CHAPTER IV.4. LIE ALGEBROIDS

Contents

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Introduction

0.1. **Who are these Lie algebroids?** In this chapter we initiate the study of Lie algebroids over prestacks (technically, over prestacks locally almost of finite type that admit deformation theory). The reason we decided to devote a chapter to this notion is that Lie algebroids provide a convenient language to discuss differential-geometric properties of prestacks, which will be studied in [Chapter IV.5].

0.1.1. In classical algebraic geometry, a Lie algebroid (over a classical scheme) X is a quasi-coherent sheaf $\mathfrak{L}$, equipped with an $\mathcal{O}_X$-linear map to the tangent sheaf and an operation of Lie bracket that satisfy some natural axioms (see Sect. 9.1).

In the setting of derived we *define* the category of Lie algebroids on $\mathcal{X}$ to be that of formal groupoids on $\mathcal{X}$. This is sensible because the category of Lie algebras in $\mathrm{IndCoh}(\mathcal{X})$ *is* equivalent to the category of formal groups over $\mathcal{X}$, due to [Chapter IV.3, Theorem 3.6.2].

The reason we call these objects 'Lie algebroids' is that we construct various forgetful functors to more linear categories and show that Lie algebroids can be described as ind-coherent sheaves with an additional structure. However, a distinctive feature of the derived story we are going to present is that the *only* description of this extra structure that we describe is in terms of geometry. I.e., we could not come up with a more 'algebraic' definition.

0.1.2. We show that with the definition of Lie algebroids as formal groupids, one can perform with them all the expected operations:

A Lie algebroid $\mathfrak{L}$ will have an associated object

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \in \mathrm{IndCoh}(\mathcal{X}),$$

equipped with a morphism $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \to T(\mathcal{X})$, called the anchor map. The kernel of the anchor map has a structure of Lie algebra in $\mathrm{IndCoh}(\mathcal{X})$, while the space of global sections of $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})$ has also a structure of Lie algebra (in Vect).

Thus, the category $\mathrm{LieAlgbroid}(\mathcal{X})$ is related to the category $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ by a pair of adjoint functors

$$\mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \rightleftarrows \mathrm{LieAlgbroid}(\mathcal{X}) : \mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T},$$

and we will show that the resulting monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$$

acting on $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ has 'the right size', see Proposition 5.3.2.

Furthermore, $\mathrm{LieAlgbroid}(\mathcal{X})$ is related to the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ by a pair of adjoint functors

$$\mathrm{diag} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{LieAlgbroid}(\mathcal{X}) : \text{ker-anch},$$

(where the meaning of diag is that an $\mathcal{O}_{\mathcal{X}}$-linear Lie algebra can be considered into a Lie algebroid with the zero anchor map, and ker-anch sends a Lie algebroid to the kernel of its anchor map[1]). The monad

$$\text{ker-anch} \circ \mathrm{diag}_{\mathcal{X}}$$

is given by the operation of *semi-direct product* with the *inertia* Lie algebra $\mathrm{inert}_{\mathcal{X}}$, which is again what one expects from a sensible definition of Lie algebroids.

0.1.3. Finally, let us comment on our inability to define Lie algebroids without resorting to geometry. In fact, this is not surprising: throughout the book the only way we access Lie algebras is via the definition of the Lie operad as the Koszul dual of the commutative operad. So, it is natural that in order to define objects that generalize Lie algebras we resort to commutative objects (in our case, prestacks).

In Sect. 5.6 we present a very general categorical framework, in which one can define 'broids' as modules over a certain monad.

0.2. **What is done in this chapter?** We should say right away that this chapter does not contain any big theorems. Mostly, it uses the material of the previous chapters to set up the theory of Lie algebroids and also sets ground for applications in [Chapter IV.5].

0.2.1. In Sect. 1 we return to the study of groupoids (in spaces and then in the framework of algebraic geometry).

Given a space (resp., prestack) X, we define two functors from the category $\mathrm{Groupoid}(X)$ to the category of groups over X.

The first of these functors, denoted Inert, sends a groupoid to its inertia group. Applying this functor to the unit groupoid (i.e., the initial object of $\mathrm{Groupoid}(X)$), we obtain the inertia group of X, denoted Inert_X.

The second functor, denoted Ω^{fake}, sends a groupoid R to $\Omega_X(R)$, where we view R as a pointed object over X via

$$\mathrm{unit} : X \rightleftarrows R : p_s.$$

The above two functors are related by a fiber sequence

$$\Omega^{\mathrm{fake}}(R) \to \mathrm{Inert}_X \to \mathrm{Inert}(R).$$

[1]Another way to look at the above adjoint pair is that the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ identifies with the over-category

$$(\mathrm{LieAlgbroid}(\mathcal{X}))_{/0},$$

where 0 is the zero Lie algebroid.

0.2.2. In Sect. 2 we introduce the notion of Lie algebroid over an object $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$, along with two pairs of adjoint functors

$$\mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \rightleftarrows \mathrm{LieAlgbroid}(\mathcal{X}) : \mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T},$$

and

$$\mathrm{diag}_{\mathcal{X}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{LieAlgbroid}(\mathcal{X}) : \text{ker-anch}\,.$$

We introduce also another functor

$$\Omega^{\mathrm{fake}} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})),$$

so that for $\mathfrak{L} \in \mathrm{LieAlgbroid}(\mathcal{X})$ we have the fiber sequence

$$\Omega^{\mathrm{fake}}(\mathfrak{L}) \to \mathrm{inert}_{\mathcal{X}} \to \text{ker-anch}(\mathfrak{L}),$$

where $\mathrm{inert}_{\mathcal{X}}$ is the Lie algebra of the inertia group of $\mathcal{X}$.

We note that

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathcal{X}}) = T(X)[-1]$$

and when we apply $\mathbf{oblv}_{\mathrm{Lie}}$ to the map $\Omega^{\mathrm{fake}}(\mathfrak{L}) \to \mathrm{inert}_{\mathcal{X}}$, we recover the shift by $[-1]$ of the anchor map, i.e., of the object

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}.$$

0.2.3. In Sect. 3 we consider the basic examples of Lie algebroids: the tangent algebroid, the zero algebroid, the Lie algebroid attached to a map of prestacks, and the Atiyah algebroid attached to an object of $\mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$.

0.2.4. In Sect. 4 we introduce the notion of module over a Lie algebroid, and define the universal enveloping algebra of a Lie algebroid.

0.2.5. In Sect. 5 we study the relationship between square-zero extensions and Lie algebroids. Recall that according to [Chapter IV.1, Theorem 2.3.2], for a given $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$, the category of formal moduli problems *under* $\mathcal{X}$ is equivalent to that of formal groupoids over $\mathcal{X}$, and thus to the category of Lie algebroids.

Using this equivalence, we construct functor

$$\mathrm{RealSqZExt} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{FormMod}_{\mathcal{X}/}$$

to correspond to the functor

$$\mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{LieAlgbroid}(\mathcal{X}).$$

We show that the functor is the left adjoint to the functor that sends $\mathcal{X} \to \mathcal{Y}$ to $T(\mathcal{X}/\mathcal{Y}) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$, i.e., it really behaves like a square-zero extension.

We also show that the notion of square-zero extension developed in the present section using Lie algebroids is equivalent to one developed in [Chapter III.1, Sect. 10], which was bootstrapped from the case of schemes.

0.2.6. In Sect. 6 we introduce the Atiyah class, which is a functorial assignment for any $\mathcal{F} \in \operatorname{IndCoh}(\mathcal{X})$ of a map

$$T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F} \overset{\alpha_{\mathcal{F}}}{\to} \mathcal{F}.$$

We show that if $i : \mathcal{X} \to \mathcal{X}'$ is a square-zero extension of $\mathcal{X}$, given by

$$\mathcal{F}' \overset{\gamma}{\to} T(\mathcal{X}),$$

then the category $\operatorname{IndCoh}(\mathcal{X}')$ can be described as the category consisting of $\mathcal{F} \in \operatorname{IndCoh}(\mathcal{X})$, equipped with a null-homotopy of the composite map

$$\mathcal{F}' \overset{!}{\otimes} \mathcal{F} \xrightarrow{\gamma \overset{!}{\otimes} \mathrm{id}} T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F} \overset{\alpha_{\mathcal{F}}}{\to} \mathcal{F}.$$

We deduce that the dualizing object $\omega_{\mathcal{X}'} \in \operatorname{IndCoh}(\mathcal{X}')$ fits into the exact triangle

$$i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}'} \to i_*^{\mathrm{IndCoh}}(\mathcal{F}'),$$

further justifying the terminology 'square-zero extension'.

0.2.7. In Sect. 7 we show that the space of global sections of a Lie algebroid carries a canonical structure of Lie algebra. (In particular, global vector fields carry a structure of Lie algebra.)

We also show that $\mathfrak{h}$ is a Lie algebra object in $\operatorname{IndCoh}(\mathcal{X})$ obtained as $\Omega^{\mathrm{fake}}(\mathfrak{L})$ for a Lie algebroid $\mathfrak{L}$, then the Lie algebra structure on the space of global sections of $\mathfrak{h}$ is the trivial one.

0.2.8. In Sect. 8 we present another point of view on the category $\operatorname{LieAlgbroid}(\mathcal{X})$. Namely, we show that the functor

$$\text{ker-anch} : \operatorname{LieAlgbroid}(\mathcal{X}) \to \operatorname{LieAlg}(\operatorname{IndCoh}(\mathcal{X}))$$

is monadic.

I.e., the category $\operatorname{LieAlgbroid}(\mathcal{X})$ can be realized as the category of modules for the monad

$$\mathsf{M}_{\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}}} := \text{ker-anch} \circ \mathrm{diag}$$

acting on the category $\operatorname{LieAlg}(\operatorname{IndCoh}(\mathcal{X}))$.

The monad $\mathsf{M}_{\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}}}$ is given by the operation of 'semi-direct product' with the inertia Lie algebra $\mathrm{inert}_{\mathcal{X}}$. So in a sense, this gives a very manageable presentation of the category $\operatorname{LieAlgbroid}(\mathcal{X})$. We learned this idea from J. Francis.

Thus, there are (at least) two ways to exhibit $\operatorname{LieAlgbroid}(\mathcal{X})$ as modules over a monad acting on some category: one is what we just said above, and another via the adjunction

$$\mathbf{free}_{\mathrm{LieAlgbroid}} : \operatorname{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \rightleftarrows \operatorname{LieAlgbroid}(\mathcal{X}) : \mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}.$$

0.2.9. Finally, in Sect. 9 we compare our definition of Lie algebroids with the usual (i.e., classical) one, when our prestack $\mathcal{X}$ is a classical scheme X.

We show (see Theorem 9.1.5) that the subcategory consisting of Lie algebroids $\mathfrak{L}$, for which the object $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \in \operatorname{IndCoh}(X)$ lies in the essential image of the functor

$$\operatorname{QCoh}(X)^{\heartsuit} \hookrightarrow \operatorname{QCoh}(X) \overset{\Upsilon_X}{\hookrightarrow} \operatorname{IndCoh}(X)$$

is canonically equivalent to that of classical Lie algebroids.

Further, we show that if $\mathfrak{L} \in \operatorname{LieAlgbroid}(\mathcal{X})$ is such that $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \in \operatorname{IndCoh}(X)$ lies in the essential image of the functor

$$\operatorname{QCoh}(X)^{\heartsuit,\mathrm{flat}} \hookrightarrow \operatorname{QCoh}(X)^{\heartsuit} \hookrightarrow \operatorname{QCoh}(X) \overset{\Upsilon_X}{\hookrightarrow} \operatorname{IndCoh}(X),$$

then the category $\mathfrak{L}$-mod(IndCoh(X)) agrees with the classical definition of the category of modules over a Lie algebroid.

1. The inertia group

In this section we return to the discussion of groupoids, first in the category Spc and then in formal geometry.

We show that there are two forgetful functors from the category of groupoids (on a given space or prestack) $\mathcal{X}$ to that of groups over $\mathcal{X}$. The first functor is given by the inertia group, i.e. the morphisms with the same source and target. The second is given by taking the relative loop space of the groupoid. We also establish a relationship between these two functors: namely, they fit into a fiber sequence with the inertia group of the identity groupoid in the middle.

1.1. **Inertia group of a groupoid.** In this subsection we work in the category of spaces. We introduce the notion of inertia group of a groupoid.

1.1.1. Recall the setting of [Chapter IV.1, Sect. 2.1]. For $X \in \mathrm{Spc}$, note that we have a tautological forgetful functor

$$\mathrm{diag} : \mathrm{Grp}(\mathrm{Spc}_{/X}) \to \mathrm{Grpoid}(X).$$

In fact,

$$\mathrm{Grp}(\mathrm{Spc}_{/X}) \simeq \mathrm{Grpoid}(X)_{/\,\mathrm{diag}_X}.$$

Hence, the functor diag admits a right adjoint, denoted Inert, given by Cartesian product (inside Grpoid(X)) with diag_X.

Concretely, as a space

$$\mathrm{Inert}(R) := X \underset{X\times X}{\times} R,$$

(we recall that $X \times X$ is the final object in Grpoid(X)).

1.1.2. For $R = \mathrm{diag}_X$ being the identity groupoid, we thus obtain an object of $\mathrm{Grp}(\mathrm{Spc}_{/X})$, denoted Inert_X.

I.e., as an object of $\mathrm{Grp}(\mathrm{Spc}_{/X})$, we have:

$$\mathrm{Inert}_X = X \underset{X\times X}{\times} X = \Omega_X(X \times X),$$

where $X \times X$ is regarded as an object of $\mathrm{Ptd}(\mathrm{Spc}_{/X})$ via the maps

$$\Delta_X : X \rightleftarrows X \times X : p_s.$$

The object $\mathrm{Inert}_X \in \mathrm{Grp}(\mathrm{Spc}_{/X})$ is called the *inertia group*[2] of X.

1.1.3. For $R = X \underset{Y}{\times} X$, we have:

$$\mathrm{Inert}(R) = X \underset{Y}{\times} \mathrm{Inert}_Y\,.$$

[2]Note that we can also think of Inert_X as X^{S^1}, i.e., the free loop space of X.

1.1.4. There is another functor

$$\Omega^{\rm fake} : \text{Grpoid}(X) \to \text{Grp}(\text{Spc}_{/X}).$$

Namely,

$$\Omega^{\rm fake}(R) := \Omega_X(R),$$

where in the left-hand side Ω_X is the loop functor $\text{Ptd}(\text{Spc}_{/X}) \to \text{Grp}(\text{Spc}_{/X})$, where we view R as as an object of $\text{Ptd}(\text{Spc}_{/X})$ via

$$\text{unit} : X \rightleftarrows R : p_s.$$

For example,

$$\text{Inert}_X = \Omega^{\rm fake}(X \times X).$$

1.1.5. Since $X \times X$ is the final object in $\text{Grpoid}(X)$, for any groupoid R we have a tautological map $R \to X \times X$, which gives rise to a map

$$\Omega^{\rm fake}(R) \to \text{Inert}_X .$$

In addition, the unit map $\text{diag}_X \to R$ gives rise to a map in $\text{Grp}(\text{Spc}_{/X})$

$$\text{Inert}_X \to \text{Inert}(R).$$

It is easy to see that

$$\Omega^{\rm fake}(R) \to \text{Inert}_X \to \text{Inert}(R)$$

is a fiber sequence in $\text{Grp}(\text{Spc}_{/X})$.

1.1.6. Note also that the composed endo-functor of $\text{Groupoid}(X)$

$$\text{diag} \circ \Omega^{\rm fake}$$

identifies with

$$R \mapsto \text{diag}_X \underset{R}{\times} \text{diag}_X,$$

where the fiber product is taken in $\text{Groupoid}(X)$.

1.2. **Infinitesimal inertia.** In this subsection we translate the material from Sect. 1.1 to the context of infinitesimal algebraic geometry. I.e., instead of Spc, we will work with the category $\text{PreStk}_{\text{laft-def}}$, and instead of groupoids we will consider objects of $\text{FormGrpoid}(\mathcal{X})$ over a given prestack $\mathcal{X}$.

1.2.1. Let $\mathcal{X}$ be an object of $\text{PreStk}_{\text{laft-def}}$. Consider the category $\text{FormGrpoid}(\mathcal{X})$.

Note that $\text{FormGrpoid}(\mathcal{X})$ admits a final object equal to $(\mathcal{X} \times \mathcal{X})^\wedge$, the formal completion of the diagonal in $\mathcal{X} \times \mathcal{X}$.

The initial object in $\text{FormGrpoid}(\mathcal{X})$ is $\text{diag}_{\mathcal{X}}$, and we have a canonical identification

$$\text{Grp}(\text{FormMod}_{/\mathcal{X}}) \simeq \text{FormGrpoid}(\mathcal{X})_{/\,\text{diag}_{\mathcal{X}}}.$$

1.2.2. Consider the forgetful functor

$$\text{diag} : \text{Grp}(\text{FormMod}_{/\mathcal{X}}) \to \text{FormGrpoid}(\mathcal{X}).$$

It admits a right adjoint, denoted $\text{Inert}^{\rm inf}$, and given by Cartesian product (inside the category $\text{FormGrpoid}(\mathcal{X})$) with the unit groupoid $\text{diag}_{\mathcal{X}}$. Explicitly,

$$\text{Inert}^{\rm inf}(\mathcal{R}) = \mathcal{X} \underset{(\mathcal{X}\times\mathcal{X})^\wedge}{\times} \mathcal{R}.$$

1.2.3. For $\mathcal{R} = \text{diag}_{\mathcal{X}}$ being the identity groupoid, we thus obtain an object of $\text{Grp}(\text{Spc}_{/X})$, denoted $\text{Inert}^{\text{inf}}_{\mathcal{X}}$. We call it the *infinitesimal inertial group* of $\mathcal{X}$.

I.e., as an object of PreStk, we have:

$$\text{Inert}^{\text{inf}}_{\mathcal{X}} = \mathcal{X} \underset{(\mathcal{X}\times\mathcal{X})^\wedge}{\times} \mathcal{X}.$$

1.2.4. We reserve the notation $\text{Inert}_{\mathcal{X}}$ for the object

$$\mathcal{X} \underset{\mathcal{X}\times\mathcal{X}}{\times} \mathcal{X} \in \text{Grp}(\text{PreStk}_{/\mathcal{X}}),$$

i.e., the *usual* (=non-infinitesimal) inertia group of $\mathcal{X}$.

It is easy to see that $\text{Inert}^{\text{inf}}_{\mathcal{X}}$ is obtained from $\text{Inert}_{\mathcal{X}}$ by completion along the unit section.

1.2.5. There is another functor

$$\Omega^{\text{fake}} : \text{FormGrpoid}(\mathcal{X}) \to \text{Grp}(\text{FormMod}_{/\mathcal{X}}).$$

Namely,

$$\Omega^{\text{fake}}(\mathcal{R}) := \Omega_{\mathcal{X}}(\mathcal{R}),$$

where in the left-hand side $\Omega_{\mathcal{X}}$ is the loop functor $\text{Ptd}(\text{FormMod}_{/\mathcal{X}}) \to \text{Grp}(\text{FormMod}_{/\mathcal{X}})$, where we view $\mathcal{R}$ as as an object of $\text{Ptd}(\text{FormMod}_{/\mathcal{X}})$ via

$$\text{unit} : \mathcal{X} \rightleftarrows \mathcal{R} : p_s.$$

For example,

$$\text{Inert}^{\text{inf}}_{\mathcal{X}} = \Omega^{\text{fake}}((\mathcal{X}\times\mathcal{X})^\wedge).$$

1.2.6. Since $(\mathcal{X}\times\mathcal{X})^\wedge$ is the final object in $\text{FormGrpoid}(\mathcal{X})$, for any groupoid $\mathcal{R}$ we have a tautological map $\mathcal{R} \to (\mathcal{X}\times\mathcal{X})^\wedge$, which gives rise to a map

$$\Omega^{\text{fake}}(\mathcal{R}) \to \text{Inert}^{\text{inf}}_{\mathcal{X}}.$$

In addition, the unit map $\mathcal{X} \to \mathcal{R}$ gives rise to a map in $\text{Grp}(\text{FormMod}_{/\mathcal{X}})$

$$\text{Inert}^{\text{inf}}_{\mathcal{X}} \to \text{Inert}^{\text{inf}}(\mathcal{R}).$$

It is easy to see that

$$\Omega^{\text{fake}}(\mathcal{R}) \to \text{Inert}^{\text{inf}}_{\mathcal{X}} \to \text{Inert}^{\text{inf}}(\mathcal{R}) \tag{1.1}$$

is a fiber sequence.

1.3. **Inertia Lie algebras.** In this subsection we will introduce Lie algebra counterparts of the constructions in Sect. 1.2.

