CHAPTER I.1. SOME HIGHER ALGEBRA

Contents

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Introduction

This Chapter is meant to provide some background on ∞-categories and *higher algebra* (the latter includes the notions of (symmetric) monoidal ∞-category, (commutative) algebras in a (symmetric) monoidal ∞-category, and modules over such algebras).

0.1. **Why $(\infty, 1)$-categories?** At this point in the development of mathematics one hardly needs to make a case for ∞-categories. Nonetheless, in this subsection we explain why they necessarily appear in this book. I.e., why we cannot remain in the world of, say, triangulated categories (if we talk about 'linear' categories).

0.1.1. In fact, there are two separate (but related) reasons that force one to work with ∞-categories, rather than triangulated ones: *extrinsic* and *intrinsic*.

The extrinsic reason has to do with the behavior of the *totality* of ∞/triangulated categories, and the intrinsic reason has to do with what is going on within a given ∞/triangulated category.

We begin by discussing the extrinsic reason, which we believe is more fundamental.

0.1.2. The extrinsic reason has to do the with the operation of *limit* of a diagram of ∞ (resp., triangulated) categories.

An example of a limit is *gluing*: imagine that you want to glue the ∞/triangulated category of quasi-coherent sheaves on a scheme X from an affine cover.

Below we will explain why just the above operation of gluing along an open cover unavoidably appears in the theory that we are trying to build. However, taking that on faith, we arrive at the necessity to work with ∞-categories: it is well-known that triangulated categories do not glue well.

For example, given a scheme/topological space X with an action of an algebraic/compact group G, there is no known way to define the G-equivariant derived category of sheaves on X while only using the *derived* category of sheaves as an input: all the existing definitions appeal to constructions that take place at the *chain level*.

But once we put ourselves in the context of ∞-categories, everything works as expected. For example, given a prestack $\mathcal{Y}$ (i.e., an arbitrary functor from the category of affine schemes to that of ∞-groupoids), one can define the category $\mathrm{Shv}(\mathcal{Y})$ of sheaves on $\mathcal{Y}$ as the limit

$$\lim_{S,y:S\to\mathcal{Y}} \mathrm{Shv}(S),$$

where the limit is taken over the category of affine schemes over $\mathcal{Y}$. As a tiny particular case of this, we can take $\mathcal{Y} = X/G$ and recover the G-equivariant derived category on X.

0.1.3. We now explain the *intrinsic* reason why one is often forced to work with ∞-categories.

It also has to do with...–*suspense*–...the operation of taking a limit (or colimit), but now within our given ∞/triangulated category $\mathbf{C}$.

But here, in a sense, we will not say anything new. A basic example of limit is the operation of fiber product of objects

$$\mathbf{c}_1 \underset{\mathbf{c}}{\times} \mathbf{c}_2.$$

When working in a triangulated category, we usually want to interpret the latter as the (shifted by $[-1]$) *cone* of the map

$$\mathbf{c}_1 \oplus \mathbf{c}_2 \to \mathbf{c},$$

and we arrive to the familiar problem that cones are not well-defined (or, rather, that they do not have a functorial description).

Of course, one can say that cones exist, even though they are not canonical. But this non-canonicity prevents one from defining more general *homotopy colimits*, e.g., geometric realizations of simplicial objects, and without that, one cannot really do algebra, of the kind that we will be doing in Part IV of this book (operads, Lie algebras and Lie algebroids, etc.)

For example, in a monoidal triangulated category, one cannot form the tensor product of a right module and a left module over an associative algebra.

0.2. **The emergence of derived algebraic geometry and why we need gluing.** We shall first explain why derived algebraic geometry enters our game (that is, even if, at the start, one tries to work in the world of usual schemes).

We will then see how objects of derived algebraic geometry necessitate a gluing procedure.

0.2.1. Let us consider the pattern of *base change* for the derived category of quasi-coherent sheaves. Let us be given a Cartesian diagram of (usual) schemes

$$\begin{array}{ccc} X_1 & \xrightarrow{g_X} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{g_Y} & Y_2. \end{array}$$

Then from the isomorphism of functors

$$(f_2)_* \circ (g_X)_* \simeq (g_Y)_* \circ (f_1)_*$$

one obtains by adjunction the natural transformation

$$(g_Y)^* \circ (f_2)_* \to (f_1)_* \circ (g_X)^*. \tag{0.1}$$

The base change theorem says that (0.1) is an isomorphism. The only problem is that this theorem is *false*.

More precisely, it is false if X_1 is taken to be the fiber product $Y_1 \underset{Y_2}{\times} X_2$ in the category of usual schemes: consider the case when all schemes are affine; $X_i = \operatorname{Spec}(A_i)$ and $Y_i = \operatorname{Spec}(B_i)$, and apply (0.1) to

$$A_2 \in A_2\text{-mod} = \operatorname{QCoh}(X_2).$$

We obtain the map

$$B_1 \underset{B_2}{\otimes} A_2 \to A_1,$$

where the tensor product is understood in the *derived* sense, while the right hand side is its top (i.e., 0-th) cohomology.

To remedy this problem, we need to take A_1 to be the full derived tensor product $B_1 \underset{B_2}{\otimes} A_2$, so that A_1 is no longer a plain commutative algebra, but what one calls a *connective commutative DG algebra*. The spectrum of such a thing is, by definition, an affine derived scheme.

0.2.2. Thus, we see that affine derived schemes are necessary if we wish to have base change. And if we want to do algebraic geometry (i.e., consider not just affine schemes), we need to introduce the notion of general derived scheme.

We will delay the discussion of what derived schemes actually are until [Chapter I.2]. However, whatever they are, a derived scheme X is glued from an open cover of affine derived schemes $U_i = \operatorname{Spec}(A_i)$, and let us try to imagine what the ∞/triangulated category $\operatorname{QCoh}(X)$ of quasi-coherent sheaves on X should be.

By definition for each element of the cover we have

$$\operatorname{QCoh}(U_i) = A_i\text{-mod},$$

i.e., this is the category of A_i-modules. Now, whatever X is, the ∞/triangulated category $\operatorname{QCoh}(X)$ should be obtained as a gluing of $\operatorname{QCoh}(U_i)$, i.e., as the limit of the diagram of categories

$$\operatorname{QCoh}(U_{i_0} \cap ... \cap U_{i_n}), \quad n = 0, 1, ...$$

Thus, we run into the problem of taking the limit of a diagram of categories, and as we said before, in order to take this limit, we should understand the above diagram as one of ∞-categories rather than triangulated ones.

0.3. **What is done in this Chapter?** This Chapter (with the exception of Sect. 9) contains no original mathematics; it is mostly a review of the foundational works of J. Lurie, [Lu1] and [Lu2].

Thematically, it can be split into the following parts:

0.3.1. Sects. 1-2 are a review of higher category theory, i.e., the theory of $(\infty, 1)$-categories, following [Lu1, Chapters 1-5].

In Sect. 1 we introduce the basic words of the vocabulary of $(\infty, 1)$-categories. In Sect. 2 we discuss some of the most frequent manipulations that one performs with $(\infty, 1)$-categories.

0.3.2. In Sects. 3-4 we review the basics of higher algebra, following [Lu2, Chapter 4].

In Sect. 3 we discuss the notions of (symmetric) monoidal $(\infty, 1)$-category, the notion of associative/commutative algebra in a given (symmetric) monoidal $(\infty, 1)$-category, and the notion of module over an algebra.

In Sect. 4 we discuss the pattern of duality in higher algebra.

0.3.3. In Sects. 5-7, we discuss *stable* $(\infty, 1)$-categories, following [Lu2, Sects. 1.1, 1.4 and 4.8] with an incursion into [Lu1, Sect. 5.4].

In Sect. 5 we introduce the notion of *stable* $(\infty, 1)$-category.

In Sect. 6 we discuss the operation of *Lurie tensor product* on *cocomplete stable* $(\infty, 1)$-*categories*.

In Sect. 7 we discuss the notions of compactness, compact generation and ind-completion (we do this in the context of stable categories, even though these notions make sense more generally, see [Lu1, Sect. 5.3]).

0.3.4. In Sects. 8-10 we start discussing *algebra*.

In Sect. 8 we specialize the general concepts of higher algebra to the case of stable categories. I.e., we will discuss stable (symmetric) monoidal categories, module categories over them, duality for such, etc.

In Sect. 9 (which is the only section that contains some original mathematics) we introduce the notion of *rigid* monoidal category. By a loose analogy, one can think of rigid monoidal categories as Frobenius algebras in the world of stable categories. These stable monoidal categories exhibit particularly strong *extrinsic* finiteness properties: i.e., properties of module categories over them.

Finally, in Sect. 10 we introduce the notion of *DG category*. This will be the world in which we will do algebra in the main body of this book.

0.4. **What do we have to say about the theory of ∞-categories?** The theory of ∞-categories, in the form that is amenable for use by non-experts, has been constructed by J. Lurie in [Lu1]. It is based on the model of ∞-categories as *quasi-categories* (a.k.a., *weak Kan simplicial sets*), developed in the foundational work of A. Joyal, [Jo].

0.4.1. The remarkable thing about this theory is that one does *not* really need to know the contents of [Lu1] in order to apply it.

What Lurie's book provides is a syntax of allowed words and sentences in the the theory of ∞-categories, and ensures that this syntax can be realized in the model of quasi-categories.

0.4.2. In Sect. 1 we make an attempt to summarize this syntax. However, we are not making a mathematical assertion here: our grammar is incomplete and suffers from circularity (e.g., we appeal to fiber products before introducing limits).

The task of actually writing down such a syntax appears to be a non-trivial problem on its own. It seems likely, however, that in order to do that, one has to completely disengage oneself from viewing objects of a (higher) category as a set.

The latter would be desirable in any case: the simplical set underlying an ∞-category is a phantom; indeed, we never use it for any 'yes or no' questions or when we need to compute something.

0.4.3. In Sect. 2, we assume that we know how to speak the language of ∞-categories, and we introduce some basic tools that one uses to create new ∞-categories from existing ones, and similarly for functors.

These have to do with the operation of taking limits and colimits (within a given ∞-category or the totality of such), and the procedure of Kan extension.

0.4.4. Our tool-kit regarding ∞-categories is far from complete.

For example, we do not define what filtered/sifted ∞-categories are.

And, quite possibly, there are multiple other pieces of terminology, common in the theory of ∞-categories, that we use without being aware of not having introduced them. Whenever this happens, the reader should go back to [Lu1], and find the definition therein.

0.4.5. *What about set theory?* As is written in [Lu1, Sect. 1.2.15], one needs to make a decision on how one treats the sizes of our categories, i.e., the distinction between 'large' and 'small' categories.

Our policy is option (3) from *loc.cit.*, i.e., we just *ignore* these issues.

One reason for this is that the mention of cardinals when stating lemmas and theorems clutters the exposition.

Another reason is that it is very difficult to make a mistake of set-theoretic nature, unless one makes a set-theoretic argument (which we never do).

So, we will assume that our reader will not be conflicted about cutting his/her own hair, and live in the happy cardinal-free world.

0.5. **What do we have to say about higher algebra?** Nothing, in fact, beyond what is written in [Lu2, Chapter 4]. But we need much less (e.g., we do need general operads), so we decided to present a concise summary of things that we will actually use.

0.5.1. We start by discussing associative and commutative structures, i.e., monoidal/symmetric monoidal ∞-categories and associative/commutative algebras in them. In fact, it all boils down to the notion of monoid/commutative monoid in a given ∞-category.

The remarkable thing is that it is easy to encode monoids/commutative monoids using functors between ∞-categories. This idea originated in Segal's foundational work [Seg], and was implemented in the present context in [Lu2, Chapter 4].

Namely, the datum of a monoid in an ∞-category $\mathbf{C}$ is encoded by a functor

$$F : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C},$$

that satisfies the following condition: $F([0]) = *$, and for every $n = 1, ...$, the map

$$F([n]) \to \underset{i=1,\ldots,n}{\Pi} F([1])$$

is an isomorphism, where the i-th map $F([n]) \to F([1])$ corresponds to the map

$$[1] \to [n], \quad 0 \mapsto i-1, 1 \mapsto i.$$

For example, the binary operation is the map

$$F([2]) \to F([1])$$

that corresponds to the map

$$[1] \to [2], \quad 0 \mapsto 0, 1 \mapsto 2.$$

Similarly, the datum of a commutative monoid is encoded by a functor

$$F : \mathrm{Fin}_* \to \mathbf{C},$$

where Fin_* is the category of *pointed finite sets.*

Once we take this point of view, the basic definitions of higher algebra roll out quite easily. This is what is done in Sect. 3.

0.5.2. In Sect. 4 we discuss the notion of duality. It appears in several flavors: the notion of left/right dual of an object in a monoidal ∞-category; the notion of dual of module over an algebra; and also as the notion of adjoint functor.

It is easy to define what it means for an object to be *dualizable.*

However, the question of *canonicity* of the dual is trickier: in what sense is the dual uniquely defined? I.e., what kind of duality datum specifies it uniquely (i.e., *up to a contractible space of choices*)?

In fact, this question can be answered precisely, but for this one needs to work in the context of $(\infty, 2)$-categories. And we actually do this, in [Chapter A.3], in the framework of discussing the notion of adjoint 1-morphism in a given $(\infty, 2)$-category.

The upshot of *loc.cit.* is that the dual is canonically defined, and one can specify (albeit not too explicitly) the data that fixes it uniquely.

0.6. **Stable ∞-categories.** In the main body of the book we will be doing algebra in DG categories (over a field k of characteristic 0). There are (at least) two routes to set this theory up.

0.6.1. One route would be to proceed directly by working with (ordinary) categories enriched over the category of complexes of vector spaces over k.

In fact, this way of approaching DG categories has been realized in [Dr]. However, one of the essential ingredients of a functioning theory is that the totality of DG categories should itself be endowed with a structure of ∞-category (in order to be able to take limits). But since the paper [Dr] appeared before the advent of the language of ∞-categories, some amount of work would be needed to explain how to organize DG categories into an ∞-category.

The situation with the operation of *tensor product* of DG categories is similar. It had been developed in [FG], prior to the appearance of [Lu2]. However, this structure had not been formulated as a symmetric monoidal ∞-category in language that we use today.

So, instead of trying to rewrite the constructions of [Dr] and [FG] in the language of (symmetric monoidal) ∞-categories, we decided to abandon this approach, and access DG categories via a more robust (=automatic, tautological) approach using the general notion of stable ∞-category and the symmetric monoidal structure on such, developed in [Lu2].

0.6.2. The definition of stable ∞-categories given in [Lu2] has the following huge advantage: being stable is *not* an additional piece of structure, but a *property* of an ∞-category.

As a consequence of this, we do not have to labor to express the fact that any stable ∞-category is *enriched* over the ∞-category Sptr of spectra (i.e., that mapping spaces in a stable ∞-category naturally lift to objects of Sptr): whatever meaning we assign to this phrase, this structure is automatic from the definition (see Sect. 0.6.5).

0.6.3. Given a stable ∞-category, one can talk about t-structures on it. We count on the reader's familiarity with this notion: a t-structure on a stable category is the same as the t-structure on the associated triangulated category.

In terms of notation, given a stable ∞-category $\mathbf{C}$ with a t-structure, we let

$$\mathbf{C}^{\leq 0} \subset \mathbf{C} \supset \mathbf{C}^{\geq 0}$$

the corresponding full subcategories of connective/coconnective objects (so that $\mathbf{C}^{>0}$ is the *right* orthogonal to $\mathbf{C}^{\leq 0}$). We let

$$\mathbf{C}^{\leq 0} \overset{\tau^{\leq 0}}{\longleftarrow} \mathbf{C} \text{ and } \mathbf{C} \overset{\tau^{\geq 0}}{\longrightarrow} \mathbf{C}^{\geq 0}$$

be the corresponding right and left adjoints (i.e., the truncation functors).

We let

$$\mathbf{C}^{\heartsuit} = \mathbf{C}^{\leq 0} \cap \mathbf{C}^{\geq 0}$$

denote the heart of the t-structure; this is an abelian category.

We will also use the notation

$$\mathbf{C}^{-} = \underset{n\geq 0}{\cup} \mathbf{C}^{\leq n} \text{ and } \mathbf{C}^{+} = \underset{n\geq 0}{\cup} \mathbf{C}^{\geq -n}.$$

We will refer to $\mathbf{C}^{-}$ (resp., $\mathbf{C}^{+}$) as the *bounded above* or *eventually connective* (resp., *bounded below* or *eventually coconnective*) subcategory of $\mathbf{C}$.

0.6.4. *The Lurie tensor product.* One of the key features of the ∞-category of stable categories $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ (here we restrict objects to be *cocomplete* stable categories, and morphisms to be colimit-preserving functors) is that it carries a symmetric monoidal structure[1], which we call the *Lurie tensor product.*

Another huge advantage of the way this theory is set up in [Lu2, Sect. 4.8] is that the definition of this structure is automatic (=obtained by passing to appropriate full subcategories) from the Cartesian symmetric monoidal structure on the ∞-category $1\text{-}\mathrm{Cat}$ of all ∞-categories.

The intuitive idea behind the Lurie tensor product is this: if A and B are associative algebras, then the tensor product of A-mod and B-mod should be $(A \otimes B)$-mod.

[1]The existence of the Lurie tensor product is yet another advantage of working with stable ∞-categories rather than triangulated ones: one cannot define the tensor product for the latter.

0.6.5. *Spectra.* The symmetric monoidal structure on $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ leads to a very concise definition of the ∞-category Sptr of spectra. Namely, this is the unit object in $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ with respect to its symmetric monoidal structure.

In particular, every (cocomplete) stable ∞-category $\mathbf{C}$ is automatically a module over Sptr. Thus, for any two objects $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$, we can consider their *relative internal Hom*

$$\underline{\text{Hom}}_{\mathbf{C},\text{Sptr}}(\mathbf{c}_0, \mathbf{c}_1) \in \text{Sptr}\,.$$

This is the enrichment structure on $\mathbf{C}$ with respect to Sptr, mentioned earlier.

0.6.6. In Sect. 7 we study a class of cocomplete stable categories that are particularly amenable to calculations: these are the compactly generated stable categories. This material is covered by [Lu1, Sect. 5.3], and a parallel theory in the framework of DG categories can be found in [Dr].

The main point is that a compactly generated stable category $\mathbf{C}$ can be obtained as the *ind-completion*[2] of its full subcategory $\mathbf{C}^c$ of compact objects. The ind-completion procedure can be thought of as formally adjoining to $\mathbf{C}^c$ all filtered colimits. However, we can also define it explicitly as the category of all exact functors

$$(\mathbf{C}^c)^{\text{op}} \to \text{Sptr}\,.$$

The advantage of compactly generated stable categories is that the data involved in describing colimit-preserving functors out of them is manageable: for a compactly generated $\mathbf{C}$ and an arbitrary cocomplete $\mathbf{D}$ we have

$$\text{Funct}_{\text{ex,cont}}(\mathbf{C}, \mathbf{D}) \simeq \text{Funct}_{\text{ex}}(\mathbf{C}^c, \mathbf{D}).$$

0.6.7. As in a symmetric monoidal ∞-category, given an object $\mathbf{C} \in 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ we can ask about its dualizability.

It is a basic fact that if $\mathbf{C}$ is compactly generated, then it is dualizable. Moreover, its dual can be described very explicitly: it is the ind-completion of $(\mathbf{C}^c)^{\text{op}}$.

In other words, $\mathbf{C}^\vee$ is also compactly generated and we have a canonical equivalence

$$(\mathbf{C}^\vee)^c \simeq (\mathbf{C}^c)^{\text{op}}. \tag{0.2}$$

0.6.8. *Categorical meaning of Verdier duality.* The equivalence (0.2) is key to the categorical understanding of such phenomena as Verdier duality. Indeed, let X be a scheme (of finite type), and consider the cocomplete stable ∞-category D-mod(X) of D-modules on X.

The subcategory $(\text{D-mod}(X))^c$ consists of those objects that have finitely many cohomologies (with respect to the usual t-structure) all of which are *coherent* D-modules. Denote this subcategory by $\text{D-mod}(X)_{\text{coh}}$.

The usual Verdier duality for D-modules defines a contravariant auto-equivalence

$$\mathbb{D}^{\text{Verdier}}_X : (\text{D-mod}(X)_{\text{coh}})^{\text{op}} \simeq \text{D-mod}(X)_{\text{coh}}.$$

Now, the above description of duality for compactly generated stable ∞-categories implies that we can perceive Verdier duality as an equivalence

$$\mathbf{D}^{\text{Verdier}}_X : (\text{D-mod}(X))^\vee \simeq \text{D-mod}(X),$$

which reduces to $\mathbb{D}^{\text{Verdier}}_X$ at the level of compact objects.

[2]Ind-completion is another operation that requires having a stable category, rather than a triangulated one.

We also obtain a more functorial understanding of expressions such as "the Verdier conjugate of the *-direct image is the !-direct image". The categorical formulation of this is the fact that for a morphism of schemes $f : X \to Y$, the functors

$$f_{\mathrm{dR},*} : \text{D-mod}(X) \to \text{D-mod}(Y) \text{ and } f^! : \text{D-mod}(Y) \to \text{D-mod}(X)$$

are each other's *duals* in terms of the identifications $\mathbf{D}_X^{\mathrm{Verdier}}$ and $\mathbf{D}_Y^{\mathrm{Verdier}}$, see Proposition 7.3.5.

0.6.9. In Sect. 8 we discuss stable monoidal ∞-categories, and algebras in them. This consists of studying the interaction of the concepts introduced in Sect. 3 with the Lurie tensor product.

Let us give one example. Let $\mathbf{A}$ be a stable monoidal ∞-category, and let $\mathbf{M}$ be a stable module category over $\mathbf{A}$. Let $\mathcal{A}$ be an algebra object in $\mathbf{A}$.

On the one hand we can consider the (stable) ∞-category $\mathcal{A}\text{-mod}(\mathbf{M})$ of $\mathcal{A}$-modules in $\mathbf{M}$. On the other hand, we can consider $\mathbf{A}$ as acting on itself on the left, and thus consider

$$\mathcal{A}\text{-mod} := \mathcal{A}\text{-mod}(\mathbf{A}).$$

The action of $\mathbf{A}$ on itself on the right makes $\mathcal{A}$-mod into a right $\mathbf{A}$-module category.

Now, the claim is (this is Corollary 8.5.7) that there is a canonical equivalence

$$\mathcal{A}\text{-mod}(\mathbf{M}) \simeq \mathcal{A}\text{-mod} \underset{\mathbf{A}}{\otimes} \mathbf{M}.$$

0.6.10. *Rigid monoidal categories.* In Sect. 9 we discuss a key technical notion of stable *rigid* monoidal ∞-category.

If a stable monoidal ∞-category $\mathbf{A}$ is compactly generated, then being rigid is equivalent to the combination of the following conditions:

- (i) the unit object in $\mathbf{A}$ is compact;
- (ii) the monoidal operation on $\mathbf{A}$ preserves compactness;
- (iii) every compact object of $\mathbf{A}$ admits a left and a right dual.

For example, the category of modules over a commutative algebra has this property.

From the point of view of its intrinsic properties, a rigid monoidal category can be as badly behaved as any other category. However, the ∞-category of its module categories satisfies very strong finiteness conditions.

For example, given a rigid symmetric monoidal ∞-category $\mathbf{A}$, we have:

(i) Any functor between $\mathbf{A}$-module categories that is lax-compatible with $\mathbf{A}$-actions, is actually strictly compatible;

(ii) The tensor product of $\mathbf{A}$-module categories is equivalent to the co-tensor product;

(iii) An $\mathbf{A}$-module category is dualizable as such if and only if it is dualizable as a plain stable category, and the duals in both senses are isomorphic.

0.6.11. *DG categories.* We can now spell out our definition of DG categories:

Let Vect be the ∞-category of chain complexes of k-vector spaces. It is stable and cocomplete, and carries a symmetric monoidal structure. We define the ∞-category $\mathrm{DGCat}_{\mathrm{cont}}$ to be that of Vect-modules in $1\text{-}\mathrm{Cat}_{\mathrm{cont}}^{\mathrm{St,cocmpl}}$.

The stable monoidal category Vect is rigid (see above), and this ensures the good behavior of $\mathrm{DGCat}_{\mathrm{cont}}$.

1. $(\infty,1)$-categories

In this section we make an attempt to write down a user guide to the theory of $(\infty,1)$-categories. In that, the present section may be regarded as a digest of [Lu1, Chapters 1-3 and Sect. 5.2], with a view to applications (i.e., we will not be interested in how to construct the theory of $(\infty,1)$-categories, but, rather, what one needs to know in order to use it).

The main difference between this section and the introductory Sect. 1 of [Lu1] is the following. In *loc.cit.* it is explained how to use quasi-categories (i.e., weak Kan simplicial sets) to capture the structures of higher category theory, the point of departure being that Kan simplicial sets incarnate *spaces*.

By contrast, we take the basic concepts of $(\infty,1)$-categories on faith, and try to show how to use them to construct further notions. In that respect we try to stay *model independent*, i.e., we try to avoid, as much as possible, referring to simplicial sets that realize our $(\infty,1)$-categories.

The reader familiar with [Lu1] can safely skip this section.

1.1. **The basics.** In most of the practical situations, when working with $(\infty,1)$-categories, one *does not need to know* what they actually are, i.e., how exactly one defines the notion of $(\infty,1)$-category.

What one does use is the syntax: one believes that the notion of $(\infty,1)$-category exists, and all one needs to know is how to use the words correctly.

Below is the summary of the few basic words of the vocabulary. However, as was mentioned in Sect. 0.4.2, this vocabulary is flawed and incomplete. So, strictly speaking, what follows does nothing more than introduce notation, because circularity appears from the start (e.g., we talk about full subcategories and adjoints).

The reference for the material here is [Lu1, Sect. 1.2].

1.1.1. We let 1-Cat denote the $(\infty,1)$-category of $(\infty,1)$-categories.

1.1.2. We let Spc denote the $(\infty,1)$-category of spaces. We have a canonical fully faithful embedding

$$\mathrm{Set} \hookrightarrow \mathrm{Spc},$$

which admits a left adjoint, denoted

$$\mathcal{S} \mapsto \pi_0(\mathcal{S}).$$

In particular, for any $\mathcal{S} \in \mathrm{Spc}$, we have a canonical map of spaces $\mathcal{S} \to \pi_0(\mathcal{S})$.

We denote by $* \in \mathrm{Set} \subset \mathrm{Spc}$ the point space.

1.1.3. We will regard Spc as a full subcategory of 1-Cat. In particular, we will regard a space as an $(\infty,1)$-category, and maps between spaces as functors between the corresponding $(\infty,1)$-categories.

We will refer to objects of the $(\infty,1)$-category corresponding to a space $\mathcal{S}$ as *points* of $\mathcal{S}$.

The inclusion $\mathrm{Spc} \hookrightarrow 1\text{-Cat}$ admits a right adjoint, denoted $\mathbf{C} \mapsto \mathbf{C}^{\mathrm{Spc}}$; it is usually referred to as 'discarding non-invertible morphisms'.

1.1.4. For an $(\infty,1)$-category $\mathbf{C}$, and objects $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$, we denote by $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Spc}$ the corresponding *mapping space*.

1.1.5. We let $1\text{-Cat}^{\text{ordn}}$ denote the full subcategory of 1-Cat formed by ordinary categories.

This inclusion admits a left adjoint, denoted $\mathbf{C} \mapsto \mathbf{C}^{\text{ordn}}$. (Sometimes, $\mathbf{C}^{\text{ordn}}$ is called the *homotopy category of* $\mathbf{C}$ and denoted $\text{Ho}(\mathbf{C})$.) The objects of $\mathbf{C}^{\text{ordn}}$ are the same as those of $\mathbf{C}$, and we have

$$\text{Hom}_{\mathbf{C}^{\text{ordn}}}(\mathbf{c}_0, \mathbf{c}_1) = \pi_0\left(\text{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)\right).$$

Warning: we need to distinguish the $(\infty, 1)$-category $1\text{-Cat}^{\text{ordn}}$ (which is in fact a $(2,1)$-category) from the ordinary category $(1\text{-Cat})^{\text{ordn}} = \text{Ho}(1\text{-Cat})$.

1.1.6. A map $\phi : \mathbf{c}_0 \to \mathbf{c}_1$ in $\mathbf{C}$ (i.e., a point in $\text{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)$) is said to be an *isomorphism* if the corresponding map in $\mathbf{C}^{\text{ordn}}$, i.e., the image of ϕ under the projection

$$\text{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \to \pi_0\left(\text{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)\right) = \text{Hom}_{\mathbf{C}^{\text{ordn}}}(\mathbf{c}_0, \mathbf{c}_1),$$

is an isomorphism.

1.1.7. For a pair of $(\infty, 1)$-categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\text{Funct}(\mathbf{D}, \mathbf{C})$ the $(\infty, 1)$-category of functors $\mathbf{D} \to \mathbf{C}$.

We have

$$\text{Funct}(*, \mathbf{C}) \simeq \mathbf{C}$$

and

$$\text{Maps}_{1\text{-Cat}}(\mathbf{D}, \mathbf{C}) = (\text{Funct}(\mathbf{D}, \mathbf{C}))^{\text{Spc}}.$$

A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be an *equivalence* if it is an isomorphism in 1-Cat, i.e., if it induces an isomorphism in $(1\text{-Cat})^{\text{ordn}}$ (which implies, but is much stronger than asking that $F^{\text{ordn}} : \mathbf{C}^{\text{ordn}} \to \mathbf{D}^{\text{ordn}}$ be an isomorphism in $1\text{-Cat}^{\text{ordn}}$).

1.1.8. For a diagram of categories

$$\mathbf{C}' \to \mathbf{C} \leftarrow \mathbf{C}'',$$

we can form their *fiber product*

$$\mathbf{C}' \underset{\mathbf{C}}{\times} \mathbf{C}'' \in 1\text{-Cat}.$$

For $\mathbf{C}' = \mathbf{C}'' = *$, $\mathbf{C} = \mathcal{S} \in \text{Spc}$, and the maps $* \to \mathcal{S} \leftarrow *$ corresponding to a particular point $s \in \mathcal{S}$, we will denote by $\Omega(\mathcal{S})$ the loop space of $\mathcal{S}$ with base point s,

$$\Omega(\mathcal{S}) = * \underset{\mathcal{S}}{\times} *.$$

For $(\mathcal{S}, s)$ as above, the homotopy groups $\pi_i(\mathcal{S}, s)$ are defined inductively by

$$\pi_i(\mathcal{S}, s) = \pi_{i-1}(\Omega(\mathcal{S})).$$

1.1.9. The $(\infty, 1)$-category 1-Cat carries a canonical involutive auto-equivalence

$$\mathbf{C} \mapsto \mathbf{C}^{\text{op}}. \tag{1.1}$$

1.1.10. For $n = 0, 1, ...$ we let $[n]$ denote the ordinary category $0 \to 1 \to ... \to n$. In particular, we have $[0] = *$.

We let $\mathbf{\Delta}$ denote the full subcategory of $1\text{-Cat}^{\text{ordn}}$, spanned by the objects $[n]$.

The category $\mathbf{\Delta}$ carries a canonical involutive auto-equivalence, denoted rev: it acts as reversal on each $[n]$, i.e.,

$$i \mapsto n - i.$$

(Note that rev acts as the identity on objects of $\mathbf{\Delta}$.)

1.2. **Some auxiliary notions.** In this subsection we introduce some terminology and notation to be used throughout the book.

1.2.1. A functor between $(\infty, 1)$-categories $F : \mathbf{D} \to \mathbf{C}$ is said to be *fully faithful* if for for every $\mathbf{d}_1, \mathbf{d}_2 \in \mathbf{D}$ the map

$$\mathrm{Maps}_{\mathbf{D}}(\mathbf{d}_1, \mathbf{d}_2) \to \mathrm{Maps}_{\mathbf{C}}(F(\mathbf{d}_1), F(\mathbf{d}_2))$$

is a *isomorphism* in Spc.

A map of spaces $F : \mathcal{S}_0 \to \mathcal{S}_1$ is said to be a *monomorphism* if it is fully faithful as a functor, when $\mathcal{S}_0$ and $\mathcal{S}_1$ are regarded as $(\infty, 1)$-categories.

Concretely, F is a monomorphism if $\pi_0(F)$ is injective, and for every point $s_0 \in S_0$, the induced map $\pi_i(S_0, s_0) \to \pi_i(S_1, F(s_0))$ is an isomorphism for all $i > 0$.

1.2.2. Let $\mathbf{C}$ be an $(\infty, 1)$-category. Then to every full subcategory $\overline{\mathbf{C}}'$ of $\mathbf{C}^{\mathrm{ordn}}$ one can attach an ∞-category $\mathbf{C}'$. It has the same objects as $\overline{\mathbf{C}}'$ and for $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}'$, we have

$$\mathrm{Maps}_{\mathbf{C}'}(\mathbf{c}_1, \mathbf{c}_2) = \mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2).$$

We shall refer to $(\infty, 1)$-categories arising in the way as *full subcategories of* $\mathbf{C}$.

A fully faithful functor is an equivalence onto a full subcategory.

1.2.3. A *full subspace* of a space $\mathcal{S}$ is the same as a full subcategory of $\mathcal{S}$, considered as an $(\infty, 1)$-category. Those are in bijection with subsets of $\pi_0(\mathcal{S})$.

A connected component of $\mathcal{S}$ is a full subspace that projects to a single point in $\pi_0(\mathcal{S})$.

1.2.4. A functor between $(\infty, 1)$-categories $F : \mathbf{D} \to \mathbf{C}$ is said to be *1-fully faithful* if for for every $\mathbf{d}_1, \mathbf{d}_2 \in \mathbf{D}$ the map

$$\mathrm{Maps}_{\mathbf{D}}(\mathbf{d}_1, \mathbf{d}_2) \to \mathrm{Maps}_{\mathbf{C}}(F(\mathbf{d}_1), F(\mathbf{d}_2))$$

is a *monomorphism* in Spc.

If $\mathbf{D}$ and $\mathbf{C}$ are ordinary categories, a functor between them is 1-fully faithful if and only if it induces an *injection* on Hom sets.

1.2.5. A functor between $(\infty, 1)$-categories $F : \mathbf{D} \to \mathbf{C}$ is said to be *1-replete* if it is 1-fully faithful, and for every $\mathbf{d}_1, \mathbf{d}_2 \in \mathbf{D}$, the connected components of $\mathrm{Maps}_{\mathbf{C}}(F(\mathbf{d}_1), F(\mathbf{d}_2))$ *that correspond to isomorphisms* are in the image of $\mathrm{Maps}_{\mathbf{D}}(\mathbf{d}_1, \mathbf{d}_2)$.

It is not difficult to show that a functor is 1-replete if and only if it is 1-fully faithful and $\mathbf{D}^{\mathrm{Spc}} \to \mathbf{C}^{\mathrm{Spc}}$ is a monomorphism.

1.2.6. Let $\overline{\mathbf{C}}$ be an ordinary category. By a *1-full subcategory* we shall mean the category obtained by choosing a sub-class $\overline{\mathbf{C}}'$ of objects in $\overline{\mathbf{C}}$, and for every $\mathbf{c}_1, \mathbf{c}_2 \in \overline{\mathbf{C}}'$ a subset $\mathrm{Hom}_{\overline{\mathbf{C}}'}(\mathbf{c}_1, \mathbf{c}_2) \subset \mathrm{Hom}_{\overline{\mathbf{C}}}(\mathbf{c}_1, \mathbf{c}_2)$, such that $\mathrm{Hom}_{\overline{\mathbf{C}}'}(\mathbf{c}_1, \mathbf{c}_2)$ contains all isomorphisms and is closed under compositions.

Let $\mathbf{C}$ be an $(\infty, 1)$-category. Then to every 1-full subcategory $\overline{\mathbf{C}}'$ of $\mathbf{C}^{\mathrm{ordn}}$ one can attach an $(\infty, 1)$-category $\mathbf{C}'$. It has the same objects as $\overline{\mathbf{C}}'$. For $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}'$, we have

$$\mathrm{Maps}_{\mathbf{C}'}(\mathbf{c}_1, \mathbf{c}_2) = \mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \underset{\mathrm{Hom}_{\mathbf{C}^{\mathrm{ordn}}}(\mathbf{c}_1, \mathbf{c}_2)}{\times} \mathrm{Hom}_{\overline{\mathbf{C}}'}(\mathbf{c}_1, \mathbf{c}_2).$$

We shall refer to $(\infty, 1)$-categories arising in the way as *1-full subcategories of* $\mathbf{C}$.

1.2.7. In the above situation, for any $\mathbf{D} \in 1\text{-Cat}$, the resulting functor

$$\mathrm{Funct}(\mathbf{D}, \mathbf{C}') \to \mathrm{Funct}(\mathbf{D}, \mathbf{C})$$

is 1-replete. I.e., if a functor $\mathbf{D} \to \mathbf{C}$ can be factored through $\mathbf{C}'$, it can be done in an essentially unique way.

Vice versa, if $F : \mathbf{D} \to \mathbf{C}$ is a functor and the corresponding functor $\mathbf{D}^{\mathrm{ordn}} \to \mathbf{C}^{\mathrm{ordn}}$ factors (automatically uniquely) through a functor $\mathbf{D}^{\mathrm{ordn}} \to \overline{\mathbf{C}}'$, then F gives rise to a well-defined functor $\mathbf{D} \to \mathbf{C}'$.

In the above situation, the functor $\mathbf{D} \to \mathbf{C}'$ is an equivalence if and only if F is 1-replete and $\mathbf{D}^{\mathrm{ordn}} \to \overline{\mathbf{C}}'$ is an equivalence.

In particular, a 1-replete functor is an equivalence onto a uniquely defined 1-full subcategory.

