INTRODUCTION TO VOLUME I:
CORRESPONDENCES AND DUALITY

In describing the contents of Volume I we will use some terminology pertaining to higher category theory, derived algebraic geometry and the (derived) category of quasi-coherent sheaves. The reader is referred to the part of this book, called Preliminaries, where the relevant notions are surveyed.

1. Ind-coherent sheaves

The goal of Volume I is to set up the machinery of ind-coherent sheaves on (derived) schemes, in order to apply it in Volume II and describe algebro-geometrically categories and functors that naturally arise in representation theory.

1.1. **How do ind-coherent sheaves arise?** We start developing the theory of ind-coherent sheaves in Chapter 4. The idea is the following:

Given a (derived) scheme X (assumed *almost of finite type*), the usual DG category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X can be realized as the ind-completion of its full subcategory $\mathrm{QCoh}(X)^{\mathrm{perf}}$ of *perfect* objects.

The category $\mathrm{IndCoh}(X)$ is defined to be the ind-completion of another subcategory of $\mathrm{QCoh}(X)$, namely $\mathrm{Coh}(X)$ that consists of objects that are cohomologically bounded (i.e., have non-zero cohomologies only in finitely many degrees) and all of whose cohomologies are *coherent* as sheaves on the classical scheme underlying X.

The first question is: why should we consider such a thing? In the next few subsections we will try to provide an answer.

1.1.1. For the needs of representation theory, we would like to study $\mathcal{O}$-modules on algebro-geometric objects much more general than schemes. In fact, we want to consider all prestacks (locally almost of finite type). A particular class of particular relevance is that of inf-schemes, a notion that will be introduced in Volume II.

To simplify the discussion, let $\mathcal{X}$ be an *ind-scheme*, i.e., a filtered colimit of schemes

$$\mathcal{X} = \underset{i}{\mathrm{colim}}\, X_i,$$

where the transition maps $X_i \xrightarrow{f_{i,j}} X_j$ are closed embeddings.

We would like to have a version of the category of $\mathcal{O}$-modules on $\mathcal{X}$ which is the *colimit* of the corresponding categories on the X_i's, where the transition functors are given by taking *direct images* with respect to the $f_{i,j}$'s. I.e., morally, an $\mathcal{O}$-module on $\mathcal{X}$ is a union of its submodules supported on the X_i's.

Let us try to interpret the category of $\mathcal{O}$-modules as $\mathrm{QCoh}(-)$ and see what we get. If we apply the definition, we obtain

$$\mathrm{QCoh}(\mathcal{X}) \simeq \lim_{i} \mathrm{QCoh}(X_i), \quad (i \to j) \rightsquigarrow (\mathrm{QCoh}(X_j) \xrightarrow{f_{i,j}^*} \mathrm{QCoh}(X_i)),$$

i.e., our category on $\mathcal{X}$ is the *limit* of the corresponding categories on the X_i's with respect to pullbacks (rather than the *colimit* of the same categories with respect to pushforwards).

So, the category that we seek on $\mathcal{X}$ is not the usual $\mathrm{QCoh}(\mathcal{X})$. Let us, however, try something else: let us try to 'force' the definition as a colimit, while still using $\mathrm{QCoh}(-)$ on the X_i's as building blocks. I.e., consider the category

$$\underset{i}{\mathrm{colim}}\ \mathrm{QCoh}(X_i), \quad (i \to j) \rightsquigarrow (\mathrm{QCoh}(X_i) \overset{(f_{i,j})_*}{\longrightarrow} \mathrm{QCoh}(X_j)). \tag{1.1}$$

The above gives a well-defined category, but the problem is that it may be quite ill-behaved. Namely, one can formally rewrite the above *colimit* as a *limit*,

$$\underset{i}{\lim}\ \mathrm{QCoh}(X_i), \quad (i \to j) \rightsquigarrow (\mathrm{QCoh}(X_j) \overset{f^{!,\mathrm{QCoh}}_{i,j}}{\longrightarrow} \mathrm{QCoh}(X_i)),$$

where $f^{!,\mathrm{QCoh}}_{i,j}$ is the functor *right* adjoint to $(f_{i,j})_*$. The problem is caused by the potential bad behavior of the functors $f^{!,\mathrm{QCoh}}_{i,j}$.

