INTRODUCTION TO VOLUME II:
DEFORMATIONS, LIE THEORY AND FORMAL GEOMETRY

1. What is done in Volume II?

In this volume, we will apply the theory of IndCoh developed in Volume I to do geometry (more precisely, to what we understand by 'doing geometry').

Namely, we will introduce the class of geometric objects of interest, called *inf-schemes* (or relative versions thereof), and study the corresponding categories of ind-coherent sheaves. Such categories are exactly what one encounters in representation-theoretic situations.

We will perform various operations with inf-schemes (such as taking the quotient with respect to a groupoid), and we will study what such operations do to the corresponding categories of ind-coherent sheaves.

1.1. As was already mentioned, our basic class of geometric objects is that of inf-schemes, denoted $\text{infSch}_{\text{laft}}$. Just as any other class of geometric objects in this book, $\text{infSch}_{\text{laft}}$ is a full subcategory in the ambient category of prestacks. I.e., an inf-scheme is a contravariant functor on Sch^{aff} that satisfies some *conditions* (rather than having some additional structure).

As is often the case in algebraic geometry, along with a particular class of objects, there is the corresponding relative notion. I.e., we also introduce what it means for a map of prestacks to be *inf-schematic*.

What are inf-schemes? The definition is surprisingly simple. These are prestacks (technically, locally almost of finite type) whose underlying reduced prestack is a (reduced) scheme[1], and *admit deformation theory*. The latter is a condition that guarantees a reasonable infinitesimal behavior; we refer the reader to Chapter 1 of this volume for the precise definition of what it means to admit deformation theory.

This, one can informally say that the class of inf-schemes contains all prestacks that are schemes 'up to something infinitesimal, but controllable'. For example, all (derived) schemes, the de Rham prestacks of schemes and formal schemes are all examples of inf-schemes.

1.2. As was explained in Volume I, Chapter 4, once we have the theory of IndCoh on schemes (almost of finite type), functorial with respect to the operation of !-pullback, we can extend it to all prestacks (locally almost of finite type). In particular, we obtain the theory of IndCoh on inf-schemes, functorial with resect to !-pullbacks.

We proceed to define the functor of IndCoh-direct image for maps between inf-schemes that satisfies base change against !-pullback. Furthermore, this leads to the operation of IndCoh-direct image for *inf-schematic* maps between prestacks (locally almost of finite type).

The IndCoh categories and the functors of !-pullback and IndCoh-direct image describe many of representation-theoretic categories and functors between them that arise in practice.

[1] I.e., when we evaluate our prestack on reduced affine schemes, the result is representable by a (reduced) scheme.

1.3. As an illustration of the technique of inf-schemes, we proceed to develop the *Lie theory* in this context.

Let $\mathcal{X}$ be a base prestack (locally almost of finite type). On the one hand, we consider the category of *formal* group-objects over $\mathcal{Y}$. I.e., these are group-objects in the category of prestacks $\mathcal{Y}$ equipped with a map $f : \mathcal{Y} \to \mathcal{X}$, such that f is inf-schematic and induces an isomorphism at the reduced level. Denote this category by $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$.

On the other hand, we consider the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ of Lie algebra objects in the (symmetric monoidal) category $\mathrm{IndCoh}(\mathcal{X})$. We establish a canonical equivalence

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \simeq \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})). \tag{1}$$

I.e., this is an equivalence between formal groups and Lie algebras in full generality. We can view the equivalence (1) as a justification for the notion of inf-scheme (or rather, inf-schematic map): we need those in order to define the category $\mathrm{FormMod}_{/\mathcal{X}}$.

We show that given a group-object $\mathcal{G} \in \mathrm{FormMod}_{/\mathcal{X}}$, one can form its *classifying space*,

$$B_{\mathcal{X}}(\mathcal{G}) \in \mathrm{FormMod}_{/\mathcal{X}},$$

which is equipped with a section $\mathcal{X} \to B_{\mathcal{X}}(\mathcal{G})$ (i.e., it is pointed), so that $\mathcal{G}$ is recovered as the loop-object of $B_{\mathcal{X}}(\mathcal{G})$ in the category $\mathrm{FormMod}_{/\mathcal{X}}$. The above functor

$$B_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

is equivalence.

