INTRODUCTION TO PART IV: FORMAL GEOMTETRY

1. What is formal geometry?

By 'formal geometry' we mean the study of the category, whose objects are $\mathrm{PreStk}_{\text{laft-def}}$, and whose morphisms are nil-isomorphisms of prestacks.

In the course of this part, we will see that this category provides a convenient and flexible framework for many geometric operations:

- Taking quotients with respect to a groupoid;
- Correspondence between group-objects (over a given base $\mathcal{X}$) and Lie algebras in the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$;
- Considering differential-geometric constructions such as Lie algebroids, their universal enveloping algebras, Hodge filtration, etc.

A feature of the theory presented in this part is that it is really very general. E.g., when establishing the correspondence between formal groups over $\mathcal{X}$ and Lie algebras in $\mathrm{IndCoh}(\mathcal{X})$, there are no additional conditions: we really take *all* group-objects and *all* Lie algebras (no finiteness conditions).

1.1. We begin this part with the short Chapter IV.1 that discusses formal moduli problems. The main theorem of this chapter says the following:

For an object $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$, consider the following two categories: one is the category

$$\mathrm{FormMod}_{\mathcal{X}/} := (\mathrm{PreStk}_{\text{laft-def}})_{\text{nil-isom from } \mathcal{X}}.$$

I.e., it consists of prestacks locally almost of finite type that admit deformation theory and receive a nil-isomorphism from $\mathcal{X}$.

Another is the category $\mathrm{FormGrpoid}(\mathcal{X})$ of groupoid objects in $(\mathrm{PreStk}_{\text{laft-def}})_{\text{nil-isom}}$ acting on $\mathcal{X}$ (i.e., the groupoids whose 0-th space is $\mathcal{X}$ itself).

The Čech nerve construction defines a functor

$$\mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{FormGrpoid}(\mathcal{X}). \tag{1.1}$$

Now, the main result of this chapter, [Chapter IV.1, Theorem 2.3.2], says that the functor (1.1) is an equivalence.

1.2. We denote by $B_{\mathcal{X}}$ the functor inverse to (1.1). This is the functor of taking the quotient with respect to a groupoid.

A feature of our proof of the equivalence (1.1) is that it is constructive. I.e., given a groupoid $\mathcal{R}^\bullet$ over $\mathcal{X}$ (i.e., $\mathcal{R}^0 = \mathcal{X}$), we explicitly describe the prestack $B_{\mathcal{X}}(\mathcal{R}^\bullet)$.

We note, however, that the natural map

$$|\mathcal{R}^\bullet| \to B_{\mathcal{X}}(\mathcal{R}^\bullet),$$

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is *not* an isomorphism, where $|\mathcal{R}^\bullet|$ is understood as a geometric realization in the category $\mathrm{PreStk}_{\mathrm{laft}}$. The problem is that $|\mathcal{R}^\bullet|$ understood in the above way will *not* in general admit deformation theory.

1.3. By a formal moduli problem *over* a given object $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ we mean an object

$$\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}},$$

such that the morphism $\mathcal{Y} \to \mathcal{X}$ is an *inf-schematic nil-isomorphism.* I.e., for any $S \to \mathcal{X}$ with $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, the prestack $S \underset{\mathcal{X}}{\times} \mathcal{Y}$ should admit deformation theory and the map

$$^{\mathrm{red}}(S \underset{\mathcal{X}}{\times} \mathcal{Y}) \to {}^{\mathrm{red}}S$$

should be an isomorphism.

Let $\mathrm{FormMod}_{/\mathcal{X}}$ denote the category of formal moduli problems over $\mathcal{X}$.

