CHAPTER IV.3. FORMAL GROUPS AND LIE ALGEBRAS

Contents

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Introduction

In this chapter we will use the notion of inf-scheme to give what may be regarded as the ultimate formulation of the correspondence between formal groups and Lie algebras.

0.1. **Why does the tangent space of a Lie group have the structure of a Lie algebra?** In classical differential geometry the process of associating a Lie algebra to a Lie group is the following:

(i) For any manifold Y, one considers the associative algebra of global differential operators, endowed with its natural filtration;

(ii) One shows that the underlying Lie algebra is compatible with the filtration, and in particular $\text{ass-gr}^1(\text{Diff}(Y)) \simeq \Gamma(Y, T_Y)$ has a structure of Lie algebra;

(iii) If $Y = G$ is a Lie group, the operation of taking differential operators/vector fields, invariant with respect to left translations preserves the pieces of structure in (i) and (ii); in particular, left-invariant vector fields form a Lie algebra.

(iv) One identifies the tangent space at the identity of G with the vector space of left-invariant vector fields.

In the context of derived algebraic geometry, the process of associating a Lie algebra to a formal group is different, and we will describe it in this subsection.

0.1.1. We will work in a relative context over a given $\mathcal{X} \in \text{PreStk}_{\text{laft}}$ (the special case of $\mathcal{X} = \text{pt}$ is still interesting and contains all the main ideas). By a *formal group* we will mean an object of the category

$$\text{Grp}(\text{FormMod}_{/\mathcal{X}}),$$

where $\text{FormMod}_{/\mathcal{X}}$ is the full subcategory of $(\text{PreStk}_{\text{laft}})_{/\mathcal{X}}$ consisting of inf-schematic nil-isomorphisms $\mathcal{Y} \to \mathcal{X}$ (when $\mathcal{X} = \text{pt}$, this is the category of inf-schemes $\mathcal{Y}$ with ${}^{\text{red}}\mathcal{Y} = \text{pt}$).

We will define a functor

$$\text{Lie}_{\mathcal{X}} : \text{Grp}(\text{FormMod}_{/\mathcal{X}}) \to \text{LieAlg}(\text{IndCoh}(\mathcal{X})),$$

and the main goal of this Chapter is to show that it is an equivalence.

When $X = \text{pt}$ we obtain an equivalence between the category of group inf-schemes whose underlying reduced scheme is pt and the category of Lie algebras in Vect. Note that we impose no conditions on the cohomological degrees in which our Lie algebras are supposed to live.

0.1.2. To explain the idea of the functor $\text{Lie}_{\mathcal{X}}$, let us first carry it out for classical Lie groups; this will be a procedure of associating a Lie algebra to a Lie group different (but, of course, equivalent) to one described above.

Namely, let G be a Lie group. The space $\text{Distr}(G)$ of distributions supported at the identity of G has a natural structure of a co-commutative Hopf algebra (which will ultimately be identified with the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ associated to G).

Now, the Lie algebra $\mathfrak{g}$ can then be described as the space of primitive elements of $\text{Distr}(G)$.

0.1.3. We will now describe how the functor $\mathrm{Lie}_{\mathcal{X}}$ is constructed in the context of derived algebraic geometry. The construction will be compatible with pullbacks, so we can assume that $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

For any

$$(\mathcal{Y} \xrightarrow{\pi} X) \in \mathrm{FormMod}_{/X}$$

we consider

$$\mathrm{Distr}(\mathcal{Y}) := \pi_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}}) \in \mathrm{IndCoh}(X),$$

and we observe that it has a structure of co-commutative co-algebra in $\mathrm{IndCoh}(X)$, viewed as a symmetric monoidal category with respect to the !-tensor product. (If $X = \mathrm{pt}$, the dual of $\mathrm{Distr}(\mathcal{Y})$ is the commutative algebra $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$.)

We denote the resulting functor $\mathrm{FormMod}_{/X} \to \mathrm{CocomCoalg}(\mathrm{IndCoh}(X))$ by $\mathrm{Distr}^{\mathrm{Cocom}}$; one shows that it sends products to products. In particular, $\mathrm{Distr}^{\mathrm{Cocom}}$ gives rise to a functor

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}}) : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{Grp}(\mathrm{CocomCoalg}(\mathrm{IndCoh}(X))) =: \\ = \mathrm{CocomHopf}(\mathrm{IndCoh}(X)).$$

0.1.4. Recall now (see [Chapter IV.2, Sect. 4.4.2]) that the category $\mathrm{CocomHopf}(\mathrm{IndCoh}(X))$ is related by a pair of adjoint functors with the category $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$:

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \rightleftarrows \mathrm{CocomHopf}(\mathrm{IndCoh}(X)) : B_{\mathrm{Lie}} \circ \mathrm{Grp}(\mathrm{coChev}^{\mathrm{enh}}),$$

with the left adjoint being fully faithful.

Finally, we set

$$\mathrm{Lie}_{\mathcal{X}} := B_{\mathrm{Lie}} \circ \mathrm{Grp}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}}). \tag{0.1}$$

0.1.5. The upshot of the above discussion is the following: the appearance of the Lie algebra structure is due to the Quillen duality at the level of operads:

$$(\mathrm{Cocom}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Lie}[-1].$$

The shift $[-1]$ is compensated by delooping–this is where the group structure is used.

0.2. **Formal moduli problems and Lie algebras.** The equivalence

$$\mathrm{Lie}_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$$

allows us to recover Lurie's equivalence ([Lu5, Theorem 2.0.2]) between Lie algebras and formal moduli problems, as we shall presently explain.

0.2.1. Let $\mathcal{X}$ be an object $\mathrm{PreStk}_{\mathrm{laft}}$. Consider the category $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ of pointed objects in $\mathrm{FormMod}_{/\mathcal{X}}$. I.e., this is the category of diagrams

$$(\pi : \mathcal{Y} \rightleftarrows \mathcal{X} : s), \quad \pi \circ s = \mathrm{id}$$

with the map π being an inf-schematic nil-isomorphism.

Recall also that according to [Chapter IV.1, Theorem 1.6.4], the loop functor

$$\Omega_X : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$$

is an equivalence, with the inverse functor denoted B_X.

0.2.2. Thus, we obtain that the composition

$$\mathrm{Lie}_{\mathcal{X}} \circ \Omega_X : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \tag{0.2}$$

is an equivalence.

0.2.3. Let us now take $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Let us comment on the behavior of the functor $\mathrm{Lie}_X \circ \Omega_X$ in this case.

By construction, the above functor is

$$B_{\mathrm{Lie}} \circ \mathrm{Grp}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}}) \circ \Omega_X,$$

i.e., it involves first looping our moduli problem and then delooping at the level of Lie algebras.

Note, however, that there is another functor

$$\mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

Namely, the functor

$$\mathrm{Distr}^{\mathrm{Cocom}} : \mathrm{FormMod}_{/X} \to \mathrm{CocomCoalg}(\mathrm{IndCoh}(X))$$

gives rise to a functor

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Composing with the functor

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)),$$

we obtain a functor

$$\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)). \tag{0.3}$$

0.2.4. Now, the point is that the functors $\mathrm{Lie}_X \circ \Omega_X$ and $\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ are *not* isomorphic.

In terms of the equivalence (0.2), the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ corresponds to

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Hence, the discrepancy between $\mathrm{Lie}_X \circ \Omega_X$ and $\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ is the endo-functor of $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$ equal to

$$\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}}.$$

In particular, the unit of the adjunction defines a natural transformation

$$\mathrm{Lie}_X \circ \Omega_X \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}. \tag{0.4}$$

0.3. **Inf-affineness.** Let $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. In Sect. 2 we will introduce the notion of *inf-affineness* for objects of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$.

0.3.1. One of the equivalent conditions for an object $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ to be inf-affine is that the map (0.4) should be an isomorphism.

Another equivalent condition for inf-affineness is that the natural map

$$T(\mathcal{Y}/X)|_X \to \mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})$$

should be an isomorphism.

0.3.2. One of the ingredients in proving that Lie_X is an equivalence is the assertion that any $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$, viewed as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$, is inf-affine.

But, in fact, a stronger assertion is true.

0.3.3. To any $\mathcal{F} \in \mathrm{IndCoh}(X)$ we attach the *vector prestack*, denoted $\mathrm{Vect}_X(\mathcal{F})$. Namely, for

$$(f : S \rightleftarrows X : s) \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{\text{nil-isom to } X}$$

we set

$$\mathrm{Maps}_{/X}(S, \mathrm{Vect}_X(\mathcal{F})) = \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(S), \mathcal{F}),$$

where

$$\mathrm{Distr}^+(S) := \mathrm{Fib}(f_*^{\mathrm{IndCoh}}(\omega_S) \to \omega_X).$$

In Corollary 2.2.3 we show that $\mathrm{Vect}_X(\mathcal{F})$ is inf-affine.

0.3.4. It follows from [Chapter IV.2, Corollary 1.7.3] that any $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$, regarded as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$, is canonically isomorphic to

$$\mathrm{Vect}_X(\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{H}))).$$

0.4. **The functor of inf-spectrum and the exponential construction.** Let $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Above we have introduced the functor

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Another crucial ingredient in the proof of the fact that the functor Lie_X of (0.1) is an equivalence is the functor

$$\mathrm{Spec}^{\mathrm{inf}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X}),$$

right adjoint to $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$.

0.4.1. In terms of the equivalence (0.2), the functor $\mathrm{Spec}^{\mathrm{inf}}$ corresponds to the functor

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

For example, we have:

$$\mathrm{Vect}_X(\mathcal{F}) \simeq \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F})).$$

0.4.2. The notions of inf-affineness and inf-spectrum are loosely analogous to those of affineness and spectrum in algebraic geometry. But the analogy is not perfect. For example, it is *not* true that the functor $\mathrm{Spec}^{\mathrm{inf}}$ is fully faithful.

Conjecturally, the functor $\mathrm{Spec}^{\mathrm{inf}}$ maps $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ to the full subcategory of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ consisting of inf-affine objects.

0.4.3. We use the functor $\mathrm{Spec}^{\mathrm{inf}}$ to construct an inverse to the functor Lie_X:

$$\exp_X : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Grp}(\mathrm{FormMod}_{/X}).$$

Namely,

$$\exp_X := \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}.$$

0.4.4. The functor $\exp_{\mathcal{X}}$ (extended from the case of schemes to that of prestacks) can be used to give the following interpretation to the construction of the functor of *split square-zero extension*

$$\text{IndCoh}(\mathcal{X}) \to \text{Ptd}(\text{FormMod}_{/\mathcal{X}}),$$

extending the functor

$$\text{RealSplitSqZ} : (\text{Coh}(X)^{\leq 0})^{\text{op}} \to \text{Ptd}((\text{Sch}^{\text{aff}}_{\text{aft}})_{/X}), \quad X \in \text{Sch}^{\text{aff}}_{\text{aft}}$$

of [Chapter III.1, Sect. 2.1]. Here we regard $(\text{Coh}(X)^{\leq 0})^{\text{op}}$ as a full subcategory of $\text{IndCoh}(X)$ by means of

$$(\text{Coh}(X)^{\leq 0})^{\text{op}} \hookrightarrow \text{Coh}(X)^{\text{op}} \xrightarrow{\mathbb{D}^{\text{Serre}}_X} \text{Coh}(X) \hookrightarrow \text{IndCoh}(X).$$

Namely, we have:

$$\Omega_{\mathcal{X}} \circ \text{RealSplitSqZ}(\mathcal{F}) := \exp_{\mathcal{X}} \circ \mathbf{free}_{\text{Lie}} \circ \Omega,$$

where Ω on $\text{IndCoh}(\mathcal{X})$ is the functor of shift $[-1]$.

0.5. **What else is done in this chapter?** In this chapter we cover two more topics: the notion of action of objects in $\text{Grp}(\text{FormMod}_{/\mathcal{X}})$ on objects of $\text{IndCoh}(\mathcal{X})$ and on objects of $(\text{PreStk}_{\text{laft}})_{/\mathcal{X}}$.

0.5.1. For $\mathcal{H} \in \text{Grp}(\text{FormMod}_{/\mathcal{X}})$ we consider its Bar complex

$$B^{\bullet}(\mathcal{H}) \in (\text{FormMod}_{/\mathcal{X}})^{\mathbf{\Delta}^{\text{op}}}.$$

We define

$$\mathcal{H}\text{-mod}(\text{IndCoh}(X)) := \text{Tot}(\text{IndCoh}^!(B^{\bullet}(\mathcal{H}))).$$

In Sect. 5 we prove that the category $\mathcal{H}\text{-mod}(\text{IndCoh}(\mathcal{X}))$ identifies canonically with

$$\mathfrak{h}\text{-mod}(\text{IndCoh}(\mathcal{X})),$$

where $\mathfrak{h} = \text{Lie}_{\mathcal{X}}(\mathcal{H})$.

0.5.2. In Sect. 6 we study the (naturally defined) notion of action of $\mathcal{H} \in \text{Grp}(\text{FormMod}_{/\mathcal{X}})$ on $(\mathcal{Y} \xrightarrow{\pi} \mathcal{X}) \in (\text{PreStk}_{\text{laft}})_{/\mathcal{X}}$.

Given an action of $\mathcal{H}$ on $\mathcal{Y}$, we construct the localization functor

$$\text{Loc}_{\mathfrak{h},\mathcal{Y}/\mathcal{X}} : \mathfrak{h}\text{-mod}(\text{IndCoh}(\mathcal{X})) \to {}^{/\mathcal{X}}\text{Crys}(\mathcal{Y}).$$

Moreover, we show that given an action of $\mathcal{H}$ on $\mathcal{Y}$, we obtain a map in $\text{IndCoh}(\mathcal{Y})$

$$\pi^!(\mathfrak{h}) \to T(\mathcal{Y}/\mathcal{X}), \quad \mathfrak{h} = \text{Lie}_{\mathcal{X}}(\mathcal{H}).$$

We also show that if $\mathfrak{h}$ is free, i.e., $\mathfrak{h} = \mathbf{free}_{\text{Lie}}(\mathcal{F})$ with $\mathcal{F} \in \text{IndCoh}(\mathcal{X})$, then the map from the space of actions of $\mathcal{H}$ on $\mathcal{Y}$ to the space of maps $\pi^!(\mathcal{F}) \to T(\mathcal{Y}/\mathcal{X})$ is an isomorphism.

1. Formal moduli problems and co-algebras

As was mentioned in the introduction, our goal in this chapter is to address the following old question: what is exacty the relationship between formal groups and Lie algebras. By a Lie group we understand an object of $\text{Grp}(\text{FormMod}_{/X})$ and by a Lie algebra an object of $\text{LieAlg}(\text{IndCoh}(X))$.

