CHAPTER I.3. QUASI-COHERENT SHEAVES ON PRESTACKS

Contents

Introduction

0.1. **What is (derived) algebraic geometry about?** Arguably, the object of study of (derived) algebraic geometry is not so much the *geometric objects* (i.e., the most general of which we call *prestacks*, see [Chapter I.2]), but quasi-coherent sheaves on these geometric objects.

This Chapter is devoted to the definition and the study of the most basic properties and structures on quasi-coherent sheaves.

0.1.1. Having at our disposal the theory of ∞-categories, the definition of the category of quasi-coherent sheaves on a prestack is very simple.

First, if our prestack is an affine scheme $S = \text{Spec}(A)$, then

$$\text{QCoh}(S) := A\text{-mod},$$

i.e., this is the DG category of A-modules.

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For a general prestack $\mathcal{Y}$, we define $\text{QCoh}(\mathcal{Y})$ to be the *limit* of the categories $\text{QCoh}(S)$ over the category of pairs

$$(S \in \text{Sch}^{\text{aff}},\ y : S \to \mathcal{Y}). \tag{0.1}$$

I.e., an object $\mathcal{F} \in \text{QCoh}(\mathcal{Y})$ is a of assignments for every (S, y) as above of $\mathcal{F}_{S,y} \in \text{QCoh}(S)$ and for every $g : S' \to S$ and $y' \sim y \circ g$ we are given an isomorphism

$$\mathcal{F}_{S',y'} \simeq g^*(\mathcal{F}_{S,y}).$$

These isomorphisms must satisfy a homotopy-coherent system of compatibilities for compositions of morphisms between affine schemes.

We note that the above limit takes place in the ∞-category $\text{DGCat}_{\text{cont}}$, so we really need to input the entire machinery of [Lu1]. We also note that it is important that we work with DG categories rather than triangulated categories: limits of the latter are known to be ill-behaved.

0.1.2. Assume for a moment that $\mathcal{Y}$ is a (derived) scheme. Then Proposition 1.4.4 shows that in considering the above limit, it is enough to consider those $(S, y : S \to \mathcal{Y})$ for which s is an open embedding. I.e., we glue the category $\text{QCoh}(\mathcal{Y})$ from the corresponding categories on its open affine subschemes.

Note that this is *not* how most textbooks define the category QCoh on a (derived) scheme. The more usual way is to consider all *sheaves of* $\mathcal{O}$*-modules* in the Zariski topology, and then pass to the subcategory consisting of objects with quasi-coherent cohomologies.

By contrast, our definition avoids any mention of sheaves that are non quasi-coherent. We regard it as an advantage: in a sense non quasi-coherent sheaves do not fully belong to algebraic geometry.

0.1.3. Generalizing from schemes to Artin stacks, we show that if $\mathcal{Y}$ is an Artin stack, when considering the limit over the category of pairs (0.1), we can replace it by its full subcategory where we require that the map y be smooth (note that since $\mathcal{Y}$ is an Artin stack, it makes sense to talk about a map to it from an affine scheme being smooth).

Moreover, we can replace the latter category by its 1-full subcategory where when considering morphisms

$$(g : S' \to S,\ y' \simeq y \circ g),$$

we only allow those g that are themselves smooth.

So, when considering QCoh on an Artin stack, we do not have to consider maps that are non-smooth.

0.1.4. Another possible approach to the definition of QCoh would have been as the derived category of an abelian category.

Although it is true that for any prestack $\mathcal{Y}$, the category $\text{QCoh}(\mathcal{Y})$ carries a canonical t-structure, the derived category of its heart is not at all equivalent to $\text{QCoh}(\mathcal{Y})$. This equivalence fails already for affine schemes that are not classical.

What one can show, however, is that when $\mathcal{Y}$ is a *classical* algebraic stack, then the bounded below part of $\text{QCoh}(\mathcal{Y})$ is equivalent to the bounded below part of $D(\text{QCoh}(\mathcal{Y})^{\heartsuit})$.

0.2. **What is done in this Chapter beyond the definition?** So far, we only have the functor
$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$
that sends a prestack $\mathcal{Y}$ to the category $\mathrm{QCoh}(\mathcal{Y})$ and a morphism $f : \mathcal{Y}' \to \mathcal{Y}$ to the pullback functor
$$f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}').$$

The rest of this chapter is devoted to exploring some very basic properties of QCoh.

0.2.1. In Sect. 2, for a morphism $f : \mathcal{Y}' \to \mathcal{Y}$ between prestacks we study the functor
$$f_* : \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y}),$$
right adjoint to f^* (which exists by the Adjoint Functor Theorem since f^* is continuous).

In general, the functor f_* is ill-behaved. For example, it does not have the *base change property* (see Proposition 2.2.2(b) for what this means). In particular, for $\mathcal{F}' \in \mathrm{QCoh}(\mathcal{Y}')$, we cannot explicitly say what is the value of $f_*(\mathcal{F}')$ on $S \xrightarrow{y} \mathcal{Y}$.

However, the situation is much better when f is schematic quasi-compact (i.e., the base change of f by an affine scheme yields a quasi-compact scheme). In this case, the direct image functor does have the base change property.

0.2.2. In Sect. 3 we show that the functor
$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$
has a natural *right-lax* symmetric monoidal structure, where the symmetric monoidal structure on $\mathrm{PreStk}^{\mathrm{op}}$ is induced by the Cartesian symmetric monoidal structure on PreStk, and on $\mathrm{DGCat}_{\mathrm{cont}}$, it is given by the Lurie tensor product.

Concretely, this means that for $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}$ we have a canonically defined functor
$$\mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2), \tag{0.2}$$
given by the *external tensor product* of quasi-coherent sheaves
$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathcal{F}_1 \boxtimes \mathcal{F}_2,$$

We give criteria for when the functor (0.2) is an equivalence. For example, a sufficient condition is that the DG category $\mathrm{QCoh}(\mathcal{Y}_1)$ (or $\mathrm{QCoh}(\mathcal{Y}_2)$) be dualizable.

0.2.3. The symmetric monoidal structure on the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ induces a symmetric monoidal structure on $\mathrm{QCoh}(\mathcal{Y})$ for an individual $\mathcal{Y}$.

We study how various conditions on $\mathrm{QCoh}(\mathcal{Y})$ (such as being dualizable, rigid or compactly generated) interact with each other.

Finally, we study the following question: let
$$\begin{array}{ccc} \mathcal{Y}'_1 & \longrightarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{Y}_1 & \longrightarrow & \mathcal{Y} \end{array}$$
be a pullback diagram of prestacks. Under what conditions is the tautological functor
$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y}'_1)$$
an equivalence?

1. The category of quasi-coherent sheaves

In this section we define the functor QCoh^* that maps $\mathrm{PreStk}^{\mathrm{op}}$ to $\mathrm{DGCat}_{\mathrm{cont}}$. We study its basic properties: behavior with respect to n-coconnectivity and finite typeness, descent and t-structure.

We then show that in the case of Artin stacks, QCoh agrees with the more familiar definition of (the derived category of) quasi-coherent sheaves.

1.1. **Setting up the theory of quasi-coherent sheaves.** The basic input we feed into the theory of QCoh is the fact that the assignment $A \mapsto A\text{-mod}$ is a functor from $(\mathrm{AssocAlg}(\mathrm{Vect}))^{\mathrm{op}}$ to 1-Cat.

1.1.1. Recall that according to [Chapter I.1, Sect. 3.5.5] we have a canonically defined functor

$$(\mathrm{AssocAlg}(\mathrm{Vect}))^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad A \mapsto A\text{-mod}$$

Composing with the forgetful functors

$$\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}) \to \mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{AssocAlg}(\mathrm{Vect})$$

we obtain the functor

$$(\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}))^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,. \tag{1.1}$$

1.1.2. We use the functor (1.1) as the initial input for QCoh.

Namely, we interpret (1.1) as a functor

$$\mathrm{QCoh}_{\mathrm{Sch}^{\mathrm{aff}}} : \mathrm{Sch}^{\mathrm{aff}} \to \mathrm{DGCat}_{\mathrm{cont}}, \tag{1.2}$$
$$S \rightsquigarrow \mathrm{QCoh}(S), \quad (S \xrightarrow{f} S') \rightsquigarrow (\mathrm{QCoh}(S) \xrightarrow{f_*} \mathrm{QCoh}(S')).$$

We will now use the fact that the structure of $(\infty,1)$-category on $\mathrm{DGCat}_{\mathrm{cont}}$ can be canonically extended to a structure of $(\infty,2)$-category, denoted $\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$ (see [Chapter I.1, Sect. 10.3.9]).

Note that for an individual morphism $f : S \to S'$ in $\mathrm{Sch}^{\mathrm{aff}}$, the functor

$$\mathrm{QCoh}(S) \xrightarrow{f_*} \mathrm{QCoh}(S')$$

admits a left adjoint in $\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$, denoted f^*.

