CHAPTER I.2. BASICS OF DERIVED ALGEBRAIC GEOMETRY

Contents

Date: November 21, 2021.

Introduction

This Chapter is meant to introduce the basic objects of study in derived algebraic geometry that will be used in the subsequent chapters.

0.1. **Why prestacks?** The most general (and, perhaps, also the most important) type of algebro-geometric object that we will introduce is the notion of *prestack*.

0.1.1. Arguably, there is an all-pervasive problem with how one introduces classical algebraic geometry. Even nowadays, any introductory book on algebraic geometry defines schemes as *locally ringed spaces*. The problem with this is that a locally ringed space is a lot of structure, so the definition is quite heavy.

However, one does not have to go this way if one adopts Grothendieck's language of points. Namely, whatever the category of schemes is, it embeds fully faithfully into the category of functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Set},$$

where $\mathrm{Sch}^{\mathrm{aff}}$ is the category of affine schemes, i.e., $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ is the category of commutative rings.

Now, it is not difficult to characterize which functors $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Set}$ correspond to schemes: essentially the functor needs to have a Zariski atlas, a notion that has an intrinsic meaning.

0.1.2. This is exactly the point of view that we will adopt in this Chapter and throughout the book, with the difference that instead of classical (=usual=ordinary) affine schemes we consider derived affine schemes, where, by definition, the category of the latter is the one opposite to the category of connective commutative DG algebras (henceforth, when we write $\mathrm{Sch}^{\mathrm{aff}}$ we will mean the derived version, and denote the category of classical affine schemes by ${}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$).

And instead of functors with values in the category Set of sets we consider the category of functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}, \tag{0.1}$$

where Spc is the category of *spaces* (a.k.a. ∞-groupoids).

We denote the category of functors (0.1) by PreStk and call its objects *prestacks*. I.e., a prestack is something that has a Grothendieck functor of points attached to it, with no further conditions or pieces of structure.

Thus, a prestack is the most general kind of space that one can have in algebraic geometry.

All other kinds of algebro-geometric objects that we will encounter will be prestacks, that *have some particular properties* (as opposed to *extra pieces of structure*). This includes schemes (considered in Sect. 3), Artin stacks (considered in Sect. 4), ind-schemes and inf-schemes (considered in [Chapter III.2]), formal moduli problems (considered in [Chapter IV.1]), etc.

0.1.3. However, the utility of the notion of prestack goes beyond being a general concept that contains the other known types of algebro-geometric objects as particular cases.

Namely, there are some algebro-geometric constructions that can be carried out in this generality, and it turns out to be convenient to do so.

The central among these is the assignment to a prestack $\mathcal{Y}$ of the category $\mathrm{QCoh}(\mathcal{Y})$ of quasi-coherent sheaves on $\mathcal{Y}$, considered in the next Chapter, [Chapter I.3]. In fact, there is a canonically defined functor

$$\mathrm{QCoh}^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}, \quad \mathcal{Y} \mapsto \mathrm{QCoh}(\mathcal{Y}).$$

The definition of $\mathrm{QCoh}^*_{\mathrm{PreStk}}$ is actually automatic: it is the *right Kan extension* of the functor

$$\mathrm{QCoh}^*_{\mathrm{Sch}^{\mathrm{aff}}} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to 1\text{-Cat}$$

that attaches to

$$\mathrm{Spec}(A) = S \in \mathrm{Sch}^{\mathrm{aff}}$$

the DG category

$$\mathrm{QCoh}(S) := A\text{-mod}$$

and to a map $f : S' \to S$ the pullback functor $f^* : \mathrm{QCoh}(S) \to \mathrm{Coh}(S')$.

In other words,

$$\mathrm{QCoh}(\mathcal{Y}) = \lim_{(S \xrightarrow{y} \mathcal{Y}) \in ((\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S). \tag{0.2}$$

So an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is a assignment

$$(S \xrightarrow{y} \mathcal{Y}) \rightsquigarrow \mathcal{F}_{S,y} \in \mathrm{QCoh}(S),$$

$$(S' \xrightarrow{f} S) \rightsquigarrow \mathcal{F}_{S',y\circ f} \simeq f^*(\mathcal{F}_{S,y}),$$

satisfying a *homotopy compatible system of compatibilities* for compositions of morphisms between affine schemes.