In this section, we take the first step towards proving this equivalence. Namely, we establish a relationship between *pointed* formal moduli problems over X and co-commutative co-algebras in $\text{IndCoh}(X)$. Specifically, we define the functor of *inf-spectrum* that assigns to a co-commutative co-algebra a pointed formal moduli problem over X.

Formal moduli problems arising in this way play a role loosely analogous to that of affine schemes in the context of usual algebraic geometry.

1.1. **Co-algebras associated to formal moduli problems.**

To any scheme (affine or not) we can attach the commutative algebra of global sections of its structure sheaf. This functor is, obviously, contravariant.

It turns out that formal moduli problems are well-adapted for a dual operation: we send a moduli problem to the co-algebra of sections of its dualizing sheaf, which can be thought of as the co-algebra of distributions. In this subsection we describe this construction.

1.1.1. Let X be an object of ${}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}}$. We regard the category $\text{IndCoh}(X)$ as endowed with the symmetric monoidal structure, given by $\overset{!}{\otimes}$.

Recall the category $\text{FormMod}_{/X}$. We have a canonically defined functor

$$\text{FormMod}_{/X} \to (\text{DGCat}^{\text{SymMon}})_{\text{IndCoh}(X)/}, \quad \mathcal{Y} \mapsto \text{IndCoh}(\mathcal{Y}). \tag{1.1}$$

1.1.2. Consider the following general situation. Let $\mathbf{O}$ be a fixed symmetric monoidal category, and consider the category $(\text{DGCat}^{\text{SymMon}})_{\mathbf{O}/}$. Let

$$'(\text{DGCat}^{\text{SymMon}})_{\mathbf{O}/} \subset (\text{DGCat}^{\text{SymMon}})_{\mathbf{O}/}$$

be the full subcategory consisting of those objects, for which the functor $\mathbf{O} \to \mathbf{O}'$ admits a left adjoint, which is compatible with the $\mathbf{O}$-module structure.

Note that for any $\phi : \mathbf{O} \to \mathbf{O}'$ as above, the object $\phi^L(\mathbf{1}_{\mathbf{O}'}) \in \mathbf{O}$ has a canonical structure of co-commutative co-algebra in $\mathbf{O}$. In particular, we obtain a canonically defined functor

$$'(\text{DGCat}^{\text{SymMon}})_{\mathbf{O}/} \to \text{CocomCoalg}(\mathbf{O}).$$

Moreover, the functor

$$\phi^L : \mathbf{O}' \to \mathbf{O}$$

canonically factors as

$$\mathbf{O}' \to \phi^L(\mathbf{1}_{\mathbf{O}'})\text{-comod}(\mathbf{O}) \overset{\mathbf{oblv}_{\phi^L(\mathbf{1}_{\mathbf{O}'})}}{\longrightarrow} \mathbf{O},$$

in a way functorial in $\mathbf{O}' \in {}'(\text{DGCat}^{\text{SymMon}})_{\mathbf{O}/}$.

1.1.3. We apply the above discussion to $\mathbf{O} = \text{IndCoh}(X)$. Base change (see [Chapter III.3, Proposition 3.1.2]) implies that the functor (1.1) factors as

$$\text{FormMod}_{/X} \to {}'(\text{DGCat}^{\text{SymMon}})_{\text{IndCoh}(X)/}, \quad \mathcal{Y} \mapsto \text{IndCoh}(\mathcal{Y}).$$

In particular, we obtain a functor

$$\text{FormMod}_{/X} \to \text{CocomCoalg}(\text{IndCoh}(X)), \quad (\mathcal{Y} \xrightarrow{\pi} X) \mapsto \pi_*^{\text{IndCoh}}(\omega_{\mathcal{Y}}).$$

We denote this functor by $\text{Distr}^{\text{Cocom}}$. We denote by

$$\text{Distr} : \text{FormMod}_{/X} \to \text{IndCoh}(X)$$

the composition of $\text{Distr}^{\text{Cocom}}$ with the forgetful functor

$$\text{CocomCoalg}(\text{IndCoh}(X)) \overset{\mathbf{oblv}_{\text{Cocom}}}{\longrightarrow} \text{IndCoh}(X).$$

The functor

$$\pi_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{Y}) \to \text{IndCoh}(X)$$

canonically factors as

$$\text{IndCoh}(\mathcal{Y}) \to \text{Distr}^{\text{Cocom}}(\mathcal{Y})\text{-comod}(\text{IndCoh}(X)) \overset{\mathbf{oblv}_{\text{Distr}^{\text{Cocom}}(\mathcal{Y})}}{\longrightarrow} \text{IndCoh}(X)$$

in a way functorial in $\mathcal{Y}$.

1.1.4. The functor $\text{Distr}^{\text{Cocom}}$ defines a functor

$$\text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd}(\text{FormMod}_{/X}) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)),$$

and the functor Distr defines a functor

$$\text{Distr}^{\text{aug}} : \text{Ptd}(\text{FormMod}_{/X}) \to \text{IndCoh}(X)_{\omega_X/}.$$

We shall denote by Distr^+ the functor $\text{Ptd}(\text{FormMod}_{/X}) \to \text{IndCoh}(X)$ that sends $\mathcal{Y}$ to

$$\text{coFib}(\omega_X \to \text{Distr}(\mathcal{Y})) \simeq \text{Fib}(\text{Distr}(\mathcal{Y}) \to \omega_X).$$

1.1.5. *An example.* Note that we have a commutative diagram

$$\begin{array}{ccc} \left(\text{Coh}(X)^{\leq 0}\right)^{\text{op}} & \longrightarrow & \text{Ptd}(({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X}) \\ \mathbb{D}_X^{\text{Serre}} \Big\downarrow & & \Big\downarrow \text{Distr}^{\text{Cocom}^{\text{aug}}} \\ \text{Coh}(X) & \xrightarrow{\mathbf{triv}_{\text{Cocom}^{\text{aug}}}} & \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)), \end{array} \tag{1.2}$$

where the top horizontal arrow is the functor of split square-zero extension.

1.1.6. The following observation will be useful:

Lemma 1.1.7.
(a) *The functors*

$$\text{Distr} : \text{FormMod}_{/X} \to \text{IndCoh}(X)$$

and

$$\text{Distr}^{\text{Cocom}} : \text{FormMod}_{/X} \to \text{CocomCoalg}(\text{IndCoh}(X))$$

are left Kan extensions of their respective restrictions to

$$({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X} \subset \text{FormMod}_{/X}\,.$$

(b) *The functors*

$$\text{Distr}^{\text{aug}} : \text{Ptd}(\text{FormMod}_{/X}) \to \text{IndCoh}(X)_{\omega_X/}$$

and

$$\text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd}(\text{FormMod}_{/X}) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)),$$

are left Kan extensions of their respective restrictions to

$$\text{Ptd}(({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X}) \subset \text{Ptd}(\text{FormMod}_{/X}).$$

Proof. We prove point (a), since point (b) is similar.

Since the forgetful functor

$$\mathbf{oblv}_{\text{Cocom}} : \text{CocomCoalg}(\text{IndCoh}(X)) \to \text{IndCoh}(X)$$

commutes with colimits, it suffices to prove the assertion for the functor

$$\text{Distr} : \text{Ptd}(\text{FormMod}_{/X}) \to \text{IndCoh}(X).$$

The required assertion follows from [Chapter IV.1, Corollary 1.5.5].

□

Remark 1.1.8. Recall (see [Chapter IV.2, Sect. 2.2]) that for a DG category $\mathbf{O}$, in addition to the category $\text{CocomCoalg}^{\text{aug}}(\mathbf{O})$, one can consider the category

$$\text{CocomCoalg}(\mathbf{O})^{\text{aug,ind-nilp}} := \text{Cocom}^{\text{aug}}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$$

of ind-nilpotent co-commutatve co-algebras. This category is endowed with a forgetful functor

$$\mathbf{res}^{\star\to *} : \text{CocomCoalg}^{\text{aug,ind-nilp}} \to \text{CocomCoalg}^{\text{aug}}(\mathbf{O}).$$

Using Lemma 1.1.7, one can refine the above functor

$$\text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd}(\text{FormMod}_{/X}) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$$

to a functor

$$\text{Distr}^{\text{Cocom}^{\text{aug,ind-nilp}}} : \text{Ptd}(\text{FormMod}_{/X}) \to \text{CocomCoalg}^{\text{aug,ind-nilp}}(\text{IndCoh}(X)).$$

Namely, for $Z \in \text{Ptd}(({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X})$, the t-structure allows to naturally upgrade the object

$$\text{Distr}^{\text{Cocom}^{\text{aug}}}(Z) \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$$

to an object

$$\text{Distr}^{\text{Cocom}^{\text{aug,ind-nilp}}}(Z) \in \text{CocomCoalg}^{\text{aug,ind-nilp}}(\text{IndCoh}(X)).$$

Now, we let $\text{Distr}^{\text{Cocom}^{\text{aug,ind-nilp}}}$ be the left Kan extension under

$$\text{Ptd}(({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X}) \hookrightarrow \text{Ptd}(\text{FormMod}_{/X})$$

of the above functor $Z \mapsto \text{Distr}^{\text{Cocom}^{\text{aug,ind-nilp}}}(Z)$.

1.2. **The monoidal structure.** In this subsection we will establish the compatibility of the functor $\mathrm{Distr}^{\mathrm{Cocom}}$ with symmetric monoidal structures. Namely, we show that $\mathrm{Distr}^{\mathrm{Cocom}}$ is a symmetric monoidal functor and commutes with the Bar-construction on group objects.

1.2.1. Let us consider $\mathrm{FormMod}_{/X}$ and $\mathrm{CocomCoalg}(\mathrm{IndCoh}(X))$ as symmetric monoidal categories with respect to the Cartesian structure. Tautologically, the functor $\mathrm{Distr}^{\mathrm{Cocom}}$ is left-lax symmetric monoidal. We claim:

Lemma 1.2.2. *The left-lax symmetric monoidal structure on* $\mathrm{Distr}^{\mathrm{Cocom}}$ *is strict.*

Proof. We need to show that for

$$\pi_1 : \mathcal{Y}_1 \to X \text{ and } \pi_2 : \mathcal{Y}_2 \to X,$$

and

$$\mathcal{Y}_1 \underset{X}{\times} \mathcal{Y}_2 =: \mathcal{Y} \overset{\pi}{\to} X,$$

the map

$$\pi_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}}) \to (\pi_1)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}_1}) \overset{!}{\otimes} (\pi_2)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}_2})$$

is an isomorphism.

However, this follows from base change for the diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}_1 \times \mathcal{Y}_2 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_1 \times \pi_2} \\ X & \longrightarrow & X \times X. \end{array}$$

□

1.2.3. Recall the notation

$$\mathrm{CocomBialg}(\mathrm{IndCoh}(X)) := \mathrm{AssocAlg}\left(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))\right);$$

this is the category of associative algebras in $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$.

Recall also that

$$\mathrm{CocomHopf}(\mathrm{IndCoh}(X)) \subset \mathrm{CocomBialg}(\mathrm{IndCoh}(X))$$

denotes the full subcategory spanned by group-like objects.

1.2.4. By Lemma 1.2.2, the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ gives rise to a functor

$$\mathrm{Grp}(\mathrm{FormMod}_{/X}) \simeq \mathrm{Monoid}(\mathrm{Ptd}(\mathrm{FormMod}_{/X})) \to \\ \to \mathrm{AssocAlg}\left(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))\right) = \mathrm{CocomBialg}(\mathrm{IndCoh}(X)),$$

which in fact factors through

$$\mathrm{CocomHopf}(\mathrm{IndCoh}(X)) \subset \mathrm{CocomBialg}(\mathrm{IndCoh}(X)).$$

We denote the resulting functor by $\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})$.

1.2.5. The following will be useful in the sequel:

Lemma 1.2.6. *Let $\mathcal{H}$ be an object of* $\mathrm{Grp}(\mathrm{FormMod}_{/X})$*. Then the canonical map*

$$\mathrm{Bar} \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\mathcal{H}) \to \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ B_X(\mathcal{H})$$

is an isomorphism.

Proof. It is enough to establish the isomorphism in question after applying the forgetful functor $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X)$.

The left-hand side is the geometric realization of the simplicial object of $\mathrm{IndCoh}(X)$ given by

$$\mathrm{Bar}^\bullet\left(\mathrm{Distr}(\mathcal{H})\right) \overset{\text{Lemma 1.2.2}}{\simeq} \mathrm{Distr}\left(B_X^\bullet(\mathcal{H})\right).$$

We can think of $B_X^\bullet(\mathcal{H})$ as the Čech nerve of the map $X \to B_X(\mathcal{H})$. Hence, by [Chapter III.3, Proposition 3.3.3(b)], the map

$$|\,\mathrm{Distr}\left(B_X^\bullet(\mathcal{H})\right)| \to \mathrm{Distr}(B_X(\mathcal{H}))$$

is an isomorphism, as required.

□

1.3. **The functor of inf-spectrum.** Continuing the parallel with usual algebraic geometry, the functor Spec provides a right adjoint to the functor

$$\mathrm{Sch} \to (\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}))^{\mathrm{op}}, \quad X \mapsto \tau^{\leq 0}(\Gamma(X, \mathcal{O}_X)).$$

In this subsection we will develop its analog for formal moduli problems. This will be a functor, denoted $\mathrm{Spec}^{\mathrm{inf}}$, right adjoint to

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

1.3.1. Starting from $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$, we first define a presheaf on the category on $\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})$, denoted $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})_{\text{nil-isom}}$, by

$$\mathrm{Maps}(Z, \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})_{\text{nil-isom}}) := \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z), \mathcal{A}).$$

Let $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A}) \in (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$ be the left Kan extension of $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})_{\text{nil-isom}}$ along the forgetful functor

$$\left(\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})\right)^{\mathrm{op}} \to \left(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X}\right)^{\mathrm{op}}. \tag{1.3}$$

We claim that $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})$ is an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$.

Indeed, this follows from [Chapter IV.1, Corollary 1.5.2(b)] and the following assertion:

Lemma 1.3.2. *Let $Z_2' := Z_1' \underset{Z_1}{\sqcup} Z_2$ be a push-out diagram in* $\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})$*, where the map $Z_1 \to Z_1'$ is a closed embedding. Then the canonical map*

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z_1') \underset{\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z_1)}{\sqcup} \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z_2) \to \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z_2')$$

is an isomorphism.

Proof. Since the forgetful functor

$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X)_{\omega_X/}$$

commutes with colimits, it is sufficient to show that the map

$$\mathrm{Distr}^{\mathrm{aug}}(Z_1') \underset{\mathrm{Distr}^{\mathrm{aug}}(Z_1)}{\sqcup} \mathrm{Distr}^{\mathrm{aug}}(Z_2) \to \mathrm{Distr}^{\mathrm{aug}}(Z_2')$$

is an isomorphism in $\mathrm{IndCoh}(X)_{\omega_X/}$. By Serre duality, this is equivalent to showing that

$$(\pi_2')_*(\mathcal{O}_{Z_2'}) \to (\pi_1')_*(\mathcal{O}_{Z_1'}) \underset{(\pi_1)_*(\mathcal{O}_{Z_1})}{\times} (\pi_2)_*(\mathcal{O}_{Z_2})$$

is an isomorphism in $\mathrm{QCoh}(X)$, and the latter follows from the assumptions.