1.2.8. A functor $F : \mathbf{D} \to \mathbf{C}$ is said to be conservative if for a morphism $\alpha \in \mathrm{Maps}_{\mathbf{D}}(\mathbf{d}_0, \mathbf{d}_1)$ the fact that $F(\alpha)$ is an isomorphism implies that α itself is an isomorphism.

1.3. **Cartesian and coCartesian fibrations.** Now that we have the basic words of the vocabulary, we want to take the theory of $(\infty, 1)$-categories off the ground. Here are two basic things that one would want to do:

(1) For an $(\infty, 1)$-category $\mathbf{C}$, define the Yoneda functor $\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \to \mathrm{Spc}$.

(2) For a functor $F : \mathbf{D} \to \mathbf{C}$ we would like to talk about its left or right adjoint.

It turns out that this is much easier said than done: the usual way of going about this in ordinary category theory uses the construction of functors by specifying what they do on objects and morphisms, something that is not allowed in higher category theory.

To overcome this, we will use the device of straightening/unstraightening, described in the next subsection. In order to explain it, we will first need to introduce the key notion of Cartesian/coCartesian fibration.

The reference for the material in this subsection is [Lu1, Sect. 2.4].

1.3.1. *Cartesian arrows.* Let $F : \mathbf{D} \to \mathbf{C}$ be a functor between $(\infty, 1)$-categories. We shall say that a morphism $\mathbf{d}_0 \overset{\alpha}{\to} \mathbf{d}_1$ in $\mathbf{D}$ is *Cartesian* over $\mathbf{C}$ if for every $\mathbf{d}' \in \mathbf{D}$, the map

$$\mathrm{Maps}_{\mathbf{D}}(\mathbf{d}', \mathbf{d}_0) \to \mathrm{Maps}_{\mathbf{D}}(\mathbf{d}', \mathbf{d}_1) \underset{\mathrm{Maps}_{\mathbf{C}}(F(\mathbf{d}'), F(\mathbf{d}_1))}{\times} \mathrm{Maps}_{\mathbf{C}}(F(\mathbf{d}'), F(\mathbf{d}_0))$$

is an isomorphism in Spc.

1.3.2. *Cartesian fibrations.* A functor $F : \mathbf{D} \to \mathbf{C}$ is said to be a *Cartesian fibration* if for every morphism $\mathbf{c}_0 \to \mathbf{c}_1$ in $\mathbf{C}$ and an object $\mathbf{d}_1 \in \mathbf{D}$ equipped with an isomorphism $F(\mathbf{d}_1) \simeq \mathbf{c}_1$, there *exists a Cartesian* morphism $\mathbf{d}_0 \to \mathbf{d}_1$ that fits into a commutative diagram

$$\begin{array}{ccc} F(\mathbf{d}_0) & \longrightarrow & F(\mathbf{d}_1) \\ \sim\downarrow & & \downarrow\sim \\ \mathbf{c}_0 & \longrightarrow & \mathbf{c}_1. \end{array}$$

1.3.3. *Cartesian fibrations in spaces.* We shall say that a functor $F : \mathbf{D} \to \mathbf{C}$ is a *Cartesian fibration in spaces* if it is a Cartesian fibration and for every $\mathbf{c} \in \mathbf{C}$, the $(\infty, 1)$-category

$$\mathbf{D}_{\mathbf{c}} := \mathbf{D} \underset{\mathbf{C}}{\times} \{\mathbf{c}\}$$

is a space.

An alternative terminology for 'Cartesian fibration in spaces' is *right fibration*, see [Lu1, Sect. 2.1].

1.3.4. *CoCartesian counterparts.* Inverting the arrows, one obtains the parallel notions of coCartesian morphism, coCartesian fibrations and coCartesian fibrations in spaces (a.k.a. *left fibration*).

1.3.5. *Over- and under-categories.* Given a functor $F : \mathbf{I} \to \mathbf{C}$ consider the corresponding over-category and under-category

$$\mathbf{C}_{/F} := \mathbf{C} \underset{\mathrm{Funct}(\mathbf{I},\mathbf{C})}{\times} \mathrm{Funct}([1] \times \mathbf{I}, \mathbf{C}) \underset{\mathrm{Funct}(\mathbf{I},\mathbf{C})}{\times} \{F\}$$

and

$$\mathbf{C}_{F/} := \{F\} \underset{\mathrm{Funct}(\mathbf{I},\mathbf{C})}{\times} \mathrm{Funct}([1] \times \mathbf{I}, \mathbf{C}) \underset{\mathrm{Funct}(\mathbf{I},\mathbf{C})}{\times} \mathbf{C},$$

where the functors

$$\mathrm{Funct}([1] \times \mathbf{I}, \mathbf{C}) \to \mathrm{Funct}(\mathbf{I}, \mathbf{C}),$$

are given by evaluation at the objects 1 and 0 in [1], respectively, and the functor

$$\mathbf{C} \to \mathrm{Funct}(\mathbf{I}, \mathbf{C})$$

corresponds to

$$\mathbf{C} \simeq \mathrm{Funct}(*, \mathbf{C}) \to \mathrm{Funct}(\mathbf{I}, \mathbf{C}).$$

For future use we mention that when $\mathbf{I} = *$ and F is given by an object $\mathbf{c} \in \mathbf{C}$, we will simply write $\mathbf{C}_{/\mathbf{c}}$ for $\mathbf{C}_{/F}$ and $\mathbf{C}_{\mathbf{c}/}$ for $\mathbf{C}_{/F}$, respectively.

The forgetful functors

$$\mathbf{C}_{/F} \to \mathbf{C} \text{ and } \mathbf{C}_{F/} \to \mathbf{C}$$

are a Cartesian and a coCartesian fibrations in spaces, respectively.

1.3.6. Note that we have the following canonical isomorphism of spaces: for $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$

$$\mathrm{Funct}([1], \mathbf{C}) \underset{\mathbf{C} \times \mathbf{C}}{\times} \{\mathbf{c}_0, \mathbf{c}_1\} \simeq \mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1), \tag{1.2}$$

where the left-hand side, although defined to be an $(\infty, 1)$-category, is actually a space.

In particular, for $\mathbf{c}, \mathbf{c}' \in \mathbf{C}$, we have

$$(\mathbf{C}_{/\mathbf{c}})_{\mathbf{c}'} \simeq \mathrm{Maps}_{\mathbf{C}}(\mathbf{c}', \mathbf{c}) \text{ and } (\mathbf{C}_{\mathbf{c}/})_{\mathbf{c}'} \simeq \mathrm{Maps}_{\mathbf{C}}(\mathbf{c}, \mathbf{c}').$$

(we remind that the superscript $\mathbf{c}'$ means taking the fiber over $\mathbf{c}'$).

Taking $\mathbf{C} = 1\text{-Cat}$, from (1.2) we obtain

$$\mathrm{Funct}([1], 1\text{-Cat}) \underset{1\text{-Cat} \times 1\text{-Cat}}{\times} \{\mathbf{C}_0, \mathbf{C}_1\} \simeq (\mathrm{Funct}(\mathbf{C}_0, \mathbf{C}_1))^{\mathrm{Spc}}. \tag{1.3}$$

1.3.7. For future reference we introduce the following notation. For a functor $F : [1] \to \mathbf{C}$ that sends $0 \mapsto \mathbf{c}_0$ and $1 \mapsto \mathbf{c}_1$ we we will denote by

$$\mathbf{C}_{\mathbf{c}_0//\mathbf{c}_1}$$

the fiber product

$$\operatorname{Funct}([2], \mathbf{C}) \underset{\operatorname{Funct}([1],\mathbf{C})}{\times} *,$$

where $* \to \operatorname{Funct}([1], \mathbf{C})$ corresponds to the initial functor F, and $\operatorname{Funct}([2], \mathbf{C}) \to \operatorname{Funct}([1], \mathbf{C})$ is given by precomposition with

$$[1] \to [2], \quad 0 \mapsto 0, \ 1 \mapsto 2.$$

This is the $(\infty, 1)$-category, whose objects are diagrams

$$\mathbf{c}_0 \to \mathbf{c} \to \mathbf{c}_1,$$

where the composition is the map $\mathbf{c}_0 \to \mathbf{c}_1$, specified by F.

1.4. **Straightening/unstraightening.** Straightening, also known as the Grothendieck construction, is the higher-categorical counterpart to the fact that the datum of a Cartesian (resp., coCartesian) fibration of ordinary categories $\mathbf{D} \to \mathbf{C}$ is equivalent to the datum of a functor from $\mathbf{C}^{\mathrm{op}}$ (resp., $\mathbf{C}$) to the category of categories.

It is hard to overestimate the importance of this assertion in higher category theory: it paves a way to constructing functors $\mathbf{C} \to 1\text{-Cat}$.

The reason being that it is usually easier to exhibit a functor $\mathbf{D} \to \mathbf{C}$ and then *check its property* of being a Cartesian/coCartesian fibration, than to construct a functor $\mathbf{C} \to 1\text{-Cat}$.

1.4.1. Fix an $(\infty, 1)$-category $\mathbf{C}$. Consider the category

$$1\text{-Cat}_{/\mathbf{C}}.$$

Note that its objects are pairs $(\mathbf{D}; \mathbf{D} \overset{F}{\to} \mathbf{C})$.

Let $\text{coCart}_{/\mathbf{C}}$ (resp., $0\text{-coCart}_{/\mathbf{C}}$) be the full subcategory of $1\text{-Cat}_{/\mathbf{C}}$ whose objects are those $(\mathbf{D}, F)$, for which F is a coCartesian fibration (resp., coCartesian fibration in spaces).

Let $(\text{coCart}_{/\mathbf{C}})_{\text{strict}}$ be the 1-full subcategory of $\text{coCart}_{/\mathbf{C}}$, where we allow as 1-morphisms those functors $\mathbf{D}_1 \to \mathbf{D}_2$ over $\mathbf{C}$ that send coCartesian arrows to coCartesian arrows. We note that the inclusion

$$(\text{coCart}_{/\mathbf{C}})_{\text{strict}} \cap 0\text{-coCart}_{/\mathbf{C}} \hookrightarrow 0\text{-coCart}_{/\mathbf{C}}$$

is an equivalence.

1.4.2. *Straightening/unstraightening for coCartesian fibrations.* The following is the basic feature of coCartesian fibrations (see [Lu1, Sect. 3.2]):

There is a canonical equivalence between $(\text{coCart}_{/\mathbf{C}})_{\text{strict}}$ *and* $\operatorname{Funct}(\mathbf{C}, 1\text{-Cat})$.

Under the above equivalence, the full subcategory

$$0\text{-coCart}_{/\mathbf{C}} \subset (\text{coCart}_{/\mathbf{C}})_{\text{strict}}$$

corresponds to the full subcategory

$$\operatorname{Funct}(\mathbf{C}, \text{Spc}) \subset \operatorname{Funct}(\mathbf{C}, 1\text{-Cat}).$$

1.4.3. Explicitly, for a coCartesian fibration $\mathbf{D} \to \mathbf{C}$, the value of the corresponding functor $\mathbf{C} \to 1\text{-Cat}$ on $\mathbf{c} \in \mathbf{C}$ equals the fiber $\mathbf{D_c}$ of $\mathbf{D}$ over $\mathbf{c}$.

Vice versa, given a functor

$$\Phi : \mathbf{C} \to 1\text{-Cat}, \quad \mathbf{c} \mapsto \Phi(\mathbf{c}), \quad (\mathbf{c}_0 \overset{f}{\to} \mathbf{c}_1) \mapsto \Phi(\mathbf{c}_0) \overset{\Phi_f}{\longrightarrow} \Phi(\mathbf{c}_1),$$

the objects of the corresponding coCartesian fibration $\mathbf{D} \to \mathbf{C}$ are pairs $(\mathbf{c} \in \mathbf{C}, \mathbf{d} \in \Phi(\mathbf{c}))$, and morphisms

$$\text{Maps}_{\mathbf{D}}((\mathbf{c}_0, \mathbf{d}_0 \in \Phi(\mathbf{c}_0)), (\mathbf{c}_1, \mathbf{d}_1 \in \Phi(\mathbf{c}_1)))$$

are pairs consisting of $f \in \text{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)$ and $g \in \text{Maps}_{\Phi(\mathbf{c}_1)}(\Phi_f(\mathbf{d}_0), \mathbf{d}_1)$.

1.4.4. One defines the $(\infty, 1)$-categories

$$0\text{-Cart}_{/\mathbf{C}} \subset (\text{Cart}_{/\mathbf{C}})_{\text{strict}} \subset \text{Cart}_{/\mathbf{C}} \subset 1\text{-Cat}_{/\mathbf{C}}$$

in a similar way.

Note that the involution (1.1) defines an equivalence $1\text{-Cat}_{/\mathbf{C}} \to 1\text{-Cat}_{/\mathbf{C}^{\text{op}}}$ that identifies

$$0\text{-coCart}_{/\mathbf{C}} \simeq 0\text{-Cart}_{/\mathbf{C}^{\text{op}}}, \quad (\text{coCart}_{/\mathbf{C}})_{\text{strict}} \simeq (\text{Cart}_{/\mathbf{C}^{\text{op}}})_{\text{strict}} \text{ and } \text{coCart}_{/\mathbf{C}} \simeq \text{Cart}_{/\mathbf{C}^{\text{op}}}.$$

1.4.5. *Straightening/unstraightening for Cartesian fibrations.* From Sect. 1.4.2, and using the involution (1.1) on 1-Cat, one obtains:

There is a canonical equivalence between $(\text{Cart}_{/\mathbf{C}})_{\text{strict}}$ *and* $\text{Funct}(\mathbf{C}^{\text{op}}, 1\text{-Cat})$.

Under the above equivalence, the full subcategory

$$0\text{-Cart}_{/\mathbf{C}} \subset (\text{Cart}_{/\mathbf{C}})_{\text{strict}}$$

corresponds to the full subcategory

$$\text{Funct}(\mathbf{C}^{\text{op}}, \text{Spc}) \subset \text{Funct}(\mathbf{C}^{\text{op}}, 1\text{-Cat}).$$

Explicitly, for a Cartesian fibration $\mathbf{D} \to \mathbf{C}$, the value of the corresponding functor

$$\mathbf{C}^{\text{op}} \to 1\text{-Cat}$$

on $\mathbf{c} \in \mathbf{C}$ still equals the fiber $\mathbf{D_c}$ of $\mathbf{D}$ over $\mathbf{c}$.

1.5. **Yoneda.** In this subsection we will illustrate how one uses straightening/unstraightening by constructing the various incarnations of the Yoneda functor.

1.5.1. For an $(\infty, 1)$-category $\mathbf{C}$, consider the $(\infty, 1)$-category $\text{Funct}([1], \mathbf{C})$, equipped with the functor

$$\text{Funct}([1], \mathbf{C}) \to \text{Funct}(*, \mathbf{C}) \times \text{Funct}(*, \mathbf{C}) \simeq \mathbf{C} \times \mathbf{C}, \tag{1.4}$$

given by evaluation on $0, 1 \in [1]$.

We can view the above functor as a morphism in the category $(\text{Cart}_{/\mathbf{C}})_{\text{strict}}$ with respect to the projection on the first factor.

1.5.2. Applying straightening, the above morphism gives rise to a morphism in the $(\infty, 1)$-category $\text{Funct}(\mathbf{C}^{\text{op}}, 1\text{-Cat})$ from the functor

$$\mathbf{c} \mapsto \mathbf{C}_{\mathbf{c}/}$$

to the functor with constant value $\mathbf{C} \in 1\text{-Cat}$.

1.5.3. For any triple of $(\infty, 1)$-categories we have a canonical isomorphism

$$\mathrm{Funct}(\mathbf{E}, \mathrm{Funct}(\mathbf{E}', \mathbf{D})) \simeq \mathrm{Funct}(\mathbf{E} \times \mathbf{E}', \mathbf{D}) \simeq \mathrm{Funct}(\mathbf{E}', \mathrm{Funct}(\mathbf{E}, \mathbf{D})).$$

In particular, taking $\mathbf{E}' = [1]$ and a fixed $F : \mathbf{E} \to \mathbf{D}$ and $\mathbf{d} \in \mathbf{D}$, using (1.2), we obtain that the datum of a morphism in $\mathrm{Funct}(\mathbf{E}, \mathbf{D})$ from F to the constant functor with value $\mathbf{d}$ is equivalent to the datum of a map

$$\mathbf{E} \to \mathbf{D}_{/\mathbf{d}},$$

whose composition with the projection $\mathbf{D}_{/\mathbf{d}} \to \mathbf{D}$, is identified with F.

1.5.4. Thus (taking $\mathbf{E} = \mathbf{C}^{\mathrm{op}}$ and $\mathbf{D} = 1\text{-Cat}$), we can view the datum of the morphism in Sect. 1.5.2 as a functor from $\mathbf{C}^{\mathrm{op}}$ to $1\text{-Cat}_{/\mathbf{C}}$.

It is easy to check that the latter functor factors through

$$0\text{-coCart}_{/\mathbf{C}} \subset 1\text{-Cat}_{/\mathbf{C}}\,.$$

1.5.5. Applying straightening again, we thus obtain a functor

$$\mathbf{C}^{\mathrm{op}} \to \mathrm{Funct}(\mathbf{C}, \mathrm{Spc}),$$

hence a functor

$$\mathrm{Yon}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathrm{Spc}\,.$$

1.6. **Enhanced version of straightening/unstraightening.** In this subsection we will discuss a version of the straightening/unstraightening equivalence that takes into account functoriality in the base $(\infty, 1)$-category $\mathbf{C}$.

1.6.1. Consider the $(\infty, 1)$-category $\mathrm{Funct}([1], 1\text{-Cat})$. Note that its objects are triples

$$\mathbf{D} \stackrel{F}{\to} \mathbf{C}.$$

Let

$$\mathrm{Funct}^{\mathrm{coCart}}([1], 1\text{-Cat}) \subset \mathrm{Funct}([1], 1\text{-Cat})$$

be the full subcategory whose objects are those $\mathbf{D} \stackrel{F}{\to} \mathbf{C}$ that are coCartesian fibrations.

Let

$$\mathrm{Funct}^{\mathrm{coCart}}([1], 1\text{-Cat})_{\mathrm{strict}} \subset \mathrm{Funct}^{\mathrm{coCart}}([1], 1\text{-Cat})$$

be the 1-full subcategory, where we only allow as morphisms those commutative diagrams

$$\begin{array}{ccc} \mathbf{D}_1 & \xrightarrow{G_{\mathbf{D}}} & \mathbf{D}_2 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \mathbf{C}_1 & \xrightarrow{G_{\mathbf{C}}} & \mathbf{C}_2 \end{array}$$

for which the functor $G_{\mathbf{D}}$ sends morphisms in $\mathbf{D}_1$ coCartesian over $\mathbf{C}_1$ to morphisms in $\mathbf{D}_2$ coCartesian over $\mathbf{C}_2$.

Evaluation on $1 \in [1]$ defines a functor

$$\mathrm{Funct}^{\mathrm{coCart}}([1], 1\text{-Cat})_{\mathrm{strict}} \to 1\text{-Cat}\,. \tag{1.5}$$

The functor (1.5) is a Cartesian fibration.

1.6.2. An enhanced version of the straightening/unstraightening equivalence says:

The functor $1\text{-Cat}^{\text{op}} \to 1\text{-Cat}$ *corresponding to the Cartesian fibration* (1.5) *is canonically isomorphic to the functor*

$$\mathbf{C} \mapsto \text{Funct}(\mathbf{C}, 1\text{-Cat}).$$

1.6.3. Again, by applying the involution $\mathbf{D} \mapsto \mathbf{D}^{\text{op}}$, we obtain a counterpart of Sect. 1.6.2 for Cartesian fibrations:

The functor $1\text{-Cat}^{\text{op}} \to 1\text{-Cat}$ *corresponding to the Cartesian fibration*

$$\text{Funct}^{\text{Cart}}([1], 1\text{-Cat})_{\text{strict}} \to 1\text{-Cat}$$

is canonically isomorphic to the functor

$$\mathbf{C} \mapsto \text{Funct}(\mathbf{C}^{\text{op}}, 1\text{-Cat}).$$

1.7. **Adjoint functors.** In this subsection we will finally introduce the notion of *adjoint functor*, following [Lu1, Sect. 5.2.1].

However, this will not be the end of the story. We will *not* describe the datum of an adjunction as pair of a unit map and a co-unit map that satisfy some natural conditions (because in the context of higher categories, there is an infinite tail of these conditions).

We will return to the latter approach to adjunction in Sect. 4.4, and more fundamentally in [Chapter A.3]: it turns out that it is most naturally described in the context of $(\infty, 2)$-categories.

1.7.1. Let $F : \mathbf{C}_0 \to \mathbf{C}_1$ be a functor. Using (1.3), we can view F as a functor $[1] \to 1\text{-Cat}$. We now apply unstraightening and regard it as a coCartesian fibration

$$\widetilde{\mathbf{C}} \to [1]. \tag{1.6}$$

We shall say that F *admits a right adjoint* if the above functor (1.6) is a *bi-Cartesian fibration*, i.e., if it happens to be a Cartesian fibration, in addition to being a coCartesian one.

In this case, viewing (1.6) as a *Cartesian fibration* and applying straightening, we transform (1.6) into to a functor

$$[1]^{\text{op}} \to 1\text{-Cat}\,. \tag{1.7}$$

The resulting functor $\mathbf{C}_1 \to \mathbf{C}_0$ (obtained by applying the equivalence (1.3) to the functor (1.7)) is called the *right adjoint* of F, and denoted F^R. By construction, F^R is uniquely determined by F.

1.7.2. Inverting the arrows, we obtain the notion of a functor $G : \mathbf{D}_0 \to \mathbf{D}_1$, *admitting a left adjoint*. We denote the left adjoint of G by G^L.

By construction, the data of realizing G as a right adjoint of F is equivalent to the data of realizing F as a left adjoint of G: both are encoded by a bi-Cartesian fibration

$$\mathbf{E} \to [1].$$

By construction, for $\mathbf{c}_0 \in \mathbf{C}_0$ and $\mathbf{c}_1 \in \mathbf{C}_1$ we have a canonical isomorphism in Spc

$$\text{Maps}_{\mathbf{C}_0}(\mathbf{c}_0, F^R(\mathbf{c}_1)) \simeq \text{Maps}_{\mathbf{C}_1}(F(\mathbf{c}_0), \mathbf{c}_1).$$

1.7.3. Let us be in the situation Sect. 1.7.1, but *without* assuming that (1.6) is bi-Cartesian. Let $\mathbf{C}'_1 \subset \mathbf{C}_1$ be the full subcategory consisting of those objects $\mathbf{c}_1 \in \mathbf{C}_1$, for which there exists a *Cartesian* morphism

$$\mathbf{c}_0 \to \mathbf{c}_1$$

in $\widetilde{\mathbf{C}}$, covering the morphism $0 \to 1$ in $[1]$.

Let $\widetilde{\mathbf{C}}' \subset \widetilde{\mathbf{C}}$ be the corresponding full subcategory of $\widetilde{\mathbf{C}}$, so that

$$\widetilde{\mathbf{C}}'_0 = \mathbf{C}_0 \text{ and } \widetilde{\mathbf{C}}'_1 = \mathbf{C}'_1.$$

The functor

$$\widetilde{\mathbf{C}}' \to [1]$$

is now a Cartesian fibration. Applying straightening, we obtain a functor $F'^R : \mathbf{C}'_1 \to \mathbf{C}_0$.

1.7.4. We will refer to F'^R as the *partially defined right adjoint of* F. By construction, we have a canonical isomorphism

$$\mathrm{Maps}_{\mathbf{C}_1}(F(\mathbf{c}_0), \mathbf{c}_1) \simeq \mathrm{Maps}_{\mathbf{C}_0}(\mathbf{c}_0, F'^R(\mathbf{c}_1)), \quad \mathbf{c}_0 \in \mathbf{C}_0, \quad \mathbf{c}_1 \in \mathbf{C}'_1.$$

The original functor F admits a right adjoint if and only if $\mathbf{C}'_1 = \mathbf{C}_1$.

1.7.5. Inverting the arrows, in a similar way we define the notion of *partially defined left adjoint of* F.

2. Basic operations with $(\infty, 1)$-categories

In this section we will assume that we 'know' what $(\infty, 1)$-categories are, as well as the basic rules of the syntax of operating with them. I.e., we know the 'theory', but what we need now is 'practice'.

Here are some of the primary practical questions that one needs to address:

(Q1) How do we produce 'new' $(\infty, 1)$-categories?

(Q2) How do we construct functors between two given $(\infty, 1)$-categories?

Of course, there are some cheap answers: for Q1 take a full subcategory of an existing $(\infty, 1)$-category; for Q2 compose two existing functors, or pass to the adjoint of a given functor. But in this way, we will not get very far.

Here are, however, some additional powerful tools:

(A1) Start a diagram of existing ones and take its limit.

(A2) Start with a given functor, and apply the procedure of Kan extension.

These answers entail the next question: how to we calculate limits when we need to?

This circle of ideas is the subject of the present section. The material here can be viewed as a user guide to (some parts of) [Lu1, Chapter 4 and Sect. 5.5].

2.1. **Left and right Kan extensions.** Let us say that at this point we have convinced ourselves that we should work with $(\infty,1)$-categories. But here comes a question: how do we ever construct functors between two given $(\infty,1)$-categories?

The difficulty is that, unlike ordinary categories, we cannot simply specify what a functor does on objects and morphisms: we would need to specify an infinite tail of compatibilities for multi-fold compositions. (Rigorously, we would have to go to the model of $(\infty,1)$-categories given by quasi-categories, and specify a map of the underlying simplicial sets, which, of course, no one wants to do in a practical situation.)

Here to our rescue comes the operation *Kan extension*: given a functor $\Phi:\mathbf{D}\to\mathbf{E}$ and a functor $F:\mathbf{D}\to\mathbf{C}$, we can (sometimes) canonically construct a functor from $\mathbf{C}\to\mathbf{E}$.

A particular case of this operation leads to the notion of limit/colimit of a functor $\mathbf{D}\to\mathbf{E}$ (we can think of such a functor as a diagram of objects in $\mathbf{E}$, parameterized by $\mathbf{D}$).

By taking $\mathbf{E}$ to be 1-Cat, we arrive to the notion of *limit of* $(\infty,1)$*-categories*, which in itself is a *key tool of constructing* $(\infty,1)$*-categories*.

The reference for this material is [Lu1, Sect. 4.3].

2.1.1. Let $F:\mathbf{D}\to\mathbf{C}$ be a functor between $(\infty,1)$-categories. For a (target) $(\infty,1)$-category $\mathbf{E}$, consider the functor

$$\mathrm{Funct}(\mathbf{C},\mathbf{E})\mapsto\mathrm{Funct}(\mathbf{D},\mathbf{E}),$$

given by restriction along F (i.e., composition with F).

Its partially defined left (resp., right) adjoint is called the functor of *left* (resp., *right*) *Kan extension along* F, and denoted LKE_F (resp., RKE_F).

2.1.2. If $\mathbf{C}=*$, the corresponding left and right Kan extension functors are the functors of *colimit* (resp., *limit*):

$$\underset{\mathbf{D}}{\mathrm{colim}}:\mathrm{Funct}(\mathbf{D},\mathbf{E})\to\mathbf{E}\text{ and }\underset{\mathbf{D}}{\lim}:\mathrm{Funct}(\mathbf{D},\mathbf{E})\to\mathbf{E}.$$

We record the following piece of terminology: colimits over the category $\mathbf{\Delta}^{\mathrm{op}}$ are called *geometric realizations*, and limits over the category $\mathbf{\Delta}$ are called *totalizations*.

2.1.3. In general, for $\Phi:\mathbf{D}\to\mathbf{E}$, suppose that for every given $\mathbf{c}\in\mathbf{C}$, the colimit

$$\underset{\mathbf{D}\underset{\mathbf{C}}{\times}\mathbf{C}_{/\mathbf{c}}}{\mathrm{colim}}\,\Phi \tag{2.1}$$

exists. Then $\mathrm{LKE}_F(\Phi)$ exists and (2.1) calculates its value on $\mathbf{c}$.

Similarly, suppose that for every given $\mathbf{c}$, the limit

$$\underset{\mathbf{D}\underset{\mathbf{C}}{\times}\mathbf{C}_{\mathbf{c}/}}{\lim}\,\Phi, \tag{2.2}$$

exists. Then $\mathrm{RKE}_F(\Phi)$ exists and (2.2) calculates its value on $\mathbf{c}$.

2.1.4. Note that by transitivity,

$$\underset{\mathbf{D}}{\mathrm{colim}}\,\Phi\simeq\underset{\mathbf{C}}{\mathrm{colim}}\,\mathrm{LKE}_F(\Phi)$$

and

$$\underset{\mathbf{D}}{\lim}\,\Phi\simeq\underset{\mathbf{C}}{\lim}\,\mathrm{RKE}_F(\Phi).$$

2.1.5. To an $(\infty,1)$-category $\mathbf{C}$ one attaches the space

$$|\mathbf{C}| := \underset{\mathbf{C}}{\operatorname{colim}} \, *, \tag{2.3}$$

where $*$ is the functor $\mathbf{C} \to \mathrm{Spc}$ with constant value $*$.

The assignment

$$\mathbf{C} \mapsto |\mathbf{C}|$$

is the functor *left adjoint* to the inclusion $\mathrm{Spc} \hookrightarrow 1\text{-Cat}$. This procedure is usually referred to as *inverting all morphisms*. In particular, for $\mathcal{S} \in \mathrm{Spc} \subset 1\text{-Cat}$, we have a canonical isomorphism in Spc

$$|\mathcal{S}| \simeq \mathcal{S}.$$

An $(\infty,1)$-category $\mathbf{C}$ is said to be *contractible* if $|\mathbf{C}|$ is isomorphic to $*$.

2.1.6. Let $\mathbf{C}$ be an $(\infty,1)$-category and let $\Phi : \mathbf{C} \to \mathrm{Spc}$ be a functor. Then it follows from Sect. 1.4.2 that there is a canonical equivalence

$$\underset{\mathbf{C}}{\operatorname{colim}} \, \Phi \simeq |\widetilde{\mathbf{C}}_\Phi|,$$

where $\widetilde{\mathbf{C}}_\Phi \to \mathbf{C}$ is the coCartesian fibration corresponding to Φ.

2.1.7. Here is a typical application of the procedure of left Kan extension:

Let $\mathbf{C}$ be an arbitrary $(\infty,1)$-category that contains colimits. We have:

Lemma 2.1.8. *Restriction and left Kan extension along* $* \hookrightarrow \mathrm{Spc}$ *define an equivalence between the subcategory of* $\mathrm{Funct}(\mathrm{Spc}, \mathbf{C})$ *consisting of colimit-preserving functors and* $\mathrm{Funct}(*, \mathbf{C}) \simeq \mathbf{C}$.

We note that the inverse functor in Lemma 2.1.8 is explicitly given as follows: it sends $\mathbf{c} \in \mathbf{C}$ to the functor $\mathrm{Spc} \to \mathbf{C}$, given by

$$(\mathcal{S} \in \mathrm{Spc}) \mapsto (\underset{\mathcal{S}}{\operatorname{colim}} \, \mathbf{c}_\mathcal{S} \in \mathbf{C}),$$

where $\mathbf{c}_\mathcal{S}$ denotes the constant functor $\mathcal{S} \to \mathbf{C}$ with value $\mathbf{c}$, where $\mathcal{S}$ is considered as an $(\infty,1)$-category.

2.2. **Cofinality.**

Many of the actual calculations that one performs in higher category theory amount to calculating limits and colimits. How does one ever do this?

A key tool here is the notion of *cofinality* that allows to replace the limit/colimit over a given index $(\infty,1)$-category, by the limit/colimit over another one, which is potentially simpler.

Iterating this procedure, one eventually arrives to a limit/colimit that can be evaluated 'by hand'. Sometimes, at the end our limit/colimit will be given just by evaluation (or a manageable fiber product/push-out). Sometimes, it will still be a limit/colimit, but in the world of ordinary categories.

The reference for the material here is [Lu1, Sect. 4.1].

2.2.1. A functor $F : \mathbf{D} \to \mathbf{C}$ is said to be cofinal if for any $\mathbf{c} \in \mathbf{C}$, the category

$$\mathbf{D} \underset{\mathbf{C}}{\times} \mathbf{C}_{\mathbf{c}/}$$

is contractible.

We have:

Lemma 2.2.2. *The following are equivalent:*

(i) $F : \mathbf{D} \to \mathbf{C}$ *is cofinal;*

(ii) *For any* $\Phi : \mathbf{C} \to \mathbf{E}$*, the natural map*

$$\underset{\mathbf{D}}{\operatorname{colim}}\, \Phi \circ F \to \underset{\mathbf{C}}{\operatorname{colim}}\, \Phi$$

is an isomorphism, whenever either side is defined;

(ii') *Same as* (ii)*, but we take* $\mathbf{E} = \operatorname{Spc}$ *(in which case, the colimits are always defined);*

(ii") *Same as* (ii')*, but we only consider the Yoneda functors* $\mathbf{c} \mapsto \operatorname{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c})$ *for* $\mathbf{c}_0 \in \mathbf{C}$*;*

(iii) *For any functor* $\Phi : \mathbf{C}^{\mathrm{op}} \to \mathbf{E}$*, the map*

$$\lim_{\mathbf{C}^{\mathrm{op}}} \Phi \to \lim_{\mathbf{D}^{\mathrm{op}}} \Phi \circ F^{\mathrm{op}}$$

is an isomorphism, whenever either side is defined;

(iii') *Same as* (iii)*, but we take* $\mathbf{E} = \operatorname{Spc}$ *(in which case, the limits are always defined);*

(iv) *For any* $\Phi : \mathbf{C} \to \mathbf{E}$ *and any functor* $\Phi' : \mathbf{C} \to \mathbf{E}$ that sends all morphisms to isomorphisms*, the map*

$$\operatorname{Maps}_{\operatorname{Funct}(\mathbf{C},\mathbf{E})}(\Phi, \Phi') \to \operatorname{Maps}_{\operatorname{Funct}(\mathbf{D},\mathbf{E})}(\Phi \circ F, \Phi' \circ F)$$

is an isomorphism.

2.2.3. For example, any functor that admits a left adjoint is cofinal. Indeed, in this case, the category $\mathbf{D} \underset{\mathbf{C}}{\times} \mathbf{C}_{\mathbf{c}/}$ admits an initial object, given by

$$\mathbf{c} \mapsto F \circ F^L(\mathbf{c}).$$

2.2.4. Let $\mathbf{D} \to \mathbf{C}$ be a coCartesian fibration. We note that in this case for any $\mathbf{c} \in \mathbf{C}$, the functor

$$\mathbf{D}_{\mathbf{c}} \to \mathbf{D} \underset{\mathbf{C}}{\times} \mathbf{C}_{/\mathbf{c}}$$

is cofinal. Hence, we obtain that for $\Phi : \mathbf{D} \to \mathbf{E}$, the value of $\operatorname{LKE}_F(\Phi)$ at $\mathbf{c} \in \mathbf{C}$ is canonically isomorphic to

$$\underset{\mathbf{D}_{\mathbf{c}}}{\operatorname{colim}}\, \Phi.$$

I.e., instead of computing the colimit over the slice category, we can do so over the fiber.

Similarly, if $\mathbf{D} \to \mathbf{C}$ is a Cartesian fibration, then for $\Phi : \mathbf{D} \to \mathbf{E}$, the value of $\operatorname{RKE}_F(\Phi)$ at $\mathbf{c} \in \mathbf{C}$ is canonically isomorphic to

$$\lim_{\mathbf{D}_{\mathbf{c}}} \Phi.$$

2.3. **Contractible functors.** The contents of this subsection can be skipped on the first pass. It is included in order to address the following question that arises naturally after introducing the notion of cofinality:

Let $F : \mathbf{D} \to \mathbf{C}$ be a functor. For $\Phi, \Phi' \in \text{Funct}(\mathbf{C}, \mathbf{E})$, consider the restriction map

$$\text{Maps}_{\text{Funct}(\mathbf{C},\mathbf{E})}(\Phi, \Phi') \to \text{Maps}_{\text{Funct}(\mathbf{D},\mathbf{E})}(\Phi \circ F, \Phi' \circ F). \tag{2.4}$$

The condition that (2.4) be an isomorphism for any Φ and Φ' that take all morphisms to isomorphisms is equivalent to the map

$$|\mathbf{D}| \to |\mathbf{C}|$$

being an isomorphism in Spc.

According to Lemma 2.2.2 the condition that (2.4) be an isomorphism for any Φ' that takes all morphisms to isomorphisms is equivalent to F being cofinal.

We will now formulate the condition that (2.4) be an isomorphism for all pairs Φ, Φ'. I.e., that the restriction functor

$$\text{Funct}(\mathbf{C}, \mathbf{E}) \to \text{Funct}(\mathbf{D}, \mathbf{E})$$

be fully faithful.

2.3.1. For $\mathbf{c}, \mathbf{c}' \in \mathbf{C}$ and a morphism $\mathbf{c} \overset{\alpha}{\to} \mathbf{c}'$, consider the $(\infty, 1)$-category $\text{Factor}_{\mathbf{D}}(\alpha)$:

$$\left((\mathbf{C}_{\mathbf{c}/} \underset{\mathbf{C}}{\times} \mathbf{D}) \underset{\mathbf{D}}{\times} (\mathbf{D} \underset{\mathbf{C}}{\times} \mathbf{C}_{/\mathbf{c}'}) \right) \underset{\text{Maps}_{\mathbf{C}}(\mathbf{c},\mathbf{c}')}{\times} \{\alpha\}.$$

I.e., this is the category, whose objects are

$$(\widetilde{\mathbf{d}} \in \mathbf{D}, \mathbf{c} \overset{\beta}{\to} F(\widetilde{\mathbf{d}}) \overset{\gamma}{\to} \mathbf{c}', \gamma \circ \beta \sim \alpha).$$

We shall say that F is *contractible* if for any $\mathbf{c} \overset{\alpha}{\to} \mathbf{c}'$, the category $\text{Factor}_{\mathbf{D}}(\alpha)$ is contractible.

