CHAPTER IV.2. LIE ALGEBRAS AND CO-COMMUTATIVE CO-ALGEBRAS

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Introduction

0.1. **Why does this chapter exist?** Only a small portion of this chapter consists of original mathematics: if anything, it would be Theorem 6.1.2 (that expresses the functor of universal enveloping Lie algebra in terms of the Chevalley functor), and perhaps also Theorem 2.9.4 (that computes primitives in 'fake' co-free co-algebras).

Our main intention in writing this chapter was to provide a reference point for [Chapter IV.3], where we will study the relation between moduli problems and Lie algebras.

0.1.1. The main actors in our study of Lie algebras will be the pair of mutually adjoint functors

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coChev}^{\mathrm{enh}} \tag{0.1}$$

that connect Lie algebras and augmented co-commutative co-algebras in a given symmetric monoidal category $\mathbf{O}$. (In our applications in the subsequent chapters we will take $\mathbf{O} = \mathrm{IndCoh}(\mathcal{X})$, where $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, equipped with the $\overset{!}{\otimes}$ symmetric monoidal structure.)

The difficulty here (and what makes the game interesting) is that the above functors are *not* fully faithful, but they are close to being such.

For example, we conjecture that the unit and the co-unit of the adjunction

$$\mathrm{Id} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}} \text{ and } \mathrm{Chev}^{\mathrm{enh}} \circ \mathrm{coChev}^{\mathrm{enh}} \to \mathrm{Id}$$

become isomorphisms when evaluated on the essential image of $\mathrm{coChev}^{\mathrm{enh}}$ and $\mathrm{Chev}^{\mathrm{enh}}$, respectively.

We will now describe the two main results of this chapter.

0.1.2. One is Theorem 4.2.4, which is a particular case of the more general Theorem 2.9.4. It says that the unit of the adjunction

$$\mathrm{Id} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}}$$

is an isomorphism, when evaluated on any trivial Lie algebra.

As a consequence we deduce (see Theorem 4.4.6) that if we precompose the Chevalley functor with the loop functor

$$\Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})),$$

and view the result as a functor

$$\mathrm{Grp}(\mathrm{coChev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}),$$

the latter will be fully faithful.

A key observation here is that for a Lie algebra $\mathfrak{h}$, if we view $\Omega_{\mathrm{Lie}}(\mathfrak{h})$ again as a mere Lie algebra (i.e., disregard the Lie algebra structure), then it will be canonically trivialized (see Proposition 1.7.2). The latter result is true for any operad[1].

0.1.3. The second main result of this chapter is Theorem 6.1.2:

It says that the functor

$$\mathrm{Grp}(\mathrm{coChev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}),$$

considered above identifies canonically with the functor that assigns to a Lie algebra its universal enveloping algebra, considered as a co-commutative Hopf algebra.

0.2. **What else is done in this chapter?**

0.2.1. In Sect. 1 we give an overview of the general theory of algebras over operads.

We show that for a given operad $\mathcal{P}$, a $\mathcal{P}$-algebra B can be canonically lifted to *non-negatively filtered* $\mathcal{P}$-algebra B^{Fil}, such that its associated graded is trivial. This construction implies that many functors from the category of $\mathcal{P}$-algebras admit filtered versions, whose associated graded is easy to control.

In addition, we prove the above-mentioned fact that the loop functor followed by the forgetful functor on the category of $\mathcal{P}$-algebras canonically produces *trivial* $\mathcal{P}$-algebras. As an application we give a simple proof of the fact that the stabilization of the category of $\mathcal{P}$-algebras (in a symmetric monoidal DG category $\mathbf{O}$) identifies with $\mathbf{O}$ itself.

0.2.2. In Sect. 2 we review the theory of Koszul duality between algebras over an operad and co-algebras over the Koszul dual operad.

One of the key points is that there are two *inequivalent* notions of co-algebra over a co-operad. One is the usual notion of co-algebra (which in the example of the co-commutative co-operad corresponds to augmented co-commutative co-algebras). And another is that of *ind-nilpotent* co-algebra. There is a naturally defined functor (denoted $\mathbf{res}^{\star\to *}$) from the category of the latter (denoted $\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$) to the category of the former (denoted $\mathcal{Q}\text{-Coalg}(\mathbf{O})$), and we conjecture that this functor is fully faithful.

The forgetful functor $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}} : \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathbf{O}$ admits a right adjoint, denoted $\mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}}$. Composing with the functor

$$\mathbf{res}^{\star\to *} : \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathcal{Q}\text{-Coalg}(\mathbf{O}),$$

[1] In this generality we learned this fact, along with its proof, from M. Kontsevich.

we obtain the functor that we denote by

$$\mathbf{cofree}^{\text{fake}}_{\mathcal{Q}} : \mathbf{O} \to \mathcal{Q}\text{-Coalg}(\mathbf{O}).$$

For example, for $\mathcal{Q} = \text{Cocom}^{\text{aug}}$, the functor $\mathbf{cofree}^{\text{fake}}_{\mathcal{Q}}$ is the functor of symmetric co-algebra $V \mapsto \text{Sym}(V)$.

If we knew that the functor $\mathbf{res}^{\star\to *}$ was fully faithful, we would know that for $V, W \in \mathbf{O}$ the composite map

$$\begin{aligned}(0.2)\quad & \text{Maps}_{\mathbf{O}}(W,V) \simeq \text{Maps}_{\mathbf{O}}(\mathbf{oblv}^{\text{ind-nilp}}_{\mathcal{Q}} \circ \mathbf{triv}^{\text{ind-nilp}}_{\mathcal{Q}}(W), V) \simeq \\ & \simeq \text{Maps}_{\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})}(\mathbf{triv}^{\text{ind-nilp}}_{\mathcal{Q}}(W), \mathbf{cofree}^{\text{ind-nilp}}_{\mathcal{Q}}(V)) \to \\ & \to \text{Maps}_{\mathcal{Q}\text{-Coalg}(\mathbf{O})}(\mathbf{res}^{\star\to *} \circ \mathbf{triv}^{\text{ind-nilp}}_{\mathcal{Q}}(W), \mathbf{res}^{\star\to *} \circ \mathbf{cofree}^{\text{ind-nilp}}_{\mathcal{Q}}(V)) \simeq \\ & \simeq \text{Maps}_{\mathcal{Q}\text{-Coalg}(\mathbf{O})}(\mathbf{triv}_{\mathcal{Q}}(W), \mathbf{cofree}^{\text{fake}}_{\mathcal{Q}}(V))\end{aligned}$$

is an isomorphism.

Unfortunately, we do not know whether $\mathbf{res}^{\star\to *}$ is fully faithful. However, we prove, and this is one of the key technical assertions, that for a certain class of co-operads (that includes $\text{Cocom}^{\text{aug}}$ and $\text{Coassoc}^{\text{aug}}$) that map in (0.2) is an isomorphism. This is Theorem 2.9.4.

0.2.3. In Sect. 3 we specialize the context of Koszul duality to the case of associative algebras.

0.2.4. In Sect. 4 we prove Theorem 4.4.6, mentioned above, which says that the functor

$$\text{Grp}(\text{coChev}^{\text{enh}}) \circ \Omega_{\text{Lie}} : \text{LieAlg}(\mathbf{O}) \to \text{CocomBialg}(\mathbf{O}),$$

is fully faithful.

We study the functor

$$\text{CocomBialg}(\mathbf{O}) \overset{\text{Monoid}(\text{coChev}^{\text{enh}})}{\longrightarrow} \text{Monoid}(\text{LieAlg}(\mathbf{O})) \simeq \text{Grp}(\text{LieAlg}(\mathbf{O})) \overset{B_{\text{Lie}}}{\longrightarrow} \text{LieAlg}(\mathbf{O}),$$

right adjoint to $\text{Grp}(\text{coChev}^{\text{enh}}) \circ \Omega_{\text{Lie}}$.

We show that it fits into a commutative diagram

$$\begin{array}{ccc} \text{CocomBialg}(\mathbf{O}) & \xrightarrow{\mathbf{oblv}_{\text{Assoc}}} & \text{CocomCoalg}^{\text{aug}}(\mathbf{O}) \\ {\scriptstyle B_{\text{Lie}}\circ\text{Monoid}(\text{coChev}^{\text{enh}})}\downarrow & & \downarrow{\scriptstyle \text{Prim}_{\text{Cocom}^{\text{aug}}}} \\ \text{LieAlg}(\mathbf{O}) & \xrightarrow{\mathbf{oblv}_{\text{Lie}}} & \mathbf{O}. \end{array}$$

I.e., we obtain that when we apply the functor

$$\text{Prim}_{\text{Cocom}^{\text{aug}}} : \text{CocomCoalg}^{\text{aug}}(\mathbf{O}) \to \mathbf{O}$$

to an object of $\text{CocomBialg}(\mathbf{O})$, the result has a natural structure of Lie algebra.

This can be regarded as an 'ultimate explanation' of why the tangent space to a Lie group at the origin has a structure of Lie algebra (one that does not use explicit formulas).

0.2.5. In Sect. 5 we recall the basic constructions associated with the functor of universal enveloping algebra of a Lie algebra.

In Sect. 6 we prove the second main result of this chapter, described in Sect. 0.1.3 above.

In Sect. 7 we give an interpretation of an equivalence

$$\mathfrak{h}\text{-mod} \simeq U(\mathfrak{h})\text{-mod}$$

(here $\mathfrak{h}$ is a Lie algebra) in terms of the incarnation of $U^{\mathrm{Hopf}}(\mathfrak{h})$ as

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}).$$

0.2.6. In Sect. A we prove Theorem 2.9.4 described in Sect. 0.2.1.

In Sect. B, we prove the PBW theorem in the setting of higher algebra.

0.2.7. In Sect. C we address the following issue: co-commutative bialgebras can be defined in two ways: as associative algebras in the category of co-commutative co-algebras or as co-commutative co-algebras in the category of associative algebras.

In the setting of higher algebra it is not obvious that these two definitions lead to the same object. However, in Proposition C.1.3 we prove that they in fact do.

1. Algebras over operads

In this section, we review the general theory of algebras over operads.

For the purposes of this chapter, we will regard operads as algebras in the category of symmetric sequences. We review this notion in Sect. 1.1.

In this section, we also review the notions of filtered and graded objects in a DG category. We show that algebras over operads have a canonical filtration and, as a result, various functors on the category of algebras over an operad obtain canonical filtrations.

Finally, for an operad $\mathcal{P}$, we consider group objects in the category of $\mathcal{P}$-algebras. We show that the underlying $\mathcal{P}$-algebra of a group object in the category of $\mathcal{P}$-algebras is *canonically* a trivial $\mathcal{P}$-algebra.

1.1. **Operads and algebras.** In this subsection we introduce operads and algebras over them (in a given DG category).

1.1.1. Let Vect^{Σ} denote the category of symmetric sequences. As a DG category, we have:

$$\mathrm{Vect}^{\Sigma} := \prod_{n\geq 1} \mathrm{Rep}(\Sigma_n),$$

i.e., consists of objects

$$\mathcal{P} := \{\mathcal{P}(n) \in \mathrm{Rep}(\Sigma_n),\ n \geq 1\}.$$

The category Vect^{Σ} has a canonical symmetric monoidal structure such that it is the free symmetric monoidal DG category on a single object. It follows by the $(\infty, 2)$-categorical Yoneda lemma [Chapter A.2, Proposition 6.3.7] that Vect^{Σ} is the category of endomorphisms of the functor

$$\mathrm{DGCat}^{\text{2-Cat,SymMon}}_{\mathrm{cont}} \to \mathbf{1\text{-}Cat}$$

Hence, the category Vect^{Σ} is endowed with another natural (non-symmetric) monoidal structure, called the *composition monoidal structure*, corresponding to composition of functors. The unit object

$$\mathbf{1}_{\mathrm{Vect}^{\Sigma}} \in \mathrm{Vect}^{\Sigma}$$

is the one given by

$$\mathbf{1}_{\mathrm{Vect}^\Sigma}(1) = k, \quad \mathbf{1}_{\mathrm{Vect}^\Sigma}(n) = 0 \text{ for } n > 1.$$

Let $\mathbf{O}$ be a symmetric monoidal DG category. The category $\mathbf{O}$ is then a module category for Vect^Σ (with the composition monoidal structure). Explicitly, given an object $\mathcal{P} \in \mathrm{Vect}^\Sigma$ and $V \in \mathbf{O}$, the action of $\mathcal{P}$ on V is given by the formula

$$\mathcal{P} \star V := \underset{n\geq 1}{\oplus} \left(\mathcal{P}(n) \otimes V^{\otimes n}\right)_{\Sigma_n}.$$

1.1.2. A (unital) operad is by definition a unital associative algebra in Vect^Σ with respect to the composition monoidal structure.

Convention: Unless explicitly stated otherwise, we will only consider operads $\mathcal{P}$, for which the unit map defines an isomorphism $k \to \mathcal{P}(1)$. In particular, such operads, viewed as associative algebras in Vect^Σ, are automatically augmented.

1.1.3. For an operad $\mathcal{P} \in \mathrm{AssocAlg}(\mathrm{Vect}^\Sigma)$, the category of $\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})$ of $\mathcal{P}$-algebras in $\mathbf{O}$ is by definition the category $\mathcal{P}\text{-mod}(\mathbf{O})$.

We shall denote by

$$\mathbf{free}_{\mathcal{P}} : \mathbf{O} \rightleftarrows \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) : \mathbf{oblv}_{\mathcal{P}}$$

the resulting pair of adjoint functors.

The functor $\mathbf{oblv}_{\mathcal{P}}$ is conservative, and being a right adjoint, it preserves limits.

The composite functor $\mathbf{oblv}_{\mathcal{P}} \circ \mathbf{free}_{\mathcal{P}}$ is given by

$$V \mapsto \mathcal{P} \star V = \underset{n\geq 1}{\oplus} \left(\mathcal{P}(n) \otimes V^{\otimes n}\right)_{\Sigma_n}.$$

In particular, it preserves *sifted* colimits. Thus, the monad on $\mathbf{O}$, defined by $\mathcal{P}$, preserves sifted colimits. Hence, the forgetful functor $\mathbf{oblv}_{\mathcal{P}}$ also preserves *sifted* colimits.

1.1.4. The augmentation on $\mathcal{P}$ gives rise to a functor

$$\mathbf{triv}_{\mathcal{P}} : \mathbf{O} \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}),$$

which is a right inverse on $\mathbf{oblv}_{\mathcal{P}}$.

1.1.5. We will consider the following operads: $\mathrm{Assoc}^{\mathrm{aug}}$, $\mathrm{Com}^{\mathrm{aug}}$ and Lie. By definition

$$\mathrm{Assoc}^{\mathrm{aug}}(n) = k^{\Sigma_n}, \quad \mathrm{Com}^{\mathrm{aug}}(n) = k.$$

By definition

$$\mathrm{Assoc}^{\mathrm{aug}}\text{-}\mathrm{Alg}(\mathbf{O}) =: \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \text{ and } \mathrm{Com}^{\mathrm{aug}}\text{-}\mathrm{Alg}(\mathbf{O}) =: \mathrm{ComAlg}^{\mathrm{aug}}(\mathbf{O})$$

are the categories of unital augmented (equivalently, non-unital) associative and commutative algebras in $\mathbf{O}$, respectively[2].

We will also consider the operad Lie; this is the classical Lie operad, where we set by definition $\mathrm{Lie}(1) = k$. We have

$$\mathrm{Lie}\text{-}\mathrm{Alg}(\mathbf{O}) =: \mathrm{LieAlg}(\mathbf{O});$$

this the category of Lie algebras in $\mathbf{O}$.

[2]Note that in the interpretation as augmented algebras, the forgetful functor $\mathbf{oblv}_{\mathcal{P}}$ corresponds to taking the augmentation ideal.

1.2. **Tensoring a $\mathcal{P}$-algebra by a commutative algebra.** Let $\mathfrak{h}$ be a Lie algebra and A is a commutative algebra. Then the vector space $\mathfrak{h} \otimes A$ has a canonical structure of a Lie algebra given by $[h_1 \otimes a_1, h_2 \otimes a_2] = [h_1, h_2] \otimes (a_1 \cdot a_2)$.

In this subsection, we describe the following generalization of this construction. Let $\mathbf{O}$ be a symmetric monoidal category, and let A be a commutative algebra in $\mathbf{O}$. For an operad $\mathcal{P}$, let B be a $\mathcal{P}$-algebra in $\mathcal{O}$. We will show that the object $A \otimes B$ has a canonical structure of a $\mathcal{P}$-algebra.

Remark 1.2.1. This construction has the following generalization (which we will not need in the sequel). The category of operads has a symmetric monoidal structure characterized by the property that if A is a $\mathcal{P}$-algebra and B is a $\mathcal{Q}$-algebra then $A \otimes B$ is a $(\mathcal{P} \otimes \mathcal{Q})$-algebra. The commutative operad is the unit object for this symmetric monoidal structure.

1.2.2. Let $\Phi : \mathbf{O} \to \mathbf{O}'$ be a homomorphism of symmetric monoidal DG categories. Then Φ induces a (strict) functor between module categories for the monoidal category Vect^{Σ}.

In particular, for any operad $\mathcal{P}$, the functor Φ induces a functor

$$\Phi : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}') \tag{1.1}$$

that makes the diagrams

$$\begin{array}{ccc} \mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\Phi} & \mathcal{P}\text{-Alg}(\mathbf{O}') \\ {\scriptstyle \mathbf{oblv}_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathcal{P}}} \\ \mathbf{O} & \xrightarrow{\Phi} & \mathbf{O}' \end{array} \tag{1.2}$$

and

$$\begin{array}{ccc} \mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\Phi} & \mathcal{P}\text{-Alg}(\mathbf{O}') \\ {\scriptstyle \mathbf{free}_{\mathcal{P}}}\uparrow & & \uparrow{\scriptstyle \mathbf{free}_{\mathcal{P}}} \\ \mathbf{O} & \xrightarrow{\Phi} & \mathbf{O}' \end{array} \tag{1.3}$$

commute.

1.2.3. Consider the right adjoint[3] Φ^R of Φ (where we view the latter as a functor between mere DG categories).

The functor Φ^R has a natural structure of *right-lax functor* of module categories over Vect^{Σ}. In particular, it induces a functor

$$\Phi^R : \mathcal{P}\text{-Alg}(\mathbf{O}') \to \mathcal{P}\text{-Alg}(\mathbf{O}),$$

right adjoint to (1.1).

By passing to right adjoints in (1.3), we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{P}\text{-Alg}(\mathbf{O}) & \xleftarrow{\Phi^R} & \mathcal{P}\text{-Alg}(\mathbf{O}') \\ {\scriptstyle \mathbf{oblv}_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathcal{P}}} \\ \mathbf{O} & \xleftarrow{\Phi^R} & \mathbf{O}'. \end{array} \tag{1.4}$$

[3]Here we do not even need to require that this right adjoint be continuous.

1.2.4. Let now A be a commutative algebra in $\mathbf{O}$. Set $\mathbf{O}' := A\text{-mod}(\mathbf{O})$. The composition

$$\Phi^R \circ \Phi : \mathbf{O} \to \mathbf{O}$$

is the functor of tensor product by A.

By the above, this functor admits a natural structure of *right-lax functor* of module categories over Vect^Σ. In particular, we obtain a well-defined functor

$$A \otimes - : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}),$$

that makes the diagram

$$\begin{array}{ccc} \mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{A\otimes -} & \mathcal{P}\text{-Alg}(\mathbf{O}') \\ {\scriptstyle \mathbf{oblv}_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathcal{P}}} \\ \mathbf{O} & \xrightarrow{A\otimes -} & \mathbf{O}' \end{array}$$

commute.

1.2.5. Note that the construction

$$A \rightsquigarrow A \otimes - : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}) \tag{1.5}$$

is functorial in A, so we obtain a functor

$$\text{ComAlg}(\mathbf{O}) \times \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}).$$

For the sequel, we note the following:

Lemma 1.2.6. *The functor* (1.5) *commutes with finite limits in each variable.*

Proof. It is enough to prove the assertion after applying the functor $\mathbf{oblv}_{\mathcal{P}}$, and then it becomes obvious, because the functor

$$- \otimes - : \mathbf{O} \times \mathbf{O} \to \mathbf{O}$$

commutes with finite limits in each variable.

□

1.3. **Digression: filtered and graded objects.** In this subsection we will make a digression and fix some notation pertaining to filtered and graded objects in a DG category.

1.3.1. For a DG category $\mathbf{C}$, we let $\mathbf{C}^{\text{Fil}}$ (resp., $\mathbf{C}^{\text{Fil},\geq 0}$, $\mathbf{C}^{\text{Fil},\leq 0}$) denote the category of filtered (resp., non-negatively filtered, non-positively filtered) objects. By definition,

$$\mathbf{C}^{\text{Fil}} := \text{Funct}(\mathbb{Z}, \mathbf{C}), \quad \mathbf{C}^{\text{Fil},\geq 0} := \text{Funct}(\mathbb{Z}^{\geq 0}, \mathbf{C}), \quad \mathbf{C}^{\text{Fil},\leq 0} := \text{Funct}(\mathbb{Z}^{\leq 0}, \mathbf{C}),$$

where $\mathbb{Z}$ is viewed as an ordered set and hence a category.

We have the natural restriction functors

$$\mathbf{C}^{\text{Fil},\geq 0} \leftarrow \mathbf{C}^{\text{Fil}} \to \mathbf{C}^{\text{Fil},\leq 0}.$$

The above functors both admit left adjoints, given by left Kan extension. The essential image of $\mathbf{C}^{\text{Fil},\geq 0}$ in $\mathbf{C}^{\text{Fil}}$ consists of functors sending the negative integers to 0. Then essential image of $\mathbf{C}^{\text{Fil},\leq 0}$ in $\mathbf{C}^{\text{Fil}}$ consists of functors that take the constant value on $\mathbb{Z}^{\geq 0}$. Thus, we obtain the usual embeddings

$$\mathbf{C}^{\text{Fil},\geq 0} \hookrightarrow \mathbf{C}^{\text{Fil}} \hookleftarrow \mathbf{C}^{\text{Fil},\leq 0}.$$

The functor of 'forgetting the filtration'

$$\mathbf{oblv}_{\text{Fil}} : \mathbf{C}^{\text{Fil}} \to \mathbf{C}$$

is by definition the functor

$$\underset{\mathbb{Z}}{\operatorname{colim}} : \operatorname{Funct}(\mathbb{Z}, \mathbf{C}) \to \mathbf{C}.$$

1.3.2. Consider also the category

$$\mathbf{C}^{\mathrm{gr}} := \mathbf{C}^{\mathbb{Z}},$$

and its subcategories

$$\mathbf{C}^{\mathrm{gr},\geq 0} \subset \mathbf{C}^{\mathrm{gr}} \supset \mathbf{C}^{\mathrm{gr},\leq 0}.$$

We have the functor of forgetting the grading $\mathbf{oblv}_{\mathrm{gr}} : \mathbf{C}^{\mathrm{gr}} \to \mathbf{C}$, given by

$$\underset{\mathbb{Z}}{\oplus} : \mathbf{C}^{\mathbb{Z}} \to \mathbf{C}.$$

For $n \in \mathbb{Z}$ we let

$$(\deg = n) : \mathbf{C} \to \mathbf{C}^{\mathrm{gr}}$$

the functor that creates an object concentrated in degree n. Sometimes, we will also use the notation

$$V^{\mathrm{deg=n}} := (\deg = n)(V).$$

1.3.3. We have a canonically defined functor

$$(\mathrm{gr} \to \mathrm{Fil}) : \mathbf{C}^{\mathrm{gr}} \to \mathbf{C}^{\mathrm{Fil}}, \tag{1.6}$$

given by left Kan extension along

$$\mathbb{Z}^{\mathrm{Spc}} \to \mathbb{Z}. \tag{1.7}$$

(I.e., the target $\mathbb{Z}$ is considered as a category with respect to its natural order, while the source copy is considered as a groupoid.)