Note that the expression in (0.2) involves taking a limit in the ∞-category 1-Cat. Thus, in order to assign a meaning to it (equivalently, the meaning to the expression 'homotopy compatible system of compatibilities') we need to input the entire machinery of ∞-categories, developed in [Lu1]. Thus, it is fair to say that Lurie gave us the freedom to consider quasi-coherent sheaves on prestacks.

Note that before the advent of the language of ∞-categories, the definition of the (derived) category of quasi-coherent sheaves on even such benign objects as algebraic stacks was quite awkward (see [LM]). Essentially, in the past, each time one needed to construct a triangulated category, one had to start from an abelian category, take its derived category, and then perform some manipulations on it in order to obtained the desired one.

As an application of the assignment

$$\mathcal{Y} \rightsquigarrow \mathrm{QCoh}(\mathcal{Y})$$

we obtain an automatic construction of the category of D-modules/crystals (see [Chapter III.4]). Namely,

$$\mathrm{D\text{-}mod}(\mathcal{Y}) := \mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}}),$$

where $\mathcal{Y}_{\mathrm{dR}}$ is the *de Rham prestack* of $\mathcal{Y}$.

0.1.4. Another example of a theory that is convenient to develop in the generality of prestacks is *deformation theory*, considered in [Chapter III.1]. Here, too, it is crucial that we work in the context of derived (as opposed to classical) algberaic geometry.

0.1.5. As yet another application of the general notion of prestack is the construction of the *Ran space* of a given scheme, along with its category of quasi-coherent sheaves or D-modules. We will not discuss it explicitly in this book, and refer the reader to, e.g., [Ga2].

0.2. **What do we say about prestacks?** The notion of prestack is so general that it is, of course, impossible to prove anything non-trivial about arbitrary prestacks.

What we do in Sect. 1 is study some very formal properties of prestacks, which will serve us in the later chapters of this book.

0.2.1. *The notion of* n*-coconnectivity.* As was said before, the category PreStk is that of functors $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$, where

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} := \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}).$$

Now, arguably, the category $\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$ is complicated, and it is natural to try to approach it via its successive approximations, namely, the categories

$$\mathrm{ComAlg}(\mathrm{Vect}^{\geq -n, \leq 0})$$

of connective commutative DG algebras that live in cohomological degrees $\geq -n$.

We denote the corresponding full subcategory in $\mathrm{Sch}^{\mathrm{aff}}$ by ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$; we call its objects n-coconnective affine schemes. We can consider the corresponding category of functors

$$({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$$

and denote it by ${}^{\leq n}\mathrm{PreStk}$.

The ∞-categories ${}^{\leq n}\mathrm{PreStk}$ and ${}^{\leq n}\mathrm{Stk}$ are related by a pair of mutually adjoint functors

$$(0.3) \qquad {}^{\leq n}\mathrm{PreStk} \rightleftarrows \mathrm{PreStk},$$

given by restriction and *left Kan extension* along the inclusion ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$, respectively, with the left adjoint in (0.3) being fully faithful.

Thus, we can think of each ${}^{\leq n}\mathrm{PreStk}$ as a full subcategory in PreStk; we referred to its objects as n-coconnective prestacks. Informally, a functor in (0.1) is n-coconnective if it is *completely determined* by its values on n-coconnective affine schemes.

The subcategories ${}^{\leq n}\mathrm{PreStk}$ form a sequence of approximations to PreStk.

0.2.2. *Convergence.* A technically convenient condition that one can impose on a prestack is that of *convergence*. By definition, a functor $\mathcal{Y}$ in (0.1) is convergent if for any $S \in \mathrm{Sch}^{\mathrm{aff}}$ the map

$$\mathcal{Y}(S) \to \lim_{n} \mathcal{Y}({}^{\leq n}S),$$

is an isomorphism, where ${}^{\leq n}S$ denotes the n-coconnective truncation of S.

Convergence is a necessary condition for a prestack to satisfy in order to *admit deformation theory*, see [Chapter III.1, Sect. 7.1].