By a *formal group* over $\mathcal{X}$ we mean an object of the category $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$. It follows formally from the equivalence (1.1) that the loop functor defines an equivalence

$$\Omega_X : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}).$$

We denote by $B_{\mathcal{X}}$ the inverse equivalence. Thus, we obtain that any $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ admits a classifying space

$$B_{\mathcal{X}}(\mathcal{H}) \in \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}).$$

1.4. Let us add a comment here that will explain the link between our theory and that developed in [Lu5].

Suppose that $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. We have the forgetful functors

$$\mathrm{FormMod}_{X/} \to (\mathrm{PreStk}_{\mathrm{laft}})_{X/} \text{ and } \mathrm{FormMod}_{/X} \to (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$$

I.e., objects of the category $\mathrm{FormMod}_{X/}$ (resp., $\mathrm{FormMod}_{/X}$) are prestacks (locally almost of finite type) under X (resp., over X) satisfying a certain condition. I.e., at the end of the day, they are functors

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

We show, however, that the information of an object of $\mathrm{FormMod}_{X/}$ (resp., $\mathrm{FormMod}_{/X}$) is completely determined by the restriction of the corresponding functor to a much smaller category. Namely, the category in question is

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom from } X})^{\mathrm{op}} \tag{1.2}$$

in the case of $\mathrm{FormMod}_{\mathcal{X}/}$ and

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})^{\mathrm{op}} \tag{1.3}$$

in the case of $\mathrm{FormMod}_{/X}$.

I.e., in order to 'know' a formal moduli problem under X, it suffices to know how it behaves on schemes infinitesimally close to X.

For example, if $X = \mathrm{pt}$, the categories (1.2) and (1.3) both identify with the category of connective k-algebras A with finite-dimensional total cohomologies, and $H^0(A)$ being local. So, functors out of this category (satisfying the appropriate deformation theory condition) are indeed what is traditionally called a 'formal moduli problem'.

2. Lie algebras

In Chapter IV.2 we make a digression to discuss the general theory of Lie algebras (in a symmetric monoidal DG category $\mathbf{O}$).

The material from this chapter will be extensively used in Chapter IV.3, where we study the relation between formal groups and Lie algebras.

2.1. The main actors in this chapter are the mutually adjoint functors

$$\text{Chev}^{\text{enh}} : \text{LieAlg}(\mathbf{O}) \rightleftarrows \text{CocomCoalg}^{\text{aug}}(\mathbf{O}) : \text{coChev}^{\text{enh}} \tag{2.1}$$

that relate the category $\text{LieAlg}(\mathbf{O})$ of Lie algebras in $\mathbf{O}$ to the category $\text{CocomCoalg}^{\text{aug}}(\mathbf{O})$ of augmented co-commutative co-algebras in $\mathbf{O}$.

The main point is that the functors in (2.1) are *not* mutually inverse equivalences. But they are close to be such.

2.2. We remind that the composition of Chev^{enh} with the forgetful functor

$$\mathbf{oblv}_{\text{Cocom}^{\text{aug}}} : \text{CocomCoalg}^{\text{aug}}(\mathbf{O}) \to \mathbf{O}$$

is the the functor, denoted Chev, which is by definition the left adjoint to

$$\mathbf{triv}_{\text{Lie}} \circ [-1] : \mathbf{O} \to \text{LieAlg}(\mathbf{O}),$$

where $\mathbf{triv}_{\text{Lie}}$ is the functor of the 'trivial Lie algebra'.

The composition of $\text{coChev}^{\text{enh}}$ with the forgetful functor

$$\mathbf{oblv}_{\text{Lie}} : \text{LieAlg}(\mathbf{O}) \to \mathbf{O}$$

is the functor, denoted coChev, which is by definition the right adjoint to the functor

$$\mathbf{triv}_{\text{Cocom}} \circ [1] : \mathbf{O} \to \text{CocomCoalg}^{\text{aug}}(\mathbf{O}),$$

where $\mathbf{triv}_{\text{Cocom}}$ is the functor of the 'trivial co-commutative co-algebra'.

In other words, the functor

$$[1] \circ \text{coChev} : \text{CocomCoalg}^{\text{aug}}(\mathbf{O}) \to \mathbf{O}$$

is the functor Prim of primitive elements.