1.1.2. Let us isolate the problem. Let $f : X \to Y$ be a closed embedding (or, more generally, a proper map). We have the usual direct image functor

$$f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y),$$

and it follows formally from Lurie's Adjoint Functor Theorem that this functor admits a right adjoint, denoted

$$f^{!,\mathrm{QCoh}} : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X).$$

The trouble is, however, that the above functor $f^{!,\mathrm{QCoh}}$ may be ill-behaved. Technically, 'ill-behaved' means that it may fail to be *continuous* (i.e., preserve colimits).

One can ask further: why is non-contuniuty a problem? The answer to this is that the world that we would like to live in is that of DG categories that are *cocomplete*, and continuous functors between them. The reason for the latter is that in this world we have a well-defined operation of tensor product of DG categories

$$\mathbf{C}, \mathbf{D} \rightsquigarrow \mathbf{C} \otimes \mathbf{D}. \tag{1.2}$$

I.e., this is the world in which we can really 'do algebra', which is exactly what we want to do in Volume II, with a view to applications to representation theory.

In addition, it is this world in which it is most convenient to talk about duality, which is something that we will discuss in the sequel.

1.1.3. Now, the obstruction to the functor $f^{!,\mathrm{QCoh}}$ being continuous is that its left adjoint, namely, f_* does not *preserve compactness*, i.e., it does not necessarily send $\mathrm{QCoh}(X)^{\mathrm{perf}}$ to $\mathrm{QCoh}(Y)^{\mathrm{perf}}$. However, it does send $\mathrm{Coh}(X)$ to $\mathrm{Coh}(Y)$, by virtue of properness. Therefore, the right adjoint to the corresponding functor

$$f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y),$$

denoted

$$f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$$

is continuous.

So, replacing $\mathrm{QCoh}(-)$ by $\mathrm{IndCoh}(-)$ fixes the bug of non-continuity of $f^{!,\mathrm{QCoh}}$.

1.1.4. In particular, going back to the case of an ind-scheme $\mathcal{X}$, we can define

$$\text{IndCoh}(\mathcal{X}) := \underset{i}{\text{colim}}\, \text{IndCoh}(X_i), \quad (i \to j) \rightsquigarrow (\text{IndCoh}(X_i) \overset{(f_{i,j})_*^{\text{IndCoh}}}{\longrightarrow} \text{IndCoh}(X_j)), \tag{1.3}$$

and we a reasonable category, which we can rewrite also as

$$\underset{i}{\lim}\, \text{QCoh}(X_i), \quad (i \to j) \rightsquigarrow (\text{IndCoh}(X_j) \overset{f_{i,j}^!}{\longrightarrow} \text{IndCoh}(X_i)), \tag{1.4}$$

The above category matches exactly the needs of representation theory and one that we will use.

1.2. **What does the theory of ind-coherent sheaves consist of?** Let us now take $\mathcal{X}$ to be a general prestack (locally almost of finite type). We would like to define the category $\text{IndCoh}(\mathcal{X})$ that reproduces the answer given above in the case when $\mathcal{X}$ is an ind-scheme. However, we no longer expect that $\text{IndCoh}(\mathcal{X})$ could be written as a *colimit*. But we can try to approach $\text{IndCoh}(\mathcal{X})$ as a *limit*, so that in the case of ind-schemes, we recover (1.4).

Thus, we would like to define

$$\text{IndCoh}(\mathcal{X}) := \underset{X_i \to \mathcal{X}}{\lim}\, \text{IndCoh}(X_i), \tag{1.5}$$

where the limit is taken over the category of *all* schemes (almost of finite type) mapping to $\mathcal{X}$, and where the transition functors $\text{IndCoh}(X_j) \to \text{IndCoh}(X_i)$ are given by

$$(X_i \overset{f_{i,j}}{\to} X_j) \rightsquigarrow (\text{IndCoh}(X_j) \overset{f_{i,j}^!}{\longrightarrow} \text{IndCoh}(X_j).$$

But we now face a new problem: the maps $f_{i,j}$ are no longer closed embeddings (or proper); they are arbitrary maps between schemes (almost of finite type). So, we need the definition of the functor

$$f^! : \text{IndCoh}(Y) \to \text{IndCoh}(X)$$

in the case of an arbitrary map $X \overset{f}{\to} Y$.