2.3.2. We have:

Lemma 2.3.3. *The following conditions are equivalent:*

(i) *F is contractible;*

(i') *$F^{\text{op}} : \mathbf{D}^{\text{op}} \to \mathbf{C}^{\text{op}}$ is contractible;*

(ii) *For any $\mathbf{E}$, the restriction functor*

$$\text{Funct}(\mathbf{C}, \mathbf{E}) \to \text{Funct}(\mathbf{D}, \mathbf{E})$$

is fully faithful;

(ii') *Same as* (ii) *but $\mathbf{E} = \text{Spc}$;*

(iii) *The unit of the adjunction*

$$\Phi \to \text{RKE}_F(\Phi \circ F), \quad \Phi \in \text{Funct}(\mathbf{C}, \mathbf{E})$$

is an isomorphism for any $\mathbf{E}$ and Φ;

(iii') *Same as (iii), but $\mathbf{E} = \text{Spc}$;*

(iv) *The counit of the adjunction*

$$\text{LKE}_F(\Phi \circ F) \to \Phi, \quad \Phi \in \text{Funct}(\mathbf{C}, \mathbf{E})$$

is an isomorphism for any $\mathbf{E}$ and Φ;

(iv') *Same as (iv), but $\mathbf{E} = \text{Spc}$;*

(iv") *Same as (iv'), but Φ are taken to be the Yoneda functors $\mathbf{c} \mapsto \text{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c})$.*

2.3.4. We also note:

Lemma 2.3.5. *Let $F : \mathbf{D} \to \mathbf{C}$ be a Cartesian or coCartesian fibration. Then it is contractible if and only if it has contractible fibers.*

2.4. **The operation of 'passing to adjoints'.** Let

$$i \mapsto \mathbf{C}_i, \quad i \in \mathbf{I}$$

be an $\mathbf{I}$-diagram of $(\infty, 1)$-categories. In this subsection we will discuss the procedure of creating a new diagram, parameterized by $\mathbf{I}^{\mathrm{op}}$, that still sends i to $\mathbf{C}_i$, but replaces the transition functors by their adjoints.

This procedure generalizes the situation of Sect. 1.7.1: in the latter our index category $\mathbf{I}$ was simply [1].

2.4.1. Let

$$\mathbf{C}_{\mathbf{I}} : \mathbf{I} \to 1\text{-Cat}, \quad i \mapsto \mathbf{C}_i, \quad (i_0 \overset{\alpha}{\to} i_1) \mapsto \left(\mathbf{C}_{i_0} \overset{F_\alpha}{\longrightarrow} \mathbf{C}_{i_1}\right)$$

be a functor, where $\mathbf{I} \in 1\text{-Cat}$. Let

$$\widetilde{\mathbf{C}} \to \mathbf{I} \tag{2.5}$$

be the coCartesian fibration corresponding to $\mathbf{C}_{\mathbf{I}}$.

Assume that for every morphism $(i_0 \overset{\alpha}{\to} i_1) \in \mathbf{I}$, the resulting functor $\mathbf{C}_{i_0} \overset{F_\alpha}{\longrightarrow} \mathbf{C}_{i_1}$ admits a right adjoint. In this case the coCartesian fibration (2.5) is bi-Cartesian.

2.4.2. Viewing (2.5) as a Cartesian fibration, and applying straightening, we transform (2.5) into a functor

$$\mathbf{C}^R_{\mathbf{I}^{\mathrm{op}}} : \mathbf{I}^{\mathrm{op}} \to 1\text{-Cat}\,.$$

On this case, we shall say that $\mathbf{C}^R_{\mathbf{I}^{\mathrm{op}}}$ is obtained from $\mathbf{C}_{\mathbf{I}}$ *by passing to right adjoints.*

By construction, the value of $\mathbf{C}^R_{\mathbf{I}^{\mathrm{op}}}$ on $i \in \mathbf{I}$ is still $\mathbf{C}_i$. However, for a morphism $i_0 \overset{\alpha}{\to} i_1$ in $\mathbf{I}$, viewed as a morphism $i_1 \to i_0$ in $\mathbf{I}^{\mathrm{op}}$, the corresponding functor

$$\mathbf{C}_{i_1} \to \mathbf{C}_{i_0}$$

is $(F_\alpha)^R$.

2.4.3. Similarly, by inverting the arrows, we talk about a functor

$$\mathbf{D}^L_{\mathbf{J}^{\mathrm{op}}} : \mathbf{J}^{\mathrm{op}} \to 1\text{-Cat}$$

being obtained from a functor $\mathbf{D}_{\mathbf{J}} : \mathbf{J} \to 1\text{-Cat}$ by *passing to left adjoints.*

For $\mathbf{J} = \mathbf{I}^{\mathrm{op}}$, the datum of realizing $\mathbf{D}_{\mathbf{J}}$ as obtained from $\mathbf{C}_{\mathbf{I}}$ by passing to right adjoints is equivalent to the datum of realizing $\mathbf{C}_{\mathbf{I}}$ as obtained from $\mathbf{D}_{\mathbf{J}}$ by passing to left adjoints: both are encoded by a bi-Cartesian fibration

$$\mathbf{E} \to \mathbf{I}.$$

2.5. **Colimits in presentable $(\infty, 1)$-categories.** As was mentioned in the introduction, the primary reason for working with $(\infty, 1)$-categories is the fact that the operation of *limit of a diagram of* $(\infty, 1)$*-categories* is well-behaved (as opposed to one within the world of triangulated categories).

But here comes a problem: while limits are, by definition, adjusted to mapping *to* them, how do we ever construct a functor *out of* an $(\infty, 1)$-category, defined as a limit? However, quite an amazing thing happens: in a wide class of situations, a *limit* in 1-Cat happens to also be the *colimit* (taken in a slightly different category).

The pattern of how this happens will be described in this subsection.

2.5.1. We let $1\text{-Cat}_{\mathrm{Prs}} \subset 1\text{-Cat}$ be the 1-full subcategory whose objects are *presentable* $(\infty, 1)$-categories contain colimits[3], and where we allow as morphisms functors that preserve colimits.

We have the following basic fact:

Lemma 2.5.2 ([Lu1], Proposition 5.5.3.13)**.**
(a) *The* $(\infty, 1)$*-category* $1\text{-Cat}_{\mathrm{Prs}}$ *contains limits and colimits.*
(b) *The inclusion functor*

$$1\text{-Cat}_{\mathrm{Prs}} \to 1\text{-Cat}$$

preserves limits.

2.5.3. Here is a version of the Adjoint Functor Theorem:

Theorem 2.5.4 ([Lu1], Corollary 5.5.2.9)**.**
(a) *Any morphism in* $1\text{-Cat}_{\mathrm{Prs}}$*, viewed as a functor between* $(\infty, 1)$*-categories, admits a right adjoint.*
(b) *If* $\mathbf{C}$ *and* $\mathbf{D}$ *are objects in* $1\text{-Cat}_{\mathrm{Prs}}$*, and* $G : \mathbf{D} \to \mathbf{C}$ *is a functor that preserves* limits[4]*, then this functor admits a left adjoint, which is a morphism in* $1\text{-Cat}_{\mathrm{Prs}}$*.*

2.5.5. Let

$$\mathbf{C}_{\mathbf{I}} : \mathbf{I} \to 1\text{-Cat}_{\mathrm{Prs}}$$

be a functor.

By the Adjoint Functor Theorem and Sect. 2.4.1, there exists a canonically defined functor

$$\mathbf{C}^R_{\mathbf{I}^{\mathrm{op}}} : \mathbf{I}^{\mathrm{op}} \to 1\text{-Cat},$$

obtained from the composition

$$\mathbf{I} \xrightarrow{\mathbf{C}_{\mathbf{I}}} 1\text{-Cat}_{\mathrm{Prs}} \hookrightarrow 1\text{-Cat}$$

by passing to right adjoints.

---

[3]Presentability is a technical condition of set-theoretic nature (see [Lu1, Sect. 5.5.]), which is necessary for the Adjoint Functor Theorem to hold. However, following our conventions (see Sect. 0.4.5), we will omit the adjective 'presentable' even when it should properly be there.

[4]One also needs to impose a condition of set-theoretic nature that G be *accessible*, see [Lu1, Defn. 5.4.2.5] for what this means.

2.5.6. Let $\mathbf{C}_*$ denote the *colimit* of $\mathbf{C}_{\mathbf{I}}$ in $1\text{-Cat}_{\text{Prs}}$.

Let $\mathbf{I}'$ be the category obtained from $\mathbf{I}$ by adjoining a final object $*$. The functor $\mathbf{C}_{\mathbf{I}}$ canonically extents to a functor

$$\mathbf{C}_{\mathbf{I}'} : \mathbf{I}' \to 1\text{-Cat}_{\text{Prs}},$$

whose value on $*$ is $\mathbf{C}_*$.

By the Adjoint Functor Theorem and Sect. 2.4.1, there exists a canonically defined functor

$$\mathbf{C}^R_{\mathbf{I}'^{\text{op}}} : \mathbf{I}'^{\text{op}} \to 1\text{-Cat},$$

obtained from the composition

$$\mathbf{I}' \overset{\mathbf{C}_{\mathbf{I}'}}{\longrightarrow} 1\text{-Cat}_{\text{Prs}} \hookrightarrow 1\text{-Cat}$$

by passing to right adjoints, and whose restriction to $\mathbf{I}^{\text{op}}$ is the functor $\mathbf{C}^R_{\mathbf{I}^{\text{op}}}$.

Note that the category $\mathbf{I}'^{\text{op}}$ is obtained from $\mathbf{I}^{\text{op}}$ by adjoining an initial object.

In particular, we obtain a canonically defined functor

$$\mathbf{C}_* \to \lim_{\mathbf{I}^{\text{op}}} \mathbf{C}^R_{\mathbf{I}^{\text{op}}}. \tag{2.6}$$

We have the following fundamental fact, which follows from [Lu1, Corollary 5.5.3.4]:

Proposition 2.5.7. *The functor* (2.6) *is an equivalence.*

The equivalence of Proposition 2.5.7 will be used all the time in this book. We emphasize that it states the equivalence

$$\underset{i\in\mathbf{I}}{\text{colim}}\, \mathbf{C}_i \simeq \lim_{i\in\mathbf{I}^{\text{op}}} \mathbf{C}_i,$$

where the colimit in the left-hand side is taken in $1\text{-Cat}_{\text{Prs}}$, and the limit in the right-hand side is taken in 1-Cat.

2.5.8. In the setting of Proposition 2.5.7, for $i \in \mathbf{I}$, we will denote by ins_i the tautological functor

$$\mathbf{C}_i \to \mathbf{C}_*.$$

In terms of the identification

$$\mathbf{C}_* \simeq \lim_{\mathbf{I}^{\text{op}}} \mathbf{C}^R_{\mathbf{I}^{\text{op}}},$$

the functor ins_i is the left adjoint of the tautological evaluation functor

$$\text{ev}_i : \lim_{\mathbf{I}^{\text{op}}} \mathbf{C}^R_{\mathbf{I}^{\text{op}}} \to \mathbf{C}_i.$$

Thus, we can restate Proposition 2.5.7 by saying that each of the functors ev_i admits a left adjoint, and the resulting family of functors

$$(\text{ev}_i)^L : \mathbf{C}_i \to \lim_{\mathbf{I}^{\text{op}}} \mathbf{C}^R_{\mathbf{I}^{\text{op}}}$$

gives rise to an equivalence

$$\underset{\mathbf{I}}{\text{colim}}\, \mathbf{C}_{\mathbf{I}} \overset{\sim}{\longrightarrow} \lim_{\mathbf{I}^{\text{op}}} \mathbf{C}^R_{\mathbf{I}^{\text{op}}},$$

where the colimit in the left-hand side is taken in $1\text{-Cat}_{\text{Prs}}$.

2.6. **Limits and adjoints.** In this subsection we will discuss two general results about the interaction of limits of $(\infty,1)$-categories with adjunctions and with limits *within* a given $(\infty,1)$-category. We will use them in multiple places in the book.

2.6.1. Let

$$\mathbf{I} \to 1\text{-Cat}, \quad i \mapsto \mathbf{C}_i$$

be a diagram of $(\infty, 1)$-categories. Set

$$\mathbf{C} := \lim_{i \in \mathbf{I}} \mathbf{C}_i.$$

Let

$$A \to \mathbf{C}, \quad a \mapsto \mathbf{c}^a$$

be a functor, where A is some other index category. Consider the corresponding functors

$$A \to \mathbf{C} \xrightarrow{\text{ev}_i} \mathbf{C}_i, \quad a \mapsto \mathbf{c}^a_i.$$

Suppose that for each i, the limit

$$\lim_{a \in A} \mathbf{c}^a_i =: \mathbf{c}_i \in \mathbf{C}_i$$

exists. Assume also that for every 1-morphism $i \to j$ in $\mathbf{I}$, the corresponding map $F_{i,j}(\mathbf{c}_i) \to \mathbf{c}_j$ happens to be an isomorphism.

We claim:

Lemma 2.6.2. *Under the above circumstances, the limit*

$$\lim_{a \in A} \mathbf{c}^a =: \mathbf{c} \in \mathbf{C}$$

exists and the natural maps $\text{ev}_i(\mathbf{c}) \to \mathbf{c}_i$ *are isomorphisms.*

2.6.3. Let now $\mathbf{I}$ be an $(\infty, 1)$-category of indices, and let be given a functor

$$\mathbf{I} \to \text{Funct}([1], 1\text{-Cat}), \quad i \mapsto (\mathbf{D}_i \xrightarrow{\Phi_i} \mathbf{C}_i).$$

Assume that for every i, the corresponding functor Φ_i admits a right adjoint. Assume also that for every map $i \to j$ in I the natural transformation

$$F^{\mathbf{D}}_{i,j} \circ (\Phi_i)^R \to (\Phi_j)^R \circ F^{\mathbf{C}}_{i,j}$$

is an isomorphism, where $F^{\mathbf{D}}_{i,j}$ (resp., $F^{\mathbf{C}}_{i,j}$) denotes the transition functor $\mathbf{C}_i \to \mathbf{C}_j$ (resp., $\mathbf{D}_i \to \mathbf{D}_j$).

Set

$$\mathbf{D} := \lim_{i \in I} \mathbf{D}_i \text{ and } \mathbf{C} := \lim_{i \in I} \mathbf{C}_i.$$

We claim:

Lemma 2.6.4. *The resulting functor* $\Phi : \mathbf{D} \to \mathbf{C}$ *admits a right adjoint, and for every* i *the natural transformation*

$$\text{ev}^{\mathbf{D}}_i \circ \Phi^R \to \Phi^R_i \circ \text{ev}^{\mathbf{C}}_i$$

is an isomorphism.

3. Monoidal structures

This section is meant to be a user guide to some aspects of Higher Algebra, roughly Sects. 4.1-4.3, 4.5 and 4.7 of [Lu2].

We discuss the notion of monoidal $(\infty, 1)$-category; the notion of module over a given monoidal category, the notion of associative algebra in a given monoidal category, and the notion of module over an algebra in a given module category.

At the end of this section, we discuss monads and the Barr-Beck-Lurie theorem.

The reader who is familiar with [Lu2] can safely skip this section.

3.1. **The notion of monoidal $(\infty, 1)$-category.** In this subsection we introduce the notion of monoidal $(\infty, 1)$-category. The idea is very simple: a monoidal $(\infty, 1)$-category will be encoded by a functor from the category $\mathbf{\Delta}^{\mathrm{op}}$ to 1-Cat.

3.1.1. Recall the category $\mathbf{\Delta}$, see Sect. 1.1.10.

We define a monoidal $(\infty, 1)$-category to be a functor

$$\mathbf{A}^{\otimes} : \mathbf{\Delta}^{\mathrm{op}} \to 1\text{-Cat},$$

subject to the following conditions:

- $\mathbf{A}^{\otimes}([0]) = *$;
- For any n, the functor, given by the n-tuple of maps in $\mathbf{\Delta}$

$$[1] \to [n], \quad 0 \mapsto i, 1 \mapsto i+1, \quad i = 0, ..., n-1, \tag{3.1}$$

  defines an equivalence

$$\mathbf{A}^{\otimes}([n]) \to \mathbf{A}^{\otimes}([1]) \times ... \times \mathbf{A}^{\otimes}([1]).$$

3.1.2. If $\mathbf{A}^{\otimes}$ is a monoidal $(\infty, 1)$-category, we shall denote by $\mathbf{A}$ the *underlying* $(\infty, 1)$-category, i.e., $\mathbf{A}^{\otimes}([1])$.

Sometimes, we will abuse the notation and say that "$\mathbf{A}$ is a monoidal $(\infty, 1)$-category". Whenever we say this we will mean that $\mathbf{A}$ is obtained in the above way from a functor $\mathbf{A}^{\otimes}$.

3.1.3. The map

$$[1] \to [2], \quad 0 \mapsto 0, 1 \mapsto 2$$

defines a functor

$$\mathbf{A} \times \mathbf{A} \to \mathbf{A}.$$

This functor is the monoidal operation on $\mathbf{A}$, corresponding to $\mathbf{A}^{\otimes}$. Unless a confusion is likely to occur, we denote the above functor by

$$\mathbf{a}_1, \mathbf{a}_2 \mapsto \mathbf{a}_1 \otimes \mathbf{a}_2.$$

The map $[1] \to [0]$ defines a functor $* \to \mathbf{A}$; the corresponding object is the unit of the monoidal structure $\mathbf{1}_{\mathbf{A}} \in \mathbf{A}$.

3.1.4. We let $1\text{-}\mathrm{Cat}^{\mathrm{Mon}}$ denote the $(\infty,1)$-category of monoidal $(\infty,1)$-categories, which is by definition a full subcategory in $\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, 1\text{-Cat})$.

The involution (1.1) on 1-Cat induces one on $\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, 1\text{-Cat})$, and the latter preserves the full subcategory $1\text{-}\mathrm{Cat}^{\mathrm{Mon}}$. At the level of underlying $(\infty,1)$-categories, this involution acts as $\mathbf{A} \mapsto \mathbf{A}^{\mathrm{op}}$.

In other words, the opposite of a monoidal $(\infty,1)$-category carries a natural monoidal structure.

Recall the involution rev on the $\mathbf{\Delta}$; see Sect. 1.1.10. This involution also induces one on $\mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, 1\text{-Cat})$, and the latter preserves also preserves $1\text{-}\mathrm{Cat}^{\mathrm{Mon}}$.

This is the operation of passing to the monoidal $(\infty,1)$-category with the *reversed multiplication*,

$$\mathbf{A} \mapsto \mathbf{A}^{\text{rev-mult}}.$$

3.1.5. *An example: endo-functors.* Let $\mathbf{C}$ be an $(\infty,1)$-category. We claim that the $(\infty,1)$-category $\mathrm{Funct}(\mathbf{C},\mathbf{C})$ acquires a natural monoidal structure. Indeed, we define the functor

$$\mathrm{Funct}(\mathbf{C},\mathbf{C})^{\otimes} : \mathbf{\Delta}^{\mathrm{op}} \to 1\text{-Cat} \tag{3.2}$$

as follows: it sends $[n]$ to

$$\mathrm{Cart}_{/[n]^{\mathrm{op}}} \underset{1\text{-Cat}\times\ldots\times 1\text{-Cat}}{\times} \{\mathbf{C}\times\ldots\times\mathbf{C}\},$$

where the functor

$$\mathrm{Cart}_{/[n]^{\mathrm{op}}} \to \mathrm{Cart}_{/*\sqcup\ldots\sqcup *} \simeq 1\text{-Cat}\times\ldots\times 1\text{-Cat}$$

is given by restriction along

$$(*\sqcup\ldots\sqcup *) = ([n]^{\mathrm{op}})^{\mathrm{Spc}} \to [n]^{\mathrm{op}}.$$

By [Chapter A.1, Corollary 2.4.4], we have

$$\mathrm{Cart}_{/[1]^{\mathrm{op}}} \underset{1\text{-Cat}\times 1\text{-Cat}}{\times} \{\mathbf{C}\times\mathbf{C}\} \simeq \mathrm{Funct}(\mathbf{C},\mathbf{C}),$$

and the functor (3.2) is easily seen to satisfy the conditions of Sect. 3.1.1.

Remark 3.1.6. The fact that (3.2) is well-defined as a functor follows from the *enhanced straightening procedure*, see Sect. 1.6.2.

3.1.7. Unstraightening defines a fully faithful embedding

$$1\text{-}\mathrm{Cat}^{\mathrm{Mon}} \hookrightarrow (\mathrm{coCart}_{/\mathbf{\Delta}^{\mathrm{op}}})_{\mathrm{strict}},$$

denoted

$$\mathbf{A}^{\otimes} \mapsto \mathbf{A}^{\otimes,\mathbf{\Delta}^{\mathrm{op}}}.$$

Its essential image is singled out by the condition in Sect. 3.1.1.

3.2. **Lax functors and associative algebras.** In this subsection we introduce the notion of associative algebra in a given monoidal $(\infty,1)$-category.

The method by which we will do it (following [Lu2, Sect. 4.2]) will exhibit the power of the idea of unstraightening.

3.2.1. We introduce another $(\infty,1)$-category, denoted $(1\text{-}\mathrm{Cat}^{\mathrm{Mon}})_{\text{right-lax}_{\text{non-untl}}}$. It will have the same objects as $1\text{-}\mathrm{Cat}^{\mathrm{Mon}}$, and will contain the latter as a 1-full subcategory.

The idea of the category $(1\text{-}\mathrm{Cat}^{\mathrm{Mon}})_{\text{right-lax}_{\text{non-untl}}}$ is that we now allow functors $\mathbf{A}_0 \to \mathbf{A}_1$ such that the diagrams

$$\begin{array}{ccc} \mathbf{A}_0 \times \mathbf{A}_0 & \longrightarrow & \mathbf{A}_0 \\ \downarrow & & \downarrow \\ \mathbf{A}_1 \times \mathbf{A}_1 & \longrightarrow & \mathbf{A}_1 \end{array}$$

no longer commute, but do so up to a natural transformation.

3.2.2. Namely, we let $(1\text{-}\mathrm{Cat}^{\mathrm{Mon}})_{\text{right-lax}_{\text{non-untl}}}$ be the 1-full subcategory of $\mathrm{coCart}_{/\mathbf{\Delta}^{\mathrm{op}}}$, whose objects are those lying in the essential image of $1\text{-}\mathrm{Cat}^{\mathrm{Mon}}$, and where we allow as morphisms functors

$$\mathbf{A}_0^{\otimes,\mathbf{\Delta}^{\mathrm{op}}} \to \mathbf{A}_1^{\otimes,\mathbf{\Delta}^{\mathrm{op}}}$$

that map morphisms in $\mathbf{A}_0^{\otimes,\mathbf{\Delta}^{\mathrm{op}}}$ that are coCartesian over morphisms in $\mathbf{\Delta}^{\mathrm{op}}$ of the form (3.1) to morphisms in $\mathbf{A}_1^{\otimes,\mathbf{\Delta}^{\mathrm{op}}}$ with the same property.

Such a functor will be called a *right-lax monoidal functor.*

3.2.3. Passing to the opposite categories, one obtains the notion of *left-lax monoidal functor*. The next assertion follows by unwinding the definitions:

Lemma 3.2.4. *Let $\mathbf{A}_1^{\otimes}$ and $\mathbf{A}_2^{\otimes}$ be a pair of monoidal $(\infty,1)$-categories, and let*

$$F : \mathbf{A}_1 \rightleftarrows \mathbf{A}_2 : G$$

be a pair of adjoint functors of the underlying plain $(\infty,1)$-categories. Then the datum of left-lax monoidal functor on F is equivalent to the datum of right-lax monoidal functor on G.

3.2.5. Let $*^{\otimes}$ be the point category, equipped with a natural monoidal structure, i.e., $*^{\otimes}([n]) = *$ for any n.

Given a monoidal $(\infty,1)$-category $\mathbf{A}^{\otimes}$, we define the notion of associative algebra in $\mathbf{A}^{\otimes}$ to be a right-lax monoidal functor

$$\mathcal{A}^{\otimes,\mathbf{\Delta}^{\mathrm{op}}} : \mathbf{\Delta}^{\mathrm{op}} = *^{\otimes,\mathbf{\Delta}^{\mathrm{op}}} \to \mathbf{A}^{\otimes,\mathbf{\Delta}^{\mathrm{op}}}.$$

We denote the $(\infty,1)$-category of associative algebras in $\mathbf{A}^{\otimes}$ by $\mathrm{AssocAlg}(\mathbf{A})$ (suppressing the $\otimes$ superscript). We let

$$\mathbf{oblv}_{\mathrm{Assoc}} : \mathrm{AssocAlg}(\mathbf{A}) \to \mathbf{A}$$

denote the tautological forgetful functor.

Given $\mathcal{A}^{\otimes,\mathbf{\Delta}^{\mathrm{op}}} \in \mathrm{AssocAlg}(\mathbf{A})$, we denote by $\mathcal{A}$ its *underlying* object of $\mathbf{A}$, i.e., the value of $\mathcal{A}^{\otimes,\mathbf{\Delta}^{\mathrm{op}}}$ on the object $[1] \in \mathbf{\Delta}^{\mathrm{op}}$.

3.2.6. Let $\mathcal{A}$ be an associative algebra in $\mathbf{A}$. Then we obtain, tautologically, an associative algebra $\mathcal{A}^{\text{rev-mult}}$ in $\mathbf{A}^{\text{rev-mult}}$, with the same underlying object of $\mathbf{A}$ as a plain $(\infty,1)$-category.

3.3. **The symmetric(!) monoidal case.** In this subsection we explain the modifications necessary in order to talk about *symmetric* monoidal $(\infty,1)$-categories, and commutative algebras inside them.

3.3.1. The definitions involving monoidal categories and associative algebras in them can be rendered into the world of *symmetric monoidal* $(\infty, 1)$-categories and *commutative algebras*, by replacing the category $\mathbf{\Delta}^{\mathrm{op}}$ by that of *finite pointed sets*, denoted Fin_*. We replace the condition in Sect. 3.1.1 by the following one:

- $\mathbf{A}^{\otimes}(\{*\}) = *$;
- For any finite pointed set $(* \in I)$ and any $i \in I - \{*\}$, we have the map
$$(* \in I) \to (* \in \{* \cup i\}),$$
given by $i \mapsto i$ and $j \mapsto *$ for $j \neq i$. We require that the induced map
$$\mathbf{A}^{\otimes}(* \in I) \to \underset{i \in I - \{*\}}{\Pi} \mathbf{A}^{\otimes}(* \in \{* \cup i\}).$$
be an equivalence.

We let $1\text{-}\mathrm{Cat}^{\mathrm{SymMon}}$ denote the $(\infty, 1)$-category of symmetric monoidal $(\infty, 1)$-categories.

Given $\mathbf{A} \in 1\text{-}\mathrm{Cat}^{\mathrm{SymMon}}$, we let $\mathrm{ComAlg}(\mathbf{A})$ denote the $(\infty, 1)$-category of commutative algebras in $\mathbf{A}$. We let
$$\mathbf{oblv}_{\mathrm{Com}} : \mathrm{Com}(\mathbf{A}) \to \mathbf{A}$$
denote the tautological forgetful functor.

3.3.2. Note that we have a canonically defined functor
$$\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fin}_* . \tag{3.3}$$

At the level of objects this functor sends $[n] \mapsto (0 \in \{0, ..., n\})$. At the level of morphisms, it sends a non-decreasing map $\phi : [m] \to [n]$ to the map $\psi : \{0, ..., n\} \to \{0, ..., m\}$ defined as follows:

For $i \in \{0, ..., n\}$ we set $\psi(i) = j$ if there exists (an automatically unique) $j \in \{0, ..., m\}$ such that $\phi(j-1) < i \leq \phi(j)$, and $\phi(i) = 0$ otherwise.

Using the functor (3.3) we obtain that any object of commutative nature (e.g., symmetric monoidal $(\infty, 1)$-category or a commutative algebra in one such) gives rise to the corresponding associative one (monoidal $(\infty, 1)$-category or associative algebra in one such).

3.3.3. Any $(\infty, 1)$-category $\mathbf{C}$ that admits Cartesian products[5] has a canonically defined (symmetric) monoidal structure.

Namely, we start with the functor
$$(\mathrm{Fin}_*)^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}, \tag{3.4}$$
given by
$$(* \in I) \mapsto \mathrm{Funct}(I, \mathbf{C}) \underset{\mathrm{Funct}(\{*\}, \mathbf{C})}{\times} \{*\},$$
where $\{*\} \to \mathrm{Funct}(\{*\}, \mathbf{C})$ is given by the functor that maps $*$ to the final object.

Now, the condition that $\mathbf{C}$ admits Cartesian products implies that the functor (3.4) satisfies the assumption of Sect. 2.4.1. Hence, we obtain a well-defined functor
$$\mathrm{Fin}_* \to 1\text{-}\mathrm{Cat}, \tag{3.5}$$
obtained from (3.4) by passing to *right adjoints*. It is easy to see that the functor (3.4) satisfies the assumptions of Sect. 3.3.1, thereby giving rise to a symmetric monoidal structure on $\mathbf{C}$.

[5]Including the empty Cartesian product, i.e., a final object.

In particular, we can talk about commutative (and if we regard $\mathbf{C}$ just a monoidal category, also associative) algebras in $\mathbf{C}$. These objects are called *commutative monoids* (resp., just *monoids*). We denote the corresponding categories by

$$\text{ComMonoid}(\mathbf{C}) \text{ and } \text{Monoid}(\mathbf{C}),$$

respectively.

Dually, if $\mathbf{C}$ admits coproducts, it has a *coCartesian* symmetric monoidal structure.

3.3.4. In particular, we can consider the $(\infty, 1)$-category 1-Cat equipped with the Cartesian symmetric monoidal structure.

Commutative (resp., associative) algebras in 1-Cat with respect to the Cartesian structure, i.e., commutative monoids (resp., just monoids) in 1-Cat are the same as symmetric monoidal (resp., monoidal) $(\infty, 1)$-categories, see [Chapter V.3, Sect. 1.3.3].

3.3.5. Let $\mathbf{A}$ be a symmetric monoidal $(\infty, 1)$-category. In this case, the $(\infty, 1)$-category $\text{AssocAlg}(\mathbf{A})$ acquires a symmetric monoidal structure, compatible with the forgetful functor $\text{AssocAlg}(\mathbf{A}) \to \mathbf{A}$, see [Lu2, Proposition 3.2.4.3 and Example 3.2.4.4].

3.3.6. Furthermore, the $(\infty, 1)$-category $\text{ComAlg}(\mathbf{A})$ also acquires a symmetric monoidal structure, and this symmetric monoidal structure equals the *coCartesian* symmetric monoidal structure on $\text{ComAlg}(\mathbf{A})$, see [Lu2, Proposition 3.2.4.7].

In particular, every object $\mathcal{A} \in \text{ComAlg}(\mathbf{A})$ has a natural structure of commutative algebra in $\text{ComAlg}(\mathbf{A})$, and hence also in $\text{AssocAlg}(\mathbf{A})$.

3.4. **Module categories.** In this section we extend the definition of monoidal $(\infty, 1)$-categories to the case of modules.

3.4.1. Let $\mathbf{\Delta}^+$ be the 1-full subcategory of $1\text{-Cat}^{\text{ordn}}$, where we allow as objects categories of the form

$$[n] = (0 \to 1 \to ... \to n), \quad n = 0, 1, ...$$

and

$$[n]^+ = (0 \to 1 \to ... \to n \to +), \quad n = 0, 1....$$

As 1-morphisms we allow:

- Arbitrary functors $[n] \to [m]$;
- Functors $[n] \to [m]^+$, whose essential image does not contain $+$;
- Functors $[n]^+ \to [m]^+$ that send $+$ to $+$ and such that the preimage of $+$ is $+$.

3.4.2. Given a monoidal $(\infty, 1)$-category $\mathbf{A}^{\otimes}$, a module for it is a datum of extension of the functor

$$\mathbf{A}^{\otimes} : \mathbf{\Delta}^{\text{op}} \to 1\text{-Cat},$$

to a functor

$$\mathbf{A}^{+,\otimes} : \mathbf{\Delta}^{+,\text{op}} \to 1\text{-Cat},$$

such that the following condition holds:

- For any $n \geq 0$, the functor

$$\mathbf{A}^{+,\otimes}([n]^+) \to \mathbf{A}^{\otimes}([n]) \times \mathbf{A}^{+,\otimes}([0]^+),$$

  given by the morphisms

$$[n] \to [n]^+, \quad i \mapsto i \text{ and } [0]^+ \to [n]^+, \quad 0 \mapsto n, + \mapsto +,$$

  is an equivalence.

3.4.3. We will think of the $(\infty, 1)$-category

$$\mathbf{M} := \mathbf{A}^{+,\otimes}([0]^+)$$

as the $(\infty, 1)$-*category underlying the module.*

Note that $\mathbf{A}^{+,\otimes}([1]^+)$ identifies with $\mathbf{A} \times \mathbf{M}$. The map

$$[0]^+ \to [1]^+, \quad 0 \mapsto 0, + \mapsto +,$$

defines a functor

$$\mathbf{A} \times \mathbf{M} \to \mathbf{M},$$

which is the functor of action of $\mathbf{A}$ on $\mathbf{M}$.

Unless a confusion is likely to occur, we denote the above functor by

$$\mathbf{a}, \mathbf{m} \mapsto \mathbf{a} \otimes \mathbf{m}.$$

3.4.4. We let $1\text{-}\mathrm{Cat}^{\mathrm{Mon}^+}$ denote the $(\infty, 1)$-category of pairs of a monoidal $(\infty, 1)$-category equipped with a module, which is a full subcategory in

$$\mathrm{Funct}(\boldsymbol{\Delta}^{+,\mathrm{op}}, 1\text{-}\mathrm{Cat}).$$

For a fixed $\mathbf{A}^{\otimes} \in 1\text{-}\mathrm{Cat}^{\mathrm{Mon}}$, we let

$$\mathbf{A}\text{-}\mathbf{mod} := 1\text{-}\mathrm{Cat}^{\mathrm{Mon}^+} \underset{1\text{-}\mathrm{Cat}^{\mathrm{Mon}}}{\times} \{\mathbf{A}^{\otimes}\}.$$

This is the $(\infty, 1)$-category of (left) $\mathbf{A}$-module categories.

3.4.5. Replacing $\mathbf{A}$ by $\mathbf{A}^{\text{rev-mult}}$ we obtain the $(\infty, 1)$-category of right $\mathbf{A}$-module categories, denoted $\mathbf{A}\text{-}\mathbf{mod}^r$.

If $\mathbf{C}$ is an $(\infty, 1)$-category with a structure of $\mathbf{A}$-module category, then $\mathbf{C}^{\mathrm{op}}$ acquires a structure of $\mathbf{A}^{\mathrm{op}}$-module category.

3.4.6. Let $\mathbf{C}$ be an $(\infty, 1)$-category. Recall that $\mathrm{Funct}(\mathbf{C}, \mathbf{C})$ acquires a natural monoidal structure (see Sect. 3.1.5). The same construction as in *loc.cit.* shows that $\mathbf{C}$ is naturally a module category for $\mathrm{Funct}(\mathbf{C}, \mathbf{C})$.

In addition, for any $\mathbf{D}$, the category $\mathrm{Funct}(\mathbf{D}, \mathbf{C})$ (resp., $\mathrm{Funct}(\mathbf{C}, \mathbf{D})$) is naturally a left (resp., right) module over $\mathrm{Funct}(\mathbf{C}, \mathbf{C})$.

3.5. **Modules for algebras.** In this subsection we will explain that, given a monoidal $(\infty, 1)$-category $\mathbf{A}$, an $\mathbf{A}$-module $\mathbf{M}$ and $\mathcal{A} \in \mathrm{AssocAlg}(\mathbf{A})$, we can talk about $\mathcal{A}$-modules in $\mathbf{M}$.

The idea is the same as that giving rise to the definition of associative algebras: we will use unstraightening.

3.5.1. Parallel to Sect. 3.2.2, we define the $(\infty, 1)$-category $(1\text{-}\mathrm{Cat}^{\mathrm{Mon}^+})_{\text{right-lax}_{\text{non-untl}}}$.

Thus, given two pairs $(\mathbf{A}_1, \mathbf{M}_1)$, $(\mathbf{A}_2, \mathbf{M}_2)$ we can talk about a pair of functors

$$F_{\mathrm{Alg}} : \mathbf{A}_1 \to \mathbf{A}_2 \text{ and } F_{\mathrm{mod}} : \mathbf{M}_1 \to \mathbf{M}_2,$$

where F_{Alg} is a right-lax monoidal functor, and F_{mod} is right-lax compatible with actions.

In particular, for a fixed $\mathbf{A}$, and $\mathbf{M}, \mathbf{N} \in \mathbf{A}\text{-}\mathbf{mod}$ we can talk about *right-lax functors* $\mathbf{M} \to \mathbf{N}$ of $\mathbf{A}$-modules.