0.2.3. *Finite typeness.* Consider the categories ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ and ${}^{\leq n}\mathrm{PreStk}$. We shall say that an object $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ (resp., $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$) is of *finite type* (resp., *locally of finite type*) if the corresponding functor (0.1) takes *filtered limits of affine schemes* to colimits in Spc.

It follows tautologically that an object $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ is locally of finite type if and only if the corresponding functor (0.1) is completely determined by its values on affine schemes of finite type.

Now, the point is that, as in the case of classical algebraic geometry, the condition on an object $\mathrm{Spec}(A) = S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ to be of finite type is very explicit: it is equivalent to $H^0(A)$ being finitely generated over our ground field, and each $H^{-i}(A)$ (where i runs from 1 to n) being finitely generated as a module over $H^0(A)$.

Thus, objects of ${}^{\leq n}\mathrm{PreStk}$ that are locally of finite type are precisely those that can be expressed via affine schemes that are 'finite dimensional'.

0.2.4. *Inserting the word 'almost'.* Consider now the category PreStk.

We shall say that a prestack is *locally almost of finite type* if it is convergent, and for any n, the functor $\leftarrow$ in (0.3) produces from it an object locally of finite type.

The class of prestacks locally almost of finite type will play a central role in this book. Namely, it is for this class of prestacks that we will develop the theory of ind-coherent sheaves and crystals.

0.3. **What else is done in this Chapter?**

0.3.1. In Sect. 2 we introduce a hierarchy of Grothendieck topologies on $\mathrm{Sch}^{\mathrm{aff}}$: flat, ppf, étale, Zariski. Each of the above choices gives rise to a full subcategory

$$\mathrm{Stk} \subset \mathrm{PreStk}$$

consisting of objects that satisfy the corresponding descent condition. We refer to the objects of Stk as *stacks*.

The primary interest in Sect. 2 is how the descent condition interacts with the conditions of n-coconnectivity, convergence and local (almost) finite typeness.

0.3.2. In the rest of this Chapter we discuss two specific classes of stacks: schemes and Artin stacks (the former being a particular case of the latter).

The corresponding sections are essentially a paraphrase of some parts of [TV1, TV2] in the language of ∞-categories.

0.3.3. In Sect. 3 we introduce the full subactegory $\mathrm{Sch} \subset \mathrm{PreStk}$ of (derived) schemes[1].

Essentially, a prestack Z is a scheme if it is a stack and admits a Zariski atlas (i.e., a collection of affine schemes S_i equipped with *open embeddings* $S_i \to \mathcal{Y}$).

We will not go deep into the study of derived schemes, but content ourselves with establishing the properties related to n-coconnectivity and finite typeness. These can be summarized by saying that a scheme is n-coconnective (resp., of finite type) if and only if some (equivalently, any) Zariski atlas consists of affine schemes that are n-coconnective (resp., of finite type).

0.3.4. In Sect. 4 we introduce the hierarchy of k-Artin stacks, $k = 0, 1, 2....$ Our definition is a variation of the notion of a k-geometric stack defined by Simpson in [Sim] and developed in the derived context in [TV2].

For an individual k, what we call a k-Artin stack may be different from what is accepted elsewhere in the literature (e.g., in our definition, only schemes that are disjoint unions of affines are 0-Artin stacks; all other schemes are 1-Artin stacks). However, the union over all k produces the same class of objects as in other definitions, called Artin stacks.

The definition of k-Artin stacks proceeds by induction on k. By definition, a k-Artin stack is an étale prestack that admits a *smooth* $(k-1)$*-representable atlas* by affine schemes.

As in the case of schemes, we will only discuss the properties of Artin stacks related to n-coconnectivity and finite typeness, with results parallel to those mentioned above: an Artin stack is n-coconnective (resp., of finite type) if and only if some (equivalently, any) smooth atlas consists of affine schemes that are n-coconnective (resp., of finite type).

[1]Henceforth we will drop the adjective 'derived'.

1. Prestacks

In this section we introduce the principal actors in derived algebraic geometry: prestacks.

We will focus on the very formal aspects of the theory, such as what it means for a prestack to be n-coconnective (for some integer n) or to be locally (almost) of finite type.