Moreover, in order for the limit (1.5) to make sense in the world of higher categories, we need the assignment

$$X \rightsquigarrow \text{IndCoh}(X), \;\; (X \overset{f}{\to} Y) \rightsquigarrow (\text{IndCoh}(Y) \overset{f^!}{\to} \text{IndCoh}(X))$$

be a functor from the category opposite to that of schemes almost of finite type to that of DG categories and continuous functors:

$$\text{IndCoh}^!_{\text{Sch}_{\text{aft}}} : (\text{Sch}_{\text{aft}})^{\text{op}} \to \text{DGCat}_{\text{cont}}\,. \tag{1.6}$$

1.2.1. The problem is that for an arbitrary map f, the functor $f^!$ is not adjoint to anything. But we can factor f as a composition $f_1 \circ f_2$, where f_2 is an open embedding, and f_1 is a proper morphism, and define

$$f^! := f_2^! \circ f_1^!,$$

where $f_1^!$ is the right adjoint to $(f_1)_*^{\text{IndCoh}}$, and $f_2^!$ is just restriction.

If we were to realize this idea, we would have to show that the above definition of $f^!$ does not depend on the factorization f as $f_1 \circ f_2$, and moreover that it upgrades to a functor (1.6). This can be handled explicitly if our target was an ordinary category (rather than $\text{DGCat}_{\text{cont}}$), but in the world of higher categories we will have to extract (1.6) using the (rather constrained) toolbox of constructions that produce functors from the already existing ones.

1.2.2. Suppose, nevertheless, that we have constructed the functor (1.6). However, we may (and do) want more: for a *schematic* map between prestacks $g : \mathcal{X} \to \mathcal{Y}$ we want to have the direct image functor

$$g_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y}),$$

determined by the requirement that for a scheme Y and a map $f_Y : Y \to \mathcal{Y}$, for the Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{f_X} & \mathcal{X} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f_Y} & \mathcal{Y}, \end{array}$$

we have an isomorphism of funtors

$$f_Y^! \circ g_*^{\mathrm{IndCoh}} \simeq (g')_*^{\mathrm{IndCoh}} \circ f_X^!. \tag{1.7}$$

In order for this to happen, at the very least, we need an analogous property for maps between schemes. I.e., we want that for a Cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f_X} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f_Y} & Y, \end{array}$$

there exists a canonical iisomorphism of funtors

$$f_Y^! \circ g_*^{\mathrm{IndCoh}} \simeq (g')_*^{\mathrm{IndCoh}} \circ f_X^!. \tag{1.8}$$

However, in order for the isomorphisms (1.8) to give rise to (1.7), the isomorphisms (1.8) themselves must be functorial with respect to compositions of the maps f and g.

Such a functoriality is easy to spell out in the world of ordinary categories, but it becomes a non-trivial problem when we are dealing with higher categories (more precisely, when the target category, which in our case is $\mathrm{DGCat}_{\mathrm{cont}}$, is a higher category).

1.2.3. This brings us to the idea of the *category of correspondences*, discussed below, and to which we devoted Chapters 7, 8 and 9 of this volume.

In Chapter 5 we prove the existence and uniqueness of IndCoh as a functor out of the category of correspondences. Moreover, it turns out that this extended formalism (rather than just the functor (1.6)) is a very natural way to construct the functor (1.6) itself.

So, the formalism of correspondences is not only needed in order to extend IndCoh to prestacks, but a necessity for the construction of the !-pullback on schemes.

1.3. Before we pass to the discussion of the formalism of correspondences, let us mention the role of Chapter 6 of this volume. In this Chapter, we undertake a systematic study of the relationship between QCoh(−) and IndCoh(−), when both are viewed as functors

$$(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}\,.$$

The upshot is that both have a natural symmetric monoidal structure, where the symmetric monoidal structure on $(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{op}}$ is given by the Cartesian product of schemes, and on $\mathrm{DGCat}_{\mathrm{cont}}$ by (1.2).

