ be the kernel.

(a) Construct a canonical map $I/I^2 \simeq I \underset{B_1}{\otimes} B_2 \to \Omega_A(B_1) \underset{B_1}{\otimes} B_2$, such that there exists an exact sequence

$$I/I^2 \to \Omega_A(B_1) \underset{B_1}{\otimes} B_2 \to \Omega_A(B_2) \to 0.$$

(b) Let $B = k[x_1, ..., x_n]/(f_1, ..., f_m)$, where $f_k \in k[x_1, ..., x_n]$ are polynomials. Show that $\Omega_k(B)$ is generated, as a B-module, by symbols $dx_1, ..., dx_n$ with the relations given by $\underset{i=1,\ldots,n}{\Sigma} \frac{\partial f_k}{\partial x_i} \cdot dx_i = 0$, $k = 1, ..., m$.

(c) Take $A = k$, $B_1 = k[x]$, $B_2 = k[x]/x^2$. Show that in this case the map $I/I^2 \to \Omega_A(B_1) \underset{B_1}{\otimes} B_2$ is not injective.

3.(a) Let $A \to B$, $A \to A'$ be homomorphisms of commutative rings, and set $B' = A' \underset{A}{\otimes} B$. Show that $\Omega_A(B) \underset{B}{\otimes} B' \simeq \Omega_{A'}(B')$.

(b) Let $A \to B_1$ and $A \to B_2$ be homomorphisms of commutative rings. Construct an isomorphism

$$\Omega_A(B_1 \underset{A}{\otimes} B_2) \simeq \Big((B_1 \underset{A}{\otimes} B_2) \underset{B_1}{\otimes} \Omega_A(B_1)\Big) \oplus \Big((B_1 \underset{A}{\otimes} B_2) \underset{B_2}{\otimes} \Omega_A(B_2)\Big).$$

4. Let $A \to B$ be a homomorphism of commutative rings and let $S \subset B - 0$ be a multiplicative set. Show that $\Omega_A(B_S) \simeq (\Omega_A(B))_S$.

5. Let A be a f.g. algebra over an alg. closed field k; let $\mathfrak{m} \in \mathrm{Specm}(A)$ be a maximal ideal.

(a) Show that $\Omega_k(A) \underset{A}{\otimes} k \simeq \mathfrak{m}/\mathfrak{m}^2$ as k-vector spaces.

(b) Show that the set of k-algebra homomorphisms $A \to k[\epsilon]/\epsilon^2$ such that the composition

$$A \to k[\epsilon]/\epsilon^2 \to k[\epsilon]/\epsilon \simeq k$$

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equals

$$A \to A/\mathfrak{m} \simeq k$$

is canonically isomorphic to $(\mathfrak{m}/\mathfrak{m}^2)^*$.

Remark: The space $\text{Spec}(k[\epsilon]/\epsilon^2)$ should thought of as the point with an infinitesimal vector attached to it. So, point (b) says that maps from $\text{Spec}(k[\epsilon]/\epsilon^2)$ to $\text{Spec}(A)$ is isomorphic to what we called "the tangent space" to $\text{Spec}(A)$ at $\dot{\mathfrak{m}}$.

6. Let $k \subset k'$ be a finite field extension. Recall that it is said to be separable if and only if $\overline{k} \underset{k}{\otimes} k'$ is reduced, where $\overline{k}$ is the algebraic closure of k. Show that separability is equivalent to the following condition: $\Omega_k(k') = 0$. Hint: use Problems 3(a) and 5(a).

7. Consider the following short exact sequence of inverse families:

$$0 \to A_n \to B_n \to C_n \to 0,$$

where $B_n = k[t]$, $\forall n$, $C_n = k[t]/t^n$ and $A_n = t^n \cdot k[t]$. Show that in this case the map $\underset{\to}{lim}\, B_n \to \underset{\to}{lim}\, C_n$ is not surjective.