1.1. **The notion of prestack.**

Derived algebraic geometry is 'born' from connective commutative DG algebras, in the same way as classical algebraic geometry (over a given ground field k) is born from commutative algebras. Following Grothendieck, we will think of algebro-geometric objects as *prestacks*, i.e., arbitrary functors from the ∞-category of connective commutative DG algebras to that of spaces.

1.1.1. Consider the stable symmetric monoidal Vect, and its full monoidal subcategory $\mathrm{Vect}^{\leq 0}$. By a connective commutative DG algebra over k we shall mean a commutative algebra object in $\mathrm{Vect}^{\leq 0}$. The totality of such algebras forms an $(\infty,1)$-category, $\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$.

Remark 1.1.2. Note that what we call a 'connective commutative DG algebra over k' is really an abstract notion: we are appealing to the general notion of commutative algebra in symmetric monoidal category from [Chapter I.1, Sect. 3.3].

However, one can show (see [Lu2, Proposition 7.1.4.11]) that the homotopy category of the ∞-category $\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$ is a familiar object: it is obtained from the category of what one classically calls 'commutative differential graded algebras over k concentrated in degrees ≤ 0' by inverting quasi-isomorphisms.

1.1.3. We define the category of (derived) affine schemes over k to be

$$\mathrm{Sch}^{\mathrm{aff}} := (\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}))^{\mathrm{op}}.$$

1.1.4. By a (derived) prestack we shall mean a functor $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$. We let PreStk denote the $(\infty,1)$-category of prestacks, i.e.,

$$\mathrm{PreStk} := \mathrm{Funct}((\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathrm{Spc}).$$

1.1.5. Yoneda defines a fully faithful embedding

$$\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}\,.$$

For $S \in \mathrm{Sch}^{\mathrm{aff}}$ and $\mathcal{Y} \in \mathrm{PreStk}$ we have, tautologically,

$$\mathrm{Maps}_{\mathrm{PreStk}}(S, \mathcal{Y}) \simeq \mathcal{Y}(S).$$

1.1.6. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map of prestacks. We shall say that f is *affine schematic* if for every $S \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}_2}$, the fiber product $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \in \mathrm{PreStk}$ is representable by an affine scheme.

1.2. **Coconnectivity conditions: affine schemes.**

Much of the analysis in derived algebraic geometry proceeds by induction on how many negative cohomological degrees we allow our DG algebras to live in. We initiate this discussion in the present subsection.

1.2.1. For $n \geq 0$, consider the full subcategory

$$\mathrm{Vect}^{\geq -n, \leq 0} \subset \mathrm{Vect}^{\leq 0}.$$

This fully faithful embedding admits a left adjoint, given by the truncation functor $\tau^{\geq -n}$. It is clear that if $V_1' \to V_1'$ is a morphism in $\mathrm{Vect}^{\leq 0}$ such that $\tau^{\geq -n}(V_1') \to \tau^{\geq -n}(V_1')$ is an isomorphism, then

$$\tau^{\geq -n}(V_1' \otimes V_2) \to \tau^{\geq -n}(V_1' \otimes V_2)$$

is an isomorphism for any $V_2 \in \mathrm{Vect}^{\leq 0}$.

This implies that the $(\infty, 1)$-category $\mathrm{Vect}^{\geq -n, \leq 0}$ acquires a uniquely defined symmetric monoidal structure for which the functor $\tau^{\geq -n}$ is symmetric monoidal. It follows from the *symmetric* monoidal version of [Chapter I.1, Lemma 3.2.4] that the embedding

$$\mathrm{Vect}^{\geq -n, \leq 0} \hookrightarrow \mathrm{Vect}^{\leq 0} \tag{1.1}$$

has a natural *right-lax* symmetric monoidal structure.

1.2.2. In particular, the functor (1.1) induces a fully faithful functor

$$\mathrm{ComAlg}(\mathrm{Vect}^{\geq -n, \leq 0}) \to \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}), \tag{1.2}$$

whose essential image consists of those objects of $\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$ that belong to $\mathrm{Vect}^{\geq -n, \leq 0}$ when regarded as plain objects of $\mathrm{Vect}^{\leq 0}$.