Moreover, there is a natural transformation between the above two functors, denoted Υ. For an individual scheme X, the corresponding functor

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X),$$

given by tensoring the dualizing object $\omega_X \in \mathrm{IndCoh}(X)$ by an object of $\mathrm{QCoh}(X)$.

2. Correspondences

Let $\mathbf{C}$ be an $(\infty,1)$-category with Cartesian product, and let $\mathbb{S}$ be a target $(\infty,1)$-category. The role of the category of correspondences $\mathrm{Corr}(\mathbf{C})$ is to encode a 'bivariant' functor from $\mathbf{C}$ to $\mathbb{S}$. The example that one should keep in mind is $\mathbf{C} = \mathrm{Sch}_{\mathrm{aft}}$, $\mathbb{S} = \mathrm{DGCat}_{\mathrm{cont}}$ and the functor in question is IndCoh, where we allow to both take the !-pullback and *-push forward.

2.1. **Why do they arise?**

Suppose that we are given functors

$$\Phi : \mathbf{C} \to \mathbb{S} \text{ and } \Phi^! : \mathbf{C}^{\mathrm{op}} \to \mathbb{S},$$

(not necessarily related to each other by any sort of adjunction) that agree on objects, and for every Cartesian square

$$\begin{array}{ccc} \mathbf{c}'_0 & \xrightarrow{\alpha_0} & \mathbf{c}_0 \\ {\scriptstyle\beta'}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathbf{c}'_1 & \xrightarrow{\alpha_1} & \mathbf{c}_1 \end{array} \tag{2.1}$$

of an isomorphism of maps $\Phi(\mathbf{c}_0) \to \Phi(\mathbf{c}'_1)$

$$\Phi^!(\alpha_1) \circ \Phi(\beta) \simeq \Phi(\beta') \circ \Phi^!(\alpha_0).$$

We want such a data to be encoded by a functor $\mathrm{Corr}(\mathbf{C}) \to \mathbb{S}$.

2.1.1. If $\mathbf{C}$ is an ordinary category, it is easy to say what $\mathrm{Corr}(\mathbf{C})$ should be. Namely, its objects are the same as those of $\mathbf{C}$, but now morphisms from $\mathbf{c}_0$ to $\mathbf{c}_1$ are diagrams

$$\begin{array}{ccc} \mathbf{c}_{0,1} & \xrightarrow{g} & \mathbf{c}_0 \\ {\scriptstyle f}\downarrow & & \\ \mathbf{c}_1, & & \end{array} \tag{2.2}$$

and the compositions are given as follows: the composition of (2.2) and

$$\begin{array}{ccc} \mathbf{c}_{1,2} & \longrightarrow & \mathbf{c}_1 \\ \downarrow & & \\ \mathbf{c}_2, & & \end{array}$$

is the diagram

$$\begin{array}{ccc} \mathbf{c}_{0,2} & \longrightarrow & \mathbf{c}_0 \\ \downarrow & & \\ \mathbf{c}_2, & & \end{array}$$

where $\mathbf{c}_{0,2} = \mathbf{c}_{1,2} \underset{\mathbf{c}_1}{\times} \mathbf{c}_{0,1}$.

A bivariant functor as above defines a functor out of the category $\mathrm{Corr}(\mathbf{C})$ by setting

$$\Phi_{\mathrm{corr}}(\mathbf{c}) := \Phi(\mathbf{c}) = \Phi^!(\mathbf{c})$$

at the level of objects, and for a morphism (2.2), the corresponding map

$$\Phi(\mathbf{c}_0) \to \Phi(\mathbf{c}_1)$$

is given by $\Phi(\beta) \circ \Phi^!(\alpha)$. The isomorphisms (2.1) ensure that Φ_{corr} respects compositions.

2.1.2. However, if $\mathbf{C}$ is a higher category, we cannot just define $\text{Corr}(\mathbf{C})$ by specifying the objects, morphisms and compositions. Instead, we need to invent a device which would produce the desired $(\infty, 1)$-category from the (rather limited) list of procedures that produce $(\infty, 1)$-categories from the existing ones. Moreover, the $(\infty, 1)$-category $\text{Corr}(\mathbf{C})$ should exactly encode what it means for the isomorphisms (2.1) to be compatible with the compositions of vertical and horizontal morphisms.