1.3.1. In what follows we denote

$$\text{inert}_{\mathcal{X}} := \text{Lie}_{\mathcal{X}}(\text{Inert}^{\text{inf}}_{\mathcal{X}}) \in \text{LieAlg}(\text{IndCoh}(\mathcal{X}).$$

Note that

$$\mathbf{oblv}_{\text{Lie}}(\text{inert}_{\mathcal{X}}) \simeq T(\mathcal{X})[-1].$$

1.3.2. For $\mathcal{R} \in \text{FormGrpoid}(\mathcal{X})$, denote

$$\text{inert}(\mathcal{R}) := \text{Lie}(\text{Inert}^{\text{inf}}(\mathcal{R})).$$

From the fiber sequence (1.1) we obtain a fiber sequence in $\text{LieAlg}(\text{IndCoh}(\mathcal{X}))$:

$$\text{Lie}(\Omega^{\text{fake}}(\mathcal{R})) \to \text{inert}_{\mathcal{X}} \to \text{inert}(\mathcal{R}). \tag{1.2}$$

1.3.3. If $\mathcal{R}$ is the groupoid corresponding to a formal moduli problem $\mathcal{X} \to \mathcal{Y}$ (i.e., $\mathcal{R} = \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}$), then

$$\text{inert}(\mathcal{R}) \simeq \text{inert}_{\mathcal{Y}} |_{\mathcal{X}}.$$

In particular,

$$\mathbf{oblv}_{\text{Lie}}(\text{inert}(\mathcal{R})) \simeq T(\mathcal{Y})|_{\mathcal{X}}[-1].$$

The canonical map

$$\mathbf{oblv}_{\text{Lie}}(\text{inert}_{\mathcal{X}}) \to \mathbf{oblv}_{\text{Lie}}(\text{inert}(\mathcal{R})),$$

induced by $\text{Inert}^{\text{inf}}_{\mathcal{X}} \to \text{Inert}^{\text{inf}}(\mathcal{R})$, is the shft by $[-1]$ of the differential $T(\mathcal{X}) \to T(\mathcal{Y})|_{\mathcal{X}}$.

Note also that

$$\mathbf{oblv}_{\text{Lie}} \circ \text{Lie}(\Omega^{\text{fake}}(\mathcal{R})) \simeq T(\mathcal{X}/\mathcal{Y})[-1].$$

Applying $\mathbf{oblv}_{\text{Lie}}$ to (1.2), we obtain the shift by $[-1]$ of the tautological exact triangle

$$T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X}) \to T(\mathcal{Y})|_{\mathcal{X}}.$$

2. Lie algebroids: definition and basic pieces of structure

In this section we introduce the category $\text{LieAlgbroid}(\mathcal{X})$ of Lie algebroids on $\mathcal{X}$ as the category of formal groupoids on $\mathcal{X}$ and study several forgetful functors to the categories $\text{IndCoh}(\mathcal{X})$ and $\text{LieAlg}(\text{IndCoh}(\mathcal{X}))$, including those induced by the functors from Sect. 1.

2.1. **Lie algebroids and the main forgetful functor.**

We *define* the category $\text{LieAlgbroid}(\mathcal{X})$ to be the same as $\text{FormGrpoid}(\mathcal{X})$. The difference will only express itself in our point of view: we will (try to) view Lie algebroids as objects of a linear category (namely, $\text{IndCoh}(X)$), equipped with an extra structure.

According to [Chapter IV.1, Theorem 2.3.2], we can also identify

$$\text{LieAlgbroid}(\mathcal{X}) \simeq \text{FormMod}_{\mathcal{X}/}.$$

2.1.1. Our 'main' forgetful functor is denoted

$$\mathbf{oblv}_{\text{LieAlgbroid}\,/T} : \text{LieAlgbroid}(\mathcal{X}) \to \text{IndCoh}(\mathcal{X}))_{/T(\mathcal{X})},$$

and is constructed as follows:

It associates to a formal moduli problem $\mathcal{X} \to \mathcal{Y}$ the object of $\text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ equal to

$$T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X}).$$

The functor $\mathbf{oblv}_{\text{LieAlgbroid}\,/T}$ is conservative by [Chapter III.1, Proposition 8.3.2].

2.1.2. We will think of a Lie algebroid $\mathfrak{L}$ as the corresponding object $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}(\mathfrak{L})$ of $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$, abusively denoted by the same character $\mathfrak{L}$, equipped with an extra structure.

We shall denote by $\mathbf{oblv}_{\mathrm{LieAlgbroid}}$ the composition of $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}$ and the forgetful functor

$$\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{IndCoh}(\mathcal{X}).$$

The corresponding map

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \overset{\mathrm{anch}}{\to} T(\mathcal{X}) \tag{2.1}$$

is usually referred to as the *anchor map*.

Proposition 2.1.3.

(a) *The category* $\mathrm{LieAlgbroid}(\mathcal{X})$ *admits sifted colimits, and the functor* $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}$ *commutes with sifted colimits.*

(b) *The functor* $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}$ *admits a left adjoint.*

Proof. Point (a) of the proposition follows from [Chapter IV.1, Corollary 2.2.4]. To prove point (b), by the Adjoint Functor Theorem, it is enough to show that the functor $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}$ commutes with limits, while the latter is obvious from the definitions.

□

We will denote the functor

$$\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{LieAlgbroid}(\mathcal{X}),$$

left adjoint to $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}$, by $\mathbf{free}_{\mathrm{LieAlgbroid}}$. In Sect. 5 we will clarify the geometric meaning of this functor.

2.1.4. Note that Corollary 2.1.3 implies:

Corollary 2.1.5. *The functor*

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$$

is monadic.

2.1.6. The above discussion can be rendered into the relative setting, where instead of the category $\mathrm{PreStk}_{\text{laft-def}}$, we consider the category $(\mathrm{PreStk}_{\text{laft-def}})_{/\mathcal{Z}}$ over a fixed $\mathcal{Z} \in \mathrm{PreStk}_{\text{laft-def}}$.

For $\mathcal{X} \in (\mathrm{PreStk}_{\text{laft-def}})_{/\mathcal{Z}}$, we denote the resulting category of relative Lie algebroids by

$$\mathrm{LieAlgbroid}(\mathcal{X}/\mathcal{Z}).$$

Its natural forgetful functor, denoted by the same symbol $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}$ takes values in the category $\mathrm{IndCoh}(\mathcal{X})_{T(/\mathcal{X}/\mathcal{Z})}$. I.e., we now take tangent spaces *relative* to $\mathcal{Z}$.

2.2. **From Lie algebroids to Lie algebras.** It turns out that there are two forgetful functors from $\mathrm{LieAlgbroid}(\mathcal{X})$ to $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$, induced by the two functors from groupoids to groups in Sect. 1. We will explore these two functors in the present subsection.

2.2.1. We define the functor

$$\text{ker-anch} : \text{LieAlgbroid}(\mathcal{X}) \to \text{LieAlg}(\text{IndCoh}(\mathcal{X}))$$

so that the diagram

$$\begin{array}{ccc} \text{FormGrpoid}(\mathcal{X}) & \xrightarrow{\sim} & \text{LieAlgbroid}(\mathcal{X}) \\ {\scriptstyle \text{Inert}^{\text{inf}}}\downarrow & & \downarrow{\scriptstyle \text{ker-anch}} \\ \text{Grp}(\text{FormMod}_{/\mathcal{X}}) & \underset{\sim}{\xrightarrow{\text{Lie}}} & \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \end{array}$$

is commutative.

I.e., if $\mathfrak{L}$ is the algebroid corresponding to the groupoid $\mathcal{R}$, we have

$$\text{ker-anch}(\mathfrak{L}) := \text{inert}(\mathcal{R}),$$

in the notation of Sect. 1.3.2.

Note that by construction, for $\mathfrak{L} \in \text{LieAlgbroid}(\mathcal{X})$, we have:

$$\mathbf{oblv}_{\text{Lie}} \circ \text{ker-anch}(\mathfrak{L}) \simeq \text{Fib}\left(\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L}) \overset{\text{anch}}{\to} T(\mathcal{X})\right),$$

functorially in $\mathfrak{L}$.

In particular, the functor ker-anch is conservative.

2.2.2. Another forgetful functor, denoted $\Omega^{\text{fake}} : \text{LieAlgbroid}(\mathcal{X}) \to \text{LieAlg}(\text{IndCoh}(\mathcal{X}))$, is defined so that the diagram

$$\begin{array}{ccc} \text{FormGrpoid}(\mathcal{X}) & \xrightarrow{\sim} & \text{LieAlgbroid}(\mathcal{X}) \\ {\scriptstyle \Omega^{\text{fake}}}\downarrow & & \downarrow{\scriptstyle \Omega^{\text{fake}}} \\ \text{Grp}(\text{FormMod}_{/\mathcal{X}}) & \underset{\sim}{\xrightarrow{\text{Lie}}} & \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \end{array}$$

commutes.

In particular, the fiber sequence (1.2) translates as

$$\Omega^{\text{fake}}(\mathfrak{L}) \to \text{inert}_{\mathcal{X}} \to \text{ker-anch}(\mathfrak{L}). \tag{2.2}$$

Note that by construction

$$\mathbf{oblv}_{\text{Lie}} \circ \Omega^{\text{fake}}(\mathfrak{L}) \simeq \mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L})[-1].$$

I.e., the object of $\text{IndCoh}(\mathcal{X})$, underlying the shift by $[-1]$ of a Lie algebroid, carries a natural structure of Lie algebra in $\text{IndCoh}(\mathcal{X})$.

The functor Ω^{fake} is also conservative.

Remark 2.2.3. In Sect. 7, we shall see that the object of Vect equal to global sections of $\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L})$ for a Lie algebroid $\mathfrak{L}$ itself carries a structure of Lie algebra.

2.2.4. We will refer to the canonical map

$$\Omega^{\text{fake}}(\mathfrak{L}) \to \text{inert}_{\mathcal{X}} \tag{2.3}$$

as the *shifted anchor map*. After applying $\mathbf{oblv}_{\text{Lie}}$, the map (2.3) becomes the shit by $[-1]$ of the map (2.1).

Applying $\mathbf{oblv}_{\text{Lie}}$ to (2.2), we obtain a fiber sequence in $\text{IndCoh}(\mathcal{X})$ that is equal to the shift by $[-1]$ of the tautological sequence

$$\mathbf{oblv}_{\text{Lie}}(\text{ker-anch}(\mathfrak{L})) \to \mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L}) \to T(\mathcal{X}).$$

2.2.5. The functor ker-anch admits a left adjoint, denoted

$$\text{diag} : \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \to \text{LieAlgbroid}(\mathcal{X}).$$

Tautologically, it makes the following diagram commute

$$\begin{array}{ccc} \text{FormGrpoid}(\mathcal{X}) & \xrightarrow{\sim} & \text{LieAlgbroid}(\mathcal{X}) \\ {\scriptstyle\text{diag}}\uparrow & & \uparrow{\scriptstyle\text{diag}} \\ \text{Grp}(\text{FormMod}_{/\mathcal{X}}) & \underset{\sim}{\xrightarrow{\text{Lie}}} & \text{LieAlg}(\text{IndCoh}(\mathcal{X})). \end{array}$$

We note:

Lemma 2.2.6. *The following diagram of functors commutes:*

$$\begin{array}{ccc} \text{IndCoh}(\mathcal{X}) & \xrightarrow{\mathbf{free}_{\text{Lie}}} & \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \\ \downarrow & & \downarrow{\scriptstyle\text{diag}} \\ \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} & \xrightarrow{\mathbf{free}_{\text{LieAlgbroid}}} & \text{LieAlgbroid}(\mathcal{X}), \end{array}$$

where the left vertical arrow sends $\mathcal{F} \in \text{IndCoh}(\mathcal{X})$ *to* $(\mathcal{F} \overset{0}{\to} T(\mathcal{X}))$.

Proof. Follows by adjunction from the commutativity of the corresponding diagram of right adjoints

$$\begin{array}{ccc} \text{IndCoh}(\mathcal{X}) & \xleftarrow{\mathbf{oblv}_{\text{Lie}}} & \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \\ \uparrow & & \uparrow{\scriptstyle\text{ker-anch}} \\ \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} & \xleftarrow{\mathbf{oblv}_{\text{LieAlgbroid}\,/T}} & \text{LieAlgbroid}(\mathcal{X}), \end{array}$$

where the left vertical arrow sends

$$(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X})) \mapsto \text{Fib}(\gamma).$$

□

3. Examples of Lie algebroids

In this section we discuss four main examples of Lie algebroids: the tangent algebroid, the zero algebroid, the Lie algebroid attached to a map, and the Atiyah algebroid attached to a perfect complex.

3.1. **The tangent and zero Lie algebroids.** In this subsection we introduce two most basic Lie algebroids.

3.1.1. The most basic example of a Lie algebroid is the final object of LieAlgbroid($\mathcal{X}$), denoted $\mathcal{T}(\mathcal{X})$. It is called the *tangent Lie algebroid.*

It corresponds to the formal moduli problem $\mathcal{X} \overset{p_{\mathrm{dR},\mathcal{X}}}{\longrightarrow} \mathcal{X}_{\mathrm{dR}}$. The corresponding groupoid is $(\mathcal{X} \times \mathcal{X})^\wedge$.

We have
$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}(\mathcal{T}(\mathcal{X})) = (T(\mathcal{X}) \overset{\mathrm{id}}{\to} T(\mathcal{X})).$$

We also have:
$$\text{ker-anch}(\mathcal{T}(\mathcal{X})) = 0 \text{ and } \Omega^{\mathrm{fake}}(\mathcal{T}(\mathcal{X})) \simeq \mathrm{inert}_{\mathcal{X}}\,.$$

3.1.2. For a Lie algebroid $\mathfrak{L}$, we define the notion of *splitting* to be the right inverse of the canonical map $\mathfrak{L} \to \mathcal{T}(\mathcal{X})$.

3.1.3. The initial object in LieAlgbroid($\mathcal{X}$) is the 'zero' Lie algebroid, denoted
$$0 \in \mathrm{LieAlgbroid}(\mathcal{X}).$$
It equals diag(0), and corresponds to the groupoid $\mathrm{diag}_{\mathcal{X}}$. The corresponding formal moduli problem is
$$\mathcal{X} \overset{\mathrm{id}}{\to} \mathcal{X}.$$

We have:
$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}(0) = (0 \to T(\mathcal{X})).$$

We also have
$$\text{ker-anch}(0) = \mathrm{inert}_{\mathcal{X}} \text{ and } \Omega^{\mathrm{fake}}(0) = 0.$$

3.1.4. Note that the composite endo-functor of LieAlgbroid($\mathcal{X}$)
$$\mathrm{diag} \circ \Omega^{\mathrm{fake}}$$
identifies with
$$\mathfrak{L} \mapsto 0 \underset{\mathfrak{L}}{\times} 0,$$
where the fiber product is taken in LieAlgbroid(X).

3.2. The Lie algebroid attached to a map.

In this subsection we discuss the Lie algebroid attached to a map of prestacks.

3.2.1. Let $\mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Consider the corresponding map
$$\mathcal{X} \to \mathcal{Y}^\wedge_{\mathcal{X}},$$
where
$$\mathcal{Y}^\wedge_{\mathcal{X}} := \mathcal{X}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y},$$
and the corresponding formal groupoid
$$(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X})^\wedge,$$
see [Chapter IV.1, Sect. 2.3.3].

We denote the corresponding algebroid by $\mathcal{T}(\mathcal{X}/\mathcal{Y})$. We have
$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}(\mathcal{T}(\mathcal{X}/\mathcal{Y})) = (T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})).$$

We also have:
$$\text{ker-anch}(\mathcal{T}(\mathcal{X}/\mathcal{Y})) \simeq \mathrm{inert}_{\mathcal{Y}}\,|_{\mathcal{X}},$$

and therefore

$$\mathbf{oblv}_{\mathrm{LieAlg}}(\text{ker-anch}(\mathcal{T}(\mathcal{X}/\mathcal{Y}))) \simeq f^!(T(\mathcal{Y}))[-1].$$

3.2.2. Note that we recover $\mathcal{T}(\mathcal{X})$ as $\mathcal{T}(\mathcal{X}/\mathrm{pt})$.

Note also that the zero Lie algebroid can be recovered as $\mathcal{T}(\mathcal{X}/\mathcal{X})$.

3.2.3. By definition, the datum of *splitting* of the Lie algebroid $\mathcal{T}(\mathcal{X}/\mathcal{Y})$ is equivalent to that of factoring the map $\mathcal{X} \to \mathcal{Y}$ as

$$\mathcal{X} \overset{p_{\mathrm{dR},\mathcal{X}}}{\longrightarrow} \mathcal{X}_{\mathrm{dR}} \to \mathcal{Y}.$$

3.3. **Digression: the universal classifying space.** In this subsection we introduce the prestack responsible for the functor that sends an affine scheme to the (space underlying) the category of perfect complexes on this scheme. We will use this prestack in the next subsection in order to construct the Atiyah algebroid of a perfect complex.

3.3.1. We define the prestack Perf by setting

$$\mathrm{Maps}(S, \mathrm{Perf}) = (\mathrm{QCoh}(S)^{\mathrm{perf}})^{\mathrm{Spc}}, \quad S \in \mathrm{Sch}^{\mathrm{aff}},$$

where we recall that the superscript 'Spc' stands for taking the space obtained from a given $(\infty, 1)$-category by discarding non-invertible morphisms.

Proposition 3.3.2. *The prestack* Perf *belongs to* $\mathrm{PreStk}_{\text{laft-def}}$.

Proof. First, we note that Perf is convergent (see [Chapter I.3, Proposition 3.6.10]). In order to prove that Perf belongs to $\mathrm{PreStk}_{\mathrm{laft}}$, it is sufficient to show that the functor

$$S \mapsto \mathrm{QCoh}(S)^{\mathrm{perf}}$$

takes filtered limits (on all of $\mathrm{Sch}^{\mathrm{aff}}$) to colimits. However, this follows from [DrGa2, Lemma 1.9.5].

Thus, it remains to show that Perf admits deformation theory. This will be done in Sect. A.2. □

3.3.3. We will now describe the Lie algebra $\mathrm{inert}_{\mathrm{Perf}}$.

Let $\mathcal{E}_{\mathrm{univ}}$ be the tautological object of $\mathrm{QCoh}(\mathrm{Perf})^{\mathrm{perf}}$. Consider the object

$$\mathrm{End}(\mathcal{E}_{\mathrm{univ}}) \in \mathrm{AssocAlg}(\mathrm{QCoh}(\mathrm{Perf})).$$

Applying the symmetric monoidal functor

$$\Upsilon : \mathrm{QCoh}(-) \to \mathrm{IndCoh}(-)$$

(see [Chapter II.3, Sect. 3.3]), we obtain an object

$$\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}})) \in \mathrm{AssocAlg}(\mathrm{IndCoh}(\mathrm{Perf})).$$

We claim:

Proposition 3.3.4. *The object* $\mathrm{inert}_{\mathrm{Perf}} \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathrm{Perf}))$ *identifies canonically with the Lie algebra obtained from* $\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}}))$ *by applying the forgetful functor*

$$\mathbf{res}^{\mathrm{Assoc}\to\mathrm{Lie}} : \mathrm{AssocAlg}(\mathrm{IndCoh}(\mathrm{Perf})) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathrm{Perf})).$$

Proof. The rest of this subsection is devoted to the proof of this proposition.

3.3.5. Consider first the object

$$\mathrm{Inert}_{\mathrm{Perf}} \in \mathrm{Grp}(\mathrm{PreStk}_{/\,\mathrm{Perf}}).$$

By definition, for $S \in \mathrm{Sch}^{\mathrm{aff}}$, the groupoid $\mathrm{Maps}(S, \mathrm{Inert}_{\mathrm{Perf}})$ consists of the data $(\mathcal{E}, g)$, where $\mathcal{E} \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ and g is an automorphism of $\mathcal{E}$.

We need to show that the Lie algebra of the completion $\mathrm{Inert}^{\mathrm{inf}}_{\mathrm{Perf}}$ of $\mathrm{Inert}_{\mathrm{Perf}}$ along the unit section (obtained by the functor $\mathrm{Lie}_{\mathrm{Perf}}$ of [Chapter IV.3, Theorem 3.6.2]) identifies canonically with

$$\mathbf{res}^{\mathrm{Assoc}\to\mathrm{Lie}}\left(\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}}))\right).$$

3.3.6. Consider

$$\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}}) \in \mathrm{IndCoh}(\mathrm{Perf}).$$

The above description of $\mathrm{Inert}_{\mathrm{Perf}}$ implies that $\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}})$ naturally lifts to an object of

$$\mathrm{Inert}_{\mathrm{Perf}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Perf}));$$

see [Chapter IV.3, Sect. 5.1.1] for the notation.