□

1.3.3. We now claim that the assignment

$$\mathcal{A} \mapsto \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})$$

provides a right adjoint to the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$. Indeed, this follows from [Chapter IV.1, Corollaries 1.5.2(a)] and Lemma 1.1.7(b).

1.3.4. We have:

Lemma 1.3.5. *For* $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$*, there is a canonical isomorphism*

$$T(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})/X)|_X \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{A}).$$

Proof. The proof is just a repeated application of definitions. Indeed, for $\mathcal{F} \in \mathrm{Coh}(X)^{\leq 0}$ we have by definition

$$\begin{aligned} &\mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), T(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})/X)|_X) = \\ &\qquad = \mathrm{Maps}_{\mathrm{Pro}(\mathrm{QCoh}(X)^-)}(T^*(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})/X)|_X), \mathcal{F}) = \mathrm{Maps}_{X//X}(X_{\mathcal{F}}, \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})), \end{aligned}$$

where $X_{\mathcal{F}}$ is the split square-zero extension corresponding to $\mathcal{F}$, see [Chapter III.1, Sect. 2.1.1].

By definition,

$$\mathrm{Maps}_{X//X}(X_{\mathcal{F}}, \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})) = \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(S_{\mathcal{F}}), \mathcal{A}).$$

Now, by (1.2)

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(S_{\mathcal{F}}) \simeq \mathbf{triv}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F})),$$

while

$$\mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathbf{triv}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F})), \mathcal{A}) = \mathrm{Maps}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{A})),$$

again by definition.

□

1.3.6. Being a right adjoint to a symmetric monoidal functor, the functor $\mathrm{Spec}^{\mathrm{inf}}$ is automatically right-lax symmetric monoidal. Hence, it gives rise to a functor

$$\mathrm{CocomBialg}(\mathrm{IndCoh}(X)) := \mathrm{AssocAlg}\left(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))\right) \to \\ \to \mathrm{Monoid}(\mathrm{Ptd}(\mathrm{FormMod}_{/X})) \simeq \mathrm{Grp}(\mathrm{FormMod}_{/X}).$$

We shall denote the above functor by $\mathrm{Monoid}(\mathrm{Spec}^{\mathrm{inf}})$.

Remark 1.3.7. If instead of the category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ one works with the category $\mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X))$, one obtains a functor

$$\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}} : \mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X}).$$

However, the functors $\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}$ and $\mathrm{Spec}^{\mathrm{inf}}$ carry the same information: it follows formally that the functor $\mathrm{Spec}^{\mathrm{inf}}$ factors as the composition

$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \overset{\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}}{\longrightarrow} \mathrm{Ptd}(\mathrm{FormMod}_{/X}),$$

where the first arrow is the right adjoint to the forgetful functor

$$\mathbf{res}^{\star\to *} : \mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Furthermore, it follows from [Chapter IV.2, Corollary 2.10.5(b)], applied in the case of the co-operad $\mathrm{Cocom}^{\mathrm{aug}}$, and Lemma 1.3.5 that the natural map from $\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}$ to the composition

$$\mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \overset{\mathbf{res}^{\star\to *}}{\longrightarrow} \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \overset{\mathrm{Spec}^{\mathrm{inf}}}{\longrightarrow} \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

is an isomorphism.

1.4. **An example: vector prestacks.** The basic example of a scheme is the scheme attached to a finite-dimensional vector space:

$$\mathrm{Maps}(S, V) = \Gamma(S, \mathcal{O}_S) \otimes V.$$

In this subsection we describe the counterpart of this construction for formal moduli problems.

Namely, for an object $\mathcal{F} \in \mathrm{IndCoh}(X)$, we will construct a formal moduli problem $\mathrm{Vect}_X(\mathcal{F})$ over X. In the case when $\mathcal{F}$ is a coherent sheaf, $\mathrm{Vect}_X(\mathcal{F})$ will be the formal completion of the zero section of the 'vector bundle' associated to $\mathcal{F}$.

1.4.1. Let $\mathcal{F}$ be an object of $\mathrm{IndCoh}(X)$ and consider the object

$$\mathrm{Sym}(\mathcal{F}) \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)),$$

where, as always, the monoidal structure on $\mathrm{IndCoh}(X)$ is given by the !-tensor product. See [Chapter IV.2, Sect. 4.2] for the notation Sym.

Consider the corresponding object

$$\mathrm{Vect}_X(\mathcal{F}) := \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F})) \in \mathrm{Ptd}(\mathrm{FormMod}_{/X}).$$

1.4.2. Recall the notation Distr^+ introduced in Sect. 1.1.4. We claim:

Proposition 1.4.3. *For* $Z \in \mathrm{Ptd}((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X}$ *the natural map*

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Z, \mathrm{Vect}_X(\mathcal{F})) \to \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F}),$$

given by the projection $\mathrm{Sym}(\mathcal{F}) \to \mathcal{F}$*, is an isomorphism.*

Proof. First, we note that the presheaf on $\mathrm{Ptd}((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X}$, given by

$$Z \mapsto \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F}),$$

gives rise to an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ for the same reason as $\mathrm{Spec}^{\mathrm{inf}}$ does. Denote this object by $\mathrm{Vect}'_X(\mathcal{F})$.

Hence, in order to prove that the map in question is an isomorphism, by [Chapter III.1, Proposition 8.3.2], its suffices to show that the map

$$T(\mathrm{Vect}_X(\mathcal{F})/X)|_X \to T(\mathrm{Vect}'_X(\mathcal{F})/X)|_X. \tag{1.4}$$

is an isomorphism.

The commutative diagram (1.2) implies that $T(\mathrm{Vect}'_X(\mathcal{F})/X)|_X$ identifies with $\mathcal{F}$.

By Lemma 1.3.5,

$$T(\mathrm{Vect}_X(\mathcal{F})/X)|_X \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Sym}(\mathcal{F})),$$

and the map (1.4) identifies with the canonical map

$$\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Sym}(\mathcal{F})) \to \mathcal{F}.$$

Now, the latter map is an isomorphism by [Chapter IV.2, Corollary 4.2.5].

□

Note that in the process of proof we have also shown:

Corollary 1.4.4. *For* $\mathcal{F} \in \mathrm{IndCoh}(X)$*, there exists a canonical isomorphism*

$$T(\mathrm{Vect}_X(\mathcal{F})/X)|_X \simeq \mathcal{F}.$$

Remark 1.4.5. The proof of Proposition 1.4.3 used the somewhat non-trivial isomorphism of [Chapter IV.2, Corollary 4.2.5]. However, if instead of the functor $\mathrm{Spec}^{\mathrm{inf}}$, one uses the functor $\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}$ (see Remark 1.3.7), then the statement that

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}\left(Z, \mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}(\mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{F}))\right) \to \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F})$$

is an isomorphism, would be tautological. Note that

$$\mathrm{Sym}(\mathcal{F}) \simeq \mathbf{res}^{\star\to *} \circ \mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{F}) =: \mathbf{cofree}^{\mathrm{fake}}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{F}).$$

Thus, we can interpret the assertion of Proposition 1.4.3 as saying that the natural map

$$\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}(\mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{F})) \to \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F})) = \mathrm{Vect}_X(\mathcal{F})$$

is an isomorphism.

Note that the latter is a particular case of the isomorphism of functors of Remark 1.3.7.

1.4.6. We now claim:

Proposition 1.4.7. *The co-unit of the adjunction*

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Vect}_X(\mathcal{F})) \to \mathrm{Sym}(\mathcal{F}) \tag{1.5}$$

is an isomorphism.

The rest of this subsection is devoted to the proof of the proposition.

1.4.8. *Step 1.* Suppose for a moment that $\mathcal{F}$ is such that $\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}) \in \mathrm{Coh}(X)^{<0}$. In this case, by Proposition 1.4.3,

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Z, \mathrm{Vect}_X(\mathcal{F})) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F}) \simeq$$
$$\simeq \mathrm{Maps}_{\mathrm{QCoh}(X)}\left(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), \mathrm{Fib}(\pi_*(\mathcal{O}_Z) \to \mathcal{O}_X)\right) \simeq$$
$$\simeq \mathrm{Maps}_{/X}\left(Z, \mathrm{Spec}_X\left(\mathbf{free}_{\mathrm{Com}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}))\right)\right)$$

(where $\mathbf{free}_{\mathrm{Com}}$ is taken in the symmetric monoidal category $\mathrm{QCoh}(X)$), so $\mathrm{Vect}_X(\mathcal{F})$ is a scheme isomorphic to $\mathrm{Spec}_X\left(\mathbf{free}_{\mathrm{Com}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}))\right)$, and the assertion is manifest.

1.4.9. *Step 2.* Now, we claim that both sides in (1.5), viewed as functors

$$\mathrm{IndCoh}(X) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)),$$

commute with filtered colimits in $\mathcal{F}$.

The commutation is obvious for the functor $\mathcal{F} \mapsto \mathrm{Sym}(\mathcal{F})$.

Since the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ is a left adjoint, it suffices to show that the functor

$$\mathcal{F} \mapsto \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F}))$$

commutes with filtered colimits.

By the construction of the functor $\mathrm{Spec}^{\mathrm{inf}}$, it suffices to show that the functor

$$\mathcal{F} \mapsto \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F}))_{\text{nil-isom}} :$$
$$\mathrm{IndCoh}(X) \to \mathrm{Funct}\left(\left(\mathrm{Ptd}((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})\right)^{\mathrm{op}}, \mathrm{Spc}\right)$$

commutes with filtered colimits.

By Proposition 1.4.3, it suffuces to show that for $Z \in \mathrm{Ptd}((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})$, the functor

$$\mathcal{F} \mapsto \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F})$$

commutes with filtered colimits. The latter follows from the fact that $\mathrm{Distr}^+(Z) \in \mathrm{Coh}(X) = \mathrm{IndCoh}(X)^c$.

1.4.10. *Step 3.* According to Step 2, we can assume that $\mathcal{F} \in \mathrm{Coh}(Z)$. Combining with Step 1, it remains to show that if the assertion of the proposition holds for $\mathcal{F}[-1]$, then it also holds for $\mathcal{F}$.

The description of $\mathrm{Vect}_X(\mathcal{F})$, given by Proposition 1.4.3 implies that there is a canonical isomorphism

$$\mathrm{Vect}_X(\mathcal{F}[-1]) \simeq \Omega_X(\mathrm{Vect}_X(\mathcal{F})) \in \mathrm{Grp}(\mathrm{FormMod}_{/X}),$$

and hence

$$B_X(\mathrm{Vect}_X(\mathcal{F}[-1])) \simeq \mathrm{Vect}_X(\mathcal{F}).$$

Note also that we have a canonical isomorphism in $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$:

$$\mathrm{Bar}(\mathrm{Sym}(\mathcal{F}[-1])) \simeq \mathrm{Sym}(\mathcal{F}),$$

where we regard $\mathrm{Sym}(\mathcal{F}[-1])$ as an object of

$$\mathrm{CocomBialg}(\mathrm{IndCoh}(X)) \simeq \mathrm{Assoc}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)))$$

via the structure on $\mathcal{F}[-1]$ of a group-object in $\mathrm{IndCoh}(X)$.

The following diagram commutes by adjunction

$$\begin{array}{ccc}
\mathrm{Bar} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Vect}_X(\mathcal{F}[-1])) & \xrightarrow[\sim]{\text{Lemma 1.2.6}} & \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ B_X(\mathrm{Vect}_X(\mathcal{F}[-1])) \\
\downarrow & & \downarrow \sim \\
\mathrm{Bar}(\mathrm{Sym}(\mathcal{F}[-1])) & & \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Vect}_X(\mathcal{F})) \\
\sim \downarrow & & \downarrow \\
\mathrm{Sym}(\mathcal{F}) & \xrightarrow{\mathrm{id}} & \mathrm{Sym}(\mathcal{F}).
\end{array}$$

By assumption, the upper left vertical arrow in this diagram is an isomorphism. Hence, so is the lower right vertical arrow.

□

2. Inf-affineness

In this section we study the notion of inf-affineness, which is a counterpart of the usual notion of affineness in algebraic geometry.

The naive expectation would be that an inf-affine formal moduli problem over X is one of the form $\mathrm{Spec}^{\mathrm{inf}}$ of a co-commutative co-algebra in $\mathrm{IndCoh}(X)$. However, this does not quite work as the analogy with the usual notion of affineness is not perfect: it is not true that the functor $\mathrm{Spec}^{\mathrm{inf}}$ identifies the category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ with that of inf-affine objects in $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$.

2.1. **The notion of inf-affineness.** In algebraic geometry a prestack $\mathcal{Y}$ is an affine scheme if and only if $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is connective and for any $S \in \mathrm{Sch}^{\mathrm{aff}}$, the map

$$\mathrm{Maps}_{\mathrm{Sch}^{\mathrm{aff}}}(S, \mathcal{Y}) \to \mathrm{Maps}_{\mathrm{ComAlg(Vect)}}(\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \Gamma(S, \mathcal{O}_S))$$

is an isomorphism.

In formal geometry we give a similar definition.

2.1.1. Let as before $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Definition 2.1.2. *An object* $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ *is* inf-affine*, if the functor* $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ *induces an isomorphism*

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Z, \mathcal{Y}) \to$$
$$\to \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z), \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})),$$

where $Z \in \mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to }X})$.

2.1.3. Here are some basic facts related to this notion:

Proposition 2.1.4. *Any object*

$$Y \in \mathrm{Ptd}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{\text{nil-isom to } X}) \subset \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

is inf-affine.

Proof. By definition, we need to show that for

$$Y_1, Y_2 \in \mathrm{Ptd}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{\text{nil-isom to } X}),$$

with Y_1 eventually coconnective, the groupoid $\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Y_1, Y_2)$ maps isomorphically to

$$\mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}((\pi_1)^{\mathrm{IndCoh}}_* (\omega_{Y_1}), (\pi_2)^{\mathrm{IndCoh}}_*(\omega_{Y_2}))\,.$$

The assertion easily reduces to the case when Y_2 is eventually coconnective. In the latter case, Serre duality identifies the above groupoid with

$$\mathrm{Maps}_{\mathrm{ComAlg}^{\mathrm{aug}}(\mathrm{QCoh}(X))}((\pi_2)_*(\mathcal{O}_{Y_2}), (\pi_1)_*(\mathcal{O}_{Y_1}))\,,$$

and the desired isomorphism is manifest.