Hence, applying [Chapter A.3, Corollary 1.3.4], by *passing to left adjoints*, from $\mathrm{QCoh}_{\mathrm{Sch}^{\mathrm{aff}}}$ we obtain a functor

$$\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \tag{1.3}$$
$$S \rightsquigarrow \mathrm{QCoh}(S), \quad (S \xrightarrow{f} S') \rightsquigarrow (\mathrm{QCoh}(S') \xrightarrow{f^*} \mathrm{QCoh}(S)).$$

1.1.3. Finally, we define the functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

to be the *right Kan extension* of the functor $\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}}$ of (1.3) along the fully faithful embedding

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}^{\mathrm{op}}\,.$$

For an individual $\mathcal{Y} \in \mathrm{PreStk}$ we denote the value of $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ on it by $\mathrm{QCoh}(\mathcal{Y})$. For a map $f : \mathcal{Y}' \to \mathcal{Y}$ we denote the corresponding 1-morphism in $\mathrm{DGCat}_{\mathrm{cont}}$ by

$$f^* : \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y}).$$

1.1.4. By definition, for an individual $\mathcal{Y} \in \mathrm{PreStk}$, we have

$$\mathrm{QCoh}(\mathcal{Y}) \simeq \lim_{(S \xrightarrow{y} \mathcal{Y})} \mathrm{QCoh}(S), \tag{1.4}$$

where the limit is taken over the category opposite to $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$.

Thus, we can think of an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ as an assignment

$$(S \xrightarrow{y} \mathcal{Y}) \rightsquigarrow \mathcal{F}_{S,y} \in \mathrm{QCoh}(S),$$

$$(S' \xrightarrow{f} S) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}} \rightsquigarrow (\mathcal{F}_{S',y\circ f} \simeq f^*(\mathcal{F}_{S,y})) \in \mathrm{QCoh}(S'),$$

satisfying a homotopy-coherent system of compatibilities for compositions of morphisms in $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$.

1.2. Basic properties of QCoh.

1.2.1. *Quasi-coherent sheaves and n-coconnective prestacks.* Assume that $\mathcal{Y}$ is n-coconnective (see [Chapter I.2], Sect. 1.3.3), i.e., that when we view $\mathcal{Y}$ as a functor $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpd}$, it is a left Kan extension along the embedding

$$^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}.$$

We have:

Lemma 1.2.2. *Under the above circumstances, the natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \lim_{(S \xrightarrow{y} \mathcal{Y}) \in ((^{\leq n}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S)$$

is an equivalence.

Proof. Follows from (1.4), since the fact that $\mathcal{Y}$ is n-coconnective exactly means that the functor

$$(^{\leq n}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}} \to (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal.

□

In other words, the above lemma says that if $\mathcal{Y}$ in n-coconnective, in the definition of quasi-coherent sheaves, it is enough to consider only those affine DG schemes mapping to $\mathcal{Y}$ that are themselves n-coconnective.

In particular, if $\mathcal{Y}$ is a classical prestack, it is sufficient to consider only classical affine schemes mapping to $\mathcal{Y}$.

1.2.3. *Quasi-coherent sheaves on stacks locally of finite type.*
Let $\mathcal{Y} \in \mathrm{PreStk}$ be n-coconnective as above, and assume, moreover, that it is locally of finite type (see [Chapter I.2], Sect. 1.6). I.e., $\mathcal{Y}|_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}$ is the left Kan extension along the embedding

$$^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}.$$

Lemma 1.2.4. *Under the above circumstances, the natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \lim_{(S \xrightarrow{y} \mathcal{Y}) \in ((^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}}))^{\mathrm{op}}} \mathrm{QCoh}(S)$$

is an equivalence.

Proof. Follows from Lemma 1.2.2, since the fact that $\mathcal{Y}$ being locally of finite type exactly means that the functor
$$({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{Y}} \to ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$$
is cofinal.

□

I.e., for n-coconnective prestacks locally of finite type, in the definition of quasi-coherent sheaves, it is enough to consider only those affine DG schemes mapping to $\mathcal{Y}$ that are themselves n-coconnective *and* are of finite type.

1.2.5. *Non-convergence.* We note, however, that for $S \in \mathrm{Sch}^{\mathrm{aff}}$ the functor
$$\mathrm{QCoh}(S) \to \lim_{n} \mathrm{QCoh}({}^{\leq n}S)$$
is *not* necessarily an equivalence. The simplest counterexample is provided by $S = \mathrm{Spec}(k[\eta])$ with $\deg(\eta) = -2$.

This means, in particular, that for $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ we cannot express $\mathrm{QCoh}(\mathcal{Y})$ in terms of the categories $\mathrm{QCoh}(S)$ with $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$.

1.3. **Descent.** In this subsection we will discuss a fundamental feature of the functor QCoh^*, namely, that it satisfies flat descent.

1.3.1. Recall what it means for a functor $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$ to satisfy descent with respect to a given topology, see [Chapter I.2, Sect. 2.3.1].

We note, however, that this notion make sense when we replace the target category Spc by any ∞-category.

We observe, however, that the notion of descent when the target is some ∞-category $\mathbf{C}$ is expressible in terms of descent with values in Spc:

Lemma 1.3.2. *Let $F : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathbf{C}$ be a functor. Then it satisfies descent if and and only if for every $\mathbf{c} \in \mathbf{C}$, the functor*
$$\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}, -) \circ F : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$$
satisfies descent.

1.3.3. The following assertion is a version of Grothendieck's flat descent (see [Lu5, Proposition 2.7.14]):

Theorem 1.3.4. *The composite functor*
$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \overset{\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}}}{\longrightarrow} \mathrm{DGCat}_{\mathrm{cont}} \to 1\text{-}\mathrm{Cat}$$
satisfies descent with respect to the flat (and hence, ppf, étale, Zariski) topology.

Since the forgetful functor
$$\mathrm{DGCat}_{\mathrm{cont}} \to 1\text{-}\mathrm{Cat}$$
preserves limits (see [Chapter I.1, Lemma 2.5.2(b)]), we obtain:

Corollary 1.3.5. *The functor*
$$\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$
satisfies descent with respect to the flat (and hence, ppf, étale, Zariski) topology.

1.3.6. Combining Corollary 1.3.5 with [Chapter I.2, Sect. 2.3.3], we obtain:

Corollary 1.3.7. *Let $f : \mathcal{Y}' \to \mathcal{Y}$ be a map in* PreStk *that is an equivalence for the flat topology. Then*

$$f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}')$$

is an equivalence.

From this, we obtain, tautologically:

Corollary 1.3.8. *For $\mathcal{Y} \in \mathrm{PreStk}$, the canonical map $\mathcal{Y} \to L(\mathcal{Y})$ induces an equivalence:*

$$\mathrm{QCoh}(L(\mathcal{Y})) \to \mathrm{QCoh}(\mathcal{Y}).$$

1.3.9. The last corollary has a two-fold significance:

First, to specify a stack we may often have to start from a prestack given explicitly, and then apply the functor L. Corollary 1.3.8 implies that in order to calculate the category QCoh of the resulting stack we can work with the initial prestack.

Secondly, we obtain that for the purposes of QCoh, we will lose no information if we work with the subcategory Stk rather than all of PreStk.

1.3.10. From [Chapter I.2, Lemma 2.3.8], we obtain:

Corollary 1.3.11. *Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be a surjection in the flat topology. Then the natural map*

$$\mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{Tot}\left(\mathrm{QCoh}(\mathcal{Y}_1^\bullet/\mathcal{Y}_2)\right)$$

is an equivalence.

1.3.12. *Quasi-coherent sheaves on n-coconnective stacks.* Recall the notion of n-coconnective stack, see [Chapter I.2, Sect. 2.6.3].

Note that if $\mathcal{Y}$ is n-coconnective as a stack, then this does *not* mean that its is n-coconnective as a prestack. However, combining Corollary 1.3.8 and Lemma 1.2.2, we obtain:

Corollary 1.3.13. *Let $\mathcal{Y}$ be an n-coconnective stack. Then the natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \lim_{(S \overset{y}{\to} Y) \in ((^{\leq n}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S)$$

is an equivalence.

1.4. **Quasi-coherent sheaves on Artin stacks.** The point of this subsection is that when $\mathcal{Y}$ is an Artin stack, in order to recover $\mathrm{QCoh}(\mathcal{Y})$, instead of considering all affine schemes mapping to $\mathcal{Y}$, it is enough to consider only ones that are smooth over $\mathcal{Y}$.

1.4.1. Let $\mathcal{Y}$ be an k-Artin stack (see [Chapter I.2, Sect. 4.1]). We claim that in this case, there is a more concise expression for $\mathrm{QCoh}(\mathcal{Y})$.

Let $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y},\mathrm{sm}}$ denote the full subcategory of $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$ consisting of those $S \overset{y}{\to} \mathcal{Y}$, for which y is smooth (as a $(k-1)$-representable map).

Let $((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}})_{/\mathcal{Y}}$ be the 1-full subcategory of $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y},\mathrm{sm}}$, where we restrict maps $f : S' \to S$ to also be smooth.