In particular, by restriction, we can view $\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}})$ as an object of

$$\mathrm{Inert}^{\mathrm{inf}}_{\mathrm{Perf}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Perf})).$$

By [Chapter IV.3, Proposition 5.1.2], we can view $\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}})$ as an object of

$$\mathrm{inert}_{\mathrm{Perf}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Perf})),$$

and by [Chapter IV.2, Sect. 7.4] as an object of

$$U(\mathrm{inert}_{\mathrm{Perf}})\text{-mod}(\mathrm{IndCoh}(\mathrm{Perf})).$$

Hence, we obtain a map of associative algebras

$$U(\mathrm{inert}_{\mathrm{Perf}}) \to \mathrm{End}(\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}})) \simeq \Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}})).$$

By adjunction, we obtain a map of Lie algebras

$$\mathrm{inert}_{\mathrm{Perf}} \to \mathbf{res}^{\mathrm{Assoc}\to\mathrm{Lie}}\left(\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}}))\right). \tag{3.1}$$

It remains to see that the latter map is an isomorphism.

3.3.7. By definition,

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathrm{Perf}}) = T(\mathrm{Inert}_{\mathrm{Perf}}\,/\,\mathrm{Perf})|_{\mathrm{Perf}},$$

and deformation theory identifies the latter with $\mathbf{oblv}_{\mathrm{Assoc}}\left(\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}})\right)$.

Morover, by unwinding the constructions, we obtain that the resulting map

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathrm{Perf}}) \to \mathbf{oblv}_{\mathrm{Assoc}}\left(\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}})\right)$$

equals the map obtained from (3.1) by applying the functor $\mathbf{oblv}_{\mathrm{Lie}}$.

Hence, we obtain that the map (3.1) induces an isomorphism of the underlying objects of $\mathrm{IndCoh}(\mathrm{Perf})$, as required.

□

3.4. **The Atiyah algebroid.** In this subsection we introduce the Atiyah algebroid corresponding to an object of $\mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$ for $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Furthermore, we show, that as in the classical case, the Atiyah algebroid controls the obstruction to giving such an object a structure of crystal on $\mathcal{X}$.

3.4.1. Recall that for $\mathcal{X} \in \text{PreStk}$ the category
$$\text{QCoh}(\mathcal{X})^{\text{perf}} \subset \text{QCoh}(\mathcal{X})$$
is defined as
$$\lim_{S \in (\text{Sch}^{\text{aff}}_{/\mathcal{X}})^{\text{op}}} \text{QCoh}(S)^{\text{perf}}.$$

Therefore,
$$\text{QCoh}(\mathcal{X})^{\text{perf}} \simeq \text{Maps}(\mathcal{X}, \text{Perf}),$$
where Perf is as in Sect. 3.3.

3.4.2. For $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$, and given an object $\mathcal{E} \in \text{QCoh}(\mathcal{X})^{\text{perf}}$, and thus a map
$$\mathcal{X} \to \text{Perf},$$
we define the Atiyah algebroid of $\mathcal{E}$, denoted $\text{At}(\mathcal{E})$, to be $\mathcal{T}(\mathcal{X}/\,\text{Perf})$.

Note that
$$\text{ker-anch}(\text{At}(\mathcal{E})) \simeq \text{inert}(\text{Perf})|_{\mathcal{X}},$$
and the latter identifies with $\Upsilon_{\mathcal{X}}(\text{End}(\mathcal{E}))$ by Proposition 3.3.4.

3.4.3. By Sect. 3.2.3, the datum of splitting of $\text{At}(\mathcal{E})$ is equivalent to that of factoring the map $\mathcal{X} \to \text{Perf}$, corresponding to $\mathcal{E}$, as
$$\mathcal{X} \overset{p_{\text{dR},\mathcal{X}}}{\longrightarrow} \mathcal{X}_{\text{dR}} \to \text{Perf}\,.$$

I.e., this is equivalent to a structure of *left crystal* on $\mathcal{E}$, see [GaRo2, Sect. 2.1] for what this means.

According to [GaRo2, Proposition 2.4.4], this is equivalent to a structure of crystal on $\Upsilon_{\mathcal{X}}(\mathcal{E})$.

4. Modules over Lie algebroids and the universal enveloping algebra

4.1. **Modules over Lie algebroids.** In this subsection we introduce the notion of module over a Lie algebroid.

In particular, we show that for $\mathcal{E} \in \text{QCoh}(\mathcal{X})^{\text{perf}}$, the ind-coherent sheaf $\Upsilon_{\mathcal{X}}(\mathcal{E}) \in \text{IndCoh}(\mathcal{X})$ has a canonical structure of a module over the Atiyah algebroid $\text{At}(\mathcal{E})$; Moreover, the Atiyah algebroid is the universal Lie algebroid that acts on $\Upsilon_{\mathcal{X}}(\mathcal{E})$; i.e. an action of an algebroid $\mathfrak{L}$ on $\Upsilon_{\mathcal{X}}(\mathcal{E})$ is equivalent to a map of Lie algebroids $\mathfrak{L} \to \text{At}(\mathcal{E})$.

4.1.1. Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X}$, corresponding to a groupoid $\mathcal{R}$. We *define* the category $\mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X}))$ to be
$$\text{IndCoh}(\mathcal{X})^{\mathcal{R}},$$
see [Chapter IV.1, Sect. 2.2.5] for the notation.

We let
$$\mathbf{ind}_{\mathfrak{L}} : \text{IndCoh}(\mathcal{X}) \rightleftarrows \mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X})) : \mathbf{oblv}_{\mathfrak{L}}$$
denote the corresponding adjoint pair of functors.

4.1.2. Let $(\mathcal{X} \overset{\pi}{\to} \mathcal{Y}) \in \text{FormMod}_{\mathcal{X}/}$ be the object corresponding to $\mathfrak{L}$. By [Chapter IV.1, Proposition 2.2.6], we have a canonical equivalence
$$\text{IndCoh}(\mathcal{Y}) \simeq \mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X})).$$

Under this equivalence, the functor $\mathbf{oblv}_{\mathfrak{L}}$ corresponds to $\pi^!$, and the functor $\mathbf{ind}_{\mathfrak{L}}$ corresponds to π^{IndCoh}_*.

4.1.3. Assume for a moment that $\mathfrak{L}$ is of the form $\mathrm{diag}(\mathfrak{h})$ for $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})$. In this case, by [Chapter IV.3, Sect. 5.2.1], we have a canonical identification

$$\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \simeq \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})).$$

Under this equivalence, the functor $\mathbf{oblv}_{\mathfrak{L}}$ goes over to $\mathbf{oblv}_{\mathfrak{h}}$, and the functor $\mathbf{ind}_{\mathfrak{L}}$ corresponds to $\mathbf{ind}_{\mathfrak{h}}$.

4.1.4. *Examples.* For $\mathfrak{L} = \mathcal{T}(\mathcal{X})$ we obtain:

$$\mathcal{T}(X)\text{-mod}(\mathrm{IndCoh}(X)) = \mathrm{IndCoh}(\mathcal{X}_{\mathrm{dR}}) =: \mathrm{Crys}(\mathcal{X}).$$

For $\mathfrak{L} = 0$, we have

$$\mathcal{T}(X)\text{-mod}(\mathrm{IndCoh}(X)) = \mathrm{IndCoh}(\mathcal{X}).$$

4.1.5. Let now $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$. By construction, $\Upsilon_{\mathcal{X}}(\mathcal{E})$ has a canonical structure of module over $\mathrm{At}(\mathcal{E})$.

Hence, for a Lie algebroid $\mathfrak{L}$, a homomorphism $\mathfrak{L} \to \mathrm{At}(\mathcal{E})$ defines on $\Upsilon_{\mathcal{X}}(\mathcal{E})$ a structure of $\mathfrak{L}$-module.

Proposition 4.1.6. *The above map from the space of homomorphisms $\mathfrak{L} \to \mathrm{At}(\mathcal{E})$ to that of structures of $\mathfrak{L}$-module on $\Upsilon_{\mathcal{X}}(\mathcal{E})$ is an isomorphism.*

Proof. Let $\mathcal{X} \overset{\pi}{\to} \mathcal{Y}$ be the object of $\mathrm{FormMod}_{\mathcal{X}/}$ corresponding to $\mathfrak{L}$. The space of homomorphisms $\mathfrak{L} \to \mathrm{At}(\mathcal{E})$ is isomorphic to the space of factorizations of the map $\mathcal{X} \to \mathrm{Perf}$, corresponding to $\mathcal{E}$ as

$$\mathcal{X} \overset{\pi}{\to} \mathcal{Y} \to \mathrm{Perf}\,.$$

I.e., this is the space of ways to write $\mathcal{E}$ as $\pi^*(\mathcal{E}')$ for $\mathcal{E}' \in \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$.

The space of structures of $\mathfrak{L}$-module on $\Upsilon_{\mathcal{X}}(\mathcal{E})$ is isomorphic to the space of ways to write $\Upsilon_{\mathcal{X}}(\mathcal{E})$ as $\pi^!(\Upsilon_{\mathcal{Y}}(\mathcal{E}'))$. I.e., we have to show that the diagram of categories

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}} & \xrightarrow{\Upsilon_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi^!} \\ \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}} & \xrightarrow{\Upsilon_{\mathcal{X}}} & \mathrm{IndCoh}(\mathcal{X}) \end{array}$$

is a pullback square. However, this follows by descent from [Chapter II.3, Lemma 3.3.7].

□

4.2. **The universal enveloping algebra.** In this subsection we associate to a Lie algebroid $\mathfrak{L}$ its universal enveloping algebra, viewed as an algebra object in the category of endo-functors of $\mathrm{IndCoh}(-)$.

4.2.1. Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X}$. Consider the monad on $\mathrm{IndCoh}(\mathcal{X})$ corresponding to the adjunction

$$\mathbf{ind}_{\mathfrak{L}} : \mathrm{IndCoh}(\mathcal{X}) \rightleftarrows \mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) : \mathbf{oblv}_{\mathfrak{L}}.$$

We denote by $U(\mathfrak{L})$ the corresponding algebra object in the monoidal DG category

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})).$$

Tautologically,

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})) = \mathbf{oblv}_{\mathfrak{L}} \circ \mathbf{ind}_{\mathfrak{L}}.$$

4.2.2. Assume for a moment that $\mathfrak{L}$ is of the form $\text{diag}(\mathfrak{h})$ for $\mathfrak{h} \in \text{LieAlg}(\text{IndCoh}(\mathcal{X}))$.

In this case, by [Chapter IV.3, Proposition 5.1.2], $U(\mathfrak{L})$ is given by tensor product with $U(\mathfrak{h})$.

Remark 4.2.3. In [Chapter IV.5] we will see that $U(\mathfrak{L})$ possesses an extra structure: namely a filtration. This extra structure will allow us to develop *infinitesimal differential geometry* on prestacks.

4.3. **The co-algebra structure.** In the classical situation, the unversal enveloping algebra of a Lie algebroid, when considered as a left $\mathcal{O}_X$-module, has a natural structure of co-commutative co-algebra. In this subsection we will establish the corresponding property in the derived setting.

4.3.1. Consider the functor

$$\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X})) \to \text{IndCoh}(\mathcal{X}),$$

given by precomposition with

$$p_{\mathcal{X}}^! : \text{Vect} \to \text{IndCoh}(\mathcal{X}).$$

Let $U(\mathfrak{L})^L \in \text{IndCoh}(\mathcal{X})$ denote the image of

$$\mathbf{oblv}_{\text{Assoc}}(U(\mathfrak{L})) \in \text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X}))$$

under this functor.

The object $U(\mathfrak{L})^L$ corresponds to the functor

$$\mathbf{oblv}_{\mathfrak{L}} \circ \mathbf{ind}_{\mathfrak{L}} \circ p_{\mathcal{X}}^! : \text{Vect} \to \text{IndCoh}(\mathcal{X}).$$

4.3.2. Note that the category $\mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X})) \simeq \text{IndCoh}(\mathcal{Y})$ carries a natural symmetric monoidal structure, and the functor $\mathbf{oblv}_{\mathfrak{L}}$ is symmetric monoidal. Hence, the functor $\mathbf{ind}_{\mathfrak{L}}$ has a natural left-lax symmetric monoidal structure.

Hence, the functor $\mathbf{oblv}_{\mathfrak{L}} \circ \mathbf{ind}_{\mathfrak{L}} \circ p_{\mathcal{X}}^!$ also has a left-lax symmetric monoidal structure. This defines on $U(\mathfrak{L})^L \in \text{IndCoh}(\mathcal{X})$ a structure of co-commutative co-algebra in the symmetric monoidal category $\text{IndCoh}(\mathcal{X})$, and the map $0 \to \mathfrak{L}$ defines an augmentation.

4.3.3. Thus, we can view $U(\mathfrak{L})^L$ as an object of

$$\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(\mathcal{X})).$$

We are going to prove:

Proposition 4.3.4. *There exists a canonical isomorphism in* $\text{CocomCoalg}(\text{IndCoh}(\mathcal{X}))$*:*

$$U(\mathfrak{L})^L \simeq \text{Chev}^{\text{enh}}(\Omega^{\text{fake}}(\mathfrak{L})).$$

Proof. Let $p_s, p_t : \mathcal{R} \rightrightarrows \mathcal{X}$ be the formal groupoid corresponding to $\mathfrak{L}$. We can rewrite the functor $\mathbf{oblv}_{\mathfrak{L}} \circ \mathbf{ind}_{\mathfrak{L}} \circ p_{\mathcal{X}}^!$ as

$$(p_s)_*^{\text{IndCoh}} \circ p_{\mathcal{R}}^!$$

(here $p_{\mathcal{R}}$ is the projection $\mathcal{R} \to \text{pt}$), where the left-lax symmetric monoidal structure comes from the symmetric monoidal structure on $p_{\mathcal{R}}^!$ and the left-lax symmetric monoidal structure on $(p_s)_*^{\text{IndCoh}}$, the latter obtained by adjunction from the symmetric monoidal structure on $p_s^!$.

Let us regard $\mathcal{R}$ as an object of $\text{Ptd}(\text{FormMod}_{/\mathcal{X}})$ via the maps $\Delta_{\mathcal{X}}$ and p_s. Now, the statement of the proposition follows from [Chapter IV.3, Sect. 5.2.2].

□

5. Square-zero extensions and Lie algebroids

In this section, we will show that under the equivalence $\mathrm{LieAlgbroid}(\mathcal{X}) \simeq \mathrm{FormMod}_{\mathcal{X}/}$, free Lie algebroids on $\mathcal{X}$ correspond to square-zero extensions.

This is parallel to [Chapter IV.3, Corollary 3.7.8], which says that *split* square zero extensions correspond to free Lie algebras.

5.1. **Square-zero extensions of prestacks.** Let X be a scheme. Consider the full subcategory

$$\mathrm{Sch}_{X/,\text{inf-closed}} \subset \mathrm{Sch}_{X/},$$

see [Chapter III.1, Sect. 5.1.2].

Recall that we have a pair of mutually adjoint functors

$$\mathrm{RealSqZ} : ((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/})^{\mathrm{op}} \rightleftarrows \mathrm{Sch}_{X/,\text{inf-closed}},$$

where the right adjoint sends $(X \to Y) \mapsto T^*(X/Y)$.

We will now carry out parallel constructions in the setting of formal moduli problems under an arbitrary object of $\mathrm{PreStk}_{\text{laft-def}}$.

5.1.1. For $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$ consider the category $\mathrm{FormMod}_{\mathcal{X}/}$.

Consider the functor

$$\mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{IndCoh}(X)_{/T(\mathcal{X})}, \quad (\mathcal{X} \to \mathcal{Y}) \mapsto (T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})). \tag{5.1}$$

Note that under the equivalence

$$\mathrm{FormMod}_{\mathcal{X}/} \simeq \mathrm{LieAlgbroid}(\mathcal{X}),$$

the functor (5.1) corresponds to $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T}$.

Hence, by Proposition 2.1.3(b), the functor (5.1) admits a left adjoint. In what follows, we shall denote the left adjoint to (5.1) by

$$\mathrm{RealSqZ} : \mathrm{IndCoh}(X)_{/T(\mathcal{X})} \to \mathrm{FormMod}_{\mathcal{X}/}\,.$$

The following diagram commutes by definition:

$$\begin{array}{ccc} \mathrm{FormMod}_{/\mathcal{X}} & \xrightarrow{\sim} & \mathrm{FormGrpoid}(\mathcal{X}) \\ {\scriptstyle\mathrm{RealSqZ}}\uparrow & & \uparrow{\scriptstyle\sim} \\ \mathrm{IndCoh}(X)_{/T(\mathcal{X})} & \xrightarrow{\mathbf{free}_{\mathrm{LieAlgbroid}}} & \mathrm{LieAlgbroid}(\mathcal{X}). \end{array} \tag{5.2}$$

5.1.2. We have:

Lemma 5.1.3. *For any* $(\mathcal{X} \overset{f}{\to} \mathcal{Z}) \in (\mathrm{PreStk}_{\text{laft-def}})_{\mathcal{X}/}$ *and* $(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X})) \in \mathrm{IndCoh}(X)_{/T(\mathcal{X})}$, *the space of extensions of* f *to a map*

$$\mathrm{RealSqZ}(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X})) \to \mathcal{Z}$$

is canonically isomorphic to that of nul-homotopies of the composed map

$$\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X}) \to T(\mathcal{Z})|_{\mathcal{X}}.$$

Proof. We can replace $\mathcal{Z}$ by

$$\mathcal{Z}' := \mathcal{Z} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{X}_{\mathrm{dR}},$$

so that $\mathcal{Z}' \in \mathrm{FormMod}_{\mathcal{X}/}$, and then the assertion follows from the definition. □

5.1.4. Recall the functor RealSplitSqZ of [Chapter IV.3, Sect. 3.7]. By [Chapter IV.3, Proposition 3.7.3], it is the left adjoint to

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X}), \quad (\mathcal{X} \to \mathcal{Y} \to \mathcal{X}) \mapsto T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}},$$

and by construction corresponds under the equivalence

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \xrightarrow{\Omega_{\mathcal{X}}} \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \xrightarrow{\mathrm{Lie}} \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$$

to the functor

$$\mathrm{IndCoh}(\mathcal{X}) \xrightarrow{[-1]} \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\mathbf{free}_{\mathrm{Lie}}} \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})).$$

The commutative diagram of Lemma 2.2.6 translates into the commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{\mathrm{RealSplitSqZ}\circ[1]} & \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \\ \downarrow & & \downarrow \\ \mathrm{IndCoh}(X)_{/T(\mathcal{X})} & \xrightarrow{\mathrm{RealSqZ}} & \mathrm{FormMod}_{\mathcal{X}/}, \end{array} \tag{5.3}$$

where the left vertical arrow sends $\mathcal{F} \mapsto (\mathcal{F} \overset{0}{\to} T(\mathcal{X}))$, and the right vertical arrow is the tautological forgetful functor.

5.2. **Tangent complex of a square-zero extension.** In this subsection we approach the following question: how to describe the relative tangent complex of a square-zero extension?

This question makes sense even for schemes, however, it turns out that it is more convenient to answer in the framework of arbitrary objects of $\mathrm{PreStk}_{\text{laft-def}}$ and formal moduli problems.

By answering this question we will also arrive to an alternative definition of Lie algebroids as modules over a certain monad.

5.2.1. From the commutative diagram (5.2) we obtain (compare with [Chapter IV.3, Corollary 3.7.6]):

Corollary 5.2.2. *The monad on* $\mathrm{IndCoh}(X)_{/T(\mathcal{X})}$ *given by the composition*

$$T(\mathcal{X}/-) \circ \mathrm{RealSqZ}$$

is canonically isomorphic to

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}.$$

In other words, Corollary 5.2.2 gives a description of the relative tangent complex of a square-zero extension in terms of the 'more linear' functor $\mathbf{free}_{\mathrm{LieAlgbroid}}$.

Remark 5.2.3. In Sect. 5.3 we will give an 'estimate' of what the monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$$

looks like when viewed as a plain endo-functor.

5.2.4. From Corollary 2.1.5, we obtain:

Corollary 5.2.5. *There exists a canonical equivalence of categories*

$$\mathrm{LieAlgbroid}(\mathcal{X}) \simeq (T(\mathcal{X}/-) \circ \mathrm{RealSqZ})\text{-mod}(\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}). \tag{5.4}$$

Note that Corollary 5.2.5 implies that we can use the right-hand side of (5.4) as an alternative definition of the category $\mathrm{LieAlgbroid}(\mathcal{X})$.

5.3. **Filtration on the free algebroid.** The main result of this subsection, Proposition 5.3.2 gives an estimate of what the monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(X)_{/T(\mathcal{X})} \to \mathrm{IndCoh}(X)_{/T(\mathcal{X})}$$

looks like as a plain endo-functor, see Proposition 5.3.2 below.

5.3.1. The goal of this subsection is to prove:

Proposition 5.3.2. *For* $(\mathcal{F} \stackrel{\gamma}{\to} T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$*, the object*

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}(\mathcal{F} \to T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$$

can be naturally lifted to

$$(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0})_{/T(\mathcal{X})}$$

(where $T(\mathcal{X})$ *is regarded as a filtered object placed in degree* 0*), such that its associated graded identifies with*

$$\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}) \stackrel{0}{\longrightarrow} T(\mathcal{X})$$

with its natural grading.