The functor (1.2) admits a left adjoint

$$\tau^{\geq -n} : \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}) \to \mathrm{ComAlg}(\mathrm{Vect}^{\geq -n, \leq 0}) \tag{1.3}$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}) & \xrightarrow{\tau^{\geq -n}} & \mathrm{ComAlg}(\mathrm{Vect}^{\geq -n, \leq 0}) \\ {\scriptstyle \mathbf{oblv}_{\mathrm{ComAlg}}} \downarrow & & \downarrow {\scriptstyle \mathbf{oblv}_{\mathrm{ComAlg}}} \\ \mathrm{Vect}^{\leq 0} & \xrightarrow{\tau^{\geq -n}} & \mathrm{Vect}^{\geq -n, \leq 0} \end{array}$$

commute.

1.2.3. We shall say that $S \in \mathrm{Sch}^{\mathrm{aff}}$ is n-coconnective if $S = \mathrm{Spec}(A)$ with A lying in the essential image of (1.2). In other words, if $H^{-i}(A) = 0$ for $i > n$.

We shall denote the full subcategory of $\mathrm{Sch}^{\mathrm{aff}}$ spanned by n-coconnective objects by ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

1.2.4. For $n = 0$ we recover

$${}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}} := {}^{\leq 0}\mathrm{Sch}^{\mathrm{aff}},$$

the category of classical affine schemes.

1.2.5. The embedding ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$ admits a right adjoint, denoted

$$S \mapsto {}^{\leq n}S,$$

and given at the level of commutative DG algebras by the functor (1.3).

Thus, ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ is a *colocalization* of $\mathrm{Sch}^{\mathrm{aff}}$. We denote the corresponding colocalization functor

$$\mathrm{Sch}^{\mathrm{aff}} \to {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$$

by $S \mapsto \tau^{\leq n}(S)$.

Remark 1.2.6. We choose to notationally distinguish objects of ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ and their images in $\mathrm{Sch}^{\mathrm{aff}}$. Doing otherwise would cause notational clashes when talking about descent conditions.

1.2.7. We will say that $S \in \mathrm{Sch}^{\mathrm{aff}}$ is *eventually coconnective* if it belongs to ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ for some n.

We denote the full subcategory of $\mathrm{Sch}^{\mathrm{aff}}$ spanned by eventually coconnective objects by ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$.

1.3. **Coconnectivity conditions: prestacks.**

1.3.1. Consider the $(\infty, 1)$-category

$$ {}^{\leq n}\mathrm{PreStk} := \mathrm{Funct}(({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathrm{Spc}). $$

Restriction defines a functor

$$ \mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}, \tag{1.4} $$

that we will denote by $\mathcal{Y} \mapsto {}^{\leq n}\mathcal{Y}$.

1.3.2. The functor (1.4) admits a *fully faithful left adjoint*, given by the left Kan extension

$$ \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}\,. $$

Thus, ${}^{\leq n}\mathrm{PreStk}$ is a colocalization of PreStk. We denote the resulting colocalization functor

$$ \mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk} $$

by $\mathcal{Y} \mapsto \tau^{\leq n}(\mathcal{Y})$.

Remark 1.3.3. The usage of the symbol $\tau^{\leq n}$ may diverge from other sources' conventions: the latter use $\tau^{\leq n}$ to denote the corresponding truncation of the Postnikov tower, whereas we denote the latter by the symbol $\mathsf{P}_{\leq n}$, see Sect. 1.8.5 below.

Tautologically, if $\mathcal{Y}$ is representable by an affine scheme $S = \mathrm{Spec}(A)$, then the above two meanings of $\tau^{\leq n}$ coincide: the prestack $\tau^{\leq n}(\mathcal{Y})$ is representable by the affine scheme $\tau^{\leq n}(S)$.

1.3.4. We shall say that $\mathcal{Y} \in \mathrm{PreStk}$ is n-coconnective if it belongs to the essential image of the functor $\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}$.

For example, an affine scheme is n-coconnective in the sense of Sect. 1.2.3 if and only if its image under the Yoneda functor is n-coconnective as a prestack.