We claim:

Proposition 1.4.2.
(a) *The natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \lim_{(S\xrightarrow{y}\mathcal{Y})\in((\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y},\mathrm{sm}})^{\mathrm{op}}} \mathrm{QCoh}(S)$$

is an equivalence.
(b) *The natural map*

$$\mathrm{QCoh}(\mathcal{Y}) \to \lim_{(S\xrightarrow{y}\mathcal{Y})\in(((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S)$$

is an equivalence.

Proof. Assume by induction that both statements are true for $k' < k$. The base case of $k = 0$ is obvious: in this case our $\mathcal{Y}$ is a disjoint union of affine schemes.

We are going to construct a map

$$\lim_{(S\xrightarrow{y}\mathcal{Y})\in(((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S) \to \mathrm{QCoh}(\mathcal{Y}), \tag{1.5}$$

inverse to the composition

$$\mathrm{QCoh}(\mathcal{Y}) \to \lim_{(S\xrightarrow{y}\mathcal{Y})\in((\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y},\mathrm{sm}})^{\mathrm{op}}} \mathrm{QCoh}(S) \to \lim_{(S\xrightarrow{y}\mathcal{Y})\in(((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S).$$

Let $f : \mathcal{Z} \to \mathcal{Y}$ be a smooth atlas, where $\mathcal{Z}$ is a $(k-1)$-Artin stack. By Corollary 1.3.11, the map

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Tot}\left(\mathrm{QCoh}(\mathcal{Z}^\bullet/\mathcal{Y})\right)$$

is an equivalence. Thus, the datum of a map in (1.5) is equivalent to a map

$$\lim_{(S\xrightarrow{y}\mathcal{Y})\in(((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S) \to \mathrm{Tot}\left(\mathrm{QCoh}(\mathcal{Z}^\bullet/\mathcal{Y})\right). \tag{1.6}$$

Note that the expression in the LHS of (1.6) equals the value on $\mathcal{Y}$ of

$$\mathrm{RKE}_{((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}})^{\mathrm{op}} \to ((\mathrm{Stk}^{k\text{-}\mathrm{Artn}})_{\mathrm{sm}})^{\mathrm{op}}}(\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}}|_{(\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}}}).$$

In the above formula, $(\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}}$ (resp., $(\mathrm{Stk}^{k\text{-}\mathrm{Artn}})_{\mathrm{sm}}$) denotes the 1-full subcategory of $\mathrm{Sch}^{\mathrm{aff}}$ (resp., $\mathrm{Stk}^{k\text{-}\mathrm{Artn}}$), where we restrict 1-morphisms to be smooth maps.

The validity of point (b) for $k-1$ is equivalent to the fact that the map

$$\mathrm{RKE}_{((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}})^{\mathrm{op}} \to ((\mathrm{Stk}^{(k-1)\text{-}\mathrm{Artn}})_{\mathrm{sm}})^{\mathrm{op}}}(\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}}|_{(\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{sm}}}) \to$$
$$\to \mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}}|_{(\mathrm{Stk}^{(k-1)\text{-}\mathrm{Artn}})_{\mathrm{sm}}}$$

is an isomorphism.

Hence, by the transitivity of the operation of the right Kan extension, we can rewrite the LHS of (1.6) as

$$\lim_{(\mathcal{Z}'\xrightarrow{y}\mathcal{Y})\in(((\mathrm{Stk}^{(k-1)\text{-}\mathrm{Artn}})_{\mathrm{sm}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(\mathcal{Z}').$$

Now, the required map

$$\lim_{(\mathcal{Z}'\xrightarrow{y}\mathcal{Y})\in(((\mathrm{Stk}^{(k-1)\text{-}\mathrm{Artn}})_{\mathrm{sm}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(\mathcal{Z}') \to \mathrm{Tot}\left(\mathrm{QCoh}(\mathcal{Z}^\bullet/\mathcal{Y})\right)$$

is given by restriction, since $\mathcal{Z}^\bullet/\mathcal{Y}$ is a simplicial object in $(\mathrm{Stk}^{(k-1)\text{-}\mathrm{Artn}})_{\mathrm{sm}}$.

□

1.4.3. Assume now that $\mathcal{Y} = Z \in \mathrm{Sch}$. Let $(\mathrm{Sch}^{\mathrm{aff}})_{Z,\mathrm{open}}$ denote the full subcategory of $(\mathrm{Sch}^{\mathrm{aff}})_{/Z}$, that consists of those $z : S \to Z$ for which z is an open embedding. Note that morphisms in this category automatically consist of open embeddings.

Then as in Proposition 1.4.2 we prove:

Proposition 1.4.4. *The natural map*

$$\mathrm{QCoh}(Z) \to \lim_{(S \overset{z}{\to} Z) \in ((\mathrm{Sch}^{\mathrm{aff}})_{/Z,\mathrm{open}})^{\mathrm{op}}} \mathrm{QCoh}(S)$$

is an equivalence.

1.5. **The t-structure.** For any prestack $\mathcal{Y}$, the category $\mathrm{QCoh}(\mathcal{Y})$ comes equipped with a t-structure. When $\mathcal{Y}$ is an Artin stack, this t-structure is quite explicit.

1.5.1. Let $\mathcal{Y}$ be an arbitrary prestack. We claim that the category $\mathrm{QCoh}(\mathcal{Y})$ carries a canonical t-structure. Namely, we declare that an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$ if for any $S \in \mathrm{Sch}^{\mathrm{aff}}$ and $S \overset{y}{\to} \mathcal{Y}$, the corresponding object $\mathcal{F}_{S,y} \in \mathrm{QCoh}(S)$ belongs to $\mathrm{QCoh}(S)^{\leq 0}$.

This indeed defines a t-structure (see [Lu2, Proposition 1.2.1.16]):

Since the subcategory $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$ is stable under colimits, by the Adjoint Functor Theorem, the embedding

$$\mathrm{QCoh}(\mathcal{Y})^{\leq 0} \hookrightarrow \mathrm{QCoh}(\mathcal{Y})$$

admits a right adjoint.

For a general prestack there is not much that one can say about this t-structure.

1.5.2. *An example.* Let $\mathcal{Y}$ be an affine scheme $\mathrm{Spec}(A)$. We have $\mathrm{QCoh}(\mathcal{Y}) = A\text{-mod}$, while

$$(A\text{-mod})^{\heartsuit} \simeq (H^0(A)\text{-mod})^{\heartsuit},$$

so heart of the t-structure depends in this case only on the underlying classical affine scheme.

1.5.3. Assume now that $\mathcal{Y}$ is a k-Artin stack. In this case one can give an explicit description of the t-structure on $\mathrm{QCoh}(\mathcal{Y})$ in terms of an atlas:

Proposition 1.5.4.
(a) *Let $\mathcal{Y}$ be a k-Artin stack and let $f_i : S_i \to \mathcal{Y}$ be a smooth atlas, where $S_i \in \mathrm{Sch}^{\mathrm{aff}}$. Then an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$ (resp., $\mathrm{QCoh}(\mathcal{Y})^{>0}$) if and only if each $f_i^*(\mathcal{F})$ belongs to $\mathrm{QCoh}(S_i)^{\leq 0}$ (resp., $\mathrm{QCoh}(S_i)^{>0}$).*
(b) *Let $\pi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a flat map between k-Artin stacks. Then the functor $\pi^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1)$ is t-exact.*

Remark 1.5.5. Since for an atlas $f_i : S_i \to \mathcal{Y}$ and $Z = \underset{i}{\sqcup} S_i$, the functor

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{QCoh}(Z^\bullet/\mathcal{Y})$$

is an equivalence, we obtain that point (a) is a particular case of point (b).

Proof. We will argue by induction, assuming that both statements are true for $k' < k$. Let us first prove point (a). It is enough to show that the functor f^* is compatible with the truncation functors.

Denote as above $Z = \underset{i}{\sqcup} S_i$. Let $\mathcal{F}$ be an object of $\mathrm{QCoh}(\mathcal{Y})$, and let

$$\mathcal{F}|_{Z^\bullet/\mathcal{Y}} \in \mathrm{QCoh}(Z^\bullet/\mathcal{Y})$$

be the corresponding object. We claim that

$$i \mapsto \tau^{\leq 0}(\mathcal{F}|_{Z^\bullet/\mathcal{Y}}) \text{ and } i \mapsto \tau^{>0}(\mathcal{F}|_{Z^\bullet/\mathcal{Y}})$$

both belong to $\mathrm{QCoh}(Z^\bullet/\mathcal{Y})$. This follows by the induction hypothesis from the fact that the face maps in the simplicial stack $Z^\bullet/\mathcal{Y}$ are flat.

It is clear that the object $\mathcal{F}' \in \mathrm{QCoh}(\mathcal{Y})$ that corresponds to $\tau^{\leq 0}(\mathcal{F}|_{Z^\bullet/\mathcal{Y}})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{\leq 0}$.