□

2.1.5. We claim:

Lemma 2.1.6. *Let* $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ *be* inf-affine. *Then for any* $\mathcal{Z} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$, *the map*

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(\mathcal{Z}, \mathcal{Y}) \to$$
$$\to \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Z}), \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}))$$

is an isomorphism.

Proof. Follows from [Chapter IV.1, Corollary 1.5.2(a)] and Lemma 1.1.7(b). □

Remark 2.1.7. It follows from Proposition 2.3.3 below, combined with [Chapter IV.2, Corollary 2.10.5(b)] for the co-operad Cocom that if instead of $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ one uses $\mathrm{Cocom}^{\text{aug,ind-nilp}}(\mathrm{IndCoh}(X))$, one obtains the *same* notion of inf-affineness.

2.2. **Inf-affineness and inf-spectrum.** As was mentioned already, it is not true that the functor $\mathrm{Spec}^{\mathrm{inf}}$ identifies the category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ with that of inf-affine objects in $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$. The problem is that the analog of Serre's theorem fails: for a connective commutative DG algebra A, the map

$$A \to \Gamma(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$$

is an isomorphism, whereas for $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$, the map

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})) \to \mathcal{A}$$

does not have to be such.

In this subsection we establish several positive facts that can be said in this direction. A more complete picture is presented in Sect. 3.3.

2.2.1. We note:

Lemma 2.2.2. *Let* $\mathcal{A} \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$ *be such that the co-unit of the adjunction*

$$\text{Distr}^{\text{Cocom}^{\text{aug}}}(\text{Spec}^{\text{inf}}(\mathcal{A})) \to \mathcal{A}$$

is an isomorphism. Then the object $\text{Spec}^{\text{inf}}(\mathcal{A})$ *is inf-affine.*

In particular, combining with Proposition 1.4.7, we obtain:

Corollary 2.2.3. *For* $\mathcal{F} \in \text{IndCoh}(X)$*, the object* $\text{Vect}_X(\mathcal{F}) \in \text{Ptd}(\text{FormMod}_{/X})$ *is inf-affine.*

Remark 2.2.4. As we shall see in Sect. 3.3.10, it is *not* true that the functor Spec^{inf} is fully faithful. I.e., the co-unit of the adjunction

$$\text{Distr}^{\text{Cocom}^{\text{aug}}}(\text{Spec}^{\text{inf}}(\mathcal{A})) \to \mathcal{A}$$

is *not* an isomorphism for all $\mathcal{A} \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$.

However, we will see that [Chapter IV.2, Conjecture 2.8.9(b)] for the symmetric monoidal DG category $\text{IndCoh}(X)$ implies that the above map is an isomorphism for $\mathcal{A}$ lying in the essential image of the functor $\text{Distr}^{\text{Cocom}^{\text{aug}}}$.

We will also see that [Chapter IV.2, Conjecture 2.8.9(a)] for $\text{IndCoh}(X)$ implies that the essential image of the functor Spec^{inf} lands in the subcategory of $\text{Ptd}(\text{FormMod}_{/X})$ spanned by inf-affine objects.

Remark 2.2.5. The same logic shows that [Chapter IV.2, Conjecture 2.6.6] for the symmetric monoidal DG category $\text{IndCoh}(X)$ and the Lie operad, implies that the functor $\text{Spec}^{\text{inf,ind-nilp}}$ is an equivalence onto the full subcategory of $\text{Ptd}(\text{FormMod}_{/X})$ spanned by objects that are inf-affine.

2.3. **A criterion for being inf-affine.** A prestack $\mathcal{Y}$ is an affine scheme if and only if $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is connective and the canonical map

$$\mathcal{Y} \to \text{Spec}(\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$$

is an isomorphism.

The corresponding assertion is true (but not completely tautological) also in formal geometry: an object $\mathcal{Y} \in \text{Ptd}(\text{FormMod}_{/X})$ is inf-affine if and only if the unit of the adjunction

$$\mathcal{Y} \to \text{Spec}^{\text{inf}} \circ \text{Distr}^{\text{Cocom}^{\text{aug}}}(\mathcal{Y})$$

is an isomorphism.

In addition, in this subsection we will give a crucial criterion for inf-affineness in terms of the tangent space of $\mathcal{Y}$ at its distinguished point, which does not have a counterpart in usual algebraic geometry.

2.3.1. Recall the commutative diagram (1.2).

We obtain that for $\mathcal{F} \in \text{Coh}(X)^{\leq 0}$ and $\mathcal{Y} \in \text{Ptd}(\text{FormMod}_{/X})$, the functor $\text{Distr}^{\text{Cocom}^{\text{aug}}}$ gives rise to a canonically defined map

$$\text{Maps}_{\text{IndCoh}(X)}(\mathbb{D}_X^{\text{Serre}}(\mathcal{F}), T(\mathcal{Y}/X)|_X) \to \\ \to \text{Maps}_{\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))}\left(\mathbf{triv}_{\text{Cocom}^{\text{aug}}}(\mathbb{D}_X^{\text{Serre}}(\mathcal{F})), \text{Distr}^{\text{Cocom}^{\text{aug}}}(\mathcal{Y})\right). \tag{2.1}$$

Consider the functor

$$\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X).$$

We can rewrite (2.1) as a map

$$\text{(2.2)} \quad \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), T(\mathcal{Y}/X)|_X) \to \\ \to \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})).$$

The map (2.2) gives rise to a well-defined map in $\mathrm{IndCoh}(X)$:

$$\text{(2.3)} \qquad T(\mathcal{Y}/X)|_X \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}).$$

2.3.2. We claim:

Proposition 2.3.3. *For an object $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ the following conditions are equivalent:*

(i) $\mathcal{Y}$ *is inf-affine;*

(ii) *The unit of the adjunction* $\mathcal{Y} \to \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})$ *is an isomorphism;*

(iii) *The map* (2.3) *is an isomorphism.*

Proof. The implication (i) ⇒ (iii) is tautological from the definition of inf-affineness.

Suppose that $\mathcal{Y}$ satisfies (ii). Then for $Z \in \mathrm{Ptd}((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})$ the map

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Z, \mathcal{Y}) \to \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}\left(Z, \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\right)$$

is an isomorphism, while its composition with the adjunction isomorphism

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}\left(Z, \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\right) \simeq \\ \simeq \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}\left(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z), \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\right)$$

equals the map induced by the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$. Hence, $\mathcal{Y}$ is inf-affine.

Finally, assume that $\mathcal{Y}$ satisfies (iii), and let us deduce (ii). By [Chapter III.1, Proposition 8.3.2], in order to show that $\mathcal{Y} \to \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})$ is an isomorphism, it suffices to show that the map

$$T(\mathcal{Y}/X)|_X \to T(\mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})/X)|_X$$

is an isomorphism in $\mathrm{IndCoh}(X)$.

Recall the isomorphism $T(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})/X)|_X \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{A})$ of Lemma 1.3.5.

Now, it is easy to see that the composed map

$$T(\mathcal{Y}/X)|_X \to T(\mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})/X)|_X \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}))$$

equals the map (2.3), implying our assertion.

□

3. From formal groups to Lie algebras

Let G be a Lie group. The tangent space at the identity of G has the structure of a Lie algebra. One way of describing this Lie algebra structure is the following: the Lie algebra of G is given by the space of primitive elements in the co-commutative co-algebra given by the space of distributions on G supported at the identity.

In this section, we implement this idea in the context of derived algebraic geometry and finally spell out the relationship between the categories $\mathrm{Grp}(\mathrm{FormMod}_{/X})$ and $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$, i.e., formal groups and Lie algebras:

To go from an object of $\mathrm{Grp}(\mathrm{FormMod}_{/X})$ to $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$, we first attach to it an object $\mathrm{Grp}(\mathrm{LieAlg}(\mathrm{IndCoh}(X)))$ via the functor

$$\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$$

(i.e., we attach to an object of $\mathrm{Grp}(\mathrm{FormMod}_{/X})$ the corresponding augmented co-commutative co-algebra and use Quillen's functor $\mathrm{coChev}^{\mathrm{enh}}$ that maps $\mathrm{CocomCoalg}^{\mathrm{aug}}$ to LieAlg), and then deloop.

To go from $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$ we use the 'exponential map', incarnated by the functor

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg} \to \mathrm{CocomHopf}$$

(the latter is canonically isomorphic to the more usual construction given by the functor U^{Hopf}, the universal enveloping algebra, viewed as a co-commutatove Hopf algebra), and then apply the functor of inf-spectrum.

3.1. **The exponential construction.** Let as before $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. The idea of the exponential construction is the following: for a Lie algebra $\mathfrak{h}$, the corresponding formal group $\exp_X(\mathfrak{h})$ is such that

$$\mathrm{Distr}(\exp_X(\mathfrak{h})) \simeq U(\mathfrak{h}).$$

3.1.1. We define the functor

$$\exp_X : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Grp}(\mathrm{FormMod}_{/X})$$

to be

$$\mathrm{Monoid}(\mathrm{Spec}^{\mathrm{inf}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}.$$

For example, for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, we have

$$\exp(\mathbf{triv}_{\mathrm{Lie}}(\mathcal{F})) = \mathrm{Monoid}(\mathrm{Spec}^{\mathrm{inf}})(\mathrm{Sym}(\mathcal{F})) = \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}),$$

equipped with its natural group structure.

Remark 3.1.2. To bring the above construction closer to the classical idea of the exponential map, let us recall that, according to [Chapter IV.2, Theorem 6.1.2], we have a canonical isomorphism in $\mathrm{CocomHopf}(\mathrm{IndCoh}(X))$

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}}.$$

3.1.3. In the next section we will prove:

Theorem 3.1.4. *The functor*

$$\exp_X : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Grp}(\mathrm{FormMod}_{/X})$$

is an equivalence.

3.2. **Corollaries of Theorem 3.1.4.** In this subsection we will show that the functor $\exp_X$ as defined above, has all the desired properties, i.e., that there are no unpleasant surprises.

3.2.1. Recall (see [Chapter IV.2, Corollary 1.7.3]) that when $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}(\mathfrak{h})$ is viewed as an object of $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$, i.e., if we forget the algebra structure, it is (canonically) isomophic to $\mathrm{Sym}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}))$.

Hence, by Corollary 2.2.3, when we view $\exp_X(\mathfrak{h})$ as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$, it is isomorphic to $\mathrm{Vect}_X(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}))$, and hence is inf-affine.

Therefore, as a consequence of Theorem 3.1.4 (plus Corollary 1.4.4), we obtain:

Corollary 3.2.2. *Every object of* $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$*, when viewed by means of the forgetful functor as an object of* $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$*, is inf-affine, and we have:*

$$\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H}) \simeq \mathrm{Vect}_X(T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/X)|_X).$$

From Proposition 1.4.7 and [Chapter IV.2, Proposition 4.3.3], we obtain:

Corollary 3.2.3. *The natural transformation*

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) \circ \exp_X \to \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \tag{3.1}$$

is an isomorphism.

Combining the isomorphism (3.1) with the isomorphism

$$B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Id} \tag{3.2}$$

of [Chapter IV.2, Theorem 4.4.6], we obtain:

Corollary 3.2.4. *There exists a canonical isomorphism of functors*

$$B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) \circ \exp_X \simeq \mathrm{Id}\,.$$

3.2.5. Let us denote by

$$\mathrm{Lie}_X : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \tag{3.3}$$

the functor

$$B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}).$$

Hence:

Corollary 3.2.6. *The functor*

$$\mathrm{Lie}_X : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X))$$

is the inverse of

$$\exp_X : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Grp}(\mathrm{FormMod}_{/X}).$$

3.2.7. By combining Corollary 3.2.2, Proposition 2.3.3 and the tautological isomorphism

$$\mathbf{oblv}_{\mathrm{Lie}} \circ B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathbf{oblv}_{\mathrm{Grp}},$$

we obtain:

Corollary 3.2.8. *There exists a canonical isomorphism of functors*

$$\mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{IndCoh}(X), \quad \mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{Lie}_X(\mathcal{H}) \simeq T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/X)|_X.$$

In other words, this corollary says that the object of IndCoh underlying the Lie algebra corresponding to a formal group indeed identifies with the tangent space at the origin.

3.2.9. The upshot of this subsection is that in derived algebraic geometry the passage from the a formal group to its Lie algebra is given by the functor

$$\mathrm{Lie}_X := B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}).$$

3.3. **Lie algebras and formal moduli problems.** In this subsection we will assume Theorem 3.1.4 and deduce some further corollaries. In particular, we will show that there is an equivalence between pointed formal moduli problems over a scheme X and Lie algebras in $\mathrm{IndCoh}(X)$.

Furthermore, we will see what the functor of inf-spectrum really does, and what it means to be inf-affine. Namely, we will show that under the equivalence above, the functor $\mathrm{Spec}^{\mathrm{inf}}$ corresponds to the functor $\mathrm{coChev}^{\mathrm{enh}}$.

3.3.1. First, we claim:

Corollary 3.3.2. *There is the following commutative diagram of functors*

$$\begin{array}{ccc} \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xleftarrow{\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}} & \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \\ {\scriptstyle \mathrm{Chev}^{\mathrm{enh}}}\uparrow & & \sim\uparrow{\scriptstyle B_X} \\ \mathrm{LieAlg}(\mathrm{IndCoh}(X)) & \xleftarrow[\sim]{\mathrm{Lie}_X} & \mathrm{Grp}(\mathrm{FormMod}_{/X}). \end{array} \tag{3.4}$$

Proof. Indeed, by Theorem 3.1.4, it suffices to construct a functorial isomorphism

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ B_X \circ \exp_X \simeq \mathrm{Chev}^{\mathrm{enh}}.$$

However, by Lemma 1.2.6 and the isomorphism (3.1), the left-hand side identifies with

$$\mathrm{Bar} \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) \circ \exp_X \simeq \mathrm{Bar} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \simeq$$
$$\simeq \mathrm{Chev}^{\mathrm{enh}} \circ B_{\mathrm{Lie}} \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Chev}^{\mathrm{enh}}.$$

□

Corollary 3.3.3. *For $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ there is a canonical isomorphism*

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}) \simeq \mathrm{Chev}^{\mathrm{enh}} \circ \mathrm{Lie}_X \circ \Omega_X(\mathcal{Y}).$$

Remark 3.3.4. The commutative diagram (3.4) implies the following:

The functor

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$$

remembers/loses as much information as does the functor

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

However, the functor

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomBialg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$$

is fully faithful, as is the functor

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{CocomBialg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

3.3.5. By passing to right adjoints in diagram (3.4) we obtain:

Corollary 3.3.6. *There is the following commutative diagram of functors*

$$\begin{array}{ccc} \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xrightarrow{\mathrm{Spec}^{\mathrm{inf}}} & \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \\ {\scriptstyle \mathrm{coChev}^{\mathrm{enh}}}\downarrow & & \sim\downarrow{\scriptstyle \Omega_X} \\ \mathrm{LieAlg}(\mathrm{IndCoh}(X)) & \xrightarrow[\sim]{\exp_X} & \mathrm{Grp}(\mathrm{FormMod}_{/X}). \end{array} \tag{3.5}$$

Remark 3.3.7. The commutative diagram (3.5) implies:

The functor

$$\mathrm{Spec}^{\mathrm{inf}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

remembers/loses as much information as does the functor

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

3.3.8. Let

$$B_X^{\mathrm{Lie}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

denote the functor $B_X \circ \exp_X$.