We will often identify ${}^{\leq n}\mathrm{PreStk}$ with its essential image under the above functor, and thus think of ${}^{\leq n}\mathrm{PreStk}$ as a full subcategory of PreStk.

1.3.5. We will say that $\mathcal{Y} \in \mathrm{PreStk}$ is *eventually coconnective* if it is n-coconnective for some n. We shall denote the full subcategory of eventually coconnective objects of PreStk by ${}^{<\infty}\mathrm{PreStk}$.

1.3.6. *Classical prestacks.* Let $n = 0$. We shall call objects of ${}^{\leq 0}\mathrm{PreStk}$ 'classical' prestacks, and use for it also the alternative notation ${}^{\mathrm{cl}}\mathrm{PreStk}$.

We will also denote the corresponding restriction functor $\mathcal{Y} \mapsto {}^{\mathrm{cl}}\mathcal{Y}$, and the corresponding colocalization functor

$$ \mathrm{PreStk} \to {}^{\mathrm{cl}}\mathrm{PreStk} \to \mathrm{PreStk} $$

by $\mathcal{Y} \mapsto \tau^{\mathrm{cl}}(\mathcal{Y})$.

1.3.7. *The right Kan extension.* The restriction functor

$$\mathcal{Y} \mapsto {}^{\leq n}\mathcal{Y} : \text{PreStk} \to {}^{\leq n}\text{PreStk}$$

admits also a right adjoint, given by *right Kan extension.*

This functor lacks a clear geometric meaning. However, it can be explicitly described: by adjunction we have

$$(\text{RKE}_{{}^{\leq n}\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}}(\mathcal{Y}))\,(S) \simeq \mathcal{Y}(\tau^{\leq n}(S)).$$

1.4. **Convergence.** The idea of the notion of convergence is that if we perceive a connective commutative DG algebra as built iteratively by adding lower and lower cohomologies, we can ask whether the value of a given prestack on such an algebra to be determined by its values on the above sequence of truncations.

Convergence is a necessary condition if we want to approach our prestack via deformation theory (see [Chapter III.1, Sect. 7.1]).

1.4.1. Le S be an object of Sch^{aff}. Note that the assignment

$$n \mapsto \tau^{\leq n}(S)$$

is naturally a functor

$$\mathbb{Z}^{\geq 0} \to (\text{Sch}^{\text{aff}})_{/S}.$$

1.4.2. Let $\mathcal{Y}$ be a prestack. We say that $\mathcal{Y}$ is convergent if for $S \in \text{Sch}^{\text{aff}}$, the map

$$\mathcal{Y}(S) \to \lim_{n} \mathcal{Y}(\tau^{\leq n}(S))$$

is an isomorphism.

1.4.3. Since for every connective commutative DG algebra A, the map

$$A \to \lim_{n} \tau^{\geq -n}(A)$$

is an isomorphism, we have:

Lemma 1.4.4. *Any prestack representable by a (derived) affine scheme is convergent.*

Remark 1.4.5. As we shall see in the sequel, all prestacks ‘of geometric nature’, such as (derived) schemes and Artin stacks (and also ind-schemes), are convergent.

Here is, however, an example of a non-convergent prestack: consider the prestack that associates to an affine scheme $S = \text{Spec}(A)$ the category $(A\text{-mod})^{\text{Spc}}$, i.e., this is the prestack

$$(\text{Sch}^{\text{aff}})^{\text{op}} \overset{\text{QCoh}^*_{\text{Sch}^{\text{aff}}}}{\longrightarrow} 1\text{-Cat} \overset{\mathbf{C} \mapsto \mathbf{C}^{\text{Spc}}}{\longrightarrow} \text{Spc},$$

where $\text{QCoh}^*_{\text{Sch}^{\text{aff}}}$ is as in [Chapter I.3, Sect. 1.1.2].

1.4.6. We have:

Proposition 1.4.7. *A prestack $\mathcal{Y}$ is convergent if and only if, when as a functor*

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

it is the right *Kan extension from the subcategory* ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{Sch}^{\mathrm{aff}}$.