Explicitly, if an object of $\mathbf{C}^{\mathrm{gr}}$ is given by $n \rightsquigarrow V_n$, the corresponding object of $\mathbf{C}^{\mathrm{Fil}}$ is given by

$$n \rightsquigarrow \underset{n' \leq n}{\oplus} V_{n'}.$$

The functor $(\mathrm{gr} \to \mathrm{Fil})$ admits a right adjoint, denoted Rees, given by restriction along (1.7).

1.3.4. We now consider the functor of associated graded

$$\text{ass-gr} : \mathbf{C}^{\mathrm{Fil}} \to \mathbf{C}^{\mathrm{gr}},$$

given by

$$n \mapsto \operatorname{coFib}(V_{n-1} \to V_n).$$

It is a basic (and obvious) fact that the functor ass-gr is *conservative when restricted to* $\mathbf{C}^{\mathrm{Fil},\geq 0}$.

We have the following (evident) isomorphism of endo-functors of $\mathbf{C}^{\mathrm{gr}}$:

$$\text{ass-gr} \circ (\mathrm{gr} \to \mathrm{Fil}) \simeq \mathrm{Id}.$$

1.3.5. The above constructions are functorial with respect to $\mathbf{C}$. In particular, if $\mathbf{O}$ is a (symmetric) monoidal category, then so are $\mathbf{O}^{\mathrm{Fil}}$ and $\mathbf{O}^{\mathrm{gr}}$, and each of the functors

$$\text{ass-gr} : \mathbf{O}^{\mathrm{Fil}} \to \mathbf{O}^{\mathrm{gr}},\ (\mathrm{gr} \to \mathrm{Fil}) : \mathbf{O}^{\mathrm{gr}} \to \mathbf{O}^{\mathrm{Fil}} \text{ and } (\deg = 0) : \mathbf{O} \to \mathbf{O}^{\mathrm{gr}}$$

has a natural (symmetric) monoidal structure.

1.4. **Adding a filtration.** Suppose that A is an *augmented* associative algebra. In this case, A has a canonical filtration given by $A_n = 0$ for $n < 0$, $A_0 = k$ and $A_n = A$ for $n \geq 1$. The corresponding associated graded algebra is given by the square zero extension (i.e. trivial augmented associative algebra) $k \oplus A^+$, where A^+ is the augmentation ideas of A.

In this subsection, we describe a generalization of this construction. Namely, we show that any[4] $\mathcal{P}$-algebra has a canonical lift to a *filtered* $\mathcal{P}$-algebra such that the associated graded is the trivial $\mathcal{P}$-algebra. Roughly speaking, at the level of the corresponding Rees algebras, this construction amounts to scaling all the operations to zero.

This is a technically important tool as it allows to reduce many statements about $\mathcal{P}$-algebras to trivial $\mathcal{P}$-algebras.

1.4.1. Consider the commutative algebra $A := k \oplus k$; we endow it with an augmentation, given by projection on the first copy of k. We also endow it with a non-negative filtration by setting

$$A_n = \begin{cases} A \text{ for } n \geq 1 \\ k \text{ for } n = 0. \end{cases}$$

By Sect. 1.2, we can regard the assignment

$$B \mapsto A \otimes B$$

as a functor

$$\mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

Using the augmentation on A, we obtain a natural transformation

$$A \otimes B \to B.$$

Here we abuse the notation slightly, and denote simply by B the object of $\mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0})$ that should properly be denoted by $(\mathrm{gr} \to \mathrm{Fil})(B^{\deg=0})$.

1.4.2. We define the functor

$$\mathrm{AddFil} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0})$$

by:

$$B \mapsto \mathrm{Fib}(A \otimes B \to B) := (A \otimes B) \underset{B}{\times} \{0\}.$$

Sometimes, we will also use the notation

$$B^{\mathrm{Fil}} := \mathrm{AddFil}(B).$$

1.4.3. Since $\mathbf{oblv}_{\mathrm{Fil}}(A) = k \times k$, by Lemma 1.2.6, we obtain an isomorphism of functors:

$$\mathbf{oblv}_{\mathrm{Fil}}(A \otimes B) \simeq B \times B.$$

From here, we obtain that the isomorphism of functors

$$\mathbf{oblv}_{\mathrm{Fil}} \circ \mathrm{AddFil} \simeq \mathrm{Id}\,.$$

So, the assignment

$$B \rightsquigarrow B^{\mathrm{Fil}}$$

can be viewed as a canonical lift of $B \in \mathcal{P}\text{-Alg}(\mathbf{O})$ to an object of $\mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0})$.

[4] Recall our conventions for operads!

1.4.4. The following diagram commutes by construction:

$$\begin{array}{ccccc} \mathcal{P}\text{-Alg}(\mathbf{O}) & & \xrightarrow{\text{AddFil}} & & \mathcal{P}\text{-Alg}(\mathbf{O}^{\text{Fil},\geq 0}) \\ {\scriptstyle \mathbf{triv}_{\mathcal{P}}}\uparrow & & & & \uparrow{\scriptstyle \mathbf{triv}_{\mathcal{P}}} \\ \mathbf{O} & \xrightarrow{\text{deg}=1} & \mathbf{O}^{\text{gr},\geq 0} & \xrightarrow{(\text{gr}\to\text{Fil})} & \mathbf{O}^{\text{Fil},\geq 0}. \end{array}$$

The following diagram also commutes:

$$\begin{array}{ccccc} \mathcal{P}\text{-Alg}(\mathbf{O}) & & \xrightarrow{\text{AddFil}} & & \mathcal{P}\text{-Alg}(\mathbf{O}^{\text{Fil},\geq 0}) \\ {\scriptstyle \mathbf{oblv}_{\mathcal{P}}}\downarrow & & & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathcal{P}}} \\ \mathbf{O} & \xrightarrow{\text{deg}=1} & \mathbf{O}^{\text{gr},\geq 0} & \xrightarrow{(\text{gr}\to\text{Fil})} & \mathbf{O}^{\text{Fil},\geq 0}. \end{array}$$

1.4.5. We now claim:

Proposition 1.4.6. *The functor*

$$\text{ass-gr} \circ \text{AddFil} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}^{\text{gr},\geq 0})$$

is canonically isomorphic to $\mathbf{triv}_{\mathcal{P}} \circ (\deg = 1) \circ \mathbf{oblv}_{\mathcal{P}}$*, i.e.,*

$$\mathcal{P}\text{-Alg}(\mathbf{O}) \xrightarrow{\mathbf{oblv}_{\mathcal{P}}} \mathbf{O} \xrightarrow{(\deg=1)} \mathbf{O}^{\text{gr},\geq 0} \xrightarrow{\mathbf{triv}_{\mathcal{P}}} \mathcal{P}\text{-Alg}(\mathbf{O}^{\text{gr},\geq 0}).$$

Let us repeat the statement of Proposition 1.4.6 in words. It says that for $B \in \mathcal{P}\text{-Alg}(\mathbf{O})$, the associated graded of B^{Fil} is canonically trivial.

Proof. We need to show that the functor $\mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}^{\text{gr},\geq 0})$, given by

$$B \mapsto \text{Fib}(\text{ass-gr}(A) \otimes B \to B)$$

is canonically isomorphic to

$$B \mapsto \mathbf{triv}_{\mathcal{P}}(\mathbf{oblv}_{\mathcal{P}}(B)^{\deg=1}).$$

We will deduce this from a particular property of the canonical action of Vect^{Σ} on $\mathbf{O}$ from Sect. 1.1.1:

Note that $\text{ass-gr}(A) \simeq k[\epsilon]/\epsilon^2$, where $\deg(\epsilon) = 1$. Consider the functor

$$V \mapsto \text{Fib}(k[\epsilon]/\epsilon^2 \otimes V^{\deg=0} \to V^{\deg=0}), \quad \mathbf{O} \to \mathbf{O}^{\text{gr},\geq 0}, \tag{1.8}$$

as a right-lax functor of modules categories over Vect^{Σ}.

For a symmetric monoidal category $\mathbf{O}'$ let us denote by $\mathbf{O}'_{\text{triv}}$ the same DG category (i.e., $\mathbf{O}'$), but equipped with the *trivial* action of Vect^{Σ}, i.e., the action that factors through the projection on the degree 1 component $\text{Vect}^{\Sigma} \to \text{Vect}$. Note that the identity functor on $\mathbf{O}'$ can be made into a right-lax functor of modules categories over Vect^{Σ} for both

$$\mathbf{O}' \to \mathbf{O}'_{\text{triv}} \tag{1.9}$$

and

$$\mathbf{O}'_{\text{triv}} \to \mathbf{O}'. \tag{1.10}$$

With these notations, the observation is that the functor (1.8) canonically factors as a composition

$$\mathbf{O} \xrightarrow{(1.9)} \mathbf{O}_{\text{triv}} \xrightarrow{(\deg=1)} (\mathbf{O}^{\text{gr},\geq 0})_{\text{triv}} \xrightarrow{(1.10)} \mathbf{O}^{\text{gr},\geq 0}.$$

Indeed, this follows for reasons of degree since the functor (1.8) sends V to $V^{\deg=1}$.

□

1.5. **Filtered objects arising from $\mathcal{P}$-algebras.** The construction in this subsection expresses the following idea: many functors from the category of $\mathcal{P}$-algebras in $\mathbf{O}$ to $\mathbf{O}$ itself automatically lift to functors with values in the category of filtered objects in $\mathbf{O}$.

The typical examples of this phenomenon that we will consider are the functors of universal envelope or Chevalley complex of a Lie algebra (see Sect. 2.5 for the latter example).

1.5.1. Let $\mathcal{C}$ be a functor

$$\mathrm{DGCat}^{\mathrm{SymMon}} \to \infty\text{-Cat},$$

and let Φ be a natural transformation

$$\mathbf{O} \quad \rightsquigarrow \quad \mathcal{P}\text{-Alg}(\mathbf{O}) \overset{\Phi|_{\mathbf{O}}}{\to} \mathcal{C}(\mathbf{O}).$$

We observe that the natural transformation Φ automatically upgrades to a natural transformation, denoted Φ^{Fil},

$$\mathbf{O} \quad \rightsquigarrow \quad \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

Indeed, we let $\Phi^{\mathrm{Fil}}|_{\mathbf{O}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0})$ be the composition

$$\mathcal{P}\text{-Alg}(\mathbf{O}) \overset{\mathrm{AddFil}}{\longrightarrow} \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \overset{\Phi|_{\mathbf{O}^{\mathrm{Fil},\geq 0}}}{\longrightarrow} \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

1.5.2. Note that we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\Phi^{\mathrm{Fil}}|_{\mathbf{O}}} & \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Fil}}} \\
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\Phi|_{\mathbf{O}}} & \mathcal{C}(\mathbf{O}).
\end{array}$$

The next diagram commutes due to Proposition 1.4.6:

$$\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\Phi^{\mathrm{Fil}}|_{\mathbf{O}}} & \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \\
{\scriptstyle \mathbf{oblv}_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle \text{ass-gr}} \\
\mathbf{O} & & \mathcal{C}(\mathbf{O}^{\mathrm{gr},\geq 0}) \\
{\scriptstyle (\deg=1)}\downarrow & & \uparrow{\scriptstyle \Phi|_{\mathbf{O}^{\mathrm{gr},\geq 0}}} \\
\mathbf{O}^{\mathrm{gr},\geq 0} & \xrightarrow{\mathbf{triv}_{\mathcal{P}}} & \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{gr},\geq 0})
\end{array} \tag{1.11}$$

In addition, we have the following commutative diagram

$$\begin{array}{ccc}
\mathbf{O} & \xrightarrow{\mathbf{triv}_{\mathcal{P}}} & \mathcal{P}\text{-Alg}(\mathbf{O}) \\
{\scriptstyle (\deg=1)}\downarrow & & \downarrow{\scriptstyle \Phi^{\mathrm{Fil}}|_{\mathbf{O}}} \\
\mathbf{O}^{\mathrm{gr},\geq 0} & & \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \\
{\scriptstyle \mathbf{triv}_{\mathcal{P}}}\downarrow & & \uparrow{\scriptstyle \mathrm{gr}\to\mathrm{Fil}} \\
\mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{gr},\geq 0}) & \xrightarrow{\Phi|_{\mathbf{O}^{\mathrm{gr},\geq 0}}} & \mathcal{C}(\mathbf{O}^{\mathrm{gr},\geq 0}).
\end{array}$$

1.6. **Group objects in the category of $\mathcal{P}$-algebras.** In this subsection we show that the category of $\mathcal{P}$-algebras has the feature that the functors of taking the loop space and the classifying space of a group-object are mutually inverse equivalences of categories.

1.6.1. Consider the categories

$$\mathrm{Grp}(\mathcal{P}\text{-Alg}(\mathbf{O})) \subset \mathrm{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O})).$$

We claim:

Lemma 1.6.2. *The inclusion* $\mathrm{Grp}(\mathcal{P}\text{-Alg}(\mathbf{O})) \subset \mathrm{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))$ *is an equivalence.*

Proof. The inclusion $\mathrm{Grp}(\mathbf{C}) \subset \mathrm{Monoid}(\mathbf{C})$ is an equivalence for any pointed category $\mathbf{C}$, for which a map $\mathbf{c}_1 \to \mathbf{c}_2$ is an isomorphism whenever $\mathbf{c}_1 \underset{\mathbf{c}_2}{\times} * \to *$ is:

Namely, recall that a monoid object $\mathbf{c} \in \mathbf{C}$ is a group object if and only if the map

$$(p_1, m) : \mathbf{c}_1 := \mathbf{c} \times \mathbf{c} \to \mathbf{c} \times \mathbf{c} =: \mathbf{c}_2$$

is an isomorphism. However, if $\mathbf{C}$ is pointed, the canonical map $* \to \mathbf{c}$ is the unit; therefore, the natural map $\mathbf{c}_1 \underset{\mathbf{c}_2}{\times} * \to *$ is an isomorphism.

□

1.6.3. Consider now the pair of adjoint functors:

$$B_{\mathcal{P}} : \mathrm{Grp}(\mathcal{P}\text{-Alg}(\mathbf{O})) \rightleftarrows \mathcal{P}\text{-Alg}(\mathbf{O}) : \Omega_{\mathcal{P}}. \tag{1.12}$$

We claim:

Proposition 1.6.4. *The functors* (1.12) *are mutually inverse equivalences.*

Proof. We have to show that the natural transformations

$$\mathrm{Id} \to \Omega_{\mathcal{P}} \circ B_{\mathcal{P}} \text{ and } B_{\mathcal{P}} \circ \Omega_{\mathcal{P}} \to \mathrm{Id}$$

are isomorphisms.

It is enough to show that the resulting natural transformations

$$\mathbf{oblv}_{\mathcal{P}} \to \mathbf{oblv}_{\mathcal{P}} \circ \Omega_{\mathcal{P}} \circ B_{\mathcal{P}} \text{ and } \mathbf{oblv}_{\mathcal{P}} \circ B_{\mathcal{P}} \circ \Omega_{\mathcal{P}} \to \mathbf{oblv}_{\mathcal{P}}$$

are isomorphisms.

The following diagram commutes tautologically

$$\begin{array}{ccc} \mathrm{Grp}(\mathcal{P}\text{-Alg}(\mathbf{O})) & \xleftarrow{\Omega_{\mathcal{P}}} & \mathcal{P}\text{-Alg}(\mathbf{O}) \\ {\scriptstyle \mathbf{oblv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathrm{Grp}}} \downarrow & & \downarrow {\scriptstyle \mathbf{oblv}_{\mathcal{P}}} \\ \mathbf{O} & \xleftarrow{[-1]} & \mathbf{O}, \end{array}$$

because the functor $\mathbf{oblv}_{\mathcal{P}}$ commutes with limits.

The next diagram, obtained from one above by passing to left adjoints along the horizontal arrows,

$$\begin{array}{ccc} \mathrm{Grp}(\mathcal{P}\text{-Alg}(\mathbf{O})) & \xrightarrow{B_{\mathcal{P}}} & \mathcal{P}\text{-Alg}(\mathbf{O}) \\ {\scriptstyle \mathbf{oblv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathrm{Grp}}} \downarrow & & \downarrow {\scriptstyle \mathbf{oblv}_{\mathcal{P}}} \\ \mathbf{O} & \xrightarrow{[1]} & \mathbf{O} \end{array}$$

also commutes, because $\mathbf{oblv}_{\mathcal{P}}$ commutes with sifted colimits.

This implies the required assertion.

□

1.7. **Forgetting the group structure.** In this subsection we show the following: if we consider a group-object of the category of $\mathcal{P}$-algebras, and forget the group structure, then the resulting $\mathcal{P}$-algebra is *canonically* trivial.

1.7.1. We will prove:

Proposition 1.7.2. *The composite functor*

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathcal{P}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O})$$

is canonically isomorphic to

$$\mathbf{triv}_{\mathcal{P}} \circ [-1] \circ \mathbf{oblv}_{\mathcal{P}}.$$

Combining with Proposition 1.6.4, we obtain:

Corollary 1.7.3. *The functor*

$$\mathbf{oblv}_{\mathrm{Grp}} : \mathrm{Grp}(\mathcal{P}\text{-Alg}(\mathbf{O})) \to \mathcal{P}\text{-Alg}(\mathbf{O})$$

is canonically isomorphic to

$$\mathbf{triv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathrm{Grp}}.$$

The rest of this subsection is devoted to the proof of Proposition 1.7.2. The idea of the proof, explained to us by M. Kontsevich, is to interpret the composite functor $\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathcal{P}}$ as tensor product by a certain (non-unital) commutative algebra.

1.7.4. *Step 1.* Consider the commutative augmented algebra A in Vect equal to

$$\mathbf{triv}_{\mathrm{Com}^{\mathrm{aug}}}(k[-1]).$$

I.e., $A = k[-1] \oplus k$, with the multiplication on $k[-1]$ is trivial. We claim that that there exists a canonical isomorphism of endo-functors of $\mathcal{P}$-Alg$(\mathbf{O})$

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathcal{P}}(B) \simeq \mathrm{Fib}(A \otimes B \to B).$$

Indeed, take $A' = k \times k$, so that $A := k \underset{A'}{\times} k$. By Lemma 1.2.6, the pullback diagram of commutative algebras

$$\begin{array}{ccc} A & \longrightarrow & k \\ \downarrow & & \downarrow \\ k & \longrightarrow & A' \end{array} \tag{1.13}$$

gives rise to a pullback diagram in $\mathcal{P}$-Alg$(\mathbf{O})$,

$$\begin{array}{ccc} A \otimes B & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \mathrm{diag}} \\ B & \xrightarrow{\mathrm{diag}} & B \times B, \end{array}$$

functorially in $B \in \mathcal{P}$-Alg$(\mathbf{O})$.

The projection on the second factor defines an augmentation $A' \to k$, thereby allowing to view (1.13) as a diagram *over* k. From here we obtain a pullback diagram

$$\begin{array}{ccc} \mathrm{Fib}(A \otimes B \to B) & \longrightarrow & \mathrm{Fib}(B \to B) \\ \downarrow & & \downarrow \\ \mathrm{Fib}(B \to B) & \longrightarrow & \mathrm{Fib}(B \times B \to B), \end{array}$$

i.e., we obtain a pullback diagram

$$\begin{array}{ccc} \mathrm{Fib}(A \otimes B \to B) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B, \end{array}$$

as desired.

1.7.5. *Step 2.* Thus, to prove Proposition 1.7.2 we need to establish a canonical isomorphism of functors

$$\mathrm{Fib}(A \otimes B \to B) \simeq \mathbf{triv}_{\mathcal{P}}(\mathbf{oblv}_{\mathcal{P}}(B)[-1]), \quad B \in \mathcal{P}\text{-Alg}(\mathbf{O}). \tag{1.14}$$

This repeats the argument of Proposition 1.4.6.

1.8. **Stabilization.** In this subsection we use Proposition 1.7.2 to give a simple proof of the fact that the stabilization of the category of $\mathcal{P}$-algebras (in a symmetric monoidal DG category $\mathbf{O}$) identifies with $\mathbf{O}$ itself.

1.8.1. For an ∞-category $\mathbf{C}$, let $\mathrm{ComMonoid}(\mathbf{C})$ denote the category of commutative monoids in $\mathbf{C}$, see [Chapter I.1, Sect. 3.3.3].

Recall also that if $\mathbf{C}$ is stable, the forgetful functor

$$\mathrm{ComMonoid}(\mathbf{C}) \to \mathbf{C}$$

is an equivalence.

1.8.2. Let $\mathbf{O}$ be a symmetric monoidal DG category. We regard it a mere ∞-category, and consider the corresponding category $\mathrm{ComMonoid}(\mathbf{O})$. Since $\mathbf{O}$ is stable, by the above we have $\mathrm{ComMonoid}(\mathbf{O}) \simeq \mathbf{O}$.

Since the functor $\mathbf{triv}_{\mathcal{P}}$ commutes with limits (and, in particular, products), it induces a functor

$$\mathbf{O} \simeq \mathrm{ComMonoid}(\mathbf{O}) \overset{\mathrm{ComMonoid}(\mathbf{triv}_{\mathcal{P}})}{\longrightarrow} \mathrm{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O})). \tag{1.15}$$

We will prove:

Proposition 1.8.3. *The functor* (1.15) *is an equivalence.*

The above proposition can be reformulated as a commutative diagram

$$\begin{array}{ccc} \mathbf{O} & \xrightarrow{\mathbf{triv}_{\mathcal{P}}} & \mathcal{P}\text{-Alg}(\mathbf{O}) \\ \sim\downarrow & & \downarrow \mathrm{Id} \\ \mathrm{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O})) & \xrightarrow{\mathbf{oblv}_{\mathrm{ComMonoid}}} & \mathcal{P}\text{-Alg}(\mathbf{O}). \end{array}$$

Hence, we obtain:

Corollary 1.8.4. *The functor*

$$\mathrm{coPrim}_{\mathcal{P}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathbf{O},$$

left adjoint to $\mathbf{triv}_{\mathcal{P}}$ *(see Sect. 2.4.1 below), identifies* $\mathbf{O}$ *with the stabilization of* $\mathcal{P}\text{-Alg}(\mathbf{O})$.

Proof of Proposition 1.8.3. Since the functor $\mathbf{oblv}_{\mathcal{P}}$ preserves limits (and, in particular, products), it induces a functor

$$\mathrm{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O})) \overset{\mathrm{ComMonoid}(\mathbf{oblv}_{\mathcal{P}})}{\longrightarrow} \mathrm{ComMonoid}(\mathbf{O}).$$

We claim that the functors $\mathrm{ComMonoid}(\mathbf{oblv}_{\mathcal{P}})$ and $\mathrm{ComMonoid}(\mathbf{triv}_{\mathcal{P}})$ are inverses of each other.

The fact that the composition $\mathrm{ComMonoid}(\mathbf{oblv}_{\mathcal{P}}) \circ \mathrm{ComMonoid}(\mathbf{triv}_{\mathcal{P}})$ is isomorphic to the identity functor follows from the fact that $\mathbf{oblv}_{\mathcal{P}} \circ \mathbf{triv}_{\mathcal{P}} \simeq \mathrm{Id}$.

To prove that the other composition is isomorphic to the identity functor, we proceed as follows. Recall that for any ∞-category $\mathbf{C}$, the forgetful functor

$$\mathrm{ComMonoid}(\mathrm{Monoid}(\mathbf{C})) \overset{\mathrm{ComMonoid}(\mathbf{oblv}_{\mathrm{Monoid}})}{\longrightarrow} \mathrm{ComMonoid}(\mathbf{C})$$

is an equivalence, see [Lu2, Theorem 5.1.2.2].