We introduce and study this device in Chapter 7 of this volume.

2.2. **How to construct functors out of a category of correspondences?** Once the category $\text{Corr}(\mathbf{C})$ is constructed, we would like to describe a mechanism that produces functors out of it (and thereby gives rise to bi-variant functors with all the necessary compatibilities).

2.2.1. Here is a construction, a generalization of which will be one of our basic tools. Let us start with a functor

$$\Phi : \mathbf{C} \to \mathbb{S},$$

where $\mathbb{S}$ is the $(\infty, 1)$-category 1-Cat. Suppose that for every 1-morphism $\mathbf{c}_0 \overset{f}{\to} \mathbf{c}_1$ in $\mathbf{C}$, the corresponding map in 1-Cat, i.e., a *functor between* $(\infty, 1)$*-categories*,

$$\Phi(\mathbf{c}_0) \to \Phi(\mathbf{c}_1),$$

admits a right adjoint.

Then the operation of *passage to the right adjoint* adjoint defines a functor

$$\Phi^! : \mathbf{C}^{\text{op}} \to \mathbb{S}.$$

Suppose now that the following condition holds: for a Cartesian diagram (2.1), the natural transformation

$$\Phi(\beta') \circ \Phi^!(\alpha_0) \to \Phi^!(\alpha_1) \circ \Phi(\beta)$$

that arises by adjunction from the isomorphism

$$\Phi(\alpha_1) \circ \Phi(\beta') \simeq \Phi(\beta) \circ \Phi(\alpha_0),$$

is an isomorphism.

In this case we do expect that the functors $(\Phi, \Phi^!)$ comprise the datum of a functor

$$\Phi_{\text{corr}} : \text{Corr}(\mathbf{C}) \to \mathbb{S}.$$

And this turns out to indeed be the case.

Let us denote the subcategory of functors $\text{Funct}(\mathbf{C}, \mathbb{S})$ satisfying the above properties by $\text{Funct}(\mathbf{C}, \mathbb{S})^{\text{BC}}$ (here 'BC' stands either for 'Beck-Chevalley' or 'base change'). Thus, we obtain a functor

$$\text{Funct}(\mathbf{C}, \mathbb{S})^{\text{BC}} \to \text{Funct}(\text{Corr}(\mathbf{C}), \mathbb{S}), \quad \mathbb{S} = \text{1-Cat}. \tag{2.3}$$

However, the functor (2.3) is not an equivalence, and it is not quite adequate for our purposes, for two reasons.

2.2.2. For one thing, we would like to 'upgrade ' (2.3) (by 'upgrading' we mean modifying the right-hand side) to make it an equivalence, in order to be more robust and suitable for applications.

But more importantly, for now, the above is not more than wishful thinking: we would not even be able to construct the functor (2.3) unless we make a sharper claim.

2.3. **The 2-categorical enhancement.** To make the sought-for sharper claim, we notice that our discussion was specific to the target $(\infty,1)$-category being 1-Cat, in that we used the notion of 'adjoint' 1-morphism.

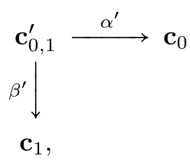
However, this is not specific just to 1-Cat, but rather is an artifact of a richer structure on the totality of $(\infty,1)$-categories: namely that 1-Cat is the $(\infty,1)$-category underlying a canonically defined $(\infty,2)$-category, denoted $\mathbf{1\text{-}Cat}$.

Thus, one expects to find a construction analogous to (2.3), where $\mathbb{S}$ is (the $(\infty,1)$-category underlying) an $(\infty,2)$-category. And such a construction is indeed possible, and can be sharpened to an equivalence, once we understand $\mathrm{Corr}(\mathbf{C})$ differently:

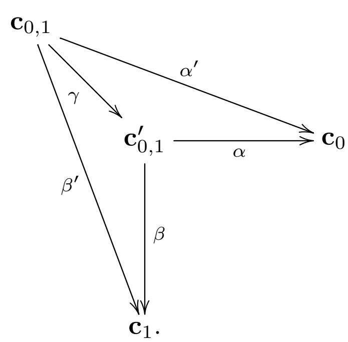
Namely, we should enhance $\mathrm{Corr}(\mathbf{C})$ itself to an $(\infty,2)$-category, denote it $\mathrm{Corr}(\mathbf{C})^{\text{2-Cat}}$.