We claim now that the object $\mathcal{F}'' \in \mathrm{QCoh}(\mathcal{Y})$ that corresponds to $\tau^{>0}(\mathcal{F}|_{Z^\bullet/\mathcal{Y}})$ belongs to $\mathrm{QCoh}(\mathcal{Y})^{>0}$. Indeed, for $\mathcal{F}''' \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$, we have

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}''', \mathcal{F}') \simeq \mathrm{Tot}\left(\mathrm{Hom}_{\mathrm{QCoh}(Z^\bullet/\mathcal{Y})}(\mathcal{F}'''|_{Z^\bullet/\mathcal{Y}}, \tau^{>0}(\mathcal{F}|_{Z^\bullet/\mathcal{Y}}))\right),$$

and the right-hand side vanishes, since $\mathcal{F}'''|_{Z^\bullet/\mathcal{Y}} \in \mathrm{QCoh}(Z^\bullet/\mathcal{Y})^{\leq 0}$.

Let us now prove point (b). By point (a), we can assume that $\mathcal{Y}_1$ is an affine scheme T (replace the initial $\mathcal{Y}_1$ by its atlas). So, we are dealing with a flat map π from an affine scheme T to a k-Artin stack $\mathcal{Y} = \mathcal{Y}_2$. Let $f_i : S_i \to \mathcal{Y}$ be an atlas with $S_i \in \mathrm{Sch}^{\mathrm{aff}}$. Consider the Cartesian square:

$$\begin{array}{ccc} T \underset{\mathcal{Y}}{\times} S_i & \xrightarrow{\pi'} & S_i \\ {\scriptstyle f_i'}\downarrow & & \downarrow{\scriptstyle f_i} \\ T & \xrightarrow{\pi} & \mathcal{Y}. \end{array}$$

Again, by point (a), it is sufficient to show that the functor

$$f'^* \circ \pi^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(T \underset{\mathcal{Y}}{\times} S_i)$$

is exact. However, $f_i'^* \circ \pi^* \simeq \pi'^* \circ f_i^*$, and f_i^* is t-exact by point (a), and π'^* is t-exact by the induction hypothesis.

□

1.5.6. Proposition 1.5.4 has the following corollary:

Corollary 1.5.7. *Let $\mathcal{Y}$ be an Artin stack.*

(a) *The t-stucture on* $\mathrm{QCoh}(\mathcal{Y})$ *is compatible with filtered colimits, i.e., the truncation functors on* $\mathrm{QCoh}(\mathcal{Y})$ *are compatible with filtered colimits (or, equivalently, the subcategory* $\mathrm{QCoh}(\mathcal{Y})^{>0}$ *is closed under filtered colimits).*

(b) *The t-structure on* $\mathrm{QCoh}(\mathcal{Y})$ *is* left-complete *and* right-complete *, i.e., for* $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, *the natural maps*

$$\mathcal{F} \to \lim_{n \in \mathbb{N}} \tau^{\geq -n}(\mathcal{F})$$

$$\operatorname{colim}_{n \in \mathbb{N}} \tau^{\leq n}(\mathcal{F}) \to \mathcal{F}$$

are isomorphisms, where τ denotes the truncation functor.

Proof. Follows from Proposition 1.4.2(b) and the fact that both assertions are true for affine schemes, using the following lemma:

Lemma 1.5.8. *Let*

$$I \to \mathrm{DGCat}_{\mathrm{cont}}, \quad i \mapsto \mathbf{C}_i$$

be a diagram of DG categories and continuous functors. Assume that each $\mathbf{C}_i$ is endowed with a t-structure, and all of the transition functors $F_{i,j} : \mathbf{C}_i \to \mathbf{C}_j$ are t-exact. Set $\mathbf{C} = \lim_{i \in I} \mathbf{C}_i$. Then:

(a) *The category* $\mathbf{C}$ *acquires a unique t-structure such that the evaluation functors* $\mathrm{ev}_i : \mathbf{C} \to \mathbf{C}_i$ *are t-exact;*

(b) *If the t-structure on each* $\mathbf{C}_i$ *is compatible with filtered colimits, then so is the one on* $\mathbf{C}$*.*

(c) *If the t-structure on each* $\mathbf{C}_i$ *is right-complete, then so is the one on* $\mathbf{C}$*.*

(d) *If the t-structure on each* $\mathbf{C}_i$ *is left-complete, then so is the one on* $\mathbf{C}$*.*

□

Proof of Lemma 1.5.8. Only the last point is potentially non-obvious (because the transition functors $F_{i,j} : \mathbf{C}_i \to \mathbf{C}_i$ are not assumed to preserve limits). However, it follows from [Chapter I.1, Lemma 2.6.2].

□

2. Direct image for QCoh

So far we only know how the form the pullback of quasi-coherent sheaves for a map between prestacks. However, in order to have a richer theory, we should also develop the operation of *direct image.*

In general, the functor of direct image is quite ill-behaved. But there are exceptions: notably, when our morphism is schematic and quasi-compact. Or when one deals with Artin stacks and restricts oneself to the *eventually coconnective* (=bounded below) subcategory.

2.1. **The functor of direct image.**

Recall that the functor QCoh^* for affine schemes was obtained by passing to left adjoints from $\mathrm{QCoh}_{\mathrm{Sch}^{\mathrm{aff}}}$, the latter being functorial with respect to the operation of direct image.

For prestacks we apply an inverse procedure: to get $\mathrm{QCoh}_{\mathrm{PreStk}}$ we pass to right adjoints in $\mathrm{QCoh}^*_{\mathrm{PreStk}}$.

2.1.1. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in PreStk, and consider the corresponding functor

$$f^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1).$$

Applying Lurie's Adjoint Functor Theorem (see [Chapter I.1, Theorem 2.5.4]), we obtain that the above functor f^* admits a *discontinuous* right adjoint, denoted

$$f_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2).$$

Remark 2.1.2. In fact, using [Chapter A.3, Corollary 1.3.4], we obtain that the assignment

$$\mathcal{Y} \rightsquigarrow \mathrm{QCoh}(\mathcal{Y}), \quad (\mathcal{Y}_1 \overset{f}{\to} \mathcal{Y}_2) \rightsquigarrow (\mathrm{QCoh}(\mathcal{Y}_1) \overset{f_*}{\longrightarrow} \mathrm{QCoh}(\mathcal{Y}_2)) \in \mathrm{DGCat}$$

extends to a functor

$$\mathrm{QCoh} : \mathrm{PreStk} \to \mathrm{DGCat},$$

whose restriction to $\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{PreStk}$ is the composition of the functor $\mathrm{QCoh}_{\mathrm{Sch}^{\mathrm{aff}}}$ of (1.2) with the forgetful functor $\mathrm{DGCat}_{\mathrm{cont}} \to \mathrm{DGCat}$.

2.1.3. Let

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g'} & \mathcal{Y}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}'_2 & \xrightarrow{g} & \mathcal{Y}_2 \end{array} \tag{2.1}$$

be a Cartesian square in PreStk. By adjunction, we obtain a natural transformation, known as the *base change morphism*

$$g^* \circ f_* \to f'_* \circ g'^*. \tag{2.2}$$

However, in general, (2.2) is *not* an isomorphism.

The simplest counter-example is provided by $\mathcal{Y}_2 = \mathrm{pt}$, $\mathcal{Y}'_2 = \mathbb{A}^1$ and $\mathcal{Y}_1$ be a countable disjoint union of copies of pt.

Remark 2.1.4. The failure of the isomorphism (2.2) says that in general, the functor f_* is difficult to calculate. Concretely, for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_1)$ and $(S \stackrel{y}{\to} \mathcal{Y}_2) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}_2}$ we do *not* have an explicit expression for $(f_*(\mathcal{F}))_{S,y} \in \mathrm{QCoh}(S)$.

2.2. **Direct image for schematic morphisms.**

2.2.1. Above we saw that the direct image functor for a general morphism between prestacks does not have good properties. However, the situation improves considerably if we consider the class of *schematic quasi-compact morphisms*, see [Chapter I.2, Sect. 3.6.1 and 4.1.9] for what this means:

Proposition 2.2.2. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks. Assume that f is schematic and quasi-compact.*

(a) *The functor f_* is continuous.*

(b) *The base change property is satisfied, i.e., for any diagram* (2.1)*, the map* (2.2) *is an isomorphism.*

Proof. Note that in order to prove point (b), it is enough to consider the case when

$$\mathcal{Y}'_2 = Z_2 \in \mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{Sch}_{\mathrm{qc}},$$

where the super-script "qc" means quasi-compact. In this case $\mathcal{Y}'_1 =: Z_1$ is also an object of $\mathrm{Sch}_{\mathrm{qc}}$, by [Chapter I.2, Proposition 3.6.2].

Note that from the transitivity of the procedure of right Kan extension, for a prestack $\mathcal{Y}$, the map

$$\mathrm{QCoh}(\mathcal{Y}) \to \lim_{(Z \to \mathcal{Y}) \in ((\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(Z)$$

is an equivalence.