2.3. Let us now describe the two main results of this chapter, Theorems 4.4.6 and 6.1.2.

Consider the composition

$$\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}} : \text{LieAlg}(\mathbf{O}) \to \text{CocomHopf}(\mathbf{O}). \tag{2.2}$$

Theorem 4.4.6 says that the functor (2.2) is fully faithful. I.e., although the functor Chev^{enh} fails to be fully faithful, if we compose it with loop functor and retain the group structure, it becomes fully faithful.

Theorem 6.1.2 says that the functor (2.2) identifies canonically with the functor U^{Hopf} of universal enveloping algebra (viewed as a Hopf algebra).

2.4. Note also that the right adjoint of the functor (2.2) is a functor

$$\text{CocomHopf}(\mathbf{O}) \to \text{LieAlg}(\mathbf{O}) \tag{2.3}$$

that makes the following diagram commutative:

$$\begin{array}{ccc} \text{CocomHopf}(\mathbf{O}) & \xrightarrow{\mathbf{oblv}_{\text{Assoc}}} & \text{CocomCoalg}^{\text{aug}}(\mathbf{O}) \\ \downarrow & & \downarrow {\scriptstyle \text{Prim}} \\ \text{LieAlg}(\mathbf{O}) & \longrightarrow & \mathbf{O}, \end{array}$$

where $\mathbf{oblv}_{\text{Assoc}}$ is the natural forgetful functor, and Prim is the functor of primitive elements.

The above commutative diagram may be viewed as an ultimate answer to the question of why the tangent space to a Lie group has a structure of Lie algebra: because given the tangent fiber of a co-commutative Hopf algebra, viewed as a mere augmented co-commutative co-algebra, has a structure of Lie algebra.

The latter observation will be extensively used in the next chapter, i.e., [Chapter IV.3].

3. Formal groups vs. Lie algebras

In [Chapter IV.3] we establish an equivalence between the category of formal groups (over a given $\mathcal{X} \in \text{PreStk}_{\text{laft}}$) and the category $\text{LieAlg}(\text{IndCoh}(\mathcal{X}))$.

3.1. Assume first that $\mathcal{X} = X \in {}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}}$. Our first step in defining the functors that connect formal groups and Lie algebras in $\text{IndCoh}(X)$ is to set up a kind of 'covariant formal algebraic geometry'.

What we mean by this is that we define a pair of mutually adjoint functors

$$\text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd}(\text{FormMod}_{/X}) \rightleftarrows \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) : \text{Spec}^{\text{inf}} . \tag{3.1}$$

The functor $\text{Distr}^{\text{Cocom}^{\text{aug}}}$ sends an object $(\mathcal{Y} \xrightarrow{f} X) \in \text{Ptd}(\text{FormMod}_{/X})$ to

$$f_*^{\text{IndCoh}}(\omega_{\mathcal{Y}}) \in \text{IndCoh}(X),$$

with the co-commutative co-algebra structure coming from the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \underset{X}{\times} \mathcal{Y}$, and the augmentation from the section $X \to \mathcal{Y}$. The functor Spec^{inf} is formally defined as the right adjoint of $\text{Distr}^{\text{Cocom}^{\text{aug}}}$.

We should warn the reader that the situation here, although formally analogous, is not totally parallel to the usual algebraic geometry. In particular, the functor $\text{Distr}^{\text{Cocom}^{\text{aug}}}$ is *not* fully faithful.

3.2. A basic example of an object in $\text{Ptd}(\text{FormMod}_{/X})$ is a vector group, denoted $\text{Vect}_X(\mathcal{F})$, associated to $\mathcal{F} \in \text{IndCoh}(X)$.