This is the functor that associates to a Lie algebra in $\mathrm{IndCoh}(X)$ the corresponding moduli problem. By Theorem 3.1.4, this functor is an equivalence, with the inverse being

$$\mathcal{Y} \mapsto \mathrm{Lie}_X \circ \Omega_X(\mathcal{Y}).$$

From Proposition 2.3.3 we obtain:

Corollary 3.3.9. *Let $\mathcal{Y}$ be an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$, and let $\mathfrak{h}$ be the corresponding object of $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$, i.e.,*

$$\mathfrak{h} = \mathrm{Lie}_X \circ \Omega_X(\mathcal{Y}) \text{ and/or } \mathcal{Y} := B_X^{\mathrm{Lie}}(\mathfrak{h}).$$

Then $\mathcal{Y}$ is inf-affine if and only if unit of the adjunction

$$\mathfrak{h} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})$$

is an isomorphism.

3.3.10. Let $\mathcal{A}$ be an object of $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$. From the diagrams (3.4) and (3.5) we obtain that the co-unit of the adjunction

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})) \to \mathcal{A} \tag{3.6}$$

identifies with the map

$$\mathrm{Chev}^{\mathrm{enh}} \circ \mathrm{coChev}^{\mathrm{enh}}(\mathcal{A}) \to \mathcal{A}. \tag{3.7}$$

In partcular, we obtain that if [Chapter IV.2, Conjecture 2.8.9(b)] holds for the symmetric monoidal DG category $\mathrm{IndCoh}(X)$ and the co-operad $\mathrm{Cocom}^{\mathrm{aug}}$, i.e., if the map (3.7) is an isomorphism for $\mathcal{A}$ lying in the essential image of the functor

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)),$$

then the map (3.6) is an isomorphism for $\mathcal{A}$ lying in the essential image of the functor

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Similarly, suppose that [Chapter IV.2, Conjecture 2.8.9(a)] holds for the symmetric monoidal DG category $\mathrm{IndCoh}(X)$ and the co-operad $\mathrm{Cocom}^{\mathrm{aug}}$, i.e., if the map

$$\mathfrak{h} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})$$

is an isomorphism for $\mathfrak{h}$ lying in the essential image of the functor

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

Then, by Corollary 3.3.9, any $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ lying in the essential image of the functor

$$\mathrm{Spec}^{\mathrm{inf}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X}),$$

is inf-affine.

3.4. **The ind-nilpotent version.** For completeness, let us explain what happens to the picture in Sect. 3.3 if we consider instead the adjoint functors

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug,ind\text{-}nil}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) : \mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}.$$

3.4.1. First, we have the commutative diagrams

$$(3.8) \qquad \begin{array}{ccc} \mathrm{CocomCoalg}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) & \xleftarrow{\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}}} & \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \\ {\scriptstyle \mathrm{Chev}^{\mathrm{enh,ind\text{-}nilp}}}\uparrow & & \sim\uparrow{\scriptstyle B_X} \\ \mathrm{LieAlg}(\mathrm{IndCoh}(X)) & \underset{\sim}{\xleftarrow{\mathrm{Lie}_X}} & \mathrm{Grp}(\mathrm{FormMod}_{/X}). \end{array}$$

and

$$(3.9) \qquad \begin{array}{ccc} \mathrm{CocomCoalg}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) & \xrightarrow{\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}} & \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \\ {\scriptstyle \mathrm{coChev}^{\mathrm{enh,ind\text{-}nilp}}}\downarrow & & \sim\downarrow{\scriptstyle \Omega_X} \\ \mathrm{LieAlg}(\mathrm{IndCoh}(X)) & \underset{\sim}{\xrightarrow{\exp_X}} & \mathrm{Grp}(\mathrm{FormMod}_{/X}). \end{array}$$

3.4.2. Let us now assume the validity of [Chapter IV.2, Conjecture 2.6.6] for the symmetric monoidal DG category $\mathrm{IndCoh}(X)$ and the co-operad $\mathrm{Cocom}^{\mathrm{aug}}$.

From it we obtain:

Conjecture 3.4.3. *The functor*

$$\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}} : \mathrm{CocomCoalg}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

is fully faithful.

3.5. **Base change.** As we saw in Proposition 2.3.3, the criterion of inf-affineness involves the operation of taking primitives in an augmented co-commutative co-algebra in $\mathrm{IndCoh}(X)$. This operation is not guaranteed to behave well with respect to the operation of pullback. The functor of inf-spectrum has a similar drawback, for the same reason.

In this subsection we will establish several positive results in this direction.

3.5.1. Let $f : X' \to X$ be a map in ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, and consider the corresponding functor

$$f^! : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{Cocom}^{\mathrm{aug}}(\mathrm{IndCoh}(X')).$$

The following diagram commutes by construction

$$\begin{array}{ccc} \mathrm{Ptd}(\mathrm{FormMod}_{/X}) & \xrightarrow{X' \underset{X}{\times} -} & \mathrm{Ptd}(\mathrm{FormMod}_{/X'}) \\ {\scriptstyle \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}} \\ \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xrightarrow{f^!} & \mathrm{Cocom}^{\mathrm{aug}}(\mathrm{IndCoh}(X')). \end{array}$$

Hence, by adjunction, for $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ we have a canonically defined map

$$X' \underset{X}{\times} \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A}) \to \mathrm{Spec}^{\mathrm{inf}}(f^!(\mathcal{A})). \tag{3.10}$$

Remark 3.5.2. It follows from Lemma 4.2.3 below and diagram (3.5), that the natural transformation (3.10) is an isomorphism if f is proper.

3.5.3. The following is immediate from Lemma 2.2.2 and Proposition 2.3.3:

Lemma 3.5.4. *Assume that $\mathcal{A}$ is such that both maps*

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A}) \to \mathcal{A} \text{ and } \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Spec}^{\mathrm{inf}}(f^!(\mathcal{A})) \to f^!(\mathcal{A})$$

are isomorphisms. Then the map (3.10) *is an isomorphism for $\mathcal{A}$.*

Corollary 3.5.5. *For $\mathcal{F} \in \mathrm{IndCoh}(X)$, the canonical map*

$$X' \underset{X}{\times} \mathrm{Vect}_X(\mathcal{F}) \to \mathrm{Vect}_{X'}(f^!(\mathcal{F}))$$

is an isomorphism.

3.5.6. By combining Corollary 3.5.5 and [Chapter IV.2, Proposition 1.7.2], we obtain:

Corollary 3.5.7. *For $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(X))$, the canonical map*

$$X' \underset{X}{\times} \exp_X(\mathfrak{h}) \to \exp_{X'}(f^!(\mathfrak{h}))$$

is an isomorphism.

Corollary 3.5.8. *For $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$, the canonical map*

$$f^!(\mathrm{Lie}_X(\mathcal{H})) \to \mathrm{Lie}_{X'}(X' \underset{X}{\times} \mathcal{H})$$

is an isomorphism.

3.6. **Extension to prestacks.** We will now extend the equivalence $\exp_X$ to the case when the base $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ is replaced by an arbitrary $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$.

3.6.1. Note that the discussion in Sect. 1.1 applies verbatim to the present situation (i.e., the base being an object of $\mathrm{PreStk}_{\mathrm{laft}}$). In particular, we obtain the functors

$$\mathrm{Distr} : \mathrm{FormMod}_{/\mathcal{X}} \to \mathrm{IndCoh}(\mathcal{X}),$$

$$\mathrm{Distr}^{\mathrm{Cocom}} : \mathrm{FormMod}_{/\mathcal{X}} \to \mathrm{CocomCoalg}(\mathrm{IndCoh}(\mathcal{X})),$$

$$\mathrm{Distr}^{+} : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X}),$$

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})),$$

and

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{CocomBial}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})).$$

We have:

Theorem 3.6.2. *The functor*

$$\mathrm{Lie}_{\mathcal{X}} := B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})$$

defines an equivalence $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$. *Furthermore, we have:*

(a) *The co-unit of the adjunction*

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \circ \mathrm{Lie}_{\mathcal{X}} \to \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})$$

is an isomorphism.

(b) *The composition*

$$\mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{Lie}_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X})$$

identifies canonically with the functor

$$\mathcal{H} \mapsto T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/\mathcal{X})|_{\mathcal{X}}.$$

Proof. Observe that for $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ the functors

$$\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \underset{X \in (({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}})^{\mathrm{op}}}{lim} \mathrm{LieAlg}(\mathrm{IndCoh}(X))$$

and

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \underset{X \in (({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}})^{\mathrm{op}}}{lim} \mathrm{Grp}(\mathrm{FormMod}_{/X})$$

are both equivalences, see [Chapter IV.1, Lemma 1.1.5] for the latter statement.

Using Theorem 3.1.4, to show that the functor $\mathrm{Lie}_{\mathcal{X}}$ is an equivalence, it remains to check that the functors Lie_X, where X is a scheme, are compatible with base change. But this follows from Corollary 3.5.8.

The isomorphisms stated in (a) and (b) follow from the case of schemes.

□

As a formal consequence we obtain:

Corollary 3.6.3. *The category* $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ *contains sifted colimits, and the functor*

$$\mathcal{H} \mapsto T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/\mathcal{X})|_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X})$$

commutes with sifted colimits.

3.6.4. Let

$$\exp_{\mathcal{X}} : \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \to \text{Grp}(\text{FormMod}_{/\mathcal{X}})$$

denote the equivalence, inverse to $\text{Lie}_{\mathcal{X}}$.

Let

$$B^{\text{Lie}}_{\mathcal{X}} : \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \to \text{Ptd}(\text{FormMod}_{/\mathcal{X}})$$

denote the resulting equivalence

$$B_{\mathcal{X}} \circ \exp_{\mathcal{X}}, \quad \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \to \text{Ptd}(\text{FormMod}_{/\mathcal{X}}).$$

Note that from Corollary 3.3.2 we obtain:

Corollary 3.6.5. *There is a canonical isomorphism of functors*

$$\text{Chev}^{\text{enh}} \simeq \text{Distr}^{\text{Cocom}^{\text{aug}}} \circ B^{\text{Lie}}_{\mathcal{X}}.$$

3.6.6. In what follows we will denote by

$$\mathcal{F} \mapsto \text{Vect}_{\mathcal{X}}(\mathcal{F}) \tag{3.11}$$

the functor $\text{IndCoh}(\mathcal{X}) \to \text{Ptd}(\text{FormMod}_{/\mathcal{X}})$, given by

$$\mathcal{F} \mapsto \text{Vect}_{\mathcal{X}}(\mathcal{F}) := \mathbf{oblv}_{\text{Grp}} \circ \exp_{\mathcal{X}} \circ \mathbf{triv}_{\text{Lie}}(\mathcal{F}) \simeq B^{\text{Lie}}_{\mathcal{X}} \circ \exp_{\mathcal{X}} \circ \mathbf{triv}_{\text{Lie}}(\mathcal{F}[-1]).$$

Note that by Corollary 3.6.5 we obtain

$$\text{Distr}^{\text{Cocom}^{\text{aug}}}(\text{Vect}_{\mathcal{X}}(\mathcal{F})) \simeq \text{Sym}(\mathcal{F}).$$

From Proposition 1.4.3, we obtain:

Corollary 3.6.7. *The functor* $\text{Vect}_{\mathcal{X}}(-) : \text{IndCoh}(\mathcal{X}) \to \text{Ptd}(\text{FormMod}_{/\mathcal{X}})$ *is the right adjoint to the functor* Distr^{+}.

Note also:

Corollary 3.6.8. *For* $\mathcal{H} \in \text{Grp}(\text{FormMod}_{/\mathcal{X}})$, *we have a canonical isomorphism*

$$\mathbf{oblv}_{\text{Grp}}(\mathcal{H}) \simeq \text{Vect}_{\mathcal{X}}(T(\mathbf{oblv}_{\text{Grp}}(\mathcal{H})/\mathcal{X})|_{\mathcal{X}}).$$

3.6.9. The functor (3.11) is easily seen to commute with products. Hence, it induces a functor

$$\text{IndCoh}(\mathcal{X}) \to \text{ComMonoid}(\text{FormMod}_{/\mathcal{X}}), \tag{3.12}$$

see [Chapter IV.2, Sect. 1.8] for the notation.

We claim:

Corollary 3.6.10. *The functor* (3.12) *is an equivalence.*

Proof. Follows from [Chapter IV.2, Proposition 1.8.3]. □

3.7. **An example: split square-zero extensions.** In [Chapter III.1, Sect. 2.1] we discussed the functor of *split square-zero extension*

$$\text{RealSplitSqZ} : (\text{Coh}(X)^{\leq 0})^{\text{op}} \to \text{Ptd}((\text{Sch}^{\text{aff}}_{\text{aft}})_{/X}), \quad X \in \text{Sch}^{\text{aff}}_{\text{aft}}.$$

In this subsection we will extend this construction to the case of arbitrary objects $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$, where instead of $(\text{Coh}(-)^{\leq 0})^{\text{op}}$ we use all of $\text{IndCoh}(\mathcal{X})$. Here for $\mathcal{X} = X \in \text{Sch}_{\text{aft}}$, we view $(\text{Coh}(X)^{\leq 0})^{\text{op}}$ as a full subcategory of $\text{IndCoh}(X)$ via

$$(\text{Coh}(X)^{\leq 0})^{\text{op}} \hookrightarrow \text{Coh}(X)^{\text{op}} \overset{\mathbb{D}^{\text{Serre}}_X}{\longrightarrow} \text{Coh}(X) \hookrightarrow \text{IndCoh}(X).$$

3.7.1. For $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$ consider the functor

$$\text{RealSplitSqZ} : \text{IndCoh}(\mathcal{X}) \to \text{Ptd}((\text{PreStk}_{\text{laft-def}})_{/\mathcal{X}}),$$

defined as follows:

We send $\mathcal{F} \in \text{IndCoh}(\mathcal{X})$ to

$$B_{\mathcal{X}} \circ \exp_{\mathcal{X}} \circ \mathbf{free}_{\text{Lie}}(\mathcal{F}[-1]) \in \text{Ptd}((\text{Sch}^{\text{aff}})_{/\mathcal{X}}) \in \text{Ptd}((\text{FormMod})_{/\mathcal{X}}) \subset \text{Ptd}((\text{PreStk}_{\text{laft-def}})_{/\mathcal{X}}).$$

We can phrase the above construction as follows: we create the free Lie algebra on $\mathcal{F}[-1]$, then we consider the corresponding object of $\text{Grp}((\text{Sch}^{\text{aff}})_{/X})$, and then take the formal classifying space of the latter.