Note also that the functor

$$(Z_2 \to \mathcal{Y}_2) \in \mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{Y}_2} \quad \rightsquigarrow Z_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \in \mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{Y}_1}$$

is cofinal. Indeed, it admits a left adjoint given by

$$(Z_1 \to \mathcal{Y}_1) \mapsto (Z_1 \to \mathcal{Y}_1 \to \mathcal{Y}_2).$$

Hence, the functor

$$\mathrm{QCoh}(\mathcal{Y}_1) \to \lim_{(Z_2 \to \mathcal{Y}_2) \in ((\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(Z_2 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1)$$

is an equivalence.

Hence, applying [Chapter I.1, Lemma 2.6.2 and 2.6.4], we obtain that it suffices to prove that the functor f_* is continuous for a morphism between quasi-compact schemes

$$W \xrightarrow{f} Z$$

and that the natural transformation (2.2) is an isomorphism when all the prestacks involved are quasi-compact schemes

$$\begin{array}{ccc} W' & \xrightarrow{g'} & W \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ Z' & \xrightarrow{g} & Z \end{array}$$

Applying [Chapter I.1, Lemma 2.6.2 and 2.6.4] again, and using the fact that the functor QCoh^* satisfies Zariski descent, we can assume that Z and Z' are affine. We will prove the assertion by induction on the number of affines by which we can cover W.

The base of the induction thus is when W (and hence also W') is affine. In this case, the functor f_* is the same functor as in (1.2), and hence is continuous. To check the isomorphism (2.2), it is enough to do so in the generator of $\mathrm{QCoh}(Z)$, i.e., on $\mathcal{O}_Z$, and this is a tautology.

Let now $W = U_1 \cup U_2$; denote $U_{1,2} := U_1 \underset{W}{\times} U_2$. Denote

$$f_1 := f|_{U_1},\ f_2 := f|_{U_2},\ f_{1,2} := f|_{U_{1,2}}.$$

By the induction hypothesis, we can assume that the assertion of the proposition holds for the morphisms $f_1, f_2, f_{1,2}$. However, it is easy to see that the functor f_* can be explicitly described as

$$\mathcal{F} \mapsto (f_1)_*(\mathcal{F}|_{U_1}) \underset{(f_{1,2})_*(\mathcal{F}|_{U_{1,2}})}{\times} (f_2)_*(\mathcal{F}|_{U_2}),$$

and this implies the required assertion for f_*.

□

Remark 2.2.3. The following strengthening of Proposition 2.2.2 is established in [DrGa1, Corollary 1.4.5]:

Instead of requiring that $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be schematic quasi-compact, it suffices to ask that the base change of f be an affine scheme yields a *QCA algebraic stack*, see [DrGa1, Definition 1.1.8] for what this means.

2.2.4. Let us denote by $\mathrm{PreStk}_{\mathrm{sch,qc}}$ the 1-subcategory of PreStk where we restrict 1-morphisms to be schematic and quasi-compact.

Consider the functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}}} : \mathrm{QCoh}^*_{\mathrm{PreStk}}|_{\mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}}} : (\mathrm{PreStk}_{\mathrm{sch,qc}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Combining Proposition 2.2.2(a) and [Chapter A.3, Corollary 1.3.4] for the target $(\infty, 2)$-category $\mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}$, we obtain that by passing to right adjoints we can obtain from the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}}}$ a canonically defined functor

$$\mathrm{QCoh}_{\mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}}} : \mathrm{PreStk}_{\mathrm{sch,qc}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

By construction, the restriction of $\mathrm{QCoh}_{\mathrm{PreStk}_{\mathrm{sch,qc\text{-}qs}}}$ to $\mathrm{Sch}^{\mathrm{aff}}$ is the functor $\mathrm{QCoh}_{\mathrm{Sch}^{\mathrm{aff}}}$ of (1.2).

2.3. **Direct image for a map between Artin stacks.** Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between Artin stacks. In general, the functor $f_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2)$ will still be discontinuous. But the situation improves if one restricts one's attention to the bounded below subcategory.

2.3.1. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between Artin stacks. Assume that π is quasi-compact and quasi-separated (see [Chapter I.2, Sect. 4.1.9] for what this means).

We have:

Proposition 2.3.2.

(a) *The restriction $f_*|_{\mathrm{QCoh}(\mathcal{Y}_1)^{\geq 0}}$ maps to $\mathrm{QCoh}(\mathcal{Y}_2)^{\geq 0}$, and commutes with filtered colimits.*

(b) *Let*

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{g'} & \mathcal{Y}_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}'_2 & \xrightarrow{g} & \mathcal{Y}_2 \end{array}$$

be a Cartesian square, where the morphism g is flat. Then the diagram of functors

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Y}'_1)^+ & \xleftarrow{g'^*} & \mathrm{QCoh}(\mathcal{Y}_1)^+ \\ {\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{QCoh}(\mathcal{Y}'_2)^+ & \xleftarrow{g^*} & \mathrm{QCoh}(\mathcal{Y}_2)^+ \end{array}$$

is commutative.

Proof. Let f be k-representable. We argue by induction on k, assuming that the assertion is true for $k' < k$. We will prove point (a); point (b) is proved similarly.

The base of the induction (i.e., the case of $k = 0$) follows from Proposition 2.2.2(a).

Let

$$Z \to \mathcal{Y}_1$$

be a smooth (or even flat) atlas with $Z \in \mathrm{Sch}$. Let f^i denote the composition

$$Z^i/\mathcal{Y}_1 \to \mathcal{Y}_1 \xrightarrow{f} \mathcal{Y}_2.$$

By Corollary 1.3.11, for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_1)$, we have:

$$f_*(\mathcal{F}) \simeq \mathrm{Tot}\left(f^\bullet_*(\mathcal{F}|_{Z^\bullet/\mathcal{Y}_1})\right).$$

Note that each f_i is $(k-1)$-representable, quasi-compact and quasi-separated.

By the induction hypothesis, for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_1)^{\geq 0}$, each term of the co-simplicial object

$$i \mapsto f^i_*(\mathcal{F}|_{Z^i/\mathcal{Y}_1}) \tag{2.3}$$

is in $\mathrm{QCoh}(\mathcal{Y}_2)^{\geq 0}$. Hence, so is $\mathrm{Tot}\left(f^\bullet_*(\mathcal{F}|_{Z^\bullet/\mathcal{Y}_1})\right)$.

Recall that the t-structure on $\mathrm{QCoh}(\mathcal{Y}_2)$ is right-complete (see Corollary 1.5.7(b)). Hence, in order to show that that $f_*|_{\mathrm{QCoh}(\mathcal{Y}_1)^{\geq 0}}$ commutes with filtered colimits, it is enough to do so for its composition with the truncation functor

$$\tau^{\leq m} : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_2)^{\leq m}$$

for every $m \geq 0$.

Note, however, that since the terms of $f^\bullet_*(\mathcal{F}|_{Z^\bullet/\mathcal{Y}_1})$ belong to $\mathrm{QCoh}(\mathcal{Y}_2)^{\geq 0}$, we have

$$\tau^{\leq m}\left(\mathrm{Tot}\left(f^\bullet_*(\mathcal{F}|_{Z^\bullet/\mathcal{Y}_1})\right)\right) \simeq \tau^{\leq m}\left(\mathrm{Tot}^{\leq m+1}\left(f^\bullet_*(\mathcal{F}|_{Z^\bullet/\mathcal{Y}_1})\right)\right),$$

where $\mathrm{Tot}^{\leq m+1}$ denotes the limit over the subcategory $\mathbf{\Delta}^{\leq m+1} \subset \mathbf{\Delta}$.

Now, $\mathrm{Tot}^{\leq m}$ is a *finite* limit, and hence it preserves filtered colimits.

□

2.4. **Classical algebraic stacks.** In this subsection for $\mathcal{Y}$ a *classical algebraic stack*, we relate the category $\mathrm{QCoh}(\mathcal{Y})$ to some (potentially) more familiar notion.

2.4.1. According to [Lu2, Sect. 1.3.3], for any cocomplete stable ∞-category $\mathbf{C}$, equipped with a left and right complete t-structure, there is a canonically defined functor

$$D(\mathbf{C}^\heartsuit) \to \mathbf{C},$$

where $D(-)$ denotes the *derived stable* ∞*-category*, attached to a given abelian category, see [Lu2, Sect. 1.3.2]. In general, this functor is very far from being an equivalence.

In particular, for any Artin stack $\mathcal{Y}$ we obtain a canonical t-exact functor

$$D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit) \to \mathrm{QCoh}(\mathcal{Y}).$$

2.4.2. Assume now that $\mathcal{Y}$ is a quasi-compact and quasi-separated algebraic stack (i.e., a 1-Artin stack), and assume that it is classical (see [Chapter I.2, Sect. 4.4.4]) for what this means.