The rest of this subsection is devoted to the proof of Proposition 5.3.2. In the proof we will appeal to the material from [Chapter IV.5, Sect. 1]. Let us explain the idea:

Given an object $(\mathcal{F} \stackrel{\gamma}{\to} T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$, scaling γ to zero gives (by applying [Chapter IV.5, Sect. 1]) a filtration on $(\mathcal{F} \stackrel{\gamma}{\to} T(\mathcal{X})$, such that the associated graded is $\mathcal{F} \stackrel{0}{\to} T(\mathcal{X})$.

The result then follows by applying $\mathbf{free}_{\mathrm{LieAlgbroid}}$ to this filtered object, because

$$\mathbf{free}_{\mathrm{LieAlgbroid}}((\mathcal{F} \stackrel{0}{\to} T(\mathcal{X}))$$

is the free Lie algebra generated by $\mathcal{F}$.

5.3.3. Consider the following presheaves of categories

$$\mathcal{P}_1 \text{ and } \mathcal{P}_2, \quad (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to \infty\text{-Cat}\,.$$

The functor $\mathcal{P}_1$ sends an affine scheme S to

$$\mathrm{IndCoh}(\mathcal{X} \times S)_{/T(\mathcal{X})|_{\mathcal{X}\times S}}.$$

The functor $\mathcal{P}_2$ sends an affine scheme S to

$$\mathrm{FormMod}_{\mathcal{X}\times S/\,/S}\,.$$

Here $\mathrm{FormMod}_{\mathcal{X}\times S/\,/S}$ stands for formal moduli problems under $X \times S$, equipped with a map of prestacks to S.

The functors

$$\mathrm{RealSqZ}_{/S} : \mathrm{IndCoh}(\mathcal{X} \times S)_{/T(\mathcal{X})|_{\mathcal{X}\times S}} \rightleftarrows \mathrm{FormMod}_{\mathcal{X}\times S/\,/S} : T(\mathcal{X} \times S/-)$$

give rise to a pair of natural transformations

$$\mathcal{P}_1 \rightleftarrows \mathcal{P}_2, \tag{5.5}$$

see Sect. 5.4.1 below for the notation.

5.3.4. We regard $\mathcal{P}_1$ and $\mathcal{P}_2$ as endowed with the *trivial* action of the monoid $\mathbb{A}^1$ (we refer the reader to [Chapter IV.5, Sect. 1.2] for the formalism of actions of monoids on presheaves of categories). The functors in (5.5) are (obviously) $\mathbb{A}^1$-equivariant.

5.3.5. We now consider the presheaf of categories $\mathcal{P}_0$, represented by the monoid $\mathbb{A}^1$, equipped with an action on itself by *multiplication.*

The object $(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X}))}$ gives rise to a natural transformation

$$\mathcal{P}_0 \to \mathcal{P}_1 \tag{5.6}$$

defined as follows: the corresponding object of $\mathcal{P}_1(\mathbb{A}^1)$ is

$$\mathcal{F}|_{\mathcal{X}\times\mathbb{A}^1} \overset{\gamma_{\mathrm{scaled}}}{\longrightarrow} T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1},$$

where the value of γ_{scaled} over $\lambda \in \mathbb{A}^1$ is $\lambda \cdot \gamma$.

It is easy to see that the above natural transformation $\mathcal{P}_0 \to \mathcal{P}_1$ has a canonical structure of *left-lax* equivariance with respect to $\mathbb{A}^1$.

Note that by [Chapter IV.5, Lemma 1.5.5(a)], the category of left-lax equivariant functors $\mathbb{A}^1 \to \mathcal{P}^1$ identifies with

$$(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0})_{/T(\mathcal{X})}.$$

Under this identification, the above functor (5.6) is given by

$$\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X}),$$

where $\mathcal{F}$ (resp., $T(\mathcal{X})$) is regarded as a filtered object placed in degree 1 (resp., 0).

5.3.6. Thus, we obtain that the composite functor

$$\mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \mathcal{P}_1$$

has a structure of left-lax equivariance with respect to $\mathbb{A}^1$.

The corresponding object of $(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0})_{/T(\mathcal{X})}$ is the desired lift of

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}(\mathcal{F} \to T(\mathcal{X})).$$

□

5.4. **Pullbacks of square-zero extensions.** In this subsection we will show that the functor

$$\mathrm{RealSqZ} : \mathrm{IndCoh}(X)_{/T(\mathcal{X})} \to \mathrm{FormMod}_{\mathcal{X}/}$$

introduced above, is compatible with base change.

This will allow us, in the next subsection, to compare RealSqZ with another notion of square-zero extension of a prestack, namely, the one from [Chapter III.1, Sect. 10.1].

5.4.1. Let $\mathcal{X}_0$ be an object of $\mathrm{PreStk}_{\text{laft-def}}$, and let $\mathcal{X} \in (\mathrm{PreStk}_{\text{laft-def}})_{/\mathcal{X}_0}$. The functor RealSqZ defines a functor

$$\mathrm{RealSqZ}_{/\mathcal{X}_0} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)} \to (\mathrm{PreStk}_{\text{laft-def}})_{\mathcal{X}/\,/\mathcal{X}_0}.$$

5.4.2. Let $f_0 : \mathcal{Y}_0 \to \mathcal{X}_0$ be a map in $\text{PreStk}_{\text{laft-def}}$, and set $\mathcal{Y} := \mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$. Let f denote the resulting map $\mathcal{X} \to \mathcal{Y}$. Tautologically,

$$f^!(T(\mathcal{X}/\mathcal{X}_0)) \simeq T(\mathcal{Y}/\mathcal{Y}_0).$$

By adjunction, for

$$(\mathcal{F}_{\mathcal{X}} \overset{\gamma_{\mathcal{X}}}{\longrightarrow} T(\mathcal{X}/\mathcal{X}_0)) \in \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)}$$

and its pullback by means of $f^!$

$$(\mathcal{F}_{\mathcal{Y}} \overset{\gamma_{\mathcal{Y}}}{\longrightarrow} T(\mathcal{Y}/\mathcal{Y}_0)) \in \text{IndCoh}(\mathcal{Y})_{/T(\mathcal{Y}/\mathcal{Y}_0)},$$

we have a canonical map in $(\text{PreStk}_{\text{laft-def}})_{\mathcal{Y}//\mathcal{Y}_0}$

$$\text{RealSqZ}_{/\mathcal{Y}_0}(\gamma_{\mathcal{Y}}) \to \mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} \text{RealSqZ}_{/\mathcal{X}_0}(\gamma_{\mathcal{X}}). \tag{5.7}$$

We claim:

Proposition 5.4.3. *The map* (5.7) *is an isomorphism.*

We can depict the assertion of Proposition 5.4.3 by the commutative diagram

$$\begin{array}{ccc} \text{IndCoh}(\mathcal{Y})_{/T(\mathcal{Y}/\mathcal{Y}_0)} & \overset{f^!}{\longleftarrow} & \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)} \\ {\scriptstyle \text{RealSqZ}_{/\mathcal{Y}_0}} \downarrow & & \downarrow {\scriptstyle \text{RealSqZ}_{/\mathcal{X}_0}} \\ (\text{PreStk}_{\text{laft-def}})_{\mathcal{Y}//\mathcal{Y}_0} & \overset{\mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} -}{\longleftarrow} & (\text{PreStk}_{\text{laft-def}})_{\mathcal{X}//\mathcal{X}_0}. \end{array} \tag{5.8}$$

5.4.4. *Proof of Proposition 5.4.3.* We have a commutative diagram

$$\begin{array}{ccc} \text{IndCoh}(\mathcal{Y}) & \overset{f^!}{\longleftarrow} & \text{IndCoh}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \text{IndCoh}(\mathcal{Y})_{/T(\mathcal{Y}/\mathcal{Y}_0)} & \overset{f^!}{\longleftarrow} & \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)}, \end{array}$$

where the vertical arrows are as in Lemma 2.2.6. Since the essential image of

$$\text{IndCoh}(\mathcal{X}) \to \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)}$$

generates the target category under sifted colimits, and since the horizontal arrows in (5.8) commute with colimits, it suffices to show that the outer diagram in

$$\begin{array}{ccc} \text{IndCoh}(\mathcal{Y}) & \overset{f^!}{\longleftarrow} & \text{IndCoh}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \text{IndCoh}(\mathcal{Y})_{/T(\mathcal{Y}/\mathcal{Y}_0)} & \overset{f^!}{\longleftarrow} & \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)} \\ {\scriptstyle \text{RealSqZ}_{/\mathcal{Y}_0}} \downarrow & & \downarrow {\scriptstyle \text{RealSqZ}_{/\mathcal{X}_0}} \\ (\text{PreStk}_{\text{laft-def}})_{\mathcal{Y}//\mathcal{Y}_0} & \overset{\mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} -}{\longleftarrow} & (\text{PreStk}_{\text{laft-def}})_{\mathcal{X}//\mathcal{X}_0} \end{array}$$

commutes.

However, by Lemma 2.2.6, the outer diagram identifies with

$$\begin{array}{ccc}
\operatorname{IndCoh}(\mathcal{Y}) & \xleftarrow{f^!} & \operatorname{IndCoh}(\mathcal{X}) \\
\Big\downarrow{\scriptstyle \mathbf{free}_{\mathrm{Lie}}} & & \Big\downarrow{\scriptstyle \mathbf{free}_{\mathrm{Lie}}} \\
\operatorname{LieAlg}(\operatorname{IndCoh}(\mathcal{Y})) & \xleftarrow{f^!} & \operatorname{LieAlg}(\operatorname{IndCoh}(\mathcal{X})) \\
\Big\downarrow{\scriptstyle B_{\mathcal{Y}}\circ\exp} & & \Big\downarrow{\scriptstyle B_{\mathcal{X}}\circ\exp} \\
\operatorname{Ptd}((\operatorname{PreStk}_{\text{laft-def}})_{/\mathcal{Y}}) & \xleftarrow{\mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} -} & \operatorname{Ptd}((\operatorname{PreStk}_{\text{laft-def}})_{/\mathcal{X}}) \\
\Big\downarrow & & \Big\downarrow \\
(\operatorname{PreStk}_{\text{laft-def}})_{\mathcal{Y}//\mathcal{Y}_0} & \xleftarrow{\mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} -} & (\operatorname{PreStk}_{\text{laft-def}})_{\mathcal{X}//\mathcal{X}_0}.
\end{array}$$

Now, the commutativity of the latter diagram is manifest, since the middle vertical arrows are equivalences.

□

5.5. **Relation to another notion of square-zero extension.** In this subsection, we will relate the category $\operatorname{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ and the functor RealSqZ to the construction considered in [Chapter III.1, Sect. 10.1].

5.5.1. Assume for a moment that $\mathcal{X} = X \in \operatorname{Sch}_{\text{aft}}$ and let us start with a map

$$T^*(X) \to \mathcal{I}, \quad \mathcal{I} \in \operatorname{Coh}(X)^{\leq -1}.$$

On the one hand, the construction of [Chapter III.1, Sect. 5.1], produces from $T^*(X) \to \mathcal{I}$ an object

$$\operatorname{RealSqZ}(T^*(X) \to \mathcal{I}) \in (\operatorname{Sch}_{\text{aft}})_{\text{nil-isom from } X} \subset (\operatorname{Sch}_{\text{aft}})_{X/}.$$

On the other hand, setting $\mathcal{F} = \mathbb{D}_X^{\text{Serre}}(\mathcal{I})$, we obtain an object

$$(\mathcal{F} \to T(X)) \in \operatorname{IndCoh}(X)_{/T(\mathcal{X})}.$$

It follows from that under the embedding

$$(\operatorname{Sch}_{\text{aft}})_{\text{nil-isom from } X} \hookrightarrow \operatorname{FormMod}_{X/},$$

we have an isomorphism

$$\operatorname{RealSqZ}(T^*(X) \to \mathcal{I}) \simeq \operatorname{RealSqZ}(\mathcal{F} \to T(X)),$$

functorially in

$$(T^*(X) \to \mathcal{I}) \in ((\operatorname{Coh}(X)^{\leq -1})_{T^*(X)/})^{\text{op}}.$$

Indeed, both objects satisfy the same universal property on the category $\operatorname{FormMod}_{X/}$.

5.5.2. Let $\mathcal{X}$ be an object of PreStk, and let $\mathcal{I} \in \operatorname{QCoh}(\mathcal{X})^{\leq 0}$. In this case, the construction of [Chapter III.1, Sect. 10.1.1] produces a category (in fact, a space) $\operatorname{SqZ}(\mathcal{X}, \mathcal{I})$, equipped with a forgetful functor

$$\operatorname{SqZ}(\mathcal{X}, \mathcal{I}) \to \operatorname{PreStk}_{\mathcal{X}/}. \tag{5.9}$$

5.5.3. Assume now that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Assume, moreover, that $\mathcal{I}$, regarded as an object of

$$\mathrm{QCoh}(\mathcal{X})^{\leq 0} \subset \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}},$$

belongs to

$$\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}} \subset \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}},$$

see [Chapter III.1, Sect. 4.3.6] for what this means.

This condition can be rewritten as follows: for any $S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}}$, the pullback $\mathcal{I}|_S \in \mathrm{QCoh}(S)^{\leq 0}$ has *coherent* cohomologies.

5.5.4. Set

$$\mathcal{F} := \mathbb{D}^{\mathrm{Serre}}_{\mathcal{X}}(\mathcal{I}[1]) \in \mathrm{IndCoh}(\mathcal{X}).$$

We claim:

Proposition 5.5.5. *There exists a canonically defined isomorphism of spaces*

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X}))$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{SqZ}(\mathcal{X}, \mathcal{I}) & \longrightarrow & \mathrm{PreStk}_{\mathcal{X}/} \\ \sim\uparrow & & \uparrow \\ \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X})) & \xrightarrow{\mathrm{RealSqZ}} & \mathrm{FormMod}_{\mathcal{X}/} \end{array}$$

commute.

The rest of this subsection is devoted to the proof of Proposition 5.5.5.

5.5.6. Note that

$$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X})) \simeq \mathrm{Maps}_{\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}}(T^*(\mathcal{X}), \mathcal{I}[1]).$$

Hence, we have a map

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \to \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X})),$$

given by the construction in [Chapter III.1, Sect. 10.2].

We will now construct the inverse map.

5.5.7. For $(\mathcal{F}_{\mathcal{X}} \overset{\gamma_{\mathcal{X}}}{\to} T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ set

$$(\mathcal{X} \hookrightarrow \mathcal{X}') := \mathrm{RealSqZ}(\gamma_{\mathcal{X}}) \in \mathrm{FormMod}_{\mathcal{X}/}.$$

We claim that the object

$$(\mathcal{X} \hookrightarrow \mathcal{X}') \in \mathrm{PreStk}_{\mathcal{X}/},$$

constructed above has a natural structure of an object of $\mathrm{SqZ}(\mathcal{X}, \mathcal{I})$.

It will be clear by unwinding the constructions that the two functors

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \leftrightarrow \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X}))$$

are inverses of each other.

5.5.8. Let S' be an object of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, equipped with a map $f' : S' \to \mathcal{X}'$. Set

$$S := S' \underset{\mathcal{X}'}{\times} \mathcal{X},$$

and let f denote the resulting map $S \to \mathcal{X}$. Denote $\mathcal{F}_S := f^!(\mathcal{F}_{\mathcal{X}})$.

Note that Proposition 5.4.3 implies that $S \to S'$ has a canonical structure of square-zero extension by means of $\mathcal{I}_S := \mathbb{D}^{\mathrm{Serre}}_S(\mathcal{F}_S)[-1]$. Hence, it remains to show that $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$.

5.5.9. To prove that $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, it is enough to show that $T^*(S)|_{^{\mathrm{red}}S} \in \mathrm{Coh}(^{\mathrm{red}}S)^{\leq 0}$. We have an exact triangle

$$T^*(S/S')|_{^{\mathrm{red}}S} \to T^*(S)|_{^{\mathrm{red}}S} \to T^*(S')|_{^{\mathrm{red}}S},$$

so it suffices to show that $T^*(S/S')|_{^{\mathrm{red}}S} \in \mathrm{Coh}(^{\mathrm{red}}S)^{\leq 0}$.

We have:

$$T^*(S/S')|_{^{\mathrm{red}}S} = \mathbb{D}^{\mathrm{Serre}}_{^{\mathrm{red}}S}(T(S/S')|_{^{\mathrm{red}}S}),$$

where $\mathbb{D}^{\mathrm{Serre}}$ is understood in the sense of [Chapter III.1, Corollary 4.3.8].

By Proposition 5.3.2, $T(S/S')|_{^{\mathrm{red}}S}$ has a canonical filtration indexed by positive integers, with the d-th sub-quotient isomorphic to the d graded component $(\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}_{^{\mathrm{red}}S}))^d$ of $\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}_{^{\mathrm{red}}S})$.

The required assertion follows now from the fact that for every d,

$$\mathbb{D}^{\mathrm{Serre}}_{^{\mathrm{red}}S}((\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}_{^{\mathrm{red}}S}))^d) \simeq \left(\mathrm{Lie}(d) \otimes \mathcal{I}_{^{\mathrm{red}}S}[1]^{\otimes d}\right)^{\Sigma_d},$$

and hence lives in cohomological degrees $\leq -d$.

5.6. **What is the general framework for the definition of Lie algebroids?** Here is a general categorical framework for the definition of 'broids' that our construction of Lie algebroids fits in.

5.6.1. Let $\mathcal{C}$ be an ∞-category with finite limits, and in particular, a final object $* \in \mathcal{C}$. Let $\mathcal{C}_*$ be the corresponding pointed category, i.e., $\mathcal{C}_* := \mathcal{C}_{*/}$.

Let $\mathcal{D}$ denote the stabilization of $\mathcal{C}_*$, i.e., the category of spectrum objects on $\mathcal{C}_*$. According to [Lu1, Corollary 1.4.2.17], this is a stable ∞-category. Let RealSplitSqZ denote the forgetful functor $\mathcal{D} \to \mathcal{C}_*$, i.e., what is usually denoted Ω^∞.

5.6.2. Consider the functor

$$\mathcal{D} \overset{\mathrm{RealSplitSqZ}}{\longrightarrow} \mathcal{C}_* \to \mathcal{C},$$

where the second arrow is the forgetful functor.

Let us assume that this functor has a left adjoint, to be denoted coTan.

5.6.3. Note that for any $y \in \mathcal{C}$ we have a tautologically defined map $\mathrm{coTan}(y) \to \mathrm{coTan}(*)$.

Consider now the functor

$$\mathrm{coTan}_{\mathrm{rel}} : \mathcal{C} \to \mathcal{D}_{\mathrm{coTan}(*)/}, \quad \mathrm{coTan}_{\mathrm{rel}}(y) := \mathrm{coFib}(\mathrm{coTan}(y) \to \mathrm{coTan}(*)). \tag{5.10}$$

Assume that this functor also admits a left adjoint, to be denoted

$$\mathrm{RealSqZ} : \mathcal{D}_{\mathrm{coTan}(*)/} \to \mathcal{C}.$$

Consider the comonad

$$\mathrm{coTan}_{\mathrm{rel}} \circ \mathrm{RealSqZ}$$

acting on $\mathcal{D}_{\mathrm{coTan}(*)/}$.

The 'broids' that we have in mind are by definition objects of the category

$$(\mathrm{coTan}_{\mathrm{rel}} \circ \mathrm{RealSqZ})\text{-comod}(\mathcal{D}_{\mathrm{coTan}(*)/}).$$

The functor $\mathrm{coTan}_{\mathrm{rel}}$ of (5.10) upgrades to a functor

$$\mathrm{coTan}^{\mathrm{enh}}_{\mathrm{rel}} : \mathcal{C} \to (\mathrm{coTan}_{\mathrm{rel}} \circ \mathrm{RealSqZ})\text{-comod}(\mathcal{D}_{\mathrm{coTan}(*)/}).$$

The above functor $\mathrm{coTan}^{\mathrm{enh}}_{\mathrm{rel}}$ is *not* an equivalence in general, but it happens to be one in our particular example, see Sect. 5.6.5.

5.6.4. By contrast, the category of 'bras' is constructed as follows. We consider the functor

$$\mathrm{RealSplitSqZ} \circ [1] : \mathcal{D} \to \mathcal{C}_*,$$

and its left adjoint

$$\mathcal{C}_* \to \mathcal{C} \overset{\mathrm{coTan}_{\mathrm{rel}}}{\longrightarrow} \mathcal{D};$$

we denote it by $\mathrm{coTan}_{\mathrm{rel}}$ by a slight abuse of notation.