For $(X' \xrightarrow{g} X) \in \text{Ptd}({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}})$, we have

$$\text{Maps}(X', \text{Vect}_X(\mathcal{F})) = \text{Maps}_{\text{IndCoh}(X)}(\text{Distr}^+(X'), \mathcal{F}),$$

where

$$\text{Distr}^+(X') := \text{Fib}(g_*^{\text{IndCoh}}(\omega_{X'}) \to \omega_X).$$

One shows that

$$\text{Vect}_X(\mathcal{F}) \simeq \text{Spec}^{\text{inf}}(\text{Sym}(\mathcal{F})).$$

3.3. We are now ready to describe the mutually inverse functors

$$\text{Lie} : \text{Grp}(\text{FormMod}_{/X}) \rightleftarrows \text{LieAlg}(\text{IndCoh}(X)) : \exp . \tag{3.2}$$

The functor Lie is the composition of the functor

$$\text{Grp}(\text{Distr}^{\text{Cocom}^{\text{aug}}}) : \text{Grp}(\text{FormMod}_{/X}) \to \text{Grp}(\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))) =: \\ = \text{CocomHopf}(\text{IndCoh}(X))$$

and the functor

$$\text{CocomHopf}(\text{IndCoh}(X)) \to \text{LieAlg}(\text{IndCoh}(X))$$

of (2.3).

3.4. One shows that the composition

$$\text{Grp}(\text{FormMod}_{/X}) \xrightarrow{\text{Lie}} \text{LieAlg}(\text{IndCoh}(X)) \xrightarrow{\mathbf{oblv}_{\text{Lie}}} \text{IndCoh}(X)$$

is the functor

$$\text{Grp}(\text{FormMod}_{/X}) \xrightarrow{\mathbf{oblv}_{\text{Grp}}} \text{Ptd}(\text{FormMod}_{/X}) \xrightarrow{\mathcal{Y} \mapsto T(\mathcal{Y}/X)|_X} \text{IndCoh}(X).$$

I.e., the object of IndCoh(X) underlying the Lie algebra of $\mathcal{H} \in \text{Grp}(\text{FormMod}_{/X})$ is the tangent space of $\mathcal{H}$ at the origin, as it should be.

3.5. The functor exp is defined as the composition of

$$\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega_{\text{Lie}} : \text{LieAlg}(\text{IndCoh}(X)) \to \text{CocomHopf}(\text{IndCoh}(X)) = \\ = \text{Grp}(\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)))$$

and the functor

$$\text{Grp}(\text{Spec}^{\text{inf}}) : \text{Grp}(\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))) \to \text{Grp}(\text{FormMod}_{/X}).$$

Knowing the equivalence (3.2), one can interpret in its terms the adjunction (3.1). Namely, it becomes the adjunction (2.1) for the category $\mathbf{O} = \text{IndCoh}(\mathcal{X})$.

3.6. We note that it follows from [Chapter IV.2, Theorem 4.2.2] that the composed functor

$$\mathbf{oblv}_{\text{Grp}} \circ \exp : \text{LieAlg}(\text{IndCoh}(X)) \to \text{Ptd}(\text{FormMod}_{/X})$$

is isomorphic to

$$\text{LieAlg}(\text{IndCoh}(X)) \xrightarrow{\mathbf{oblv}_{\text{Lie}}} \text{IndCoh}(X) \xrightarrow{\text{Vect}_X(-)} \text{Ptd}(\text{FormMod}_{/X}).$$

I.e., the object of $\text{Ptd}(\text{FormMod}_{/X})$ underlying a formal group $\mathcal{H}$ is canonically isomorphic to the vector group $\text{Vect}_X(T(\mathcal{H}/X)|_X)$.

Using this fact, one shows that the functors exp, and hence Lie, are compatible with base change with respect to X, and thus give rise to an equivalence

$$\text{Lie} : \text{Grp}(\text{FormMod}_{/\mathcal{X}}) \rightleftarrows \text{LieAlg}(\text{IndCoh}(\mathcal{X})) : \exp$$

for any $\mathcal{X} \in \text{PreStk}_{\text{laft}}$.

4. Lie algebroids

In Chapter IV.4 we initiate the study of Lie algebroids.

4.1. Lie algebroids are defined classically as quasi-coherent sheaves with some extra structure, while this structure involves a differential operator of order one. Because the definition of Lie algebroids involves explicit formulas, it is difficult to render it directly to the the world of derived algebraic geometry.