Hence, it suffices to construct an isomorphism between the composition

$$\mathrm{ComMonoid}(\mathrm{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))) \overset{\mathrm{ComMonoid}(\mathbf{oblv}_{\mathrm{Monoid}})}{\longrightarrow} \mathrm{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O})) \to$$
$$\overset{\mathrm{ComMonoid}(\mathbf{oblv}_{\mathcal{P}})}{\longrightarrow} \mathrm{ComMonoid}(\mathbf{O}) \overset{\mathrm{ComMonoid}(\mathbf{triv}_{\mathcal{P}})}{\longrightarrow} \mathrm{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))$$

and

$$\mathrm{ComMonoid}(\mathrm{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))) \overset{\mathrm{ComMonoid}(\mathbf{oblv}_{\mathrm{Monoid}})}{\longrightarrow} \mathrm{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O})).$$

However, this follows by applying ComMonoid to the isomorphism between

$$\mathrm{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))) \overset{\mathbf{oblv}_{\mathrm{Monoid}}}{\longrightarrow} \mathcal{P}\text{-Alg}(\mathbf{O}) \overset{\mathbf{oblv}_{\mathcal{P}}}{\longrightarrow} \mathbf{O} \overset{\mathbf{triv}_{\mathcal{P}}}{\longrightarrow} \mathcal{P}\text{-Alg}(\mathbf{O})$$

and

$$\mathrm{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O})) \overset{\mathbf{oblv}_{\mathrm{Monoid}}}{\longrightarrow} \mathcal{P}\text{-Alg}(\mathbf{O}),$$

the latter given by Corollary 1.7.3.

□

1.8.5. We can use Proposition 1.7.2 also to describe the co-stabilization of $\mathcal{P}\text{-Alg}(\mathbf{O})$, i.e., the stabilization of $\mathcal{P}\text{-Alg}(\mathbf{O})^{\mathrm{op}}$.

Proposition 1.8.6. *The suspension functor* $\Sigma_{\mathcal{P}}$ *on* $\mathcal{P}\text{-Alg}(\mathbf{O})$ *identifies with*

$$\mathbf{free}_{\mathcal{P}} \circ [1] \circ \mathrm{coPrim}_{\mathcal{P}},$$

where $\mathrm{coPrim}_{\mathcal{P}}$ *is as in Sect. 2.4.1.*

Proof. Follows by adjunction from Proposition 1.7.2.

□

Corollary 1.8.7. *The functor*

$$(\mathbf{free}_{\mathcal{P}})^{\mathrm{op}} : \mathbf{O}^{\mathrm{op}} \to (\mathcal{P}\text{-Alg}(\mathbf{O}))^{\mathrm{op}}$$

identifies $\mathbf{O}^{\mathrm{op}}$ *with the stabilization of* $(\mathcal{P}\text{-Alg}(\mathbf{O}))^{\mathrm{op}}$.

Proof. We have to show that the functor $(\mathbf{free}_{\mathcal{P}})^{\mathrm{op}}$ identifies $\mathbf{O}^{\mathrm{op}}$ with the category of *spectrum objects* in $(\mathcal{P}\text{-Alg}(\mathbf{O}))^{\mathrm{op}}$, i.e., with the category of sequences

$$A_0 \simeq \Omega(A_1),\ A_1 \simeq \Omega(A_2)...,\quad A_i \in (\mathcal{P}\text{-Alg}(\mathbf{O}))^{\mathrm{op}},$$

where Ω is the loop functor on $(\mathbf{free}_{\mathcal{P}})^{\mathrm{op}}$, i.e., when we regard A_i as $\mathcal{P}$-algebras in $\mathbf{O}$, we have

$$A_0 \simeq \Sigma_{\mathcal{P}}(A_1),\ A_1 \simeq \Sigma_{\mathcal{P}}(A_2)...$$

We claim that any such sequence is canonically of the form

$$A_i = \mathbf{free}_{\mathcal{P}} \circ [-i] \circ \mathrm{coPrim}_{\mathcal{P}}(A_0). \tag{1.16}$$

Indeed, it follows from Proposition 1.8.6 that for every $i \geq 0$ we have

$$A_i \simeq \mathbf{free}_{\mathcal{P}} \circ [1] \circ \mathrm{coPrim}_{\mathcal{P}}(A_{i+1}), \tag{1.17}$$

hence

$$\mathrm{coPrim}_{\mathcal{P}}(A_{i+1}) \simeq \mathrm{coPrim}_{\mathcal{P}}(A_i)[-1],$$

and hence

$$\mathrm{coPrim}_{\mathcal{P}}(A_{i+1}) \simeq \mathrm{coPrim}_{\mathcal{P}}(A_0)[-(i+1)]. \tag{1.18}$$

Combining (1.18) and (1.17) we obtain (1.16).

□

2. Koszul duality

In this section we review the general theory of Koszul duality that relates algebras over an operad with co-algebras over the Koszul dual co-operads.

The main point of this section is that there are two *inequivalent* notions of co-algebra over a co-operad: the usual one, and what we call a *ind-nilpotent co-algebra.* There is a forgetful functor from the latter to the former, which we conjecture to be fully faithful.

The Koszul duality functors naturally connect $\mathcal{P}$-algebras and ind-nilpotent co-algebras for $\mathcal{P}^\vee$. We propose some conjectures to the effect of what fully-faithfulness properties we can expect from the Koszul duality functors.

2.1. **Co-operads.** In this subsection we introduce the notion of co-operad. There are no surprises here, but there will be some when we will consider the corresponding notion of co-algebra over a co-operad.

2.1.1. By a *co-operad* we shall mean a co-associative co-algebra object in Vect^Σ.

As in the case of operads (see Sect. 1.1.2), we will only consider co-operads $\mathcal{Q}$ for which the co-unit defines an isomorphism $\mathcal{Q}(1) \to k$. (In particular, all our co-operads are augmented.)

2.1.2. Let $\mathrm{Vect}^\Sigma_{\text{f.d.}} \subset \mathrm{Vect}^\Sigma$ be the full subcategory spanned by those objects $\mathcal{P}$, for which $\mathcal{P}(n) \in \mathrm{Vect}$ is finite-dimensional in each cohomological degree for every n.

Term-wise dualization $\mathcal{P} \mapsto \mathcal{P}^*$ defines a monoidal equivalence

$$(\mathrm{Vect}^\Sigma_{\text{f.d.}})^{\mathrm{op}} \to \mathrm{Vect}^\Sigma_{\text{f.d.}}\,.$$

In particular, it defines an anti-equivalence between the subcategories of operads and co-operads that belong to $\mathrm{Vect}^\Sigma_{\text{f.d.}}$.

2.1.3. We set

$$\mathrm{Coassoc}^{\mathrm{aug}} := (\mathrm{Assoc}^{\mathrm{aug}})^*.$$

This is the co-operad responsible for unital and augmented (or, equivalently, non-unital) co-associative co-algebras.

We set

$$\mathrm{Cocom}^{\mathrm{aug}} := (\mathrm{Com}^{\mathrm{aug}})^*.$$

This is the co-operad responsible for unital and augmented (or, equivalently, non-unital) co-commutative co-algebras.

2.2. **Ind-nilpotent co-algebras over a co-operad.** It turns out that there are two (and, outside of characteristic 0, four) *inequivalent* notions of co-algebra over a given co-operad. In this subsection we will study one of them: the notion of ind-nilpotent co-algebra.

2.2.1. Recall the action of Vect^{Σ} on $\mathbf{O}$, considered in Sect. 1.1.1.

Let $\mathcal{Q}$ be a co-operad. By definition, the category

$$\mathcal{Q}\text{-}\mathrm{Coalg}^{\text{ind-nilp}}(\mathbf{O})$$

is that of $\mathcal{Q}$-comodules in $\mathbf{O}$ with respect to the $\star$-action.

Remark 2.2.2. Modules for the above monad should be more properly called 'ind-nilpotent co-algebras with *divided powers*', see [FraG, Sect. 3.5]. However, we shall omit the reference to divided powers from the notation because we are working over a field of characteristic zero.

2.2.3. We have the pair of adjoint functors

$$\mathbf{oblv}^{\text{ind-nilp}}_{\mathcal{Q}} : \mathcal{Q}\text{-}\mathrm{Coalg}^{\text{ind-nilp}}(\mathbf{O}) \rightleftarrows \mathbf{O} : \mathbf{cofree}^{\text{ind-nilp}}_{\mathcal{Q}},$$

with $\mathbf{oblv}^{\text{ind-nilp}}_{\mathcal{Q}}$ being co-monadic. In particular, $\mathbf{oblv}^{\text{ind-nilp}}_{\mathcal{Q}}$ is conservative, preserves all colimits and totalizations of $\mathbf{oblv}^{\text{ind-nilp}}_{\mathcal{Q}}$-split co-simplicial objects.

2.2.4. The augmentation on $\mathcal{Q}$ gives rise to a functor

$$\mathbf{triv}^{\text{ind-nilp}}_{\mathcal{Q}} : \mathbf{O} \to \mathcal{Q}\text{-}\mathrm{Coalg}^{\text{ind-nilp}}(\mathbf{O}),$$

right inverse to $\mathbf{oblv}^{\text{ind-nilp}}_{\mathcal{Q}}$.

2.3. **The Koszul dual (co)-operad.** In this subsection we introduce the Koszul duality functor that relates operads and co-operads.

2.3.1. Let $\mathbf{O}'$ be a (not necessarily symmetric) monoidal category with limits and colimits. We will assume that the monoidal operation on $\mathbf{O}'$ commutes with *sifted* colimits in each variable (but not not necessarily all colimits).

In this case we have a pair of mutually adjoint functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}') \rightleftarrows \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}') : \mathrm{coBar}^{\mathrm{enh}}, \tag{2.1}$$

referred to as Koszul duality, see Sect. 3.2.7 below.

(Note, however, that since the monoidal operation on $\mathbf{O}'$ is not assumed to commute with coproducts, augmented associatove/co-associative algebras/co-algebras in $\mathbf{O}'$ are *not* the same as algebras/co-algebras over the $\mathrm{AssocAlg}^{\mathrm{aug}}$-operad.)

We apply this to $\mathbf{O}' := \mathrm{Vect}^{\Sigma}$. In this case, the resulting functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{Operads} \rightleftarrows \mathrm{coOperads} : \mathrm{coBar}^{\mathrm{enh}}$$

are mutually inverse equivalences. One can prove this by adapting the argument of [FraG, Proposition 4.1.2].

2.3.2. Let $\mathcal{P}$ be an operad. We denote

$$\mathcal{P}^\vee := \mathrm{Bar}^{\mathrm{enh}}(\mathcal{P}),$$

and for a co-operad $\mathcal{Q}$ we denote

$$\mathcal{Q}^\vee := \mathrm{coBar}^{\mathrm{enh}}(\mathcal{Q}).$$

If $\mathcal{P} \in \mathrm{Vect}^{\Sigma}_{\mathrm{f.d.}}$, then $\mathcal{P}^\vee$ has the same property, and vice versa.

2.3.3. It is known that

$$(\mathrm{Coassoc}^{\mathrm{aug}})^\vee \simeq \mathrm{Assoc}^{\mathrm{aug}}[-1],$$

and hence

$$(\mathrm{Assoc}^{\mathrm{aug}})^\vee \simeq \mathrm{Coassoc}^{\mathrm{aug}}[1].$$

It is also known that

$$(\mathrm{Cocom}^{\mathrm{aug}})^\vee \simeq \mathrm{Lie}[-1],$$

and hence

$$\mathrm{Lie}^\vee \simeq \mathrm{Cocom}^{\mathrm{aug}}[1].$$

2.4. **Koszul duality functors.** In this subsection we will define the operation central for this section (and the entire chapter): the functors of Koszul duality that relate algebras over an operad to co-algebras over the Koszul dual co-operad.

The exposition here follows closely [FraG, Sect. 3].

2.4.1. Let $\mathcal{P}$ be an operad. Recall the functor

$$\mathbf{triv}_{\mathcal{P}} : \mathbf{O} \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}).$$

It preserves limits (since its composition with the conservative limit-preserving functor $\mathbf{oblv}_{\mathcal{P}}$ does). Hence, by the Adjoint Functor Theorem, it admits a left adjoint:

$$\mathrm{coPrim}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathbf{O}.$$

By adjunction,

$$\mathrm{coPrim}_{\mathcal{P}} \circ \mathbf{free}_{\mathcal{P}} \simeq \mathrm{Id}\,.$$

2.4.2. We calculate the functor $\mathrm{coPrim}_{\mathcal{P}}$ as the Bar-construction of the augmented associative algebra $\mathcal{P}$ in $\mathrm{AssocAlg}(\mathrm{Vect}^{\Sigma})$ acting on a $\mathcal{P}$-module in $\mathbf{O}$:

$$\mathrm{coPrim}_{\mathcal{P}} \simeq \mathrm{Bar}(\mathcal{P}, -).$$

It follows from the definition of the Koszul duality functor that for an operad $\mathcal{P}$ we have a canonical isomorphism of co-monads acting on $\mathbf{O}$:

$$\mathrm{coPrim}_{\mathcal{P}} \circ \mathbf{triv}_{\mathcal{P}} \simeq \mathcal{P}^\vee \star -,$$

see [FraG, Lemma 3.3.4] (or Sect. 7.1.3 later in this chapter).

2.4.3. Hence, we obtain that the functor $\mathrm{coPrim}_{\mathcal{P}}$ canonically lifts to a functor

$$\mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}),$$

where $\mathcal{Q} = \mathcal{P}^\vee$, so that

$$\mathrm{coPrim}_{\mathcal{P}} \simeq \mathbf{oblv}^{\mathrm{ind\text{-}nilp}}_{\mathcal{Q}} \circ \mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathcal{P}},$$

and

$$\mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathcal{P}} \circ \mathbf{triv}_{\mathcal{P}} \simeq \mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathcal{Q}}, \tag{2.2}$$

see [FraG, Corollary 3.3.5 and Sect. 3.3.6].

We also have:

$$\mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathcal{P}} \circ \mathbf{free}_{\mathcal{P}} \simeq \mathbf{triv}^{\mathrm{ind\text{-}nilp}}_{\mathcal{Q}} \tag{2.3}$$

2.5. **A digression: the filtered version.** In this subsection we observe that the functors $\mathrm{coPrim}_{\mathcal{P}}$ and $\Phi = \mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathcal{P}}$ naturally admit filtered versions.

2.5.1. In the context of Sect. 1.5 let us take

(i) $\mathcal{C}(\mathbf{O}) = \mathbf{O}$, $\Phi = \mathrm{coPrim}_{\mathcal{P}}$.

(ii) $\mathcal{C}(\mathbf{O}) = \mathcal{P}^\vee\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$, $\Phi = \mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathcal{P}}$.

2.5.2. Thus, we obtain that the functor

$$\mathrm{coPrim}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathbf{O}$$

canonically lifts to a functor

$$\mathrm{coPrim}^{\mathrm{Fil}}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathbf{O}^{\mathrm{Fil},\geq 0},$$

and the functor

$$\mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}^\vee\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$$

canonically lifts to a functor

$$\mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp,Fil}}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}^\vee\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}^{\mathrm{Fil},\geq 0})$$

so that

$$\mathrm{coPrim}^{\mathrm{Fil}}_{\mathcal{P}} \simeq \mathbf{oblv}^{\mathrm{ind\text{-}nilp}}_{\mathcal{P}^\vee} \circ \mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp,Fil}}_{\mathcal{P}}.$$

We have a canonical isomorphism

$$\text{ass-gr} \circ \mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp,Fil}}_{\mathcal{P}} \simeq \mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathcal{P}^\vee} \circ (\deg = 1) \circ \mathbf{oblv}_{\mathcal{P}}, \tag{2.4}$$

as functors $\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}^\vee\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}^{\mathrm{gr},\geq 0})$.

Remark 2.5.3. One can show that the composite functor

$$\mathrm{coPrim}^{\mathrm{enh,ind\text{-}nilp,Fil}}_{\mathcal{P}} \circ \mathbf{triv}_{\mathcal{P}} : \mathbf{O} \to \mathcal{P}^\vee\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}^{\mathrm{Fil},\geq 0})$$

identifies canonically with

$$(\mathrm{gr} \to \mathrm{Fil}) \circ \mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathcal{P}^\vee} \circ (\deg = 1).$$

2.6. **The adjoint Koszul duality functors.** In this subsection we describe the construction of the adjoint Koszul duality functor: it goes from the category of (ind-nilpotent) co-algebras over a given co-operad $\mathcal{Q}$ to the category of algebras over the Koszul dual of $\mathcal{Q}$.

2.6.1. Let $\mathcal{Q}$ be a co-operad. The functor $\mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}$ preserves colimits, since its composition with $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$ does. Hence, by the Adjoint Functor Theorem, it admits a right adjoint

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} : \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathbf{O}.$$

By adjunction, $\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}} \simeq \mathrm{Id}$.

2.6.2. We calculate the functor $\mathrm{Prim}_{\mathcal{P}}^{\text{ind-nilp}}$ as the coBar-construction of the augmented co-associative co-algebra $\mathcal{Q}$ in $\mathrm{CoassocCoalg}(\mathrm{Vect}^{\Sigma})$ acting on a $\mathcal{Q}$-comodule in $\mathbf{O}$:

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \simeq \mathrm{coBar}(\mathcal{Q}, -).$$

In addition, for a co-operad $\mathcal{Q}$, we have a canonical *morphism* (but not an isomorphism) of monads

$$(\mathcal{Q}^{\vee} \star -) \to \mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \circ \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}, \tag{2.5}$$

see [FraG, Lemma 3.3.9].

2.6.3. Hence, we obtain that the functor $\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}}$ canonically lifts to a functor

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} : \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}),$$

where $\mathcal{P} = \mathcal{Q}^{\vee}$, so that

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \simeq \mathbf{oblv}_{\mathcal{P}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}},$$

see [FraG, Corollary 3.3.11], and

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}} \simeq \mathbf{triv}_{\mathcal{P}}.$$

The map (2.5) gives rise to a natural transformation of functors $\mathbf{O} \to \mathcal{P}\text{-Alg}(\mathbf{O})$, namely,

$$\mathbf{free}_{\mathcal{P}} \to \mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \circ \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}.$$

2.6.4. Furthermore, according to [FraG, Corollary 3.3.13] the functors

$$\mathrm{coPrim}_{\mathcal{P}}^{\text{enh,ind-nilp}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \rightleftarrows \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) : \mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \tag{2.6}$$

are mutually adjoint.

2.6.5. The following is part of [FraG, Conjecture 3.4.5]:

Conjecture 2.6.6. *The functor*

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} : \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O})$$

is fully faithful.

In the sequel, we will relate Conjecture 2.6.6 to several other plausible conjectures, see Sect. 2.11.

Remark 2.6.7. Added in November 2021: It turns out that Conjecture 2.6.6 is false. Namely, the co-unit map

$$\mathrm{coPrim}_{\mathcal{P}}^{\text{enh,ind-nilp}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \to \mathrm{Id}$$

fails to be an isomorphism for $\mathcal{P} = \mathrm{Com}^{\mathrm{aug}}$ (and so $\mathcal{Q}$ is the shifted Lie operad), when evaluated on

$$A := \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}(V),$$

where V is an infinite-dimensional vector space. We are grateful to J. Lurie for pointing this out to us.

Indeed, in this case $\mathrm{Prim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathcal{Q}}(A)$ is the completed polynomial algebra on the vector space V as generators, and its classical cotangent space has a non-trivial kernel when mapping to V.

2.7. **(Usual) co-algebras over a co-operad.** In this subsection we will define another notion of co-algebra over a given co-operad. It is this notion that in the case of $\mathrm{CoAssoc}^{\mathrm{aug}}$ (resp., $\mathrm{Cocom}^{\mathrm{aug}}$) recovers co-associative co-algebras (resp., co-commutative co-algebras).

2.7.1. We have another *right-lax* action of Vect^{Σ} on $\mathbf{O}$, given by

$$\mathcal{P} * V = \prod_{n\geq 1} (\mathcal{P}(n) \otimes V^{\otimes n})^{\Sigma_n}.$$

2.7.2. For a co-operad $\mathcal{Q}$, the category $\mathcal{Q}\text{-Coalg}(\mathbf{O})$ of augmented $\mathcal{Q}$-co-algebras is that of $\mathcal{Q}$-co-modules in $\mathbf{O}$ with respect to the $*$-action.

Remark 2.7.3. Note, however, that since the $*$-action of Vect^{Σ} on $\mathbf{O}$ is only right-lax, the functor $\mathbf{O} \to \mathbf{O}$, defined by $\mathcal{Q}$, is *not* a co-monad.

2.7.4. For example, for $\mathcal{Q} = \mathrm{Coassoc}^{\mathrm{aug}}$, we obtain the usual category $\mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O})$ of co-unital augmented (or, equivalently, non co-unital) co-associative co-algebras.

Similarly, for $\mathcal{Q} = \mathrm{Cocom}^{\mathrm{aug}}$, we obtain the usual category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$ of co-unital augmented (or, equivalently, non co-unital) co-commutative co-algebras.

2.7.5. We let

$$\mathbf{oblv}_{\mathcal{Q}} : \mathcal{Q}\text{-Coalg}(\mathbf{O}) \to \mathbf{O}$$

denote the corresponding forgetful functor.

The functor $\mathbf{oblv}_{\mathcal{Q}}$ is conservative and preserves all colimits (in fact, one can show that $\mathbf{oblv}_{\mathcal{Q}}$ admits a right adjoint, but it is not easy to describe this right adjoint explicitly).

In addition, it is known that the functor $\mathbf{oblv}_{\mathcal{Q}}$ commutes with totalizations of $\mathbf{oblv}_{\mathcal{Q}}$-split co-simplicial objects.

Remark 2.7.6. From the above it follows that the functor

$$\mathbf{oblv}_{\mathcal{Q}} : \mathcal{Q}\text{-Coalg}(\mathbf{O}) \to \mathbf{O}$$

is co-monadic. Yet, as was noted in Remark 2.7.3, the corresponding endo-functor of $\mathbf{O}$ is *not* the one, given by the $*$-action of $\mathcal{Q}$.

2.7.7. The augmentation on $\mathcal{Q}$ defines the functor

$$\mathbf{triv}_{\mathcal{Q}} : \mathbf{O} \to \mathcal{Q}\text{-Coalg}(\mathbf{O}),$$

right inverse to $\mathbf{oblv}_{\mathcal{Q}}$.

The functor $\mathbf{triv}_{\mathcal{Q}}$ preserves colimits, since its composition with $\mathbf{oblv}_{\mathcal{Q}}$ does. Hence, the functor $\mathbf{triv}_{\mathcal{Q}}$ admits a right adjoint

$$\mathrm{Prim}_{\mathcal{Q}} : \mathcal{Q}\text{-Coalg}(\mathbf{O}) \to \mathbf{O}.$$

In Sect. A.2 we will describe the functor $\mathrm{Prim}_{\mathcal{Q}}$ a little more explicitly.

2.8. **Relation between two types of co-algebras.** In this subsection we will study the relationship between the notions of co-algebra over a co-operad and that of ind-nilpotent co-algebra.

2.8.1. Note that we have the following natural transformation between the two right-lax actions of Vect^{Σ} on $\mathbf{O}$:

$$\mathcal{P}\star V\to\mathcal{P}*V. \tag{2.7}$$

Remark 2.8.2. Note that the natural transformation (2.7) involves the operation of averaging with respect to symmetric groups, see [FraG, Sect. 3.5.5].