Proposition 2.4.3. *Under the above circumstances, the functor* $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)^+ \to \mathrm{QCoh}(\mathcal{Y})^+$ *is an equivalence.*

Remark 2.4.4. The above proposition implies that, under the specified assumptions, the category $\mathrm{QCoh}(\mathcal{Y})$ identifies with the left-completion of $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)$. We do not know what are the general conditions that guarantee that $D(\mathrm{QCoh}(\mathcal{Y})^\heartsuit)$ itself is left-complete. For example, this is true for quasi-compact schemes. It is also easy to see that this is true for algebraic stacks of the form Z/G, where Z is a quasi-projective DG scheme and G an algebraic group acting linearly on Z (recall that we are working over a field of characteristic 0).

Proof of Proposition 2.4.3. The proof will follow from the following general lemma:

Lemma 2.4.5. *Let* $\mathbf{C}$ *be a DG category equipped with a t-structure compatible with filtered colimits, and which is right-complete. Assume that for every object* $\mathbf{c} \in \mathbf{C}^\heartsuit$ *there exists an injection* $\mathbf{c} \to \mathbf{c}_0$*, where* $\mathbf{c}_0$ *is an injective object in* $\mathbf{C}^\heartsuit$*, and such that* $\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}', \mathbf{c}_0[n]) = 0$ *for* $n > 0$ *and all* $\mathbf{c}' \in \mathbf{C}^\heartsuit$*. Then the natural functor*

$$D(\mathbf{C}^\heartsuit)^+ \to \mathbf{C}^+$$

is an equivalence.

We apply this lemma to $\mathbf{C} = \mathrm{QCoh}(\mathcal{Y})$. Let $f : S \to \mathcal{Y}$ be a map, where S is a classical affine scheme. Since the diagonal morphism of $\mathcal{Y}$ is affine, the map f itself is affine. Hence, by Proposition 1.5.4(a), if $\mathcal{F}_S \in \mathrm{QCoh}(S)^\heartsuit$, then $f_*(\mathcal{F}_S) \in \mathrm{QCoh}(\mathcal{Y})^\heartsuit$. Moreover, by Proposition 1.5.4(b), if f is flat and $\mathcal{F}_S \in \mathrm{QCoh}(S)^\heartsuit$ is injective, we have

$$\mathrm{Hom}_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{F}', f_*(\mathcal{F}_S)[n]) = 0, \ \forall\, \mathcal{F}' \in \mathrm{QCoh}(\mathcal{Y})^\heartsuit, \ \forall\, n > 0.$$

If $\mathcal{F}$ is an object of $\mathrm{QCoh}(\mathcal{Y})^\heartsuit$, let $f : S \to \mathcal{Y}$ be a flat atlas with $S \in \mathrm{Sch}^{\mathrm{aff}}$. Since $\mathcal{Y}$ was assumed classical, S is classical as well. Choose an injective $f^*(\mathcal{F}) \hookrightarrow \mathcal{F}_S$, and $\mathcal{F}$ embed into $f_*(\mathcal{F}_S)$.

□

Remark 2.4.6. The same proof shows that (the homotopy category of) $D(\mathrm{QCoh}(\mathcal{Y})^{\heartsuit})^{+}$ identifies with the eventually coconnective part of the quasi-coherent derived category of $\mathcal{Y}$ as defined in [LM].

3. The symmetric monoidal structure

In this section we will study the symmetric monoidal structure on QCoh^* as a functor, and the symmetric monoidal structure on $\mathrm{QCoh}(\mathcal{Y})$ as a category for a given prestack $\mathcal{Y}$.

3.1. **The symmetric monoidal structure on** QCoh **as a functor.** In this subsection we return to the setting of Sect. 1.1. We will show that the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ has a natural *right-lax symmetric monoidal structure*.

3.1.1. First, according to [Chapter I.1, Sect. 8.5.10], the functor

$$A \mapsto A\text{-mod}, \quad (\mathrm{AssocAlg}(\mathrm{Vect}))^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

has a natural symmetric monoidal structure, where AssocAlg(Vect) is viewed as a symmetric monoidal category via the operation of tensor product of algebras, and $\mathrm{DGCat}_{\mathrm{cont}}$ is viewed as a symmetric monoidal category via the Lurie tensor product.

Composing with the forgetful functors

$$\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}) \to \mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{AssocAlg}(\mathrm{Vect}),$$

we obtain that the functor

$$(\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}))^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad A \mapsto A\text{-mod} \tag{3.1}$$

has a natural symmetric monoidal structure. Note that according to [Chapter I.1, Sect. 3.6.6], the symmetric monoidal structure on $(\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}))^{\mathrm{op}}$ is *Cartesian*.

3.1.2. Thus, we obtain that the functor

$$\mathrm{QCoh}_{\mathrm{Sch}^{\mathrm{aff}}} : \mathrm{Sch}^{\mathrm{aff}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

has a naturally defined symmetric monoidal structure, where $\mathrm{Sch}^{\mathrm{aff}}$ is endowed with the Cartesian symmetric monoidal structure.

Applying [Chapter V.3, Sect. 3.1] (in the simplest case of *vert* = *horiz* = *adm* = all, *co-adm* = isom) we obtain that the functor

$$\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

also acquires a symmetric monoidal structure, where the symmetric monoidal structure on $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ is induced by the Cartesian symmetric monoidal structure on $\mathrm{Sch}^{\mathrm{aff}}$.

3.1.3. Finally, applying [Chapter V.3, Sect. 3.2] (in the simplest case of *vert* = *adm* = isom) we obtain that the functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : \mathrm{PreStk}^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

acquires a *right-lax* symmetric monoidal structure, where the symmetric monoidal structure on $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ is induced by the Cartesian symmetric monoidal structure on PreStk.

3.1.4. In concrete terms, the meaning of the above construction is that for $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}$ we have a canonically defined functor

$$\mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2), \tag{3.2}$$

denoted

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathcal{F}_1 \boxtimes \mathcal{F}_2.$$

By construction, the functor (3.2) is an equivalence if $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{Sch}^{\mathrm{aff}}$.

3.1.5. The functor (3.2) can be explicitly described as follows.

$$\mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(\mathcal{Y}_2) \simeq \left(\underset{S_1 \overset{y_1}{\to} \mathcal{Y}_1}{\underset{\longleftarrow}{\lim}} \mathrm{QCoh}(S_1) \right) \otimes \left(\underset{S_2 \overset{y_2}{\to} \mathcal{Y}_2}{\underset{\longleftarrow}{\lim}} \mathrm{QCoh}(S_2) \right), \tag{3.3}$$

whereas

$$\mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2) \simeq \underset{S \overset{y}{\to} \mathcal{Y}_1 \times \mathcal{Y}_2}{\underset{\longleftarrow}{\lim}} \mathrm{QCoh}(S).$$

However, the functor

$$(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}_1} \times (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}_2} \to (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}, \quad S_1, S_2 \mapsto S_1 \times S_2$$

is cofinal, so we can rewrite

$$\mathrm{QCoh}(\mathcal{Y}_1 \times \mathcal{Y}_2) \simeq \lim_{S_1 \overset{y_1}{\to} \mathcal{Y}_1, S_2 \overset{y_2}{\to} \mathcal{Y}_2} \mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2). \tag{3.4}$$

Now, the map (3.2) is the tautological map from (3.3) to (3.4) (swapping the limit with the tensor product).

3.1.6. We claim:

Proposition 3.1.7. *Assume that $\mathcal{Y}_1$ is such that the category $\mathrm{QCoh}(\mathcal{Y}_1)$, viewed as an object of $\mathrm{DGCat}_{\mathrm{cont}}$, is dualizable. Then for any $\mathcal{Y}_2$, the functor (3.2) is an equivalence.*

Proof. We need to show that the map from (3.3) to (3.4) is an isomorphism. We can write it as a composition

$$\mathrm{QCoh}(\mathcal{Y}_1) \otimes \left(\lim_{S_2 \overset{y_2}{\to} \mathcal{Y}_2} \mathrm{QCoh}(S_2) \right) \to \lim_{S_2 \overset{y_2}{\to} \mathcal{Y}_2} \mathrm{QCoh}(\mathcal{Y}_1) \otimes \mathrm{QCoh}(S_2) \simeq$$
$$\simeq \lim_{S_2 \overset{y_2}{\to} \mathcal{Y}_2} \left(\left(\lim_{S_1 \overset{y_1}{\to} \mathcal{Y}_1} \mathrm{QCoh}(S_1) \right) \otimes \mathrm{QCoh}(S_2) \right) \to$$
$$\to \lim_{S_2 \overset{y_2}{\to} \mathcal{Y}_2} \left(\lim_{S_1 \overset{y_1}{\to} \mathcal{Y}_1} (\mathrm{QCoh}(S_1) \otimes \mathrm{QCoh}(S_2)) \right).$$

The first arrow is an isomorphism since tensoring with a dualizable category commutes with limits (see [Chapter I.1, Sect. 4.3.2]). The third arrow is an isomorphism for the same reason, as $\mathrm{QCoh}(S)$ for an affine scheme S is dualizable.