2.3.1. If $\mathbf{C}$ was an ordinary category, then $\mathrm{Corr}(\mathbf{C})^{\text{2-Cat}}$ would be an ordinary 2-category, where we introduce 2-morphisms as follows:

For a morphism $\mathbf{c}_0 \to \mathbf{c}_1$, given by (2.2), and another one, given by



the set of maps between them is that of commutative diagrams



When $\mathbf{C}$ is a genuine $(\infty,1)$-category, we construct $\mathrm{Corr}(\mathbf{C})^{\text{2-Cat}}$ using a device that we call 'Segal categories'.

2.3.2. Thus, in order to have an adequate theory of categories of correspondences, one has to venture into the (so far, not so well explored) world of $(\infty,2)$-categories. Once we do this, we will have a naturally defined map

$$\mathrm{Funct}(\mathrm{Corr}(\mathbf{C})^{\text{2-Cat}}, \mathbb{S}) \to \mathrm{Funct}(\mathbf{C}, \mathbb{S}),$$

whose essential image is $\mathrm{Funct}(\mathbf{C}, \mathbb{S})^{\mathrm{BC}}$, i.e., we obtain the sought-for equivalence

$$\mathrm{Funct}(\mathbf{C}, \mathbb{S})^{\mathrm{BC}} \simeq \mathrm{Funct}(\mathrm{Corr}(\mathbf{C})^{\text{2-Cat}}, \mathbb{S}), \tag{2.4}$$

refining (2.3).

2.3.3. In addition to defining the $(\infty, 1)$-category $\mathrm{Corr}(\mathbf{C})^{2\text{-Cat}}$ and also the $(\infty, 2)$-category $\mathrm{Corr}(\mathbf{C})^{2\text{-Cat}}$, in Chapter 7 we prove an extension theorem, that allows us to construct IndCoh as a functor out of the category of correspondences to $\mathrm{DGCat}_{\mathrm{cont}}$.

In Chapter 8 we prove two more extension theorems that allow us to extend a functor from one category of correspondences to a 'bigger' one. These theorems will be applied in Volume II to extending IndCoh from schemes to *inf-schemes*.

2.4. **Correspondences and duality.** It turns out that the formalism of functors out of $\mathrm{Corr}(\mathbf{C})$ is an efficient way to encode the idea of duality.

2.4.1. Let us observe that the Cartesian product on $\mathbf{C}$ makes $\mathrm{Corr}(\mathbf{C})$ into a symmetric monoidal category. We note, however, that every object $\mathbf{c} \in \mathrm{Corr}(\mathbf{C})$ is canonically self-dual. Namely, the unit and co-unit maps are given by the diagrams

$$\begin{array}{ccc} \mathbf{c} & \longrightarrow & * \\ \downarrow & & \\ \mathbf{c} \times \mathbf{c} & & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{c} & \longrightarrow & \mathbf{c} \times \mathbf{c} \\ \downarrow & & \\ *, & & \end{array}$$

where $*$ denote the final object of $\mathbf{C}$, and $\mathbf{c} \to \mathbf{c} \times \mathbf{c}$ is the diagonal map.

2.4.2. Suppose that we are given a functor

$$\Phi_{\mathrm{corr}} : \mathrm{Corr}(\mathbf{C}) \to \mathbb{S},$$

where both $\mathbb{S}$ and Φ_{corr} are equipped with symmetric monoidal structures.

Then we obtain that for any $\mathbf{c} \in \mathbf{C}$, the corresponding object $\Phi_{\mathrm{corr}}(\mathbf{c}) \in \mathbb{S}$ is canonically self-dual.