By construction, we have a commutative diagram:

$$\begin{array}{ccc} \text{IndCoh}(\mathcal{X}) & \xrightarrow{\text{RealSplitSqZ}} & \text{Ptd}(\text{FormMod}_{/\mathcal{X}}) \\ {\scriptstyle \mathbf{free}_{\text{Lie}}\circ[-1]}\downarrow & & \downarrow{\scriptstyle \Omega_{\mathcal{X}}}\,\sim \\ \text{LieAlg}(\text{IndCoh}(\mathcal{X})) & \xrightarrow{\exp_{\mathcal{X}}} & \text{Grp}(\text{FormMod}_{/\mathcal{X}}). \end{array} \tag{3.13}$$

3.7.2. We claim that the functor RealSplitSqZ can also be described as a left adjoint:

Proposition 3.7.3. *The functor* RealSplitSqZ *is the left adjoint of the functor*

$$\text{Ptd}((\text{PreStk}_{\text{laft-def}})_{/\mathcal{X}}) \to \text{IndCoh}(\mathcal{X}), \quad \mathcal{Y} \mapsto T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}}.$$

Proof. Given $\mathcal{Y} \in \text{Ptd}((\text{PreStk}_{\text{laft-def}})_{/\mathcal{X}})$ and $\mathcal{F} \in \text{IndCoh}(\mathcal{X})$ we need to establish a canonical isomorphism

$$\text{Maps}_{\text{Ptd}((\text{PreStk}_{\text{laft-def}})_{/\mathcal{X}})}(\text{RealSplitSqZ}(\mathcal{F}), \mathcal{Y}) \simeq \text{Maps}_{\text{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}}). \tag{3.14}$$

Note that the left-hand side receives an isomorphism from $\text{Maps}(\text{RealSplitSqZ}(\mathcal{F}), \mathcal{Y}^{\wedge}_{\mathcal{X}})$, where $\mathcal{Y}^{\wedge}_{\mathcal{X}}$ is the formal completion of $\mathcal{Y}$ along the map $\mathcal{X} \to \mathcal{Y}$. So, with no restriction of generailty, we can assume that $\mathcal{Y} \in \text{Ptd}((\text{FormMod})_{/\mathcal{X}})$.

In this case, by [Chapter IV.1, Theorem 1.6.4], we can further rewrite the left-hand side in (3.14) as

$$\text{Maps}_{\text{Grp}((\text{FormMod})_{/\mathcal{X}})}(\exp_{\mathcal{X}} \circ \mathbf{free}_{\text{Lie}}(\mathcal{F}[-1]), \Omega_{\mathcal{X}}(\mathcal{Y})),$$

and then as

$$\text{Maps}_{\text{LieAlg}(\text{IndCoh}(\mathcal{X}))}\left(\mathbf{free}_{\text{Lie}}(\mathcal{F}[-1]), \text{Lie}_{\mathcal{X}} \circ \Omega_{\mathcal{X}}(\mathcal{Y})\right) \simeq$$
$$\simeq \text{Maps}_{\text{IndCoh}(\mathcal{X})}\left(\mathcal{F}[-1], \mathbf{oblv}_{\text{Lie}} \circ \text{Lie}_{\mathcal{X}} \circ \Omega_{\mathcal{X}}(\mathcal{Y})\right).$$

However, by Corollary 3.2.8, we have

$$\mathbf{oblv}_{\text{Lie}} \circ \text{Lie}_{\mathcal{X}} \circ \Omega_{\mathcal{X}}(\mathcal{Y}) \simeq T(\Omega_{\mathcal{X}}(\mathcal{Y})/\mathcal{X})|_{\mathcal{X}} \simeq T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}}[-1].$$

Thus, the left-hand side in (3.14) identifies with

$$\text{Maps}_{\text{IndCoh}(\mathcal{X})}\left(\mathcal{F}[-1], T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}}[-1]\right),$$

as required.

□

Remark 3.7.4. The above verification of the adjunction can be summarized by the commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{X}) & \xleftarrow{T(-/\mathcal{X})|_{\mathcal{X}}} & \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \\ {\scriptstyle [1]\circ \mathbf{oblv}_{\mathrm{Lie}}}\uparrow & & \downarrow{\scriptstyle \Omega_{\mathcal{X}}}\,\sim \\ \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \xleftarrow{\mathrm{Lie}_{\mathcal{X}}} & \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}). \end{array}$$

3.7.5. As a corollary of Proposition 3.7.3, we obtain:

Corollary 3.7.6. *The monad on* $\mathrm{IndCoh}(\mathcal{X})$*, given by the composition*

$$([-1] \circ T(-/\mathcal{X})|_{\mathcal{X}}) \circ (\mathrm{RealSplitSqZ} \circ [1])$$

is canonically isomorphic to $\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}$.

3.7.7. The next property of the functor RealSplitSqZ follows formally from Proposition 3.7.3:

Corollary 3.7.8. *For* $\mathcal{Y} \in (\mathrm{PreStk}_{\text{laft-def}})_{\mathcal{X}/}$ *and* $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ *there is a canonical isomorphism*

$$\mathrm{Maps}_{(\mathrm{PreStk}_{\text{laft-def}})_{\mathcal{X}/}}(\mathrm{RealSplitSqZ}(\mathcal{F}), \mathcal{Y}) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{Y})|_{\mathcal{X}}).$$

In the above corollary, by a slight abuse of notation, we view $\mathrm{RealSplitSqZ}(\mathcal{F})$ as an object of $(\mathrm{PreStk}_{\text{laft-def}})_{\mathcal{X}/}$ rather than $\mathrm{Ptd}((\mathrm{PreStk}_{\text{laft-def}})_{/\mathcal{X}})$.

Proof. Set $\mathcal{Y}' := \mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{Y}$, and apply the adjunction of Proposition 3.7.3. □

3.7.9. Let us now compare the functor RealSplitSqZ as defined above with its version introduced in [Chapter III.1, Sect. 2.1]:

Corollary 3.7.10. *For* $X \in \mathrm{Sch}_{\mathrm{aft}}$ *we have a commutative diagram*

$$\begin{array}{ccc} (\mathrm{Coh}(X)^{\leq 0})^{\mathrm{op}} & \xrightarrow{\mathrm{RealSplitSqZ}} & \mathrm{Ptd}((\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom to }X}) \\ {\scriptstyle \mathbb{D}^{\mathrm{Serre}}_X}\downarrow & & \downarrow \\ \mathrm{IndCoh}(X) & \xrightarrow{\mathrm{RealSplitSqZ}} & \mathrm{Ptd}((\mathrm{FormMod})_{/X}). \end{array}$$

Proof. Follows from Corollary 3.7.8, since the split square-zero construction of [Chapter III.1, Sect. 2.1] has the same universal property.

□

3.7.11. It follows from the equivalence

$$B^{\mathrm{Lie}}_{\mathcal{X}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

that the functor RealSplitSqZ takes coproducts in $\mathrm{IndCoh}(X)$ to coproducts in the category $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$. In particular, it defines a functor

$$\mathrm{IndCoh}(\mathcal{X})^{\mathrm{op}} \simeq \mathrm{ComMonoid}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{op}}) \to \mathrm{ComMonoid}(\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})^{\mathrm{op}}). \tag{3.15}$$

We claim:

Proposition 3.7.12. *The functor* (3.15) *is an equivalence.*

Proof. Follows from the fact that $B^{\mathrm{Lie}}_{\mathcal{X}}$ is an equivalence, combined with [Chapter IV.2, Corollary 1.8.7]. □

4. Proof of Theorem 3.1.4

4.1. **Step 1.** In this subsection we will prove that the functor $\exp_X$ defines an equivalence from $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$ to the full subcategory of $\mathrm{Grp}(\mathrm{FormMod}_{/X})$, spanned by objects that are inf-affine when viewed as objects of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ (i.e., after forgetting the group structure).

We denote this category by $\mathrm{Grp}(\mathrm{FormMod}_{/X})'$.

4.1.1. First, we note that by Proposition 1.4.7 and [Chapter IV.2, Proposition 1.7.2], for any $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(X))$, the object

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \exp_X(\mathfrak{h}) \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

is inf-affine, and the canonical map

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\exp_X(\mathfrak{h})) \to \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}(\mathfrak{h}) \tag{4.1}$$

is an isomorphism.

4.1.2. We claim that the functor Lie_X of Sect. 3.3, restricted to $\mathrm{Grp}(\mathrm{FormMod}_{/X})'$, provides a right adjoint to $\exp_X$. In other words, we claim that for $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(X))$ and $\mathcal{H}' \in \mathrm{Grp}(\mathrm{FormMod}_{/X})'$, there is a canonical isomorphism:

$$\mathrm{Maps}_{\mathrm{Grp}(\mathrm{FormMod}_{/X})}(\exp_X(\mathfrak{h}), \mathcal{H}') \simeq \mathrm{Maps}_{\mathrm{LieAlg}(\mathrm{IndCoh}(X))}(\mathfrak{h}, \mathrm{Lie}_X(\mathcal{H}')).$$

Indeed, by Lemma 2.1.6 and (4.1), we rewrite the left-hand side as

$$\mathrm{Maps}_{\mathrm{CocomHopf}(\mathrm{IndCoh}(X))}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}(\mathfrak{h}), \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\mathcal{H}')),$$

and, further, using [Chapter IV.2, Sect. 4.4.2] as

$$\mathrm{Maps}_{\mathrm{LieAlg}(\mathbf{O})}\left(\mathfrak{h}, B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\mathcal{H}')\right),$$

as required.

4.1.3. We claim that the unit of the adjunction

$$\mathrm{Id} \to \mathrm{Lie}_X \circ \exp_X$$

is an isomorphism.

Indeed, this follows from (4.1) and (3.2).

4.1.4. Hence, it remains to show that the functor Lie_X, restricted to $\mathrm{Grp}(\mathrm{FormMod}_{/X})'$, is conservative. I.e., we need to show that if $\mathcal{H}_1 \to \mathcal{H}_2$ is a map in $\mathrm{Grp}(\mathrm{FormMod}_{/X})'$, such that $\mathrm{Lie}_X(\mathcal{H}_1) \to \mathrm{Lie}_X(\mathcal{H}_2)$ is an isomorphism, then the original map is also an isomorphism.

More generally, we claim that if $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a map between two inf-affine objects of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$, such that the induced map

$$\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}_1) \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}_2)$$

is an isomorphism in $\mathrm{IndCoh}(X)$, then the original map is also an isomorphism.

Indeed, this follows from Proposition 2.3.3 and [Chapter III.1, Proposition 8.3.2].

4.2. **Step 2.** In this subsection we will reduce the assertion of Theorem 3.1.4 to the case when X is reduced.

4.2.1. Taking into account Step 1, the assertion of Theorem 3.1.4 is equivalent to the fact that every object $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$ is inf-affine, when we consider it as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$.

Thus, by Proposition 2.1.4, we need to show that for any $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$, the canonical map

$$T(\mathcal{H}/X)|_X \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{H})$$

is an isomorphism.

4.2.2. Let $f : X' \to X$ be a map in $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X}$. We have the symmetric monoidal functor

$$f^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X'),$$

which makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{f^!} & \mathrm{IndCoh}(X') \\ {\scriptstyle \mathbf{triv}_{\mathrm{Cocom}}}\downarrow & & \downarrow{\scriptstyle \mathbf{triv}_{\mathrm{Cocom}}} \\ \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xrightarrow{f^!} & \mathrm{Cocom}^{\mathrm{aug}}(\mathrm{IndCoh}(X')) \end{array}$$

Hence, by adjunction, we obtain a natural transformation:

$$f^! \circ \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ f^!. \tag{4.2}$$

We claim:

Lemma 4.2.3. *Assume that f is proper. Then the natural transformation* (4.2) *is an isomorphism*

Proof. Follows by the $(f^{\mathrm{IndCoh}}_*, f^!)$-adjunction from the commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xleftarrow{f^{\mathrm{IndCoh}}_*} & \mathrm{IndCoh}(X') \\ {\scriptstyle \mathbf{triv}_{\mathrm{Cocom}}}\downarrow & & \downarrow{\scriptstyle \mathbf{triv}_{\mathrm{Cocom}}} \\ \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xleftarrow{f^{\mathrm{IndCoh}}_*} & \mathrm{Cocom}^{\mathrm{aug}}(\mathrm{IndCoh}(X')). \end{array}$$

□

4.2.4. Let i denote the canonical map $X' := {}^{\mathrm{red}}X \to X$. From Lemma 4.2.3 we obtain that for $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$, we have a commutative diagram with vertical arrows being isomorphisms

$$\begin{array}{ccc} i^!(T(\mathcal{Y}/X)|_X) & \longrightarrow & i^!\left(\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\right) \\ \downarrow & & \downarrow \\ T(\mathcal{Y}'/X')|_{X'} & \longrightarrow & \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}'), \end{array}$$

where $\mathcal{Y}' := X' \underset{X}{\times} \mathcal{Y}$.

Since the functor $i^!$ is conservative (see [Chapter II.1, Corollary 6.1.5]), we obtain that if

$$T(\mathcal{Y}'/X')|_{X'} \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}')$$

is an isomorphism, then so is

$$T(\mathcal{Y}/X)|_X) \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}).$$

Hence, the assertion of Theorem 3.1.4 for ${}^{\mathrm{red}}X$ implies that for X.

4.3. **Step 3.** We will now show that the functor $\exp_X$ is essentially surjective onto the entire category $\text{Grp}(\text{FormMod}_{/X})$. By Step 2, we can assume that X is reduced.

4.3.1. For $\mathcal{H} \in \text{Grp}(\text{FormMod}_{/X})$ set

$$\mathcal{Y} := B_X(\mathcal{H}) \in \text{Ptd}(\text{FormMod}_{/X}).$$

Using [Chapter IV.1, Corollary 1.5.2(a)], we can write

$$\mathcal{Y} \simeq \underset{\alpha \in A}{\text{colim}}\, Z_\alpha,$$

where the index category A is

$$\Big(\text{Ptd}(({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{nil-isom to } X})\Big)_{/\mathcal{Y}},$$

and where the colimit is taken in the category $\text{PreStk}_{\text{laft}}$.

We make the following observation:

Lemma 4.3.2. *If the scheme X is reduced, then the category A is sifted.*

Proof. We claim that the diagonal functor $A \to A \times A$ admits a left adjoint. Namely, it is given by sending

$$Z_1, Z_2 \to Z_1 \underset{X}{\sqcup} Z_2,$$

see [Chapter III.1, Proposition 7.2.2].