□

3.2. **The symmetric monoidal structure on** QCoh **of a prestack.** In this subsection we will use the right-lax symmetric monoidal structure on the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ to construct a symmetric monoidal structure on each $\mathrm{QCoh}(\mathcal{Y})$ for $\mathcal{Y} \in \mathrm{PreStk}$.

3.2.1. Being right-lax symmetric monoidal, the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ sends commutative algebra objects in $\mathrm{PreStk}^{\mathrm{op}}$ to commutative algebra objects on $\mathrm{DGCat}_{\mathrm{cont}}$.

However, since the symmetric monoidal structure on $\mathrm{PreStk}^{\mathrm{op}}$ is coCartesian, the forgetful functor

$$\mathrm{ComAlg}(\mathrm{PreStk}^{\mathrm{op}}) \to \mathrm{PreStk}^{\mathrm{op}}$$

is an equivalence (see [Lu2, Corollary 2.4.3.10]).

We obtain that the functor $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ naturally lifts to a functor

$$\mathrm{PreStk}^{\mathrm{op}} \to \mathrm{ComAlg}(\mathrm{DGCat}_{\mathrm{cont}}) =: \mathrm{DGCat}^{\mathrm{SymMon}}_{\mathrm{cont}}.$$

3.2.2. In other words, for any $\mathcal{Y} \in \mathrm{PreStk}$, the DG category $\mathrm{QCoh}(\mathcal{Y})$ acquires a canonical symmetric monoidal structure, explicitly given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathcal{F}_1 \otimes \mathcal{F}_2 := (\mathrm{diag}_{\mathcal{Y}})^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

Furthermore, for a morphism $f : \mathcal{Y}' \to \mathcal{Y}$ in PreStk, the functor

$$f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}')$$

is naturally symmetric monoidal.

3.2.3. Consider again the direct image functor

$$f_* : \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y}).$$

Being a right adjoint to a symmetric monoidal functor, the functor f_* is *right-lax symmetric monoidal* (this is the commutative version of [Chapter I.1, Lemma 3.2.4]).

In particular, for $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, $\mathcal{F}' \in \mathrm{QCoh}(\mathcal{Y}')$, we have a canonically defined map

$$\mathcal{F} \otimes f_*(\mathcal{F}') \to f_*(f^*(\mathcal{F}) \otimes \mathcal{F}')), \tag{3.5}$$

called the *projection formula map.*

In general, the map (3.5) is *not* an isomorphism. However, as in Proposition 2.2.2, one shows:

Lemma 3.2.4. *Assume that f is schematic quasi-compact. Then the map* (3.5) *is an isomorphism.*

3.2.5. Let $f : \mathcal{Y}' \to \mathcal{Y}$ be again a map between prestacks. Since the functor f_* is right-lax (symmetric) monoidal, it maps algebras in $\mathrm{QCoh}(\mathcal{Y}')$ to algebras in $\mathrm{QCoh}(\mathcal{Y})$.

In particular, the object

$$f_*(\mathcal{O}_{\mathcal{Y}'}) = \mathcal{A} \in \mathrm{QCoh}(\mathcal{Y})$$

has a natural structure of commutative algebra.

Moreover, by [Chapter I.1, Sect. 3.7.3], the functor f_* naturally factors as

$$\mathrm{QCoh}(\mathcal{Y}') \to \mathcal{A}\text{-mod}(\mathrm{QCoh}(\mathcal{Y})) \xrightarrow{\mathbf{oblv}_{\mathcal{A}}} \mathrm{QCoh}(\mathcal{Y}).$$

In general, the above functor

$$\mathrm{QCoh}(\mathcal{Y}') \to \mathcal{A}\text{-mod}(\mathrm{QCoh}(\mathcal{Y})) \tag{3.6}$$

is *not* an equivalence.

3.2.6. Let

$$\begin{array}{ccc} \mathcal{Y}'_1 & \longrightarrow & \mathcal{Y}_1 \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ \mathcal{Y}'_2 & \xrightarrow{g} & \mathcal{Y}_2 \end{array}$$

be a Cartesian square of prestacks.

Note that we have a canonical map:

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}'_2) \to \mathrm{QCoh}(\mathcal{Y}'_1). \tag{3.7}$$

In general, the functor (3.7) is *not* an equivalence. Here is a counter-example:

Take $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathrm{pt}$, and $\mathcal{Y} = \mathrm{pt}/A$, where A is an abelian variety. Then $\mathcal{Y}_1 \underset{\mathcal{Y}}{\times} \mathcal{Y}_2 \simeq A$, while

$$\mathrm{QCoh}(\mathrm{pt}/A) \simeq H\text{-mod},$$

where $H = (\Gamma(A, \mathcal{O}_A))^\vee$ is an algebra with respect to convolution, and is isomorphic to $\mathrm{Sym}(H^1(X, \mathcal{O}_A)^\vee[1])$. So

$$\mathrm{Vect} \underset{H\text{-mod}}{\otimes} \mathrm{Vect} \simeq \mathrm{Sym}(H^1(X, \mathcal{O}_A)^\vee[2])\text{-mod}.$$

3.3. **The quasi-affine case.**

3.3.1. We shall say that an object $X \in \mathrm{Sch}$ is *quasi-affine* if it quasi-compact and admits an open embedding into an affine scheme.

We shall say that a morphism $f : \mathcal{Y}' \to \mathcal{Y}$ in PreStk is *quasi-affine* if its base change by an affine scheme yields a quasi-affine scheme.

3.3.2. We claim:

Proposition 3.3.3. *Let $f : \mathcal{Y}' \to \mathcal{Y}$ be quasi-affine. Then the functor* (3.6) *is an equivalence.*

Proof. Let first f be arbitrary. We note that we have a canonical homomorphism of monads acting on $\mathrm{QCoh}(\mathcal{Y})$.

$$\mathcal{A} \otimes - \to f_* \circ f^*.$$

Assume now that f is schematic and quasi-compact. In this case, by Lemma 3.2.4, the above map of monads is an isomorphism. Hence, in order to prove the proposition, it remains to show that the functor f_* satisfies the hypothesis of the Barr-Beck-Lurie theorem, see [Chapter I.1, Proposition 3.7.7].

Now, since f was assumed schematic quasi-compact, the functor f_* commutes with all colimits, by Proposition 2.2.2(a). Thus, it remains to show that f_* is conservative. By Proposition 2.2.2(b), the latter assertion reduces to the case when $\mathcal{Y} \in \mathrm{Sch}^{\mathrm{aff}}$.

Thus, it remains to show that the functor of *global sections* on QCoh of a quasi-affine scheme X is conservative. Let $j : X \hookrightarrow S$ be an open embedding, where $S \in \mathrm{Sch}^{\mathrm{aff}}$. Since the functor of global sections over S is conservative, it remains to show that the functor j_* is fully faithful.

However, we claim that j_* admits a left inverse, namely, j^*. Indeed, this follows from Proposition 2.2.2(b) for the Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{j} & S. \end{array}$$

□

3.3.4. Here is another favorable feature of quasi-affine maps:

Proposition 3.3.5. *Assume in the situation of Sect. 3.2.6, the map f (and hence f') is quasi-affine. Then then map* (3.7) *is an equivalence.*

Proof. By Proposition 3.3.3
$$\mathrm{QCoh}(\mathcal{Y}'_1) \simeq f'_*(\mathcal{O}_{\mathcal{Y}'_1})\text{-mod}(\mathrm{QCoh}(\mathcal{Y}'_2)).$$

Now, again by Proposition 3.3.3 and [Chapter I.1, Corollary 8.5.7],
$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}'_2) \simeq f_*(\mathcal{O}_{\mathcal{Y}_1})\text{-mod}(\mathrm{QCoh}(\mathcal{Y}_2)) \underset{\mathrm{QCoh}(\mathcal{Y}_2)}{\otimes} \mathrm{QCoh}(\mathcal{Y}'_2) \simeq$$
$$\simeq g^*(f_*(\mathcal{O}_{\mathcal{Y}_1}))\text{-mod}(\mathrm{QCoh}(\mathcal{Y}'_2)).$$

Finally, by Proposition 2.2.2(2),
$$f'_*(\mathcal{O}_{\mathcal{Y}'_1}) \simeq g^*(f_*(\mathcal{O}_{\mathcal{Y}_1}))$$
as algebras in $\mathrm{QCoh}(\mathcal{Y}'_2)$.

□

3.4. **When is** QCoh **rigid?**

3.4.1. Recall the notion of *rigid* stable monoidal category, [Chapter I.1, Sect. 9.1].

The following assertion provides a partial converse to Proposition 3.1.7:

Proposition 3.4.2. *Let $\mathcal{Y}$ be a prestack, such that the diagonal map* $\mathrm{diag}_{\mathcal{Y}}$ *is schematic and quasi-compact, and such that the object $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ is compact. Then the following conditions are equivalent:*

- (i*) The functor* $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}') \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}')$ *is an equivalence for any $\mathcal{Y}'$.*
- (ii) *The functor* $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y})$ *is an equivalence.*
- (iii) *The category $\mathcal{Y}$ is rigid as a stable monoidal category.*
- (iv) *The category* $\mathrm{QCoh}(\mathcal{Y})$ *is dualizable.*

Proof. The implicatios (i) ⇒ (ii) is tautological, and the implication (iii) ⇒ (iv) follows froom [Chapter I.1, Sect. 9.2.1].