2.4.3. Applying this observation to the IndCoh functor, we will obtain that Serre duality is a formal consequence of the existence of IndCoh as a functor out of the category of correspondences. Namely, we will see that for $X \in \mathrm{Sch}_{\mathrm{aft}}$, the DG category $\mathrm{IndCoh}(X)$ is equipped with a canonical identification

$$\mathbf{D}^{\mathrm{Serre}} : \mathrm{IndCoh}(X) \simeq \mathrm{IndCoh}(X)^{\vee}, \tag{2.5}$$

where $\mathrm{IndCoh}(X)^{\vee}$ is the dual category, i.e., $\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(X), \mathrm{Vect})$.

At the level of compact objects, the equivalence (2.5) gives rise to an equivalence

$$\mathbb{D}^{\mathrm{Serre}} : \mathrm{Coh}(X)^{\mathrm{op}} \simeq \mathrm{Coh}(X),$$

which is the usual Serre duality.

A similar reasoning will lead to Verdier duality for D-modules, which we will develop in Volume II of the book.

3. The appendix on $(\infty, 2)$-categories

As has been mentioned above, in order to establish, or even formulate, the isomorphism (2.4), one needs to venture into the world of $(\infty, 2)$-categories.

Some of the foundations of the theory of $(\infty, 2)$-categories can be found in the existing literature. However, the theory developed so far does quite meet our needs. For this reason, we have decided to include Part A, which lays out this theory the way we would like to see it (albeit, omitting some proofs).

3.1. **Setting up the theory.** We approach $(\infty, 2)$-categories by imitating the complete Segal space approach to $(\infty, 1)$-categories.

3.1.1. Namely, we recall that the datum of an $(\infty, 1)$-category $\mathbf{C}$ is completely recorded from the datum of the simplicial space $\mathrm{Seq}_\bullet(\mathbf{C})$ that sends $[n] \in \mathbf{\Delta}$ to the space of strings of objects of $\mathbf{C}$

$$\mathbf{c}_0 \to \mathbf{c}_1 \to \ldots \to \mathbf{c}_n. \tag{3.1}$$

Thus, we obtain a functor

$$\mathrm{Seq}_\bullet : \text{1-Cat} \to \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}},$$

which is fully faithful, and one can explicitly describe its essential image.

3.1.2. We would like to define $(\infty, 2)$-categories similarly. Namely, we wish to define the $(\infty, 1)$-category 2-Cat a certain full subcategory in $\text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$. However, one immediately runs into the following dilemma: if $\mathbb{S}$ is an $(\infty, 2)$-category (whatever this notion is), there are two possibilities of what the $(\infty, 1)$-category of length n strings objects could be.

In both cases, the objects of our category are strings as in (3.1), where the arrows are 1-morphisms. But there is a choice involved in how we define morphisms between such objects. In one case, we ask for diagrams

$$\begin{array}{ccccccccc} \mathbf{c}_0 & \longrightarrow & \mathbf{c}_1 & \longrightarrow & \cdots & \longrightarrow & \mathbf{c}_{n-1} & \longrightarrow & \mathbf{c}_n \\ \downarrow & \swarrow & \downarrow & \swarrow & & \swarrow & \downarrow & \swarrow & \downarrow \\ \mathbf{c}'_0 & \longrightarrow & \mathbf{c}'_1 & \longrightarrow & \cdots & \longrightarrow & \mathbf{c}'_{n-1} & \longrightarrow & \mathbf{c}'_n, \end{array} \tag{3.2}$$

where the slanted arrows stand for 2-morphisms.

In the other case, we ask for diagrams

$$\mathbf{c}_0 \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \mathbf{c}_1 \cdots \; \mathbf{c}_{n-1} \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \mathbf{c}_n.$$

I.e., these are the same as diagrams (3.2), but with the vertical 1-morphisms being isomorphisms.

3.1.3. What we obtain is that there is whatever the $(\infty, 1)$-category 2-Cat is, it is equipped with two functors

$$\mathrm{Seq}_\bullet : \text{2-Cat} \to \text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$$

(corresponding to the second kind of 1-morphisms on n-simplices), and

$$\mathrm{Seq}^{\mathrm{ext}}_\bullet : \text{2-Cat} \to \text{1-Cat}^{\mathbf{\Delta}^{\mathrm{op}}},$$

(corresponding to the first kind of 1-morphisms on n-simplices), both of which are supposed to be fully faithful, with an explicitly described essential image.