The category of 'bras' is:

$$(\mathrm{coTan}_{\mathrm{rel}} \circ \mathrm{RealSplitSqZ} \circ [1])\text{-comod}(\mathcal{D}).$$

The functor $\mathrm{coTan}_{\mathrm{rel}}$ upgrades to a functor

$$\mathrm{coTan}^{\mathrm{enh}}_{\mathrm{rel}} : \mathcal{C}_* \to \mathrm{RealSplitSqZ} \circ [1])\text{-comod}(\mathcal{D}).$$

This functor $\mathrm{coTan}^{\mathrm{enh}}_{\mathrm{rel}}$ is also *not* an equivalence in general, but it happens to be one in the example of Sect. 5.6.5.

5.6.5. In our case, we apply the above discussion to $\mathcal{C} = (\mathrm{FormMod}_{\mathcal{X}/})^{\mathrm{op}}$, so that

$$\mathcal{C}_* = \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}).$$

Recall that by [Chapter IV.3, Proposition 3.7.12], the functor

$$\mathrm{RealSplitSqZ} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

identifies $(\mathrm{IndCoh}(\mathcal{X}))^{\mathrm{op}}$ with the stabilization of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})^{\mathrm{op}}$.

Now, we claim that the notion of 'broid' (resp. 'bra') defined above recovers the notion of Lie algebroid on $\mathcal{X}$ (resp., Lie algebra in $\mathrm{IndCoh}(\mathcal{X})$). Indeed, this follows from Corollary 5.2.5 (resp., [Chapter IV.3, Corollary 3.7.6]).

6. IndCoh of a square-zero extension

The goal of this section is to describe the category of ind-coherent sheaves on a square-zero extension.

First, we show that every ind-coherent sheaf on $\mathcal{X}$ has a canonical action of the Lie algebra inert_{X}. We then use this fact to give an algebraic description of the category of ind-coherent sheaves on a square-zero extension.

Subsequently, we show that the dualizing sheaf of a square-zero extension of $\mathcal{X}$ is naturally an extension of the direct image of the 'defining ideal' by the direct image image of the dualizing sheaf of $\mathcal{X}$.

6.1. **Modules for the inertia Lie algebra.** In this subsection we observe that any object of $\mathrm{IndCoh}(-)$ acquires a canonical action of the inertia Lie algebra.

6.1.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\text{laft-def}}$. Recall the infinitesimal inertia group $\mathrm{Inert}^{\mathrm{inf}}_{\mathcal{X}}$ and its Lie algebra $\mathrm{inert}_{\mathcal{X}}$.

By [Chapter IV.3, Sect. 5.2.1], we have:

$$\mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \simeq \mathrm{IndCoh}((\mathcal{X}\times\mathcal{X})^\wedge),$$

where the forgetful functor

$$\mathbf{oblv}_{\mathrm{inert}_{\mathcal{X}}} : \mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{IndCoh}(\mathcal{X})$$

corresponds to

$$\Delta^!_{\mathcal{X}} : \mathrm{IndCoh}((\mathcal{X}\times\mathcal{X})^\wedge) \to \mathrm{IndCoh}(\mathcal{X}),$$

and the functor

$$\mathbf{triv}_{\mathrm{inert}_{\mathcal{X}}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$$

corresponds to

$$p_s^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}((\mathcal{X}\times\mathcal{X})^\wedge).$$

6.1.2. Note, however, that the functor

$$p_t^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}((\mathcal{X}\times\mathcal{X})^\wedge)$$

gives rise to another symmetric monoidal functor, denoted

$$\mathbf{can} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})),$$

equipped with an isomorphism

$$\mathbf{oblv}_{\mathrm{inert}_{\mathcal{X}}} \circ \mathbf{can} = \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X})}\,. \tag{6.1}$$

The datum of the functor **can** and the isomorphism (6.1) is equivalent to a functorial assignment to any $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ of a structure of $\mathrm{inert}_{\mathcal{X}}$-module.

6.1.3. By construction, for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, a datum of isomorphism

$$\mathbf{can}(\mathcal{F}) \simeq \mathbf{triv}_{\mathrm{inert}_{\mathcal{X}}}(\mathcal{F}) \in \mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$$

is equivalent to that of an isomorphism

$$p_s^!(\mathcal{F}) \simeq p_t^!(\mathcal{F}) \in \mathrm{IndCoh}((\mathcal{X}\times\mathcal{X})^\wedge).$$

This datum is strictly weaker than that of descent of $\mathcal{F}$ with respect to the groupoid $(\mathcal{X}\times\mathcal{X})^\wedge$, i.e., a structure of crystal.

6.1.4. Assume for a moment that $\mathcal{F} = \Upsilon_{\mathcal{X}}(\mathcal{E})$ for $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$.

Consider the canonical map in LieAlg(IndCoh($\mathcal{X}$)):

$$\mathrm{inert}_{\mathcal{X}} \to \text{ker-anch}(\mathrm{At}(\mathcal{E})) \simeq \Upsilon_{\mathcal{X}}(\mathrm{End}(\mathcal{E})).$$

By Proposition 4.1.6, the datum of such a map is equivalent to that of structure of $\mathrm{inert}_{\mathcal{X}}$-module on $\Upsilon_{\mathcal{X}}(\mathcal{E})$. One can show that this is the same structure as given by the functor **can**, applied to $\Upsilon_{\mathcal{X}}(\mathcal{E})$.

6.2. **The canonical split square-zero extension.** In this section we observe that for any object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ there exists a canonical (a.k.a. Atiyah) map $T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F} \to \mathcal{F}$.

We will see that this map is induced by the action of the Lie algebra $\mathrm{inert}_{\mathcal{X}}$ on $\mathcal{F}$, using the fact that $\mathbf{oblv}_{\mathrm{LieAlg}}(\mathrm{inert}_{\mathcal{X}}) = T(\mathcal{X})[-1]$.

6.2.1. Consider again the object

$$\Delta_{\mathcal{X}} : \mathcal{X} \to (\mathcal{X} \times \mathcal{X})^\wedge : p_s$$

in $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$. We have

$$T((\mathcal{X} \times \mathcal{X})^\wedge/\mathcal{X})_{\mathcal{X}} \simeq T(\mathcal{X}).$$

Hence, applying [Chapter IV.3, Proposition 3.7.3] to the identity map $T(\mathcal{X}) \to T(\mathcal{X})$, we obtain a canonically defined map

$$\mathrm{RealSplitSqZ}(T(\mathcal{X})) \to (\mathcal{X} \times \mathcal{X})^\wedge,$$

such that the composition

$$\mathrm{RealSplitSqZ}(T(\mathcal{X})) \to (\mathcal{X} \times \mathcal{X})^\wedge \overset{p_s}{\to} \mathcal{X}$$

is the tautological projection $\mathrm{RealSplitSqZ}(T(\mathcal{X})) \to \mathcal{X}$.

6.2.2. Consider now the composition

$$\mathrm{RealSplitSqZ}(T(\mathcal{X})) \to (\mathcal{X} \times \mathcal{X})^\wedge \overset{p_t}{\to} \mathcal{X};$$

we denote it by $\mathfrak{d}$ (cf. [Chapter III.1, Sect. 4.5.1]).

By Lemma 5.1.3, the map $\mathfrak{d}$ corresponds to a particular choice of the null-homotopy of the map

$$T(\mathcal{X})[-1] \overset{0}{\to} T(\mathcal{X}) \overset{\mathrm{id}}{\longrightarrow} T(\mathcal{X}).$$

Unwinding the definitions, the above null-homotopy is given by the identity map on $T(\mathcal{X})$.

6.2.3. Identifying

$$\mathrm{IndCoh}(\mathrm{RealSplitSqZ}(T(\mathcal{X}))) \simeq \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X))$$

(see [Chapter IV.3, Sect. 5.2.1]), we obtain a functor

$$\mathrm{IndCoh}(X) \overset{\mathbf{can}}{\longrightarrow} \mathrm{IndCoh}((\mathcal{X} \times \mathcal{X})^\wedge) \to$$
$$\to \mathrm{IndCoh}(\mathrm{RealSplitSqZ}(T(\mathcal{X}))) \simeq \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X)).$$

We denote this functor by

$$\mathbf{can}_{\mathbf{free}} : \mathrm{IndCoh}(X) \to \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X)).$$

Its composition with the forgetful functor

$$\mathbf{oblv}_{\mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])} : \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(\mathcal{X})$$

is the identity functor, i.e.,

$$\mathbf{oblv}_{\mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])} \circ \mathbf{can}_{\mathbf{free}} \simeq \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X})}. \tag{6.2}$$

6.2.4. The datum of the functor $\mathbf{can}_{\mathbf{free}}$ and the isomorphism (6.2) is equivalent to a functorial assignment to any $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ of a map

$$\alpha_{\mathcal{F}} : T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F} \to \mathcal{F}. \tag{6.3}$$

Note that by construction, for $\mathcal{F}' \in \mathrm{IndCoh}(\mathcal{X}_{\mathrm{dR}})$, the map

$$\alpha_{(p_{\mathcal{X},\mathrm{dR}})^!(\mathcal{F}')} : T(\mathcal{X})[-1] \overset{!}{\otimes} (p_{\mathcal{X},\mathrm{dR}})^!(\mathcal{F}') \to (p_{\mathcal{X},\mathrm{dR}})^!(\mathcal{F}') \tag{6.4}$$

is canonically trivialized.

6.2.5. By construction, the map

$$\mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1]) \to \mathrm{inert}_{\mathcal{X}}$$

coming from the identification $\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathcal{X}}) \simeq T(\mathcal{X})[-1]$, induces a commutative diagram

$$\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{\mathbf{can}} & \mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \\
\Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow \\
\mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{\mathbf{can}_{\mathbf{free}}} & \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X)) \\
\Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \mathbf{oblv}_{\mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])}} \\
\mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{\mathrm{Id}} & \mathrm{IndCoh}(\mathcal{X}).
\end{array}$$

6.3. **Description of** IndCoh **of a square-zero extension.** In this subsection we will give an explicit description of the category IndCoh(−) on a square-zero extension.

6.3.1. Let $\gamma : \mathcal{F} \to T(\mathcal{X})$ be an object of $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$. Consider the following category, denoted $\mathrm{Annul}(\mathcal{F}, \gamma)$:

It consists of objects $\mathcal{F}' \in \mathrm{IndCoh}(\mathcal{F})$, equipped with a null-homotopy for the map

$$\mathcal{F}[-1] \overset{!}{\otimes} \mathcal{F}' \to T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F}' \overset{\alpha'_{\mathcal{F}}}{\to} \mathcal{F}'.$$

We have a tautological forgetful functor

$$\mathrm{Annul}(\mathcal{F}, \gamma) \to \mathrm{IndCoh}(\mathcal{X}).$$

6.3.2. Consider now the object

$$\mathrm{RealSqZ}(\mathcal{F}, \gamma) \in \mathrm{FormMod}_{\mathcal{X}/}.$$

In this subsection we will prove (cf. [Chapter III.1, Sect. 5.1.1]):

Theorem 6.3.3. *There exists a canonically defined equivalence of categories*

$$\mathrm{Annul}(\mathcal{F}, \gamma) \simeq \mathrm{IndCoh}(\mathrm{RealSqZ}(\mathcal{F}, \gamma))$$

that commutes with the forgetful functors to $\mathrm{IndCoh}(\mathcal{X})$.

The rest of this subsection is devoted to the proof of Theorem 6.3.3.

6.3.4. *Step 1.* We first construct the functor

$$\text{IndCoh}(\text{RealSqZ}(\mathcal{F},\gamma)) \to \text{Annul}(\mathcal{F},\gamma). \tag{6.5}$$

Let $f:\mathcal{X}\to\mathcal{Y}$ be an object of $\text{FormMod}_{\mathcal{X}/}$. It follows from the definitions, that for $\mathcal{F}'\in \text{IndCoh}(\mathcal{Y})$, the map

$$T(\mathcal{X}/\mathcal{Y})[-1] \overset{!}{\otimes} f^!(\mathcal{F}') \to T(\mathcal{X})[-1] \overset{!}{\otimes} f^!(\mathcal{F}') \overset{\alpha_{f^!(\mathcal{F})}}{\longrightarrow} f^!(\mathcal{F}')$$

is equipped with a canonical null-homotopy.

Applying this to $\mathcal{Y}:=\text{RealSqZ}(\mathcal{F},\gamma)$, and composing with the tautological map

$$\mathcal{F}\to T(\mathcal{X}/\,\text{RealSqZ}(\mathcal{F},\gamma)),$$

we obtain the desried functor (6.5).

6.3.5. *Step 2.* It is easy to see that the forgetful functor

$$\text{Annul}(\mathcal{F},\gamma)\to \text{IndCoh}(\mathcal{X})$$

is monadic. Let $\mathsf{M}_{\mathcal{F},\gamma}$ denote the corresponding monad.

By Step 1, we obtain a map of monads

$$\mathsf{M}_{\mathcal{F},\gamma}\to U(\mathbf{free}_{\text{LieAlgbroids}}(\mathcal{F},\gamma)). \tag{6.6}$$

To prove the proposition, it remains to show that the map (6.6) is an isomorphism.

6.3.6. *Step 3.* We claim that both sides in (6.6), and the map between them, can be naturally upgraded to the category

$$\text{AssocAlg}\left((\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}),\text{IndCoh}(\mathcal{X})))^{\text{Fil},\geq 0}\right).$$

Indeed, this enhancement corresponds to the $\mathbb{A}^1$-family that deforms γ to the 0 map, as in Sect. 5.3.5.

Since the functor ass. gr. is conservative on $(\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X},\text{IndCoh}(\mathcal{X})))^{\text{Fil},\geq 0}$, it suffices to show that the map (6.6) induces an isomorphism at the associated graded level.

This reduces the verification of the isomorphism (6.6) to the case when γ is the 0 map.

6.3.7. *Step 4.* Note that when $\gamma=0$, the category $\text{Annul}(\mathcal{F},\gamma)$ identifies with that of objects $\mathcal{F}'\in\text{IndCoh}(\mathcal{X})$, equipped with a map

$$\mathcal{F}\overset{!}{\otimes}\mathcal{F}'\to\mathcal{F}'.$$

I.e., $\text{Annul}(\mathcal{F},0)\simeq \mathbf{free}_{\text{Assoc}}(\mathcal{F})\text{-mod}(\text{IndCoh}(\mathcal{X}))$, and the monad $\mathsf{M}_{\mathcal{F},\gamma}$ is given by tensor product with $\mathbf{free}_{\text{Assoc}}(\mathcal{F})$.

Similarly, the monad $U(\mathbf{free}_{\text{LieAlgbroids}}(\mathcal{F},0))$ is given by tensor product with $U(\mathbf{free}_{\text{Lie}}(\mathcal{F}))$.

Unwinding the definitions, we obtain that the map (6.6) corresponds to the map

$$\mathbf{free}_{\text{Assoc}}(\mathcal{F})\to U(\mathbf{free}_{\text{Lie}}(\mathcal{F})),$$

and hence is an isomorphism.

6.4. **The dualizing sheaf of a square-zero extension.** As a corollary of Theorem 6.3.3 we obtain the following fact that justifies the terminology ‘square-zero extension’.

6.4.1. Let $\mathcal{X}, \mathcal{F}, \gamma$ be as above. Denote

$$\mathcal{X}' := \text{RealSqZ}(\mathcal{F}, \gamma) \in \text{FormMod}_{\mathcal{X}/},$$

Let $i : \mathcal{X} \to \mathcal{X}'$ denote the canonical map.

We claim:

Proposition 6.4.2. *There is a canonical fiber sequence* IndCoh$(\mathcal{X}')$

$$i_*^{\text{IndCoh}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}'} \to i_*^{\text{IndCoh}}(\mathcal{F})[1]. \tag{6.7}$$

The rest of this subsection is devoted to the proof of the proposition.

6.4.3. *Step 1.* We will construct a fiber sequence

$$i_*^{\text{IndCoh}}(\mathcal{F}) \to i_*^{\text{IndCoh}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}'}.$$

We interpret the category IndCoh$(\mathcal{X}')$ as

$$\text{Annul}(\mathcal{F}, \gamma) \simeq \mathsf{M}_{\mathcal{F},\gamma}\text{-mod}(\text{IndCoh}(\mathcal{X})).$$

Under this identification, the functor i_*^{IndCoh} corresponds to $\mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}$.

The object $\omega_{\mathcal{X}'}$ corresponds to $\omega_{\mathcal{X}} \in \text{IndCoh}(\mathcal{X})$, where the null-homotopy for

$$\mathcal{F}[-1] \overset{!}{\otimes} \omega_{\mathcal{X}} \to T(\mathcal{X})[-1] \overset{!}{\otimes} \omega_{\mathcal{X}} \xrightarrow{\alpha_{\mathcal{X}}} \omega_{\mathcal{X}}$$

comes from (6.4).

6.4.4. *Step 2.* The datum of a map $i_*^{\text{IndCoh}}(\mathcal{F}) \to i_*^{\text{IndCoh}}(\omega_{\mathcal{X}})$ is equivalent to that of a map

$$\mathcal{F} \to \mathsf{M}_{\mathcal{F},\gamma}(\omega_{\mathcal{X}})$$

in IndCoh$(\mathcal{X})$.

Consider the canonical filtration on $\mathsf{M}_{\mathcal{F},\gamma}$, see Sect. 6.3.6. We have a fiber sequence

$$\omega_{\mathcal{X}} \to \mathsf{M}_{\mathcal{F},\gamma}^{\text{Fil},\leq 1}(\omega_{\mathcal{X}}) \to \mathcal{F}.$$

Moreover, the composition

$$\omega_{\mathcal{X}} \to \mathsf{M}_{\mathcal{F},\gamma}^{\text{Fil},\leq 1}(\omega_{\mathcal{X}}) \to \mathsf{M}_{\mathcal{F},\gamma}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}},$$

(where the last arrow is obtained by adjunction from $i_*^{\text{IndCoh}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}'}$), is the identity map.

Hence, we obtain a splitting

$$\mathsf{M}_{\mathcal{F},\gamma}^{\text{Fil},\leq 1}(\omega_{\mathcal{X}}) \simeq \mathcal{F} \oplus \omega_{\mathcal{X}},$$

and in particular a map

$$\mathcal{F} \to \mathsf{M}_{\mathcal{F},\gamma}(\omega_{\mathcal{X}}),$$

whose composition with $\mathsf{M}_{\mathcal{F},\gamma}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}}$ is zero.

This gives rise to a map

$$\mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\mathcal{F}) \to \mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\omega_{\mathcal{X}})$$

in $\mathsf{M}_{\mathcal{F},\gamma}$-mod(IndCoh$(\mathcal{X})$), whose composition with the map

$$\mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}}$$

is zero.

6.4.5. *Step 3.* Thus, it remains to show that

$$\mathbf{oblv}_{\mathsf{M}_{\mathcal{F},\gamma}} \circ \mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\mathcal{F}) \to \mathbf{oblv}_{\mathsf{M}_{\mathcal{F},\gamma}} \circ \mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}}$$

is an exact triangle.

It is enough to establish the exactness at the associated graded level. However, in this case, the maps in question identity with

$$(\mathbf{oblv}_{\mathrm{Assoc}} \circ \mathbf{free}_{\mathrm{Assoc}}(\mathcal{F})) \overset{!}{\otimes} \mathcal{F} \to (\mathbf{oblv}_{\mathrm{Assoc}} \circ \mathbf{free}_{\mathrm{Assoc}}(\mathcal{F})) \to \omega_{\mathcal{X}},$$

and the exactness is manifest.

7. Global sections of a Lie algebroid

In this section we address the following question: one expects that global sections of a Lie algebroid form a Lie algebra. This is done in two steps:

First for the tangent Lie algebroid and then in general. For the tangent Lie algebroid, the idea is that its global sections can be identified with the Lie algebra of the group of (formal) automorphisms of $\mathcal{X}$. To implement the second step, we relate actions of a free Lie algebra to free Lie algebroids.

7.1. **Action of the free Lie algebra and Lie algebroids.** In this subsection we show that the quotient of a prestack with respect to an action of a free Lie algebra is given by a square-zero extension of that prestack.

7.1.1. For $V \in \mathrm{Vect}$, consider $\mathbf{free}_{\mathrm{Lie}}(V) \in \mathrm{LieAlg}(\mathrm{Vect})$. Consider the corresponding object

$$\exp(\mathbf{free}_{\mathrm{Lie}}(V)) \in \mathrm{Grp}(\mathrm{FormMod}_{/\,\mathrm{pt}}).$$

Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Recall that according to [Chapter IV.3, Theorem 6.1.5], the datum of an action of $\exp(\mathbf{free}_{\mathrm{Lie}}(V))$ on $\mathcal{X}$ is equivalent to that of map

$$V \otimes \omega_{\mathcal{X}} \to T(\mathcal{X})$$

in $\mathrm{IndCoh}(\mathcal{X})$.

7.1.2. Given an action of $\exp(\mathbf{free}_{\mathrm{Lie}}(V))$ on $\mathcal{X}$, consider

$$\exp(\mathbf{free}_{\mathrm{Lie}}(V)) \times \mathcal{X}$$

as a formal groupoid over $\mathcal{X}$.