For this reason, we take a different approach and define Lie algebroids via geometry. Namely, we let the category of Lie algebroids $\mathrm{LieAlgbroid}(\mathcal{X})$ on $\mathcal{X} \in \mathrm{PreStk}_{\text{laft-def}}$ be, by definition, equivalent to that of formal groupoids over $\mathcal{X}$.

The reason why this definition has a chance to be reasonable is the equivalence (3.2) between formal groups and Lie algebras.

Much of this chapter is devoted to the explanation of why Lie algebroids defined in the above way really behave as Lie algebroids should.

4.2. We define the forgetful functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})},$$

that sends a Lie algebroid $\mathfrak{L}$ to the underlying quasi-coherent sheaf $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})$, equipped with the anchor map

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \to T(\mathcal{X}).$$

We show that this functor is monadic; in particular, it admits a left adjoint, denoted

$$\mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{LieAlgbroid}(\mathcal{X}).$$

We show that the endo-functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$$

of $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ has the 'right size', i.e., what one expect from a reasonable definition of Lie algebroids (it has a canonical filtration with the expected form of the associated graded).

4.3. Thus, we have the equivalences

$$\mathrm{LieAlgbroid}(\mathcal{X}) \simeq \mathrm{FormGrpoid}(\mathcal{X}) \simeq \mathrm{FormMod}_{\mathcal{X}/}\,. \tag{4.1}$$

We show that the functor $\mathbf{free}_{\mathrm{LieAlgbroid}}$ translates into the functor of the *square-zero extension*

$$\mathrm{RealSqZ} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{FormMod}_{\mathcal{X}/}\,.$$

4.4. The category of Lie algebroids on $\mathcal{X}$ is related to the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ by a pair of adjoint functors

$$\mathrm{diag} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \rightleftarrows \mathrm{LieAlgbroid}(\mathcal{X}) : \text{ker-anch}\,. \tag{4.2}$$

The meaning of the functor diag should be clear: a Lie algebra on $\mathcal{X}$ can be viewed as a Lie algebroid with the trivial anchor map. The functor ker-anch sends a Lie algebroid to the kernel of the anchor map.

We show that the adjoint pair (4.2) is also monadic. The corresponding monad

$$\mathrm{LieAlgbroid}(\mathcal{X}) \circ \mathrm{diag}$$

on the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ is given by *semi-direct product* with the inertia Lie algebra $\mathrm{inert}_{\mathcal{X}}$ (the Lie algebra of the inertia group $\mathrm{Inert}_{\mathcal{X}} := \mathcal{X} \underset{\mathcal{X}\times\mathcal{X}}{\times} \mathcal{X}$).

We learned about this way of realizing Lie algebroids from J. Francis.

5. Infinitesimal differential geometry

In Chapter [Chapter IV.5] we develop the ideas from [Chapter IV.4] to set up constructions of differential nature on objects $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$.

5.1. The key construction in [Chapter IV.5] is that of *deformation to the normal bundle.*

We start with $(\mathcal{X} \to \mathcal{Y}) \in \text{FormMod}_{\mathcal{X}/}$ and we define an $\mathbb{A}^1$-family

$$(\mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}_{\text{scaled}})$$

of formal moduli problems under $\mathcal{X}$.

A crucial piece of structure that $\mathcal{Y}_{\text{scaled}}$ has is that of *left-lax equivariance* with respect to $\mathbb{A}^1$ that acts on itself *by multiplication*.

The structure of equivariance with respect to $\mathbb{G}_m \subset \mathbb{A}^1$ implies that the fibers $\mathcal{Y}_a$ of $\mathcal{Y}_{\text{scaled}}$ at $0 \neq a \in \mathbb{A}^1$ are all canonically isomorphic to $\mathcal{X}$.

The fiber at $0 \in \mathbb{A}^1$ identifies with $\text{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1])$, i.e., the formal version of the total space of the normal to $\mathcal{X}$ inside $\mathcal{Y}$.