3.5.2. Passing to opposite categories, we obtain the corresponding notion of left-lax functor. The following is not difficult to obtain from the definitions (see also [Lu2, Corollary 7.3.2.7]):

Lemma 3.5.3. *Let* $\mathbf{A}$ *be a monoidal* $(\infty,1)$*-category, and let* $\mathbf{M},\mathbf{N}\in\mathbf{A}\text{-}\mathbf{mod}$*. Let*

$$F:\mathbf{M}\rightleftarrows\mathbf{N}:G$$

be a pair of adjoint functors as plain $(\infty,1)$*-categories. Then the structure on* F *of left-lax functor of* $\mathbf{A}$*-modules is equivalent to the structure on* G *of right-lax functor of* $\mathbf{A}$*-modules.*

3.5.4. Consider the point-object

$$*^{+,\otimes}\in 1\text{-}\mathrm{Cat}^{\mathrm{Mon}^+}.$$

Given $\mathbf{A}^{+,\otimes}\in 1\text{-}\mathrm{Cat}^{\mathrm{Mon}^+}$ with the corresponding $\mathbf{A},\mathbf{M}$ we let $\mathrm{AssocAlg}+\mathrm{mod}(\mathbf{A},\mathbf{M})$ denote the resulting category of right-lax functors

$$*^{+,\otimes}\to\mathbf{A}^{+,\otimes}.$$

This is, by definition, the category of pairs $\mathcal{A}\in\mathrm{AssocAlg}(\mathbf{A})$ and $\mathcal{M}\in\mathcal{A}\text{-mod}(\mathbf{M})$. The fiber of the forgetful functor

$$\mathrm{AssocAlg}+\mathrm{mod}(\mathbf{A},\mathbf{M})\to\mathrm{AssocAlg}(\mathbf{A}) \tag{3.6}$$

over a given $\mathcal{A}\in\mathrm{AssocAlg}(\mathbf{A})$ is the category of $\mathcal{A}$-modules in $\mathbf{M}$, denoted $\mathcal{A}\text{-mod}(\mathbf{M})$.

3.5.5. The forgetful functor (3.6) is a Cartesian fibration via the operation of *restricting the module structure.*

If $\mathbf{M}$ admits geometric realizations, then the functor (3.6) is also a coCartesian fibration via the operation of *inducing the module structure.*

3.5.6. Note that we have a naturally defined functor

$$\mathbf{\Delta}^+\to\mathbf{\Delta},\quad [n]\mapsto[n],[n]^+\mapsto[n+1].$$

Restriction along this functor shows that for any $\mathbf{A}^{\otimes}\in 1\text{-}\mathrm{Cat}^{\mathrm{Mon}}$, the underlying $(\infty,1)$-category $\mathbf{A}$ is naturally a module for $\mathbf{A}^{\otimes}$.

Thus, we can talk about the category

$$\mathcal{A}\text{-mod}:=\mathcal{A}\text{-mod}(\mathbf{A})$$

of $\mathcal{A}$-modules in $\mathbf{A}$ itself.

3.5.7. For example, taking $\mathbf{A}$ equal to 1-Cat with the Cartesian monoidal structure, and $\mathcal{A}$ being an associative algebra object in 1-Cat, i.e., a monoidal $(\infty,1)$-category $\mathbf{O}$, the resulting $(\infty,1)$-category

$$\mathbf{O}\text{-mod}=\mathbf{O}\text{-mod}(1\text{-Cat})$$

is the same thing as what we denoted earlier by $\mathbf{O}$-mod, i.e., this is the $(\infty,1)$-category of $\mathbf{O}$-module categories.

3.5.8. Similarly, we obtain the $(\infty,1)$-category $\mathcal{A}\text{-mod}^r$ of *right* $\mathcal{A}$-modules, i.e.,

$$\mathcal{A}\text{-mod}^r:=\mathcal{A}^{\text{rev-mult}}\text{-mod}(\mathbf{A}^{\text{rev-mult}}).$$

Tensor product *on the right* makes $\mathcal{A}$-mod into a right $\mathbf{A}$-module category, and tensor product *on the left* makes $\mathcal{A}\text{-mod}^r$ into a left $\mathbf{A}$-module category.

3.5.9. By a pattern similar to Sect. 3.5.6, for $\mathcal{A}\in\mathrm{AssocAlg}(\mathbf{A})$, the object $\mathcal{A}\in\mathbf{A}$ has a natural structure of an object of $\mathcal{A}$-mod (resp., $\mathcal{A}\text{-mod}^r$).

3.6. **The relative inner Hom.**

3.6.1. Let $\mathbf{A}$ be a monoidal $(\infty,1)$-category, and let $\mathbf{M}$ an $\mathbf{A}$-module $(\infty,1)$-category.

Given two objects $\mathbf{m}_0, \mathbf{m}_1 \in \mathbf{M}$, consider the functor

$$\mathbf{A}^{\mathrm{op}} \to \mathrm{Spc}, \quad \mathbf{a} \mapsto \mathrm{Maps}_{\mathbf{M}}(\mathbf{a} \otimes \mathbf{m}_0, \mathbf{m}_1).$$

If this functor is representable, we will denote the representing object by

$$\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{m}_0, \mathbf{m}_1) \in \mathbf{A}.$$

This is the *relative inner* Hom.

3.6.2. In particular we can take $\mathbf{M} = \mathbf{A}$, regarded as a module over itself. In this case, for $\mathbf{a}_0, \mathbf{a}_1 \in \mathbf{A}$, we obtain the notion of usual inner Hom

$$\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}_0, \mathbf{a}_1) \in \mathbf{A}.$$

3.6.3. For example, let us take $\mathbf{A} = 1\text{-Cat}$, equipped with the Cartesian monoidal structure. Then for $\mathbf{C}_0, \mathbf{C}_1 \in 1\text{-Cat}$, the resulting object

$$\underline{\mathrm{Hom}}_{1\text{-Cat}}(\mathbf{C}_0, \mathbf{C}_1) \in 1\text{-Cat}$$

identifies with $\mathrm{Funct}(\mathbf{C}_0, \mathbf{C}_1)$.

3.6.4. Let $\mathcal{A} \in \mathbf{A}$ be an associative algebra. Following Sect. 3.5.8, we consider the $(\infty,1)$-category $\mathcal{A}\text{-mod}^r$ as a (left) module category over $\mathbf{A}$.

Thus, for two objects $\mathcal{M}_0, \mathcal{M}_1 \in \mathcal{A}\text{-mod}^r$, it makes sense to ask about the existence of their inner Hom as an object of $\mathbf{A}$. We shall denote it by

$$\underline{\mathrm{Hom}}_{\mathbf{A},\mathcal{A}}(\mathcal{M}_0, \mathcal{M}_1).$$

3.6.5. Assume now that $\mathbf{A}$ is symmetric monoidal, and that $\mathcal{A}$ is a commutative algebra in $\mathbf{A}$. In this case, for $\mathcal{M}, \mathcal{N} \in \mathcal{A}\text{-mod}$, the above object

$$\underline{\mathrm{Hom}}_{\mathbf{A},\mathcal{A}}(\mathcal{M}, \mathcal{N}) \in \mathbf{A}$$

naturally acquires a structure of $\mathcal{A}$-module.

3.6.6. Let $\mathbf{A}$ be again a monoidal $(\infty,1)$-category, and let $\mathbf{M}$ be an $\mathbf{A}$-module $(\infty,1)$-category.

Let $\mathbf{m} \in \mathbf{M}$ be an object. Suppose that the relative inner Hom object $\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{m}, \mathbf{m}) \in \mathbf{A}$ exists.

Then $\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{m}, \mathbf{m})$ has a natural structure of associative algebra in $\mathbf{A}$. This is the unique algebra structure, for which the tautological map

$$\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{m}, \mathbf{m}) \otimes \mathbf{m} \to \mathbf{m}$$

extends to a structure of $\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{m}, \mathbf{m})$-module on $\mathbf{m}$, see [Lu2, Sect. 4.7.1].

3.7. **Monads and Barr-Beck-Lurie.**

3.7.1. Let $\mathbf{C}$ be an $(\infty,1)$-category. Recall that $\mathrm{Funct}(\mathbf{C}, \mathbf{C})$ has a natural structure of monoidal category, and $\mathbf{C}$ that of $\mathrm{Funct}(\mathbf{C}, \mathbf{C})$-module, see Sects. 3.1.5 and 3.4.6.

By definition, a *monad* acting on $\mathbf{C}$ is an associative algebra $\mathcal{A} \in \mathrm{Funct}(\mathbf{C}, \mathbf{C})$.

3.7.2. Given a monad $\mathcal{A}$, we can consider the category $\mathcal{A}\text{-mod}(\mathbf{C})$. We denote by

$$\mathbf{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}(\mathbf{C}) \to \mathbf{C}$$

the tautological forgetful functor.

The functor $\mathbf{oblv}_{\mathcal{A}}$ admits a left adjoint, denoted

$$\mathbf{ind}_{\mathcal{A}} : \mathbf{C} \to \mathcal{A}\text{-mod}(\mathbf{C}).$$

The composite functor

$$\mathbf{oblv}_{\mathcal{A}} \circ \mathbf{ind}_{\mathcal{A}} : \mathbf{C} \to \mathbf{C}$$

identifies with the functor

$$\mathbf{c} \mapsto \mathcal{A}(\mathbf{c}),$$

where we view $\mathcal{A}$ as an endo-functor of $\mathbf{C}$, see [Lu2, Corollary 4.2.4.8].

3.7.3. Recall now that for any $(\infty, 1)$-category $\mathbf{D}$, the $(\infty, 1)$-category $\mathrm{Funct}(\mathbf{D}, \mathbf{C})$ is also a module over $\mathrm{Funct}(\mathbf{C}, \mathbf{C})$, see Sect. 3.4.6.

One can deduce from the construction that for a given $G \in \mathrm{Funct}(\mathbf{D}, \mathbf{C})$, a structure on G of $\mathcal{A}$-module, i.e., that of object in

$$\mathcal{A}\text{-mod}(\mathrm{Funct}(\mathbf{D}, \mathbf{C}))$$

is equivalent to that of factoring G as

$$\mathbf{D} \to \mathcal{A}\text{-mod}(\mathbf{C}) \overset{\mathbf{oblv}_{\mathcal{A}}}{\to} \mathbf{C}.$$

3.7.4. Let G be a functor $\mathbf{D} \to \mathbf{C}$. It is easy to see that if G admits a left adjoint, then the inner Hom object

$$\underline{\mathrm{Hom}}_{\mathrm{Funct}(\mathbf{C}, \mathbf{C})}(G, G) \in \mathrm{Funct}(\mathbf{C}, \mathbf{C})$$

exists and identifies with $G \circ G^L$ (see [Lu2, Lemma 4.7.3.1].).

Note that according to Sect. 3.6.6,

$$\mathcal{A} := G \circ G^L \in \mathrm{Funct}(\mathbf{C}, \mathbf{C})$$

acquires a structure of associative algebra.

By the above, the functor G canonically factors as

$$\mathbf{D} \overset{G^{\mathrm{enh}}}{\longrightarrow} \mathcal{A}\text{-mod}(\mathbf{C}) \overset{\mathbf{oblv}_{\mathcal{A}}}{\to} \mathbf{C}.$$

Definition 3.7.5. *We shall say that G is* monadic *if the above functor*

$$G^{\mathrm{enh}} : \mathbf{D} \to \mathcal{A}\text{-mod}(\mathbf{C})$$

is an equivalence.

3.7.6. Here is the statement of a simplified version of the Barr-Beck-Lurie theorem (see [Lu2, Theorem 4.7.3.5] for the general statement):

Proposition 3.7.7. *Suppose that in the above situation both categories* $\mathbf{C}$ *and* $\mathbf{D}$ *contain geometric realizations. Then the functor* G *is monadic provided that the following two conditions hold:*

(1) G *is conservative;*
(2) G *preserves geometric realizations.*

Proof. By assumption, the functor G^{enh} is conservative. Hence, it suffices to show that G^{enh} admits a left adjoint, to be denoted F^{enh}, and that the natural transformation

$$\mathbf{oblv}_{\mathcal{A}} \to \mathbf{oblv}_{\mathcal{A}} \circ G^{\text{enh}} \circ F^{\text{enh}} \simeq G \circ F^{\text{enh}} \tag{3.7}$$

is an isomorphism.

It is clear that the (a priori partially defined) left adjoint F^{enh} is defined on objects of the form $\mathbf{ind}_{\mathcal{A}}(\mathbf{c})$ for $\mathbf{c} \in \mathbf{C}$, and by transitivity $F^{\text{enh}} \circ \mathbf{ind}_{\mathcal{A}} = G^L$. The corresponding map

$$\mathbf{oblv}_{\mathcal{A}} \circ \mathbf{ind}_{\mathcal{A}} \to G \circ F^{\text{enh}} \circ \mathbf{ind}_{\mathcal{A}} \tag{3.8}$$

is the tautological isomorphism $\mathbf{oblv}_{\mathcal{A}} \circ \mathbf{ind}_{\mathcal{A}} \to G \circ G^L$.

Now, every object of $\mathcal{A}$-mod$(\mathbf{C})$ can be obtained as a geometric realization of a simplicial object, whose terms are of the form $\mathbf{ind}_{\mathcal{A}}(\mathbf{c})$ for $\mathbf{c} \in \mathbf{C}$. Hence, the fact that F^{enh} is defined on such objects implies that it is defined on all of $\mathcal{A}$-mod$(\mathbf{C})$. Given that (3.8) is an isomorphism, in order to deduce the corresponding fact for (3.7), it suffices to show that both sides in (3.7) preserve geometric realizations.

This is clear for the right-hand side in (3.7), since G preserves geometric realizations. The fact that $\mathbf{oblv}_{\mathcal{A}}$ preserves geometric realizations follows from the fact that the functor

$$\mathcal{A} \otimes - \simeq G \circ G^L$$

has this property.

□

3.7.8. Here is a typical situation in which Proposition 3.7.7 applies. Let $\mathbf{A}$ be a monoidal $(\infty, 1)$-category, $\mathcal{A} \in \text{Assoc}(\mathbf{A})$, and $\mathbf{M} \in \mathbf{A}\text{-}\mathbf{mod}$. Then the forgetful functor

$$\mathbf{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}(\mathbf{M}) \to \mathbf{M}$$

is monadic, and the corresponding monad on $\mathbf{M}$ is given by

$$\mathbf{m} \mapsto \mathcal{A} \otimes \mathbf{m}.$$

Consistently with Sect. 3.7.2, we denote the corresponding left adjoint $\mathbf{M} \to \mathcal{A}$-mod$(\mathbf{M})$ by $\mathbf{ind}_{\mathcal{A}}$.

4. Duality

In this section we will discuss the general pattern of duality. It will apply to the notion of dualizable object in a monoidal $(\infty, 1)$-category, dualizable module over an algebra, and also to that of adjoint functor.

The material in this section can be viewed as a user guide to (some parts) of [Lu2, Sects. 4.4 and 4.6].

4.1. **Dualizability.** In this subsection we introduce the notion of dualizability of an object in a monoidal $(\infty, 1)$-category.

4.1.1. Let $\mathbf{A}$ be a monoidal $(\infty,1)$-category. We shall say that an object $\mathbf{a}\in\mathbf{A}$ is *right-dualizable* if it is so as an object of $\mathbf{A}^{\mathrm{ordn}}$.

I.e., $\mathbf{a}$ admits a *right dual* if there exists an object $\mathbf{a}^{\vee,R}\in\mathbf{A}$ equipped with 1-morphisms

$$\mathbf{a}\otimes\mathbf{a}^{\vee,R}\overset{\text{co-unit}}{\longrightarrow}\mathbf{1}_{\mathbf{A}} \text{ and } \mathbf{1}_{\mathbf{A}}\overset{\text{unit}}{\longrightarrow}\mathbf{a}^{\vee,R}\otimes\mathbf{a},$$

such that the composition

$$\mathbf{a}\overset{\mathrm{id}\otimes\text{unit}}{\longrightarrow}\mathbf{a}\otimes\mathbf{a}^{\vee,R}\otimes\mathbf{a}\overset{\text{co-unit}\otimes\mathrm{id}}{\longrightarrow}\mathbf{a} \tag{4.1}$$

projects to the identity element in $\pi_0(\mathrm{Maps}_{\mathbf{A}}(\mathbf{a},\mathbf{a}))$, and the composition

$$\mathbf{a}^{\vee,R}\overset{\text{unit}\otimes\mathrm{id}}{\longrightarrow}\mathbf{a}^{\vee,R}\otimes\mathbf{a}\otimes\mathbf{a}^{\vee,R}\overset{\mathrm{id}\otimes\text{co-unit}}{\longrightarrow}\mathbf{a}^{\vee,R} \tag{4.2}$$

projects to the identity element in $\pi_0(\mathrm{Maps}_{\mathbf{A}}(\mathbf{a}^{\vee,R},\mathbf{a}^{\vee,R}))$.

Similarly, one defines the notion of being left-dualizable.

If $\mathbf{A}$ is *symmetric* monoidal, then there is no difference between being right or left dualizable.

4.1.2. Let us be given $\mathbf{a}\in\mathbf{A}$ that admits a right dual. Consider the corresponding data

$$(\mathbf{a}^{\vee,R},\mathbf{a}\otimes\mathbf{a}^{\vee,R}\overset{\text{co-unit}}{\longrightarrow}\mathbf{1}_{\mathbf{A}}). \tag{4.3}$$

We obtain that for any $\mathbf{a}'\in\mathbf{A}$, the composite map

$$\mathrm{Maps}_{\mathbf{A}}(\mathbf{a}',\mathbf{a}^{\vee,R})\to\mathrm{Maps}_{\mathbf{A}}(\mathbf{a}\otimes\mathbf{a}',\mathbf{a}\otimes\mathbf{a}^{\vee,R})\overset{\text{co-unit}}{\longrightarrow}\mathrm{Maps}_{\mathbf{A}}(\mathbf{a}\otimes\mathbf{a}',\mathbf{1}_{\mathbf{A}}).$$

is an isomorphism.

From here, we obtain that the data of (4.3) is *uniquely defined*.

Similarly, the data of

$$(\mathbf{a}^{\vee,R},\mathbf{1}_{\mathbf{A}}\overset{\text{unit}}{\longrightarrow}\mathbf{a}^{\vee,R}\otimes\mathbf{a}) \tag{4.4}$$

is uniquely defined.

Furthermore, we can fix both (4.3) and (4.4) uniquely by choosing a path between (4.1) with $\mathrm{id}_{\mathbf{a}}$ *or* a path between (4.2) with $\mathrm{id}_{\mathbf{a}^{\vee,R}}$.

4.1.3. A convenient framework for viewing the notions of right or left dual is that of adjunction of 1-morphisms in an $(\infty,2)$-category, developed in [Chapter A.3]: the datum of a monoidal $(\infty,1)$-category is equivalent to that of an $(\infty,2)$-category with a single object.

In particular, it follows from [Chapter A.3, Sect. 1], that given an object $\mathbf{a}\in\mathbf{A}$ that admits a right dual there exists a canonically defined $\mathbf{a}^{\vee,R}$, equipped with the data of (4.3) and (4.4), as well as paths connecting (4.1) with $\mathrm{id}_{\mathbf{a}}$ *and* (4.2) with $\mathrm{id}_{\mathbf{a}^{\vee,R}}$. These data are fixed uniquely by requiring that they satisfy a certain *infinite* set of compatibility conditions, specified in *loc.cit.*

Thus, we can talk about *the* right dual of an object $\mathbf{a}\in\mathbf{A}$.

A similar discussion applies to the word 'right' replaced by 'left'. By construction, the datum of making $\mathbf{a}'$ the right dual of $\mathbf{a}$ is equivalent to the datum of making $\mathbf{a}$ the left dual of $\mathbf{a}'$.

4.1.4. Let $\mathbf{A}^{\text{right-dualizable}}$ (resp., $\mathbf{A}^{\text{left-dualizable}}$) denote the full subcategory spanned by right (resp., left) dualizable objects. Applying [Chapter A.3, Corollary 1.3.6] we obtain that dualization defines an equivalence of monoidal $(\infty,1)$-categories

$$(\mathbf{A}^{\text{right-dualizable}})^{\text{op}} \simeq (\mathbf{A}^{\text{left-dualizable}})^{\text{rev-mult}}.$$

For a morphism $\phi : \mathbf{a}_1 \to \mathbf{a}_2$ we denote by $\phi^{\vee,R}$ (resp., $\phi^{\vee,L}$) the corresponding morphism $\mathbf{a}_2^{\vee,R} \to \mathbf{a}_1^{\vee,R}$ (resp., $\mathbf{a}_2^{\vee,L} \to \mathbf{a}_1^{\vee,L}$).

If $\mathbf{A}$ is symmetric monoidal we denote

$$\mathbf{A}^{\text{right-dualizable}} =: \mathbf{A}^{\text{dualizable}} := \mathbf{A}^{\text{left-dualizable}}.$$

For a morphism $\phi : \mathbf{a}_1 \to \mathbf{a}_2$ we let $\phi^\vee : \mathbf{a}_2^\vee \to \mathbf{a}_1^\vee$ denote its dual.

4.1.5. Consider $\mathbf{A}$ as a module over itself, and for two objects $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}$ recall the notation

$$\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}_1, \mathbf{a}_2) \in \mathbf{A}$$

(see Sect. 3.6.1). I.e., this is an object of $\mathbf{A}$ (if it exists) such that

$$\mathrm{Maps}_{\mathbf{A}}(\mathbf{a}', \underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}_1, \mathbf{a}_2)) \simeq \mathrm{Maps}_{\mathbf{A}}(\mathbf{a}' \otimes \mathbf{a}_1, \mathbf{a}_2).$$

Assume that $\mathbf{a}_1 \in \mathbf{A}$ is left dualizable. Then it is easy to see that $\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}_1, \mathbf{a}_2)$ exists and we have a canonical isomorphism

$$\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}_1, \mathbf{a}_2) := \mathbf{a}_2 \otimes \mathbf{a}_1^{\vee,L}.$$

Lemma 4.1.6. *Let $\mathbf{A}$ be a monoidal $(\infty,1)$-category.*

(a) *Suppose that the functor $\mathrm{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, -)$ is conservative. Then if $\mathbf{a} \in \mathbf{A}$ is right dualizable, then the functor $\mathbf{a}' \mapsto \mathbf{a}' \otimes \mathbf{a}$ commutes with limits.*

(b) *Let $\mathbf{I}$ be an index category, and suppose that the functor $\otimes : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ preserves colimits in each variable indexed by $\mathbf{I}$. Assume also that the functor $\mathrm{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, -)$ preserves colimits indexed by $\mathbf{I}$. Then for any $\mathbf{a} \in \mathbf{A}$ that is left or right dualizable, the functor $\mathrm{Maps}_{\mathbf{A}}(\mathbf{a}, -)$ preserves colimits indexed by $\mathbf{I}$.*

Proof. For (a), we rewrite the functor $\mathbf{a}' \mapsto \mathbf{a}' \otimes \mathbf{a}$ as $\mathbf{a}' \mapsto \underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}^{\vee,R}, \mathbf{a}')$, so it is sufficient to show that the latter functor preserves limits. Since the functor $\mathrm{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, -)$ commutes with limits and is conservative (by assumption), it is enough to show that the functor

$$\mathbf{a}' \mapsto \mathrm{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, (\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}^{\vee,R}, \mathbf{a}')))$$

preserves limits. However, the latter functor is isomorphic to $\mathrm{Maps}_{\mathbf{A}}(\mathbf{a}^{\vee,R}, \mathbf{a}')$.

For (b) we give a proof when $\mathbf{a}$ is left-dualizable. Indeed, the functor

$$\mathbf{a}' \mapsto \mathrm{Maps}_{\mathbf{A}}(\mathbf{a}, \mathbf{a}')$$

is the composition of the functor

$$\mathbf{a}' \mapsto \underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}, \mathbf{a}') \simeq \mathbf{a}' \otimes \mathbf{a}^L,$$

followed by the functor $\mathrm{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, -)$.

□

4.1.7. Let $\mathcal{A}$ be an associative algebra in $\mathbf{A}$, and $\mathbf{a} \in \mathbf{A}$ an $\mathcal{A}$-module, which is left-dualizable as a plain object of $\mathbf{A}$.

In this case, the left dual $\mathbf{a}^{\vee,L}$ of $\mathbf{a}$ acquires a natural structure of right $\mathcal{A}$-module.

The corresponding action map $\mathbf{a}^{\vee,L} \otimes \mathcal{A} \to \mathbf{a}^{\vee,L}$ is explicitly given by

$$\mathbf{a}^{\vee,L} \otimes \mathcal{A} \overset{\mathrm{id}_{\mathbf{a}^{\vee,L}} \otimes \mathrm{id}_{\mathcal{A}} \otimes \mathrm{unit}}{\longrightarrow} \mathbf{a}^{\vee,L} \otimes \mathcal{A} \otimes \mathbf{a} \otimes \mathbf{a}^{\vee,L} \to \mathbf{a}^{\vee,L} \otimes \mathbf{a} \otimes \mathbf{a}^{\vee,L} \overset{\text{co-unit} \otimes \mathrm{id}_{\mathbf{a}^{\vee,L}}}{\longrightarrow} \mathbf{a}^{\vee,L},$$

where the middle arrow is given by the action map $\mathcal{A} \otimes \mathbf{a} \to \mathbf{a}$.

More generally, for any $\mathcal{A}$-module $\mathbf{a}$ and $\mathbf{a}' \in \mathbf{A}$ for which $\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}, \mathbf{a}') \in \mathbf{A}$ exists, the object $\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}, \mathbf{a}')$ is naturally a right $\mathcal{A}$-module.

4.2. **Tensor products of modules.** In this subsection we will make a digression and discuss the operation of tensor product of modules over an associative (resp., commutative) algebra.

4.2.1. Assume now that $\mathbf{A}$ contains geometric realizations that distribute over the monoidal operation in $\mathbf{A}$. We claim that in this case there exists a canonically defined functor

$$\mathcal{A}\text{-mod}^r \times \mathcal{A}\text{-mod} \to \mathbf{A}, \quad \mathcal{N}, \mathcal{M} \mapsto \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M},$$

see [Lu2, Sect. 4.4].

Indeed, it is *uniquely* defined by the following conditions:

- It preserves geometric realizations in each variable;
- It is a functor of $\mathbf{A}$-bimodule categories;
- It sends
$$(\mathcal{A} \times \mathcal{A} \in \mathcal{A}\text{-mod}^r \times \mathcal{A}\text{-mod}) \mapsto (\mathcal{A} \in \mathbf{A}),$$
in a way compatible with the homomorphisms

$$\mathcal{A} \times \mathcal{A}^{\text{rev-mult}} \to \underline{\mathrm{Hom}}_{\mathbf{A} \times \mathbf{A}^{\text{rev-mult}}}(\mathcal{A} \times \mathcal{A}, \mathcal{A} \times \mathcal{A}) \text{ and } \mathcal{A} \times \mathcal{A}^{\text{rev-mult}} \to \underline{\mathrm{Hom}}_{\mathbf{A} \times \mathbf{A}^{\text{rev-mult}}}(\mathcal{A}, \mathcal{A}).$$

4.2.2. Let now $\mathbf{A}$ be a symmetric monoidal $(\infty,1)$-category. In this case, the $(\infty,1)$-category

$$\mathrm{AssocAlg} + \mathrm{mod}(\mathbf{A}) := \mathrm{AssocAlg} + \mathrm{mod}(\mathbf{A}, \mathbf{A})$$

has a natural symmetric monoidal structure, so that the forgetful functor

$$\mathrm{AssocAlg} + \mathrm{mod}(\mathbf{A}) \to \mathrm{AssocAlg}(\mathbf{A}) \tag{4.5}$$

is symmetric monoidal, see [Lu2, Proposition 3.2.4.3].

4.2.3. Assume now that $\mathbf{A}$ contains geometric realizations that distribute over the monoidal operation in $\mathbf{A}$. In this case (4.5) is a coCartesian fibration.

4.2.4. Let now $\mathcal{A}$ be a commutative algebra in $\mathbf{A}$, viewed as a commutative algebra object in $\mathrm{AssocAlg}(\mathbf{A})$, see Sect. 3.3.6.

Combining with the above, we obtain that the $(\infty,1)$-category $\mathcal{A}$-mod acquires a canonically defined symmetric monoidal structure (thought of as given by tensor product over $\mathcal{A}$), see [Lu2, Theorem 4.5.2.1].

4.3. **Duality for modules over an algebra.** In this subsection we will discuss the notion of duality between left and right modules over a given associative algebra.

4.3.1. Let $\mathbf{A}$ be a monoidal $(\infty,1)$-category, and $\mathcal{A}$ an associative algebra in $\mathbf{A}$. Let $\mathcal{N}$ and $\mathcal{M}$ be a right and left $\mathcal{A}$-modules in $\mathbf{A}$. A duality datum between $\mathcal{N}$ and $\mathcal{M}$ is a pair of morphisms

$$\text{unit} : \mathbf{1}_{\mathbf{A}} \to \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M}$$

and

$$\text{co-unit} : \mathcal{M} \otimes \mathcal{N} \to \mathcal{A},$$

the latter being a map of $\mathcal{A} \otimes \mathcal{A}^{\text{rev-mult}}$-modules, such that the composition

$$\mathcal{M} \overset{\text{id} \otimes \text{unit}}{\longrightarrow} \mathcal{M} \otimes (\mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M}) \simeq (\mathcal{M} \otimes \mathcal{N}) \underset{\mathcal{A}}{\otimes} \mathcal{M} \overset{\text{co-unit} \otimes \text{id}}{\longrightarrow} \mathcal{A} \underset{\mathcal{A}}{\otimes} \mathcal{M} \simeq \mathcal{M}$$

projects to the identity element in $\pi_0(\text{Maps}_{\mathbf{A}}(\mathcal{M},\mathcal{M}))$, and the composition

$$\mathcal{N} \overset{\text{unit} \otimes \text{id}}{\longrightarrow} (\mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M}) \otimes \mathcal{N} \simeq \mathcal{N} \underset{\mathcal{A}}{\otimes} (\mathcal{M} \otimes \mathcal{N}) \overset{\text{id} \otimes \text{co-unit}}{\longrightarrow} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{A} \simeq \mathcal{N}$$

projects to the identity element in $\pi_0(\text{Maps}_{\mathbf{A}}(\mathcal{N},\mathcal{N}))$.

Thus, it makes sense to talk about dualizable left or right $\mathcal{A}$-modules.

The discussion in Sect. 4.1.2 regarding the canonicity of the dual and the duality data applies *mutatis mutandis* to the present setting.

4.3.2. Consider $\mathcal{A}\text{-mod}^r$ as a $\mathbf{A}$-module category. Let $\mathcal{M}$ and $\mathcal{N}$ be a pair of objects of $\mathcal{A}\text{-mod}^r$. Assume that $\mathcal{M}$ is dualizable, and let $\mathcal{M}^\vee \in \mathcal{A}\text{-mod}$ denote its dual. In this case, it is easy to see that the object

$$\underline{\text{Hom}}_{\mathbf{A},\mathcal{A}}(\mathcal{M},\mathcal{N}) \in \mathbf{A}$$

exists and identifies canonically with $\mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M}^\vee$.

In particular, we obtain that in the situation of Lemma 4.1.6(a), the functor

$$\mathcal{N} \mapsto \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M}^\vee$$

preserves $\mathbf{I}$-limits.

4.3.3. Assume now that $\mathbf{A}$ is symmetric monoidal and $\mathcal{A}$ is commutative. Recall that in this case, the $(\infty,1)$-category $\mathcal{A}$-mod itself carries a symmetric monoidal structure.

In this case, the duality datum between $\mathcal{M}$ and $\mathcal{N}$ in the sense of Sect. 4.3.1 is equivalent to the duality datum between them as objects of $\mathcal{A}$-mod as a symmetric monoidal $(\infty,1)$-category.

Furthermore, in this case, if $\mathcal{M}$ is dualizable, the isomorphism

$$\underline{\text{Hom}}_{\mathbf{A},\mathcal{A}}(\mathcal{M},\mathcal{N}) \simeq \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M}^\vee$$

upgrades to one in the category $\mathcal{A}$-mod.

4.4. **Adjoint functors, revisited.** We will now make a digression and discuss the point of view on the notion of adjoint functor parallel to that of the dual object. (This is, of course, more than an analogy: the two are part of the same paradigm–the notion of adjunction for 1-morphisms in an $(\infty,2)$-category.)

The reference for the material here is [Lu1, Sect. 5.2].

4.4.1. Let

$$F : \mathbf{C}_0 \rightleftarrows \mathbf{C}_1 : G$$

be a pair of functors between $(\infty, 1)$-categories.

An adjunction datum between F and G is the datum of natural transformations

$$\text{unit} : \mathrm{Id}_{\mathbf{C}_0} \to G \circ F \text{ and } \text{co-unit} : F \circ G \to \mathrm{Id}_{\mathbf{C}_1}, \tag{4.6}$$

such that the composition

$$F \overset{\mathrm{id} \circ \mathrm{unit}}{\longrightarrow} F \circ G \circ F \overset{\text{co-unit} \circ \mathrm{id}}{\longrightarrow} F \tag{4.7}$$

maps to the identity element in $\pi_0(\mathrm{Maps}_{\mathrm{Funct}(\mathbf{C}_0, \mathbf{C}_1)}(F, F))$, and the composition

$$G \overset{\mathrm{unit} \circ \mathrm{id}}{\longrightarrow} G \circ F \circ G \overset{\mathrm{id} \circ \text{co-unit}}{\longrightarrow} F \tag{4.8}$$

maps to the identity element in $\pi_0(\mathrm{Maps}_{\mathrm{Funct}(\mathbf{C}_1, \mathbf{C}_0)}(G, G))$.

4.4.2. Given F (resp., G) that can be complemented to an adjunction datum, the discussion in Sect. 4.1.2 applies as to the canonicity of the data of $(G, \mathrm{unit}, \text{co-unit})$ (resp., $(F, \mathrm{unit}, \text{co-unit})$).

4.4.3. Suppose that F and G are mutually adjoint in the sense of Sect. 1.7.1. Then F and G can be *canonically* equipped with the adjunction datum; moreover there exists a canonical choice for a path between (4.7) (resp., (4.8)) and the identity endomorphism of F (resp., G).

Vice versa, a functor F (resp., G) that can be complemented to an adjunction datum admits a right (resp., left) adjoint in the sense of Sect. 1.7.1.

5. Stable $(\infty, 1)$-categories

In this section we study the notion of *stable* $(\infty, 1)$-category. This is the ∞-categorical enhancement of the notion of triangulated category.

The main point of difference between these two notions is that stable categories are *much better behaved* when it comes to such operations as taking the limit of a diagram of categories.

Related to this is the fact that given a pair of stable categories, we can form their *tensor product*, discussed in the next section.

5.1. **The notion of stable category.** In this subsection we define the notion of stable $(\infty, 1)$-category.

In a way parallel to abelian categories, the additive structure carried by $(\infty, 1)$-categories is in fact *not* an additional piece of structure, but rather a property of an $(\infty, 1)$-category.

The material here follows [Lu2, Sect. 1.1].

5.1.1. Let $\mathbf{C}$ be an $(\infty, 1)$-category. We say that $\mathbf{C}$ is *stable* if:

- It contains fiber products and push-outs[6];
- The map from the initial object to the final object is an isomorphism; we will henceforth denote it by 0;

[6]Including the empty ones, i.e., a final and and an initial objects.

- A diagram
$$\begin{array}{ccc} \mathbf{c}_0 & \longrightarrow & \mathbf{c}_1 \\ \downarrow & & \downarrow \\ \mathbf{c}_2 & \longrightarrow & \mathbf{c}_3 \end{array}$$
is a pullback square if and only if it is a push-out square.

Clearly, $\mathbf{C}$ is stable if and only if $\mathbf{C}^{\text{op}}$ is.

5.1.2. Let $\mathbf{C}$ be a stable category. For $\mathbf{c} \in \mathbf{C}$ we will use the short-hand notation $\mathbf{c}[-1]$ and $\mathbf{c}[1]$ for
$$\Omega(\mathbf{c}) := 0 \underset{\mathbf{c}}{\times} 0 \text{ and } \Sigma(\mathbf{c}) := 0 \underset{\mathbf{c}}{\sqcup} 0,$$
respectively. It follows from the axioms that the functors [1] and [−1], which are a priori are mutually adjoint, are actually mutually inverse.

Consider the homotopy category $\text{Ho}(\mathbf{C})$ of $\mathbf{C}$, i.e., in our notation $\mathbf{C}^{\text{ordn}}$. Then $\mathbf{C}^{\text{ordn}}$ has a structure of triangulated category: its distinguished triangles are images of fiber sequences
$$\mathbf{c}_1 \to \mathbf{c}_2 \to \mathbf{c}_3,$$
i.e.,
$$\mathbf{c}_1 \simeq 0 \underset{\mathbf{c}_3}{\times} \mathbf{c}_2.$$

The map $\mathbf{c}_3[-1] \to \mathbf{c}_1$ comes from the tautological map
$$0 \underset{\mathbf{c}_3}{\sqcup} 0 \to \mathbf{c}_1 \simeq 0 \underset{\mathbf{c}_3}{\times} \mathbf{c}_2.$$

5.1.3. A functor between stable categories is said to be *exact* if it preserves pullbacks (equivalently, push-outs).

We let 1-Cat^{St} denote the 1-full subcategory of 1-Cat, whose objects are stable categories and whose morphisms are exact functors.

It is clear that the inclusion functor
$$1\text{-Cat}^{\text{St}} \to 1\text{-Cat}$$
preserves limits.

5.1.4. For a pair of stable categories $\mathbf{C}$ and $\mathbf{D}$, let
$$\text{Funct}_{\text{ex}}(\mathbf{C}, \mathbf{D})$$
denote the full subcategory of $\text{Funct}(\mathbf{C}, \mathbf{D})$ spanned by exact functors. We have
$$(\text{Funct}_{\text{ex}}(\mathbf{C}, \mathbf{D}))^{\text{Spc}} = \text{Maps}_{1\text{-Cat}^{\text{St}}}(\mathbf{C}, \mathbf{D}).$$

The $(\infty, 1)$-category $\text{Funct}_{\text{ex}}(\mathbf{C}, \mathbf{D})$ is itself stable.

5.1.5. We shall say that a stable category is *cocomplete*[7] if it contains filtered colimits. This condition is equivalent to the (seemingly stronger) condition of containing arbitrary colimits, and also to the (seemingly weaker) condition of containing direct sums.