Let

$$\mathcal{X}/\exp(\mathbf{free}_{\mathrm{Lie}}(V))$$

denote the corresponding object of $\mathrm{FormMod}_{\mathcal{X}/}$.

We claim:

Proposition 7.1.3. *There is a canonical isomorphism in* $\mathrm{FormMod}_{\mathcal{X}/}$

$$\mathcal{X}/\exp(\mathbf{free}_{\mathrm{Lie}}(V)) \simeq \mathrm{RealSqZ}(V \otimes \omega_{\mathcal{X}} \to T(\mathcal{X})).$$

The above proposition can be reformulated as follows.

Corollary 7.1.4. *The Lie algebroid corresponding to the formal groupoid* $\exp(\mathbf{free}_{\mathrm{Lie}}(V)) \times \mathcal{X}$ *identifies canonically with*

$$\mathbf{free}_{\mathrm{LieAlgbroid}}(V \otimes \omega_{\mathcal{X}} \to T(\mathcal{X})).$$

7.1.5. *Proof of Proposition 7.1.3.* Let $f : \mathcal{X} \to \mathcal{Y}$ be an object of $\text{FormMod}_{\mathcal{X}/}$. We need to show that the datum of a map

$$\mathcal{X}/\exp(\mathbf{free}_{\text{Lie}}(V)) \to \mathcal{Y}$$

in $\text{FormMod}_{\mathcal{X}/}$ is canonically equivalent to that of a map

$$(V \otimes \omega_{\mathcal{X}} \to T(\mathcal{X})) \to (T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X}))$$

in $\text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$.

However, the latter follows from [Chapter IV.3, Theorem 6.1.5], applied to $\mathcal{X}$, viewed as an object of $(\text{PreStk}_{\text{laft-def}})_{/\mathcal{Y}}$.

□

7.2. **The Lie algebra of vector fields.** In this subsection we will show that global vector fields on prestack form a Lie algebra.

7.2.1. Let $\mathcal{X}$ be an object of $\text{PreStk}_{\text{laft-def}}$.

Consider the (discontinuous) functor

$$(p^!_{\mathcal{X}})^R : \text{IndCoh}(X) \to \text{Vect},$$

right adjoint to $p^!_{\mathcal{X}}$.

Remark 7.2.2. Note that when $\mathcal{X}$ is an eventually coconnective scheme X, the functor $(p^!_X)^R$ is continuous and identifies with

$$\Gamma(X, -) \circ \Upsilon^R_X,$$

where Υ^R_X is the right adjoint of the functor

$$\Upsilon_X : \text{QCoh}(X) \to \text{IndCoh}(X), \quad \mathcal{E} \mapsto \mathcal{E} \otimes \omega_X.$$

7.2.3. Consider the object $(p^!_{\mathcal{X}})^R(T(\mathcal{X})) \in \text{Vect}$. We claim:

Proposition-Construction 7.2.4. *The object* $(p^!_{\mathcal{X}})^R(T(\mathcal{X}))$ *can be canonically lifted to an object* $\text{VF}(\mathcal{X}) \in \text{LieAlg}(\text{Vect})$.

Proof. Recall the object

$$\text{Aut}^{\text{inf}}(\mathcal{X}) \in \text{Grp}((\text{FormMod}_{\text{laft}})_{/\,\text{pt}}),$$

see [Chapter IV.3, Sect. 6.2.1].

Define

$$\text{VF}(\mathcal{X}) := \text{Lie}_{\text{pt}}(\text{Aut}^{\text{inf}}(\mathcal{X})).$$

We need to show that

$$\mathbf{oblv}_{\text{Lie}}(\text{VF}(\mathcal{X})) \simeq (p^!_{\mathcal{X}})^R(T(\mathcal{X})).$$

This is equivalent to showing that for $V \in \text{Vect}$,

$$\text{Maps}_{\text{LieAlg(Vect)}}(\mathbf{free}_{\text{Lie}}(V), \text{VF}(\mathcal{X})) \simeq \text{Maps}_{\text{Vect}}(V, (p^!_{\mathcal{X}})^R(T(\mathcal{X}))).$$

However, the latter follows from [Chapter IV.3, Theorem 6.1.5].

□

Remark 7.2.5. Note that by the construction of $\text{Aut}^{\text{inf}}(\mathcal{X})$, for $\mathfrak{h} \in \text{LieAlg}(\text{Vect})$, the space

$$\text{Maps}_{\text{LieAlg(Vect)}}(\mathfrak{h}, \text{VF}(\mathcal{X}))$$

identifies canonically with that of actions of the formal group $\exp(\mathfrak{h})$ on $\mathcal{X}$.

7.3. **Construction of the Lie algebra structure.** In this subsection we will finally construct a structure of Lie algebra on global sections of a Lie algebroid, see Proposition 7.3.3.

7.3.1. Let $\mathcal{X}$ be an object of $\text{PreStk}_{\text{laft-def}}$. We define a functor

$$p^!_{\mathcal{X}} : \text{LieAlg}(\text{Vect})_{/\,\text{VF}(\mathcal{X})} \to \text{LieAlgbroid}(\mathcal{X}) \tag{7.1}$$

as follows.

By definition, we can think of an object

$$(\mathfrak{h} \to \text{VF}(\mathcal{X})) \in \text{LieAlg}(\text{Vect})_{/\,\text{VF}(\mathcal{X})}$$

as a datum of action of $\exp(\mathfrak{h})$ on $\mathcal{X}$.

We let $p^!_{\mathcal{X}}(\mathfrak{h} \to \text{VF}(\mathcal{X})) \in \text{LieAlgbroid}(\mathcal{X})$ be the Lie algebroid corresponding to the formal groupoid $\exp(\mathfrak{h}) \times \mathcal{X}$.

7.3.2. We claim:

Proposition 7.3.3. *The functor $p^!_{\mathcal{X}}$ of* (7.1) *admits a right adjoint, denoted $(p^!_{\mathcal{X}})^R_{/\,\text{VF}(\mathcal{X})}$. The composition*

$$\text{LieAlgbroid}(\mathcal{X}) \overset{(p^!_{\mathcal{X}})^R{/\,\text{VF}(\mathcal{X})}}{\longrightarrow} \text{LieAlg}(\text{Vect})_{/\,\text{VF}(\mathcal{X})} \overset{\mathbf{oblv}_{\text{Lie}}}{\longrightarrow} \text{Vect}_{/(p^!_{\mathcal{X}})^R(T(\mathcal{X}))}$$

is the functor

$$\text{LieAlgbroid}(\mathcal{X}) \overset{\mathbf{oblv}_{\text{LieAlgbroid}\,/T}}{\longrightarrow} \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} \text{Vect}_{/(p^!_{\mathcal{X}})^R(T(\mathcal{X}))}.$$

Proof. Follows immediately from Corollary 7.1.4. □

7.3.4. Note that by construction, we have a commutative diagram

$$\begin{array}{ccc}
\text{LieAlgbroid}(\mathcal{X}) & \xrightarrow{(p^!_{\mathcal{X}})^R_{/\,\text{VF}(\mathcal{X})}} & \text{LieAlg}(\text{Vect})_{/\,\text{VF}(\mathcal{X})} \\
{\scriptstyle\text{ker-anch}}\downarrow & & \downarrow \\
\text{LieAlg}(\text{IndCoh}(\mathcal{X})) & \xrightarrow{(p^!_{\mathcal{X}})^R} & \text{LieAlg}(\text{Vect})
\end{array}$$

where the right vertical arrow is the functor

$$(\mathfrak{h} \overset{\gamma}{\to} \text{VF}(\mathcal{X})) \mapsto \text{Fib}(\gamma).$$

It is easy to see, however, that the diagram, obtained from the above one by passing to left adjoints along the vertical arrows, is also commutative:

$$\begin{array}{ccc}
\text{LieAlgbroid}(\mathcal{X}) & \xrightarrow{(p^!_{\mathcal{X}})^R_{/\,\text{VF}(\mathcal{X})}} & \text{LieAlg}(\text{Vect})_{/\,\text{VF}(\mathcal{X})} \\
{\scriptstyle\text{diag}}\uparrow & & \uparrow \\
\text{LieAlg}(\text{IndCoh}(\mathcal{X})) & \xrightarrow{(p^!_{\mathcal{X}})^R} & \text{LieAlg}(\text{Vect}),
\end{array} \tag{7.2}$$

where the right vertical arrow sends

$$\mathfrak{h} \mapsto (\mathfrak{h} \overset{0}{\to} \text{VF}(\mathcal{X})).$$

7.3.5. Let us denote by $(p^!_{\mathcal{X}})^R$ the composition

$$\text{LieAlgbroid}(\mathcal{X}) \overset{(p^!_{\mathcal{X}})^R_{/\,\text{VF}(\mathcal{X})}}{\longrightarrow} \text{LieAlg}(\text{Vect})_{/\,\text{VF}(\mathcal{X})} \to \text{LieAlg}(\text{Vect}),$$

where the second arrow is the forgetful functor.

From (7.3), we obtain a commutative diagram

$$\begin{array}{ccc} \text{LieAlgbroid}(\mathcal{X}) & \xrightarrow{(p^!_{\mathcal{X}})^R} & \text{LieAlg}(\text{Vect}) \\ {\scriptstyle\text{diag}}\uparrow & & \uparrow{\scriptstyle\text{id}} \\ \text{LieAlg}(\text{IndCoh}(\mathcal{X})) & \xrightarrow{(p^!_{\mathcal{X}})^R} & \text{LieAlg}(\text{Vect}), \end{array} \tag{7.3}$$

7.3.6. Consider now the functor

$$\text{LieAlgbroid}(\mathcal{X}) \overset{\Omega^{\text{fake}}}{\longrightarrow} \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} \text{LieAlg}(\text{Vect}). \tag{7.4}$$

We claim:

Proposition 7.3.7. *The functor* (7.4) *identifies canonically with*

$$\text{LieAlgbroid}(\mathcal{X}) \overset{\mathbf{oblv}_{\text{LieAlgbroid}}}{\longrightarrow} \text{IndCoh}(\mathcal{X}) \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} \text{Vect} \overset{[-1]}{\longrightarrow} \text{Vect} \overset{\mathbf{triv}_{\text{Lie}}}{\longrightarrow} \text{LieAlg}(\text{Vect}).$$

Proof. Using (7.3), we rewrite the functor (7.4) as

$$\text{LieAlgbroid}(\mathcal{X}) \overset{\Omega^{\text{fake}}}{\longrightarrow} \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \overset{\text{diag}}{\longrightarrow} \text{LieAlgbroid}(\mathcal{X}) \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} \text{LieAlg}(\text{Vect}).$$

Using Sect. 3.1.4, we further rewrite this as

$$\text{LieAlgbroid}(\mathcal{X}) \to \text{LieAlgbroid}(\mathcal{X}) \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} \text{LieAlg}(\text{Vect}) \tag{7.5}$$

where the first arrow is

$$\mathfrak{L} \mapsto 0 \underset{\mathfrak{L}}{\times} 0.$$

This the functor $\text{LieAlgbroid}(\mathcal{X}) \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} \text{LieAlg}(\text{Vect})$ commutes with fiber products, the functor in (7.5) identifies with

$$\mathbf{oblv}_{\text{Grp}} \circ \Omega_{\text{Lie}} \circ (p^!_{\mathcal{X}})^R. \tag{7.6}$$

Now, recall that according to [Chapter IV.2, Proposition 1.7.2], we have

$$\mathbf{oblv}_{\text{Grp}} \circ \Omega_{\text{Lie}} \simeq \mathbf{triv}_{\text{Lie}} \circ [-1] \circ \mathbf{oblv}_{\text{Lie}}.$$

Hence, (7.6) identifies with

$$\mathbf{triv}_{\text{Lie}} \circ [-1] \circ \mathbf{oblv}_{\text{Lie}} \circ (p^!_{\mathcal{X}})^R \simeq \mathbf{triv}_{\text{Lie}} \circ [-1] \circ \mathbf{oblv}_{\text{Lie}} \circ (p^!_{\mathcal{X}})^R \circ \mathbf{oblv}_{\text{LieAlgbroid}},$$

as required.

□

Remark 7.3.8. Propositions 7.3.3 and 7.3.7 can be summarized as follows: for a Lie algebroid $\mathfrak{L}$ on $\mathcal{X}$, consider the corresponding object $\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L}) \in \text{IndCoh}(\mathcal{X})$. Of course, it does not have a structure of Lie algebra in $\text{IndCoh}(\mathcal{X})$. Yet, $(p^!_{\mathcal{X}})^R(\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L}))$ does have a structure of Lie algebra.

Now, $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[-1]$ does have a structure of Lie algebra, but it is not obtained by looping another object in $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$. Despite this, the Lie algebra of global sections of $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[-1]$ is obtained by looping the Lie algebra of global sections of $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})$.

8. Lie algebroids as modules over a monad

In this section we develop the idea borrowed from [Fra]:

Lie algebroids on $\mathcal{X}$ can be expressed as modules over a certain canonically defined monad acting on the category $\mathrm{LieAlgbroid}(\mathrm{IndCoh}(\mathcal{X}))$. This monad is given by the operation of 'semi-direct product' with the inertia Lie algebra $\mathrm{inert}_{\mathcal{X}}$.

8.1. **The inertia monad.** In this subsection we will work in the category of spaces. Given a space X, we will define a monad acting on the category $\mathrm{Grp}(\mathrm{Spc}_{/X})$, modules for which 'almost' reproduce the category $\mathrm{Grpoid}(X)$.

8.1.1. For $X \in \mathrm{Spc}$, consider the above pair of adjoint functors

$$\mathrm{diag} : \mathrm{Grp}(\mathrm{Spc}_{/X}) \rightleftarrows \mathrm{Grpoid}(X) : \mathrm{Inert}\,.$$

It gives rise to a monad on $\mathrm{Grp}(\mathrm{Spc}_{/X})$ that we will denote by $\mathsf{M}_{\mathrm{Inert}_X}$, and refer to it as the *inertia monad* on X.

8.1.2. For $H \in \mathrm{Grp}(\mathrm{Spc}_{/X})$, the object $\mathsf{M}_{\mathrm{Inert}_X}(H) \in \mathrm{Grp}(\mathrm{Spc}_{/X})$ has the following pieces of structure:

- We have a map $H \to \mathsf{M}_{\mathrm{Inert}_X}(H)$, corresponding to the unit in $\mathsf{M}_{\mathrm{Inert}_X}$;
- We have a map $\mathsf{M}_{\mathrm{Inert}_X}(H) \to \mathrm{Inert}_X$, corresponding to the map $H \to X$ and the identification
$$\mathsf{M}_{\mathrm{Inert}_X}(X) = \mathrm{Inert}(\mathrm{diag}_X) = \mathrm{Inert}_X;$$
- A right inverse $\mathrm{Inert}_X \to \mathsf{M}_{\mathrm{Inert}_X}(H)$ of the above map $\mathsf{M}_{\mathrm{Inert}_X}(H) \to \mathrm{Inert}_X$, corresponding to the map $X \to H$.

It is easy to see that the maps

$$H \to \mathsf{M}_{\mathrm{Inert}_X}(H) \to \mathrm{Inert}_X$$

form a fiber sequence in $\mathrm{Grp}(\mathrm{Spc}_{/X})$.

Monads having these properties will be axiomatized in Sect. 8.2 under the name *special monads.*

8.1.3. Note that the fiber sequence and the section of the second arrow

$$H \to \mathsf{M}_{\mathrm{Inert}_X}(H) \rightleftarrows \mathrm{Inert}_X$$

makes $\mathsf{M}_{\mathrm{Inert}_X}(H)$ look like a semi-direct product

$$\mathrm{Inert}_X \ltimes H.$$

In particular, we obtain a canonically defined action of Inert_X on any $H \in \mathrm{Grp}(\mathrm{Spc}_{/X})$.

8.1.4. Consider the category

$$\mathsf{M}_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X})),$$

equipped with a pair of adjoint functors

$$\mathbf{ind}_{\mathsf{M}_{\mathrm{Inert}_X}} : \mathrm{Grp}(\mathrm{Spc}_{/X}) \rightleftarrows \mathsf{M}_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X})) : \mathbf{oblv}_{\mathsf{M}_{\mathrm{Inert}_X}}.$$

As we shall presently see, the category $\mathsf{M}_{\mathrm{Inert}_X}$-mod(Grp($\mathrm{Spc}_{/X}$)) is 'almost equivalent' to Grpoid(X).

8.1.5. By construction, the functor Inert factors though a canonically defined functor

$$\mathrm{Inert}^{\mathrm{enh}} : \mathrm{Grpoid}(X) \to \mathsf{M}_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X})),$$

so that

$$\mathrm{Inert}(R) = \mathbf{oblv}_{\mathsf{M}_{\mathrm{Inert}_X}}(\mathrm{Inert}^{\mathrm{enh}}(R)).$$

It is easy to see that the above functor $R \mapsto \mathrm{Inert}^{\mathrm{enh}}(R)$ admits a left adjoint; we will denote it by

$$\mathrm{diag}^{\mathrm{enh}} : \mathsf{M}_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X})) \to \mathrm{Grpoid}(X).$$

Proposition 8.1.6. *The functor* $\mathrm{diag}^{\mathrm{enh}}$ *is fully faithful. Its essential image consists of those* $R \in \mathrm{Grpoid}(X)$, *for which the map*

$$\pi_0(\mathrm{Inert}(R)) \to \pi_0(R)$$

is surjective.

Proof. First, we have the following general claim:

Lemma 8.1.7. *Let* $\mathsf{F} : \mathbf{C} \rightleftarrows \mathbf{D} : \mathsf{G}$ *be a pair of adjoint functors between* ∞*-categories, where* G *commutes* G*-split geometric realizations. Then the resulting functor*

$$\mathsf{F}^{\mathrm{enh}} : (\mathsf{G} \circ \mathsf{F})\text{-mod}(\mathbf{C}) \to \mathbf{D}$$

is fully faithful.

The fact that $\mathrm{diag}^{\mathrm{enh}}$ is fully faithful follows immediately from the lemma. The essential image of $\mathrm{diag}^{\mathrm{enh}}$ lies in the specified subcategory of Grpoid(X) because this is so for diag, and because this subcategory is closed under colimits.

To prove the proposition it remains to show that the functor Inert is conservative on the specified subcategory of Grpoid(X) and commutes with geometric realizations. The former is straightforward. The latter follows from [Chapter IV.1, Lemma 2.1.3].

□

8.2. **Special monads.** In this subsection we introduce a certain class of monads that we call *special*. They will be useful in studying Lie algebroids. However, we believe that this notion has other applications as well.

8.2.1. *Assumption on the category.* Let $\mathfrak{T}$ be a pointed $(\infty, 1)$-category; denote its final/initial object by $* \in \mathfrak{T}$.

We shall make the following general assumptions:

- (i) $\mathfrak{T}$ admits limits;
- (ii) Sifted colimits in $\mathfrak{T}$ exist and are universal (=commute with base change);
- (iii) Groupoids in $\mathfrak{T}$ are universal (see [Lu1, Definition 6.1.2.14] for what this means).

Note that for any $\widetilde{t} \to t$, the map

$$|\widetilde{t}^\bullet/t| \to t \tag{8.1}$$

is a *monomorphism* (here $\widetilde{t}^\bullet/t$ is the simplicial object of $\mathfrak{T}$ equal to the Čech nerve of $\widetilde{t} \to t$).

We shall say that $\widetilde{t} \to t$ is an *effective epimorphism* if the map (8.1) is an isomorphism. Let $(\mathfrak{T}_{/t})_{\text{epi}}$ be the full subcategory of $\mathfrak{T}_{/t}$ spanned by effective epimorphisms.

8.2.2. We shall now make the following additional assumption on $\mathfrak{T}$:

For any $t \in \mathfrak{T}$, the functor

$$(\mathfrak{T}_{/t})_{\text{epi}} \to \mathfrak{T}, \quad (\widetilde{t} \to t) \mapsto \widetilde{t} \underset{t}{\times} *$$

is conservative.

8.2.3. *Examples.* Here are two examples of this situation:

One is $\text{Grp}(\text{Spc}_{/X})$, where $X \in \text{Spc}$.

Another is $\text{LieAlg}(\mathbf{O})$, where $\mathbf{O}$ is a symmetric monoidal DG category.

8.2.4. One corollary of the property in Sect. 8.2.2 is that the inclusion

$$\text{Grp}(\mathfrak{T}) \hookrightarrow \text{Monoid}(\mathfrak{T})$$

is an equality.

Indeed, for $t \in \text{Monoid}(\mathfrak{T})$, we need to show that the map

$$t \times t \overset{(\text{id},\text{mult})}{\longrightarrow} t \times t$$

is an isomorphism. However, the above map is a map on $(\mathfrak{T}_{/t})_{\text{epi}}$, where both sides map to t via the first projection, while the base change of the above map with respect to $* \to t$ is the identity map.

8.2.5. *Definition of special monad.* Let $(\mathfrak{T}, *)$ be as above. Let $\text{Monad}(\mathfrak{T})$ denote the category of all monads acting on $\mathfrak{T}$.