2.8.3. The natural transformation (2.7) gives rise to the forgetful functor

$$\mathbf{res}^{\star\to *}:\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})\to\mathcal{Q}\text{-Coalg}(\mathbf{O}). \tag{2.8}$$

We propose[5]:

Conjecture 2.8.4. *The functor*

$$\mathbf{res}^{\star\to *}:\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})\to\mathcal{Q}\text{-Coalg}(\mathbf{O})$$

of (2.8) *is fully faithful.*

2.8.5. We have:

$$\mathbf{oblv}_{\mathcal{Q}}\circ\mathbf{res}^{\star\to *}\simeq\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}},\quad\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})\to\mathbf{O} \tag{2.9}$$

and

$$\mathbf{triv}_{\mathcal{Q}}\simeq\mathbf{res}^{\star\to *}\circ\mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}},\quad\mathbf{O}\to\mathcal{Q}\text{-Coalg}(\mathbf{O}). \tag{2.10}$$

We shall denote

$$\mathbf{cofree}_{\mathcal{Q}}^{\text{fake}}:=\mathbf{res}^{\star\to *}\circ\mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}}:\mathbf{O}\to\mathcal{Q}\text{-Coalg}(\mathbf{O}).$$

2.8.6. Let $\mathcal{P}:=\mathcal{Q}^{\vee}$ be the Koszul dual operad. We denote

$$\mathrm{coPrim}_{\mathcal{P}}^{\text{enh}}:=\mathbf{res}^{\star\to *}\circ\mathrm{coPrim}_{\mathcal{P}}^{\text{enh,ind-nilp}},\quad\mathcal{P}\text{-Alg}(\mathbf{O})\to\mathcal{Q}\text{-Coalg}(\mathbf{O}).$$

By (2.2), we have

$$\mathrm{coPrim}_{\mathcal{P}}^{\text{enh}}\circ\mathbf{triv}_{\mathcal{P}}\simeq\mathbf{cofree}_{\mathcal{Q}}^{\text{fake}} \tag{2.11}$$

and by (2.3), we have

$$\mathrm{coPrim}_{\mathcal{P}}^{\text{enh}}\circ\mathbf{free}_{\mathcal{P}}\simeq\mathbf{triv}_{\mathcal{Q}}. \tag{2.12}$$

2.8.7. It follows from (2.9) that the functor $\mathbf{res}^{\star\to *}$ commutes with colimits. Hence, it admits a right adjoint, denoted $(\mathbf{res}^{\star\to *})^R$.

We define

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{enh}}:=\mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}}\circ(\mathbf{res}^{\star\to *})^R:\mathcal{Q}\text{-Coalg}(\mathbf{O})\to\mathcal{P}\text{-Alg}(\mathbf{O}). \tag{2.13}$$

By adjunction, the functors

$$\mathrm{coPrim}_{\mathcal{P}}^{\text{enh}}:\mathcal{P}\text{-Alg}(\mathbf{O})\rightleftarrows\mathcal{Q}\text{-Coalg}(\mathbf{O}):\mathrm{Prim}_{\mathcal{Q}}^{\text{enh}}$$

form an adjoint pair.

By passing to right adjoints in (2.10), we obtain an isomorphism:

$$\mathrm{Prim}_{\mathcal{Q}}\simeq\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}}\circ(\mathbf{res}^{\star\to *})^R, \tag{2.14}$$

[5]In [FraG, Remark 3.5.3] it was erroneously stated that the authors knew how to prove this statement. Unfortunately, this turned out not be the case.

and applying the definition of $\mathrm{Prim}^{\mathrm{enh}}_{\mathcal{Q}}$

$$\mathbf{oblv}_{\mathcal{Q}} \circ \mathrm{Prim}^{\mathrm{enh}}_{\mathcal{Q}} \simeq \mathrm{Prim}_{\mathcal{Q}}\,.$$

2.8.8. We propose the following variant of Conjecture 2.6.6:

Conjecture 2.8.9.
(a) *The unit of the adjunction*

$$\mathrm{Id} \to \mathrm{Prim}^{\mathrm{enh}}_{\mathcal{Q}} \circ \mathrm{coPrim}^{\mathrm{enh}}_{\mathcal{P}}$$

is an isomorphism, when evaluated on objects lying in the essential image of the functor $\mathrm{Prim}^{\mathrm{enh}}_{\mathcal{Q}}$*.*
(b) *The co-unit of the adjunction*

$$\mathrm{coPrim}^{\mathrm{enh}}_{\mathcal{P}} \circ \mathrm{Prim}^{\mathrm{enh}}_{\mathcal{Q}} \to \mathrm{Id}$$

is an isomorphism, when evaluated on objects lying in the essential image of the functor $\mathrm{coPrim}^{\mathrm{enh}}_{\mathcal{P}}$*.*

Remark 2.8.10. Added in November 2021: just like Conjecture 2.6.6, the same counterexample disproves point (b) of Conjecture 2.8.9.

As of now, point (a) of Conjecture 2.8.9 still stands, but we are highly dubious of its validity.

2.9. **Calculation of primitives.** In this subsection we will be concerned with the functor

$$\mathrm{Prim}_{\mathcal{Q}} \circ \mathbf{cofree}^{\mathrm{fake}}_{\mathcal{Q}} : \mathbf{O} \to \mathbf{O},$$

where we recall that $\mathbf{cofree}^{\mathrm{fake}}_{\mathcal{Q}}$ is the functor

$$\mathbf{res}^{\star\to *} \circ \mathbf{cofree}^{\text{ind-nilp}}_{\mathcal{Q}} : \mathbf{O} \to \mathcal{Q}\text{-}\mathrm{Coalg}^{\text{ind-nilp}}(\mathbf{O}).$$

2.9.1. Consider the unit of the adjunction

$$\mathrm{Id} \to (\mathbf{res}^{\star\to *})^R \circ \mathbf{res}^{\star\to *}.$$

Composing with $\mathrm{Prim}^{\text{enh,ind-nilp}}_{\mathcal{Q}}$ and pre-composing with $\mathbf{cofree}^{\mathrm{fake}}_{\mathcal{Q}}$, we obtain a natural transformation

$$\mathbf{triv}_{\mathcal{P}} \simeq \mathrm{Prim}^{\text{enh,ind-nilp}}_{\mathcal{Q}} \circ \mathbf{cofree}^{\text{ind-nilp}}_{\mathcal{Q}} \to \\ \to \mathrm{Prim}^{\text{enh,ind-nilp}}_{\mathcal{Q}} \circ (\mathbf{res}^{\star\to *})^R \circ \mathbf{res}^{\star\to *} \circ \mathbf{cofree}^{\text{ind-nilp}}_{\mathcal{Q}} \simeq \mathrm{Prim}^{\mathrm{enh}}_{\mathcal{Q}} \circ \mathbf{cofree}^{\mathrm{fake}}_{\mathcal{Q}},$$

where $\mathcal{P} := \mathcal{Q}^\vee$. I.e., we have a natural transformation:

$$\mathbf{triv}_{\mathcal{P}} \to \mathrm{Prim}^{\mathrm{enh}}_{\mathcal{Q}} \circ \mathbf{cofree}^{\mathrm{fake}}_{\mathcal{Q}}, \quad \mathbf{O} \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}). \tag{2.15}$$

Composing further with the forgetful functor $\mathbf{oblv}_{\mathcal{P}}$

$$\mathrm{Id} \to \mathrm{Prim}_{\mathcal{Q}} \circ \mathbf{cofree}^{\mathrm{fake}}_{\mathcal{Q}}, \tag{2.16}$$

as endo-functors of $\mathbf{O}$.

2.9.2. The following conjecture follows tautologically from Conjecture 2.8.4:

Conjecture 2.9.3. *Then the natural transformation* (2.16) *is an isomorphism.*

Since the functor $\mathbf{oblv}_{\mathcal{P}}$ is conservative, Conjecture 2.9.3 is equivalent to the natural transformation (2.15) being an isomorphism.

In Sect. A we will prove:

Theorem 2.9.4. *Conjecture 2.9.3 holds if the co-operad $\mathcal{Q}$ is such that $\mathcal{Q}$ and $\mathcal{Q}^\vee[1]$ are both classical and finite-dimensional.*

2.10. **Some implications.** In this subsection we will assume that Conjecture 2.9.3 holds for a given co-operad $\mathcal{Q}$ (in particular, it applies to $\mathcal{Q} := \mathrm{Cocom}^{\mathrm{aug}}$ and $\mathcal{Q} := \mathrm{Coassoc}^{\mathrm{aug}}$), and derive some corollaries.

2.10.1. Note that the fact that the natural transformation (2.16) is an isomorphism can be reformulated as saying that the functor $\mathbf{res}^{\star\to *}$ induces an isomorphism

$$\mathrm{Maps}_{\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})}\left(\mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}(V), \mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}}(W)\right) \to \tag{2.17}$$
$$\to \mathrm{Maps}_{\mathcal{Q}\text{-Coalg}(\mathbf{O})}\left(\mathbf{triv}_{\mathcal{Q}}(V), \mathbf{cofree}_{\mathcal{Q}}^{\text{fake}}(W)\right)$$

is an isomorphism for any $V, W \in \mathbf{O}$.

2.10.2. We claim:

Proposition 2.10.3. *The functor $\mathbf{res}^{\star\to *}$ defines an isomorphism*

$$\mathrm{Maps}_{\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})}\left(\mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}(V), A\right) \to \mathrm{Maps}_{\mathcal{Q}\text{-Coalg}(\mathbf{O})}\left(\mathbf{triv}_{\mathcal{Q}}(V), \mathbf{res}^{\star\to *}(A)\right)$$

for any $V \in \mathbf{O}$ and $A \in \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$.

Proof. For the proof we will need the following lemma:

Lemma 2.10.4. *The functor $\mathbf{res}^{\star\to *}$ preserves totalizations of co-simplicial objects that are $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$-split.*

Proof. Follows from the combination of the following three facts:

(1) the functor $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$ commutes with totalizations of $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$-split co-implicial objects;

(2) the functor $\mathbf{res}^{\star\to *}$ sends $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$-split co-simplicial objects to co-simplicial objects that are $\mathbf{oblv}_{\mathcal{Q}}$-split;

(3) the functor $\mathbf{oblv}_{\mathcal{Q}}$ commutes with totalizations of $\mathbf{oblv}_{\mathcal{Q}}$-split co-implicial objects. □

Now, the assertion of the proposition follows from the fact that every object

$$A \in \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$$

can be written as such a totalization as in Lemma 2.10.4, whose terms are objects of the form $\mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}}(W)$ for $W \in \mathbf{O}$. □

Corollary 2.10.5.

(a) *The natural transformation* $\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \to \mathrm{Prim}_{\mathcal{Q}} \circ \mathbf{res}^{\star\to *}$ *is an isomorphism.*

(b) $\mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \to \mathrm{Prim}_{\mathcal{Q}}^{\text{enh}} \circ \mathbf{res}^{\star\to *}$ *is an isomorphism.*

2.10.6. As another corollary of Proposition 2.10.3, we obtain:

Corollary 2.10.7. *The functor* $\mathbf{res}^{\star\to *}$ *defines an isomorphism*

$$\mathrm{Maps}_{\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})}(A', A) \to \mathrm{Maps}_{\mathcal{Q}\text{-Coalg}(\mathbf{O})}(\mathbf{res}^{\star\to *}(A'), \mathbf{res}^{\star\to *}(A))$$

for any A' *lying in the essential image of the functor* $\mathrm{coPrim}^{\text{enh,ind-nilp}}_{\mathcal{P}}$, *where* $\mathcal{P} := \mathcal{Q}^\vee$.

Proof. Follows from the fact that any object of $\mathcal{P}$-Alg($\mathbf{O}$) can be written as a colimit of ones of the form $\mathbf{free}_{\mathcal{P}}(V)$, while

$$\mathrm{coPrim}^{\text{enh,ind-nilp}}_{\mathcal{P}} \circ \mathbf{free}_{\mathcal{P}}(V) \simeq \mathbf{triv}^{\text{ind-nilp}}_{\mathcal{Q}}(V).$$

□

2.11. **Some implications between the conjectures.** In this subsection we continue to assume that $\mathcal{Q}$ is such that Conjecture 2.9.3 holds. We will prove that Conjecture 2.6.6 implies Conjectures 2.8.4 and 2.8.9.

2.11.1. First, we claim:

Theorem 2.11.2. *Conjecture 2.6.6 (for the co-operad* $\mathcal{Q}$*) implies Conjecture 2.8.4.*

Proof. Taking into account Corollary 2.10.7, it suffices to know that the functor

$$\mathrm{coPrim}^{\text{enh,ind-nilp}}_{\mathcal{P}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$$

is essentially surjective. However, the latter follows from Conjecture 2.6.6. □

2.11.3. Next, we claim:

Theorem 2.11.4. *Conjecture 2.6.6 (for the co-operad* $\mathcal{Q}$*) implies Conjecture 2.8.9.*

Proof. For point (b) of Conjecture 2.8.9, we claim that a stronger statement follows from Conjecture 2.6.6. Namely, we claim that the natural transformation

$$\mathrm{coPrim}^{\text{enh}}_{\mathcal{P}} \circ \mathrm{Prim}^{\text{enh}}_{\mathcal{Q}} \to \mathrm{Id}$$

is an isomorphism on the essential image of $\mathbf{res}^{\star\to *}$. Indeed, the composition

$$\mathbf{res}^{\star\to *} \circ \mathrm{coPrim}^{\text{enh,ind-nilp}}_{\mathcal{P}} \circ \mathrm{Prim}^{\text{enh,ind-nilp}}_{\mathcal{Q}} \simeq \mathrm{coPrim}^{\text{enh}}_{\mathcal{P}} \circ \mathrm{Prim}^{\text{enh,ind-nilp}}_{\mathcal{Q}} \to \\ \to \mathrm{coPrim}^{\text{enh}}_{\mathcal{P}} \circ \mathrm{Prim}^{\text{enh}}_{\mathcal{Q}} \circ \mathbf{res}^{\star\to *} \to \mathbf{res}^{\star\to *}$$

equals the natural transformation obtained from the co-unit of the adjunction

$$\mathrm{coPrim}^{\text{enh,ind-nilp}}_{\mathcal{P}} \circ \mathrm{Prim}^{\text{enh,ind-nilp}}_{\mathcal{Q}} \to \mathrm{Id}$$

by composing with $\mathbf{res}^{\star\to *}$. Hence, it is an isomorphism, by assumption.

Now, the second arrow in the above composition is an isomorphism by Corollary 2.10.5(b). Hence, so is the third arrow.

The unit of the adjunction

$$\mathrm{Id} \to \mathrm{Prim}^{\text{enh}}_{\mathcal{Q}} \circ \mathrm{coPrim}^{\text{enh}}_{\mathcal{P}}$$

identifies with the composition

$$\mathrm{Id} \to \mathrm{Prim}^{\text{enh,ind-nilp}}_{\mathcal{Q}} \circ \mathrm{coPrim}^{\text{enh,ind-nilp}}_{\mathcal{P}} \to \\ \to \mathrm{Prim}^{\text{enh,ind-nilp}}_{\mathcal{Q}} \circ (\mathbf{res}^{\star\to *})^R \circ \mathbf{res}^{\star\to *} \circ \mathrm{coPrim}^{\text{enh,ind-nilp}}_{\mathcal{P}} \simeq \mathrm{Prim}^{\text{enh}}_{\mathcal{Q}} \circ \mathrm{coPrim}^{\text{enh}}_{\mathcal{P}}.$$

Now, since we already know that Conjecture 2.6.6 implies Conjecture 2.8.4, it suffices to show that the map

$$\mathrm{Id} \to \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}}$$

is an isomorphism on the essential image of $\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}}$. However, this is a formal consequence of the fact that $\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}}$ is fully faithful.

□

3. Associative algebras

In this section we specialize the notions from Sects. 1 and 2 to the case of the associative operads, and point out some specifics.

In particular, we will see that the (augmented) associative operad is self Koszul-dual and we will give more explicit descriptions of the Koszul duality functors between augmented associative algebras and co-algebras.

3.1. Associative algebras and co-algebras.

In this subsection we recall some basic concepts related to the notion of associative algebra in a given monoidal category.

3.1.1. Let $\mathbf{O}$ be a monoidal category. We let $\mathrm{AssocAlg}(\mathbf{O})$ denote the category of unital associative algebras in $\mathbf{O}$. We let $\mathbf{oblv}_{\mathrm{Assoc}}$ denote the forgetful functor $\mathrm{AssocAlg}(\mathbf{O}) \to \mathbf{O}$. The functor $\mathbf{oblv}_{\mathrm{Assoc}}$ is conservative and commutes with limits.

Since $\mathbf{1}_{\mathbf{O}} \in \mathbf{O}$ is the initial object in $\mathrm{AssocAlg}(\mathbf{O})$, the functor $\mathbf{oblv}_{\mathrm{Assoc}}$ canonically factors as

$$\mathrm{AssocAlg}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1}_{\mathbf{O}}/} \to \mathbf{O}.$$

We will denote the resulting functor $\mathrm{AssocAlg}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1}_{\mathbf{O}}/}$ by $\mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}/}$.

3.1.2. Assume that $\mathbf{O}$ admits colimits, and that the monoidal operation preserves *sifted* colimits in each variable. Then the category $\mathrm{AssocAlg}(\mathbf{O})$ also admits colimits, and the functor $\mathbf{oblv}_{\mathrm{Assoc}}$ commutes *sifted* colimits, see [Lu2, Proposition 3.2.3.1].

Moreover, in this case $\mathbf{oblv}_{\mathrm{Assoc}}$ admits a left adjoint, denoted

$$\mathbf{free}_{\mathrm{Assoc}} : \mathbf{O} \to \mathrm{AssocAlg}(\mathbf{O}).$$

When the monoidal operation on $\mathbf{O}$ commutes with coproducts, the composition $\mathbf{oblv}_{\mathrm{Assoc}} \circ \mathbf{free}_{\mathrm{Assoc}}$ is canonically isomorphic to the functor

$$V \mapsto \underset{n\geq 0}{\sqcup} V^{\otimes n},$$

see [Lu2, Proposition 4.1.1.14].

Remark 3.1.3. Note that the adjoint pair

$$\mathbf{free}_{\mathrm{Assoc}} : \mathbf{O} \rightleftarrows \mathrm{AssocAlg}(\mathbf{O}) : \mathbf{oblv}_{\mathrm{Assoc}}$$

does *not* fit into the paradigm of algebras over operads as defined in Sect. 1.1.2. This is because in our definition of operads we did not allow 0-ary operations.

3.1.4. Assume that $\mathbf{O}$ is *symmetric* monoidal. In this case, the category $\mathrm{AssocAlg}(\mathbf{O})$ has a natural symmetric monoidal structure (given by tensor product) and the functor $\mathbf{oblv}_{\mathrm{Assoc}}$ is naturally symmetric monoidal, see [Chapter I.1, Sect. 3.3.5].

Since the initial object of $\mathrm{AssocAlg}(\mathbf{O})$, i.e., $\mathbf{1_O}$, is the unit of $\mathrm{AssocAlg}(\mathbf{O})$ with respect to its symmetric monoidal structure, the identity functor on $\mathrm{AssocAlg}(\mathbf{O})$ has a natural right-lax symmetric monoidal structure, when considered as a functor from $\mathrm{AssocAlg}(\mathbf{O})$ equipped with the tensor product structure to $\mathrm{AssocAlg}(\mathbf{O})$ equipped with the co-Cartesian symmetric monoidal structure.

I.e., we have a compatible system of natural transformations:

$$A_1 \sqcup ... \sqcup A_n \to A_1 \otimes ... \otimes A_n, \tag{3.1}$$

given as the coproduct of the maps

$$A_i \simeq \mathbf{1_O} \otimes ... \otimes A_i \otimes ... \otimes \mathbf{1_O} \to A_1 \otimes ... \otimes A_n.$$

3.1.5. Let $\mathbf{O} = \mathbf{C}$ be a category with finite limits, viewed as a symmetric monoidal category with respect to the Cartesian symmetric monoidal structure. In this case we have, by definition,

$$\mathrm{AssocAlg}(\mathbf{C}) = \mathrm{Monoid}(\mathbf{C}).$$

3.1.6. Let $\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$ denote the category $(\mathrm{AssocAlg}(\mathbf{O}))_{/\mathbf{1_O}}$. This is the category of augmented associative algebras on $\mathbf{O}$. The category $\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$ has several forgetful functors, denoted

$$\mathbf{oblv}_{\mathrm{Assoc}},\ \mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}/},\ \mathbf{oblv}_{\mathrm{Assoc},/\mathbf{1}},\ \mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}//\mathbf{1}},$$

with values in

$$\mathbf{O},\ \mathbf{O}_{\mathbf{1_O}/},\ \mathbf{O}_{/\mathbf{1_O}},\ \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}},$$

respectively.

3.1.7. In this sub-subsection, we shall assume that $\mathbf{O}$ is a symmetric monoidal *DG* category. (In particular, the monoidal operation on $\mathbf{O}$ commutes with all colimits.)

We have a canonical equivalence

$$\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \simeq \mathrm{Assoc}^{\mathrm{aug}}\text{-}\mathrm{Alg}(\mathbf{O}), \tag{3.2}$$

where the latter is the category of algebras over the $\mathrm{Assoc}^{\mathrm{aug}}$ operad. Thus, we obtain yet another forgetful functor

$$\mathbf{oblv}_{\mathrm{AssocAlg},+} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O},$$

equal in the notion of Sect. 1.1.3 to $\mathbf{oblv}_{\mathrm{Assoc}^{\mathrm{aug}}}$. It equals the composition of $\mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}//\mathbf{1}}$ with the functor $\mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \to \mathbf{O}$, inverse to the equivalence

$$V \mapsto \mathbf{1_O} \oplus V, \quad \mathbf{O} \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}, \tag{3.3}$$

i.e. it is given by the fiber of the augmentation map $V \to \mathbf{1_O}$.

The functor $\mathbf{free}_{\mathrm{Assoc}}$ is naturally isomorphic to the composition

$$\mathbf{O} \overset{\mathbf{free}_{\mathrm{AssocAlg}^{\mathrm{aug}}}}{\longrightarrow} \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{AssocAlg}(\mathbf{O}),$$

where the second arrow is the forgetful functor.

3.1.8. By reversing the arrows, we obtain the corresponding definitions and pieces of notation of co-associative co-algebras.

3.1.9. The following observation will be used repeatedly. Let $\mathbf{O} = \mathbf{C}$ be as in Sect. 3.1.5. Then the forgetful functor

$$\mathbf{oblv}_{\text{Coassoc}} : \text{Coassoc}(\mathbf{C}) \to \mathbf{C}$$

is an equivalence, see [Lu2, Proposition 2.4.3.9].

Informally, every object $\mathbf{c}$ of $\mathbf{C}$ canonically lifts to one in Coassoc($\mathbf{C}$) via the diagonal map

$$\mathbf{c} \to \mathbf{c} \times \mathbf{c}.$$

3.2. **The Bar construction.** In this section we let $\mathbf{O}$ be a monoidal category with limits and colimits.

We will review the general Bar-construction that relates augmented associative algebras and co-algebras in $\mathbf{O}$.

3.2.1. We have a canonically defined functor

$$\text{Bar}^\bullet : \text{AssocAlg}^{\text{aug}}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}^{\text{op}}},$$

see [Lu2, Sect. 5.2.2].

The functor $\text{Bar}^\bullet$ lifts to a functor

$$\text{Bar}^\bullet_{\mathbf{1}//\mathbf{1}} : \text{AssocAlg}^{\text{aug}}(\mathbf{O}) \to (\mathbf{O}^{\mathbf{\Delta}^{\text{op}}})_{\mathbf{1_O}//\mathbf{1_O}},$$

where $\mathbf{1_O} \in \mathbf{O}^{\mathbf{\Delta}^{\text{op}}}$ is the constant simplicial object with value $\mathbf{1_O}$.

If the monoidal structure on $\mathbf{O}$ is symmetric, then the above functors have a natural symmetric monoidal structure.

3.2.2. We define the functors

$$\text{Bar}_{\mathbf{1}//\mathbf{1}} : \text{AssocAlg}^{\text{aug}}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \text{ and } \text{Bar} : \text{AssocAlg}^{\text{aug}}(\mathbf{O}) \to \mathbf{O}$$

to be the compositions of $\text{Bar}^\bullet_{\mathbf{1}//\mathbf{1}}$ (resp., $\text{Bar}^\bullet$) with the functor of colimit over $\mathbf{\Delta}^{\text{op}}$ (a.k.a, geometric realization)

$$\mathbf{O}^{\mathbf{\Delta}^{\text{op}}} \to \mathbf{O}.$$

If the monoidal structure on $\mathbf{O}$ is symmetric, then the symmetric monoidal structure on $\text{Bar}^\bullet_{\mathbf{1}//\mathbf{1}}$ (resp., $\text{Bar}^\bullet$) induces one on $\text{Bar}_{\mathbf{1}//\mathbf{1}}$ (resp., Bar).