We let $1\text{-Cat}^{\text{St,cocmpl}} \subset 1\text{-Cat}^{\text{St}}$ be the full subcategory of 1-Cat^{St} spanned by cocomplete stable categories.

[7]When talking about cocomplete categories we will always assume that they are presentable.

5.1.6. Let $\mathbf{C}$ and $\mathbf{D}$ be a pair of cocomplete stable categories, and let $F : \mathbf{D} \to \mathbf{C}$ be an exact functor.

We shall say that F is *continuous* if it preserves filtered colimits. This condition is equivalent to the (seemingly stronger) condition of preserving arbitrary colimits, and also to the (seemingly weaker) condition of preserving direct sums.

We let

$$1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}} \subset 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}$$

denote the 1-full subcategory where we restrict morphisms to continuous functors.

5.1.7. Let $\mathbf{C}$ and $\mathbf{D}$ be a pair of stable categories. Consider the stable category $\mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}, \mathbf{D})$. If $\mathbf{D}$ is cocomplete, then $\mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}, \mathbf{D})$ is also cocomplete (this follows from the definition of cocompleteness via direct sums).

Assume now that $\mathbf{C}$ and $\mathbf{D}$ are cocomplete. We let

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{C}, \mathbf{D}) \subset \mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}, \mathbf{D})$$

be the full subcategory spanned by continuous functors. We have

$$(\mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{C}, \mathbf{D}))^{\mathrm{Spc}} = \mathrm{Maps}_{1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{C}, \mathbf{D}).$$

The $(\infty, 1)$-category $\mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{C}, \mathbf{D})$ is stable and cocomplete. The inclusion

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{C}, \mathbf{D}) \hookrightarrow \mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}, \mathbf{D})$$

is continuous.

5.1.8. Let $\mathbf{C}$ be any $(\infty, 1)$-category that contains coproducts. We equip $\mathbf{C}$ with the *coCartesian* symmetric monoidal structure. Then the forgetful functor

$$\mathrm{ComAlg}(\mathbf{C}) \overset{\mathbf{oblv}_{\mathrm{ComAlg}}}{\longrightarrow} \mathbf{C}$$

is an equivalence, see [Lu2, Corollary 2.4.3.10]. (Informally, every object $\mathbf{c} \in \mathbf{C}$ has a uniquely defined structure of commutative algebra, given by $\mathbf{c} \sqcup \mathbf{c} \to \mathbf{c}$).

Let now $\mathbf{C}$ be stable. In this case, the *coCartesian* symmetric monoidal structure coincides with the Cartesian one. Hence, we obtain that the forgetful functor

$$\mathrm{ComMonoid}(\mathbf{C}) \overset{\mathbf{oblv}_{\mathrm{ComMonoid}}}{\longrightarrow} \mathbf{C} \tag{5.1}$$

is an equivalence, where the notation $\mathrm{ComMonoid}(-)$ is as in Sect. 3.3.3.

Let $\mathrm{ComGrp}(\mathbf{C}) \subset \mathrm{ComMonoid}(\mathbf{C})$ be the full subcategory of group-like objects[8]. The following assertion is immediate (it happens at the level of the underlying triangulated category):

Lemma 5.1.9. *For a stable category* $\mathbf{C}$*, the inclusion* $\mathrm{ComGrp}(\mathbf{C}) \hookrightarrow \mathrm{ComMonoid}(\mathbf{C})$ *is an equivalence.*

5.1.10. Let now F be a functor $\mathbf{C} \to \mathbf{D}$, where $\mathbf{C}$ is stable and $\mathbf{D}$ is an $(\infty, 1)$-category with Cartesian products. Assume that F preserves finite products. We obtain that F canonically factors as

$$\mathbf{C} \to \mathrm{ComGrp}(\mathbf{D}) \overset{\mathbf{oblv}_{\mathrm{ComGrp}}}{\longrightarrow} \mathbf{D},$$

where $\mathbf{oblv}_{\mathrm{ComGrp}}$ denotes the tautological forgetful functor.

[8]A (commutative) monoid in an $(\infty, 1)$-category is said to be *group-like*, if it is is such in the corresponding ordinary category.

5.2. **The 2-categorical structure.** In the later chapters in this book (specifically, for the formalism of IndCoh as a functor out of the category of correspondences), we will need to consider the $(\infty,2)$-categorical enhancement of the totality of stable categories.

We refer the reader to [Chapter A.1, Sect. 2], where the notion of $(\infty,2)$-category is introduced, along with the corresponding terminology.

This subsection could (and, probably, should) be skipped on the first pass.

5.2.1. The structure of $(\infty,1)$-category on 1-Cat naturally upgrades to a structure of $(\infty,2)$-category, denoted **1 - Cat**, see [Chapter A.1, Sect. 2.4].

We let $\mathbf{1\text{-}Cat}^{\mathrm{St}}$ be the 1-full subcategory of **1 - Cat**, where we restrict objects to stable categories, and 1-morphisms to exact functors.

We let

$$\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}} \subset \mathbf{1\text{-}Cat}^{\mathrm{St}}$$

be the full subcategory where we restrict objects to be cocomplete stable categories.

Let

$$\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}} \subset \mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}$$

be the 1-full subcategory, where we restrict 1-morphisms to exact functors that are continuous.

5.2.2. Explicitly, the $(\infty,2)$-category **1 - Cat** is defined in [Chapter A.1, Sect. 2.4] as follows:

The simplicial $(\infty,1)$-category $\mathrm{Seq}_\bullet(\mathbf{1\text{-}Cat})$ is defined so that each $\mathrm{Seq}_n(\mathbf{1\text{-}Cat})$ is the 1-full subcategory of $\mathrm{Cart}_{/[n]^{\mathrm{op}}}$, where we restrict 1-morphisms to those functors between $(\infty,1)$-categories over $[n]^{\mathrm{op}}$ that induce an equivalence on the fiber over each $i \in [n]^{\mathrm{op}}$.

Then

$$\mathrm{Seq}_n(\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}) \subset \mathrm{Seq}_n(\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}) \subset \mathrm{Seq}_n(\mathbf{1\text{-}Cat}^{\mathrm{St}})$$

are the full subcategories of $\mathrm{Seq}_n(\mathbf{1\text{-}Cat})$ defined by the following conditions:

We take those $(\infty,1)$-categories $\mathbf{C}$ equipped with a Cartesian fibration over $[n]^{\mathrm{op}}$ for which:

- In all three cases, we require that for every $i = 0, ..., n$, the $(\infty,1)$-category $\mathbf{C}_i$ be stable, and in the case of $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}$ and $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ that it be cocomplete;
- In all three cases, we require that for every $i = 1, ..., n$ the corresponding functor $\mathbf{C}_{i-1} \to \mathbf{C}_i$ be exact, and in the case of $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ that it be continuous.

5.2.3. By construction, we have:

$$\mathbf{Maps}_{\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{D},\mathbf{C}) = \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{D},\mathbf{C}), \quad \mathbf{C},\mathbf{D} \in 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$$

and

$$\mathbf{Maps}_{\mathbf{1\text{-}Cat}^{\mathrm{St},}}(\mathbf{D},\mathbf{C}) = \mathrm{Funct}_{\mathrm{ex}}(\mathbf{D},\mathbf{C}), \quad \mathbf{C},\mathbf{D} \in 1\text{-}\mathrm{Cat}^{\mathrm{St}}.$$

5.3. **Some residual 2-categorical features.** The $(\infty,2)$-categories introduced in Sect. 5.2.1 allow to assign an intrinsic meaning to the notion of adjunction of (various classes of) functors between (various classes of) stable categories.

We will exploit this in the present subsection.

We note, however, that, unlike Sect. 5.2, the constructions here are not esoteric, but are of direct practical import (e.g., the notion of exact monad).

5.3.1. According to [Chapter A.3, Sect. 1], it makes sense to ask whether a 1-morphism $F : \mathbf{C} \to \mathbf{D}$ in $\mathbf{1\text{-}Cat}^{\mathrm{St}}$ (resp., $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}$, $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$) admits a right adjoint 1-morphism within the corresponding $(\infty, 2)$-category.

The following results from Theorem 2.5.4(a):

Lemma 5.3.2. *Let $F : \mathbf{C} \to \mathbf{D}$ be a morphism in $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.*

(a) *The right adjoint of F always exists as a 1-morphism in $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}$.*

(b) *The right adjoint from* (a)*, when viewed as a functor between plain $(\infty, 1)$-categories, is the right adjoint F^R of F, when the latter is viewed also as a functor between plain $(\infty, 1)$-categories.*

(c) *The right adjoint of F exists in $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ if and only if F^R preserves filtered colimits (equivalently, all colimits or direct sums).*

(d) *If a 1-morphism in $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}$ admits a left adjoint (as a plain functor), then this left adjoint is automatically a 1-morphism in $\mathbf{1\text{-}Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.*

5.3.3. In particular, to any functor

$$\mathbf{C}_{\mathbf{I}} : \mathbf{I} \to 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$$

one can associate a functor

$$\mathbf{C}^R_{\mathbf{I}^{\mathrm{op}}} : \mathbf{I}^{\mathrm{op}} \to 1\text{-Cat}^{\mathrm{St,cocmpl}},$$

obtained by passing to right adjoints.

The following is a formal consequence of Lemma 2.5.2 and Proposition 2.5.7:

Corollary 5.3.4.

(a) *The $(\infty, 1)$-category $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ contains limits and colimits, and the functor*

$$1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}} \to 1\text{-Cat}^{\mathrm{St,cocmpl}}$$

preserves limits.

(b) *Let $\mathbf{C}_{\mathbf{I}} : \mathbf{I} \to 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ be a functor and let $\mathbf{C}_*$ denote its colimit in $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$. Let $\mathbf{C}^R_{\mathbf{I}^{\mathrm{op}}} : \mathbf{I}^{\mathrm{op}} \to 1\text{-Cat}^{\mathrm{St,cocmpl}}$ be the functor obtained from $\mathbf{C}_{\mathbf{I}}$ by passing to right adjoints. Then the resulting map in $1\text{-Cat}^{\mathrm{St,cocmpl}}$*

$$\mathbf{C}_* \to \lim_{\mathbf{I}^{\mathrm{op}}} \mathbf{C}^R_{\mathbf{I}^{\mathrm{op}}}$$

is an isomorphism.

5.3.5. Given $\mathbf{C} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ we can consider the monoidal $(\infty, 1)$-category

$$\mathbf{Maps}_{1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{C}, \mathbf{C}), \tag{5.2}$$

which is equipped with an action on

$$\mathbf{Maps}_{1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{D}, \mathbf{C}),$$

for any $\mathbf{D} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, see [Chapter V.3, Sect. 4.1.1] where the general paradigm is explained.

In particular, we can talk about the $(\infty, 1)$-category of *exact continuous* monads acting on $\mathbf{C}$, which are by definition associative algebra objects in the monoidal $(\infty, 1)$-category (5.2).

5.3.6. Given a monad $\mathcal{A}$, we can consider the $(\infty, 1)$-category

$$\mathcal{A}\text{-mod}(\mathbf{C})$$

in the sense of Sect. 3.7.2.

The category $\mathcal{A}$-mod$(\mathbf{C})$ is itself an object of $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ and the adjoint pair

$$\mathbf{ind}_{\mathcal{A}} : \mathbf{C} \rightleftarrows \mathcal{A}\text{-mod}(\mathbf{C}) : \mathbf{oblv}_{\mathcal{A}}$$

takes place in $\mathbf{1\text{-}Cat}^{\text{St,cocmpl}}_{\text{cont}}$.

For a 1-morphism $G \in \mathbf{Maps}_{1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}}(\mathbf{D}, \mathbf{C})$ the datum of action of $\mathcal{A}$ on G is equivalent to that of factoring G as

$$\mathbf{oblv}_{\mathcal{A}} \circ G^{\text{enh}}, \quad G^{\text{enh}} \in \mathbf{Maps}_{1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}}(\mathbf{D}, \mathcal{A}\text{-mod}(\mathbf{C})).$$

5.3.7. Let $G : \mathbf{D} \to \mathbf{C}$ be as above, and assume that it admits a left adjoint G^L as a plain functor. Recall (see Lemma 5.3.2) that G^L is then automatically a 1-morphism in $\mathbf{1\text{-}Cat}^{\text{St,cocmpl}}_{\text{cont}}$. Consider the corresponding monad

$$\mathcal{A} := G \circ G^L,$$

see Sect. 3.7.4, so that G gives rise to a 1-morphism in $\mathbf{1\text{-}Cat}^{\text{St,cocmpl}}_{\text{cont}}$:

$$G^{\text{enh}} : \mathbf{D} \to \mathcal{A}\text{-mod}(\mathbf{C}).$$

The following is an immediate consequence of Proposition 3.7.7:

Corollary 5.3.8. *Suppose that G does not send non-zero objects to zero. Then G^{enh} is an equivalence.*

5.4. **Generation.**

5.4.1. Let $\mathbf{C}$ be an object of $1\text{-Cat}^{\text{St,cocmpl}}$. A collection of objects $\{\mathbf{c}_\alpha\}$ is said to *generate* if

$$\text{Maps}_{\mathbf{C}}(\mathbf{c}_\alpha[-i], \mathbf{c}) = *, \ \forall \alpha, \ \forall i = 0, 1, ... \ \Rightarrow \ \mathbf{c} = 0,$$

where $[-i]$ denotes the shift functor on $\mathbf{C}$, i.e., the i-fold loop functor Ω^i.

5.4.2. The following is tautological:

Lemma 5.4.3. *Let $\mathbf{D}$ be a (not necessarily stable) $(\infty, 1)$-category, and let $F : \mathbf{D} \to \mathbf{C}$ be a functor that admits a right adjoint, and whose essential image is preserved by the loop functor. Then the essential image of F generates $\mathbf{C}$ if and only if its right adjoint F^R is conservative.*

5.4.4. We have the following basic statement:

Proposition 5.4.5. *A collection $\{\mathbf{c}_\alpha\}$ of objects generates $\mathbf{C}$ if and only if $\mathbf{C}$ does not properly contain a cocomplete stable subcategory that contains all $\mathbf{c}_\alpha$.*

Proof. Let $\mathbf{C}'$ be the smallest cocomplete stable full subcategory of $\mathbf{C}$ that contains the objects of the form $\mathbf{c}_\alpha$. The inclusion $\iota : \mathbf{C}' \hookrightarrow \mathbf{C}$ admits an (*a priori* non-continuous) right adjoint, denoted ι^R. Set $\mathbf{C}'' := \ker(\iota^R)$.

The inclusion

$$\mathbf{C} \overset{\jmath}{\hookleftarrow} \mathbf{C}''$$

admits a left adjoint $\jmath^L$, given by

$$\mathbf{c} \mapsto \text{coFib}(\iota \circ \iota^R(\mathbf{c}) \to \mathbf{c}).$$

By definition,

$$\mathbf{c} \in \mathbf{C}'' \Leftrightarrow \mathbf{c} \in (\mathbf{C}')^{\perp} \Leftrightarrow \mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_\alpha[-i], \mathbf{c}) = 0, \ \forall \alpha, \ \forall i = 0, 1, ...$$

Now it is clear that the inclusion $\imath$ is an equivalence if and only if $\jmath^L$ is zero if and only if $\mathbf{C}'' = 0$.

□

5.4.6. Finally, we have:

Proposition 5.4.7. *Let $F : \mathbf{D} \to \mathbf{C}$ be a continuous functor. Then its essential image generates the target (i.e., $\mathbf{C}$) if and only if for any continuous functor $G : \mathbf{C} \to \mathbf{C}'$ with $G \circ F = 0$ we have $G = 0$.*

Proof. We first prove the 'only if' direction. Assume that the essential image of F generates $\mathbf{C}$, and let $G : \mathbf{C} \to \mathbf{C}'$ be such that $F \circ G = 0$. Since G is continuous it admits a (possibly discontinuous) right adjoint G^R, and it suffices to show that $G^R = 0$. Since F^R is conservative, it suffices to show that $F^R \circ G^R = 0$. However, the latter identifies with $(G \circ F)^R$, which vanishes by assumption.

We now prove the 'if' direction. Let $\mathbf{C}' \subset \mathbf{C}$ be the full subcategory, generated by the essential image of F (i.e., the smallest stable cocomplete subcategory of $\mathbf{C}$ that contains the essential image of F). Let $(\mathbf{C}'', \imath, \jmath)$ be as in the proof of Proposition 5.4.5.

Being a left adjoint, $\jmath^L$ preserves colimits. Hence, the fact that $\mathbf{C}$ is cocomplete implies that $\mathbf{C}''$ is cocomplete (and $\jmath^L$ is continuous).

Now, by the construction of $\mathbf{C}'$, the composition $F \circ \jmath^L$ is zero. Hence, $\jmath^L = 0$, i.e., $\imath$ is an equivalence.

□

6. The symmetric monoidal structure on $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$

In this section we will discuss some of the key features of the $(\infty, 1)$-category $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$: the symmetric monoidal structure, given by *tensor product of stable categories*, which we call the *Lurie tensor product*, and the notion of dualizable stable category.

In the process we will encounter the most basic stable category–that of *spectra*.

This section can be regarded as a user guide to [Lu2, 1.4 and 4.8].

6.1. **The Lurie tensor product.** In this subsection we introduce the Lurie tensor product.

It is quite remarkable that one does not have to work very hard in order to characterize it uniquely: for a pair of stable categories $\mathbf{C}_1$ and $\mathbf{C}_2$ and a third one $\mathbf{D}$, the space of exact continuous functors

$$\mathbf{C}_1 \otimes \mathbf{C}_2 \to \mathbf{D}$$

is a full subspace in

$$\mathrm{Maps}_{1\text{-Cat}}(\mathbf{C}_1 \times \mathbf{C}_2, \mathbf{D})$$

that consists of functors that are exact and continuous in each variable.

I.e., one does not need to introduce any additional pieces of structure, but rather impose conditions.

6.1.1. Consider the coCartesian fibration

$$1\text{-Cat}^{\times,\mathrm{Fin}_*} \to \mathrm{Fin}_*,$$

corresponding to the Cartesian symmetric monoidal structure on 1-Cat.

We let

$$(1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}})^{\otimes,\mathrm{Fin}_*} \subset 1\text{-Cat}^{\times,\mathrm{Fin}_*}$$

be the 1-full subcategory, where:

- We restrict objects to those

$$(I,*), \quad (i \in I - \{*\}) \mapsto (\mathbf{C}_i \in 1\text{-Cat}),$$

  where each $\mathbf{C}_i$ is stable and cocomplete;

- We restrict morphisms to those

$$\phi : (I,*) \to (J,*), \quad (j \in J - \{*\}) \mapsto (\underset{i\in\phi^{-1}(j)}{\Pi} \mathbf{C}_i \overset{F_j}{\to} \mathbf{C}_j),$$

  where each F_j is exact and continuous in each variable.

Theorem 6.1.2. *The composite functor* $(1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}})^{\otimes,\mathrm{Fin}_*} \to \mathrm{Fin}_*$ *is a coCartesian fibration, that lies in the essential image of the fully faithful functor*

$$1\text{-}\mathrm{Cat}^{\mathrm{SymMon}} \to (\mathrm{coCart}_{/\,\mathrm{Fin}_*})_{\mathrm{strict}}.$$

This theorem is a combination of [Lu2, Propositions 4.8.1.3, 4.8.1.14 and 4.8.1.18].

6.1.3. It follows from Theorem 6.1.2 that the $(\infty,1)$-category $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ of stable categories acquires a symmetric monoidal structure. We will refer to it as the *Lurie symmetric monoidal structure.*

The corresponding monoidal operation, denoted

$$(\mathbf{C}_i, i\in I) \mapsto \underset{i\in I}{\otimes} \mathbf{C}_i$$

is the *Lurie tensor product.*

6.1.4. By construction, for $\mathbf{D} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ the space of exact continuous functors

$$\underset{i\in I}{\otimes} \mathbf{C}_i \to \mathbf{D}$$

is the full subspace in the space of functors

$$\underset{i\in I}{\Pi} \mathbf{C}_i \to \mathbf{D}$$

that are exact and continuous in each variable.

It follows from the above description and Proposition 2.5.7 that the monoidal operation:

$$\{\mathbf{C}_i\} \mapsto \underset{i\in I}{\otimes} \mathbf{C}_i$$

preserves colimits (taken in $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$) in each variable.

Remark 6.1.5. A remarkable aspect of this theory is that Theorem 6.1.2 is not very hard. The existence of the tensor product $\underset{i\in I}{\otimes}\mathbf{C}_i$ follows from the Adjoint Functor Theorem. The fact tat the canonical functor

$$\mathbf{C}_1 \otimes (\mathbf{C}_2 \otimes \mathbf{C}_3) \to \mathbf{C}_1 \otimes \mathbf{C}_2 \otimes \mathbf{C}_3$$

follows by interpreting exact continuous functors

$$\mathbf{C}_1 \otimes (\mathbf{C}_2 \otimes \mathbf{C}_3) \to \mathbf{D}$$

as exact continuous functors

$$\mathbf{C}_2 \otimes \mathbf{C}_3 \to \text{Funct}_{\text{ex,cont}}(\mathbf{C}_1, \mathbf{D}).$$

6.1.6. By construction, we have a tautological functor

$$\prod_{i\in I} \mathbf{C}_i \to \underset{i\in I}{\otimes} \mathbf{C}_i, \tag{6.1}$$

which is exact and continuous in each variable.

For $\mathbf{c}_i \in \mathbf{C}_i$, we let

$$\underset{i}{\boxtimes}\, \mathbf{c}_i \in \underset{i\in I}{\otimes} \mathbf{C}_i$$

denote the image of the object $(\underset{i}{\times}\, \mathbf{c}_i) \in \prod_{i\in I} \mathbf{C}_i$ under the functor (6.1).

6.1.7. Note that for $\mathbf{C}, \mathbf{D} \in 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ the object

$$\text{Funct}_{\text{ex,cont}}(\mathbf{D}, \mathbf{C}) \in 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$$

(see Sect. 5.1.7) identifies with the inner Hom object

$$\underline{\text{Hom}}_{1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}}(\mathbf{D}, \mathbf{C}).$$

I.e., for $\mathbf{E} \in 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ we have a canonical isomorphism

$$\text{Maps}_{1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}}(\mathbf{E} \otimes \mathbf{D}, \mathbf{C}) \simeq \text{Maps}_{1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}}(\mathbf{E}, \text{Funct}_{\text{ex,cont}}(\mathbf{D}, \mathbf{C})).$$

6.1.8. *The symmetric monoidal structure on* $\mathbf{1\text{-}Cat}^{\text{St,cocmpl}}_{\text{cont}}$. For future use, we note that the structure of symmetric monoidal $(\infty, 1)$-category on $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ canonically extends to that of symmetric monoidal $(\infty, 2)$-category on the 2-categorical enhancement of $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$, i.e., $\mathbf{1\text{-}Cat}^{\text{St,cocmpl}}_{\text{cont}}$ (see [Chapter V.3, Sect. 1.4] for the notion of symmetric monoidal structure on an $(\infty, 2)$-category.

Indeed, we start with the 2-coCartesian fibration $\mathbf{1\text{-}Cat}^{\otimes,\text{Fin}_*} \to \text{Fin}_*$ that defines the Cartesian symmetric monoidal structure on $\mathbf{1\text{-}Cat}$.

Note that

$$\left(\mathbf{1\text{-}Cat}^{\otimes,\text{Fin}_*}\right)^{1\text{-Cat}} \simeq 1\text{-Cat}^{\otimes,\text{Fin}_*}.$$

We let

$$\left(\mathbf{1\text{-}Cat}^{\text{St,cocmpl}}_{\text{cont}}\right)^{\otimes,\text{Fin}_*}$$

be the 1-full subcategory of $\mathbf{1\text{-}Cat}^{\otimes,\text{Fin}_*}$ that corresponds to

$$\left(1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}\right)^{\otimes,\text{Fin}_*} \subset 1\text{-Cat}^{\otimes,\text{Fin}_*}.$$

One checks that the composite functor

$$\left(\mathbf{1\text{-}Cat}^{\text{St,cocmpl}}_{\text{cont}}\right)^{\otimes,\text{Fin}_*} \to \mathbf{1\text{-}Cat}^{\otimes,\text{Fin}_*} \to \text{Fin}_*$$

is a 2-coCartesian fibration, and as such defines a symmetric monoidal structure on the $(\infty, 2)$-category $\mathbf{1}\text{-}\mathbf{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

6.2. **The $(\infty, 1)$-category of spectra.** The symmetric monoidal structure on $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ leads to a concise definition of the $(\infty, 1)$-category of spectra, along with (some of) its key features.

6.2.1. The $(\infty, 1)$-category Sptr of spectra can be defined as the unit object in the symmetric monoidal $(\infty, 1)$-category $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

Let $\mathbf{1}_{\mathrm{Sptr}}$ denote the unit object in Sptr. This is the *sphere spectrum*.

6.2.2. Recall the setting of Lemma 2.1.8. We obtain that the object $\mathbf{1}_{\mathrm{Sptr}} \in \mathrm{Sptr}$ gives rise to a functor

$$\mathrm{Spc} \to \mathrm{Sptr}\,.$$

We denote this functor by Σ^∞.

6.2.3. The functor Σ^∞ has the following universal property (see [Lu2, Corollary 1.4.4.5]):

Lemma 6.2.4. *For $\mathbf{C} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, restriction and left Kan extension along Σ^∞ define an equivalence between $\mathrm{Funct}_{\mathrm{ex,cont}}(\mathrm{Sptr}, \mathbf{C})$ and the full subcategory of $\mathrm{Funct}(\mathrm{Spc}, \mathbf{C})$ consisting of colimit-preserving functors.*

The above lemma expresses the universal property of the category Sptr as the *stabilization* of Spc.

6.2.5. Combining Lemmas 6.2.4 and 2.1.8 we obtain:

Corollary 6.2.6. *For $\mathbf{C} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, restriction and left Kan extension along*

$$\{\mathbf{1}_{\mathrm{Sptr}}\} \hookrightarrow \mathrm{Sptr}$$

define an equivalence

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathrm{Sptr}, \mathbf{C}) \simeq \mathbf{C}.$$

6.2.7. The functor Σ^∞ admits a right adjoint, denoted Ω^∞. By Sect. 5.1.10, the functor Ω^∞ canonically factors via a functor

$$\mathrm{Sptr} \to \mathrm{ComGrp}(\mathrm{Spc}), \tag{6.2}$$

followed by the forgetful functor

$$\mathrm{ComGrp}(\mathrm{Spc}) \overset{\mathbf{oblv}_{\mathrm{ComGrp}}}{\longrightarrow} \mathrm{Spc}\,.$$

The functor Ω^∞ preserves filtered colimits.

6.2.8. The stable category Sptr has a t-structure, uniquely determined by the condition that an object $\mathcal{S} \in \mathrm{Sptr}$ is strictly *coconnective*, i.e., belongs to $\mathrm{Sptr}^{>0}$, if and only if $\Omega^\infty(\mathcal{S}) = *$; see [Lu2, Proposition 1.4.3.6].

The t-structure on Sptr is both left and right complete. This means that for $\mathcal{S} \in \mathrm{Sptr}$ the canonical maps

$$\mathcal{S} \mapsto \lim_n \tau^{\geq -n}(\mathcal{S}) \text{ and } \operatorname*{colim}_n \tau^{\leq n}(\mathcal{S}) \to \mathcal{S}$$

are isomorphisms.

6.2.9. The restriction of the functor (6.2) to the full subcategory of *connective spectra*

$$\mathrm{Sptr}^{\leq 0} \subset \mathrm{Sptr}$$

defines an equivalence

$$\mathrm{Sptr}^{\leq 0} \to \mathrm{ComGrp}(\mathrm{Spc}); \tag{6.3}$$

see [Lu2, Theorem 5.2.6.10] (this statement goes back to [May] and [BoV]).

6.2.10. Let $\mathbf{C}$ be an object of $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$. Since Sptr is the unit object in the symmetric monoidal category $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, our $\mathbf{C}$ has a canonical structure of Sptr-module category.

For $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$, consider the corresponding relative inner Hom object

$$\underline{\mathrm{Hom}}_{\mathrm{Sptr}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Sptr}$$

see Sect. 3.6.1.

We will also use the notation

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) := \underline{\mathrm{Hom}}_{\mathrm{Sptr}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Sptr}\,.$$

I.e., for $\mathcal{S} \in \mathrm{Sptr}$ we have

$$\mathrm{Maps}_{\mathrm{Sptr}}(\mathcal{S}, \mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)) \simeq \mathrm{Maps}_{\mathbf{C}}(\mathcal{S} \otimes \mathbf{c}_0, \mathbf{c}_1).$$

By adjunction, we have

$$\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \simeq \Omega^\infty(\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)).$$

6.3. **Duality of stable categories.** Since $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ has a symmetric monoidal structure, we can talk about dualizable objects in it, see Sect. 4.1.1. Thus, we arrive at the notion of *dualizable cocomplete stable category*. In the same vein, we can talk about the datum of duality between two objects of $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

These notions turn out to be immensely useful in practice.

6.3.1. By definition, a duality datum between $\mathbf{C}$ and $\mathbf{D}$ is the datum of a morphism

$$\epsilon : \mathbf{C} \otimes \mathbf{D} \to \mathrm{Sptr} \text{ and } \mu : \mathrm{Sptr} \to \mathbf{D} \otimes \mathbf{C},$$

such that the composition

$$\mathbf{C} \overset{\mathrm{Id}_{\mathbf{C}} \otimes \mu}{\longrightarrow} \mathbf{C} \otimes \mathbf{D} \otimes \mathbf{C} \overset{\epsilon \otimes \mathrm{Id}_{\mathbf{C}}}{\longrightarrow} \mathbf{C}$$

is isomorphic to $\mathrm{Id}_{\mathbf{C}}$, and the composition

$$\mathbf{D} \overset{\mu \otimes \mathrm{Id}_{\mathbf{D}}}{\longrightarrow} \mathbf{D} \otimes \mathbf{C} \otimes \mathbf{D} \overset{\mathrm{Id}_{\mathbf{C}} \otimes \epsilon}{\longrightarrow} \mathbf{D}$$

is isomorphic to $\mathrm{Id}_{\mathbf{D}}$.

6.3.2. Let $\mathbf{C}$ and $\mathbf{D}$ be dualizable objects in $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, and let $\mathbf{C}^\vee$ and $\mathbf{D}^\vee$ denote their respective duals.

For a continuous functor $F : \mathbf{C} \to \mathbf{D}$, we denote by $F^\vee : \mathbf{D}^\vee \to \mathbf{C}^\vee$ the dual functor (see Sect. 4.1.4). Explicitly, $F^\vee$ is given as the composition

$$\mathbf{D}^\vee \overset{\mu_{\mathbf{C}} \otimes \mathrm{Id}_{\mathbf{D}^\vee}}{\longrightarrow} \mathbf{C}^\vee \otimes \mathbf{C} \otimes \mathbf{D}^\vee \overset{\mathrm{Id}_{\mathbf{C}^\vee} \otimes F \otimes \mathrm{Id}_{\mathbf{D}^\vee}}{\longrightarrow} \mathbf{C}^\vee \otimes \mathbf{D} \otimes \mathbf{D}^\vee \overset{\mathrm{Id}_{\mathbf{C}^\vee} \otimes \epsilon_{\mathbf{D}}}{\longrightarrow} \mathbf{C}^\vee.$$

By Sect. 4.1.5 and Sect. 6.1.7, we have the canonical isomorphisms

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{C}, \mathbf{D}) \simeq \mathbf{C}^\vee \otimes \mathbf{D} \simeq \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{D}^\vee, \mathbf{C}^\vee).$$

6.3.3. Let us again be in the situation of Sect. 5.3.3. Assume that all the objects $\mathbf{C}_i$ are dualizable, and that the right adjoints of the transition functors $\mathbf{C}_i \to \mathbf{C}_j$ are continuous.

By applying the dualization functor (see Sect. 4.1.4), from

$$\mathbf{C}_{\mathbf{I}} : \mathbf{I} \to 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}},$$

we obtain another functor, denoted

$$\mathbf{C}^{\vee}_{\mathbf{I}^{\text{op}}} : \mathbf{I}^{\text{op}} \to 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}, \quad i \mapsto \mathbf{C}^{\vee}_i.$$

We claim:

Proposition 6.3.4. *Under the above circumstances, the object $\mathbf{C}_*$ is dualizable, and the dual of the colimit diagram*

$$\underset{\mathbf{I}}{\text{colim}}\, \mathbf{C}_{\mathbf{I}} \to \mathbf{C}_*$$

is a limit diagram, i.e., the map

$$(\mathbf{C}_*)^{\vee} \to \underset{\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_{\mathbf{I}^{\text{op}}}$$

is an equivalence.

The rest of this subsection is devoted to the proof of this proposition.

6.3.5. We will construct the duality datum between $\underset{\mathbf{I}}{\text{colim}}\, \mathbf{C}_{\mathbf{I}}$ and $\underset{\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_{\mathbf{I}^{\text{op}}}$.

The functor

$$\epsilon : \left(\underset{\mathbf{I}}{\text{colim}}\, \mathbf{C}_{\mathbf{I}}\right) \otimes \left(\underset{\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_{\mathbf{I}^{\text{op}}}\right) \to \text{Sptr}$$

is given as follows:

Since

$$\left(\underset{i\in\mathbf{I}}{\text{colim}}\, \mathbf{C}_i\right) \otimes \left(\underset{j\in\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_j\right) \simeq \underset{i\in\mathbf{I}}{\text{colim}}\, \left(\mathbf{C}_i \otimes \left(\underset{j\in\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_j\right)\right),$$

the datum of ϵ is equivalent to a compatible family of functors

$$\mathbf{C}_i \otimes \left(\underset{j\in\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_j\right) \to \text{Sptr}, \quad i \in \mathbf{I}.$$

The latter are given by

$$\mathbf{C}_i \otimes \left(\underset{j\in\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_j\right) \overset{\text{Id}_{\mathbf{C}_i} \otimes \text{ev}_i}{\longrightarrow} \mathbf{C}_i \otimes \mathbf{C}^{\vee}_i \overset{\epsilon_{\mathbf{C}_i}}{\to} \text{Sptr},$$

where ev_i denotes the evaluation functor $\underset{j\in\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_j \to \mathbf{C}^{\vee}_i$.

6.3.6. Let us now construct the functor

$$\mu : \text{Sptr} \to \left(\underset{\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_{\mathbf{I}^{\text{op}}}\right) \otimes \left(\underset{\mathbf{I}}{\text{colim}}\, \mathbf{C}_{\mathbf{I}}\right).$$

For this we note that the functor $\mathbf{C}^{\vee}_{\mathbf{I}^{\text{op}}}$ is obtained by passing to *right adjoints* from a functor $\mathbf{I} \to 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$, which in turn is given by passing to right adjoints in $\mathbf{C}_{\mathbf{I}}$, and then passing to the duals.

Hence, by Corollary 5.3.4, the limit $\underset{\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_{\mathbf{I}^{\text{op}}}$ can be rewritten as a colimit. Hence, for any $\mathbf{D} \in 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ the natural map

$$\mathbf{D} \otimes \left(\underset{j\in\mathbf{I}^{\text{op}}}{\lim}\, \mathbf{C}^{\vee}_j\right) \to \underset{j\in\mathbf{I}^{\text{op}}}{\lim}\, \left(\mathbf{D} \otimes \mathbf{C}^{\vee}_j\right)$$

is an equivalence.

Hence,

$$\left(\lim_{j\in\mathbf{I}^{\mathrm{op}}} \mathbf{C}_j^\vee\right)\otimes\left(\operatorname{colim}_{i\in\mathbf{I}} \mathbf{C}_i\right)\simeq \lim_{j\in\mathbf{I}^{\mathrm{op}}}\left(\mathbf{C}_j^\vee\otimes\left(\operatorname{colim}_{i\in\mathbf{I}} \mathbf{C}_i\right)\right).$$

Therefore, the datum of μ amounts to a compatible family of functors

$$\mathrm{Sptr}\to \mathbf{C}_j^\vee\otimes\left(\operatorname{colim}_{i\in\mathbf{I}} \mathbf{C}_i\right),\quad j\in\mathbf{I}.$$

The latter are given by

$$\mathrm{Sptr}\overset{\mu_{\mathbf{C}_j}}{\longrightarrow}\mathbf{C}_j^\vee\otimes\mathbf{C}_j\overset{\mathrm{Id}_{\mathbf{C}_j}\otimes\,\mathrm{ins}_j}{\longrightarrow}\mathbf{C}_j^\vee\otimes\left(\operatorname{colim}_{i\in\mathbf{I}} \mathbf{C}_i\right),$$

where ins_j denotes the insertion functor $\mathbf{C}_j\to\operatorname{colim}_{i\in\mathbf{I}}\mathbf{C}_i$.

6.3.7. The fact that the functors ϵ and μ constructed above satisfy the adjunction identities is a straightforward verification.

6.4. **Generation of tensor products.**

6.4.1. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be objects of $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}$, and consider their tensor product

$$\mathbf{C}_1\otimes\mathbf{C}_2.$$

We have the following basic fact:

Proposition 6.4.2. *Let $F_i:\mathbf{D}_i\to\mathbf{C}_i$, $i=1,2$ be continuous functors, such that their respective essential images generate the target. The essential image of the tautological functor*

$$\otimes:\mathbf{C}_1\times\mathbf{C}_2\to\mathbf{C}_1\otimes\mathbf{C}_2,\quad \mathbf{c}_1\times\mathbf{c}_2\mapsto\mathbf{c}_1\boxtimes\mathbf{c}_2$$

generates the target.