Proof. We claim that the functor of right Kan extension along ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{Sch}^{\mathrm{aff}}$ is given by sending

$$\mathcal{Z}' \in \mathrm{Funct}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathrm{Spc}) \mapsto \mathcal{Z} \in \mathrm{Funct}((\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathrm{Spc}),$$

with

$$\mathcal{Z}(S) = \lim_n \mathcal{Z}'(\tau^{\leq n}(S)).$$

Indeed, a priori, the value of $\mathcal{Z}$ on S is given by

$$\lim_{S' \to S} \mathcal{Z}'(S'),$$

where the limit is taken over the category opposite to $({}^{<\infty}\,\mathrm{Sch}^{\mathrm{aff}})_{/S}$. Now, the assertion follows from the fact that the functor

$$\mathbb{Z}^{\geq 0} \to ({}^{<\infty}\,\mathrm{Sch}^{\mathrm{aff}})_{/S}, \quad n \mapsto \tau^{\leq n}(S)$$

is cofinal.

□

1.4.8. Let ${}^{\mathrm{conv}}\mathrm{PreStk} \subset \mathrm{PreStk}$ denote the full subcategory of convergent prestacks. This embedding admits a left adjoint, which we call the *convergent completion* and denote by

$$\mathcal{Y} \mapsto {}^{\mathrm{conv}}\mathcal{Y}.$$

According to Proposition 1.4.7, we have:

$${}^{\mathrm{conv}}\mathcal{Y} \simeq \mathrm{RKE}_{{}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}|_{{}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}}).$$

Explicitly,

$${}^{\mathrm{conv}}\mathcal{Y}(S) = \lim_n \mathcal{Y}(\tau^{\leq n}(S)).$$

1.4.9. Consider the canonical map

$$\operatorname*{colim}_n \tau^{\leq n}(\mathcal{Y}) \to \mathcal{Y}.$$

Tautologically, $\mathcal{Y}_1 \in \mathrm{PreStk}$ is convergent if and only if for every $\mathcal{Y}$, the map

$$\mathrm{Maps}(\mathcal{Y}, \mathcal{Y}_1) \to \mathrm{Maps}(\operatorname*{colim}_n \tau^{\leq n}(\mathcal{Y}), \mathcal{Y}_1) = \lim_n \mathrm{Maps}(\tau^{\leq n}(\mathcal{Y}), \mathcal{Y}_1)$$

is an isomorphism.

Remark 1.4.10. Note that the *left* Kan extension functor

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}$$

does *not* map into ${}^{\mathrm{conv}}\mathrm{PreStk}$.

1.5. **Affine schemes of finite type (the eventually coconnective case).** We will now introduce the notion of what it means for a (derived) affine scheme to be of finite type. This generalizes the usual notion of being of finite type over a field. As in classical algebraic geometry, *finite typeness* puts us in the context of finite-dimensional geometry.

1.5.1. We say that an object $S = \operatorname{Spec}(A) \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ is *of finite type* if $H^0(A)$ is of finite type over k, and each $H^{-i}(A)$ is finitely generated as a module over $H^0(A)$.

Let ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ denote the full subcategory of ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ consisting of affine schemes of finite type.

1.5.2. Denote by ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ the intersection ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \cap {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

The following theorem is proved by induction on n using deformation theory (but we will not do it here, but see [Lu2, Proposition 7.2.5.31]):

Theorem 1.5.3.
(a) *The objects of* $({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}$ *are compact in* $({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$.
(b) *For every object* $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, *the category* opposite to $({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{S/}$ *is filtered, and the map*

$$S \mapsto \lim_{S_0 \in ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{S/}} S_0$$

is an isomorphism.

Remark 1.5.4. We note that the filteredness assertion in Theorem 1.5.3(b) is easy: it follows from the fact that the category $({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{S/}$ has fiber products.

1.5.5. By [Lu1, Proposition 5.3.5.11], the assertion of Theorem 1.5.3 is equivalent to the following:

Corollary 1.5.6. *We have a canonical equivalence:*

$${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \simeq \mathrm{Pro}({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}).$$

1.5.7. Since ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ is closed under retracts, using [Lu1, Lemma 5.4.2.4], from Corollary 1.5.6 we obtain:

Corollary 1.5.8. *The inclusion* $({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \subset (({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}})^c$ *of Theorem 1.5.6(a) is an equality.*

1.6. **Prestacks locally of finite type (the eventually coconnective case).** In this subsection we will make precise the following idea: a prestack is locally of finite type if and only if it is completely determined by its values on affine schemes of finite type.