We take the first realization (i.e., one with $\text{Seq}_\bullet$) as the definition of 2-Cat, and prove that the other realization (i.e., one with $\text{Seq}_\bullet^{\text{ext}}$) has the expected properties. This is done in Chapter 10 of this volume.

In turns out that the first realization is more convenient for taking the theory off the ground, while the second one is necessary for our treatment of adjunctions, see below.

3.1.4. In Chapter 11 we study some basic constructions associated with $(\infty,2)$-categories, namely, the straightening/unstraightening equivalence (generalizing the familiar construction in the context of $(\infty,1)$-categories), and the Yoneda embedding.

3.2. **Adjunctions.** The main reason we need to develop the theory of $(\infty,2)$-categories is to have a theory of adjunctions, adequate for establishing the equivalence (2.4).

3.2.1. Let $\mathbf{C}$ an $(\infty,2)$-category. Then for a 1-morphism $\alpha : \mathbf{c}_0 \to \mathbf{c}_1$, there is a notion of what it means to *admit a right adjoint*.

If $\mathbf{C}$ is an ordinary category, if a right adjoint of a 1-morphism exists, it is defined up to a canonical isomorphism. More generally, if $F : \mathbf{I} \to \mathbf{C}$ is a functor, if for every arrow $i_0 \to i_1$ in $\mathbf{I}$, the corresponding 1-morphism $F(i_0) \to F(i_1)$ admits a right adjoint, we can canonically construct a functor

$$G : \mathbf{I}^{\text{op}} \to \mathbf{C},$$

which is the same as F at the level of objects, but which at the level of morphisms is obtained from F by replacing each $F(i_0) \to F(i_1)$ by its right adjoint. In this case we will say that G *is obtained from F by passing to right adjoints.*

Let $\text{Funct}(\mathbf{I},\mathbf{C})^L$ be the full subcategory of $\text{Funct}(\mathbf{I},\mathbf{C})$ consisting of those functors $F : \mathbf{I} \to \mathbf{C}$ such that for every 1-morphism $i_0 \to i_1$ in $\mathbf{I}$, the corresponding 1-morphism $F(i_0) \to F(i_1)$ admits a right adjoint. Let $\text{Funct}(\mathbf{I}^{\text{op}},\mathbf{C})^R$ be the corresponding full subcategory of $\text{Funct}(\mathbf{I}^{\text{op}},\mathbf{C})$ (replace 'right' by 'left'). Then the assignment $F \mapsto G$ defines an equivalence

$$\text{Funct}(\mathbf{I},\mathbf{C})^L \simeq \text{Funct}(\mathbf{I}^{\text{op}},\mathbf{C})^R. \tag{3.3}$$

3.2.2. The same assertions–canonicity of the adjoint for an individual morphism and the equivalence (3.3)– –remain true in the context of $(\infty,2)$-categories, but it is a non-trivial task to formulate, and subsequently prove them.

We develop the theory of adjunctions in Chapter 12. We show that for any $\mathbf{I}$ there exists an $(\infty,2)$-category, $\mathbf{I}^R$, equipped with a pair of functors

$$\mathbf{I} \to \mathbf{I}^R \leftarrow \mathbf{I}^{\text{op}},$$

such that for any target $(\infty,2)$-category $\mathbf{C}$, restrictions along the above functors define equivalences

$$\text{Funct}(\mathbf{I},\mathbf{C})^R \xleftarrow{\sim} \text{Funct}(\mathbf{I}^R,\mathbf{C}) \xrightarrow{\sim} \text{Funct}(\mathbf{I}^{\text{op}},\mathbf{C})^L. \tag{3.4}$$

Moreover, it turns out that the equivalences (3.4) are precisely adjusted to establishing the equivalence (2.4).

The construction of $\mathbf{I}^R$ is based on the 'second' realization of 2-Cat as a full subcategory of $1\text{-Cat}^{\boldsymbol{\Delta}^{\text{op}}}$, i.e., one using the functor $\text{Seq}_\bullet^{\text{ext}}$.