Proof. Let $\mathbf{C}'$ be the smallest cocomplete stable full subcategory of $\mathbf{C}:=\mathbf{C}_1\otimes\mathbf{C}_2$ that contains the objects of the form $\mathbf{c}_1\boxtimes\mathbf{c}_2$. Recall the notations in the proof of Proposition 5.4.5.

Being a left adjoint, $\jmath^L$ preserves colimits. Hence, the fact that $\mathbf{C}$ is cocomplete implies that $\mathbf{C}''$ is cocomplete.

We need to show that $\mathbf{C}''=0$, which is equivalent to the functor $\jmath^L$ being zero. By the universal property of $\mathbf{C}_1\otimes\mathbf{C}_2$, the latter is equivalent to the fact that the composition

$$\mathbf{C}_1\times\mathbf{C}_2\to\mathbf{C}_1\otimes\mathbf{C}_2=\mathbf{C}\overset{\jmath^L}{\longrightarrow}\mathbf{C}''$$

maps to the zero object of $\mathbf{C}''$.

However, the latter composition factors as

$$\mathbf{C}_1\times\mathbf{C}_2\to\mathbf{C}'\overset{\iota}{\hookrightarrow}\mathbf{C}\overset{\jmath^L}{\longrightarrow}\mathbf{C}'',$$

while $\jmath^L\circ\iota$ is tautologically 0.

□

In addition, we have:

Proposition 6.4.3. *Let $F_i:\mathbf{D}_i\to\mathbf{C}_i$, $i=1,2$ be continuous functors, such that their respective essential images generate the target. Then the same is true for*

$$F_1\otimes F_2:\mathbf{D}_1\otimes\mathbf{D}_2\to\mathbf{C}_1\otimes\mathbf{C}_2.$$

Proof. By Proposition 5.4.7, it is enough to show that for a continuous functor

$$G : \mathbf{C}_1 \otimes \mathbf{C}_2 \to \mathbf{C}',$$

if the composition $G \circ (F_1 \otimes F_2)$ is zero, then $G = 0$. Thus, we have to show that for a fixed $\mathbf{c}_1 \in \mathbf{C}_1$, the functor

$$G(\mathbf{c}_1 \otimes -) : \mathbf{C}_2 \to \mathbf{C}'$$

is zero. By Proposition 5.4.7, it suffices to show that $G(\mathbf{c}_1 \otimes F_2(\mathbf{d}_2)) = 0$ for any $\mathbf{d}_2 \in \mathbf{D}_2$. I.e., it suffices to show that the functor

$$G(- \otimes F_2(\mathbf{d}_2)) : \mathbf{C}_1 \to \mathbf{C}'$$

is zero (for a fixed $\mathbf{d}_2 \in \mathbf{D}_2$).

Applying Proposition 5.4.7 again, we obtain that it suffices to show that $G(F_1(\mathbf{d}_1) \otimes F_2(\mathbf{d}_2))$ is zero for any $\mathbf{d}_1 \in \mathbf{D}_1$. However, the latter is just the assumption that $G \circ (F_1 \otimes F_2) = 0$. □

6.4.4. Consider the following situation: let $\mathbf{C}_i$, $i = 1, 2$ be objects of $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$, and let

$$\mathcal{A}_i \in \text{AssocAlg}(\text{Funct}_{\text{ex,cont}}(\mathbf{C}_i, \mathbf{C}_i))$$

be a monad acting on $\mathbf{C}$.

Consider the monad

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \in \text{AssocAlg}(\text{Funct}_{\text{ex,cont}}(\mathbf{C}_1 \otimes \mathbf{C}_2, \mathbf{C}_1 \otimes \mathbf{C}_2)).$$

The tautological action of $\mathcal{A}_i$ on $\mathbf{oblv}_{\mathcal{A}_i} \in \text{Funct}_{\text{ex,cont}}(\mathcal{A}_i\text{-mod}(\mathbf{C}_i), \mathbf{C}_i)$ induces an action of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on

$$\mathbf{oblv}_{\mathcal{A}_1} \otimes \mathbf{oblv}_{\mathcal{A}_2} \in \text{Funct}_{\text{ex,cont}}(\mathcal{A}_1\text{-mod}(\mathbf{C}_1) \otimes \mathcal{A}_2\text{-mod}(\mathbf{C}_2), \mathcal{A}_1\text{-mod}(\mathbf{C}_1) \otimes \mathcal{A}_2\text{-mod}(\mathbf{C}_2)).$$

Hence, by Sect. 5.3.6, the functor $\mathbf{oblv}_{\mathcal{A}_1} \otimes \mathbf{oblv}_{\mathcal{A}_2}$ upgrades to a functor

$$\mathcal{A}_1\text{-mod}(\mathbf{C}_1) \otimes \mathcal{A}_2\text{-mod}(\mathbf{C}_2) \to (\mathcal{A}_1 \otimes \mathcal{A}_2)\text{-mod}(\mathbf{C}_1 \otimes \mathbf{C}_2). \tag{6.4}$$

We claim:

Lemma 6.4.5. *The functor* (6.4) *is an equivalence.*

Proof. The left adjoint of $\mathbf{oblv}_{\mathcal{A}_1} \otimes \mathbf{oblv}_{\mathcal{A}_2}$ is provided by

$$\mathbf{ind}_{\mathcal{A}_1} \otimes \mathbf{ind}_{\mathcal{A}_2},$$

and hence, the canonical map from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to the monad on $\mathbf{C}_1 \otimes \mathbf{C}_2$, corresponding to $\mathbf{oblv}_{\mathcal{A}_1} \otimes \mathbf{oblv}_{\mathcal{A}_2}$, is an isomorphism.

Hence, by Corollary 5.3.8, it suffices to show that the functor $\mathbf{oblv}_{\mathcal{A}_1} \otimes \mathbf{oblv}_{\mathcal{A}_2}$ is conservative. By Lemma 5.4.3, this is equivalent to the fact that the essential image of

$$\mathbf{ind}_{\mathcal{A}_1} \otimes \mathbf{ind}_{\mathcal{A}_2}$$

generates the target. This is true for each $\mathbf{ind}_{\mathcal{A}_i}$ (since $\mathbf{oblv}_{\mathcal{A}_i}$ are conservative), and hence the required assertion follows from Proposition 6.4.2. □

7. Compactly generated stable categories

Among all stable categories one singles out a class of those that are particularly manageable: these are the *compactly generated stable categories.*

One favorable property of compactly generated stable categories is that they are dualizable with a very explicit description of the dual.

Another is that the tensor product of two compactly generated categories can also be described rather explicitly.

The material in Sects. 7.1 and 7.2 is based on [Lu1, Sect. 5.3].

7.1. **Compactness.**

The notion of compactness is key for doing computations in a given stable category: we usually can calculate the mapping spaces out of compact objects.

For a related reason, compactly generated stable categories are those that we know how to calculate functors *from.*

7.1.1. Let $\mathbf{C}$ be an object of $1\text{-Cat}^{\mathrm{St,cocmpl}}$. An object $\mathbf{c} \in \mathbf{C}$ is said to be *compact* if the functor

$$\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Spc}$$

preserves filtered colimits.

Equivalently, $\mathbf{c}$ is compact if the functor

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Sptr}$$

preserves filtered colimits (equivalently, all colimits or direct sums).

We let $\mathbf{C}^c \subset \mathbf{C}$ denote the full subcategory spanned by compact objects. We have $\mathbf{C}^c \in 1\text{-Cat}^{\mathrm{St}}$.

7.1.2. We give the following definition:

Definition 7.1.3. *An object* $\mathbf{C} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}$ *is said to be* compactly generated *of it admits a set of compact generators.*

7.1.4. Let $F : \mathbf{C} \to \mathbf{D}$ be a morphism in $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, and assume that $\mathbf{C}$ is compactly generated.

In this case one can give an easy criterion for when the right adjoint F^R of F, which is *a priori* a morphism in $1\text{-Cat}^{\mathrm{St,cocmpl}}$ (see Lemma 5.3.2), is in fact a morphism in $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

Namely, we have the following (almost immediate) assertion:

Lemma 7.1.5. *Under the above circumstance,* F^R *is continuous if and only if* F *sends* $\mathbf{C}^c$ *to* $\mathbf{D}^c$.

7.2. **The operation of ind-completion.**

In the previous subsection we attached to a cocomplete stable category $\mathbf{C}$ its full subcategory consisting of compact objects.

In this subsection we will discuss the inverse procedure: starting from a non-cocomplete stable category $\mathbf{C}_0$ we will be able to canonically produce a cocomplete one by 'adding all filtered colimits'. This is the operation of *ind-completion.*

7.2.1. Let $\mathbf{C}_0$ be an object of $1\text{-Cat}^{\mathrm{St}}$. Consider the following $(\infty,1)$-categories:

(1) The full subcategory of $\mathrm{Funct}(\mathbf{C}_0, \mathrm{Spc})$ that consists of functors that preserve fiber products.

(2) The full subcategory of $\mathrm{Funct}(\mathbf{C}_0, \mathrm{ComGrp}(\mathrm{Spc}))$ that consists of functors that preserve fiber products.

(3) The full subcategory of $\mathrm{Funct}(\mathbf{C}_0, \mathrm{Sptr})$ that consists of functors that preserve fiber products, i.e., $\mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}_0, \mathrm{Sptr})$.

The functors

$$\mathrm{Sptr} \xrightarrow{\tau^{\leq 0}} \mathrm{Sptr}^{\leq 0} \simeq \mathrm{ComGrp}(\mathrm{Spc}) \overset{\mathbf{oblv}_{\mathrm{ComGrp}}}{\longrightarrow} \mathrm{Spc}$$

define functors (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

We have (see [Lu2, Corollary 1.4.2.23]):

Lemma 7.2.2. *The above functors (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are equivalences.*

7.2.3. For $\mathbf{C}_0 \in 1\text{-Cat}^{\mathrm{St}}$, we define the $(\infty,1)$-categpry

$$\mathrm{Ind}(\mathbf{C}_0) := \mathrm{Funct}_{\mathrm{ex}}((\mathbf{C}_0)^{\mathrm{op}}, \mathrm{Sptr}).$$

According to Sect. 5.1.7, $\mathrm{Ind}(\mathbf{C}_0)$ is stable and cocomplete. Yoneda defines a fully faithful functor

$$\mathbf{C}_0 \to \mathrm{Ind}(\mathbf{C}_0). \tag{7.1}$$

We have (see [Lu1, 5.3.5] and [Lu2, Remark 1.4.2.9])

Lemma 7.2.4.

(1) *The essential image of* (7.1) *is contained in* $\mathrm{Ind}(\mathbf{C}^0)^c$.

(1') *The essential image of* (7.1) *generates* $\mathrm{Ind}(\mathbf{C}^0)$. *Moreover, any object of* $\mathrm{Ind}(\mathbf{C}^0)$ *can be written as a* filtered *colimit of objects from* $\mathbf{C}_0$.

(1") *Any compact object in* $\mathrm{Ind}(\mathbf{C}^0)$ *is a direct summand of one in the essential image of* (7.1).

(2) *For* $\mathbf{C} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}$, *restriction along* (7.1) *defines an equivalence*

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathrm{Ind}(\mathbf{C}^0), \mathbf{C}) \to \mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}^0, \mathbf{C}).$$

(3) *Let* $\mathbf{C}$ *be an object of* $1\text{-Cat}^{\mathrm{St,cocmpl}}$, *and let* $\mathbf{C}^0 \subset \mathbf{C}^c$ *be a full subcategory that generates* $\mathbf{C}$. *Then the functor* $\mathrm{Ind}(\mathbf{C}^0) \to \mathbf{C}$, *arising from* (2), *is an equivalence.*

(3') *For a compactly generated* $\mathbf{C} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}$, *the functor* $\mathrm{Ind}(\mathbf{C}^c) \to \mathbf{C}$ *is an equivalence.*

7.2.5. Note that point (2) in Lemma 7.2.4 says that the assignment

$$\mathbf{C}_0 \mapsto \mathrm{Ind}(\mathbf{C}_0) \tag{7.2}$$

provides a functor $1\text{-Cat}^{\mathrm{St}} \to 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, left adjoint to the inclusion

$$1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}} \hookrightarrow 1\text{-Cat}^{\mathrm{St}}.$$

7.2.6. Let us return to the setting of Sect. 5.3.3. Assume that each of the categories $\mathbf{C}_i$ is compactly generated, and that each of the functors $\mathbf{C}_i \to \mathbf{C}_j$ preserves compactness.

In this case, the functor $\mathbf{C}_\mathbf{I} : \mathbf{I} \to 1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ gives rise to a functor

$$\mathbf{C}^c_\mathbf{I} : \mathbf{I} \to 1\text{-Cat}^{\mathrm{St}}, \quad i \mapsto \mathbf{C}^c_i.$$

We have a tautological exact functor

$$\underset{\mathbf{I}}{\mathrm{colim}}\, \mathbf{C}^c_\mathbf{I} \to \mathbf{C}_*,$$

where the colimit in the left-hand side is taken in $1\text{-Cat}^{\mathrm{St}}$. Using Lemma 7.2.4(2), we obtain a morphism in $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$

$$\mathrm{Ind}\left(\underset{\mathbf{I}}{\mathrm{colim}}\, \mathbf{C}^c_\mathbf{I}\right) \to \mathbf{C}_*. \tag{7.3}$$

Since the functor (7.2) is a left adjoint, it preserves colimits. Hence, combining with Lemma 7.2.4(3'), we obtain:

Corollary 7.2.7.
(a) *The functor* (7.3) *is an equivalence.*
(b) *The category* $\mathbf{C}_*$ *is compactly generated by the essential images of the functors*

$$\mathbf{C}^c_i \to \mathbf{C}_i \xrightarrow{\mathrm{ins}_i} \mathbf{C}_*.$$

Remark 7.2.8. Note that in the present situation, the assertion of Corollary 5.3.4 becomes particularly obvious. Namely, we have

$$\mathbf{C}_* \simeq \mathrm{Ind}\left(\underset{i\in\mathbf{I}}{\mathrm{colim}}\, \mathbf{C}^c_i\right) = \mathrm{Funct}_{\mathrm{ex}}\left(\left(\underset{i\in\mathbf{I}}{\mathrm{colim}}\, \mathbf{C}^c_i\right)^{\mathrm{op}}, \mathrm{Sptr}\right) \simeq$$
$$\simeq \mathrm{Funct}_{\mathrm{ex}}\left(\underset{i\in\mathbf{I}}{\mathrm{colim}}\, (\mathbf{C}^c_i)^{\mathrm{op}}, \mathrm{Sptr}\right) \simeq \underset{i\in\mathbf{I}^{\mathrm{op}}}{\lim}\, \mathrm{Funct}_{\mathrm{ex}}\left((\mathbf{C}^c_i)^{\mathrm{op}}, \mathrm{Sptr}\right) \simeq$$
$$\simeq \underset{i\in\mathbf{I}^{\mathrm{op}}}{\lim}\, \mathrm{Ind}(\mathbf{C}^c_i) \simeq \underset{i\in\mathbf{I}^{\mathrm{op}}}{\lim}\, \mathbf{C}_i.$$

7.3. **The dual of a compactly generated category.** One of the key features of compactly generated stable categories is that they are dualizable in the sense of the Lurie symmetric monoidal structure. Moreover, the dual can be described very explicitly:

$$(\mathrm{Ind}(\mathbf{C}_0))^\vee \simeq \mathrm{Ind}((\mathbf{C}_0)^\vee).$$

As was mentioned in the introduction, the latter equivalence provides a framework for such phenomena as Verdier duality: rather than talking about a contravariant self-equivalence (on a small category of compact objects), we talk about the datum of self-duality on the entire category.

7.3.1. Let $\mathbf{C} \in 1\text{-Cat}^{\mathrm{St,cocmpl}}$ be of the form $\mathrm{Ind}(\mathbf{C}_0)$ for some $\mathbf{C}_0 \in 1\text{-Cat}^{\mathrm{St}}$. In particular, $\mathbf{C}$ is compactly generated, and any compactly generated object of $1\text{-Cat}^{\mathrm{St,cocmpl}}$ is of this form.

The assignment

$$(\mathbf{c}, \mathbf{c}' \in \mathbf{C}_0) \mapsto \mathcal{M}aps_\mathbf{C}(\mathbf{c}, \mathbf{c}')$$

defines a functor

$$(\mathbf{C}_0)^{\mathrm{op}} \times (\mathbf{C}_0) \to \mathrm{Sptr},$$

which is exact in each variable.

Applying left Kan extension along

$$(\mathbf{C}_0)^{\mathrm{op}} \times (\mathbf{C}_0) \hookrightarrow \mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}) \times \mathrm{Ind}(\mathbf{C}_0),$$

we obtain a functor

$$\mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}) \times \mathrm{Ind}(\mathbf{C}_0) \to \mathrm{Sptr},$$

which is exact and continuous in each variable. Hence, it gives rise to a functor

$$\mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathbf{C}_0) \to \mathrm{Sptr}\,. \tag{7.4}$$

Proposition 7.3.2. *The functor* (7.4) *provides the co-unit map of an adjunction data, thereby identifying* $\mathrm{Ind}(\mathbf{C}_0)$ *and* $\mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}})$ *as each other's duals.*

The proof given below essentially copies [Lu2, Proposition 4.8.1.16].

Proof. We have

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathrm{Ind}(\mathbf{C}_0), \mathrm{Sptr}) \simeq \mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}_0, \mathrm{Sptr}) \simeq \mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}).$$

Hence, it suffices to show that for $\mathbf{D} \in 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}$, the tautological functor

$$\mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}) \otimes \mathbf{D} \to \mathrm{Funct}_{\mathrm{ex,cont}}(\mathrm{Ind}(\mathbf{C}_0), \mathbf{D})$$

is an equivalence.

Thus, we need to show that for $\mathbf{E} \in 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}$, the space of continuous functors

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathrm{Ind}(\mathbf{C}_0), \mathbf{D}) \to \mathbf{E},$$

which is is the same as the space of continuous functors

$$\mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}_0, \mathbf{D}) \to \mathbf{E},$$

maps isomorphically to the space of continuous functors

$$\mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}) \otimes \mathbf{D} \to \mathbf{E},$$

while the latter identifies with the space of exact functors

$$(\mathbf{C}_0)^{\mathrm{op}} \to \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{D}, \mathbf{E}).$$

We will use the following observation:

Lemma 7.3.3. *For* $\mathbf{F}_1, \mathbf{F}_2 \in 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}$*, the passage to the right adjoint functor and the opposite category defines an equivalence*

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{F}_1, \mathbf{F}_2) \to \mathrm{Funct}_{\mathrm{ex,cont}}((\mathbf{F}_2)^{\mathrm{op}}, (\mathbf{F}_1)^{\mathrm{op}}).$$

Applying the lemma, we rewrite

$$\begin{aligned}\mathrm{Maps}_{1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}_0, \mathbf{D}), \mathbf{E}) &\simeq \mathrm{Maps}_{1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{E}^{\mathrm{op}}, (\mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}_0, \mathbf{D}))^{\mathrm{op}}) \simeq \\ &\simeq \mathrm{Maps}_{1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{E}^{\mathrm{op}}, \mathrm{Funct}_{\mathrm{ex}}((\mathbf{C}_0)^{\mathrm{op}}, \mathbf{D}^{\mathrm{op}})) \simeq \\ &\simeq \mathrm{Maps}_{1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}}((\mathbf{C}_0)^{\mathrm{op}}, \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{E}^{\mathrm{op}}, \mathbf{D}^{\mathrm{op}})) \simeq \\ &\simeq \mathrm{Maps}_{1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}}((\mathbf{C}_0)^{\mathrm{op}}, \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{D}, \mathbf{E}))\,,\end{aligned}$$

as required.

□

7.3.4. Let $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ be an exact functor between stable categories. Set

$$\mathbf{C} := \mathrm{Ind}(\mathbf{C}_0), \quad \mathbf{D} := \mathrm{Ind}(\mathbf{D}_0).$$

Let $F : \mathbf{C} \to \mathbf{D}$ be the left Kan extension along $\mathbf{C}_0 \to \mathbf{C}$ of the composite functor

$$\mathbf{C}_0 \stackrel{F_0}{\to} \mathbf{D}_0 \to \mathbf{D}.$$

We can also think about F as being obtained from F_0 by applying the functor

$$\mathrm{Ind} : 1\text{-}\mathrm{Cat}^{\mathrm{St}} \to 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}.$$

Note that according to Proposition 7.3.2, we have a canonical identification

$$\mathbf{C}^\vee \simeq \mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}) \text{ and } \mathbf{D}^\vee \simeq \mathrm{Ind}((\mathbf{D}_0)^{\mathrm{op}}).$$

Consider the functor

$$(F_0)^{\mathrm{op}} : (\mathbf{C}_0)^{\mathrm{op}} \to (\mathbf{D}_0)^{\mathrm{op}},$$

and let

$$F^{\text{fake-op}} : \mathbf{C}^\vee \to \mathbf{D}^\vee$$

denote its ind-extension.

Proposition 7.3.5. *The functor $F^{\text{fake-op}}$ is the dual of the right adjoint F^R of F. I.e.,*

$$F^{\text{fake-op}} \simeq (F^R)^\vee.$$

Remark 7.3.6. The functor $F^{\text{fake-op}}$ is in no sense the opposite of F; the latter would be a (not necessarily continuous) functor

$$\mathbf{C}^{\mathrm{op}} \to \mathbf{D}^{\mathrm{op}}.$$

However, the two agree on the full subcategory

$$\mathbf{C}^{\mathrm{op}} \supset (\mathbf{C}_0)^{\mathrm{op}} \subset \mathbf{C}^\vee,$$

which they both map to

$$\mathbf{D}^{\mathrm{op}} \supset (\mathbf{D}_0)^{\mathrm{op}} \subset \mathbf{D}^\vee.$$

Proof. We need to show that the functor

$$\mathbf{C}^\vee \times \mathbf{D} \to \mathbf{C}^\vee \otimes \mathbf{D} \stackrel{F^{\text{fake-op}} \otimes \mathrm{Id}_{\mathbf{D}}}{\longrightarrow} \mathbf{D}^\vee \otimes \mathbf{D} \stackrel{\epsilon_{\mathbf{D}}}{\longrightarrow} \mathrm{Sptr} \tag{7.5}$$

identifies with

$$\mathbf{C}^\vee \times \mathbf{D} \to \mathbf{C}^\vee \otimes \mathbf{D} \stackrel{\mathrm{Id}_{\mathbf{C}^\vee} \otimes F^R}{\longrightarrow} \mathbf{C}^\vee \otimes \mathbf{C} \stackrel{\epsilon_{\mathbf{C}}}{\longrightarrow} \mathrm{Sptr}\,. \tag{7.6}$$

Both functors are left Kan extensions from their respective restrictions to

$$(\mathbf{C}_0)^{\mathrm{op}} \times \mathbf{D}_0 \subset \mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}) \times \mathrm{Ind}(\mathbf{D}_0) \simeq \mathbf{C}^\vee \times \mathbf{D},$$

and are uniuquely recovered from their respective compositions with $\Omega^\infty : \mathrm{Sptr} \longrightarrow \mathrm{Spc}$.

The functor $(\mathbf{C}_0)^{\mathrm{op}} \times \mathbf{D}_0 \to \mathrm{Spc}$ obtained from (7.5) is

$$(\mathbf{C}_0)^{\mathrm{op}} \times \mathbf{D}_0 \stackrel{(F_0)^{\mathrm{op}} \times \mathrm{Id}_{\mathbf{D}_0}}{\longrightarrow} (\mathbf{D}_0)^{\mathrm{op}} \times \mathbf{D}_0 \stackrel{\mathrm{Yon}_{\mathbf{D}_0}}{\longrightarrow} \mathrm{Spc},$$

which we can further rewrite as

$$(\mathbf{C}_0)^{\mathrm{op}} \times \mathbf{D}_0 \to \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \stackrel{F^{\mathrm{op}} \times \mathrm{Id}_{\mathbf{D}}}{\longrightarrow} \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \stackrel{\mathrm{Yon}_{\mathbf{D}}}{\longrightarrow} \mathrm{Spc},$$

where the first arrow is obtained from the embeddings $\mathbf{C}_0 \to \mathbf{C}$ and $\mathbf{D}_0 \to \mathbf{D}$.

The functor $(\mathbf{C}_0)^{\mathrm{op}} \times \mathbf{D}_0 \to \mathrm{Spc}$, obtained from (7.6), is

$$(\mathbf{C}_0)^{\mathrm{op}} \times \mathbf{D}_0 \;\to\; \mathbf{C}^{\mathrm{op}} \times \mathbf{D} \;\to\; \mathrm{Funct}(\mathbf{C}, \mathrm{Spc}) \times \mathbf{D} \;\to\; \mathrm{Funct}(\mathbf{C}, \mathrm{Spc}) \times \mathbf{C} \;\to\; \mathrm{Spc},$$

where the first arrow is obtained from the embeddings $\mathbf{C}_0 \to \mathbf{C}$ and $\mathbf{D}_0 \to \mathbf{D}$, respectively, the second arrow is obtained from the Yoneda embedding for $\mathbf{C}$, the third arrow from F^R, and the last arrow is evaluation.

Thus, it suffices to see that the functors

$$\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \overset{F^{\mathrm{op}} \times \mathrm{Id}_{\mathbf{D}}}{\longrightarrow} \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \overset{\mathrm{Yon}_{\mathbf{D}}}{\longrightarrow} \mathrm{Spc}$$

and

$$\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathrm{Funct}(\mathbf{C}, \mathrm{Spc}) \times \mathbf{D} \overset{\mathrm{Id}_{\mathrm{Funct}(\mathbf{C},\mathrm{Spc})} \times F^R}{\longrightarrow} \mathrm{Funct}(\mathbf{C}, \mathrm{Spc}) \times \mathbf{C} \to \mathrm{Spc},$$

are canonically identified. However, the latter fact expresses the adjunction between F and F^R. □

7.4. **Compact generation of tensor products.** In this subsection we will discuss a variant of Proposition 6.4.2 in the compactly generated case. This turns out to be a more explicit statement, which will tell us 'what the tensor product actually looks like'.

7.4.1. Let us be given a pair of compactly generated stable categories $\mathbf{C}$ and $\mathbf{D}$. We claim:

Proposition 7.4.2.
(a) *The tensor product* $\mathbf{C} \otimes \mathbf{D}$ *is compactly generated by objects of the form* $\mathbf{c}_0 \boxtimes \mathbf{d}_0$ *with* $\mathbf{c}_0 \in \mathbf{C}^c$ *and* $\mathbf{d}_0 \in \mathbf{D}^c$.
(b) *For* $\mathbf{c}_0, \mathbf{d}_0$ *as above, and* $\mathbf{c} \in \mathbf{C}$, $\mathbf{d} \in \mathbf{D}$, *we have a canonical isomorphism*

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}) \otimes \mathcal{M}aps_{\mathbf{D}}(\mathbf{d}_0, \mathbf{d}) \simeq \mathcal{M}aps_{\mathbf{C} \otimes \mathbf{D}}(\mathbf{c}_0 \boxtimes \mathbf{d}_0, \mathbf{c} \boxtimes \mathbf{d}).$$

The rest of this subsection is devoted to the proof of this proposition.

7.4.3. To prove the proposition we will give an alternative description of the tensor product $\mathbf{C} \otimes \mathbf{D}$.

Set

$$\mathbf{C} = \mathrm{Ind}(\mathbf{C}_0) \text{ and } \mathbf{D} = \mathrm{Ind}(\mathbf{D}_0).$$

Note that we have a canonically defined functor

$$\mathbf{C} \otimes \mathbf{D} \to \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{C}^\vee \otimes \mathbf{D}^\vee, \mathrm{Sptr}) \to \mathrm{Funct}((\mathbf{C}_0)^{\mathrm{op}} \times (\mathbf{D}_0)^{\mathrm{op}}, \mathrm{Sptr}), \tag{7.7}$$

defined so that the corresponding functor

$$\mathbf{C} \times \mathbf{D} \to \mathrm{Funct}((\mathbf{C}_0)^{\mathrm{op}} \times (\mathbf{D}_0)^{\mathrm{op}}, \mathrm{Sptr})$$

is given by

$$(\mathbf{c}, \mathbf{d}) \mapsto ((\mathbf{c}_0, \mathbf{d}_0) \mapsto \mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}) \otimes \mathcal{M}aps_{\mathbf{D}}(\mathbf{d}_0, \mathbf{d}))\,.$$

7.4.4. By construction, the essential image of (7.7) is contained in the full subcategory

$$\mathrm{Funct}_{\text{bi-ex}}((\mathbf{C}_0)^{\mathrm{op}} \times (\mathbf{D}_0)^{\mathrm{op}}, \mathrm{Sptr}) \subset \mathrm{Funct}((\mathbf{C}_0)^{\mathrm{op}} \times (\mathbf{D}_0)^{\mathrm{op}}, \mathrm{Sptr})$$

that consists of functors that are exact in each variable.

Denote the resulting functor

$$\mathbf{C} \otimes \mathbf{D} \to \mathrm{Funct}_{\text{bi-ex}}((\mathbf{C}_0)^{\mathrm{op}} \times (\mathbf{D}_0)^{\mathrm{op}}, \mathrm{Sptr}) \tag{7.8}$$

by $h_{\mathbf{C},\mathbf{D}}$.

7.4.5. We claim that the functor $h_{\mathbf{C},\mathbf{D}}$ is an equivalence. Indeed, this follows from the interpretation of (7.8) as the composition

$$\mathbf{C}\otimes\mathbf{D}\simeq \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{C}^\vee,\mathbf{D})\simeq \mathrm{Funct}_{\mathrm{ex}}((\mathbf{C}_0)^{\mathrm{op}},\mathbf{D})\simeq$$
$$\simeq \mathrm{Funct}_{\mathrm{ex}}((\mathbf{C}_0)^{\mathrm{op}},\mathrm{Funct}_{\mathrm{ex}}((\mathbf{D}_0)^{\mathrm{op}},\mathrm{Sptr}))\simeq \mathrm{Funct}_{\text{bi-ex}}((\mathbf{C}_0)^{\mathrm{op}}\times(\mathbf{D}_0)^{\mathrm{op}},\mathrm{Sptr}).$$

7.4.6. Now, an analog of Yoneda's lemma for $h_{\mathbf{C},\mathbf{D}}$ says that for $\mathbf{c}_0\in\mathbf{C}_0$ and $\mathbf{d}_0\in\mathbf{D}_0$, and any $F\in \mathrm{Funct}_{\text{bi-ex}}((\mathbf{C}_0)^{\mathrm{op}}\times(\mathbf{D}_0)^{\mathrm{op}},\mathrm{Sptr})$ we have a canonical isomorphism

$$\mathcal{M}aps_{\mathrm{Funct}_{\text{bi-ex}}((\mathbf{C}_0)^{\mathrm{op}}\times(\mathbf{D}_0)^{\mathrm{op}},\mathrm{Sptr})}(h_{\mathbf{C},\mathbf{D}}(\mathbf{c}_0\boxtimes\mathbf{d}_0),F)\simeq F(\mathbf{c}_0\times\mathbf{d}_0).$$

This implies that the objects

$$h_{\mathbf{C},\mathbf{D}}(\mathbf{c}_0\boxtimes\mathbf{d}_0)\in \mathrm{Funct}_{\text{bi-ex}}((\mathbf{C}_0)^{\mathrm{op}}\times(\mathbf{D}_0)^{\mathrm{op}},\mathrm{Sptr})$$

are compact, generate $\mathrm{Funct}_{\text{bi-ex}}((\mathbf{C}_0)^{\mathrm{op}}\times(\mathbf{D}_0)^{\mathrm{op}},\mathrm{Sptr})$, and

$$\mathcal{M}aps_{\mathrm{Funct}_{\text{bi-ex}}((\mathbf{C}_0)^{\mathrm{op}}\times(\mathbf{D}_0)^{\mathrm{op}},\mathrm{Sptr})}(h_{\mathbf{C},\mathbf{D}}(\mathbf{c}_0\boxtimes\mathbf{d}_0),h_{\mathbf{C},\mathbf{D}}(\mathbf{c}\boxtimes\mathbf{d}))\simeq$$
$$\simeq \mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0,\mathbf{c})\otimes\mathcal{M}aps_{\mathbf{D}}(\mathbf{d}_0,\mathbf{d}).$$

8. Algebra in stable categories

In this section we apply the theory developed above to study *stable monoidal categories*, which are by definition associative algebra objects in $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

Our particular points of interest are how the behavior of modules over stable monoidal categories interacts with such notions as the Lurie tensor product, duality and compactness.

8.1. **Modules over a stable monoidal category.** We consider the symmetric monoidal category $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$. Our interest in this and the next few sections are associative and commutative algebra objects $\mathbf{A}$ in $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$. We will refer to them as *stable monoidal* (resp., *symmetric monoidal*) *categories*.

In this subsection we summarize and adapt some pieces of notation introduced earlier to the present context.

8.1.1. Note that the 1-fully faithful embedding $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\to 1\text{-}\mathrm{Cat}$ induces 1-fully faithful embeddings

$$\mathrm{AssocAlg}(1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}})\to 1\text{-}\mathrm{Cat}^{\mathrm{Mon}} \text{ and } \mathrm{ComAlg}(1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}})\to 1\text{-}\mathrm{Cat}^{\mathrm{SymMon}},$$

respectively.

I.e., a monoidal (resp., symmetric monoidal) cocomplete stable category is a particular case of a monoidal (symmetric monoidal) $(\infty,1)$-category.

So, we can talk about right-lax functors between monoidal (resp., symmetric monoidal) cocomplete stable categories.

In particular, given $\mathbf{A}$, we can talk about associative (resp., commutative) algebras in $\mathbf{A}$.

8.1.2. Given $\mathbf{A}$, following Sect. 3.4.4, we can consider the corresponding $(\infty,1)$-category of $\mathbf{A}$-modules in $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$, i.e., $\mathbf{A}\text{-mod}(1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}})$, for which we will also use the notation

$$\mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}.$$

Note that $\mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}$ is a 1-full subcategory in $\mathbf{A}\text{-}\mathbf{mod}$, the latter being the $(\infty,1)$-category of module categories over $\mathbf{A}$, when the latter is considered as a plain monoidal $(\infty,1)$-category.

Namely, an object $\mathbf{M} \in \mathbf{A}\text{-}\mathbf{mod}$ belongs to $\mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}$ if and only if $\mathbf{M}$ is a cocomplete stable category, and the action functor

$$\mathbf{A} \times \mathbf{M} \to \mathbf{M}$$

is exact and continuous in each variable.

A morphism $F : \mathbf{M}_0 \to \mathbf{M}_1$ in $\mathbf{A}\text{-}\mathbf{mod}$ belongs to $\mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}$ if and only if, when viewed as a plain functor, F is exact and continuous.

8.1.3. We have a pair of adjoint functors

$$\mathbf{ind}_{\mathbf{A}} : 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}} \rightleftarrows \mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}} : \mathbf{oblv}_{\mathbf{A}},$$

and the corresponding monad on $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ is given by tensor product with $\mathbf{A}$.

The functor $\mathbf{oblv}_{\mathbf{A}}$ preserves limits (being a right adjoint), and also colimits (because $\mathbf{A} \otimes -$ does).

8.1.4. Let $F : \mathbf{M} \to \mathbf{N}$ be a morphism in $\mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}$, and suppose that F, *when viewed as a plain functor between* $(\infty,1)$*-categories*, admits a right adjoint, F^R.

Then, according to Lemma 3.5.3, F^R has a natural structure of *right-lax* functor between $\mathbf{A}$-module categories.

8.1.5. According to Sect. 3.5.1, given $\mathbf{A} \in \text{AssocAlg}(1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}})$, $\mathbf{M} \in \mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}$ and $\mathcal{A} \in \text{AssocAlg}(\mathbf{A})$, we can consider the $(\infty,1)$-category

$$\mathcal{A}\text{-mod}(\mathbf{M}).$$

8.2. **Inner Hom and tensor products.**

8.2.1. According to Sect. 3.6, for a given $\mathbf{A} \in \text{AssocAlg}(1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}})$ and a pair of objects $\mathbf{M}, \mathbf{N} \in \mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}$, we can consider their relative inner Hom

$$\underline{\text{Hom}}_{1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}},\mathbf{A}}(\mathbf{M},\mathbf{N}) \in 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}.$$

We will use the notation

$$\text{Funct}_{\mathbf{A}}(\mathbf{M},\mathbf{N}) := \underline{\text{Hom}}_{1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}},\mathbf{A}}(\mathbf{M},\mathbf{N}) \in 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}.$$

We have:

$$(\text{Funct}_{\mathbf{A}}(\mathbf{M},\mathbf{N}))^{\text{Spc}} \simeq \text{Maps}_{\mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}}(\mathbf{M},\mathbf{N}).$$

8.2.2. By Corollary 6.2.6, evaluation at $\mathbf{1}_{\mathbf{A}}$ defines an equivalence of stable categories.

$$\text{Funct}_{\mathbf{A}}(\mathbf{A},\mathbf{M}) \simeq \mathbf{M}.$$

8.2.3. According to Sect. 3.6.6, in the case $\mathbf{N} = \mathbf{M}$, the object $\text{Funct}_{\mathbf{A}}(\mathbf{M},\mathbf{M})$ has a natural structure of associative algebra, i.e., a structure of stable monoidal category.

8.2.4. According to Sect. 3.6.5, if $\mathbf{A}$ is a stable *symmetric* monoidal category, then for $\mathbf{M},\mathbf{N}\in \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ as above, the object $\mathrm{Funct}_{\mathbf{A}}(\mathbf{M},\mathbf{N})$ has a natural structure of $\mathbf{A}$-module in $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, i.e., lifts to $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

8.2.5. According to Sect. 4.2.1, for a given $\mathbf{A}\in \mathrm{AssocAlg}(1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}})$, we have a well-defined functor

$$\mathbf{A}^{\text{rev-mult}}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\times \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\to 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}},\quad \mathbf{N},\mathbf{M}\mapsto \mathbf{N}\underset{\mathbf{A}}{\otimes}\mathbf{M}.$$

Lemma 8.2.6. *For $\mathbf{M}$ and $\mathbf{N}$ as above, the image of the tautological functor of stable categories*

$$\mathbf{N}\otimes\mathbf{M}\to \mathbf{N}\underset{\mathbf{A}}{\otimes}\mathbf{M}$$

generates the target.