We let $\text{Monad}(\mathfrak{T})^{\text{spl}} \subset \text{Monad}(\mathfrak{T})$ denote the full subcategory spanned by monads M satisfying the following condition:

For every $t \in \mathfrak{T}$, the maps

$$t \to \mathsf{M}(t) \to \mathsf{M}(*)$$

form a fiber sequence, i.e., the map

$$t \to \mathsf{M}(t) \underset{\mathsf{M}(*)}{\times} *$$

is an isomorphism.

Here $t \to \mathsf{M}(t)$ is given by the unit of the monad M, and $\mathsf{M}(t) \to \mathsf{M}(*)$ is given by the canonical map $t \to *$. We will refer to such monads as *special monads*.

8.2.6. Note that for any $t \in \mathfrak{T}$, the above map

$$\mathsf{M}(t) \to \mathsf{M}(*)$$

admits a section, given by applying M to the canonical map $t \leftarrow *$. So, we have a diagram

$$t \to \mathsf{M}(t) \rightleftarrows \mathsf{M}(*). \tag{8.2}$$

8.2.7. *Basic properties of special monads.* Note that (8.2) implies that for $t \in \mathfrak{T}$, the map $\mathsf{M}(t) \to \mathsf{M}(*)$ is an effective epimorphism. From here, we obtain:

Lemma 8.2.8. *The monad* M*, considered as a mere endo-functor of* $\mathfrak{T}$*, commutes with sifted colimits.*

Proof. We have to show that for a sifted family t_i the map

$$\operatorname{colim} \mathsf{M}(t_i) \to \mathsf{M}(\operatorname{colim} t_i)$$

is an isomorphism. By Sect. 8.2.2, it is enough to show that

$$(\operatorname{colim} \mathsf{M}(t_i)) \underset{\mathsf{M}(*)}{\times} * \to \mathsf{M}(\operatorname{colim} t_i) \underset{\mathsf{M}(*)}{\times} * \simeq \operatorname{colim} t_i$$

is an isomorphism. However, since sifted colimits in $\mathfrak{T}$ are universal,

$$(\operatorname{colim} \mathsf{M}(t_i)) \underset{\mathsf{M}(*)}{\times} * \simeq \operatorname{colim} \left(\mathsf{M}(t_i) \underset{\mathsf{M}(*)}{\times} *\right) \simeq \operatorname{colim} t_i,$$

as required. □

Corollary 8.2.9. *The category* $\mathsf{M}\text{-mod}(\mathfrak{T})$ *admits sifted colimits and the forgetful functor*

$$\mathbf{oblv}_{\mathsf{M}} : \mathsf{M}\text{-mod}(\mathfrak{T}) \to \mathfrak{T}$$

commutes with sifted colimits.

8.3. **Infinitesimal inertia monad.** We will now adapt the material in Sect. 8.1 to the setting of formal geometry.

8.3.1. As in Sect. 8.1, the pair of adjoint functors

$$\operatorname{diag} : \operatorname{Grp}(\operatorname{FormMod}_{/\mathcal{X}}) \rightleftarrows \operatorname{FormGrpoid}(\mathcal{X}) : \operatorname{Inert}^{\operatorname{inf}}$$

defines a monad, denoted $\mathsf{M}_{\operatorname{Inert}^{\operatorname{inf}}_{\mathcal{X}}}$ on $\operatorname{Grp}(\operatorname{FormMod}_{/\mathcal{X}})$.

Moreover, is easy to see that $\mathsf{M}_{\operatorname{Inert}^{\operatorname{inf}}_{\mathcal{X}}}$ is *special*.

8.3.2. Consider the resulting pair of adjoint functors

$$\operatorname{diag}^{\operatorname{enh}} : \mathsf{M}_{\operatorname{Inert}^{\operatorname{inf}}_{\mathcal{X}}}\text{-mod}\left(\operatorname{Grp}(\operatorname{FormMod}_{/\mathcal{X}})\right) \rightleftarrows \operatorname{FormGrpoid}(\mathcal{X}) : \operatorname{Inert}^{\operatorname{inf,enh}}. \tag{8.3}$$

We now claim:

Proposition 8.3.3. *The functor* $\operatorname{diag}^{\operatorname{enh}}$ *and* $\operatorname{Inert}^{\operatorname{inf,enh}}$ *of* (8.3) *are mutually inverse equivalences of categories.*

Proof. We need to show that the functor $\operatorname{Inert}^{\operatorname{inf}}$ satisfies the conditions of the Barr-Beck-Lurie theorem. The fact that the functor $\operatorname{Inert}^{\operatorname{inf}}$ commutes with sifted colimits (and, in particular, geometric realizations) follows from [Chapter IV.1, Corollary 2.2.4]. Hence, it remains to see that $\operatorname{Inert}^{\operatorname{inf}}$ is conservative. This follows, e.g., from the fact that the functor $\Omega_{\mathcal{X}}$ is conservative, via the fiber sequence (1.1). □

8.4. **The inertia monad on Lie algebras and Lie algebroids.** In this subsection we show that the category LieAlg(IndCoh($\mathcal{X}$)) carries a canonical monad, given by semi-direct product with the inertia Lie algebra, and that Lie algebroids identify with the category of modules over this monad.

8.4.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Recall the equivalence

$$\mathrm{Lie}_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \rightleftarrows \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) : \exp$$

of [Chapter IV.3, Theorem 3.6.2].

Hence, the monad $\mathsf{M}_{\mathrm{Inert}^{\mathrm{inf}}_{\mathcal{X}}}$ acting on $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ defines a special monad, denoted $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}$, on LieAlg(IndCoh($\mathcal{X}$)).

8.4.2. From Proposition 8.3.3, we obtain:

Corollary 8.4.3. *The category* LieAlgbroid($\mathcal{X}$)*, equipped with the forgetful functor* ker. anch. *is canonically equivalent to the category* $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}$-mod(LieAlg(IndCoh($\mathcal{X}$)))*, equipped with the forgetful functor* $\mathbf{oblv}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}}$.

8.4.4. By adjunction, under the identification of Corollary 8.4.3, the functor

$$\mathrm{diag} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{LieAlgbroid}(\mathcal{X})$$

identifies with

$$\mathbf{ind}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}\text{-mod}(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))).$$

The zero Lie algebroid, i.e., the initial object of LieAlgbroid($\mathcal{X}$), corresponds to

$$\mathbf{ind}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}}(0) \in \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}\text{-mod}(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))).$$

Under the identification of Corollary 8.4.3 the tangent algebroid $\mathcal{T}(\mathcal{X})$ (i.e., the final object in LieAlgbroid($\mathcal{X}$)) corresponds to

$$0 \in \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}\text{-mod}(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))).$$

8.4.5. Note that

$$\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}(0) = \mathbf{oblv}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}} \circ \mathbf{ind}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}}(0) = \mathrm{inert}_{\mathcal{X}}.$$

As was mentioned already, the monad $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}$ is special. Hence, for $\mathfrak{h} \in$ LieAlg(IndCoh($\mathcal{X}$)), from (8.2) we obtain a split fiber sequence

$$\mathfrak{h} \to \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}(\mathfrak{h}) \rightleftarrows \mathrm{inert}_{\mathcal{X}}. \tag{8.4}$$

Hence, we can think of $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}(\mathfrak{h})$ as a semi-direct product

$$\mathrm{inert}_{\mathcal{X}} \ltimes \mathfrak{h}$$

for a canonically defined action of $\mathrm{inert}_{\mathcal{X}}$ on $\mathfrak{h}$.

Remark 8.4.6. When we forget the Lie algebra structure on $\mathfrak{h}$, we recover the canonical action of $\mathrm{inert}_{\mathcal{X}}$ on objects of IndCoh($\mathcal{X}$) from Sect. 6.1.2.

Vice versa, since the functor **can** of Sect. 6.1.2 is symmetric monoidal, it defines an action of $\mathrm{inert}_{\mathcal{X}}$ on every $\mathfrak{h} \in$ LieAlg(IndCoh($\mathcal{X}$)), and one can show that this is the same action as defined above.

8.4.7. Recall the functor

$$\Omega^{\text{fake}} : \text{LieAlgbroid}(\mathcal{X}) \to \text{LieAlg}(\text{IndCoh}(\mathcal{X})).$$

In terms of the equivalence of Corollary 8.4.3, it sends $\mathfrak{L} \in \text{LieAlgbroid}(\mathcal{X})$, to the fiber of the composite map

$$\text{inert}_{\mathcal{X}} \to \mathsf{M}_{\text{inert}_{\mathcal{X}}}(\mathfrak{h}) \to \mathfrak{h}, \tag{8.5}$$

where the first arrow is the canonical splitting of (8.4), and the second arrow is given by the action of $\mathsf{M}_{\text{inert}_{\mathcal{X}}}$ on $\mathfrak{h}$.

8.4.8. We have the following identifications

$$\text{ker-anch} \circ \text{diag}(\mathfrak{h}) \simeq \mathsf{M}_{\text{inert}_{\mathcal{X}}}(\mathfrak{h}) \simeq \text{inert}_{\mathcal{X}} \ltimes \mathfrak{h};$$

$$\Omega^{\text{fake}} \circ \text{diag}(\mathfrak{h}) \simeq \Omega_{\text{Lie}}(\mathfrak{h});$$

$$\mathbf{oblv}_{\text{LieAlgbroid}\,/T} \circ \text{diag}(\mathfrak{h}) \simeq (\mathbf{oblv}_{\text{Lie}}(\mathfrak{h}) \overset{0}{\to} T(\mathcal{X})).$$

Remark 8.4.9. Note that there are the following two ways to relate the category LieAlgbroid($\mathcal{X}$) to a more linear category.

One is given by Corollary 8.4.3, which implies that we can interpret LieAlgbroid($\mathcal{X}$) as $\mathsf{M}_{\text{inert}_{\mathcal{X}}}$-mod(LieAlg(IndCoh($\mathcal{X}$))).

The other is as modules for the monad

$$\mathbf{oblv}_{\text{LieAlgbroid}\,/T} \circ \mathbf{free}_{\text{LieAlgbroid}} \simeq T(\mathcal{X}/-) \circ \text{RealSqZ}$$

in the category $\text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$.

This former has the advantage that the monad involved, i.e., $\mathsf{M}_{\text{inert}_{\mathcal{X}}}$, is 'smaller': it is given by semi-direct product with $\text{inert}_{\mathcal{X}}$.

The latter has the advantage that the recipient category, i.e., $\text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ is more elementary than LieAlg(IndCoh($\mathcal{X}$)).

9. Relation to classical Lie algebroids

In this section we let X be a *classical* scheme locally of finite type. Our goal is to show that Lie algebroids, as defined in Sect. 2.1, whose underlying object of IndCoh is 'classical' are the same as classical Lie algebroids.

9.1. **Classical Lie algebroids.** In this subsection we recall the notion of classical Lie algebroid on a classical scheme and state the main result of this section, Theorem 9.1.5.

9.1.1. First, we introduce the object $T^{\text{naive}}(X) \in \text{QCoh}(X)^{\heartsuit}$ as follows.

Recall the functor

$$\Upsilon_X : \text{QCoh}(X) \to \text{IndCoh}(X)$$

(see [Chapter II.3, Sect. 3.2.5]). Let Υ_X^R denote its right adjoint. [3]

We start with $T(X) \in \text{IndCoh}(X)$, and consider the object

$$\Upsilon_X^R(T(X)) \in \text{QCoh}(X).$$

[3]Since X is classical, and in particular, eventually coconnective, the functor Υ_X^R is continuous, see [Ga1, Corollary 9.6.3].

It follows from the definitions that

$$\Upsilon_X^R(T(X)) \simeq \underline{\mathrm{Hom}}(T^*(X), \mathcal{O}_X),$$

where $\underline{\mathrm{Hom}}$ is internal Hom in the symmetric monoidal category $\mathrm{QCoh}(X)$.

In particular, $\Upsilon_X^R(T(X)) \in \mathrm{QCoh}(X)^{\geq 0}$. Finally, we set

$$T^{\mathrm{naive}}(X) := H^0(\Upsilon_X^R(T(X))).$$

I.e., $T^{\mathrm{naive}}(X)$ is the usual naive tangent sheaf of a classical scheme.

9.1.2. Let us recall the notion of *classical* Lie algebroid over X (see [BB, Sect. 2]).

By definition, this is a data of

(1) $\mathfrak{L}^{\mathrm{cl}} \in \mathrm{QCoh}(X)^{\heartsuit}$;
(2) a map $\mathrm{anch} : \mathfrak{L}^{\mathrm{cl}} \to T^{\mathrm{naive}}(X)$;
(3) a Lie bracket on $\mathfrak{L}^{\mathrm{cl}}$, which is a differential operator of order 1,

such that

- The map anch is compatible with the Lie brackets;
- The $[\xi_1, f \cdot \xi_2] = f \cdot [\xi_1, \xi_2] + (\mathrm{anch}(\xi_1)(f)) \cdot \xi_2$.

9.1.3. Let $\mathrm{LieAlgbroid}(X)^{\mathrm{cl}}$ denote the category of classical Lie algebroids on X. We have a tautological forgetful functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}} : \mathrm{LieAlgbroid}(X)^{\mathrm{cl}} \to (\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)},$$

and it is easy to see that it admits a left adjoint, denoted $\mathbf{free}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}}$.

The pair

$$\mathbf{free}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}} : (\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} \rightleftarrows \mathrm{LieAlgbroid}(X)^{\mathrm{cl}} : \mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}}$$

is easily seen to be monadic.

9.1.4. The goal of this section is to prove the following:

Theorem 9.1.5. *There exists a canonical equivalence between* $\mathrm{LieAlgbroid}(X)^{\mathrm{cl}}$ *and the full subcategory of* $\mathrm{LieAlgbroid}(X)$ *that consists of those objects for which* $\mathbf{oblv}_{\mathrm{Algbroid}}(\mathfrak{L})$ *belongs to the essential image of* $\mathrm{QCoh}(X)^{\heartsuit}$ *under the (fully faithful) functor*

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X).$$

This equivalence makes the diagram

$$\begin{array}{ccc} \mathrm{LieAlgbroid}(X)^{\mathrm{cl}} & \longrightarrow & \mathrm{LieAlgbroid}(X) \\ {\scriptstyle \mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{LieAlgbroid}/T}} \\ (\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} & \xrightarrow{\Upsilon_X} & \mathrm{IndCoh}(X)_{/T(X)} \end{array}$$

commute.

9.2. **The locally projective case.** In this subsection we consider a special case of Theorem 9.1.5 where the groupoid corresponding to the algebroid in question is itself classical and formally smooth over $\mathcal{X}$.

9.2.1. Let $\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0} \subset \mathrm{QCoh}(X)^{\heartsuit}$ be the full subcategory consisting of objects that are Zariski-locally projective and countably generated.

As a first step towards the proof of Theorem 9.1.5 we will establish its particular case:

Theorem 9.2.2. *The following four categories are naturally equivalent:*

(a) *The full subcategory of* $\mathrm{LieAlgbroid}(X)^{\mathrm{cl}}$*, consisting of those* $\mathfrak{L}^{\mathrm{cl}}$*, for which the object*

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}}(\mathfrak{L}^{\mathrm{cl}}) \in \mathrm{QCoh}(X)^{\heartsuit}$$

belongs to $\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}$

(a') *The full subcategory of* $\mathrm{FormGrpoid}(X)$*, spanned by those objects* $\mathcal{R}$ *that:*

- $\mathcal{R}$ *is an indscheme, which is* classical *and* $\aleph_0$ (*see* [GaRo1, Sect. 1.4.11] *for what this means*);
- $\mathcal{R}$ *is* classically formally smooth (*see* [GaRo1, Defn. 8.1.1] *for what this means*) *relative to* X *with respect to the projection* $p_s : \mathcal{R} \to X$.

(b) *The full subcategory of* $\mathrm{LieAlgbroid}(X)$*, consisting of those objects* $\mathfrak{L}$*, for which*

$$\mathbf{oblv}_{\mathrm{Algbroid}}(\mathfrak{L}) \in \mathrm{IndCoh}(X),$$

belongs to the essential image under Υ_X *of the full subcategory*

$$\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0} \subset \mathrm{QCoh}(X).$$

(b') *The full subcategory of* $\mathrm{FormGrpoid}(X)$*, spanned by those objects* $\mathcal{R}$ *that:*

- $\mathcal{R}$ *is an indscheme, which is weakly* $\aleph_0$ (*see* [GaRo1, Sect. 1.4.11] *for what this means*)*;*
- $\mathcal{R}$ *is formally smooth relative to* X (*see* [Chapter III.1, Sect. 7.3.1] *for what this means*) *with respect to the projection* $p_s : \mathcal{R} \to X$.

The rest of the subsection is devoted to the proof of Theorem 9.2.2.

9.2.3. *The equivalence of (a) and (a').* This is standard in the theory of classical Lie algebroids.

9.2.4. *The equivalence of (b) and (b').* Follows by combining [Chapter III.2, Corollary 3.3.5], [GaRo1, Corollary 8.3.6] and the following fact (see [BD, Proposition 7.12.6 and Theorem 7.12.8]):

Lemma 9.2.5. *Let* $\mathcal{F} \in \mathrm{QCoh}(X)^{\heartsuit}$ *be Zariski-locally countably generated. Then the following conditions are equivalent:*

(i) $\mathcal{F}$ *is Zariski-locally projective.*

(ii) *The functor*

$$\mathrm{QCoh}(X)^{\heartsuit} \to \mathrm{Vect}^{\heartsuit}, \quad \mathcal{F}' \mapsto H^0(\Gamma(X, \mathcal{F} \otimes \mathcal{F}'))$$

can be written as

$$\underset{i \in \mathbb{Z}^{\geq 0}}{\mathrm{colim}}\, \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}'),$$

where the maps $\mathcal{F}_i \to \mathcal{F}_j$ *for* $j \geq i$ *are surjective.*

9.2.6. *The equivalence of (a') and (b').* This is a relative version of [GaRo1, Corollary 9.1.7].

9.3. **The general case.** In this subsection we will finish the proof of Theorem 9.1.5 by reducing the general case to the projective one by a trick that involves monads.

9.3.1. As will be evident from the proof, the assertion of Theorem 9.1.5 is Zariski-local on X. So, henceforth, we will assume that X is affine.

Consider the full subcategories

$$(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0})_{/T^{\mathrm{naive}}(X)} \subset (\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} \subset (\mathrm{QCoh}(X)^{\leq 0})_{/\Upsilon_X^R(T(X))}$$

and

$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)} \subset \\ \subset \mathrm{IndCoh}(X)_{/T(X)}.$$

The functor Υ_X defines equivalences

$$\begin{array}{ccc} (\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0})_{/T^{\mathrm{naive}}(X)} & \overset{\sim}{\longrightarrow} & (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}))_{/T(X)} \\ \downarrow & & \downarrow \\ (\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} & \overset{\sim}{\longrightarrow} & (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)} \\ \downarrow & & \downarrow \\ (\mathrm{QCoh}(X)^{\leq 0})_{/T^{\mathrm{naive}}(X)} & \overset{\sim}{\longrightarrow} & (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)} \end{array}$$

Note also that the inclusions

$$(\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} \subset (\mathrm{QCoh}(X)^{\leq 0})_{/T^{\mathrm{naive}}(X)}$$

and

$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}$$

admit left adjoints, given by truncation. We denote these functors in both contexts by $\tau^{\geq 0}_{\mathrm{QCoh}}$.

9.3.2. Consider the monad $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$ acting on $\mathrm{IndCoh}(X)_{/T(X)}$. We have:

Lemma 9.3.3. *The monad* $\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$ *preserves the full subcategories*

$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)} \subset \mathrm{IndCoh}(X)_{/T(X)}.$$

The map of functors

$$\tau^{\geq 0}_{\mathrm{QCoh}} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}) \to \\ \to \tau^{\geq 0}_{\mathrm{QCoh}} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}) \circ \tau^{\geq 0}_{\mathrm{QCoh}}$$

is an isomorphism.

Proof. Follows from Proposition 5.3.2. □

9.3.4. From Lemma 9.3.3 we obtain that the endo-functor

$$\tau^{\geq 0}_{\mathrm{QCoh}} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}})$$

of

$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)} \to (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)}$$

has a natural structure of monad, and the category

$$\tau^{\geq 0}_{\mathrm{QCoh}} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}})\text{-mod}((\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)}) \tag{9.1}$$

identifies canonically with the full subcategory of

$$(\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}})\text{-mod}((\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}),$$

equal to the preimage of

$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}$$

under the forgetful functor

$$(\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}})\text{-mod}((\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}) \to \\ \to (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}.$$

Thus, we obtain that the full subcategory of LieAlgbroid(X) appearing in Theorem 9.1.5, identifies canonically with the category (9.1).