NB: the fact that X is reduced is used to ensure that the maps $X \to Z_i$ are closed (and hence, nilpotent embeddings).

□

4.3.3. Set

$$\mathcal{H}_\alpha := \Omega_X(Z_\alpha).$$

Since $\mathcal{H}_\alpha$ is a scheme, it is inf-affine, by Proposition 2.1.4. Hence, there exists a canonically defined functor

$$A \to \text{LieAlg}(\text{IndCoh}(X)), \quad \alpha \mapsto \mathfrak{h}_\alpha,$$

so that $\mathcal{H}_\alpha = \exp_X(\mathfrak{h}_\alpha)$.

Set

$$\mathfrak{h} := \underset{\alpha \in A}{\text{colim}}\, \mathfrak{h}_\alpha \in \text{LieAlg}(\text{IndCoh}(X)).$$

We are going to construct an isomorphism $\mathcal{H} \simeq \exp_X(\mathfrak{h})$.

4.3.4. By [Chapter IV.1, Theorem 1.6.4], it suffices to construct an isomorphism

$$\mathcal{Y} \simeq B_X \circ \exp_X(\mathfrak{h})$$

in $\text{Ptd}(\text{FormMod}_{/X})$.

We let $\mathcal{Y} \to B_X \circ \exp_X(\mathfrak{h})$ be the map, given by the compatible system of maps

$$Z_\alpha \to B_X \circ \exp_X(\mathfrak{h})$$

that correspond under the equivalence Ω_X to the maps

$$\mathcal{H}_\alpha \simeq \exp_X(\mathfrak{h}_\alpha) \to \exp_X(\mathfrak{h}).$$

To prove that the resulting map $\mathcal{Y} \to B_X \circ \exp_X(\mathfrak{h})$ is an isomorphism, by [Chapter III.1, Proposition 8.3.2], it suffices to show that the induced map

$$T(\mathcal{Y}/X)|_X \to T(B_X \circ \exp_X(\mathfrak{h})/X)|_X$$

is an isomorphism in $\mathrm{IndCoh}(X)$.

4.3.5. We have a commutative diagram

$$\begin{array}{ccc} \underset{\alpha\in A}{\mathrm{colim}}\, T(Z_\alpha/X)|_X & \xrightarrow{\mathrm{id}} & \underset{\alpha\in A}{\mathrm{colim}}\, T(Z_\alpha/X)|_X \\ \downarrow & & \downarrow \\ T(\mathcal{Y}/X)|_X & \longrightarrow & T(B_X \circ \exp_X(\mathfrak{h})/X)|_X. \end{array}$$

We note that the left vertical arrow is an isomorphism by [Chapter III.1, Proposition 2.5.3], since the category of indices A is sifted (see Lemma 4.3.2).

Hence, it remains to show that the right vertical arrow is an isomorphism.

4.3.6. The corresponding map

$$\underset{\alpha\in A}{\mathrm{colim}}\, T(Z_\alpha/X)|_X[-1] \to T(B_X \circ \exp_X(\mathfrak{h})/X)|_X[-1]$$

identifies with

$$\underset{\alpha\in A}{\mathrm{colim}}\, T(\mathcal{H}_\alpha/X)|_X \to T(\exp_X(\mathfrak{h})/X)|_X,$$

and, further, by Proposition 2.3.3, with

$$\underset{\alpha\in A}{\mathrm{colim}}\, \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_\alpha) \to \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}). \tag{4.3}$$

Since the category A is sifted, in the commutative diagram

$$\begin{array}{ccc} \underset{\alpha\in A}{\mathrm{colim}}\, \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_\alpha) & \longrightarrow & \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}) \\ \downarrow & & \downarrow \mathrm{id} \\ \mathbf{oblv}_{\mathrm{Lie}}\left(\underset{\alpha\in A}{\mathrm{colim}}\, \mathfrak{h}_\alpha\right) & \xrightarrow{\sim} & \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}) \end{array}$$

the vertical arrows are isomorphisms.

Hence, the map (4.3) is an isomorphism, as required.

5. Modules over formal groups and Lie algebras

In the previous sections we have constructed an equivalence between formal groups and Lie algebras. In this section we will show that under this equivalence, the datum of action of a formal group on a given object of IndCoh is equivalent to that of action of the corresponding Lie algebra.

5.1. **Modules over formal groups.**

5.1.1. Let $\mathcal{H}$ be an object of $\mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathcal{X}})$. We define the category $\mathcal{H}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$ as

$$\mathrm{Tot}\left(\mathrm{IndCoh}^!(B^\bullet(\mathcal{H}))\right),$$

where $\mathrm{IndCoh}^!(B^\bullet(\mathcal{H}))$ is the co-simplicial category, obtained by applying the (contravariant) functor $\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ to the simplicial object $B^\bullet(\mathcal{H})$ of $\mathrm{PreStk}_{\mathrm{laft}}$.

Denote $\mathfrak{h} := \mathrm{Lie}_{\mathcal{X}}(\mathcal{H})$. The goal of this subsection is to prove the following:

Proposition-Construction 5.1.2. *There exists a canonical equivalence of categories*

$$\mathcal{H}\text{-mod}(\text{IndCoh}(\mathcal{X})) \simeq \mathfrak{h}\text{-mod}(\text{IndCoh}(\mathcal{X})) \tag{5.1}$$

that commutes with the forgetful functor to IndCoh($\mathcal{X}$)*, and is functorial with respect to* $\mathcal{X}$*.*

The rest of this subsection is devoted to the proof of Proposition 5.1.2.

Without loss of generality, we can assume that $\mathcal{X} = X \in {}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}}$.

5.1.3. Consider the object

$$\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}) \in \text{AssocAlg}(\text{CocomCoalg}(\text{IndCoh}(X))).$$

Consider the corresponding simplicial object

$$\text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})) \in \text{CocomCoalg}(\text{IndCoh}(X))^{\mathbf{\Delta}^{\text{op}}},$$

and the simplicial category

$$\text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\text{IndCoh}(X)).$$

According to [Chapter IV.2, Proposition 7.2.2 and Sect. 7.4], there exist canonical equivalences

$$\mathfrak{h}\text{-mod}(\text{IndCoh}(X)) \simeq \Big(\text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})\Big)\text{-mod}(\text{IndCoh}(X)) \simeq$$
$$\simeq |\,\text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\text{IndCoh}(X))|.$$

5.1.4. By [Chapter I.1, Proposition 2.5.7], we have

$$\mathcal{H}\text{-mod}(\text{IndCoh}(X)) = \text{Tot}\left(\text{IndCoh}^!(B^\bullet(\mathcal{H}))\right) \simeq |\,(\text{IndCoh}_*(B^\bullet(\mathcal{H})))\,|,$$

where $\text{IndCoh}_*(B^\bullet(\mathcal{H}))$ is the simplicial category, obtained by applying the functor

$$\text{IndCoh}_{\text{PreStk}_{\text{laft}}} : \text{PreStk}_{\text{laft}} \to \text{DGCat}_{\text{cont}}$$

to the simplicial object $B^\bullet(\mathcal{H})$ of $\text{PreStk}_{\text{laft}}$.

We will construct a functor between simplicial categories

$$\text{IndCoh}_*(B^\bullet(\mathcal{H})) \to \text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\text{IndCoh}(X)), \tag{5.2}$$

and show that it induces an equivalence on geometric realizations.

5.1.5. Let $\pi^\bullet$ denote the augmentation $B^\bullet(\mathcal{H}) \to X$. By Sect. 1.1.3, The functor $(\pi^\bullet)_*^{\text{IndCoh}}$ defines a map of simplicial categories

$$\text{IndCoh}_*(B^\bullet(\mathcal{H})) \to \text{Distr}^{\text{Cocom}}(B^\bullet(\mathcal{H}))\text{-comod}(\text{IndCoh}(X)). \tag{5.3}$$

Note that by Lemma 1.2.2, we have:

$$\text{Distr}^{\text{Cocom}}(B^\bullet(\mathcal{H})) \simeq \text{Bar}^\bullet(\text{Grp}(\text{Distr}^{\text{Cocom}^{\text{aug}}})(\mathcal{H})).$$

Since

$$\text{Grp}(\text{Distr}^{\text{Cocom}^{\text{aug}}})(\mathcal{H}) \simeq \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}),$$

we obtain

$$\text{Distr}^{\text{Cocom}}(B^\bullet(\mathcal{H})) \simeq \text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})).$$

Combining with (5.3), we obtain the desired functor (5.2).

5.1.6. It remains to show that the induced functor

$$|\mathrm{IndCoh}_*(B^\bullet(\mathcal{H}))| \to |\,\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X))|$$

is an equivalence.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{\mathrm{id}} & \mathrm{IndCoh}(X) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{IndCoh}_*(B^0(\mathcal{H})) & \longrightarrow & \mathrm{Bar}^0(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X)) \\ \downarrow & & \downarrow \\ |\mathrm{IndCoh}_*(B^\bullet(\mathcal{H}))| & \longrightarrow & |\,\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X))|. \end{array}$$

The functor corresponding to the composite left vertical arrow is monadic by [Chapter III.3, Proposition 3.3.3(a)].

The functor corresponding to the composite left vertical arrow is monadic by [Chapter IV.2, Proposition 7.2.2].

Hence, it remains to check that the resulting map of monads on $\mathrm{IndCoh}(X)$ induces an isomorphism at the level of the underlying endo-functors.

By [Chapter III.3, Proposition 3.3.3(a)], the former endo-functor is given by !-tensor product with $\pi_*(\omega_{\mathcal{H}})$, while the latter is given by !-tensor product with

$$\mathbf{oblv}_{\mathrm{Cocom}} \circ \mathbf{oblv}_{\mathrm{Assoc}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}) \simeq \mathbf{oblv}_{\mathrm{Cocom}} \circ \mathrm{Distr}^{\mathrm{Cocom}}(\mathcal{H}) \simeq \mathrm{Distr}(\mathcal{H}) \simeq \pi_*(\omega_{\mathcal{H}}).$$

Now, it is easy to see that the resulting map of endo-functors is the identity map on $\pi_*(\omega_{\mathcal{H}})$.

5.2. **Relation to nil-isomorphisms.** Let

$$\pi : \mathcal{Y} \leftrightarrows \mathcal{X} : s$$

be an object of $\mathrm{Ptd}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathcal{X}})$, and set $\mathcal{H} = \Omega_{\mathcal{X}}(\mathcal{Y})$.

In this subsection we will interpret various functors between the categories $\mathrm{IndCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(X)$ in terms of the equivalence of Proposition 5.1.2.

5.2.1. Set $\mathcal{H} := \Omega_{\mathcal{X}}(\mathcal{Y})$. By [Chapter III.3, Proposition 3.3.3(b)], there is a canonical equivalence

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq \mathrm{Tot}\,(\mathrm{IndCoh}(B^\bullet(\mathcal{H}))) = \mathcal{H}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})), \tag{5.4}$$

and thus

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})).$$

Under this identification, the forgetful functor $\mathbf{oblv}_{\mathfrak{h}} : \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{IndCoh}(\mathcal{X})$ corresponds to $s^!$, and the functor

$$\mathbf{triv}_{\mathfrak{h}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$$

corresponds to $\pi^!$.

5.2.2. The functor
$$\pi_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X}),$$
being the left adjoint of $\pi^!$, identifies with
$$\mathbf{coinv}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{IndCoh}(\mathcal{X}).$$

The functor π_*^{IndCoh} naturally lifts to a functor
$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Distr}^{\mathrm{Cocom}}(\mathcal{Y})\text{-comod}(\mathrm{IndCoh}(X)),$$
and the latter can be identified with
$$\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathrm{IndCoh}(\mathcal{X})),$$
see [Chapter IV.2, Sect. 7.3.4] for the notation.

5.2.3. The functor
$$s_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y}),$$
being the left adjoint of $s^!$, identifies with
$$\mathbf{free}_{\mathfrak{h}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})).$$

In particular, we obtain:

Corollary 5.2.4. *The monad $s^! \circ s_*^{\mathrm{IndCoh}}$ on $\mathrm{IndCoh}(\mathcal{X})$ is canonically isomorphic to the monad $U(\mathfrak{h}) \overset{!}{\otimes} (-)$, where $\mathfrak{h} := \mathrm{Lie}_{\mathcal{X}}(\mathcal{H})$.*

5.3. **Compatibility with colimits.** In this subsection we will prove the following technically important assertion: the assignment $\mathcal{Y} \rightsquigarrow \mathrm{IndCoh}(\mathcal{Y})$ commutes with sifted colimits in $\mathrm{FormMod}_{\mathcal{X}/}$. This is not tautological because the forgetful functor
$$\mathrm{FormMod}_{\mathcal{X}/} \to (\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/}$$
does *not* commute with sifted colimits[1].

5.3.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Let $i \mapsto \mathcal{Y}_i$ be a sifted diagram in $\mathrm{FormMod}_{\mathcal{X}/}$, and let $\mathcal{Y}$ be its colimit. Denote by f_i the canonical map $\mathcal{Y}_i \to \mathcal{Y}$.

Under the above circumstances, we have:

Proposition 5.3.2. *The functor*
$$\mathrm{IndCoh}(\mathcal{Y}) \to \lim_i \mathrm{IndCoh}(\mathcal{Y}_i),$$
given by the compatible collection of functors $(f_i)^!$, is an equivalence.

As a formal consequence of Proposition 5.3.2

Corollary 5.3.3. *Under the assumptions of Proposition 5.3.2 we have:*

(a) *The functor*
$$\underset{i}{\mathrm{colim}}\, \mathrm{IndCoh}(\mathcal{Y}_i) \to \mathrm{IndCoh}(\mathcal{Y}),$$
defined by the compatible collection of functors $(f_i)_*^{\mathrm{IndCoh}}$, is an equivalence.

(b) *The natural map*
$$\underset{i}{\mathrm{colim}}\, (f_i)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}_i}) \to \omega_{\mathcal{Y}}$$
is an isomorphism in $\mathrm{IndCoh}(\mathcal{Y})$.

The rest of this subsection is devoted to the proof of Proposition 5.3.2.

[1]Note, however, that it does commute with filtered colimits, by [Chapter III.1].

5.3.4. *Step 1.* We will first treat the case when the diagram $i \mapsto \mathcal{Y}_i$ is in $\text{Ptd}(\text{FormMod}_{/\mathcal{X}})$. In this case $\mathcal{Y}$ also naturally an object of $\text{Ptd}(\text{FormMod}_{/\mathcal{X}})$, and identifies with the colimit of $\mathcal{Y}_i$ *in* $\text{Ptd}(\text{FormMod}_{/\mathcal{X}})$.