3.2.3. The functor $\text{Bar}_{\mathbf{1}//\mathbf{1}}$ can be also thought of as follows:

We have a naturally defined functor

$$\mathbf{triv}_{\text{Assoc}^{\text{aug}}} : \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \to \text{AssocAlg}^{\text{aug}}(\mathbf{O}).$$

The functor $\text{Bar}_{\mathbf{1}//\mathbf{1}}$ is the left adjoint of the composition

$$\mathbf{triv}_{\text{Assoc}^{\text{aug}}} \circ \Omega_{\mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}}.$$

3.2.4. Suppose for a moment that the monoidal structure on $\mathbf{O}$ is Cartesian. Then

$$\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) = \mathrm{Monoid}(\mathbf{O}),$$

and the corresponding functor

$$\mathrm{Bar}^{\bullet} : \mathrm{Monoid}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}$$

is fully faithful, see [Lu2, Proposition 4.1.2.6].

Its essential image consists of those simplicial objects $n \mapsto V^n$, for which the maps for every n the maps

$$[1] \to [n], \quad (\{0\} \mapsto \{i-1\}, \{1\} \mapsto \{i\}), \quad i = 1, ..., n$$

define an isomorphism

$$V^n \to (V^1)^{\times n}.$$

The functor Bar identifies with the classifying space functor

$$B : \mathrm{Monoid}(\mathbf{O}) \to \mathbf{O}.$$

3.2.5. A key feature of the functor

$$\mathrm{Bar}_{\mathbf{1}//\mathbf{1}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}$$

is that it canonically lifts to a functor

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}),$$

i.e.,

$$\mathrm{Bar}_{\mathbf{1}//\mathbf{1}} \simeq \mathbf{oblv}_{\mathrm{Coassoc},\mathbf{1}//\mathbf{1}} \circ \mathrm{Bar}^{\mathrm{enh}},$$

see [Lu2, Theorem 5.2.2.17].

If $\mathbf{O}$ is symmetric, then the functor $\mathrm{Bar}^{\mathrm{enh}}$ also acquires a left-lax symmetric monoidal structure, extending that on $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$. This structure is strict if the monoidal operation on $\mathbf{O}$ preserves colimits.

3.2.6. Reversing the arrows, we obtain the corresponding functors

$$\mathrm{coBar}^{\bullet}_{\mathbf{1}//\mathbf{1}} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to (\mathbf{O}^{\mathbf{\Delta}})_{\mathbf{1_O}//\mathbf{1_O}},$$

$$\mathrm{coBar}^{\bullet} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}},$$

$$\mathrm{coBar}_{\mathbf{1}//\mathbf{1}} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}},$$

$$\mathrm{coBar} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O},$$

and

$$\mathrm{coBar}^{\mathrm{enh}} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}).$$

3.2.7. It is another basic fact that the functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \rightleftarrows \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coBar}^{\mathrm{enh}} \tag{3.4}$$

form an adjoint pair.

In general, neither of the functors (3.4) is fully faithful.

3.3. **Koszul duality functors: associative case.** In this subsection we let $\mathbf{O}$ be a symmetric monoidal DG category.

We will specialize the paradigm of Koszul duality functors

$$\mathrm{coPrim}^{\mathrm{enh}}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \rightleftarrows \mathcal{P}^{\vee}\text{-}\mathrm{Coalg}(\mathbf{O}) : \mathrm{Prim}^{\mathrm{enh}}_{\mathcal{P}^{\vee}}$$

to the case $\mathcal{P} = \mathrm{Assoc}^{\mathrm{aug}}$.

3.3.1. According to Sect. 3.2, we have a canonically defined functor

$$\mathrm{Bar}_{\mathbf{1}//\mathbf{1}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}.$$

Let

$$\mathrm{Bar}_{/\mathbf{1}},\ \mathrm{Bar}_{\mathbf{1}/}$$

and Bar_+ denote the composition of $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$ with the functors

$$\mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \to \mathbf{O}_{/\mathbf{1_O}},\ \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \to \mathbf{O}_{\mathbf{1_O}/}$$

and

$$\mathbf{O} \simeq \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}, \quad V \mapsto \mathbf{1_O} \oplus V,$$

respectively.

By Sect. 3.2.3, the functor Bar_+ is the left adjoint to the functor

$$\mathbf{O} \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathbf{triv}_{\mathrm{AssocAlg}^{\mathrm{aug}}} \circ [-1].$$

I.e.,

$$\mathrm{Bar}_+ \simeq [1] \circ \mathrm{coPrim}_{\mathrm{Assoc}^{\mathrm{aug}}}.$$

3.3.2. Similarly, we have the functors

$$\mathrm{coBar}_{/\mathbf{1}},\ \mathrm{coBar}_{\mathbf{1}/},\ \mathrm{coBar}\ \text{ and }\ \mathrm{coBar}_+,$$

where coBar_+ is the right adjoint to the functor

$$\mathbf{O} \to \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathbf{triv}_{\mathrm{CoassocCoalg}^{\mathrm{aug}}} \circ [1].$$

Hence,

$$\mathrm{coBar}_+ \simeq [-1] \circ \mathrm{Prim}_{\mathrm{Coassoc}^{\mathrm{aug}}}.$$

3.3.3. As was mentioned in Sect. 2.3.3, we have canonical isomorphisms of operads

$$(\mathrm{Coassoc}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Assoc}^{\mathrm{aug}}[-1]. \tag{3.5}$$

Hence, the functors $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$ and $\mathrm{coBar}_{\mathbf{1}//\mathbf{1}}$ lift to functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathrm{Bar}_{\mathbf{1}//\mathbf{1}} \simeq \mathbf{oblv}_{\mathrm{Coassoc},\mathbf{1}//\mathbf{1}} \circ \mathrm{Bar}^{\mathrm{enh}}$$

and

$$\mathrm{coBar}^{\mathrm{enh}} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathrm{coBar}_{\mathbf{1}//\mathbf{1}} \simeq \mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}//\mathbf{1}} \circ \mathrm{coBar}^{\mathrm{enh}},$$

respectively.

3.3.4. It is a basic feature of the isomorphism (3.5) that the above functors $\mathrm{Bar}^{\mathrm{enh}}$ and $\mathrm{coBar}^{\mathrm{enh}}$ are canonically the same as those in Sect. 3.2.5.

In particular, the functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \rightleftarrows \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coBar}^{\mathrm{enh}}$$

are mutually adjoint, and the functor

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O})$$

is naturally symmetric monoidal.

By adjunction, we obtain that the functor $\mathrm{coBar}^{\mathrm{enh}}$ is naturally right-lax symmetric monoidal.

4. Lie algebras and co-commutative co-algebras

In this section we study of the relationship between Lie algebras and co-commutative co-algebras.

The main result of this section is Theorem 4.4.6, which says that, although the Chevalley functor from Lie algebras to co-commutative co-algebras is not fully faithful, its cousin, obtained by first looping our Lie algebra, and regarding the output as a co-commutative Hopf algebra, is fully faithful.

4.1. **Koszul duality functors: commutative vs. Lie case.** In this subsection we continue to suppose that $\mathbf{O}$ is a symmetric monoidal DG category. We will specialize the paradim of Koszul duality functors

$$\mathrm{coPrim}^{\mathrm{enh}}_{\mathcal{P}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \rightleftarrows \mathcal{P}^\vee\text{-Coalg}(\mathbf{O}) : \mathrm{Prim}^{\mathrm{enh}}_{\mathcal{P}^\vee}$$

to the case $\mathcal{P} = \mathrm{Lie}$.

4.1.1. First, we remark that the discussion in Sect. 3.1 renders verbatim to the situation when instead of associative algebras on $\mathbf{O}$ we talk about (co-)commutative (co-)algebras.

We note, however, the following feature of the symmetric monoidal structure on $\mathrm{ComAlg}(\mathbf{O})$: the corresponding natural transformations (3.1) are isomorphisms.

I.e., the symmetric monoidal structure on $\mathrm{ComAlg}(\mathbf{O})$, given by tensor product, equals the co-Cartesian symmetric monoidal structure, see [Chapter I.1, Sect. 3.3.6]. Similarly, the symmetric monoidal structure on $\mathrm{CocomCoalg}(\mathbf{O})$, given by tensor product, equals the Cartesian symmetric monoidal structure.

Note that the forgetful functors

$$\mathbf{res}^{\mathrm{Com}\to\mathrm{Assoc}} : \mathrm{ComAlg}(\mathbf{O}) \to \mathrm{AssocAlg}(\mathbf{O})$$

and

$$\mathbf{res}^{\mathrm{Cocom}\to\mathrm{Coassoc}} : \mathrm{CocomCoalg}(\mathbf{O}) \to \mathrm{CoassocCoalg}(\mathbf{O})$$

both have a natural symmetric monoidal structure.

4.1.2. We let Chev_+ denote the functor

$$\mathrm{LieAlg}(\mathbf{O}) \to \mathbf{O}, \quad [1] \circ \mathrm{coPrim}_{\mathrm{Lie}}.$$

I.e., this is the functor, left adjoint to the functor

$$\mathbf{O} \to \mathrm{LieAlg}(\mathbf{O}), \quad \mathbf{triv}_{\mathrm{Lie}} \circ [-1].$$

We let

$$\mathrm{Chev}_{\mathbf{1}//\mathbf{1}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}$$

denote the composition of Chev_+ with the equivalence (3.3).

Composing further with the forgetful functors from $\mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}$, we obtain the corresponding functors, denoted

$$\mathrm{Chev}_{/\mathbf{1}},\ \mathrm{Chev}_{\mathbf{1}/},\ \mathrm{Chev},$$

from $\mathrm{LieAlg}(\mathbf{O})$ to

$$\mathbf{O}_{/\mathbf{1_O}},\ \mathbf{O}_{\mathbf{1_O}/},\ \mathbf{O},$$

respectively.

4.1.3. We denote by coChev the functor

$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}, \quad [-1] \circ \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}.$$

I.e., this is the functor, right adjoint to the functor

$$\mathbf{O} \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathbf{triv}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ [1].$$

4.1.4. As was mentioned in Sect. 2.3.3, we have canonical isomorphisms of operads

$$(\mathrm{Cocom}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Lie}[-1]. \tag{4.1}$$

Hence, the functors $\mathrm{Chev}_{\mathbf{1}//\mathbf{1}}$ and coChev lift to functors

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathrm{Chev}_{\mathbf{1}//\mathbf{1}} \simeq \mathbf{oblv}_{\mathrm{Cocom},\mathbf{1}//\mathbf{1}} \circ \mathrm{Chev}^{\mathrm{enh}}$$

and

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}), \quad \mathrm{coChev} \simeq \mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{coChev}^{\mathrm{enh}},$$

respectively.

Furthermore, the functors

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coChev}^{\mathrm{enh}}$$

are mutually adjoint.

In particular, we obtain a canonical natural transformation

$$\mathrm{Id} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}},$$

and by applying the forgetful functor $\mathbf{oblv}_{\mathrm{Lie}}$, also the natural transformation

$$[1] \circ \mathbf{oblv}_{\mathrm{Lie}} \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Chev}^{\mathrm{enh}}. \tag{4.2}$$

4.1.5. The functor

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

has a natural left-lax symmetric monoidal structure, when we consider both categories as endowed with Cartesian symmetric monoidal structure. (Recall, however, that the Cartesian symmetric monoidal structure on $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$ equals one given by the tensor product, see Sect. 4.1.1.)

In particular, we obtain that the functor

$$\mathrm{Chev} : \mathrm{LieAlg}(\mathbf{O}) \to \mathbf{O}$$

inherits a left-lax symmetric monoidal structure.

In Sect. 4.2.6 we will prove:

Lemma 4.1.6. *The left-lax symmetric monoidal structure on* $\mathrm{Chev}^{\mathrm{enh}}$ *is strict.*

Corollary 4.1.7. *The left-lax symmetric monoidal structure on* Chev *is strict.*

4.2. **The symmetric co-algebra.** The symmetric (co)algebra construction

$$V \rightsquigarrow \underset{n\geq 0}{\oplus} \mathrm{Sym}^n(V)$$

is ubiquitous in algebra.

In this subsection we initiate its study in its incarnation as a *co-commutative co-algebra.*

4.2.1. We denote by

$$\mathrm{Sym} : \mathbf{O} \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

the functor $\mathbf{cofree}^{\mathrm{fake}}_{\mathrm{Cocom}^{\mathrm{aug}}}$, see Sect. 2.8.5 for the notation.

4.2.2. Let us denote by $\underline{\mathrm{Sym}}$ (resp., $\underline{\mathrm{Sym}}_+$) the functor of $\mathbf{O} \to \mathbf{O}$ equal to the composition $\mathbf{oblv}_{\mathrm{Cocom}} \circ \mathrm{Sym}$ (resp., $\mathbf{oblv}_{\mathrm{Cocom},+} \circ \mathrm{Sym}$).

By definition, the endo-functor $\underline{\mathrm{Sym}}_+$ of $\mathbf{O}$ is one given by

$$V \mapsto \mathrm{Cocom}^{\mathrm{aug}} \star V.$$

Explicitly,

$$\underline{\mathrm{Sym}}(V) = \underset{n\geq 0}{\oplus} \mathrm{Sym}^n(V) \text{ and } \underline{\mathrm{Sym}}_+(V) = \underset{n\geq 1}{\oplus} \mathrm{Sym}^n(V).$$

4.2.3. By (2.11) we have

$$\mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \simeq \mathrm{Sym}, \tag{4.3}$$

and

$$\mathrm{Chev} \circ \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \simeq \underline{\mathrm{Sym}}.$$

By adjunction from (4.3), we obtain canonical natural transformations

$$\mathbf{triv}_{\mathrm{Lie}} \circ [-1] \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Sym}, \quad \mathbf{O} \to \mathrm{LieAlg}(\mathbf{O}) \tag{4.4}$$

and by applying $\mathbf{oblv}_{\mathrm{Lie}}$ the natural transformation

$$\mathrm{Id} \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Sym}\,. \tag{4.5}$$

By Theorem 2.9.4, we have:

Theorem 4.2.4. *The natural transformation* (4.4) *is an isomorphism.*

Corollary 4.2.5. *The natural transformation* (4.5) *is an isomorphism.*

4.2.6. *Proof of Lemma 4.1.6.* We need to show that for $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathrm{LieAlg}(\mathbf{O})$, the map

$$\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_1 \times \mathfrak{h}_2) \to \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_1) \sqcup \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_2) \simeq \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_1) \otimes \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_2)$$

is an isomorphism.

By Sect. 2.5.2, the above map lifts to a map

$$\mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_1 \times \mathfrak{h}_2) \to \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_1) \otimes \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_2),$$

and it suffices to show that this map is an isomorphism in $\mathrm{CocomCoalg}(\mathbf{O}^{\mathrm{Fil},\geq 0})$.

Since the functor ass-gr is conservative on non-negatively graded objects, in order to show that the latter map is an isomorphism, it suffices to show that the induced map

$$\text{ass-gr} \circ \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_1 \times \mathfrak{h}_2) \to \text{ass-gr} \circ \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_1) \otimes \text{ass-gr} \circ \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_2)$$

is an isomorphism in $\mathrm{CocomCoalg}(\mathbf{O}^{\mathrm{gr},\geq 0})$.

By (2.4), the latter map identifies with the canonical map

$$\mathrm{Sym}^{\mathrm{gr}}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_1) \oplus \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_2)) \simeq \mathrm{Sym}^{\mathrm{gr}}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_1 \times \mathfrak{h}_2)) \to$$
$$\to \mathrm{Sym}^{\mathrm{gr}}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_1)) \otimes \mathrm{Sym}^{\mathrm{gr}}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_2)),$$

where

$$\mathrm{Sym}^{\mathrm{gr}} : \mathbf{O} \to \mathrm{CocomCoalg}(\mathbf{O}^{\mathrm{gr},\geq 0})$$

is the graded version of the functor Sym of Sect. 4.2.1, i.e.,

$$\mathrm{Sym}^{\mathrm{gr}} = \mathrm{Sym} \circ (\deg = 1).$$

Now, the fact that for $V_1, V_2 \in \mathbf{O}$ the map

$$\mathrm{Sym}^{\mathrm{gr}}(V_1 \oplus V_2) \to \mathrm{Sym}^{\mathrm{gr}}(V_1) \otimes \mathrm{Sym}^{\mathrm{gr}}(V_2)$$

is an isomorphism, is straightforward.

□

4.3. **Chevalley complex and the loop functor.** The principal actor in this chapter will be the functor[6]

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}).$$

We will see (Theorem 4.4.6) that, unlike the functor $\mathrm{Chev}^{\mathrm{enh}}$, the above functor is fully faithful (i.e., looping helps to preserve structure).

4.3.1. Recall that by Lemma 4.1.6, the functor

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

has a symmetric monoidal structure, when we consider both $\mathrm{LieAlg}(\mathbf{O})$ and $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$ as symmetric monoidal categories with respect to Cartesian product.

In particular, we obtain that $\mathrm{Chev}^{\mathrm{enh}}$ gives rise to a functor

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) : \mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})) \simeq \mathrm{Monoid}(\mathrm{LieAlg}(\mathbf{O})) \to \\ \to \mathrm{Monoid}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) =: \mathrm{CocomBialg}(\mathbf{O}).$$

Moreover, its essential image automatically lies in

$$\mathrm{CocomHopf}(\mathbf{O}) := \mathrm{Grp}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \subset \\ \subset \mathrm{Monoid}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) = \mathrm{CocomBialg}(\mathbf{O}).$$

4.3.2. Consider now the composite functor

$$\mathbf{oblv}_{\mathrm{Monoid}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}).$$

We claim:

Proposition 4.3.3. *The functor* $\mathbf{oblv}_{\mathrm{Monoid}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{Lie}$ *identifies canonically with* $\mathrm{Sym} \circ \mathbf{oblv}_{\mathrm{Lie}}$.

Proof. First, we note that we have a tautological isomorphism

$$\mathbf{oblv}_{\mathrm{Monoid}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}}.$$

Now, by Proposition 1.7.2, we have

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}} \simeq \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \circ \mathbf{oblv}_{\mathrm{Lie}},$$

so

$$\mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \circ \mathbf{oblv}_{\mathrm{Lie}} \simeq \mathrm{Sym} \circ \mathbf{oblv}_{\mathrm{Lie}}.$$

□

[6]As we will see in Sect. 6, the functor $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$ identifies with another familiar functor, namely, that of the universal enveloping algebra.

4.4. **Primitives in bialgebras.** Let $\mathfrak{h}$ be a Lie algebra. Then the universal enveloping algebra $U(\mathfrak{h})$ is naturally a cocommutative Hopf algebra. Moreover, $\mathfrak{h}$ can be recovered as the subspace of primitive elements of $U(\mathfrak{h})$.

In this subsection, we will give a higher algebra version of this statement. We show that the space of primitives of a cocommutative bi-algebra has a canonical structure of a Lie algebra and that it gives a left inverse to the functor $\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}$, (while the latter identifies with the universal enveloping algebra by Theorem 6.1.2).

The key actor in this subsection we be the functor right adjoint to

$$\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}} : \text{LieAlg}(\mathbf{O}) \to \text{CocomBialg}(\mathbf{O}).$$

We will see that this right adjoint provides a lift of the functor

$$\text{Prim}_{\text{Cocom}^{\text{aug}}} \circ \mathbf{oblv}_{\text{Monoid}} : \text{CocomBialg}(\mathbf{O}) \to \mathbf{O}$$

to a functor

$$\text{CocomBialg}(\mathbf{O}) \to \text{LieAlg}(\mathbf{O}).$$

4.4.1. Consider again the functor

$$\text{coChev}^{\text{enh}} : \text{CocomCoalg}^{\text{aug}}(\mathbf{O}) \to \text{LieAlg}(\mathbf{O}).$$

Being the right adjoint of a symmetric monoidal functor (namely, Chev^{enh}), the functor $\text{coChev}^{\text{enh}}$ acquires a natural right-lax symmetric monoidal structure. In particular, it gives rise to a functor, denoted $\text{Monoid}(\text{coChev}^{\text{enh}})$:

$$\text{CocomBialg}(\mathbf{O}) = \text{Monoid}(\text{CocomCoalg}^{\text{aug}}(\mathbf{O})) \to \text{Monoid}(\text{LieAlg}(\mathbf{O})) \simeq \text{Grp}(\text{LieAlg}(\mathbf{O})).$$

By construction, the functor $\text{Monoid}(\text{coChev}^{\text{enh}})$ is the right adjoint of the functor

$$\text{Grp}(\text{LieAlg}(\mathbf{O})) \overset{\text{Grp}(\text{Chev}^{\text{enh}})}{\longrightarrow} \text{Grp}(\text{CocomCoalg}^{\text{aug}}(\mathbf{O})) \hookrightarrow \text{Monoid}(\text{CocomCoalg}^{\text{aug}}(\mathbf{O})).$$

4.4.2. Since B_{Lie} and Ω_{Lie} are mutually inverse equivalences (see Proposition 1.6.4), the functor $\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}$ is the left adjoint of the functor

$$B_{\text{Lie}} \circ \text{Monoid}(\text{coChev}^{\text{enh}}), \quad \text{CocomBialg}(\mathbf{O}) \to \text{LieAlg}(\mathbf{O}).$$

4.4.3. Note that

$$\mathbf{oblv}_{\text{Lie}} \circ B_{\text{Lie}} \circ \text{Monoid}(\text{coChev}^{\text{enh}}) \simeq \text{Prim}_{\text{Cocom}^{\text{aug}}} \circ \mathbf{oblv}_{\text{Monoid}}, \quad \text{CocomBialg}(\mathbf{O}) \to \mathbf{O},$$

where

$$\mathbf{oblv}_{\text{Monoid}} : \text{CocomBialg}(\mathbf{O}) \to \text{CocomCoalg}^{\text{aug}}(\mathbf{O})$$

is the functor of forgetting the monoid structure.

So, the functor $B_{\text{Lie}} \circ \text{Monoid}(\text{coChev}^{\text{enh}})$ can be viewed as one upgrading the functor

$$\text{Prim}_{\text{Cocom}^{\text{aug}}} \circ \mathbf{oblv}_{\text{Monoid}} : \text{CocomBialg}(\mathbf{O}) \to \mathbf{O}$$

to a functor $\text{CocomBialg}(\mathbf{O}) \to \text{LieAlg}(\mathbf{O})$.

Remark 4.4.4. Let us repeat the last observation in words:

For a co-commutative bi-algebra A, the space of primitives of A considered just as an augmented co-commutative co-algebra, has a natural structure of Lie algebra.

This is a higher algebra version of the motto 'the tangent space of a Lie group has a structure of a Lie algebra'.

Note, however, that we defined this Lie algebra structure not be explicitly writing down the Lie bracket, but by appealing the Koszul duality of the corresponding operads:

$$(\mathrm{Cocom}^{\mathrm{aug}})^\vee \simeq \mathrm{Lie}[-1].$$

4.4.5. We now claim:

Theorem 4.4.6. *The functor*

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O})$$

is fully faithful.

Proof. We need to show that the unit of the adjunction

$$\mathrm{Id} \to \Big(B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}})\Big) \circ \Big(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}\Big)$$

is an isomorphism.

Since B_{Lie} and Ω_{Lie} are mutually inverse equivalences, it suffices to show that the natural transformation

$$\Omega_{\mathrm{Lie}} \to \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}},$$

obtained by applying the unit of the $(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}), \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}))$-adjunction to Ω_{Lie}, is an isomorphism.