The latter observation allows to reduce many isomorphism questions regarding formal moduli problems to the simplest situation, when our moduli problem is a vector group $\text{Vect}_{\mathcal{X}}(\mathcal{F})$ for $\mathcal{F} \in \text{IndCoh}(\mathcal{X})$.

5.2. If $\mathcal{X} = X$ is a classical scheme, and $\mathcal{Y}$ is the formal completion of X in Y, where $X \to Y$ is a regular embedding, then $\mathcal{Y}_{\text{scaled}}$ is the completion of $X \times \mathbb{A}^1$ in the usual deformation of Y to the normal cone.

For the final object in the category $\text{FormMod}_{\mathcal{X}/}$, i.e.,

$$\mathcal{X} \to \mathcal{X}_{\text{dR}},$$

the deformation $(\mathcal{X}_{\text{dR}})_{\text{scaled}}$ is the Dolbeault deformation of $\mathcal{X}_{\text{dR}}$ to $\text{Vect}_{\mathcal{X}}(T(\mathcal{X})[1])$.

5.3. The relevance of the $\mathbb{A}^1$ left-lax equivariant family $\mathcal{Y}_{\text{scaled}}$ is the following: functors from $\text{FormMod}_{\mathcal{X}/}$ with values in a DG category $\mathbf{C}$ will automatically upgrade to functors with values in the category

$$\mathbf{C}^{\text{Fil},\geq 0}.$$

This is due to the equivalence

$$\mathbf{C}^{\text{Fil},\geq 0} \simeq (\mathbf{C} \otimes \text{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}},$$

see [Chapter IV.2, Lemma 2.5.5(a)].

5.4. As a first application of the deformation $\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\text{scaled}}$ we construct a canonical filtration on the universal enveloping algebra

$$U(\mathfrak{L}) \in \text{AssocAlg}(\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X}))$$

of a Lie algebroid $\mathfrak{L}$.

This approach to the filtration on the universal enveloping algebra is natural from the point of view of classical algebraic and smooth schemes: the canonical filtration on the algebra of differential operators is closely related to the Dolbeault deformation.

5.5. Another central construction in chapter [Chapter IV.5] is that of the n-th infinitesimal neighborhood

$$\mathcal{X} \to \mathcal{X}^{(n)} \to \mathcal{Y}$$

for $\mathcal{Y} \in \text{FormMod}_{\mathcal{X}/}$.

Again, this construction is not at all straightforward in the generality of we considering it: nil-isomorphisms between objects of $\text{PreStk}_{\text{laft-def}}$.

We construct the n-th infinitesimal neighborhood inductively, with $\mathcal{X}^{(n)}$ being a square-zero extension of $\mathcal{X}^{(n-1)}$ by means of $\text{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$.

In the process of construction of this extension we crucially rely on the deformation

$$\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\text{scaled}}.$$

5.6. We show that the natural map

$$\underset{n}{\text{colim}}\, \mathcal{X}^{(n)} \to \mathcal{Y}$$

is an isomorphism.

In particular, we obtain that the dualizing sheaf $\omega \in \text{IndCoh}(\mathcal{Y})$ has a canonical filtration whose n-th term is the direct image of $\omega_{\mathcal{X}^{(n)}}$ under $\mathcal{X}^{(n)} \to \mathcal{Y}$.

Translating to the language of Lie algebroids via (4.1), the above filtration can be interpreted as the de Rham resolution of the unit module over a Lie algebroid $\mathfrak{L}$, with the n-th associated graded being the induced module from

$$\text{Sym}^n(\mathbf{oblv}_{\text{LieAlgebroid}}(\mathfrak{L})).$$

For $(\mathcal{X} \to \mathcal{X}_{\text{dR}}) \in \text{FormMod}_{\mathcal{X}/}$ we recover the Hodge filtration on the unit crystal (D-module) $\omega_{\mathcal{X}_{\text{dR}}}$.