Hence, to prove Theorem 9.1.5, it suffices to show that under the equivalence (of ordinary (!) categories)

$$(\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} \simeq (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)},$$

the monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}\,/T^{\mathrm{naive}}} \circ \mathbf{free}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}}$$

identifies with the monad

$$\tau^{\geq 0}_{\mathrm{QCoh}} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}).$$

9.3.5. Note, however, that from Theorem 9.2.2, we obtain that the two monads are canonically identified when restricted to

$$(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0})_{/T^{\mathrm{naive}}(X)} \simeq (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}))_{/T(X)}.$$

Moreover, it is easy to see that the monad $\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}\,/T^{\mathrm{naive}}} \circ \mathbf{free}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}}$ commutes with sifted colimits. The corresponding fact holds also for the monad

$$\tau^{\geq 0}_{\mathrm{QCoh}} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}),$$

by Corollary 8.2.9.

9.3.6. Now, the desired isomorphism of monads follows from the following fact: for any object $\gamma \in (\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)}$, the category

$$\left((\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0})_{/T^{\mathrm{naive}}(X)}\right)_{/\gamma}$$

is sifted and the canonical map

$$\underset{\gamma' \in \left((\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0})_{/T^{\mathrm{naive}}(X)}\right)_{/\gamma}}{\mathrm{colim}} \to \gamma$$

is an isomorphism.

9.4. **Modules over classical Lie algebroids.** In this subsection we compare we will compare the category $\mathfrak{L}$-mod(IndCoh(X)), as defined above, with the corresponding category for a classical Lie algebroid on a classical scheme.

9.4.1. Let X be a classical scheme of finite type, and let $\mathfrak{L}^{\mathrm{cl}}$ be a classical Lie algebroid on X. Throughout this subsection we wil assume that $\mathfrak{L}^{\mathrm{cl}}$ is *flat* as an $\mathcal{O}_X$-module.

Let

$$(\mathrm{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\mathrm{rel.flat}}$$

be the monoidal category introduced in [Chapter III.4, Sect. 4.1.1].

According to [BB, Sect. 2], to $\mathfrak{L}^{\mathrm{cl}}$ one associates its universal enveloping algebra $U(\mathfrak{L}^{\mathrm{cl}})$ which is an associative algebra object in $(\mathrm{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\mathrm{rel.flat}}$.

9.4.2. We have a canonically defined fully faithful monoidal functor

$$(\mathrm{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\mathrm{rel.flat}} \to \mathrm{QCoh}(X \times X)$$

and a monoidal equivalence

$$\mathrm{QCoh}(X \times X) \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)).$$

Composing, we obtain a fully faithful functor

$$\mathrm{AssocAlg}((\mathrm{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\mathrm{rel.flat}}) \to \mathrm{AssocAlg}\left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X))\right). \tag{9.2}$$

Hence, we obtain that $U(\mathfrak{L}^{\mathrm{cl}})$ gives rise to a monad acting on $\mathrm{QCoh}(X)$. In particular, it makes sense to talk about the category

$$U(\mathfrak{L}^{\mathrm{cl}})\text{-mod}(\mathrm{QCoh}(X)).$$

This is, by definition, the category of modules over the classical Lie algebroid $\mathfrak{L}^{\mathrm{cl}}$, denoted $\mathfrak{L}^{\mathrm{cl}}$-mod(QCoh($X$)).

Remark 9.4.3. The category $\mathfrak{L}^{\mathrm{cl}}$-mod(QCoh($X$)) has a t-structure uniquely characterized by the property that the forgetful functor to QCoh(X) is t-exact. Now, as in [GaRo2, Proposition 4.7.3] one can show that if $\mathfrak{L}^{\mathrm{cl}}$ is flat as an object of QCoh(X), then the naturally defined functor

$$D((\mathfrak{L}^{\mathrm{cl}}\text{-mod}(\mathrm{QCoh}(X)))^{\heartsuit}) \to \mathfrak{L}^{\mathrm{cl}}\text{-mod}(\mathrm{QCoh}(X))$$

is an equivalence.

9.4.4. Let $\mathfrak{L}$ be the object of LieAlgbroid(X), corresponding to $\mathfrak{L}^{\mathrm{cl}}$ under the equivalence of Theorem 9.1.5.

The next assertion follows from [Chapter IV.5, Theorem 6.1.2] (which will be proved independently):

Lemma 9.4.5. *For $\mathfrak{L}^{\mathrm{cl}}$ flat as an $\mathcal{O}_X$-module, the endo-functor $\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))$ preserves the essential image of the (fully faithful) functor $\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$.*

Hence, we obtain that $U(\mathfrak{L})$ defines a monad, denoted $U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$, on QCoh($X$). Moreover, the functor Υ_X gives rise to a fully faithful functor

$$U(\mathfrak{L})\text{-mod}(\mathrm{QCoh}(X)) \to U(\mathfrak{L})\text{-mod}(\mathrm{IndCoh}(X)) := \mathfrak{L}\text{-mod}(\mathrm{IndCoh}(X)).$$

9.4.6. We are going to prove:

Theorem 9.4.7. *The monads $U(\mathfrak{L}^{\mathrm{cl}})$ and $U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$ on QCoh(X) are canonically isomorphic.*

As a corollary, we obtain:

Corollary 9.4.8. *The category $\mathfrak{L}^{\mathrm{cl}}$-mod(QCoh($X$)) is canonically equivalent to the full subcategory of $\mathfrak{L}$-mod(IndCoh(X)), consisting of objects, whose image under the forgetful functor*

$$\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X)$$

lies in the essential image of $\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$.

9.4.9. *Proof of Theorem 9.4.7, Step 1.* First, the assumption on $\mathfrak{L}^{\mathrm{cl}}$ and [Chapter IV.5, Theorem 6.1.2] imply that $U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$ lies in the essential image of the functor (9.2).

Hence, the assertion of the the theorem is about comparison of associative algebras in the *ordinary* monoidal category $(\mathrm{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\mathrm{rel.flat}}$.

In particular, the assertion is Zariski-local on X, and hence we can assume that X is affine.

9.4.10. *Proof of Theorem 9.4.7, Step 2.* We claim that the stated isomorphism of associative algebras holds when

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}}(\mathfrak{L}^{\mathrm{cl}}) \in \mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}.$$

Indeed, this follows by unwinding the construction of the equivalence in Theorem 9.2.2.

9.4.11. *Proof of Theorem 9.4.7, Step 3.* We claim that the assignments

$$\mathfrak{L} \rightsquigarrow U(\mathfrak{L}^{\mathrm{cl}}) \text{ and } \mathfrak{L} \rightsquigarrow U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$$

commute with sifted colimits.

Indeed, for $U(\mathfrak{L}^{\mathrm{cl}})$ this follows from the construction. For $U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$, this follows from Proposition 2.1.3(a) and [Chapter IV.5, Theorem 6.1.2].

9.4.12. *Proof of Theorem 9.4.7, Step 4.* The required isomorphism follows from Step 3, since our $\mathfrak{L}^{\mathrm{cl}}$ can be written as a sifted colimit of Lie algebroids as in Step 2, see Sect. 9.3.6.

Appendix A. An application: ind-coherent sheaves on push-outs

In this section we will use the material from Sect. 6.3 to show that the categories IndCoh(−) and $\mathrm{QCoh}(-)^{\mathrm{perf}}$ behave well with respect to push-outs of affine schemes.

A.1. **Behavior of ind-coherent sheaves with respect to push-outs.** In this subsection we will consider the case of IndCoh.

A.1.1. Let

$$\begin{array}{ccc} X_1' & \xrightarrow{f'} & X_2' \\ {\scriptstyle g_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array} \tag{A.1}$$

be a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, where the vertical maps are closed embeddings, and the horizontal maps are finite. Consider the corresponding commutative diagram of categories

$$\begin{array}{ccc} \mathrm{IndCoh}(X_1') & \xleftarrow{(f')^!} & \mathrm{IndCoh}(X_2') \\ {\scriptstyle g_1^!}\downarrow & & \downarrow{\scriptstyle g_2^!} \\ \mathrm{IndCoh}(X_1) & \xleftarrow{f^!} & \mathrm{IndCoh}(X_2). \end{array} \tag{A.2}$$

The goal of this subsection is to prove the following result:

Theorem A.1.2. *The diagram* (A.2) *is a pullback square.*

The rest of this subsection is devoted to the proof of Theorem A.1.2.

A.1.3. *Reduction step 1.* Note that in [Chapter III.1, Proposition 1.4.5] we showed that the functor

$$\text{IndCoh}(X'_2) \to \text{IndCoh}(X'_1) \underset{\text{IndCoh}(X_1)}{\times} \text{IndCoh}(X_2) \tag{A.3}$$

is fully faithful. So, it remains to show that the functor (A.3) is essentially surjective.

Let $\text{IndCoh}(X'_1)_{X_1} \subset \text{IndCoh}(X'_1)$ (resp., $\text{IndCoh}(X'_2)_{X_2} \subset \text{IndCoh}(X'_2)$) be the full subcategory consisting of objects with set-theoretic support on X_1 (resp., X_2). It is easy to see that it is sufficient to show that the corresponding functor

$$\text{IndCoh}(X'_2)_{X_2} \to \text{IndCoh}(X'_1)_{X_1} \underset{\text{IndCoh}(X_1)}{\times} \text{IndCoh}(X_2) \tag{A.4}$$

is an equivalence.

Indeed, the essential surjectivity of (A.4) will imply the same property of (A.3), which follows from the localization sequences of DG categories

$$\text{IndCoh}(X'_2)_{X_2} \to \text{IndCoh}(X'_2) \to \text{IndCoh}(X'_2 \setminus X_2)$$

and

$$\text{IndCoh}(X'_1)_{X_1} \underset{\text{IndCoh}(X_1)}{\times} \text{IndCoh}(X_2) \to \text{IndCoh}(X'_1) \underset{\text{IndCoh}(X_1)}{\times} \text{IndCoh}(X_2) \to \text{IndCoh}(X'_2 \setminus X_2).$$

A.1.4. *Reduction step 2.* The formal completion of X_1 in X'_1 can be written as a *filtered* colimit of schemes $X'_{1,\alpha}$, where each $X_1 \to X'_{1,\alpha}$ is a nilpotent embedding. Then the formal completion of X_2 in X'_2 can be written as the colimit of the schemes

$$X'_{2,\alpha} := X'_{1,\alpha} \underset{X_1}{\sqcup} X_2,$$

see [GaRo1, Proposition 6.7.4].

The functors

$$\text{IndCoh}(X'_2)_{X_2} \to \lim_{\alpha} \text{IndCoh}(X'_{2,\alpha}) \text{ and } \text{IndCoh}(X'_1)_{X_1} \to \lim_{\alpha} \text{IndCoh}(X'_{1,\alpha})$$

are both equivalences (see [GaRo1, Proposition 7.4.5]).

This reduces us to the case when $X_1 \to X'_1$ is a nilpotent embedding.

A.1.5. *Reduction step 3.* Using [Chapter III.1, Proposition 5.5.3] and the convergence property of IndCoh (see [Chapter II.2, Proposition 6.4.3]) we can further reduce to the case when the map

$$X_1 \to X'_1$$

has a structure of a square-zero extension.

A.1.6. *Proof in the case when $X_1 \to X'_1$ is a square-zero extension.* Let the square-zero extension $X_1 \to X'_1$ be given by a map

$$T^*(X_1) \to \mathcal{F}, \quad \mathcal{F}[-1] \in \text{Coh}(X_1).$$

Then $X_2 \to X'_2$ is also a square-zero extension, given by

$$T^*(X_2) \xrightarrow{(df)^*} f_*(T^*(X_1)) \to f_*(\mathcal{F}).$$

Denote

$$\widetilde{\mathcal{F}}_1 := \mathbb{D}^{\text{Serre}}_{X_1}(\mathcal{F}), \quad \widetilde{\mathcal{F}}_2 := \mathbb{D}^{\text{Serre}}_{X_2}(f_*(\mathcal{F})).$$

Since f is finite, we have

$$f^{\text{IndCoh}}_*(\widetilde{\mathcal{F}}_1) \simeq \widetilde{\mathcal{F}}_2.$$

According to Theorem 6.3.3, the category $\operatorname{IndCoh}(X_1')$ can be described as consisting of pairs $\mathcal{F}_1' \in \operatorname{IndCoh}(X_1)$, equipped with a null-homotopy of the composition

$$\widetilde{\mathcal{F}}_1[-1] \overset{!}{\otimes} \mathcal{F}_1' \to T(X_1)[-1] \overset{!}{\otimes} \mathcal{F}_1' \to \mathcal{F}_1',$$

and similarly for $\operatorname{IndCoh}(X_2')$.

Now, this makes the assertion of Theorem A.1.2 manifest: an object of the fiber product $\operatorname{IndCoh}(X_1') \underset{\operatorname{IndCoh}(X_1)}{\times} \operatorname{IndCoh}(X_2)$ is an object $\mathcal{F}_2' \in \operatorname{IndCoh}(X_2)$, equipped with a null-homotopy for the composition

$$\widetilde{\mathcal{F}}_1[-1] \overset{!}{\otimes} f^!(\mathcal{F}_2') \to T(X_1)[-1] \overset{!}{\otimes} f^!(\mathcal{F}_2') \to f^!(\mathcal{F}_2'),$$

which by adjunction is the same as a null-homotopy of the map

$$f_*^{\operatorname{IndCoh}}(\widetilde{\mathcal{F}}_1[-1] \overset{!}{\otimes} f^!(\mathcal{F}_2')) \to f_*^{\operatorname{IndCoh}}(T(X_1)[-1] \overset{!}{\otimes} f^!(\mathcal{F}_2')) \to \mathcal{F}_2',$$

while the latter, by the projection formula identifies with the map

$$f_*^{\operatorname{IndCoh}}(\mathcal{F}_1)[-1] \overset{!}{\otimes} \mathcal{F}_2' \to f_*^{\operatorname{IndCoh}}(T(X_1))[-1] \overset{!}{\otimes} \mathcal{F}_2' \to \mathcal{F}_2',$$

and the latter map identifies with

$$\widetilde{\mathcal{F}}_2[-1] \overset{!}{\otimes} \mathcal{F}_2' \to T(X_2)[-1] \overset{!}{\otimes} \mathcal{F}_2' \to \mathcal{F}_2'.$$

□

A.2. **Deformation theory for the functor** $\operatorname{QCoh}(-)^{\operatorname{perf}}$**.** In this subsection we will study the behavior of the category $\operatorname{QCoh}(-)^{\operatorname{perf}}$ with respect to push-outs.

A.2.1. First, we claim that Theorem A.1.2 admits the following corollary:

Corollary A.2.2. *Under the assumptions of Theorem A.1.2, the diagram*

$$\begin{array}{ccc} \operatorname{QCoh}(X_1')^{\operatorname{perf}} & \xleftarrow{(f')^*} & \operatorname{QCoh}(X_2')^{\operatorname{perf}} \\ {\scriptstyle g_1^*}\downarrow & & \downarrow{\scriptstyle g_2^*} \\ \operatorname{QCoh}(X_1)^{\operatorname{perf}} & \xleftarrow{f^*} & \operatorname{QCoh}(X_2)^{\operatorname{perf}} \end{array} \tag{A.5}$$

is a pullback square.

Proof. Follows from the fact that we have a commutative diagram of *symmetric monoidal* functors

$$\begin{array}{ccc} \operatorname{QCoh}(X_2') & \longrightarrow & \operatorname{QCoh}(X_1') \underset{\operatorname{QCoh}(X_1)}{\times} \operatorname{QCoh}(X_2) \\ \Upsilon\downarrow & & \downarrow\Upsilon \\ \operatorname{IndCoh}(X_2') & \longrightarrow & \operatorname{IndCoh}(X_1') \underset{\operatorname{IndCoh}(X_1)}{\times} \operatorname{IndCoh}(X_2), \end{array}$$

combined with [Chapter II.3, Lemma 3.3.7]:

Indeed, Theorem A.1.2 implies that the bottom horizontal arrow identifies the category of dualizable objects in $\operatorname{IndCoh}(X_2')$ with

$$\operatorname{IndCoh}(X_1')^{\operatorname{dualizable}} \underset{\operatorname{IndCoh}(X_1)^{\operatorname{dualizable}}}{\times} \operatorname{IndCoh}(X_2)^{\operatorname{dualizable}}.$$

□

A.2.3. We now claim that the diagram (A.5) is a pullback square for any diagram of affine schemes (A.2), in which the vertical arrows are closed embeddings and horizontal maps finite.

Indeed, the laft property of the functor $\mathrm{QCoh}(-)^{\mathrm{perf}}$, we reduce the assertion to the case when X_1, X_1' and X_2 belong to $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$.

A.2.4. We are now ready to finish the proof of the fact that the prestack Perf admits deformation theory.

From Corollary A.2.2 it follows that Perf admits pro-cotangent spaces and is infinitesimally cohesive. Hence, it remains to show that it admits a pro-cotangent complex.

By [Chapter III.1, Lemma 4.2.4(b)], it suffices to prove the following. Let $f : X_1 \to X_2$ be a map in $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, and let $\mathcal{F}_1$ be an object of $\mathrm{Coh}(X_1)^{\leq 0}$, and let $(X_1)_{\mathcal{F}_1}$ denote the corresponding split square-zero extension of X_1.

For every $\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}$ equipped with a map $f^*(\mathcal{F}_2) \to \mathcal{F}_1$, consider the map

$$(X_1)_{\mathcal{F}_1} \to (X_2)_{\mathcal{F}_2},$$

and the corresponding functor

$$\mathrm{QCoh}((X_2)_{\mathcal{F}_2})^{\mathrm{perf}} \to \mathrm{QCoh}((X_1)_{\mathcal{F}_1})^{\mathrm{perf}} \underset{\mathrm{QCoh}(X_1)^{\mathrm{perf}}}{\times} \mathrm{QCoh}(X_2)^{\mathrm{perf}}.$$

We need to show that the functor

$$\underset{\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, f^*(\mathcal{F}_2) \to \mathcal{F}_1}{\mathrm{colim}} \mathrm{QCoh}((X_2)_{\mathcal{F}_2})^{\mathrm{perf}} \to \mathrm{QCoh}((X_1)_{\mathcal{F}_1})^{\mathrm{perf}} \underset{\mathrm{QCoh}(X_1)^{\mathrm{perf}}}{\times} \mathrm{QCoh}(X_2)^{\mathrm{perf}}$$

is an equivalence.

We will deduce this from Theorem 6.3.3 and [Chapter II.3, Lemma 3.3.7].

A.2.5. We rewrite the category $\mathrm{QCoh}((X_1)_{\mathcal{F}_1})^{\mathrm{perf}}$ as consisting of pairs $\mathcal{F}' \in \mathrm{QCoh}(X_1)^{\mathrm{perf}}$, equipped with a map

$$\mathbb{D}^{\mathrm{Serre}}_{X_1}(\mathcal{F}_1) \overset{!}{\otimes} \Upsilon_{X_1}(\mathcal{F}') \to \Upsilon_{X_1}(\mathcal{F}')$$

in $\mathrm{IndCoh}(X_1)$, which is equivalent to a map

$$\mathrm{End}(\mathcal{F}') \to \mathcal{F}_1,$$

in $\mathrm{QCoh}(X_1)$, and similarly for $\mathrm{QCoh}((X_2)_{\mathcal{F}_2})^{\mathrm{perf}}$.

For a given $\mathcal{F}' \in \mathrm{QCoh}(X_2)^{\mathrm{perf}}$, denote $\mathcal{E} := \mathrm{End}(\mathcal{F}') \in \mathrm{QCoh}(X_2)^{\mathrm{perf}}$. Thus, we have to show that the map

$$\underset{\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, f^*(\mathcal{F}_2) \to \mathcal{F}_1}{\mathrm{colim}} \mathrm{Maps}_{\mathrm{QCoh}(X_2)}(\mathcal{E}, \mathcal{F}_2) \to \\ \to \mathrm{Maps}_{\mathrm{QCoh}(X_1)}(f^*(\mathcal{E}), \mathcal{F}_1) \simeq \mathrm{Maps}_{\mathrm{QCoh}(X_2)}(\mathcal{E}, f_*(\mathcal{F}_1)) \tag{A.6}$$

is an isomorphism.

A.2.6. We note that the index category

$$\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, f^*(\mathcal{F}_2) \to \mathcal{F}_1$$

that appears in the above formula identifies by adjunction with

$$\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, \mathcal{F}_2 \to f_*(\mathcal{F}_1),$$

i.e., with $(\mathrm{Coh}(X_2)^{\leq 0})_{/f_*(\mathcal{F}_1)}$.

Since $f_*(\mathcal{F}_1) \in \mathrm{QCoh}(X_2)^{\geq 0}$, this category is filtered and the map

$$\underset{\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, \mathcal{F}_2 \to f_*(\mathcal{F}_1)}{\mathrm{colim}} \mathcal{F}_2 \to f_*(\mathcal{F}_1)$$

is an isomorphism.

Now, the isomorphism in (A.6) follows from the fact that $\mathcal{E} \in \mathrm{QCoh}(X_2)$ is compact.