Let

$$i \mapsto \mathfrak{h}_i$$

be the diagram in $\text{LieAlg}(\text{IndCoh}(\mathcal{X}))$ so that $\mathcal{Y}_i = B^{\text{Lie}}_{\mathcal{X}}(\mathfrak{h}_i)$. Denote

$$\mathfrak{h} := \underset{i}{\text{colim}}\, \mathfrak{h}_i \in \text{LieAlg}(\text{IndCoh}(\mathcal{X})),$$

so that $\mathcal{Y} \simeq B^{\text{Lie}}_{\mathcal{X}}(\mathfrak{h})$.

By Proposition 5.1.2, it suffices to show that the functor

$$\mathfrak{h}\text{-mod}(\text{IndCoh}(\mathcal{X})) \to \underset{i}{\lim}\, \mathfrak{h}_i\text{-mod}(\text{IndCoh}(\mathcal{X})),$$

given by restriction, is an equivalence. However, this is true for any sifted diagram of Lie algebras.

5.3.5. *Step 2.* Let us now return to the general case of a sifted diagram $i \mapsto \mathcal{Y}_i$ in $\text{FormMod}_{\mathcal{X}/}$. Consider the corresponding diagram

$$i \mapsto \mathcal{R}^\bullet_i$$

in $\text{FormGrpoid}(\mathcal{X})$. Let $\mathcal{R}^\bullet$ be the formal groupoid corresponding to $\mathcal{Y}$.

By [Chapter IV.1, Corollary 2.2.4], for every n, the map

$$\underset{i}{\text{colim}}\, \mathcal{R}^n_i \to \mathcal{R}^n$$

is an isomorphism in $\text{Ptd}(\text{FormMod}_{/\mathcal{X}})$.

Applying Step 1, we obtain that for every n, the functor

$$\text{IndCoh}(\mathcal{R}^n) \to \underset{i}{\lim}\, \text{IndCoh}(\mathcal{R}^n_i)$$

is an equivalence.

Now, the equivalence

$$\text{IndCoh}(\mathcal{Y}) \to \underset{i}{\lim}\, \text{IndCoh}(\mathcal{Y}_i)$$

follows by descent, i.e., [Chapter IV.1, Proposition 2.2.6].

6. Actions of formal groups on prestacks

The goal of this section is to make precise the following idea: an action of a Lie algebra on a prestack is equivalent to that of action of the corresponding formal group.

The first difficulty that we have to grapple with is to define what we mean by an action of a Lie algebra on a prestack. For now we will skirt this question by considering free Lie algebras; we will return to it in [Chapter IV.4, Sect. 7].

6.1. **Action of groups vs. Lie algebras.** In this subsection we will make precise the following construction:

If a formal group $\mathcal{H}$ acts on a prestack $\mathcal{Y}$, then the Lie algebra of $\mathcal{H}$ maps to global vector fields on $\mathcal{Y}$.

6.1.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$. Let $\mathcal{H} \in \mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathcal{X}})$; denote $\mathfrak{h} := \mathrm{Lie}_{\mathcal{X}}(\mathcal{H})$.

Let $\pi : \mathcal{Y} \to \mathcal{X}$ be an object of $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$, equipped with an action of $\mathcal{H}$. Let us assume that $\mathcal{Y}$ admits deformation theory relative to $\mathcal{X}$ (see [Chapter III.1, Sect. 7.1.6] for what this means).

6.1.2. We claim that the data of action gives rise to a map in $\mathrm{IndCoh}(\mathcal{Y})$;

$$\pi^!(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h})) \to T(\mathcal{Y}/\mathcal{X}). \tag{6.1}$$

Indeed, if act denotes the action map

$$\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y} \to \mathcal{Y},$$

then we have a canonically map

$$T((\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y})/\mathcal{X}) \to \mathrm{act}^!(T(\mathcal{Y}/\mathcal{X})).$$

Pulling back along the unit section of $\mathcal{H}$, and composing with the canonical map

$$\pi^!(T(\mathcal{H}/X)|_X) \to T(\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y}/\mathcal{X})|_{\mathcal{Y}},$$

and using the isomorphism $T(\mathcal{H}/X)|_X \simeq \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h})$ of Corollary 3.2.8, we obtain the desired map

$$\pi^!(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h})) \simeq \pi^!(T(\mathcal{H}/X)|_X) \to T(\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y}/\mathcal{X})|_{\mathcal{Y}} \to \mathrm{act}^!(T(\mathcal{Y}/\mathcal{X}))|_{\mathcal{Y}} \simeq T(\mathcal{Y}/\mathcal{X}).$$

6.1.3. Assume now that $\mathfrak{h}$ is of the form $\mathbf{free}_{\mathrm{Lie}}(\mathcal{F})$ for some $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$. Note that by adjunction we have a canonical map

$$\mathcal{F} \to \mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}).$$

Composing with (6.1), we obtain a map

$$\pi^!(\mathcal{F}) \to T(\mathcal{Y}/\mathcal{X}). \tag{6.2}$$

6.1.4. The above construction defines a map from the groupoid of actions of $\mathcal{H}$ on $\mathcal{Y}$ to

$$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{Y})}(\pi^!(\mathcal{F}), T(\mathcal{Y}/\mathcal{X})).$$

The goal of this section is to prove the following assertion:

Theorem 6.1.5. *For $\mathcal{Y}$ and $\mathcal{F}$ as above, the map from groupoid of data of actions of $\mathcal{H}$ on $\mathcal{Y}$ to $\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{Y})}(\pi^!(\mathcal{F}), T(\mathcal{Y}/\mathcal{X}))$ is an isomorphism.*

6.2. Proof of Theorem 6.1.5.

6.2.1. *Idea of proof.* The statement of the theorem readily reduces to the case when $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Let $(\pi^!)^R$ denote the (discontinuous) right adjoint of $\pi^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(\mathcal{Y})$, so that

$$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{Y})}(\pi^!(\mathcal{F}), T(\mathcal{Y}/\mathcal{X})) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(X)}\left(\mathcal{F}, (\pi^!)^R(T(\mathcal{Y}/\mathcal{X}))\right).$$

Starting from $\mathcal{Y}$ as above, we will construct an object

$$\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X) \in \mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathcal{X}}),$$

such that for any $\mathcal{H}' \in \mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathcal{X}})$, the data of action of $\mathcal{H}'$ on $\mathcal{Y}$ is equivalent to that of a homomorphism

$$\mathcal{H}' \to \mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X).$$

Moreover, we will show that the map

$$\mathbf{oblv}_{\mathrm{Lie}}\left(\mathrm{Lie}_X(\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X))\right) \to (\pi^!)^R(T(\mathcal{Y}/X)), \tag{6.3}$$

arising by adjunction from (6.1), is an isomorphism.

This will prove Theorem 6.1.5, since the functor Lie_X is an equivalence.

6.2.2. By [Chapter IV.1, Proposition 1.2.2], in order to construct $\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X)$ as an object of

$$\mathrm{Monoid}((\mathrm{FormMod}_{\mathrm{laft}})_{/X}),$$

it suffices to define it as a presheaf with values in Monoid(Spc) on the category

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to }X},$$

so that it satisfies the deformation theory conditions of [Chapter IV.1, Proposition 1.2.2(b)].

For $Z \in ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to }X}$, we set

$$\mathrm{Maps}_{/X}(Z, \mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X)) := \mathrm{Maps}_{/X}(Z \underset{X}{\times} \mathcal{Y}, \mathcal{Y}) \underset{\mathrm{Maps}_{/X}({}^{\mathrm{red}}Z \underset{X}{\times} \mathcal{Y}, \mathcal{Y})}{\times} * \simeq$$

$$\simeq \mathrm{Maps}_{/Z}(\mathcal{Y}_Z, \mathcal{Y}_Z) \underset{\mathrm{Maps}_{/{}^{\mathrm{red}}Z}(\mathcal{Y}_{{}^{\mathrm{red}}Z}, \mathcal{Y}_{{}^{\mathrm{red}}Z})}{\times} *,$$

(here $\mathcal{Y}_Z := Z \underset{X}{\times} \mathcal{Y}$ and $\mathcal{Y}_{{}^{\mathrm{red}}Z} = {}^{\mathrm{red}}Z \underset{X}{\times} \mathcal{Y}$).

The deformation theory conditions of [Chapter IV.1, Proposition 1.2.2(b)] follow from the fact that $\mathcal{Y}$ admits deformation theory.

Remark 6.2.3. The prestack $\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X)$ constructed above is the formal completion of the full automorphism prestack $\mathrm{Aut}(\mathcal{Y}/X)$ along the identity.

6.2.4. Thus, we have constructed $\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X)$ as an object of $\mathrm{AssocAlg}((\mathrm{FormMod}_{\mathrm{laft}})_{/X})$. It belongs to $\mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/X})$ by [Chapter IV.1, Lemma 1.6.2].

It remains to show that the map (6.3) is an isomorphism. By construction, for $\mathcal{F} \in \mathrm{Coh}(X)$ such that $\mathbb{D}^{\mathrm{Serre}}_X(\mathcal{F}) \in \mathrm{Coh}(X)^{\leq 0}$, we have

$$\mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathcal{F}, T(\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X))|_X) \simeq \mathrm{Maps}_{/X}(\mathrm{RealSplitSqZ}(\mathbb{D}^{\mathrm{Serre}}_X(\mathcal{F})) \underset{X}{\times} \mathcal{Y}, \mathcal{Y}).$$

By the deformation theory of $\mathcal{Y}$, the latter maps isomorphically to

$$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{Y})}(\pi^!(\mathcal{F}), T(\mathcal{Y}/X))),$$

and by adjunction, further (still isomorphically) to

$$\mathrm{Maps}_{\mathrm{IndCoh}(X)}\left(\mathcal{F}, (\pi^!)^R(T(\mathcal{Y}/X))\right).$$

Furthermore, it follows from the construction that the resulting map

$$\mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathcal{F}, T(\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X))|_X) \to \mathrm{Maps}_{\mathrm{IndCoh}(X)}\left(\mathcal{F}, (\pi^!)^R(T(\mathcal{Y}/X))\right)$$

is the one induced by (6.2).

This implies the required assertion, as $\mathrm{IndCoh}(X)$ is generated by the above objects of $\mathrm{Coh}(X)$ under colimits.

□

6.3. **Localization of Lie algebra modules.** In this subsection we show how to construct crystals on a given prestack starting from modules over a Lie algebra that acts on this prestack.

6.3.1. Let $f : \mathcal{Y} \to \mathcal{X}$ and $\mathcal{H}$ be as in Sect. 6.1.1. Consider the prestack

$$\mathcal{Y}_{/\mathcal{X}_{\mathrm{dR}}} := \mathcal{Y}_{\mathrm{dR}} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X},$$

see [Chapter III.4, Sect. 3.3.2] for the notation.

Recall also the notation

$$^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}) := \mathrm{IndCoh}(\mathcal{Y}_{\mathrm{dR}} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X}).$$

In this subsection we will construct the localization functor

$$\mathrm{Loc}_{\mathfrak{h},\mathcal{Y}/\mathcal{X}} : \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to {}^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}).$$

6.3.2. The action of $\mathcal{H}$ on $\mathcal{Y}$ defines an object

$$\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y} \in \mathrm{FormGrpoid}(\mathcal{Y}),$$

see [Chapter IV.1, Sect. 2.2.1] for the notation.

By (the relative over $\mathcal{X}$ version of) [Chapter IV.1, Theorem 2.3.2], the corresponding quotient

$$\mathcal{Y}/\mathcal{H} \in \mathrm{FormMod}_{\mathcal{Y}/}$$

is well-defined.

We have canonically defined maps of prestacks

$$\begin{array}{ccc} \mathcal{Y}/\mathcal{H} & \xrightarrow{f/\mathcal{H}} & B_X(\mathcal{H}) \\ {\scriptstyle g}\downarrow & & \\ \mathcal{Y}_{/\mathcal{X}_{\mathrm{dR}}}. & & \end{array}$$

6.3.3. We define the sought-for functor $\mathrm{Loc}_{\mathfrak{h},\mathcal{Y}/\mathcal{X}}$ as

$$g_*^{\mathrm{IndCoh}} \circ (f/\mathcal{H})^!,$$

where we identify

$$\mathrm{IndCoh}(B_{\mathcal{X}}(\mathcal{H})) \simeq \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$$

by means of (5.4).

6.3.4. Note that the functor $\mathrm{Loc}_{\mathfrak{h},\mathcal{Y}/\mathcal{X}}$ is by construction the left adjoint of the (in general, discontinuous) functor

$$((f/\mathcal{H})_*^{\mathrm{IndCoh}})^R \circ g^! : {}^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}) \to \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})).$$

We claim that the functor $((f/\mathcal{H})_*^{\mathrm{IndCoh}})^R \circ g^!$ makes the following diagram commutative:

$$\begin{array}{ccc} {}^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}) & \xrightarrow{\mathbf{oblv}_{/\mathcal{X}_{\mathrm{dR}},\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}) \\ {\scriptstyle ((f/\mathcal{H})_*^{\mathrm{IndCoh}})^R \circ g^!}\downarrow & & \downarrow{\scriptstyle (f^!)^R} \\ \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) & \xrightarrow{\mathbf{oblv}_{\mathfrak{h}}} & \mathrm{IndCoh}(\mathcal{X}), \end{array}$$

where $\mathbf{oblv}_{/\mathcal{X}_{\mathrm{dR}},\mathcal{Y}}$ is by definition the !-pullback functor along

$$p_{/\mathcal{X}_{\mathrm{dR}},\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}_{/\mathcal{X}_{\mathrm{dR}}}.$$

6.3.5. Indeed, we need to establish the commutativity of the diagram

$$\begin{array}{ccc} \operatorname{IndCoh}(\mathcal{Y}) & \xrightarrow{(f_*^{\operatorname{IndCoh}})^R} & \operatorname{IndCoh}(\mathcal{X}) \\ \uparrow & & \uparrow \\ \operatorname{IndCoh}(\mathcal{Y}/\mathcal{H}) & \xrightarrow{((f/\mathcal{H})_*^{\operatorname{IndCoh}})^R} & \operatorname{IndCoh}(B_{\mathcal{X}}(\mathcal{H})), \end{array}$$

where the vertical arrows are given by !-pullback.

However, this follows by passing to right adjoints in the commutative diagram,

$$\begin{array}{ccc} \operatorname{IndCoh}(\mathcal{Y}) & \xleftarrow{f_*^{\operatorname{IndCoh}}} & \operatorname{IndCoh}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \operatorname{IndCoh}(\mathcal{Y}/\mathcal{H}) & \xleftarrow{(f/\mathcal{H})_*^{\operatorname{IndCoh}}} & \operatorname{IndCoh}(B_{\mathcal{X}}(\mathcal{H})), \end{array}$$

given by base-change.