Proof. The object $\mathbf{N}\underset{\mathbf{A}}{\otimes}\mathbf{M}$ can be calculated as the geometric realization of a simplicial object of $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ with terms given by $\mathbf{N}\otimes\mathbf{A}^{\otimes n}\otimes\mathbf{M}$ (see [Lu2, Theorem 4.4.2.8]). By Corollary 5.3.4, this geometric realization can be rewritten as a totalization (taken in $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}$) of the corresponding co-simplcial object. By Lemma 5.4.3, we need to show that the functor of evaluation on 0-simplices

$$\mathbf{N}\underset{\mathbf{A}}{\otimes}\mathbf{M}\to \mathbf{N}\otimes\mathbf{M}$$

is conservative. However, this follows from the fact that every object in $\mathbf{\Delta}$ admits a morphism from [0]. □

8.2.7. According to Sect. 4.2.4, if $\mathbf{A}$ is an object of $\mathrm{ComAlg}(1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}})$, then the operation of tensor product of modules extends to a structure of symmetric monoidal $(\infty,1)$-category on $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

8.2.8. According to Sect. 4.3.1, given a right $\mathbf{A}$-module $\mathbf{M}$ and a left $\mathbf{A}$-module $\mathbf{N}$, we can talk about the data of duality between them.

According to Sect. 4.3.3, if $\mathbf{A}$ is symmetric monoidal, a datum of duality between $\mathbf{M}$ and $\mathbf{N}$ in the above sense is equivalent to that in the sense of objects of $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ as a symmetric monoidal $(\infty,1)$-category.

8.3. **The 2-categorical structure.** The material in this subsection is an extension of that in Sect. 5.2; it will be needed in later Chapters in the book.

8.3.1. Let $\mathbf{A}$ be a stable monoidal category. We claim that the structure of $(\infty,1)$-category on $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ can be naturally upgraded to that of $(\infty,2)$-category, to be denoted

$$\left(\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\right)^{2\text{-}\mathrm{Cat}}.$$

Namely, we define the corresponding simplicial $(\infty,1)$-category

$$\mathrm{Seq}_\bullet\left(\left(\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\right)^{2\text{-}\mathrm{Cat}}\right)$$

as follows.

We let $\mathrm{Seq}_n\left(\left(\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\right)^{2\text{-}\mathrm{Cat}}\right)$ be the full $(\infty,1)$-category in

$$\mathbf{A}\text{-}\mathbf{mod}\underset{1\text{-}\mathrm{Cat}}{\times}\mathrm{Seq}_n(\mathbf{1}\text{-}\mathbf{Cat})\subset \mathbf{A}\text{-}\mathbf{mod}\underset{1\text{-}\mathrm{Cat}}{\times}\mathrm{Cart}_{/[n]^{\mathrm{op}}},$$

singled out by the following conditions:

We take those $(\infty,1)$-categories $\mathbf{C}$, equipped with an action of $\mathbf{A}$ (regarded as a monoidal $(\infty,1)$-category), and a Cartesian fibration $\mathbf{C}\to [n]^{\mathrm{op}}$ for which:

- We require that for every $i=0,...,n$, the $(\infty,1)$-category $\mathbf{C}_i$ be stable and cocomplete;
- For every $i=1,...,n$ the corresponding functor $\mathbf{C}_{i-1}\to\mathbf{C}_i$ be exact and continuous;
- For every i, the action morphism $\mathbf{A}\times\mathbf{C}_i\to\mathbf{C}_i$ be exact and continuous in each variable;
- The action functor $\mathbf{A}\times\mathbf{C}\to\mathbf{C}$ should be a morphism in $(\mathrm{Cart}_{/[n]^{\mathrm{op}}})_{\mathrm{strict}}$.

8.3.2. One checks that the object

$$\mathrm{Seq}_\bullet\left(\left(\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\right)^{2\text{-Cat}}\right)\in 1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$$

defined above indeed lies in the essential image of the functor

$$\mathrm{Seq}_\bullet : 2\text{-Cat}\to 1\text{-Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$$

and thus defines an object

$$\left(\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\right)^{2\text{-Cat}}\in 2\text{-Cat},$$

whose underlying $(\infty,1)$-category is $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

8.3.3. By construction, for $\mathbf{M},\mathbf{N}\in\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, we have

$$\mathbf{Maps}_{\left(\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\right)^{2\text{-Cat}}}(\mathbf{M},\mathbf{N})=\mathrm{Funct}_{\mathbf{A}}(\mathbf{M},\mathbf{N}).$$

8.3.4. Finally, we note that by repeating Sects. 4.2.4 and 4.2.2, we obtain that if $\mathbf{A}$ is a stable *symmetric* monoidal category, then the $(\infty,2)$-category $\left(\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\right)^{2\text{-Cat}}$ acquires a natural symmetric monoidal structure.

8.4. **Some residual 2-categorical features.** The material in this subsection is an extension of that in Sect. 5.3.

8.4.1. Let $F:\mathbf{M}\to\mathbf{N}$ be a morphism in $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, and suppose that F, when viewed as a plain functor between $(\infty,1)$-categories, admits a right adjoint, F^R. According to Sect. 8.1.4, the functor F^R has a natural structure of *right-lax* functor between $\mathbf{A}$-module categories.

It follows from the definitions that F^R is a *strict* functor between $\mathbf{A}$-module categories if and only if F, when viewed as a 1-morphism in $\left(\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}\right)^{2\text{-Cat}}$, admits a right adjoint.

8.4.2. *Limits and colimits.* Let $\mathbf{I}$ be an index category, and let

$$\mathbf{C}_{\mathbf{I}}:\mathbf{I}\to\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}.$$

Denote

$$\mathbf{C}_*:=\underset{\mathbf{I}}{\mathrm{colim}}\,\mathbf{C}_{\mathbf{I}}\in\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}.$$

Assume that for every arrow $i\to j$ in $\mathbf{I}$, the corresponding 1-morphism $\mathbf{C}_i\to\mathbf{C}_j$ admits a right adjoint. Then, the procedure of passage to right adjoints (see [Chapter A.3, Corollary 1.3.4]) gives rise to a functor

$$\mathbf{C}^R_{\mathbf{I}^{\mathrm{op}}}:\mathbf{I}^{\mathrm{op}}\to\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}.$$

The following results from Proposition 2.5.7:

Corollary 8.4.3. *The canonically defined morphism*

$$\mathbf{C}_* \to \lim_{\mathbf{I}^{\mathrm{op}}} \mathbf{C}_{\mathbf{I}^{\mathrm{op}}}$$

is an equivalence, where the above limit is taken in $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

8.4.4. Let $\mathbf{C}$ be an object of $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$. Then in the way parallel to Sect. 5.3.5, we can consider the monoidal $(\infty,1)$-category

$$\mathbf{Maps}_{\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{C},\mathbf{C}), \tag{8.1}$$

which is equipped with an action on

$$\mathbf{Maps}_{\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{D},\mathbf{C}),$$

for any $\mathbf{D} \in 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

In particular, we can talk about the $(\infty,1)$-category of $\mathbf{A}$-linear monads acting on $\mathbf{C}$, which are by definition associative algebra objects in the monoidal $(\infty,1)$-category (8.1).

Given an $\mathbf{A}$-linear monad $\mathcal{B}$, we can consider the $(\infty,1)$-category

$$\mathcal{B}\text{-mod}(\mathbf{C})$$

in the sense of Sect. 3.7.2.

The category $\mathcal{B}\text{-mod}(\mathbf{C})$ is itself an object of $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ and the adjoint pair

$$\mathbf{ind}_{\mathcal{B}} : \mathbf{C} \rightleftarrows \mathcal{B}\text{-mod}(\mathbf{C}) : \mathbf{oblv}_{\mathcal{B}}$$

takes place in $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

For a 1-morphism $G \in \mathbf{Maps}_{\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{D},\mathbf{C})$ the datum of action of $\mathcal{B}$ on G is equivalent to that of factoring G as

$$\mathbf{oblv}_{\mathcal{B}} \circ G^{\mathrm{enh}}, \quad G^{\mathrm{enh}} \in \mathbf{Maps}_{\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}}(\mathbf{D},\mathcal{B}\text{-mod}(\mathbf{C})).$$

Let $G : \mathbf{D} \to \mathbf{C}$ be as above, and assume that it admits a left adjoint G^L. The functor G^L acquires a natural left-lax functor between $\mathbf{A}$-module categories. Assume, however, that this left-lax structure is strict. Then

$$\mathcal{B} := G \circ G^L$$

acquires a natural structure of $\mathbf{A}$-linear monad. The functor G gives rise to a 1-morphism in $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$

$$G^{\mathrm{enh}} : \mathbf{D} \to \mathcal{B}\text{-mod}(\mathbf{C}).$$

8.5. **Modules over an algebra.** In this subsection we will start combining the general features of modules over algebras with the specifics of dealing with cocomplete stable categories.

8.5.1. Let $\mathbf{A}$ be an object of $\mathrm{AssocAlg}(1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}})$. Fix also $\mathbf{M} \in \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ and $\mathcal{A} \in \mathrm{AssocAlg}(\mathbf{A})$.

Consider the category $\mathcal{A}\text{-mod}(\mathbf{M})$. Recall that we have a pair of adjoint functors

$$\mathbf{ind}_{\mathcal{A}} : \mathbf{M} \rightleftarrows \mathcal{A}\text{-mod}(\mathbf{M}) : \mathbf{oblv}_{\mathcal{A}},$$

where $\mathbf{oblv}_{\mathcal{A}}$ is monadic, and the corresponding monad on $\mathbf{M}$ is given by

$$\mathbf{m} \mapsto \mathcal{A} \otimes \mathbf{m}.$$

8.5.2. Suppose that we have two such triples $(\mathbf{A}_1, \mathbf{M}_1, \mathcal{A}_1)$ and $(\mathbf{A}_2, \mathbf{M}_2, \mathcal{A}_2)$. By Sect. 4.2.4, we can regard $\mathbf{A}_1 \otimes \mathbf{A}_2$ as an object of $\text{AssocAlg}(1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}})$, and

$$\mathcal{A}_1 \boxtimes \mathcal{A}_2 \in \mathbf{A}_1 \otimes \mathbf{A}_2$$

has a natural structure of object in $\text{Assoc}(\mathbf{A}_1 \otimes \mathbf{A}_2)$.

Furthermore, $\mathbf{M}_1 \otimes \mathbf{M}_2$ has a natural structure of object of $(\mathbf{A}_1 \otimes \mathbf{A}_2)\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}$.

8.5.3. Suppose that in the above situation, the $\mathbf{A}_1$-module structure on $\mathbf{M}_1$ (resp., $\mathbf{A}_2$-module structure on $\mathbf{M}_2$) has been extended to a structure of module over $\mathbf{A}_1 \otimes \mathbf{A}^{\text{rev-mult}}$ (resp., $\mathbf{A}_2 \otimes \mathbf{A}$), where $\mathbf{A}$ is yet another monoidal stable category.

In this case, we can form $\mathbf{M}_1 \underset{\mathbf{A}}{\otimes} \mathbf{M}_2$, which is a module over $\mathbf{A}_1 \otimes \mathbf{A}_2$. In addition, $\mathcal{A}_1\text{-mod}(\mathbf{M}_1)$ (resp., $\mathcal{A}_2\text{-mod}(\mathbf{M}_2)$) is a right (resp., left) module over $\mathbf{A}$, so we can form

$$\mathcal{A}_1\text{-mod}(\mathbf{M}_1) \underset{\mathbf{A}}{\otimes} \mathcal{A}_2\text{-mod}(\mathbf{M}_2).$$

We have a canonically defined functor

$$\mathcal{A}_1\text{-mod}(\mathbf{M}_1) \underset{\mathbf{A}}{\otimes} \mathcal{A}_2\text{-mod}(\mathbf{M}_2) \to \mathbf{M}_1 \underset{\mathbf{A}}{\otimes} \mathbf{M}_2,$$

on which $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ acts as a monad. Hence, we obtain a functor

$$\mathcal{A}_1\text{-mod}(\mathbf{M}_1) \underset{\mathbf{A}}{\otimes} \mathcal{A}_2\text{-mod}(\mathbf{M}_2) \to (\mathcal{A}_1 \boxtimes \mathcal{A}_2)\text{-mod}(\mathbf{M}_1 \underset{\mathbf{A}}{\otimes} \mathbf{M}_2). \tag{8.2}$$

Proposition 8.5.4. *The functor* (8.2) *is an equivalence.*

Proof. Follows in the same way as Lemma 6.4.5 from Corollary 5.3.8. □

Here are some particular cases of Proposition 8.5.4.

8.5.5. First, let us take $\mathbf{A} = \text{Sptr}$. In this case, Proposition 8.5.4 says that the functor

$$\mathcal{A}_1\text{-mod}(\mathbf{M}_1) \otimes \mathcal{A}_2\text{-mod}(\mathbf{M}_2) \to (\mathcal{A}_1 \boxtimes \mathcal{A}_2)\text{-mod}(\mathbf{M}_1 \otimes \mathbf{M}_2)$$

is an equivalence. Note this is also a corollary of Lemma 6.4.5.

8.5.6. Let us now take $\mathbf{A}_1 = \mathbf{A}$, $\mathcal{A}_1 =: \mathcal{A}$ and $\mathbf{M}_1 = \mathbf{A}$ with its natural structure of $\mathbf{A}$-bimodule. Take $\mathbf{A}_2 = \text{Sptr}$, $\mathcal{A}_2 = \mathbf{1}_{\text{Sptr}}$ and $\mathbf{M}_2 =: \mathbf{M} \in \mathbf{A}\text{-}\mathbf{mod}^{\text{St,cocmpl}}_{\text{cont}}$. Thus, from Proposition 8.5.4 we obtain:

Corollary 8.5.7. *The functor*

$$\mathcal{A}\text{-mod} \underset{\mathbf{A}}{\otimes} \mathbf{M} \to \mathcal{A}\text{-mod}(\mathbf{M})$$

is an equivalence.

8.5.8. Let now take $(\mathbf{A}_1, \mathcal{A}_1, \mathbf{M}_2) = (\mathbf{A}, \mathcal{A}, \mathbf{A})$ as above, and let

$$(\mathbf{A}_2, \mathcal{A}_2, \mathbf{M}) = (\mathbf{A}^{\text{mult-rev}}, \mathcal{A}^{\text{mult-rev}}, \mathbf{A}^{\text{mult-rev}}).$$

We obtain:

Corollary 8.5.9. *The functor*

$$\mathcal{A}\text{-mod} \underset{\mathbf{A}}{\otimes} \mathcal{A}\text{-mod}^r \to (\mathcal{A} \boxtimes \mathcal{A}^{\text{mult-rev}})\text{-mod}(\mathbf{A})$$

is an equivalence.

8.5.10. Let $\mathbf{A}$ be a stable symmetric monoidal category. Recall the Cartesian fibration

$$\mathrm{AssocAlg} + \mathrm{mod}(\mathbf{A}) \to \mathrm{AssocAlg}(\mathbf{A}),$$

of (4.5), and the corresponding functor

$$(\mathrm{AssocAlg}(\mathbf{A}))^{\mathrm{op}} \to 1\text{-Cat}, \quad \mathcal{A} \mapsto \mathcal{A}\text{-mod}. \tag{8.3}$$

It follows from Proposition 8.5.4 that (8.3) upgrades to a symmetric monoidal functor

$$(\mathrm{AssocAlg}(\mathbf{A}))^{\mathrm{op}} \to \mathbf{A}\text{-mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}.$$

8.6. **Duality for module categories.**

8.6.1. Consider $\mathbf{A}$ as an associative algebra object in the monoidal category $1\text{-Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$. Hence, it makes sense to talk about duality between left and right $\mathbf{A}$-modules, see Sect. 4.3.1.

8.6.2. From Corollary 8.5.9 we will now deduce:

Corollary 8.6.3. *The left* $\mathbf{A}$*-module category* $\mathcal{A}\text{-mod}^r$ *is naturally dual to the right* $\mathbf{A}$*-module category* $\mathcal{A}\text{-mod}$*.*

Proof. We will construct explicitly the duality datum. The functor

$$\text{co-unit} : \mathcal{A}\text{-mod}^r(\mathbf{A}) \otimes \mathcal{A}\text{-mod}(\mathbf{A}) \to \mathbf{A}$$

corresponds to the functor of tensor product

$$\mathcal{A}\text{-mod}^r(\mathbf{A}) \times \mathcal{A}\text{-mod}(\mathbf{A}) \to \mathbf{A}$$

of Sect. 4.2.1.

The functor

$$\text{unit} : \mathrm{Sptr} \to \mathcal{A}\text{-mod}(\mathbf{A}) \underset{\mathbf{A}}{\otimes} \mathcal{A}\text{-mod}^r(\mathbf{A})$$

is constructed as follows. Under the identification

$$\mathcal{A}\text{-mod}(\mathbf{A}) \underset{\mathbf{A}}{\otimes} \mathcal{A}\text{-mod}^r(\mathbf{A}) \simeq (\mathcal{A} \boxtimes \mathcal{A}^{\text{mult-rev}})\text{-mod}(\mathbf{A})$$

of Corollary 8.5.9, it corresponds to the object

$$\mathcal{A} \in (\mathcal{A} \boxtimes \mathcal{A}^{\text{mult-rev}})\text{-mod}(\mathbf{A}).$$

□

Corollary 8.6.4. *For* $\mathbf{M} \in \mathbf{A}\text{-}\mathbf{mod}$ *there is a canonical equivalence*

$$\mathrm{Funct}_{\mathbf{A}}(\mathcal{A}\text{-mod}^r, \mathbf{M}) \simeq \mathcal{A}\text{-mod}(\mathbf{M}).$$

8.7. **Compact generation of tensor products.**

8.7.1. Let $\mathbf{A}$ be a monoidal stable category, and let $\mathbf{M}$ and $\mathbf{N}$ be a left and a right $\mathbf{A}$-modules, respectively.

Assume that the monoidal operation $\mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ admits a continuous right adjoint, and that so do the action functors $\mathbf{A} \otimes \mathbf{M} \to \mathbf{M}$ and $\mathbf{N} \otimes \mathbf{A} \to \mathbf{N}$.

We will prove:

Proposition 8.7.2. *Under the above circumstances, the right adjoint to the tautological functor*

$$\mathbf{N} \otimes \mathbf{M} \to \mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M}$$

is continuous.

Proof. The category $\mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M}$ is given as the geometric realization of the simplicial category

$$i \mapsto \mathbf{N} \otimes \mathbf{A}^{\otimes i} \otimes \mathbf{M}.$$

Hence, applying Corollary 5.3.4, it is enough to show that the functor

$$\mathbf{\Delta}^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}, \quad [n] \mapsto \mathbf{N} \otimes \mathbf{A}^{\otimes n} \otimes \mathbf{M}$$

has the property that it sends every morphism in $\mathbf{\Delta}^{\mathrm{op}}$ to a 1-morphism in $\mathbf{1}\text{-}\mathbf{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ that admits a continuous right adjoint. However, this follows from the assumption on the monoidal operation on $\mathbf{A}$ and the action functors.

□

8.7.3. Combining with Lemma 8.2.6 and Proposition 7.4.2, we obtain:

Corollary 8.7.4. *Assume that $\mathbf{A}$, $\mathbf{M}_1$, $\mathbf{M}_2$ are compactly generated, and that the functors*

$$\mathbf{A} \otimes \mathbf{A} \to \mathbf{A},\ \mathbf{A} \otimes \mathbf{M} \to \mathbf{M},\ \mathbf{N} \otimes \mathbf{A} \to \mathbf{N}$$

preserve compact objects. Then the functor

$$\mathbf{N} \otimes \mathbf{M} \to \mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M}$$

sends compact objects to compact ones. In particular, $\mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M}$ is compactly generated.

8.8. **Compactness and relative compactness.**

8.8.1. Let $\mathbf{A}$ be a stable monoidal category, and let $\mathbf{M}$ be an object of $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

For an object $\mathbf{m} \in \mathbf{M}$ consider the functor

$$\mathbf{M} \to \mathbf{A}, \quad \mathbf{m}' \mapsto \underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{m}, \mathbf{m}'). \tag{8.4}$$

Definition 8.8.2. *We shall say that* $\mathbf{m}$ *is* compact relative to $\mathbf{A}$ *if the functor* (8.4) *preserves filtered colimits (equivalently, all colimits or direct sums).*

8.8.3. The following is immediate:

Lemma 8.8.4.

(a) *Suppose that $\mathbf{A}$ is compactly generated, and that the action functor $\mathbf{A} \times \mathbf{M} \to \mathbf{M}$ sends $\mathbf{A}^c \times \mathbf{M}^c$ to $\mathbf{M}^c$. Then every compact object in $\mathbf{M}$ is compact relative to $\mathbf{A}$.*

(b) *Suppose that $\mathbf{1_A} \in \mathbf{A}$ is compact. Then every object in $\mathbf{M}$ that is compact relative to $\mathbf{A}$ is compact.*

8.8.5. Let us now take $\mathbf{M} = \mathbf{A}$. It is clear that if $\mathbf{a} \in \mathbf{A}$ is left-dualizable (see Sect. 4.1.1 for what this means), then it is compact relative to $\mathbf{A}$: indeed

$$\underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}, \mathbf{a}') \simeq \mathbf{a}' \otimes \mathbf{a}^{\vee, L},$$

while the monoidal operation on $\mathbf{A}$ distributes over colimits.

We have the following partial converse to this statement:

Lemma 8.8.6. *Suppose that* $\mathbf{A}$ *is generated by left-dualizable objects. Then every object of* $\mathbf{A}$ *that is compact relative to* $\mathbf{A}$ *is left-dualizable.*

Proof. To show that an object $\mathbf{a} \in \mathbf{A}$ is left-dualizable, it suffices to show that for any $\mathbf{a}' \in \mathbf{A}$, the natural map

$$\mathbf{a}' \otimes \underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}, \mathbf{1}_{\mathbf{A}}) \to \underline{\mathrm{Hom}}_{\mathbf{A}}(\mathbf{a}, \mathbf{a}') \tag{8.5}$$

is an isomorphism.

Let $\mathbf{a} \in \mathbf{A}$ be compact relative to $\mathbf{A}$. By assumption, both sides in (8.5) preserve colimits in $\mathbf{a}'$. Hence, it suffices to show that (8.5) is an isomorphism for $\mathbf{a}'$ taken from a generating collection of objects of $\mathbf{A}$. We take this collection to be that left-dualizable obejcts. However, (8.5) is an isomorphism for any $\mathbf{a}$, provided that $\mathbf{a}'$ is left-dualizable.

□

9. Rigid monoidal categories

This section contains, what probably is, the only piece of original mathematics in this chapter–the notion of *rigid monoidal category.* These are stable monoidal categories with particularly strong finiteness properties.

9.1. **The notion of rigid monoidal category.**

9.1.1. Let $\mathbf{A}$ be a stable monoidal category. Let $\mathrm{mult}_{\mathbf{A}}$ denote the tensor product functor $\mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$.

Definition 9.1.2. *We shall say that* $\mathbf{A}$ *is* rigid *if the following conditions hold:*

- *The object* $\mathbf{1}_{\mathbf{A}} \in \mathbf{A}$ *is compact;*
- *The right adjoint of* $\mathrm{mult}_{\mathbf{A}}$*, denoted* $(\mathrm{mult}_{\mathbf{A}})^R$*, is continuous;*
- *The functor* $(\mathrm{mult}_{\mathbf{A}})^R : \mathbf{A} \to \mathbf{A} \otimes \mathbf{A}$ *is a functor of* $\mathbf{A}$*-bimodule categories (a priori it is only a right-lax functor);*

A tautological example of a rigid stable monoidal category is $\mathbf{A} = \mathrm{Sptr}$.

9.1.3. *An example.* Let $\mathcal{A}$ be a commutative algebra object in the stable symmetric monoidal category Sptr. Then the stable (symmetric) monoidal category $\mathcal{A}$-mod is rigid.

More generally, let $\mathbf{A}$ be a rigid symmetric monoidal category, and let $\mathcal{A}$ be a commutative algebra in $\mathbf{A}$. Then the stable (symmetric) monoidal category $\mathcal{A}$-mod is rigid.

9.1.4. Here is the link to the more familiar definition of rigidity:

Lemma 9.1.5. *Suppose that* $\mathbf{A}$ *is compactly generated. Then* $\mathbf{A}$ *is rigid if and only if the following conditions hold:*

- *The object* $\mathbf{1}_{\mathbf{A}}$ *is compact;*
- *The functor* $\mathrm{mult}_{\mathbf{A}}$ *sends* $\mathbf{A}^c \times \mathbf{A}^c$ *to* $\mathbf{A}^c$*;*
- *Every compact object in* $\mathbf{A}$ *admits both a left and a right dual.*

Proof. First, the fact that $\mathbf{A}$, and hence $\mathbf{A} \otimes \mathbf{A}$, is compactly generated implies that $\mathrm{mult}_{\mathbf{A}}$ preserves compactness if and only if $(\mathrm{mult}_{\mathbf{A}})^R$ is continuous.

Assume that every compact object in $\mathbf{A}$ admits a left dual. We claim that in this case, every right-lax functor between $\mathbf{A}$-module categories $F : \mathbf{M} \to \mathbf{N}$ is strict. Indeed, it suffices to show that for every $\mathbf{m} \in \mathbf{M}$ and $\mathbf{a} \in \mathbf{A}^c$, the map

$$\mathbf{a} \otimes F(\mathbf{m}) \to F(\mathbf{a} \otimes \mathbf{m})$$

is an isomorphism. However, the above map admits an explicit inverse, given by

$$F(\mathbf{a} \otimes \mathbf{m}) \to \mathbf{a} \otimes \mathbf{a}^{\vee,L} \otimes F(\mathbf{a} \otimes \mathbf{m}) \to \mathbf{a} \otimes F(\mathbf{a}^{\vee,L} \otimes \mathbf{a} \otimes \mathbf{m}) \to \mathbf{a} \otimes F(\mathbf{m}).$$

Suppose, vice versa, that $\mathbf{A}$ is rigid. Let us show that every object $\mathbf{a} \in \mathbf{A}^c$ admits a left dual. For that end, it suffices to show that the functor

$$\mathbf{a}' \mapsto \mathbf{a}' \otimes \mathbf{a}, \quad \mathbf{A} \to \mathbf{A}$$

admits a right adjoint, and this right adjoint is a strict (as opposed to right-lax) functor between left $\mathbf{A}$-modules. However, the right adjoint in question is given by

$$\mathbf{A} \overset{(\mathrm{mult}_{\mathbf{A}})^R}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \overset{\mathrm{Id} \otimes \mathcal{M}aps_{\mathbf{A}}(\mathbf{a},-)}{\longrightarrow} \mathbf{A} \otimes \mathrm{Sptr} \simeq \mathbf{A}.$$

The situation with right duals is similar.

□

9.1.6. As a corollary, we obtain:

Corollary 9.1.7. *Let* $\mathbf{A}$ *be rigid and compactly generated. Then an object of* $\mathbf{A}$ *is compact if and only if it is left-dualizable and if and only if it is right-dualizable.*

9.2. **Basic properties of rigid monoidal categories.** A fundamental property of a rigid monoidal category (and one that entails the multiple properties of its modules) is that it is *canonically self-dual* when viewed as a plain stable category.

Moreover, this self-duality interacts in a very explicit way with many operations (such as the monoidal operation on $\mathbf{A}$ or monoidal functors between rigid monoidal categories).

9.2.1. Suppose that $\mathbf{A}$ is rigid. In this case, it is easy to see that the data of

$$\epsilon : \mathbf{A} \otimes \mathbf{A} \overset{\mathrm{mult}_{\mathbf{A}}}{\longrightarrow} \mathbf{A} \overset{\mathcal{M}aps_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}},-)}{\longrightarrow} \mathrm{Sptr}$$

and

$$\mu : \mathrm{Sptr} \overset{\mathbf{1}_{\mathbf{A}}}{\longrightarrow} \mathbf{A} \overset{(\mathrm{mult}_{\mathbf{A}})^R}{\longrightarrow} \mathbf{A} \otimes \mathbf{A}$$

define an isomorphism

$$\mathbf{A} \to \mathbf{A}^{\vee,R} = \mathbf{A}^{\vee}.$$

We denote the above isomorphism by $\phi_{\mathbf{A}}$.

9.2.2. *An example.* Consider again the example from Sect. 9.1.3. The co-unit of the above self-duality data on $\mathcal{A}$-mod is given by

$$\mathcal{A}\text{-mod} \otimes \mathcal{A}\text{-mod} \overset{\otimes}{\longrightarrow} \mathcal{A}\text{-mod} \overset{\mathcal{M}aps_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}},-)}{\longrightarrow} \text{Sptr}\,.$$

9.2.3. Let us regard $\mathbf{A}$ as a bimodule over itself. Then, according to Sect. 4.1.7, $\mathbf{A}^\vee$ also acquires a structure of $\mathbf{A}$-bimodule. It is easy to see that the isomorphism

$$\phi_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^\vee$$

is compatible with the *left* $\mathbf{A}$-module structure.

Lemma 9.2.4. *Suppose that* $\mathbf{A}$ *is compactly generated. Then the equivalence* $(\mathbf{A}^c)^{\text{op}} \to \mathbf{A}^c$*, induced by* $\phi_{\mathbf{A}}$*, identifies with* $\mathbf{a} \mapsto \mathbf{a}^{\vee,R}$.

Proof. We need to construct a functorial isomorphism

$$\text{Maps}_{\mathbf{A}}(\phi_{\mathbf{A}}(\mathbf{a}), \mathbf{a}') \simeq \text{Maps}_{\mathbf{A}}(\mathbf{a}^{\vee,R}, \mathbf{a}'), \quad \mathbf{a}' \in \mathbf{A}.$$

By definition,

$$\text{Maps}_{\mathbf{A}}(\phi_{\mathbf{A}}(\mathbf{a}), \mathbf{a}') = \epsilon(\mathbf{a}' \boxtimes \mathbf{a}) \simeq \text{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, \mathbf{a}' \otimes \mathbf{a}),$$

while $\text{Maps}_{\mathbf{A}}(\mathbf{a}^{\vee,R}, \mathbf{a}')$ also identifies with $\text{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, \mathbf{a}' \otimes \mathbf{a})$, as required. □

9.2.5. The following is obtained by diagram chase:

Lemma 9.2.6. *Let* $\mathbf{A}$ *be a rigid monoidal* $(\infty,1)$*-category. Then:*

(a) *The following diagram commutes:*

$$\begin{array}{ccc} \mathbf{A}^\vee & \xrightarrow{(\text{mult}_{\mathbf{A}})^\vee} & \mathbf{A}^\vee \otimes \mathbf{A}^\vee \\ {\scriptstyle \phi_{\mathbf{A}}}\uparrow & & \uparrow{\scriptstyle \phi_{\mathbf{A}} \otimes \phi_{\mathbf{A}}} \\ \mathbf{A} & \xrightarrow{(\text{mult}_{\mathbf{A}})^R} & \mathbf{A} \otimes \mathbf{A} \end{array}$$

commutes.

(b) *Let* $F : \mathbf{A}_1 \to \mathbf{A}_2$ *be a monoidal functor between rigid monoidal* $(\infty,1)$*-categories. Then its right adjoint* F^R *is continuous and the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{A}_2^\vee & \xrightarrow{F^\vee} & \mathbf{A}_1^\vee \\ {\scriptstyle \phi_{\mathbf{A}_2}}\uparrow & & \uparrow{\scriptstyle \phi_{\mathbf{A}_1}} \\ \mathbf{A}_2 & \xrightarrow{F^R} & \mathbf{A}_1. \end{array}$$

9.2.7. It is clear that $\mathbf{A}$ is rigid if and only if $\mathbf{A}^{\text{rev-mult}}$ is. Reversing the multiplication on $\mathbf{A}$ we obtain another identification $\mathbf{A} \to \mathbf{A}^\vee$, denoted $\phi_{\mathbf{A}^{\text{rev-mult}}}$.

We have $\phi_{\mathbf{A}^{\text{rev-mult}}} = \phi_{\mathbf{A}} \circ \varphi_{\mathbf{A}}$, where $\varphi_{\mathbf{A}}$ is an automorphism of $\mathbf{A}$.

It is easy to see, however, that $\varphi_{\mathbf{A}}$ is naturally an automorphism[9] of $\mathbf{A}$ as a monoidal $(\infty,1)$-category.

If $\mathbf{A}$ is symmetric monoidal, then $\varphi_{\mathbf{A}}$ is canonically isomorphic to the identity functor.

Unwinding the definitions, we obtain:

[9]We are grateful to J. Lurie for pointing this out to us.

Lemma 9.2.8. *Suppose that* $\mathbf{A}$ *is compactly generated. Then* $\varphi_{\mathbf{A}}$ *is induced by the automorphism*

$$\mathbf{a} \mapsto (\mathbf{a}^{\vee,L})^{\vee,L}$$

of $\mathbf{A}^c$.

9.3. **Modules over rigid categories.** It turns out that modules over rigid monoidal categories exhibit some very special features:

- For a $\mathbf{A}$-module $\mathbf{M}$, the action map $\mathrm{act}_{\mathbf{A},\mathbf{M}} : \mathbf{A} \otimes \mathbf{M} \to \mathbf{M}$ admits a continuous right adjoint, and this right adjoint identifies with the *dual* of $\mathrm{act}_{\mathbf{A},\mathbf{M}}$ with respect to the self-duality on $\mathbf{A}$;
- Any right-lax (or left-lax) functor between $\mathbf{A}$-module categories is *strict*;
- The tensor product of modules over $\mathbf{A}$ is isomorphic to the co-tensor product;
- An $\mathbf{A}$-module is dualizable if and only if it is such as a plain stable category, and the stable category underlying the dual of an $\mathbf{A}$-module $\mathbf{M}$ identifies with the dual of $\mathbf{M}$ as a plain stable category.

9.3.1. Throughout this subsection we let $\mathbf{A}$ be a rigid monoidal category. Let $\mathbf{M}$ be an $\mathbf{A}$-module. Let

$$\mathrm{act}_{\mathbf{A},\mathbf{M}} : \mathbf{A} \otimes \mathbf{M} \to \mathbf{M}$$

denote the action functor.

Lemma 9.3.2. *The action functor* $\mathrm{act}_{\mathbf{A},\mathbf{M}}$ *admits a continuous right adjoint, which is given by the composition*

$$\mathbf{M} \simeq \mathrm{Sptr} \otimes \mathbf{M} \overset{\mu \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M}. \tag{9.1}$$

Proof. We construct the adjunction data as follows. The composition

$$\mathbf{M} \overset{(9.1)}{\longrightarrow} \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{M}$$

identifies with

$$\mathbf{M} \simeq \mathrm{Sptr} \otimes \mathbf{M} \overset{\mathbf{1}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M} \overset{(\mathrm{mult}_{\mathbf{A}})^R \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{mult}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{M},$$

which, by virtue of the $(\mathrm{mult}_{\mathbf{A}}, (\mathrm{mult}_{\mathbf{A}})^R)$-adjunction, admits a canonically defined map to

$$\mathbf{M} \simeq \mathrm{Sptr} \otimes \mathbf{M} \overset{\mathbf{1}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{M},$$

the latter being the identity map on $\mathbf{M}$.

The composition

$$\mathbf{A} \otimes \mathbf{M} \overset{\mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{M} \overset{(9.1)}{\longrightarrow} \mathbf{A} \otimes \mathbf{M}$$

identifies with

$$\mathbf{A} \otimes \mathbf{M} \simeq \mathrm{Sptr} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathbf{1}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{(\mathrm{mult}_{\mathbf{A}})^R \otimes \mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow}$$
$$\to \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{mult}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M},$$

and the latter, in turn identifies with

$$\mathbf{A} \otimes \mathbf{M} \overset{(\mathrm{mult}_{\mathbf{A}})^R \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M},$$

which by adjunction receives a map from

$$\mathbf{A} \otimes \mathbf{M} \simeq \mathbf{A} \otimes \mathrm{Sptr} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{1}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M},$$

while the latter is the identity functor on $\mathbf{A} \otimes \mathbf{M}$.

□

Combining with Proposition 8.7.2, we obtain:

Corollary 9.3.3. *For a left* $\mathbf{A}$*-module* $\mathbf{M}$ *and a right* $\mathbf{A}$*-module* $\mathbf{N}$*, the right adjoint to the tautological functor*

$$\mathbf{N} \otimes \mathbf{M} \to \mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M}$$

is continuous.

Combining with Lemma 8.8.4, we obtain:

Corollary 9.3.4. *Let* $\mathbf{M}$ *be an* $\mathbf{A}$*-module category. Then an object* $\mathbf{m} \in \mathbf{M}$ *is compact relative to* $\mathbf{A}$ *if and only if it is compact.*

9.3.5. We also claim:

Lemma 9.3.6. *Any right-lax or (left-lax) functor between* $\mathbf{A}$*-module categories is strict.*

Remark 9.3.7. Note that if $\mathbf{A}$ is compactly generated, the assertion of Lemma 9.3.6 has been established in the course of the proof of Lemma 9.1.5.

Proof. Let $F : \mathbf{M} \to \mathbf{N}$ be a right-lax functor between $\mathbf{A}$-module categories. We need to show that the (given) natural transformation from

$$\mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes F}{\longrightarrow} \mathbf{A} \otimes \mathbf{N} \overset{\mathrm{act}_{\mathbf{A},\mathbf{N}}}{\longrightarrow} \mathbf{N} \tag{9.2}$$

to

$$\mathbf{A} \otimes \mathbf{M} \overset{\mathrm{act}_{\mathbf{A},\mathbf{N}}}{\longrightarrow} \mathbf{M} \overset{F}{\longrightarrow} \mathbf{N} \tag{9.3}$$

is an isomorphism. We will construct an explicit inverse natural transformation.