We show that there is a canonical equivalence

$$\mathrm{IndCoh}(B_{\mathcal{X}}(\mathcal{G})) \simeq \mathfrak{g}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})), \tag{2}$$

where the latter is the category of $\mathfrak{g}$-modules in the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$, where $\mathfrak{g}$ is the object of $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ corresponding to $\mathcal{G}$ via (1). I.e., the Lie theory works at the level of representations as it should.

1.4. Let $\mathcal{X}$ be a prestack (locally almost of finite type that admits deformation theory). One of the basic objects involved in doing 'differential geometry' on $\mathcal{X}$ is that of Lie algebroid. Here comes an unpleasant surprise, though:

We have not been able to define the notion of Lie algebroid *algebraically*. Namely, the classical definition of Lie algebroid involves some binary operations that satisfy some relations, and we were not able to make sense of those in our context of derived algebraic geometry.

Instead, we *define* the notion of Lie algebroid via geometry: we set the category $\mathrm{LieAlgbroid}(\mathcal{X})$ to be, by definition, that of formal groupoids over $\mathcal{X}$. This definition is justified by the equivalence (1), which says that Lie algebras are the same as formal groups.

We show that Lie algebroids defined in this way indeed behave in the way we expect Lie algebroids to behave. For example, we have a pair of adjoint functors

$$\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \rightleftarrows \mathrm{LieAlgbroid}(\mathcal{X}),$$

(here $T(\mathcal{X}) \in \mathrm{IndCoh}(\mathcal{X})$ is the tangent complex[2] of $\mathcal{X}$), where the right adjoint forgets the algebroid structure, and the left adjoint is the functor of *free Lie algebroid.*

[2]Another advantage of the theory of ind-coherent sheaves is that a prestack $\mathcal{X}$ (locally almost of finite type that admits deformation theory) admit a *tangent complex*, which is an object of $\mathrm{IndCoh}(\mathcal{X})$, while the more traditional *cotangent complex* is an object of the pro-category, and thus is more difficult to work with.

We also show that the category $\text{LieAlgbroid}(\mathcal{X})$ is equivalent to the category $\text{FormMod}_{\mathcal{X}/}$ of formal moduli problems *under* $\mathcal{X}$, i.e., to the category of prestacks $\mathcal{Y}$ (locally almost of finite type that admit deformation theory) equipped with a map $f : \mathcal{X} \to \mathcal{Y}$ such that f is inf-schematic and induces an isomorphism at the reduced level.

For example, we show that under the equivalence

$$\text{LieAlgbroid}(\mathcal{X}) \simeq \text{FormMod}_{\mathcal{X}/}, \tag{3}$$

the functor of free Lie algebroid corresponds to the functor of square-zero extension.

Generalizing (2), we show that if $\mathcal{L} \in \text{LieAlgbroid}(\mathcal{X})$ corresponds to $\mathcal{Y} \in \text{FormMod}_{\mathcal{X}/}$, we have a canonical equivalence

$$\text{IndCoh}(\mathcal{Y}) \simeq \mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X})), \tag{4}$$

where the left-hand side is the appropriately defined category of objects of $\text{IndCoh}(\mathcal{X})$, equipped with an action of $\mathfrak{L}$.

Here is a typical construction from representation theory that uses the theory of Lie algebroids. Let $\mathcal{X}$ be as above, and let $\mathfrak{g}$ be a Lie algebra (i.e., a Lie algebra object in the category Vect of chain complexes of vector spaces over our ground field) that *acts* on $\mathcal{X}$. In this case, we can form a Lie algebroid $\mathfrak{g}_{\mathcal{X}}$ on $\mathcal{X}$, and a functor

$$\mathfrak{g}\text{-mod}(\text{Vect}) \to \mathfrak{g}_{\mathcal{X}}\text{-mod}(\text{IndCoh}(\mathcal{X})).$$

Composing with the induction functor for the map $\mathfrak{g}_{\mathcal{X}} \to \mathcal{T}(\mathcal{X})$, where $\mathcal{T}(\mathcal{X})$ is the *tangent* Lie algebroid, and using the equivalence (4), we construct the *localization functor*

$$\mathfrak{g}\text{-mod}(\text{Vect}) \to \text{IndCoh}(\mathcal{X}_{\text{dR}}),$$

where $\text{IndCoh}(\mathcal{X}_{\text{dR}})$ identifies, by definition, with the category of D-modules on $\mathcal{X}$.