For the latter, it suffices to show that the natural transformation

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}} \to \mathbf{oblv}_{\mathrm{Monoid}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$$

is an isomorphism. Note, however, that the latter natural transformation identifies with

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}},$$

obtained by applying the unit of the $(\mathrm{Chev}^{\mathrm{enh}}, \mathrm{coChev}^{\mathrm{enh}})$-adjunction to $\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}}$.

However, by Proposition 1.7.2, the essential image of the functor $\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}}$ belongs to the essential image of the functor $\mathbf{triv}_{\mathrm{Lie}}$, and the assertion follows from Theorem 4.2.4.

□

5. The universal enveloping algebra

In this section we recall some basic facts about the functor of universal enveloping algebra in the setting of higher algebra.

5.1. **Universal enveloping algebra: definition.**

In this subsection we recollect the main constructions related to the functor of universal envelope of a Lie algebra.

5.1.1. There is a canonical map of operads

$$\mathrm{Lie} \to \mathrm{Assoc}^{\mathrm{aug}}. \tag{5.1}$$

From this map we obtain the restriction functor

$$\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Lie}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}).$$

The functor

$$U : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$$

is defined to be the left adjoint of $\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Lie}}$.

5.1.2. The map (5.1) has the following additional structure: the functor $\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}}\to\mathrm{Lie}}$ has a natural right-lax symmetric monoidal structure, where $\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$ is a symmetric monoidal category via the tensor product, and $\mathrm{LieAlg}(\mathbf{O})$ a symmetric monoidal category via the Cartesian product.

Hence, the functor U acquires a natural left-lax symmetric monoidal structure (as we shall see shortly, this left-lax symmetric monoidal structure is actually symmetric monoidal).

Finally, we will need one more piece of structure on (5.1):

The above left-lax symmetric monoidal structure on U makes the following diagram of left-lax symmetric monoidal functors commute:

$$\begin{array}{ccc} \mathrm{LieAlg}(\mathbf{O}) & \xrightarrow{\mathrm{Chev}^{\mathrm{enh}}} & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \\ {\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{res}^{\mathrm{Cocom}^{\mathrm{aug}}\to\mathrm{Coassoc}^{\mathrm{aug}}}} \\ \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) & \xrightarrow{\mathrm{Bar}^{\mathrm{enh}}} & \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}), \end{array} \tag{5.2}$$

such that the induced isomorphism of functors

$$\begin{aligned} (5.3)\quad & \mathrm{Bar}_+ \circ U \simeq \mathbf{oblv}_{\mathrm{Coassoc},+} \circ \mathrm{Bar}^{\mathrm{enh}} \circ U \simeq \\ & \simeq \mathbf{oblv}_{\mathrm{Coassoc},+} \circ \mathbf{res}^{\mathrm{Cocom}^{\mathrm{aug}}\to\mathrm{Coassoc}^{\mathrm{aug}}} \circ \mathrm{Chev}^{\mathrm{enh}} \simeq \mathbf{oblv}_{\mathrm{Cocom},+} \circ \mathrm{Chev}^{\mathrm{enh}} \simeq \mathrm{Chev}_+ \end{aligned}$$

is the tautological isomorphism arising by adjunction from

$$\mathbf{res}^{\mathrm{Coassoc}^{\mathrm{aug}}\to\mathrm{Lie}} \circ \mathbf{triv}_{\mathrm{Assoc}^{\mathrm{aug}}} \simeq \mathbf{triv}_{\mathrm{Lie}}.$$

Remark 5.1.3. In fact, one can obtain the map (5.1), along with the above properties, by defining it as corresponding to the map of co-operads

$$\mathrm{Cocom}^{\mathrm{aug}} \to \mathrm{Coassoc}^{\mathrm{aug}}$$

via the isomorphisms

$$(\mathrm{Cocom}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Lie}[-1] \text{ and } (\mathrm{Coassoc}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Assoc}^{\mathrm{aug}}[-1].$$

5.1.4. Being (left-lax) monoidal, the functor U gives rise to a functor

$$\mathrm{CocomCoalg}(\mathrm{LieAlg}(\mathbf{O})) \to \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})).$$

Pre-composing with the equivalence $\mathrm{LieAlg}(\mathbf{O}) \simeq \mathrm{CocomCoalg}(\mathrm{LieAlg}(\mathbf{O}))$ (see Sect. 3.1.9, applied to the commutative case), we obtain a functor:

$$\mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{LieAlg}(\mathbf{O})).$$

Composing with the equivalence

$$\mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})) \simeq \mathrm{CocomBialg}(\mathbf{O})$$

of Proposition C.1.3, we obtain a functor

$$U^{\mathrm{Hopf}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}).$$

5.1.5. By Sect. 1.5, we can upgrade the functor U^{Hopf} to a functor

$$(U^{\mathrm{Hopf}})^{\mathrm{Fil}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

We will also consider the functor

$$U^{\mathrm{Fil}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

5.2. **The PBW theorem.** In this subsection we will first give a somewhat non-standard formulation of the PBW theorem, Theorem 5.2.4.

Subsequently, we will deduce from it the usual form of the PBW theorem, Corollary 5.2.6.

5.2.1. We claim that there exists a canonically defined natural transformation

$$U \circ \mathbf{triv}_{\mathrm{Lie}} \to \mathbf{res}^{\mathrm{Com}^{\mathrm{aug}} \to \mathrm{Assoc}^{\mathrm{aug}}} \circ \mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}, \tag{5.4}$$

as functors $\mathbf{O} \to \mathrm{AssocAlg}(\mathbf{O})$.

The datum of a map (5.4) is equivalent, by adjunction, to that of a natural transformation

$$\mathbf{triv}_{\mathrm{Lie}} \to \mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Lie}} \circ \mathbf{res}^{\mathrm{Com}^{\mathrm{aug}} \to \mathrm{Assoc}^{\mathrm{aug}}} \circ \mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}} \tag{5.5}$$

as functors $\mathbf{O} \to \mathrm{LieAlg}(\mathbf{O})$.

5.2.2. We construct the natural transformation (5.5) as follows.

We note that map of operads

$$\mathrm{Lie} \to \mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Com}^{\mathrm{aug}}$$

equals

$$\mathrm{Lie} \to \mathbf{1}_{\mathrm{Vect}^{\Sigma}} \to \mathrm{Com}^{\mathrm{aug}}.$$

Hence, the functor $\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Lie}} \circ \mathbf{res}^{\mathrm{Com}^{\mathrm{aug}} \to \mathrm{Assoc}^{\mathrm{aug}}}$ is canonically isomorphic to

$$\mathbf{triv}_{\mathrm{Lie}} \circ \mathbf{oblv}_{\mathrm{Com}^{\mathrm{aug}}}.$$

Now, the datum of the natural transformation in (5.5) is obtained by applying $\mathbf{triv}_{\mathrm{Lie}}$ to the natural transformation

$$\mathrm{Id} \to \underline{\mathrm{Sym}}_{+}$$

as functors $\mathbf{O} \to \mathbf{O}$.

5.2.3. The PBW theorem says:

Theorem 5.2.4. *The natural transformation* (5.4) *is an isomorhism.*

We will prove Theorem 5.2.4 in Sect. B. See Corollary 5.2.6 below for the relation with the more usual version of the PBW theorem.

5.2.5. Recall the symmetric monoidal functor

$$\text{ass-gr} : \mathbf{O}^{\mathrm{Fil}} \to \mathbf{O}^{\mathrm{gr}},$$

and the corresponding functor

$$\mathrm{Assoc}^{\mathrm{aug}}(\text{ass-gr}) : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}^{\mathrm{Fil}}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}^{\mathrm{gr}}).$$

Consider the functor

$$U^{\mathrm{gr}} := \mathrm{Assoc}^{\mathrm{aug}}(\text{ass-gr}) \circ U^{\mathrm{Fil}}, \quad \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}^{\mathrm{gr}}).$$

We claim:

Corollary 5.2.6. *There exists a canonical isomorphism of functors*

$$\mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}^{\mathrm{gr}})$$

between U^{gr} *and the composition*

$$\mathrm{LieAlg}(\mathbf{O}) \overset{\mathbf{oblv}_{\mathrm{Lie}}}{\longrightarrow} \mathbf{O} \overset{\deg=1}{\longrightarrow} \mathbf{O}^{\mathrm{gr}} \longrightarrow \mathrm{AssocAlg}(\mathbf{O}^{\mathrm{gr}}),$$

where the last arrow is $\mathbf{res}^{\mathrm{Com}^{\mathrm{aug}} \to \mathrm{Assoc}^{\mathrm{aug}}} \circ \mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}$*.*

Proof. By (1.11), the functor U^{gr} identifies canonically with

$$\mathrm{LieAlg}(\mathbf{O}) \xrightarrow{\mathbf{oblv}_{\mathrm{Lie}}} \mathbf{O} \xrightarrow{\deg=1} \mathbf{O}^{\mathrm{gr}} \xrightarrow{\mathbf{triv}_{\mathrm{Lie}}} \mathrm{LieAlg}(\mathbf{O}^{\mathrm{gr}}) \xrightarrow{U} \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}^{\mathrm{gr}}).$$

Hence, the assertion of Corollary 5.2.6 follows from that of Theorem 5.2.4. □

Corollary 5.2.6 is the usual formulation of the PBW theorem: the associated graded of the universal enveloping algebra is the symmetric algebra.

5.2.7. From Corollary 5.2.6 we shall now deduce:

Lemma 5.2.8. *The left-lax symmetric monoidal structure on the functor*

$$U : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}(\mathbf{O})$$

is symmetric monoidal.

Proof. We have to show that for $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathrm{LieAlg}(\mathbf{O})$, the morphism

$$U(\mathfrak{h}_1 \times \mathfrak{h}_2) \to U(\mathfrak{h}_1) \otimes U(\mathfrak{h}_2)$$

is an isomorphism.

It is enough to prove the corresponding fact for the functor U^{Fil}, and hence also for the functor U^{gr}. Now the assertion follows via Corollary 5.2.6 from the fact that the functor $\mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}$ is symmetric monoidal. □

5.3. **The Bar complex of the universal envelope.** Recall the isomorphism

$$\mathrm{Bar}_+ \circ U \simeq \mathrm{Chev}_+$$

of (5.3), and the resulting isomorphism

$$\mathrm{Bar} \circ U \simeq \mathrm{Chev}\,. \tag{5.6}$$

In this subsection we will upgrade the latter isomorphism to one between functors taking values in $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$.

5.3.1. Consider the functor

$$\mathrm{Bar} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

and note that the following diagram commutes

$$\begin{array}{ccc} \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})) & \xrightarrow{\mathrm{Bar}} & \mathrm{CocomCoalg}(\mathbf{O}) \\ {\scriptstyle \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{oblv}_{\mathrm{Cocom}})}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Cocom}}} \\ \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) & \xrightarrow{\mathrm{Bar}} & \mathbf{O}, \end{array}$$

since the functor $\mathbf{oblv}_{\mathrm{Cocom}} : \mathrm{CocomCoalg}(\mathbf{O}) \to \mathbf{O}$ is symmetric monoidal.

5.3.2. We claim:

Proposition 5.3.3. *There exists a canonical isomorphism*

$$\mathrm{Bar} \circ U^{\mathrm{Hopf}} \simeq \mathrm{Chev}^{\mathrm{enh}}$$

as functors $\mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$*, such that the induced isomorphism*

$$\mathrm{Bar} \circ U \simeq \mathrm{Bar} \circ \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ U^{\mathrm{Hopf}} \simeq \mathbf{oblv}_{\mathrm{Cocom}} \circ \mathrm{Bar} \circ U^{\mathrm{Hopf}} \simeq$$
$$\simeq \mathbf{oblv}_{\mathrm{Cocom}} \circ \mathrm{Chev}^{\mathrm{enh}} \simeq \mathrm{Chev}$$

identifies with (5.6).

Proof. Recall the commutative diagram (5.2), from which we produce the inner square in the next commutative diagram

$$\begin{array}{ccc}
\mathrm{LieAlg}(\mathbf{O}) & \xrightarrow{\mathrm{Chev}^{\mathrm{enh}}} & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{LieAlg}(\mathbf{O})) & \xrightarrow{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{Chev}^{\mathrm{enh}})} & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \\
{\scriptstyle \mathrm{CocomCoalg}^{\mathrm{aug}}(U)}\downarrow & & \downarrow{\scriptstyle \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{res}^{\mathrm{Cocom}^{\mathrm{aug}} \to \mathrm{Coassoc}^{\mathrm{aug}}})} \\
\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})) & \xrightarrow{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{Bar}^{\mathrm{enh}})} & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O})) \\
\sim\downarrow & & \downarrow{\scriptstyle \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{oblv}_{\mathrm{Coassoc}^{\mathrm{aug}}})} \\
\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) & \xrightarrow{\mathrm{Bar}} & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}).
\end{array}$$

In the above diagram, the composite left vertical arrow is, by definition, the functor U^{Hopf}, and the composite right vertical arrow is the identity functor.

□

6. The universal envelope via loops

In this section we establish the main result of this chapter, Theorem 6.1.2. It says that the universal enveloping algebra of a Lie algebra can be expressed via the Chevalley functor, namely, we have a canonical isomorphism of functors

$$U^{\mathrm{Hopf}} \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}.$$

6.1. **The main result.** In this subsection we state the main result of this chapter, Theorem 6.1.2.

6.1.1. Our main result is the following:

Theorem 6.1.2. *There exists a canonical isomorphism of functors*

$$U^{\mathrm{Hopf}} \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}, \quad \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}).$$

Several remarks are in order:

Remark 6.1.3. The proof of Theorem 6.1.2 is such that the isomorphism stated in the theorem automatically upgrades to an isomorphism at the filtered level:

$$(U^{\mathrm{Hopf}})^{\mathrm{Fil}} \simeq \left(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}\right)^{\mathrm{Fil}}.$$

Remark 6.1.4. One can generalize the proof of Theorem 6.1.2 to estabish the isomorphims of functors

$$U_{\mathbb{E}_n} \simeq \mathbb{E}_n\text{-Alg}^{\text{aug}}(\text{Chev}) \circ \Omega_{\text{Lie}}^{\times n}, \tag{6.1}$$

where $U_{\mathbb{E}_n}$ is the left adjoint to the forgetful functor

$$\mathbf{res}^{\mathbb{E}_n^{\text{aug}} \to \text{Lie}} : \mathbb{E}_n\text{-Alg}^{\text{aug}}(\mathbf{O}) \to \text{LieAlg}(\mathbf{O}),$$

arising from the corresponding map of operads.

Moreover, the isomorphism (6.1) automatically upgrades to an isomorphism of the corresponding functors

$$\text{LieAlg}(\mathbf{O}) \to \text{CocomCoalg}(\mathbb{E}_n^{\text{aug}}\text{-Alg}(\mathbf{O})) \simeq \mathbb{E}_n\text{-Alg}(\text{CocomCoalg}^{\text{aug}}(\mathbf{O})),$$

$$U_{\mathbb{E}_n}^{\text{Hopf}} \simeq \mathbb{E}_n\text{-Alg}^{\text{aug}}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}}^{\times n}. \tag{6.2}$$

Furthermore, the isomorphism (6.2) can be upgraded to an isomorphism of functors with values in $\text{CocomCoalg}(\mathbb{E}_n\text{-Alg}(\mathbf{O}^{\text{Fil},\geq 0}))$.

Remark 6.1.5. A very natural proof of the isomorphism (6.1) can be given using the language of factorization algebras. In the context of algebraic geometry, this is done in [FraG, Proposition 6.1.2].

6.1.6. Note that by combining Theorem 6.1.2 with Proposition 4.3.3 we obtain:

Corollary 6.1.7. *There exists a canonical isomorphism of functors*

$$\mathbf{oblv}_{\text{Assoc}} \circ U^{\text{Hopf}} \simeq \text{Sym} \circ \mathbf{oblv}_{\text{Lie}}, \quad \text{LieAlg}(\mathbf{O}) \to \text{CocomCoalg}^{\text{aug}}(\mathbf{O})$$

Remark 6.1.8. The assertion of Corollary 6.1.7 is of course well-known. The curious aspect of our proof is that it does not use the symmetrization map from the symmetric algebra to the tensor algebra, although one can show that the latter map gives the same isomorphism.

6.2. **Proof of Theorem 6.1.2.** The idea of the proof is the following: we consider the functor

$$\text{Assoc}^{\text{aug}}(U^{\text{Hopf}}) \circ \Omega_{\text{Lie}} : \text{LieAlg}(\mathbf{O}) \to \text{AssocAlg}^{\text{aug}}\left(\text{AssocAlg}^{\text{aug}}(\text{CocomCoalg}(\mathbf{O}))\right),$$

and we will compose it with two different versions of the Bar-construction: the 'inner' and the 'outer':

$$\text{AssocAlg}^{\text{aug}}\left(\text{AssocAlg}^{\text{aug}}(\text{CocomCoalg}(\mathbf{O}))\right) \to \text{AssocAlg}^{\text{aug}}(\text{CocomCoalg}(\mathbf{O})) \simeq$$
$$\simeq \text{CocomBialg}(\mathbf{O}).$$

6.2.1. The left-lax symmetric monoidal structure on the functor

$$U : \text{LieAlg}(\mathbf{O}) \to \text{AssocAlg}^{\text{aug}}(\mathbf{O})$$

gives rise to one on the functor

$$U^{\text{Hopf}} : \text{LieAlg}(\mathbf{O}) \to \text{CocomCoalg}(\text{AssocAlg}^{\text{aug}}(\mathbf{O})) \simeq \text{AssocAlg}^{\text{aug}}(\text{CocomCoalg}(\mathbf{O})).$$

However, since the left-lax symmetric monoidal structure on U is strict (see Lemma 5.2.8), so is one on U^{Hopf}. Hence, the functor U^{Hopf} gives rise to a functor that we denote $\text{Assoc}^{\text{aug}}(U^{\text{Hopf}})$:

$$\text{Monoid}(\text{LieAlg}(\mathbf{O})) \to \text{AssocAlg}\left(\text{AssocAlg}^{\text{aug}}(\text{CocomCoalg}(\mathbf{O}))\right) \simeq$$
$$\simeq \text{AssocAlg}^{\text{aug}}\left(\text{AssocAlg}^{\text{aug}}(\text{CocomCoalg}(\mathbf{O}))\right).$$

Remark 6.2.2. We can think of the category $\text{AssocAlg}^{\text{aug}}\left(\text{AssocAlg}^{\text{aug}}(\text{CocomCoalg}(\mathbf{O}))\right)$ as that of augmented $\mathbb{E}_2$-algebras in $\text{CocomCoalg}(\mathbf{O})$.

6.2.3. Consider the resulting functor

$$\mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right). \tag{6.3}$$

We consider the two functors,

$$\mathrm{AssocAlg}^{\mathrm{aug}}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})),$$

denoted $\mathrm{Assoc}^{\mathrm{aug}}(\mathrm{Bar})$ and Bar, corresponding to taking the Bar-complex with respect to the 'inner' and 'outer' associative algebra structure, respectively.

We claim:

$$\mathrm{Bar} \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}} \tag{6.4}$$

and

$$\mathrm{Assoc}^{\mathrm{aug}}(\mathrm{Bar}) \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \tag{6.5}$$

as functors

$$\mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})).$$

6.2.4. Indeed, since the functor U^{Hopf} is symmetric monoidal, we have

$$\mathrm{Bar} \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}} \circ B_{\mathrm{Lie}} \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}}, \tag{6.6}$$

which gives the isomorphism in (6.4).

To establish the isomorphism in (6.5), we note that the isomorphism of Proposition 5.3.3 is compatible with the symmetric monoidal structures, and, hence, gives rise to an isomorphism

$$\mathrm{Assoc}^{\mathrm{aug}}(\mathrm{Bar}) \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}})$$

as functors

$$\mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})) \simeq \mathrm{Monoid}(\mathrm{LieAlg}(\mathbf{O})) \to$$
$$\to \mathrm{AssocAlg}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \simeq \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})).$$

This gives rise to the isomorphism in (6.5) by precomposing with Ω_{Lie}.

6.2.5. Recall that the symmetric monoidal structure on $\mathrm{CocomCoalg}(\mathbf{O})$ is Cartesian. In particular, we can consider the full subcategories

$$\mathrm{CocomHopf}(\mathbf{O}) := \mathrm{Grp}(\mathrm{CocomCoalg}(\mathbf{O})) \subset \mathrm{Monoid}(\mathrm{CocomCoalg}(\mathbf{O})) =$$
$$= \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})),$$

and

$$\mathrm{Grp}(\mathrm{Grp}(\mathrm{CocomCoalg}(\mathbf{O}))) \subset \mathrm{Monoid}(\mathrm{Monoid}(\mathrm{CocomCoalg}(\mathbf{O}))) =$$
$$= \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))).$$

We have the following basic fact proved below:

Proposition 6.2.6. *For an ∞-category $\mathbf{C}$ endowed with the Cartesian symmetric monoidal structure, there exists a canonical isomorphism of functors*

$$\mathrm{Grp}(B) \simeq B, \quad \mathrm{Grp}(\mathrm{Grp}(\mathbf{C})) \to \mathrm{Grp}(\mathbf{C}).$$

We compose the isomorphism of Proposition 6.2.6 with the functor (6.3), and obtain an isomorphism

$$\mathrm{Assoc}^{\mathrm{aug}}(\mathrm{Bar}) \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Bar} \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}}. \tag{6.7}$$

Combining the isomorphism (6.7) with the isomorphisms (6.4) and (6.5), we arrive at the conclusion of the theorem.

□

6.3. **Proof of Proposition 6.2.6.**

6.3.1. By adjunction, the assertion of the proposition amounts to a canonical isomorphism of functors

$$\Omega \simeq \mathrm{Grp}(\Omega) : \mathrm{Grp}(\mathbf{C}) \to \mathrm{Grp}(\mathrm{Grp}(\mathbf{C})). \tag{6.8}$$

The latter reduces the assertion to the proposition when $\mathbf{C} = \mathrm{Spc}$ is the category of spaces, by the Yoneda lemma.

6.3.2. We start with the tautological isomorphism of functors

$$\mathrm{Grp}(\Omega) \circ \Omega \simeq \Omega \circ \Omega, \quad \mathrm{Spc}_{\{*\}/} \to \mathrm{Grp}(\mathrm{Grp}(\mathrm{Spc})). \tag{6.9}$$

By adjunction, we obtain a natural transformation

$$B \circ \mathrm{Grp}(\Omega) \to \Omega \circ B \simeq \mathrm{Id}, \quad \mathrm{Grp}(\mathrm{Spc}) \to \mathrm{Grp}(\mathrm{Spc}). \tag{6.10}$$

Applying $\Omega : \mathrm{Grp}(\mathrm{Spc}) \to \mathrm{Grp}(\mathrm{Grp}(\mathrm{Spc}))$ to (6.10), we obtain the desired natural transformation

$$\mathrm{Grp}(\Omega) \simeq \Omega \circ B \circ \mathrm{Grp}(\Omega) \to \Omega.$$

6.3.3. To show that the resulting map $\mathrm{Grp}(\Omega) \to \Omega$ is an isomorphism, it is enough to do so after precomposing with $\Omega : \mathrm{Spc}_{\{*\}/} \to \mathrm{Grp}(\mathrm{Spc})$. However, the resulting map

$$\mathrm{Grp}(\Omega) \circ \Omega \to \Omega \circ \Omega$$

equals that of (6.9), and hence is an isomorphism.

7. Modules

The goal of this section is to give a new perspective on the equivalence of categories

$$\mathfrak{h}\text{-mod} \simeq U(\mathfrak{h})\text{-mod}$$

for a Lie algebra $\mathfrak{h}$. We will do so using the isomorphism

$$U(\mathfrak{h})\text{-mod} \simeq \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}),$$

given by Theorem 6.1.2.

In a sense, the upshot of this section is that one does not really need the definition of the functor

$$U : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$$

as the left adjoint of the restriction functor $\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Lie}}$. Namely, all the essential features of this functor are more conveniently expressed through its incarnation as $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega$.