We consider two more functors $\mathbf{A} \otimes \mathbf{M} \to \mathbf{N}$. One is

$$\mathbf{A} \otimes \mathbf{M} \overset{\mathbf{1}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{(\mathrm{mult}_{\mathbf{A}})^R \otimes \mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow}$$
$$\to \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{A}} \otimes F}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{N} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{A},\mathbf{N}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{N} \overset{\mathrm{act}_{\mathbf{A},\mathbf{N}}}{\longrightarrow} \mathbf{N}. \tag{9.4}$$

The other is

$$\mathbf{A} \otimes \mathbf{M} \overset{\mathbf{1}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{(\mathrm{mult}_{\mathbf{A}})^R \otimes \mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow}$$
$$\to \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{A},\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes F}{\longrightarrow} \mathbf{A} \otimes \mathbf{N} \overset{\mathrm{act}_{\mathbf{A},\mathbf{N}}}{\longrightarrow} \mathbf{N}. \tag{9.5}$$

The unit of the $(\mathrm{mult}_{\mathbf{A}}, {\mathrm{mult}_{\mathbf{A}}}^R)$-adjunction gives rise to a natural transformation from (9.3) to (9.4). The right-lax structure on F gives rise to a natural transformation from (9.4) to (9.5). Finally, the co-unit of the $(\mathrm{mult}_{\mathbf{A}}, {\mathrm{mult}_{\mathbf{A}}}^R)$-adjunction gives rise to a natural transformation from (9.5) to (9.2). Combining, we obtain the desired natural transformation from (9.3) to (9.2).

The case of a left-lax functor is treated similarly.

□

9.4. **Duality for modules over rigid categories–the commutative case.** In this subsection we will show that the theory of duality for modules over a rigid category is particularly transparent.

9.4.1. Recall (see Sect. 4.1.7) that if $\mathbf{A}$ is a stable monoidal category, and $\mathbf{M}$ is a left (resp., right) $\mathbf{A}$-module, and $\mathbf{M}$ is dualizable as a plain stable category, then $\mathbf{M}^\vee$ is naturally a right (resp., left) $\mathbf{A}$-module.

More generally, if $\mathbf{M}$ is a left (resp., right) $\mathbf{A}$-module, and $\mathbf{C}$ is a stable category, then $\mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{M}, \mathbf{C})$ is naturally a right (resp., left) $\mathbf{A}$-module.

9.4.2. For the duration of this subsection we let $\mathbf{A}$ be a rigid *symmetric* monoidal category, so that there is no distinction between left and right modules.

Recall that in this case, the automorphism $\varphi_{\mathbf{A}}$ of $\mathbf{A}$ is canonically the identity map. So, $\mathbf{A}^\vee$ identifies with $\mathbf{A}$ as an $\mathbf{A}$-*bimodule.*

9.4.3. We claim:

Proposition 9.4.4. *Let* $\mathbf{M}$ *be an* $\mathbf{A}$*-module. Then* $\mathbf{M}$ *is dualizable as an* $\mathbf{A}$*-module if and only if* $\mathbf{M}$ *is dualizable as a plain stable category. In this case, the dual of* $\mathbf{M}$ *as a* $\mathbf{A}$*-module identifies canonically with* $\mathbf{M}^\vee$ *with its natural* $\mathbf{A}$*-module structure.*

Proof. Let first $\mathbf{A}$ be any stable monoidal category such that the underlying stable category is dualizable. We consider $\mathbf{A}^\vee$ equipped with a natural structure of bimodule over $\mathbf{A}$, see Sect. 9.4.1.

Note that if $\mathbf{M}$ is an $\mathbf{A}$-module and $\mathbf{C} \in 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, we have

$$\mathrm{Funct}_{\mathbf{A}}(\mathbf{M}, \mathbf{C} \otimes \mathbf{A}^\vee) \simeq \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{M}, \mathbf{C}), \tag{9.6}$$

as right $\mathbf{A}$-modules.

Assume that $\mathbf{M}$ is dualizable as a left $\mathbf{A}$-module. In this case, from (9.6) we obtain that for any $\mathbf{C}, \mathbf{D} \in 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$, the functor

$$\mathbf{D} \otimes \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{M}, \mathbf{C}) \to \mathbf{D} \otimes \mathrm{Funct}_{\mathrm{ex,cont}}(\mathbf{M}, \mathbf{D} \otimes \mathbf{C})$$

is an equivalence. Hence, $\mathbf{M}$ is dualizable as a plain stable category.

Let us now restore the assumption that $\mathbf{A}$ be rigid. The dual of $\mathbf{M}$ as a $\mathbf{A}$-module is given by $\mathrm{Funct}_{\mathbf{A}}(\mathbf{M}, \mathbf{A})$. Using the equivalence $\mathbf{A} \simeq \mathbf{A}^\vee$ and (9.6), we obtain the stated description of the dual of $\mathbf{M}$.

It remains to show that if $\mathbf{M}$ is dualizable as a plain stable category, then it is dualizable as an $\mathbf{A}$-module. For that it suffices to show that the functor

$$\mathbf{N} \mapsto \mathrm{Funct}_{\mathbf{A}}(\mathbf{M}, \mathbf{N}), \quad \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}} \to 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$$

preserves sifted colimits and the operation of tensoring up by an object of $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$.

We note that $\mathrm{Funct}_{\mathbf{A}}(\mathbf{M}, \mathbf{N})$ is given as the totalization of the co-simplicial category with terms

$$\mathbf{M}^\vee \otimes (\mathbf{A}^\vee)^{\otimes n} \otimes \mathbf{N}.$$

Now, by Lemma 9.3.2, the transition maps in this cosimplicial category are *continuous* functors, and hence, by Corollary 5.3.4, the above totalization can be rewritten as a geometric realization. This implies the required assertion.

□

9.4.5. *Digression: the co-tensor product.*

Let $\mathcal{O}$ be an associative algebra in a *symmetric* monoidal category $\mathbf{O}$. Note that in this case we can regard $\mathcal{O}^{\text{rev-mult}}$ also as an associative algebra in $\mathbf{O}$. Furthermore, the category of $\mathcal{O}$-bimodules in $\mathbf{O}$ identifies with

$$(\mathcal{O} \otimes \mathcal{O}^{\text{rev-mult}})\text{-mod}.$$

Let $\mathcal{M}$ and $\mathcal{N}$ be a left and a right $\mathcal{O}$-modules in $\mathbf{O}$, respectively. We regard $\mathcal{M} \otimes \mathcal{N}$ as a $(\mathcal{O} \otimes \mathcal{O}^{\text{rev-mult}})$-module in $\mathbf{O}$.

We let

$$\mathcal{M} \overset{\mathcal{O}}{\otimes} \mathcal{N} \in \mathbf{O}$$

denote the object

$$\underline{\text{Hom}}_{\mathbf{O},\mathcal{O}\otimes\mathcal{O}^{\text{rev-mult}}}(\mathcal{O}, \mathcal{M} \otimes \mathcal{N}),$$

provided that the latter exists.

9.4.6. Applying this to $\mathbf{O} = 1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$ and

$$\mathcal{O} = \mathbf{A} \in \text{AssocAlg}(1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}),$$

we obtain the notion of *co-tensor* product of $\mathbf{A}$-module categories.

9.4.7. We claim:

Proposition 9.4.8. *Suppose that* $\mathbf{A}$ *is rigid. Then for* $\mathbf{A}$*-modules* $\mathbf{M}$ *and* $\mathbf{N}$*, we have a canonical isomorphism in* $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$

$$\mathbf{M} \overset{\mathbf{A}}{\otimes} \mathbf{N} \simeq \mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M}.$$

Proof. First we note that the assumption that $\mathbf{A}$ is rigid implies that $\mathbf{A} \otimes \mathbf{A}^{\text{rev-mult}}$ is also rigid. Consider $\mathbf{A}$ as a module over $\mathbf{A} \otimes \mathbf{A}^{\text{rev-mult}}$. From Proposition 9.4.4, it follows that the dual of $\mathbf{A}$ as a module over $\mathbf{A} \otimes \mathbf{A}^{\text{rev-mult}}$ identifies with $\mathbf{A}$.

Hence,

$$\mathbf{M} \overset{\mathbf{A}}{\otimes} \mathbf{N} := \text{Funct}_{\mathbf{A}\otimes\mathbf{A}^{\text{rev-mult}}}(\mathbf{A}, \mathbf{M} \otimes \mathbf{N}) \simeq \mathbf{A} \underset{\mathbf{A}\otimes\mathbf{A}^{\text{rev-mult}}}{\otimes} (\mathbf{M} \otimes \mathbf{N}) \simeq \mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M},$$

as required.

□

9.5. **Duality for modules over rigid categories–the general case.** In this subsection we will explain the (minor) modifications needed to generalize the results from Sect. 9.4 to the case when $\mathbf{A}$ is just a rigid monoidal category (i.e., *not necessarily* symmetric monoidal).

These modifications will amount to a twist by the automorphism $\varphi_{\mathbf{A}}$ of $\mathbf{A}$.

9.5.1. In what follows, for a right $\mathbf{A}$-module $\mathbf{N}$, we denote by $\mathbf{N}_\varphi$ the $\mathbf{A}$-module, with the same underlying category, but where the action of $\mathbf{A}$ is obtained by pre-composing with the *inverse* of the automorphism $\varphi_{\mathbf{A}}$ of $\mathbf{A}$.

9.5.2. Then we have the following variant of Proposition 9.4.4 (with the same proof):

Proposition 9.5.3. *Let* $\mathbf{M}$ *be a left* $\mathbf{A}$*-module. Then* $\mathbf{M}$ *is dualizable as a left* $\mathbf{A}$*-module if and only if* $\mathbf{M}$ *is dualizable as a plain stable category. In this case, the dual of* $\mathbf{M}$ *as a right* $\mathbf{A}$*-module identifies canonically with* $(\mathbf{M}^\vee)_\varphi$.

As a corollary we obtain:

Corollary 9.5.4. *Let* $\mathbf{M}$ *(resp.,* $\mathbf{N}$*) be a dualizable left (resp., right)* $\mathbf{A}$*-module. Then* $\mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M} \in 1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$ *is dualizable, and its dual is given by*

$$(\mathbf{M}^\vee)_\varphi \underset{\mathbf{A}}{\otimes} \mathbf{N}^\vee.$$

Proof. Follows from the fact that for any stable monoidal category $\mathbf{A}$, if $\mathbf{M}$ is a dualizable left $\mathbf{A}$-module with dual $\mathbf{L}$, and $\mathbf{N}$ is a right module, dualizable as a plain stable category, then the tensor product $\mathbf{N} \underset{\mathbf{A}}{\otimes} \mathbf{M}$ is dualizable with dual given by

$$\mathbf{L} \underset{\mathbf{A}}{\otimes} \mathbf{N}^\vee.$$

□

9.5.5. We will now consider the co-tensor product of modules over $\mathbf{A}$. We have the following variant of Proposition 9.4.8, with the same proof:

Proposition 9.5.6. *Suppose that* $\mathbf{A}$ *is rigid. Then for a left* $\mathbf{A}$*-module* $\mathbf{M}$ *and a right* $\mathbf{A}$*-module* $\mathbf{N}$*, we have a canonical isomorphism in* $1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}$

$$\mathbf{M} \overset{\mathbf{A}}{\otimes} \mathbf{N} \simeq \mathbf{N}_\varphi \underset{\mathbf{A}}{\otimes} \mathbf{M}.$$

10. DG categories

10.1. The $(\infty,1)$-category of vector spaces.

10.1.1. Throughout this book we will be working over a ground field k of characteristic 0. To k we can attach the $(\infty,1)$-category Vect of complexes of vector spaces over k.

This is the *derived* ∞*-category* attached to the abelian category of vector spaces, in the sense of [Lu2, Sect. 1.3.2].

This $(\infty,1)$-category is endowed with a t-structure, and the corresponding abelian category $\mathrm{Vect}^\heartsuit$ is the usual abelian category of vector spaces over k.

The $(\infty,1)$-category Vect is *stable* and *cocomplete*.

Remark 10.1.2. Starting from an abelian category with enough projectives $\mathcal{A}$, the definition in [Lu2, Sect. 1.3.2.7] produces the 'bounded above' derived $(\infty,1)$-category $\mathcal{D}^-(\mathcal{A})$. In the case of $\mathcal{A} = \mathrm{Vect}^\heartsuit$, one recovers the entire Vect as the *right completion* of Vect^- with respect to its t-structure. I.e., Vect is the unique stable category equipped with a t-structure such that its bounded above part is Vect^- and for any $V \in \mathrm{Vect}$, the tautological map

$$\underset{n}{\mathrm{colim}}\, \tau^{\leq n}(V) \to V$$

is an isomorphism.

The construction of the derived $(\infty,1)$-category $\mathcal{D}^-(\mathcal{A})$ given in [Lu2, Sect. 1.3.2.7] appeals to an explicit procedure called 'the differential graded nerve'. We have no desire to reproduce it here because this construction appeals to a particular model of $(\infty,1)$-category (namely, quasi-categories): the explicit knowledge of what it is does not usually add any information of practical import. What is important to know is that the homotopy category of $\mathcal{D}^-(\mathcal{A})$, i.e., $(\mathcal{D}^-(\mathcal{A}))^{\text{ordn}}$, is the usual triangulated bounded above derived category of $\mathcal{A}$.

The good news, however, is that the derived $(\infty,1)$-category $\mathcal{D}^-(\mathcal{A})$ can be characterized by a universal property, see [Lu2, Theorem 1.3.3.2] or the less heavy looking [Lu2, Proposition 1.3.3.7].

10.1.3. We let

$$\text{Vect}^{\text{f.d.}} \subset \text{Vect},$$

denote the full subcategory of *finite complexes of finite-dimensional vector spaces* over k.

The corresponding abelian category $(\text{Vect}^{\text{f.d.}})^{\heartsuit}$ is that of usual finite-dimensional vector spaces over k.

We have

$$\text{Vect}^{\text{f.d.}} = \text{Vect}^c,$$

and Vect is compactly generated by $\text{Vect}^{\text{f.d.}}$.

10.1.4. The fact of crucial importance is that the stable category Vect carries a symmetric monoidal structure uniquely characterized by the following conditions ([Lu2, Theorems 4.5.2.1 and 7.1.2.13]):

- It is compatible with the (usual) symmetric monoidal structure on $\text{Vect}^{\heartsuit} \subset \text{Vect}$.
- The monoidal operation $\text{Vect} \times \text{Vect} \to \text{Vect}$ preserves colimits in each variable.

The second of the above conditions means that Vect is a commutative algebra object in $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$.

10.1.5. The symmetric monoidal structure on Vect induces one on its full subcategory $\text{Vect}^{\text{f.d.}}$.

Every object in the symmetric monoidal category $\text{Vect}^{\text{f.d.}}$ is dualizable. Hence, by Sect. 4.1.4, the functor of dualization defines an equivalence

$$(\text{Vect}^{\text{f.d.}})^{\text{op}} \to \text{Vect}^{\text{f.d.}}.$$

From Lemma 9.1.5 we obtain:

Corollary 10.1.6. *The stable symmetric monoidal* $(\infty,1)$*-category* Vect *is rigid.*

10.2. **The Dold-Kan functor(s).**

10.2.1. Since Sptr is the unit object in the symmetric monoidal $(\infty,1)$-category $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$, we have a canonically defined symmetric monoidal functor

$$\text{Sptr} \to \text{Vect}. \tag{10.1}$$

This functor admits a right adjoint, denoted

$$\text{Vect} \overset{\text{Dold-Kan}^{\text{Sptr}}}{\longrightarrow} \text{Sptr}.$$

The functor $\text{Dold-Kan}^{\text{Sptr}}$ is continuous (e.g., by Lemma 9.2.6(b)).

10.2.2. The functor $\text{Dold-Kan}^{\text{Sptr}}$ has the following additional property: it is *t-exact* (i.e., compatible with the t-structures).

In particular, $\text{Dold-Kan}^{\text{Sptr}}$ restricts to a functor

$$\text{Vect}^{\leq 0} \overset{\text{Dold-Kan}^{\text{ComGrp}}}{\longrightarrow} \text{ComGrp}(\text{Spc}),$$

where we recall that ComGrp(Spc) identifies with $\text{Sptr}^{\leq 0}$.

10.2.3. The composition

$$\text{Vect}^{\leq 0} \overset{\text{Dold-Kan}^{\text{ComGrp}}}{\longrightarrow} \text{ComGrp}(\text{Spc}) \overset{\mathbf{oblv}_{\text{ComGrp}}}{\longrightarrow} \text{Spc},$$

or, which is the same

$$\Omega^\infty \circ \text{Dold-Kan}^{\text{Sptr}},$$

is the usual Dold-Kan functor

$$\text{Vect}^{\leq 0} \overset{\text{Dold-Kan}}{\longrightarrow} \text{Spc}\,.$$

The functor Dold-Kan preserves filtered colimits and all limits. In addition, Dold-Kan commutes with *sifted* colimits (because the forgetful functor $\mathbf{oblv}_{\text{ComGrp}}$ does, see [Chapter IV.2, Sect. 1.1.3]).

For $V \in \text{Vect}^{\leq 0}$ we have

$$\pi_i(\text{Dold-Kan}(V)) = H^{-i}(V), \quad i = 0, 1, ...$$

10.2.4. By construction, the functor Dold-Kan is the right adjoint to the composition

$$\text{Spc} \overset{\Sigma^\infty}{\longrightarrow} \text{Sptr} \overset{(10.1)}{\longrightarrow} \text{Vect}\,. \tag{10.2}$$

In terms of the equivalence of Lemma 2.1.8, the above functor (10.2) corresponds to the object $k \in \text{Vect}$, and can be thought of as the functor of *chains with coefficients in* k.

$$\mathcal{S} \mapsto \text{C}_\bullet(\mathcal{S}, k).$$

10.3. **The notion of DG category.** In the rest of this chapter we will develop the theory of modules (in 1-Cat^{St}) over the (symmetric) monoidal categories $\text{Vect}^{\text{f.d.}}$ and Vect.

However, the entire discussion is equally applicable, when we replace the pair $\text{Vect}^{\text{f.d.}} \subset \text{Vect}$ by

$$(\mathbf{A}^c \subset \mathbf{A}),$$

where $\mathbf{A}$ is a *rigid symmetric monoidal category* that satisfies the equivalent conditions of Lemma 9.1.5.

10.3.1. We let $\text{DGCat}^{\text{non-cocmpl}}$ denote the full subcategory in the $(\infty, 1)$-category

$$\text{Vect}^{\text{f.d.}}\text{-}\mathbf{mod} = \text{Vect}^{\text{f.d.}}\text{-}\mathbf{mod}(1\text{-Cat})$$

(see Sect. 3.5.7 for the notation), consisting of those $\text{Vect}^{\text{f.d.}}$-modules $\mathbf{C}$, for which:

- $\mathbf{C}$ is stable;
- The action functor $\text{Vect}^{\text{f.d.}} \times \mathbf{C} \to \mathbf{C}$ is *exact* in both variables.

10.3.2. The identification $(\text{Vect}^{\text{f.d.}})^{\text{op}} \simeq \text{Vect}^{\text{f.d.}}$ induces an involution

$$\mathbf{C} \mapsto \mathbf{C}^{\text{op}}$$

on $\text{DGCat}^{\text{non-cocmpl}}$.

10.3.3. We let $\mathrm{DGCat}_{\mathrm{cont}}$ denote the $(\infty,1)$-category

$$\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{St,cocmpl}}_{\mathrm{cont}} := \mathrm{Vect}\text{-}\mathbf{mod}(1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}}).$$

By unwinding the definitions, we obtain:

Lemma 10.3.4.

(a) *The functor*

$$\mathrm{DGCat}_{\mathrm{cont}} \to \mathrm{DGCat}^{\text{non-cocmpl}},$$

given by restriction of action along $\mathrm{Vect}^{\mathrm{f.d.}} \hookrightarrow \mathrm{Vect}$ *is* 1-replete*, i.e., is an equivalence on a 1-full subcategory.*

(b) *An object of* $\mathrm{DGCat}^{\text{non-cocmpl}}$ *lies in the essential image of the functor from* (a) *if and only if the underlying stable category is cocomplete.*

(c) *A morphism in* $\mathrm{DGCat}^{\text{non-cocmpl}}$ *between objects in the essential image of* $\mathrm{DGCat}_{\mathrm{cont}}$ *comes from a morphism in* $\mathrm{DGCat}_{\mathrm{cont}}$ *if and only if the underlying functor between the corresponding stable categories is continuous.*

The $(\infty,1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$ will be the principal actor in this book.

We introduce one more notion: we let $\mathrm{DGCat} \subset \mathrm{DGCat}^{\text{non-cocmpl}}$ be the full subcategory equal to the essential image of the functor $\mathrm{DGCat}_{\mathrm{cont}} \to \mathrm{DGCat}^{\text{non-cocmpl}}$.

Thus, $\mathrm{DGCat}_{\mathrm{cont}}$ is a 1-full subcategory of DGCat with the same class of objects (i.e., cocomplete DG categories), but in the latter we allow non-continuous functors.

10.3.5. Let $\mathbf{C}, \mathbf{D}$ be two objects of $\mathrm{DGCat}^{\text{non-cocmpl}}$. By Sect. 3.6.5, we can associate to them an object

$$\underline{\mathrm{Hom}}_{1\text{-}\mathrm{Cat},\mathrm{Vect}^{\mathrm{f.d.}}}(\mathbf{D},\mathbf{C}) \in \mathrm{Vect}^{\mathrm{f.d.}}\text{-}\mathbf{mod}(1\text{-}\mathrm{Cat}).$$

It is easy to see, however, that $\underline{\mathrm{Hom}}_{1\text{-}\mathrm{Cat},\mathrm{Vect}^{\mathrm{f.d.}}}(\mathbf{D},\mathbf{C})$ belongs to the full subcategory

$$\mathrm{DGCat}^{\text{non-cocmpl}} \subset \mathrm{Vect}^{\mathrm{f.d.}}\text{-}\mathbf{mod}(1\text{-}\mathrm{Cat}).$$

We will use the notation:

$$\mathrm{Funct}_k(\mathbf{D},\mathbf{C}) := \underline{\mathrm{Hom}}_{1\text{-}\mathrm{Cat},\mathrm{Vect}^{\mathrm{f.d.}}}(\mathbf{D},\mathbf{C}).$$

This is the DG category of exact k-linear functors from $\mathbf{D}$ to $\mathbf{C}$. We have

$$(\mathrm{Funct}_k(\mathbf{D},\mathbf{C}))^{\mathrm{Spc}} = \mathrm{Maps}_{\mathrm{DGCat}^{\text{non-cocmpl}}}(\mathbf{D},\mathbf{C}).$$

Note that if $\mathbf{C}$ is cocomplete, then so is $\mathrm{Funct}_k(\mathbf{D},\mathbf{C})$, i.e., in this case it is an object of DGCat.

10.3.6. Let now $\mathbf{C}, \mathbf{D}$ be two objects of $\mathrm{DGCat}_{\mathrm{cont}}$. By Sect. 8.2.1, we can consider the object

$$\underline{\mathrm{Hom}}_{1\text{-}\mathrm{Cat}^{\mathrm{St,cocmpl}}_{\mathrm{cont}},\mathrm{Vect}}(\mathbf{D}, \mathbf{C}) =: \mathrm{Funct}_{\mathrm{Vect}}(\mathbf{D}, \mathbf{C}) \in \mathrm{DGCat}_{\mathrm{cont}}.$$

We will use the notation:

$$\mathrm{Funct}_{k,\mathrm{cont}}(\mathbf{D}, \mathbf{C}) := \mathrm{Funct}_{\mathrm{Vect}}(\mathbf{D}, \mathbf{C}).$$

This is the DG category of continuous exact k-linear functors $\mathbf{D}$ to $\mathbf{C}$. We have

$$(\mathrm{Funct}_{k,\mathrm{cont}}(\mathbf{D}, \mathbf{C}))^{\mathrm{Spc}} = \mathrm{Maps}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathbf{D}, \mathbf{C}).$$

By construction, we have a map in $\mathrm{DGCat}_{\mathrm{cont}}$:

$$\mathrm{Funct}_{k,\mathrm{cont}}(\mathbf{D}, \mathbf{C}) \to \mathrm{Funct}_k(\mathbf{D}, \mathbf{C}), \tag{10.3}$$

which is fully faithful at the level of the underlying $(\infty, 1)$-categories.

10.3.7. Let $\mathbf{C}$ be an object of $\mathrm{DGCat}^{\text{non-cocmpl}}$. For a pair of objects $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$ we introduce the object

$$\mathcal{M}aps_{k,\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Vect}$$

by

$$\mathrm{Maps}_{\mathrm{Vect}}(V, \mathcal{M}aps_{k,\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)) \simeq \mathrm{Maps}_{\mathbf{C}}(V \otimes \mathbf{c}_0, \mathbf{c}_1), \quad V \in \mathrm{Vect}^{\mathrm{f.d.}}. \tag{10.4}$$

It is easy to see that $\mathcal{M}aps_{k,\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)$ always exists.

10.3.8. If $\mathbf{C} \in \mathrm{DGCat}$, then we have a canonical isomorphism

$$\mathcal{M}aps_{k,\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \simeq \underline{\mathrm{Hom}}_{\mathrm{Vect}}(\mathbf{c}_0, \mathbf{c}_1),$$

i.e., the isomorphism (10.4) holds for $V \in \mathrm{Vect}$ (and not just $\mathrm{Vect}^{\mathrm{f.d.}}$).

Finally, we note that we have

$$\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \simeq \text{Dold-Kan}^{\mathrm{Sptr}}\left(\mathcal{M}aps_{k,\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)\right).$$

10.3.9. *The 2-categorical structure.* According to Sect. 8.3, the structure of $(\infty, 1)$-category on $\mathrm{DGCat}_{\mathrm{cont}}$ can be naturally upgraded to a structure of $(\infty, 2)$-category.

We denote the resulting $(\infty, 2)$-category by $\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$. By Sect. 8.3.3, we have

$$\mathbf{Maps}_{\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}}(\mathbf{D}, \mathbf{C}) \simeq \mathrm{Funct}_{k,\mathrm{cont}}(\mathbf{D}, \mathbf{C}).$$

We also note (see Lemma 9.3.6) that if a morphism in $\mathrm{DGCat}_{\mathrm{cont}}$, when viewed as plain stable categories, admits a continuous right adjoint, then the initial 1-morphism admits a right adjoint in the $(\infty, 2)$-category $\mathrm{DGCat}^{2\text{-Cat}}_{\mathrm{cont}}$.

10.4. **The symmetric monoidal structure on DG categories.**

10.4.1. According to Sect. 8.2.7, the $(\infty, 1)$-category $\mathrm{DGCat}_{\mathrm{cont}}$ is equipped with a symmetric monoidal structure. We will denote the corresponding monoidal operation by

$$\mathbf{C}, \mathbf{D} \mapsto \mathbf{C} \underset{\mathrm{Vect}}{\otimes} \mathbf{D}.$$

For $\mathbf{c} \in \mathbf{C}$ and $\mathbf{d} \in \mathbf{D}$ we let denote by $\mathbf{c} \underset{k}{\boxtimes} \mathbf{d} \in \mathbf{C} \underset{\mathrm{Vect}}{\otimes} \mathbf{D}$ the image of $\mathbf{c} \times \mathbf{d} \in \mathbf{C} \times \mathbf{D}$ under the tautological functor

$$\mathbf{C} \times \mathbf{D} \to \mathbf{C} \underset{\mathrm{Vect}}{\otimes} \mathbf{D}.$$

10.4.2. In particular, given $\mathbf{C}, \mathbf{D} \in \text{DGCat}_{\text{cont}}$, we can talk about the datum of *duality* between them, the latter being the datum of functors

$$\mu : \text{Vect} \to \mathbf{C} \underset{\text{Vect}}{\otimes} \mathbf{D} \text{ and } \mathbf{D} \underset{\text{Vect}}{\otimes} \mathbf{C} \to \text{Vect}$$

such that the corresponding identities hold.

10.4.3. According to Proposition 9.4.4, a DG category $\mathbf{C}$ is dualizable as an object of $\text{DGCat}_{\text{cont}}$ if and only if it is dualizable as a plain stable category.

Moreover, again by Proposition 9.4.4, the datum of duality between $\mathbf{C}$ and $\mathbf{D}$ as DG categories is equivalent to the datum of duality between $\mathbf{C}$ and $\mathbf{D}$ as plain stable categories.

10.4.4. Explicitly, if

$$\mathbf{C} \underset{\text{Vect}}{\otimes} \mathbf{D} \to \text{Vect}$$

is the co-unit of a duality in $\text{DGCat}_{\text{cont}}$, then the composition

$$\mathbf{C} \otimes \mathbf{D} \to \mathbf{C} \underset{\text{Vect}}{\otimes} \mathbf{D} \to \text{Vect} \overset{\text{Dold-Kan}^{\text{Sptr}}}{\longrightarrow} \text{Sptr}$$

is the co-unit of a duality in $1\text{-Cat}^{\text{St,cocmpl}}_{\text{cont}}$.

In fact, for any $\mathbf{C} \in \text{DGCat}_{\text{cont}}$, the composed functor

$$\text{Funct}_{k,\text{cont}}(\mathbf{C}, \text{Vect}) \to \text{Funct}_{\text{ex,cont}}(\mathbf{C}, \text{Vect}) \overset{\text{Dold-Kan}^{\text{Sptr}}}{\longrightarrow} \text{Funct}_{\text{ex,cont}}(\mathbf{C}, \text{Sptr})$$

is an equivalence, see the proof of Proposition 9.4.4.

10.4.5. Finally, we mention that according to Sect. 8.3.4, the above symmetric monoidal structure on the $(\infty,1)$-category $\text{DGCat}_{\text{cont}}$ naturally upgrades to a symmetric monoidal structure on the $(\infty,2)$-category

$$\text{DGCat}^{2\text{-Cat}}_{\text{cont}}.$$

10.5. **Compact objects and ind-completions.**

10.5.1. Let $\mathbf{C}$ be an object of DGCat. We note that, according to Corollary 9.3.4, an object $\mathbf{c} \in \mathbf{C}$ is compact if and only if it is compact relative to Vect, i.e., the functor

$$\mathcal{M}aps_{k,\mathbf{C}}(\mathbf{c}, -), \quad \mathbf{C} \to \text{Vect}$$

preserves filtered colimits (equivalently, direct sums or all colimits).

Alternatively, the equivalence of the two notions follows from the fact that the functor

$$\text{Dold-Kan}^{\text{Sptr}} : \text{Vect} \to \text{Sptr}$$

is continuous and conservative.

10.5.2. The full subcategory $\mathbf{C}^c \subset \mathbf{C}$ is preserved by the monoidal operation

$$\text{Vect}^{\text{f.d}} \times \mathbf{C} \to \mathbf{C}.$$

Hence, $\mathbf{C}^c$ naturally acquires a structure of object of $\text{DGCat}^{\text{non-cocmpl}}$.

10.5.3. Vice versa, let $\mathbf{C}_0$ be an object of $\text{DGCat}^{\text{non-cocmpl}}$. Consider the corresponding object

$$\text{Ind}(\mathbf{C}_0) = \text{Funct}_{\text{ex}}((\mathbf{C}_0)^{\text{op}}, \text{Sptr}).$$

The action of $\text{Vect}^{\text{f.d.}}$ on $\mathbf{C}_0$ defines an action of $\text{Vect}^{\text{f.d.}}$ on $\text{Ind}(\mathbf{C}_0)$ by Sect. 4.1.7.

By Lemma 10.3.4(b), since $\text{Ind}(\mathbf{C}_0)$ is cocomplete, we obtain that $\text{Ind}(\mathbf{C}_0)$ is an object of DGCat.

10.5.4. By construction, the tautological functor

$$\mathbf{C}_0 \to \mathrm{Ind}(\mathbf{C}_0)$$

is a functor of $\mathrm{Vect}^{\mathrm{f.d.}}$-module categories.

For $\mathbf{C} \in \mathrm{DGCat}_{\mathrm{cont}}$, the equivalence

$$\mathrm{Funct}_{\mathrm{ex,cont}}(\mathrm{Ind}(\mathbf{C}_0), \mathbf{C}) \to \mathrm{Funct}_{\mathrm{ex}}(\mathbf{C}_0, \mathbf{C})$$

is a functor of bimodule categories over $\mathrm{Vect}^{\mathrm{f.d.}}$. Hence, combining with Lemma 10.3.4(c), we obtain an equivalence

$$\mathrm{Funct}_{k,\mathrm{cont}}(\mathrm{Ind}(\mathbf{C}_0), \mathbf{C}) \simeq \mathrm{Funct}_{k}(\mathbf{C}_0, \mathbf{C}). \tag{10.5}$$

In other words, we obtain that the ind-completion of $\mathbf{C}_0$ as a plain stable category is also the ind-completion as a DG category.

10.5.5. We claim that the DG category $\mathrm{Ind}(\mathbf{C}_0)$ can also be described as $\mathrm{Funct}_k((\mathbf{C}_0)^{\mathrm{op}}, \mathrm{Vect})$. More precisely:

Lemma 10.5.6. *The functor*

$$\mathrm{Funct}_k((\mathbf{C}_0)^{\mathrm{op}}, \mathrm{Vect}) \to \mathrm{Funct}_{\mathrm{ex}}((\mathbf{C}_0)^{\mathrm{op}}, \mathrm{Vect}) \overset{\text{Dold-Kan}^{\mathrm{Sptr}}}{\longrightarrow} \mathrm{Funct}_{\mathrm{ex}}((\mathbf{C}_0)^{\mathrm{op}}, \mathrm{Sptr}) = \mathrm{Ind}(\mathbf{C}_0)$$

is an equivalence.

Proof. By (10.5), we have

$$\mathrm{Funct}_k((\mathbf{C}_0)^{\mathrm{op}}, \mathrm{Vect}) \simeq \mathrm{Funct}_{k,\mathrm{cont}}(\mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}), \mathrm{Vect}),$$

which by Sect. 10.4.3 identifies with

$$(\mathrm{Ind}((\mathbf{C}_0)^{\mathrm{op}}))^{\vee} \simeq \mathrm{Ind}(\mathbf{C}_0),$$

as required.

□

10.5.7. It follows from Corollary 8.7.4 that if $\mathbf{C}$ and $\mathbf{D}$ are compactly generated DG categories, then the same is true for $\mathbf{C} \underset{\mathrm{Vect}}{\otimes} \mathbf{D}$. Moreover, objects of the form

$$\mathbf{c} \underset{k}{\boxtimes} \mathbf{d} \in \mathbf{C} \underset{\mathrm{Vect}}{\otimes} \mathbf{D}, \quad \mathbf{c} \in \mathbf{C}^c, \mathbf{d} \in \mathbf{D}^d$$

are the compact generators of $\mathbf{C} \underset{\mathrm{Vect}}{\otimes} \mathbf{D}$.

In addition, the following is obtained by repeating the proof of Proposition 7.4.2:

Proposition 10.5.8. *For $\mathbf{c}_0, \mathbf{c} \in \mathbf{C}$ and $\mathbf{d}_0, \mathbf{d} \in \mathbf{D}$ with $\mathbf{c}_0, \mathbf{d}_0$ compact, we have a canonical isomorphism*

$$\mathcal{M}aps_{k,\mathbf{C}}(\mathbf{c}_0, \mathbf{c}) \underset{k}{\otimes} \mathcal{M}aps_{k,\mathbf{D}}(\mathbf{d}_0, \mathbf{d}) \simeq \mathcal{M}aps_{k,\mathbf{C} \underset{\mathrm{Vect}}{\otimes} \mathbf{D}}(\mathbf{c}_0 \underset{k}{\boxtimes} \mathbf{d}_0, \mathbf{c} \underset{k}{\boxtimes} \mathbf{d}).$$

10.6. **Change of notations.** In the main body of the book, the only stable categories that we will ever encounter will be DG categories. For this reason we will simplify our notations as follows:

- For $\mathbf{C} \in \mathrm{DGCat}^{\text{non-cocmpl}}$ and $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$ we will write $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)$ instead of $\mathcal{M}aps_{k,\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)$ (i.e., our $\mathcal{M}aps_{\mathbf{C}}(-,-)$ is an object of Vect, rather than Sptr; the latter is obtained by applying the functor $\text{Dold-Kan}^{\text{Sptr}}$);
- For $\mathbf{C}, \mathbf{D} \in \mathrm{DGCat}^{\text{non-cocmpl}}$ we will write $\mathrm{Funct}(\mathbf{D}, \mathbf{C})$ instead of $\mathrm{Funct}_k(\mathbf{D}, \mathbf{C})$;
- For $\mathbf{C}, \mathbf{D} \in \mathrm{DGCat}$ we will write $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}, \mathbf{C})$ instead of $\mathrm{Funct}_{k,\mathrm{cont}}(\mathbf{D}, \mathbf{C})$;
- For $\mathbf{C}, \mathbf{D} \in \mathrm{DGCat}$ we will write $\mathbf{C} \otimes \mathbf{D}$ instead of $\mathbf{C} \underset{\mathrm{Vect}}{\otimes} \mathbf{D}$;
- For $\mathbf{C}, \mathbf{D} \in \mathrm{DGCat}$ and $\mathbf{c} \in \mathbf{C}, \mathbf{d} \in \mathbf{D}$ we will write $\mathbf{c} \boxtimes \mathbf{d}$ instead of $\mathbf{c} \underset{k}{\boxtimes} \mathbf{d}$.