1.5. At the end of this volume, we develop some elements of 'infinitesimal differential geometry'. Namely, we address the following question:

Many objects of differential-geometric nature come equipped with natural filtrations. For example, we have the filtration on the ring of differential operators (according to the order), or the filtration on a formal completion of a scheme along a subscheme (the n-th infinitesimal neighborhoods). We wish to have similar pieces of structure in the general context of prestacks (locally almost of finite type that admit deformation theory). However, as is always the case in higher category theory and derived algebraic geometry, we cannot define these filtrations 'by hand'.

Instead, we use the following idea: a filtered object (of linear nature) is the same as a family of such objects over $\mathbb{A}^1$ which is equivariant with respect to the action of $\mathbb{G}_m$ by dilations.

We implement this idea in geometry. Namely, we show that given a $\mathcal{Y} \in \text{FormMod}_{\mathcal{X}/}$, we can canonically construct its *deformation to the normal cone*, which is a family

$$\mathcal{Y}_t \in \text{FormMod}_{\mathcal{X}/}, \quad t \in \mathbb{A}^1, \tag{5}$$

that deforms the original $\mathcal{Y}$ to a vector bundle situation.

We show (which by itself might not be so well-known even in the context of usual algebraic geometry) that this deformation gives rise to all the various filtrations that we are interested in.

In its turn, the deformation (5) also needs to be constructed by a functorial procedure (rather than an explicit formula). We construct it using a certain geometric device, explained in Chapter 9, Sect. 2.

2. What do we use from Volume I?

2.1. One thing is unavoidable: we use the language of higher category theory. So, the reader in encouraged to familiarize him/herself with the contents of Chapter 1, Sects. 1 and 2.

Here are some of the most essential pieces of notation.

We denote by 1-Cat the $(\infty,1)$-category of $(\infty,1)$-categories, and by Spc its full subcategory that consists of spaces (a.k.a. ∞-groupoids).

For a pair of $(\infty,1)$-categories $\mathbf{C}$ and $\mathbf{D}$, we let $\mathrm{Funct}(\mathbf{C},\mathbf{D})$ denote the $(\infty,1)$-category of functors between them.

For an $(\infty,1)$-category $\mathbf{C}$, and a pair objects $\mathbf{c}_0,\mathbf{c}_1\in\mathbf{C}$, we denote by $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_0,\mathbf{c}_1)$ the space of maps from $\mathbf{c}_0$ to $\mathbf{c}_1$.

2.2. A lot of the time, we will work in a linear context, by which we mean the world of DG categories. We refer the reader to Chapter 1, Sect. 10 for the definition.

We let Vect denote the DG category of chain complexes of vector spaces (over a fixed ground field k, assumed to be of characteristic zero).

Given a pair of DG categories $\mathbf{C}$ and $\mathbf{D}$, we will write $\mathrm{Funct}(\mathbf{C},\mathbf{D})$ for the DG category of k-linear functors between them; this clashes with the symbol introduced earlier that denoted all functors, but we believe that this is unlikely to result in a confusion.

We write $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C},\mathbf{D})$ for the full subcategory of $\mathrm{Funct}(\mathbf{C},\mathbf{D})$ that consists of *continuous* (i.e., colimit preserving) functors. We denote by $\mathrm{DGCat}_{\mathrm{cont}}$ the $(\infty,1)$-category formed by DG categories and continuous linear functors between them.

For a given DG category $\mathbf{C}$ and a pair of objects $\mathbf{c}_0,\mathbf{c}_1\in\mathbf{C}$, we will write $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0,\mathbf{c}_1)\in$ Vect for the chain complex of maps between them. The objects $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0,\mathbf{c}_1)\in\mathrm{Vect}$ and $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_0,\mathbf{c}_1)\in\mathrm{Spc}$ are related by the Dold-Kan functor, see *loc. cit.*

For a pair of DG categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\mathbf{C}\otimes\mathbf{D}$ their tensor product.

2.3. The basic notions pertaining to derived algebraic geometry are set up in Chapter 2.

We denote by PreStk the $(\infty,1)$-category of all prestacks, i.e., accessible functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}\to\mathrm{Spc}\,.$$

For this volume, it is of crucial importance to know the definition of the full subcategory

$$\mathrm{PreStk}_{\mathrm{laft}}\subset\mathrm{PreStk}$$

that consists of prestacks that are locally almost of finite type, see Chapter 2, Sect. 1.7 for the definition.

Derived schemes are introduced in Chapter 2, Sect. 3; the corresponding category is denoted by Sch. Henceforth, we will omit the word 'derived' and refer to derived schemes as schemes.