7.1. **Left modules for associative algebras.** In this subsection we recall some basic pieces of structure pertaining to left modules over associative algebras and to the Koszul duality functor in this case.

7.1.1. Let $\mathbf{O}$ be a monoidal category. Let A be an object of $\mathrm{AssocAlg}(\mathbf{O})$. We let $A\text{-mod}(\mathbf{O})$ denote the category of left A-modules on $\mathbf{O}$. We have a tautological pair of adjoint functors

$$\mathbf{free}_A : \mathbf{O} \rightleftarrows A\text{-mod}(\mathbf{O}) : \mathbf{oblv}_A.$$

The monad $\mathbf{oblv}_A \circ \mathbf{free}_A$ is given by tensor product with A.

Reversing the arrows, we obtain the corresponding pieces of notations for comodules:

$$\mathbf{oblv}_B : B\text{-comod}(\mathbf{O}) \rightleftarrows \mathbf{O} : \mathbf{cofree}_B, \quad B \in \mathrm{CoassocCoalg}(\mathbf{O}).$$

7.1.2. Let now A be an *augmented* associative algebra. We have a canonically defined functor

$$\mathrm{Bar}^\bullet(A, -) : A\text{-mod}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}},$$

see [Lu2, Sect. 4.4.2.7].

We denote by

$$\mathrm{Bar}(A, -) : A\text{-mod}(\mathbf{O}) \to \mathbf{O}$$

the composition of $\mathrm{Bar}^\bullet(A, -)$, followed by the functor of geometric realization $\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathbf{O}$, provided that the latter is defined.

The functor $\mathrm{Bar}(A, -)$ is the left adjoint of the functor

$$\mathbf{triv}_A : \mathbf{O} \to A\text{-mod}(\mathbf{O}),$$

given by the augmentation on A.

7.1.3. We have the following additional crucial piece of structure on the adjoint pair

$$\mathrm{Bar}(A, -) : A\text{-mod}(\mathbf{O}) \rightleftarrows \mathbf{O} : \mathbf{triv}_A.$$

Namely, the co-monad $\mathrm{Bar}(A, -) \circ \mathbf{triv}_A$ on $\mathbf{O}$ identifies canonically with one given by tensor product with the co-associative co-algebra $\mathrm{Bar}^{\mathrm{enh}}(A)$, see [Lu2, Sect. 5.2.2].

In particular, we have a canonically defined functor

$$\mathrm{Bar}^{\mathrm{enh}}(A, -) : A\text{-mod}(\mathbf{O}) \to \mathrm{Bar}^{\mathrm{enh}}(A)\text{-comod}(\mathbf{O}),$$

making the following diagrams commutative:

$$\begin{array}{ccc} A\text{-mod}(\mathbf{O}) & \xrightarrow{\mathrm{Bar}^{\mathrm{enh}}(A,-)} & \mathrm{Bar}^{\mathrm{enh}}(A)\text{-comod}(\mathbf{O}) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Bar}^{\mathrm{enh}}(A)}} \\ A\text{-mod}(\mathbf{O}) & \xrightarrow{\mathrm{Bar}(A,-)} & \mathbf{O} \end{array} \tag{7.1}$$

and

$$\begin{array}{ccc} A\text{-mod}(\mathbf{O}) & \xrightarrow{\mathrm{Bar}^{\mathrm{enh}}(A,-)} & \mathrm{Bar}^{\mathrm{enh}}(A)\text{-comod}(\mathbf{O}) \\ {\scriptstyle \mathbf{triv}_A}\uparrow & & \uparrow{\scriptstyle \mathbf{cofree}_{\mathrm{Bar}^{\mathrm{enh}}(A)}} \\ \mathbf{O} & \xrightarrow{\mathrm{Id}} & \mathbf{O}. \end{array} \tag{7.2}$$

7.2. **Modules over co-commutative Hopf algebras.** Let $\mathbf{O}$ be a symmetric monoidal category.

The goal of this subsection is to establish the following basic fact: given a co-commutative *Hopf* algebra A, the category of modules over A as an associative algebra is equivalent to the totalization of the co-simplicial category of co-modules over $\mathrm{Bar}^\bullet(A)$, where $\mathrm{Bar}^\bullet(A)$ is considered as a simplicial co-algebra.

7.2.1. Let A be a co-commutative bi-algebra in $\mathbf{O}$. Consider the corresponding object

$$\mathrm{Bar}^\bullet(A) \in \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}.$$

Consider the resulting simplicial category

$$\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O}),$$

i.e., the simplicial category formed by co-modules in $\mathbf{O}$ over the terms of $\mathrm{Bar}^\bullet(A)$, viewed as a simplicial co-algebra.

Passing to right adjoints, we obtain a co-simpicial category

$$\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R.$$

The goal of this subsection is to establish the following:

Proposition-Construction 7.2.2. *Assume that* $A \in \mathrm{CocomHopf}(\mathbf{O})$*, and let*

$$\widetilde{A} := \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}})(A)$$

be the underlying associative algebra. Then there is a canonical equivalence of categories:

$$\widetilde{A}\text{-mod} \simeq \mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right). \tag{7.3}$$

The construction of the equivalence in Proposition 7.2.2 will have the following features.

7.2.3. Recall the equivalence

$$\mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right) \simeq |\,\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})|$$

of [Chapter I.1, Proposition 2.5.7]. Thus, we obtain a functor

$$|\,\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})| \simeq \mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right) \simeq \widetilde{A}\text{-mod} \overset{\mathrm{Bar}(\widetilde{A},-)}{\longrightarrow} \mathbf{O}. \tag{7.4}$$

Corollary 7.2.4. *The functor* (7.4) *is given by the simplex-wise forgetful functors*

$$\mathbf{oblv}_{\mathrm{Bar}^m(A)} : \mathrm{Bar}^m(A)\text{-comod}(\mathbf{O}) \to \mathbf{O}.$$

Proof. Follows by considering the corresponding right adjoints. □

7.2.5. Upgrading of $\widetilde{A}$ to an object of $\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O}))$ defines on the category $\widetilde{A}$-mod a symmetric monoidal structure. Similarly, the category $\mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right)$ is naturally symmetric monoidal.

It will follow from the construction, given below, that the equivalence (7.3) is naturally compatible with the above symmetric monoidal structures.

7.2.6. The rest of this subsection is devoted to the proof of Proposition 7.2.2.

Using Proposition C.1.3, to A we can canonically attach an object

$$A' \in \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})),$$

so that

$$\widetilde{A} \simeq \mathbf{oblv}_{\mathrm{Cocom}}(A').$$

Moreover, by construction, under the equivalence

$$\mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}} \simeq \mathrm{CocomCoalg}(\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}})$$

the object

$$\mathrm{Bar}^\bullet(A) \in \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}$$

identifies with the corresponding object

$$\mathrm{Cocom}(\mathrm{Bar}^\bullet)(A') \in \mathrm{CocomCoalg}(\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}).$$

7.2.7. Consider the category P that consists of pairs (B, M), where $B \in \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$ and $M \in B$-mod. This is a symmetric monoidal category under the operation of tensor product.

If B upgrades to an object $B' \in \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}))$, then the object $(B, \mathbf{1_O}) \in \mathsf{P}$ has a natural structure of object of $\mathrm{CocomCoalg}(\mathsf{P})$, denoted $(B', \mathbf{1_O})$. Moreover, we have a naturally defined functor

$$B\text{-mod}(\mathbf{O}) \to (B', \mathbf{1_O})\text{-comod}(\mathsf{P}), \quad M \mapsto (B, M). \tag{7.5}$$

We have a naturally defined symmetric monoidal functor

$$\mathrm{Bar}^\bullet_{\text{with module}} : \mathsf{P} \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}, \quad (B, M) \mapsto \mathrm{Bar}^\bullet(B, M), \tag{7.6}$$

so that for $B \in \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$, we have

$$\mathrm{Bar}^\bullet_{\text{with module}}(B, \mathbf{1_O}) \simeq \mathrm{Bar}^\bullet(B),$$

and for $B' \in \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}))$

$$\mathrm{Cocom}(\mathrm{Bar}^\bullet_{\text{with module}})(B', \mathbf{1_O}) \simeq \mathrm{Cocom}(\mathrm{Bar}^\bullet)(B'),$$

as objects of $\mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}$.

7.2.8. Combining (7.5) and (7.6) we obtain a functor

$$\widetilde{A}\text{-mod}(\mathbf{O}) \to \mathrm{Sect}\left(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})\right), \tag{7.7}$$

where $\mathrm{Sect}(\mathbf{\Delta}^{\mathrm{op}}, -)$ denotes the category of (not necessarily co-Cartesian) sections of a given simplicial category. Specifically, this functor maps an $\widetilde{A}$-module M to the section which assigns to $[n]$, the $\mathrm{Bar}^n(A)$-comodule given by $\mathrm{Bar}^n(A, M)$ and maps given by restriction of comodules.

Lemma 7.2.9. *If the bi-algebra A is a* Hopf *algebra, then for $M \in \widetilde{A}\text{-mod}(\mathbf{O})$, the section* (7.7) *defines, by passing to right adjoints, an object of*

$$\mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right)$$

(i.e., the corresponding morphisms are isomorphisms for every arrow in $\mathbf{\Delta}$).

Proof. For an $\widetilde{A}$-module M, we have the action map

$$A \otimes M \to M.$$

Applying the coinduction functor (right adjoint to restriction of comodules) to M, this gives a map

$$A \otimes M \to A \otimes M$$

of A-comodules. Unraveling the definitions, the statement of the lemma reduces to the statement that the above map is an isomorphism. This follows from the fact that A is a group object is the category of cocommutative coalgebras. □

7.2.10. Thus, by Lemma 7.2.9, we obtain the desired functor

$$\widetilde{A}\text{-mod}(\mathbf{O}) \to \mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right). \tag{7.8}$$

Let us now show that the functor (7.8) is an equivalence.

7.2.11. Let

$$\mathrm{ev}^0 : \mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right) \to \mathbf{O}$$

denote the functor of evaluation on 0-simplices.

It is easy to see that the co-simplicial category $\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R$ satisfies the monadic Beck-Chevalley condition (see [Ga3, Defn. C.1.2] for what this means). Hence, the functor ev^0 is monadic, and the resulting monad on $\mathbf{O}$, regarded as a plain endo-functor, is given by tensor product with

$$\mathbf{oblv}_{\mathrm{Cocom}} \circ \mathbf{oblv}_{\mathrm{Assoc}}(A) \simeq \mathbf{oblv}_{\mathrm{Assoc}}(\widetilde{A}).$$

By construction, the composite functor

$$\widetilde{A}\text{-mod}(\mathbf{O}) \to \mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right) \xrightarrow{\mathrm{ev}^0} \mathbf{O}$$

is the tautological forgetful functor $\mathbf{oblv}_{\widetilde{A}} : \widetilde{A}\text{-mod}(\mathbf{O}) \to \mathbf{O}$. Hence, it is also monadic, and the resulting monad on $\mathbf{O}$, regarded as a plain endo-functor, is given by tensor product with $\mathbf{oblv}_{\mathrm{Assoc}}(\widetilde{A})$.

Hence, it remains to see that the homomorphism of monads on $\mathbf{O}$, induced by (7.8), is an isomorphism as plain endo-functors of $\mathbf{O}$. However, it follows from the construction that the map in question is the identity map on the functor $\mathbf{oblv}_{\mathrm{Assoc}}(\widetilde{A}) \otimes -$.

7.3. **Modules for Lie algebras.** Let $\mathbf{O}$ be a symmetric monoidal DG category.

In this subsection we recall some basic pieces of structure pertaining to modules over Lie algebras and the Koszul duality functor.

7.3.1. For a Lie algebra $\mathfrak{h}$ in $\mathbf{O}$ we let $\mathfrak{h}\text{-mod}(\mathbf{O})$ the category of (operadic) $\mathfrak{h}$-modules on $\mathbf{O}$. We let

$$\mathbf{oblv}_{\mathfrak{h}} : \mathfrak{h}\text{-mod}(\mathbf{O}) \to \mathbf{O}$$

denote the tautological forgetful functor.

7.3.2. The map from $\mathfrak{h}$ to the zero Lie algebra defines a functor

$$\mathbf{triv}_{\mathfrak{h}} : \mathbf{O} \to \mathfrak{h}\text{-mod}(\mathbf{O}).$$

This functor admits a left adjoint, denoted

$$\mathbf{coinv}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathbf{O}) \to \mathbf{O}.$$

7.3.3. In the sequel we will need the following additional piece of structure on the adjoint pair

$$\mathbf{coinv}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathbf{O}) \rightleftarrows \mathbf{O} : \mathbf{triv}_{\mathfrak{h}}.$$

Namely, the co-monad $\mathbf{coinv}(\mathfrak{h}, -) \circ \mathbf{triv}_{\mathfrak{h}}$ on $\mathbf{O}$ identifies canonically with one given by tensor product with $\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})$.

NB: Here we are abusing the notation slightly: we view $\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})$ as an object of the category $\mathrm{CoassocCoalg}(\mathbf{O})$; properly, we should have written

$$\mathbf{res}^{\mathrm{Cocom}\to\mathrm{Coassoc}}(\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})).$$

7.3.4. In particular, we have a canonically defined functor

$$\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h},-) : \mathfrak{h}\text{-mod}(\mathbf{O}) \to \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathbf{O}),$$

making the following diagrams commutative:

$$\begin{array}{ccc} \mathfrak{h}\text{-mod}(\mathbf{O}) & \xrightarrow{\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h},-)} & \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathbf{O}) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})}} \\ \mathfrak{h}\text{-mod}(\mathbf{O}) & \xrightarrow{\mathbf{coinv}(\mathfrak{h},-)} & \mathbf{O} \end{array} \tag{7.9}$$

and

$$\begin{array}{ccc} \mathfrak{h}\text{-mod}(\mathbf{O}) & \xrightarrow{\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h},-)} & \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathbf{O}) \\ {\scriptstyle \mathbf{triv}_{\mathfrak{h}}}\uparrow & & \uparrow{\scriptstyle \mathbf{cofree}_{\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})}} \\ \mathbf{O} & \xrightarrow{\mathrm{Id}} & \mathbf{O}. \end{array} \tag{7.10}$$

7.3.5. From the commutative diagram (7.10) it follows that for $M_1, M_2 \in \mathfrak{h}\text{-mod}(\mathbf{O})$, the map

$$\mathrm{Maps}_{\mathfrak{h}\text{-mod}(\mathbf{O})}(M_1, M_2) \to \mathrm{Maps}_{\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathbf{O})}(\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h}, M_1), \mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h}, M_2)), \tag{7.11}$$

induced by the functor $\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h},-)$, is an isomorphism whenever M_2 lies in the essential image of the functor $\mathbf{triv}_{\mathfrak{h}}$.

7.4. **Modules for a Lie algebra and its universal envelope.** Let $\mathfrak{h} \in \mathrm{LieAlg}(\mathbf{O})$ be as above. In this subsection we will construct a canonical equivalence

$$\mathfrak{h}\text{-mod}(\mathbf{O}) \simeq U(\mathfrak{h})\text{-mod}(\mathbf{O}) \tag{7.12}$$

that makes the following diagrams commute:

$$\begin{array}{ccc} \mathfrak{h}\text{-mod}(\mathbf{O}) & \longrightarrow & U(\mathfrak{h})\text{-mod}(\mathbf{O}) \\ {\scriptstyle \mathbf{oblv}_{\mathfrak{h}}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{U(\mathfrak{h})}} \\ \mathbf{O} & \xrightarrow{\mathrm{Id}} & \mathbf{O} \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{h}\text{-mod}(\mathbf{O}) & \longrightarrow & U(\mathfrak{h})\text{-mod}(\mathbf{O}) \\ {\scriptstyle \mathbf{coinv}(\mathfrak{h},-)}\downarrow & & \downarrow{\scriptstyle \mathrm{Bar}(U(\mathfrak{h}),-)} \\ \mathbf{O} & \xrightarrow{\mathrm{Id}} & \mathbf{O}. \end{array} \tag{7.13}$$

In constructing (7.12) we will use the incarnation of $U(\mathfrak{h})$ as

$$\mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}),$$

given by Theorem 6.1.2.

Note that using Sect. 7.2.5, this will endow the category $\mathfrak{h}\text{-mod}(\mathbf{O})$ with a symmetric monoidal structure, compatible with the forgetful functor $\mathbf{oblv}_{\mathfrak{h}}$.

7.4.1. We start with the object

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}) \in \mathrm{CocomHopf}(\mathbf{O}) \subset \mathrm{AssocAlg}(\mathrm{CococomCoalg}(\mathbf{O})),$$

and form the object

$$\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})) \in \mathrm{CococomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}.$$

Consider the resulting simplicial category

$$\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}),$$

and the co-simplicial category

$$\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^{R},$$

obtained by passing to right adjoints.

According to Proposition 7.2.2, the category

$$\mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})$$

identifies with

$$\mathrm{Tot}\left(\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^{R}\right),$$

in such a way that the forgetful functor

$$\mathbf{oblv}_{\mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}})\circ\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}})\circ\Omega(\mathfrak{h})} : \\ \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \to \mathbf{O}$$

identifies with the functor of evaluation on zero-simplices.

7.4.2. Note that the simplicial co-algebra $\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))$ identifies with

$$\mathrm{Chev}^{\mathrm{enh}}(\mathrm{Bar}^{\bullet} \circ \Omega(\mathfrak{h})),$$

where

$$\mathrm{Bar}^{\bullet} \circ \Omega(\mathfrak{h}) \in \mathrm{LieAlg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}$$

is the Čech nerve in $\mathrm{LieAlg}(\mathbf{O})$ of the map $0_{\mathbf{O}} \to \mathfrak{h}$.

Consider the co-simplicial category

$$\mathrm{Bar}^{\bullet} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}). \tag{7.14}$$

Since

$$\underset{\mathbf{\Delta}^{\mathrm{op}}}{\mathrm{colim}}\, \mathrm{Bar}^{\bullet} \circ \Omega(\mathfrak{h}) \simeq \mathfrak{h},$$

we have

$$\mathfrak{h}\text{-mod}(\mathbf{O}) \simeq \mathrm{Tot}\left(\mathrm{Bar}^{\bullet} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})\right).$$

7.4.3. We will construct the sought-for equivalence

$$\mathfrak{h}\text{-mod}(\mathbf{O}) \simeq \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})$$

by constructing an equivalence

$$\mathrm{Tot}\left(\mathrm{Bar}^{\bullet} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})\right) \simeq \mathrm{Tot}\left(\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^{R}\right).$$

To do so, it is sufficient to show that the corresponding *semi-totalizations* are equivalent.

7.4.4. Let
$$\mathrm{Bar}^\bullet \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^L$$
be the simplicial category obtained by passing to left adjoints in (7.14).

The functor $\mathbf{coinv}^{\mathrm{enh}}(-,-)$ gives rise to a functor of simplicial categories

$$\mathrm{Bar}^\bullet \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^L \to \mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}), \tag{7.15}$$

and, in particular, a functor between the underlying semi-simplicial categories.

We have:

Lemma 7.4.5. *For an* injective *map* $[m_1] \to [m_2]$*, the diagram of obtained by passing to right adjoints along the vertical arrows in*

$$\begin{array}{ccc} \mathrm{Bar}^{m_1} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) & \longrightarrow & \mathrm{Bar}^{m_1}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}) \\ \downarrow & & \downarrow \\ \mathrm{Bar}^{m_2} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) & \longrightarrow & \mathrm{Bar}^{m_2}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}) \end{array}$$

commutes.

From Lemma 7.4.5 we obtain that the term-wise application of the functor $\mathbf{coinv}^{\mathrm{enh}}(-,-)$, gives rise to a functor from the co-semisimplicial category underlying $\mathrm{Bar}^\bullet \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})$ to that underlying $\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^R$.

To prove that the resulting functor between co-semisimplicial categories induces an equivalence of semi-totalizations, it is sufficient to show that for every m, the corresponding functor

$$\mathbf{coinv}(\mathrm{Bar}^m \circ \Omega(\mathfrak{h}), -) : \mathrm{Bar}^m \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \to \mathrm{Bar}^m(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})$$

is fully faithful on the essential image of all the face maps $[0] \to [m]$.

However, this follows from Sect. 7.3.5.

7.4.6. It remains to establish the commutativity of the diagram (7.13).

According to Corollary 7.2.4, under the identification

$$\mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \simeq \\ \simeq \mathrm{Tot}\left(\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^R\right)$$

of Proposition 7.2.2, the functor

$$\mathrm{Bar}(\mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}), -) : \\ \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \to \mathbf{O}$$

corresponds to the functor

$$\mathrm{Tot}\left(\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^R\right) \to \\ \to |\,\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})| \to \mathbf{O},$$

given by the forgetful functors

$$\mathbf{oblv}_{\mathrm{Bar}^m(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}})\circ\Omega(\mathfrak{h}))} : \mathrm{Bar}^m(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}) \to \mathbf{O}.$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}\,(\mathrm{Bar}^\bullet \circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})) & \longrightarrow & \mathrm{Tot}\left(\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^R\right) \\ \sim\uparrow & & \uparrow\sim \\ |\,\mathrm{Bar}^\bullet \circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^L| & \longrightarrow & |\,\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})|, \end{array}$$

where the lower horizontal arrow comes from the map of simplicial categories (7.15).

Hence, we need to show that the functor

$$|\,\mathrm{Bar}^\bullet \circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^L| \to \mathbf{O},$$

given by

$$\mathbf{coinv}(\mathrm{Bar}^m \circ\Omega(\mathfrak{h}), -) : \mathrm{Bar}^m \circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^L \to \mathbf{O},$$

corresponds under

$$|\,\mathrm{Bar}^\bullet \circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^L| \simeq \mathrm{Tot}\,(\mathrm{Bar}^\bullet \circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})) \simeq \mathfrak{h}\text{-mod}(\mathbf{O}) \tag{7.16}$$

to the functor

$$\mathbf{coinv}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathbf{O}) \to \mathbf{O}.$$

However, this follows from the fact that the functor (7.16) is given by the functors,

$$\mathrm{Bar}^m \circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \to \mathfrak{h}\text{-mod}(\mathbf{O})$$

left adjoint to those given by restriction.

Remark 7.4.7. An alternate proof of the equivalence $\mathfrak{h}\text{-mod}(\mathbf{O}) \simeq U(\mathfrak{h})\text{-mod}(\mathbf{O})$ can be given as follows. Given an object $M \in \mathbf{O}$, one has the relative inner Hom

$$\underline{\mathrm{End}}_{\mathbf{O}}(M) := \underline{\mathrm{Hom}}_{\mathbf{O}}(M, M)$$

which is an associative algebra in $\mathbf{O}$ (see [Chapter I.1, Sect. 3.6.6]). For any associative algebra A in $\mathbf{O}$, the structure of an A-module on M is equivalent to a map of associative algebras $A \to \underline{\mathrm{End}}_{\mathbf{O}}(M)$ [Lu2, Corollary 4.7.2.41]. Similarly, one can prove that for any Lie algebra $\mathfrak{h}$, the structure of an $\mathfrak{h}$-module on M is equivalent to a map of Lie algebras $\mathfrak{h} \to \underline{\mathrm{End}}_{\mathbf{O}}(M)$. The equivalence then follows from the description of $U(\mathfrak{h})$ as the algebra induced from $\mathfrak{h}$ along the map of operads $\mathrm{Lie} \to \mathrm{Assoc}^{\mathrm{aug}}$.