The implication (iv) ⇒ (i) is the content of Proposition 3.1.7. It remains to show (ii) ⇒ (iii).

Given (ii), we can identify the map $\mathrm{mult}^*_{\mathrm{QCoh}(\mathcal{Y})}$ (we are using the notation of [Chapter I.1, Sect. 9.1.1]) with
$$(\mathrm{diag}_{\mathcal{Y}})^* : \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}).$$
The fact that it satisfies the assumptions of *loc. cit.* follows from Proposition 2.2.2(b).

□

3.4.3. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between prestacks, such that both $\mathrm{QCoh}(\mathcal{Y}_1)$ and $\mathrm{QCoh}(\mathcal{Y}_2)$ are rigid. From [Chapter I.1, Lemma 9.2.6(2)] we obtain:

Lemma 3.4.4. *The functor $f_* : \mathrm{QCoh}(\mathcal{Y}_1) \to \mathrm{QCoh}(\mathcal{Y}_2)$ is continuous, and under the identifications*
$$\mathrm{QCoh}(\mathcal{Y}_i)^\vee \simeq \mathrm{QCoh}(\mathcal{Y}_i),$$
we have $f_* \simeq (f^*)^\vee$.

3.5. **Passable prestacks.**

3.5.1. We shall say that a prestack $\mathcal{Y}$ is *passable* if

(1) The diagonal morphism of $\mathcal{Y}$ is quasi-affine;
(2) $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ is compact;
(3) The category $\mathrm{QCoh}(\mathcal{Y})$ is dualizable.

For example, any stack which is perfect (see Sect. 3.7.1) below is passable. In particular, any quasi-compact scheme is passable when viewed as a prestack.

By Proposition 3.4.2, we obtain that if $\mathcal{Y}$ is passable, then $\mathrm{QCoh}(\mathcal{Y})$ is rigid as a monoidal category.

3.5.2. We are going to show that passable prestacks are adapted to having the map in (3.7) an equivalence:

Let $\mathcal{Y}$ be a passable prestack. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be prestacks mapping to $\mathcal{Y}$.

Proposition 3.5.3. *If under the above circumstances* $\mathrm{QCoh}(\mathcal{Y}_1)$ *is dualizable as a category, the natural functor*

$$\mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1 \underset{\mathcal{Y}}{\times} \mathcal{Y}_2)$$

is an equivalence.

Proof. By [Chapter I.1, Propositions 9.4.4 and Sect. 4.3.2], the rigidity of $\mathrm{QCoh}(\mathcal{Y})$ and the fact that $\mathrm{QCoh}(\mathcal{Y}_1)$ is dualizable imply that the operation

$$\mathbf{C} \mapsto \mathrm{QCoh}(\mathcal{Y}_1) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}$$

preserves limits.

This allows to replace $\mathcal{Y}_2$ by $S \in \mathrm{Sch}^{\mathrm{aff}}$. But then the map $S \to \mathcal{Y}$ is quasi-compact and quasi-affine, and we find ourselves in the situation of Proposition 3.3.5.

□

3.6. **The perfect subcategory.**

3.6.1. Recall the notion of *dualizable* object in a symmetric monoidal category, see [Chapter I.1, Sect. 4.1].

For a prestack $\mathcal{Y}$, we let

$$\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}} \subset \mathrm{QCoh}(\mathcal{Y})$$

denote the full subcategory consisting of dualizable objects.

For a map $f : \mathcal{Y}' \to \mathcal{Y}$, the functor $f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}')$ clearly sends $\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$ to $\mathrm{QCoh}(\mathcal{Y}')^{\mathrm{perf}}$.

Thus, we obtain a well-defined functor

$$\mathcal{Y} \mapsto \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}, \quad \mathrm{PreStk}^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}\,. \tag{3.8}$$

3.6.2. We have the following basic assertion:

Lemma 3.6.3. *Let*

$$I \to 1\text{-}\mathrm{Cat}^{\mathrm{Mon}}, \quad i \mapsto \mathbf{A}_i$$

be a functor, and denote $\mathbf{A} := \lim_i \mathbf{A}_i$. *Then an object* $\mathbf{a} \in \mathbf{A}$ *is left dualizable if and only if* $\mathrm{ev}_i(\mathbf{a}) \in \mathbf{A}_i$ *is dualizable for every* i.

As a corollary we obtain:

Corollary 3.6.4.

(a) *An object* $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ *is perfect if and only if for every* $(S \xrightarrow{y} Y) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$, *the corresponding* $\mathcal{F}_{S,y} \in \mathrm{QCoh}(S)$ *is perfect.*

(b) *The functor* (3.8) *maps isomorphically to the right Kan extension to its restriction to* $\mathrm{Sch}^{\mathrm{aff}}$.

Moreover, combing with Theorem 1.3.4, we obtain:

Corollary 3.6.5. *The restriction of the functor* (3.8) *to* $\mathrm{Sch}^{\mathrm{aff}}$ *satisfies flat descent.*

3.6.6. *Perfectness and compactness.* Let S be an affine DG scheme. We recall the following (this is a particular case of [Chapter I.1, Corollary 9.1.7]):

Lemma 3.6.7. *For* $M \in \mathrm{QCoh}(S)$ *the following conditions are equivalent:*

(i) M *is compact;*

(ii) M *is dualizable as an object of the symmetric monoidal category* $\mathrm{QCoh}(S)$.

We now claim:

Proposition 3.6.8.

(1) *Suppose that the diagonal morphism of* $\mathcal{Y}$ *is schematic and quasi-compact. Then any compact object of* $\mathrm{QCoh}(\mathcal{Y})$ *is perfect.*

(2) *Suppose that* $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ *is compact. Then any perfect object of* $\mathrm{QCoh}(\mathcal{Y})$ *is compact.*

Proof. Point (2) follows from [Chapter I.1, Lemma 8.8.4]: dualizability implies compactness in any symmetric monoidal stable category in which the unit is compact.

To prove point (1), taking into account Lemma 3.6.7, we have to show that the functor $\mathcal{F} \mapsto f^*(\mathcal{F})$ for $f : S \to \mathcal{Y}$ with S affine, sends compact objects to compact ones. However, this is true, since the right adjoint of f^*, i.e., the functor f_* is continuous, by Proposition 2.2.2(a). □

3.6.9. Finally, we note:

Proposition 3.6.10. *The functor*

$$S \mapsto \mathrm{QCoh}(S)^{\mathrm{perf}}, \quad (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}$$

is convergent.

Proof. By definition, we need to show that for $S \in \mathrm{Sch}^{\mathrm{aff}}$, the family of functors

$$\mathcal{F} \mapsto \mathcal{F}|_{\leq n S},$$

given by restriction, defines an equivalence

$$\mathrm{QCoh}(S)^{\mathrm{perf}} \to \lim_n \mathrm{QCoh}({}^{\leq n}S)^{\mathrm{perf}}.$$

However, we claim that more is true: namely, the functor

$$\mathrm{QCoh}(S)^{-} \to \lim_n \mathrm{QCoh}({}^{\leq n}S)^{-}$$

is an equivalence[1]. Indeed, its inverse is given by sending a compatible family $\mathcal{F}_n \in \mathrm{QCoh}({}^{\leq n}S)^-$ to

$$\lim_n (i_n)_*(\mathcal{F}_n),$$

where i_n denotes the tautological map ${}^{\leq n}S \to S$.

□

3.7. **Perfect prestacks.**

3.7.1. Following [BFN], we shall say that a prestack $\mathcal{Y}$ is perfect if

(1) The diagonal morphism of $\mathcal{Y}$ is affine;
(2) The functor $\mathrm{Ind}(\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}) \to \mathrm{QCoh}(\mathcal{Y})$ is an equivalence.

3.7.2. By Proposition 3.6.8, the above conditions can be reformulated as follows:

(1) The diagonal morphism of $\mathcal{Y}$ is affine,
(2) $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$ is compact,
(3) $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated.

3.7.3. Since every compactly generated category is dualizable, we obtain that every perfect stack is passable, see Sect. 3.5.1 for what this means.

3.7.4. *Examples.* In [BFN], following the arguments of [TT] and [Ne], it is shown that any quasi-compact scheme, considered as a prestack, is perfect.

Moreover, in [BFN] it is shown that if $\mathcal{Y}$ is of the form X/G, where G is an algebraic group and X is a scheme endowed with a G-equivariant ample line bundle, then $\mathcal{Y}$ is perfect (under our assumption that the ground field k is of char. 0).

[1]Note, however, that the corresponding fact is *false* for all of QCoh.