For a prestack $\mathcal{X}$, we denote by $\mathrm{QCoh}(\mathcal{X})$ the DG category of quasi-coherent sheaves on it; see Chapter 3, Sect. 1 for the definition.

2.4. One cannot read this volume without knowing what ind-coherent sheaves are. For an individual scheme X (almost of finite type), the category $\operatorname{IndCoh}(X)$ is defined in Chapter 4, Sect. 1. The basic functoriality is given by the functor of IndCoh-direct image

$$f_*^{\operatorname{IndCoh}} : \operatorname{IndCoh}(X) \to \operatorname{IndCoh}(Y),$$

for a map $f : X \to Y$, see Chapter 4, Sect. 2.

The machinery of the IndCoh functor is fully developed in Chapter 5, where we construct the !-pullback functor

$$f^! : \operatorname{IndCoh}(Y) \to \operatorname{IndCoh}(X),$$

for f as above.

We denote by $\omega_X \in \operatorname{IndCoh}(X)$ the dualizing object, i.e., the pullback pf the generator

$$k \in \operatorname{Vect} \simeq \operatorname{IndCoh}(\mathrm{pt})$$

under the tautological projection $X \to \mathrm{pt}$.

The assignment $X \rightsquigarrow \operatorname{IndCoh}(X)$ (with !-pullbacks) is extended from schemes (almost of finite type) to all of $\mathrm{PreStk}_{\mathrm{laft}}$ by the procedure of left Kan extension. Moreover, we make IndCoh a functor out of the *2-category of correspondences*,

$$\operatorname{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{sch}\,\&\,\mathrm{proper}}_{\mathrm{sch\text{-}qc;all}} \to \mathrm{DGCat}^{\mathrm{2\text{-}Cat}}_{\mathrm{cont}}.$$

In the above formula, $\mathrm{DGCat}^{\mathrm{2\text{-}Cat}}_{\mathrm{cont}}$ is the 2-categorical enhancement of $\mathrm{DGCat}_{\mathrm{cont}}$, see Chapter 1, Sect. 10.3.9.

One piece of notation from Chapter 6 that the reader might need is the functor

$$\Upsilon_{\mathcal{X}} : \operatorname{QCoh}(\mathcal{X}) \to \operatorname{IndCoh}(\mathcal{X}),$$

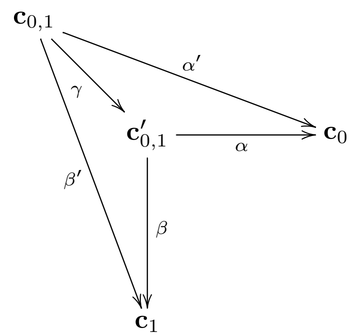
defined for any $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, and given by tensoring a given object of $\operatorname{QCoh}(\mathcal{X})$ by the dualizing object $\omega_{\mathcal{X}} \in \operatorname{IndCoh}(\mathcal{X})$.

2.5. Chapter 7 introduces the category of correspondences. The idea is that given an $(\infty,1)$-category $\mathbf{C}$ and three classes of morphisms *vert*, *horiz* and *adm*, we introduce an $(\infty,2)$-category $\operatorname{Corr}(\mathbf{C})^{adm}_{vert,horiz}$, whose objects are the same as those of $\mathbf{C}$, but where 1-morphisms from $\mathbf{c}_0$ to $\mathbf{c}_1$ are *correspondences*

$$\begin{array}{ccc} \mathbf{c}_{0,1} & \xrightarrow{\alpha} & \mathbf{c}_0 \\ \downarrow{\scriptstyle\beta} & & \\ \mathbf{c}_1 & & \end{array}$$

where $\alpha \in horiz$ and $\beta \in vert$. For a pair of correspondences $(\mathbf{c}_{0,1}, \alpha, \beta)$ and $(\mathbf{c}'_{0,1}, \alpha', \beta')$, the space of 2-morphisms between them is that of commutative diagrams

For a pair of correspondences $(\mathbf{c}_{0,1}, \alpha, \beta)$ and $(\mathbf{c}'_{0,1}, \alpha', \beta')$, we want the space of maps between them to be that of commutative diagrams



(6)

with $\gamma \in adm$.

2.6. The use of the rest of the notions and notations from Volume I is sporadic and the reader can look it up, using the index of notation of Volume I when needed.