Appendix A. Proof of Theorem 2.9.4

Recall that Theorem 2.9.4 says that (under a certain hypothesis on the co-operad $\mathcal{Q}$) if we compute primitives in an object of $\mathcal{Q}^\vee\text{-Coalg}(\mathbf{O})$ of the form

$$\mathbf{cofree}^{\mathrm{fake}}_{\mathcal{Q}}(V), \quad V \in \mathbf{O},$$

we recover V. The non-triviality here lies in the fact that $\mathbf{cofree}^{\mathrm{fake}}_{\mathcal{Q}}$ is not the co-free $\mathcal{Q}$ co-algebra; rather, it comes from the corresponding co-free object under the functor

$$\mathcal{Q}^\vee\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathcal{Q}^\vee\text{-Coalg}(\mathbf{O}).$$

So, we are dealing with the difference between direct sums and direct products. At the end of the day the proof will consist of showing that a certain spectral sequence converges, and that will be achieved by taking into account t-structures (hence the assumption on $\mathcal{Q}$).

A.1. **Calculation of co-primitives.** Let $\mathcal{P}$ be an operad. In this subsection we will give an expression for the functor

$$\mathrm{coPrim}_{\mathcal{P}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathbf{O}$$

in terms of the Koszul dual co-operad.

A.1.1. For $n \geq 1$, let

$$\iota_n : \mathrm{Vect} \to \mathrm{Vect}^{\Sigma}$$

be the tautological functor that produces symmetric sequences with only the n-th non-zero component.

We have the following basic fact:

Lemma A.1.2. *For an operad $\mathcal{P}$, the object $\mathbf{1}_{\mathrm{Vect}^{\Sigma}} \in \mathrm{Vect}^{\Sigma}$, regarded as a right $\mathcal{P}$-module in the monoidal category Vect^{Σ}, can be canonically written as a colimit*

$$\underset{n\geq 1}{\mathrm{colim}}\, \mathcal{M}_n$$

with

$$\mathrm{coFib}(\mathcal{M}_{n-1} \to \mathcal{M}_n) \simeq \iota_n(\mathcal{P}^{\vee}(n)) \star \mathcal{P}, \quad n \geq 1.$$

A.1.3. The assertion of Lemma A.1.2 gives rise to the following more explicit way to express the functor $\mathrm{coPrim}_{\mathcal{P}}$:

Corollary A.1.4. *The functor*

$$A \mapsto \mathrm{coPrim}_{\mathcal{P}}(A), \quad \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathbf{O}$$

admits a canonical filtration by functors of the form

$$A \mapsto \mathcal{M}_n \underset{\mathcal{P}}{\star} A,$$

where $\mathcal{M}_n$ are right $\mathcal{P}$-modules, such that the associated graded of this filtration is canonically identified with

$$n \mapsto \mathcal{P}^{\vee}(n) \star \mathbf{oblv}_{\mathcal{P}}(A).$$

A.2. **Computation of primitives.** Our current goal is to formulate and prove an analog of Corollary A.1.4 for co-algebras over a co-operad, namely Proposition A.2.3 below.

A.2.1. Let $\mathcal{Q}$ be a co-operad, $\mathcal{N}$ a right $\mathcal{Q}$-comodule in Vect^{Σ}, and $A \in \mathcal{Q}\text{-Coalg}(\mathbf{O})$.

We can form a co-simpicial object $\mathrm{coBar}^{\bullet}_{*}(\mathcal{N}, \mathcal{Q}, A)$ of $\mathbf{O}$ with the n-th term

$$\mathrm{coBar}^{n}_{*}(\mathcal{N}, \mathcal{Q}, A) := \left(\mathcal{N} \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{n} \right) * A.$$

We define

$$\mathcal{N} \overset{\mathcal{Q}}{*} A := \mathrm{Tot}\left(\mathrm{coBar}^{\bullet}_{*}(\mathcal{N}, \mathcal{Q}, A)\right).$$

A.2.2. We are going to prove:

Proposition A.2.3. *The functor*

$$A \mapsto \mathrm{Prim}_{\mathcal{Q}}(A), \quad \mathcal{Q}\text{-Coalg}(\mathbf{O}) \to \mathbf{O}$$

can be canonically written as an inverse limit of functors of the form

$$\mathcal{N}_n \overset{\mathcal{Q}}{*} A, \quad n \geq 1,$$

where $\mathcal{N}_n$ are right $\mathcal{Q}$-comodules in Vect^Σ *with*

$$\mathrm{Fib}(\mathcal{N}_n \to \mathcal{N}_{n-1}) \simeq \iota_n(\mathcal{Q}^\vee(n)) \star \mathcal{Q}, \quad n > 1.$$

The rest of this subsection is devoted to the proof of this proposition.

A.2.4. By definition, the functor $\mathrm{Prim}_{\mathcal{Q}}$ is calculated as

$$A \mapsto \mathbf{1}_{\mathrm{Vect}^\Sigma} \overset{\mathcal{Q}}{*} A.$$

Now, we have the following assertion, which is an analog of Lemma A.1.2 for co-operads:

Lemma A.2.5. *For a co-operad $\mathcal{Q}$, the object $\mathbf{1}_{\mathrm{Vect}^\Sigma} \in \mathrm{Vect}^\Sigma$, regarded as a right $\mathcal{Q}$-comodule in the monoidal category Vect^Σ, can be canonically written as a limit*

$$\lim_{n \geq 1} \mathcal{N}_n,$$

with

$$\mathrm{Fib}(\mathcal{N}_n \to \mathcal{N}_{n-1}) \simeq \iota_n(\mathcal{Q}^\vee(n)) \star \mathcal{Q}, \quad n > 1.$$

A.2.6. Since functor of totalization commutes with the formation of limits of terms, in order to prove Proposition A.2.3, it suffices to show that for every $m \geq 0$, the natural map

$$\mathrm{coBar}^m_*(\mathbf{1}_{\mathrm{Vect}^\Sigma}, \mathcal{Q}, A) \to \lim_n \mathrm{coBar}^m_*(\mathcal{N}_n, \mathcal{Q}, A)$$

is an isomorphism.

For the latter, by the definition of the $*$-action, it suffices to show that for any $i \geq 0$, the map

$$\left(\mathbf{1}_{\mathrm{Vect}^\Sigma} \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{m}\right)(i) \otimes A^{\otimes i} \to \lim_n \left(\mathcal{N}_n \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{m}\right)(i) \otimes A^{\otimes i}$$

is an isomorphism.

However, the required isomorphism follows from the fact that for every given i, the family

$$n \mapsto \left(\mathcal{N}_n \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{m}\right)(i)$$

stabilizes to

$$\left(\mathbf{1}_{\mathrm{Vect}^\Sigma} \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{m}\right)(i)$$

for $n \geq i$.

A.3. **Proof of Theorem 2.9.4.**

A.3.1. *Strategy of the proof.* We take $A := \mathbf{cofree}^{\text{fake}}_{\mathcal{Q}}(V)$ for $V \in \mathbf{O}$. We need to show that the natural map

$$V \to \mathrm{Prim}_{\mathcal{Q}} \circ \mathbf{cofree}^{\text{fake}}_{\mathcal{Q}}(V)$$

is an isomorphism.

We calculate the right-hand side via Proposition A.2.3. We will prove that for every $n \geq 1$, the map

$$\mathrm{coFib}\left(V \to \mathcal{N}_n \overset{\mathcal{Q}}{*} \mathbf{cofree}^{\text{fake}}_{\mathcal{Q}}(V)\right) \to \mathrm{coFib}\left(V \to \mathcal{N}_{n-1} \overset{\mathcal{Q}}{*} \mathbf{cofree}^{\text{fake}}_{\mathcal{Q}}(V)\right)$$

is zero. This will prove the required assertion.

A.3.2. *Step 0.* For a right $\mathcal{Q}$-comodule $\mathcal{N}$ in Vect^{Σ}, and $A \in \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$, consider the co-simplicial object $\mathrm{coBar}^{\bullet}_{\star}(\mathcal{N}, \mathcal{Q}, A)$ of $\mathbf{O}$ with terms

$$\mathrm{coBar}^n_*(\mathcal{N}, \mathcal{Q}, A) := \left(\mathcal{N} \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{n}\right) \star A.$$

Set

$$\mathcal{N} \overset{\mathcal{Q}}{\star} A := \mathrm{Tot}\left(\mathrm{coBar}^{\bullet}_{\star}(\mathcal{N}, \mathcal{Q}, A)\right).$$

Note that for $A \in \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$, from (2.7) we obtain a map

$$\mathcal{N} \overset{\mathcal{Q}}{\star} A \to \mathcal{N} \overset{\mathcal{Q}}{*} \mathbf{res}^{\star \to *}(A). \tag{A.1}$$

We observe:

Lemma A.3.3. *Let $\mathcal{N}$ be* cofree*, i.e., of the form $\mathcal{N}' \star \mathcal{Q}$ for $\mathcal{N}' \in \mathrm{Vect}^{\Sigma}$. Then we have a commutative diagram with vertical arrows being isomorphisms:*

$$\begin{array}{ccc} \mathcal{N} \overset{\mathcal{Q}}{\star} A & \xrightarrow{\text{(A.1)}} & \mathcal{N} \overset{\mathcal{Q}}{*} \mathbf{res}^{\star \to *}(A) \\ \uparrow & & \uparrow \\ \mathcal{N}' \star \mathbf{oblv}^{\text{ind-nilp}}_{\mathcal{Q}}(A) & \xrightarrow{(2.7)} & \mathcal{N}' * \mathbf{oblv}^{\text{ind-nilp}}_{\mathcal{Q}}(A). \end{array}$$

Corollary A.3.4. *Let $\mathcal{N}$ be of the form $\iota_n(V) \star \mathcal{Q}$ for some n and $V \in \mathbf{O}$. Then the map* (A.1) *is an isomorphism.*

A.3.5. *Step 1.* We return to the proof of Theorem 2.9.4. We note that for any $n \geq 1$, the object $\mathcal{N}_n$ has a finite filtration by objects of the form $\iota_m(\mathcal{Q}^{\vee}(m)) \star \mathcal{Q}$, $m \leq n$.

By Corollary A.3.4, we obtain that for any $A \in \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$ the map

$$\mathcal{N}_n \overset{\mathcal{Q}}{\star} A \to \mathcal{N}_n \overset{\mathcal{Q}}{*} \mathbf{res}^{\star \to *}(A)$$

of (A.1) is an isomorphism.

Hence, we obtain that it suffices to show that the map

$$\mathrm{coFib}\left(V \to \mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)\right) \to \mathrm{coFib}\left(V \to \mathcal{N}_{n-1} \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)\right)$$

is zero.

A.3.6. *Step 2.* Note that each $\mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}\text{ind-nilp}}(V)$ is naturally graded by integers $d \geq 1$, such that the map

$$V \to \mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}\text{ind-nilp}}(V)$$

is an isomorphism on the degree 1 part for all n.

Hence, it remains to show that for all $d > 1$, the map

$$\left(\mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}\text{ind-nilp}}(V)\right)^d \to \left(\mathcal{N}_{n-1} \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}\text{ind-nilp}}(V)\right)^d \tag{A.2}$$

is zero, where the superscript d indicates the degree d part.

A.3.7. *Step 3.* Note now that the functor

$$V \mapsto \left(\mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}\text{ind-nilp}}(V)\right)^d$$

(resp., the natutal transformation (A.2)) is given by

$$V \mapsto (\mathcal{K}_n^d \otimes V^{\otimes d})_{\Sigma_d}$$

for some $\mathcal{K}_n^d \in \mathrm{Rep}(\Sigma_d)$ (resp., a map $\mathcal{K}_n^d \to \mathcal{K}_{n-1}^d$).

Hence, it remains to show that for every $d > 1$ and every n, the map

$$\mathcal{K}_n^d \to \mathcal{K}_{n-1}^d \tag{A.3}$$

is zero.

A.3.8. *Step 4.* Since the category $\mathrm{Rep}(\Sigma_d)$ is semi-simple, the fact that (A.3) is equivalent to the map in question inducing the zero map on cohomology.

The latter reduces the assertion of the theorem to the case of $\mathbf{O} = \mathrm{Vect}$. Namely, it suffices to show that for some/any $V \in \mathrm{Vect}^{\heartsuit}_{\mathrm{f.d.}}$ with $\dim(V) \geq d$, the map (A.2) induces the zero map on cohomology.

A.3.9. *Step 5.* Consider the operad $\mathcal{Q}^*$, and set $\mathcal{M}_n := \mathcal{N}_n^*$. We obtain that the object

$$\left(\mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}\text{ind-nilp}}(V)\right)^d$$

is the linear dual of the object

$$\left(\mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d. \tag{A.4}$$

Hence, is is enough to show that the map

$$\left(\mathcal{M}_{n-1} \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d \to \left(\mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d \tag{A.5}$$

induces a zero map on cohomology for all $n \geq 1$ and $d > 1$.

A.3.10. *Step 6.* We note that $(\mathcal{Q}^*)^\vee \simeq (\mathcal{Q}^\vee)^*$. So, by the assumption that $\mathcal{Q}^\vee[1]$ and $\mathcal{Q}^*$ are classical,

$$\text{coFib}\left(\mathcal{M}_{n-1} \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*) \to \mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)$$

is concentrated in cohomological degree $-n$.

Hence,

$$\text{coFib}\left(\left(\mathcal{M}_{n-1} \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d \to \left(\mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d\right)$$

is also concentrated in cohomological degree $-n$.

Therefore, in order to show that (A.5) induces a zero map on cohomology, it suffices to show that the colimit

$$\underset{n}{\text{colim}} \left(\mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d. \tag{A.6}$$

is acyclic.

A.3.11. *Step 7.* By Corollary A.1.4, the colimit (A.6) identifies with the degree d part of

$$\text{coPrim}_{\mathcal{Q}^*} \circ \mathbf{free}_{\mathcal{Q}^*}(V^*).$$

However,

$$\text{coPrim}_{\mathcal{Q}^*} \circ \mathbf{free}_{\mathcal{Q}^*}(V^*) \simeq V^*$$

and hence its degree d part for $d \neq 1$ vanishes.

□

Appendix B. Proof of the PBW theorem

In this section we will prove the version of the PBW theorem stated in the main body of the paper as Theorem 5.2.4.

B.1. **The PBW theorem at the level of operads.** In this subsection we formulate a version of Theorem 5.2.4 that takes place within the category Vect^Σ.

B.1.1. We have the canonical maps

$$\phi : \text{Lie} \to \text{Assoc}^{\text{aug}} \text{ and } \psi : \text{Assoc}^{\text{aug}} \to \text{Com}^{\text{aug}},$$

such that the composition $\psi \circ \phi$ factors through the augmentation/unit

$$\text{Lie} \to \mathbf{1}_{\text{Vect}^\Sigma} \to \text{Com}\,.$$

B.1.2. The map ϕ gives rise to the forgetful functor

$$\mathbf{res}^{\text{Assoc}^{\text{aug}} \to \text{Lie}} : \text{AssocAlg}^{\text{aug}}(\mathbf{O}) \to \text{LieAlg}(\mathbf{O}),$$

and the map ψ gives rise to the forgetful functor

$$\mathbf{res}^{\text{Com}^{\text{aug}} \to \text{Assoc}^{\text{aug}}} : \text{Com}^{\text{aug}}(\mathbf{O}) \to \text{AssocAlg}^{\text{aug}}(\mathbf{O}).$$

The functor

$$U : \text{LieAlg}(\mathbf{O}) \to \text{AssocAlg}^{\text{aug}}(\mathbf{O})$$

is given by

$$\mathfrak{h} \mapsto \text{Assoc}^{\text{aug}} \underset{\text{Lie}}{\star} \mathfrak{h}.$$

B.1.3. The functor

$$U \circ \mathbf{triv}_{\mathrm{Lie}} : \mathbf{O} \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$$

is given by

$$V \mapsto (\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}}) \star V.$$

The canonical map

$$U \circ \mathbf{triv}_{\mathrm{Lie}}(V) \to \mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}(V)$$

comes from the map in Vect^{Σ}:

$$\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}} \to \mathrm{Com}^{\mathrm{aug}}, \tag{B.1}$$

which arises via the description of the map $\psi \circ \phi$ in Sect. B.1.1.

B.1.4. The operadic PBW theorem says:

Theorem B.1.5. *The map* (B.1) *is an isomorphism in* Vect^{Σ}.

It is clear that Theorem B.1.5 implies Theorem 5.2.4.

B.2. **Proof of Theorem B.1.5.**

B.2.1. We have the natural map in Vect^{Σ}

$$\mathrm{Com}^{\mathrm{aug}} \to \mathrm{Assoc}^{\mathrm{aug}}$$

which realizes the symmetrization map at the level of functors. This gives a map of right Lie-modules in Vect^{Σ}

$$\mathrm{Com}^{\mathrm{aug}} \star \mathrm{Lie} \to \mathrm{Assoc}^{\mathrm{aug}}.$$

It follows from the classical PBW theorem applied to a free Lie algebra on a vector space that this map is an isomorphism.

Hence, we have an isomorphism between $\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}}$ and $\mathrm{Com}^{\mathrm{aug}}$. In particular, for every n, we have:

$$\left(\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}}\right)(n) \in \mathrm{Vect}^{\heartsuit}.$$

B.2.2. It remains to show that for any $V \in \mathrm{Vect}^{\heartsuit}$, the map

$$H^0\left(\left(\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}}\right) \star V\right) \to \mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}(V)$$

is an isomorphism.

Note, however, that the object $H^0\left(\left(\mathrm{Assoc} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}}\right) \star V\right)$ identifies with

$$H^0\left(U \circ \mathbf{triv}_{\mathrm{Lie}}(V)\right),$$

i.e., the universal enveloping algebra of the trivial Lie algebra, taken in the world of *classical* associative algebras.

However, the latter is easily seen to map isomorphically to $\mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}(V)$.

Appendix C. Commutative co-algebras and bialgebras

Let H be a classical co-commutative bialgebra. We can regard H as either an associative algebra in the category of co-commutative co-algebras or, equivalently, a co-commutative co-algebra in the category of associative algebras.

In this section, we establish the corresponding fact in the context of higher algebra, i.e., an equivalence of $(\infty, 1)$-categories $\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \simeq \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O}))$. The latter is not altogether obvious, as the corresponding classical assertion is proved by 'an explicit formula'.

C.1. **Two incarnations of co-commutative bialgebras.**

Co-commutative bialgebras can be thought of in two different ways: as co-commutative co-algebras in the category of associative algebras, or as associative algebras in the category of co-commutative co-algebras. In this subsection we show that the two are equivalent.

C.1.1. In this subsection we let $\mathbf{O}$ be a symmetric monoidal category, which contains colimits, and for which the functor of tensor product preserves colimits in each variable.

The category $\mathrm{CocomBialg}(\mathbf{O})$ is defined as

$$\mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})) \simeq \mathrm{AssocAlg}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})), \tag{C.1}$$

where the (symmetric) monoidal structure on $\mathrm{CocomCoalg}(\mathbf{O})$ is given by tensor product, which coincides with the Cartesian product in $\mathrm{CocomCoalg}(\mathbf{O})$.

Consider now the category $\mathrm{AssocAlg}(\mathbf{O})$, endowed with a symmetric monoidal structure given by tensor product. Consider the category

$$\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \simeq \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})). \tag{C.2}$$

C.1.2. In this section we will prove:

Proposition-Construction C.1.3. *There exists a canonical equivalence of categories*

$$\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \simeq \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O}))$$

that makes the diagram

$$\begin{array}{ccc}
\mathrm{CocomCoalg}(\mathbf{O}) & \xrightarrow{\mathrm{Id}} & \mathrm{CocomCoalg}(\mathbf{O}) \\
{\scriptstyle \mathrm{CocomCoalg}(\mathbf{oblv}_{\mathrm{Assoc}})}\uparrow & & \uparrow{\scriptstyle \mathbf{oblv}_{\mathrm{Assoc}}} \\
\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) & \longrightarrow & \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})) \\
{\scriptstyle \mathbf{oblv}_{\mathrm{Cocom}}}\downarrow & & \downarrow{\scriptstyle \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}})} \\
\mathrm{AssocAlg}(\mathbf{O}) & \xrightarrow{\mathrm{Id}} & \mathrm{AssocAlg}(\mathbf{O})
\end{array}$$

commute.

C.2. **Proof of Proposition C.1.3.**

C.2.1. *Step 1.* We have a canonically defined symmetric monoidal functor

$$\mathrm{Bar}^\bullet : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

In particular, we obtain a functor

$$\mathrm{Cocom}(\mathrm{Bar}^\bullet) : \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})) \to \mathrm{CocomCoalg}(\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}) \simeq \\ \simeq \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}.$$

Combining with (C.2), we obtain a functor

$$\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \to \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}. \tag{C.3}$$

C.2.2. *Step 2.* Since the symmetric monoidal structure on CocomCoalg($\mathbf{O}$) is Cartesian, the functor

$$\mathrm{Bar}^\bullet : \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})) \to \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}} \tag{C.4}$$

is fully faithful.

Now, it is easy to see that the essential image of the functor (C.3) lies in that of (C.4).

This defines a functor in one direction:

$$\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \to \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})). \tag{C.5}$$

C.2.3. *Step 3.* Let us now prove that the functor (C.5) is an equivalence. By construction, the composite functor

$$\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \to \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})) \overset{\mathbf{oblv}_{\mathrm{Assoc}}}{\longrightarrow} \mathrm{CocomCoalg}(\mathbf{O})$$

is the tautological functor

$$\mathrm{Cocom}(\mathbf{oblv}_{\mathrm{Assoc}}) : \mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \to \mathrm{CocomCoalg}(\mathbf{O}). \tag{C.6}$$

It suffices to show that the functor (C.6) and

$$\mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})) \overset{\mathbf{oblv}_{\mathrm{Assoc}}}{\longrightarrow} \mathrm{CocomCoalg}(\mathbf{O}) \tag{C.7}$$

are both monadic, and that the map of monads, induced by (C.5), is an isomorphism as plain endo-functors of CocomCoalg($\mathbf{O}$).

C.2.4. *Step 4.* The functor

$$\mathrm{AssocAlg}(\mathbf{O}') \overset{\mathbf{oblv}_{\mathrm{Assoc}}}{\longrightarrow} \mathbf{O}'$$

is monadic for any monoidal category $\mathbf{O}'$ (satisfying the same assumtion as $\mathbf{O}$); its left adjoint is given by

$$V \mapsto \mathbf{free}_{\mathrm{Assoc}}(V).$$

In particular, the functor (C.7) is monadic: take $\mathbf{O}' := \mathrm{CocomCoalg}(\mathbf{O})$.

C.2.5. *Step 5.* We have a pair of adjoint functors

$$\mathbf{free}_{\mathrm{Assoc}} : \mathbf{O} \rightleftarrows \mathrm{AssocAlg}(\mathbf{O}) : \mathbf{oblv}_{\mathrm{Assoc}},$$

with the right adjoint being symmetric monoidal.

Hence, the above pair induces an adjoint pair

$$\mathrm{CocomCoalg}(\mathbf{O}) \rightleftarrows \mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})).$$

Hence, we obtain that the functor (C.6) is also monadic.

C.2.6. *Step 6.* To show that the map of monads on CocomCoalg($\mathbf{O}$), induced by (C.5) is an isomorphism as plain endo-functors, it is enough to do so after composing with the (conservative) forgetful functor $\mathbf{oblv}_{\mathrm{Cocom}} : \mathrm{CocomCoalg}(\mathbf{O}) \to \mathbf{O}$.

By construction, it suffices to prove that the natural transformation

$$\mathbf{free}_{\mathrm{Assoc}} \circ \mathbf{oblv}_{\mathrm{Cocom}} \to \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathbf{free}_{\mathrm{Assoc}},$$

coming by adjunction from the isomorphism

$$\mathbf{oblv}_{\mathrm{Cocom}} \circ \mathbf{oblv}_{\mathrm{Assoc}} \simeq \mathbf{oblv}_{\mathrm{Assoc}} \circ \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}),$$

is itself an isomorphism.

However, this follows from the fact that the functor

$$\mathbf{oblv}_{\mathrm{Cocom}} : \mathrm{CocomCoalg}(\mathbf{O}) \to \mathbf{O}$$

is symmetric monoidal and preserves coproducts.

□