1.6.1. Let $\mathcal{Y}$ be an object of ${}^{\leq n}\mathrm{PreStk}$ for some n. We say that it is *locally of finite type* if it is the left Kan extension (of its own restriction) along the embedding

$$({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}.$$

We denote the resulting full subcategory of ${}^{\leq n}\mathrm{PreStk}$ by ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$.

1.6.2. In other words, we can identify ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ with the category of functors

$$({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Spc},$$

and we have a pair of mutually adjoint functors

$${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \rightleftarrows {}^{\leq n}\mathrm{PreStk},$$

given by restriction and left Kan extension along ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, respectively, where the left Kan extension functor is fully faithful.

1.6.3. Now, using [Lu1, Proposition 5.3.5.10], from Corollary 1.5.6, we obtain:

Corollary 1.6.4. *An object* $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ *belongs to* ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ *if and only if it takes filtered limits in* ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ *to colimits in* Spc.

1.6.5. Combining Corollaries 1.6.4 and 1.5.8, we obtain:

Lemma 1.6.6. *Let S be an object of ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$. It belongs to ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ if and only if the prestack that it represents belongs to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$.*

1.6.7. Evidently, the restriction functor ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \leftarrow {}^{\leq n}\mathrm{PreStk}$ commutes with limits and colimits. The functor

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk},$$

being a left adjoint commutes with colimits.

In addition, we have the following:

Lemma 1.6.8. *The functor* $\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}$ *commutes with finite limits.*

Proof. This follows from Corollary 1.6.4: indeed, the condition of taking filtered limits in ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ to colimits in Spc is preserved by the operation of taking finite limits of prestacks. □

1.7. **The 'locally almost of finite type' condition.** In Sect. 1.6 we introduced the 'locally of finite type' condition for n-coconnective prestacks. In this subsection we will give a definition crucial for the rest of the book: what it means for an object of PreStk to be locally *almost* of finite type (=laft). This will be the class of prestacks for which we will develop the theory of ind-coherent sheaves.

1.7.1. We say that an affine (derived) scheme S is *almost of finite type* if ${}^{\leq n}S$ is of finite type for every n.

I.e., $S = \mathrm{Spec}(A)$ is almost of finite type if $H^0(A)$ is of finite type over k, and each $H^{-i}(A)$ is finitely generated as a module over $H^0(A)$.

Let $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ denote the full subcategory of $\mathrm{Sch}^{\mathrm{aff}}$ consisting of affine schemes almost of finite type.

1.7.2. We say that $\mathcal{Y} \in \mathrm{PreStk}$ is *locally almost of finite type* if the following conditions hold:

(1) $\mathcal{Y}$ is convergent.
(2) For every n, we have ${}^{\leq n}\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$

We denote the corresponding full subcategory by

$$\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}.$$

By Lemma 1.6.6, we have

$$\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} = \mathrm{Sch}^{\mathrm{aff}} \cap \mathrm{PreStk}_{\mathrm{laft}}.$$

1.7.3. In particular, if $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, then ${}^{\mathrm{cl}}\mathcal{Y}$ is an object of ${}^{\mathrm{cl}}\mathrm{PreStk}$ locally of finite type, i.e., it is a classical prestack locally of finite type.

Remark 1.7.4. Note that by Remark 1.4.10, the left Kan extension functor does *not* send ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ to $\mathrm{PreStk}_{\mathrm{laft}}$: the resulting prestack will satisfy the second condition, but in general, not the first one.

However, if $\mathcal{Y} \in \mathrm{PreStk}$ is obtained as a left Kan extension functor of an object of ${}^{\leq n}\mathrm{PreStk}$ that belongs to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$, then its convergent completion ${}^{\mathrm{conv}}\mathcal{Y}$ will belong to $\mathrm{PreStk}_{\mathrm{laft}}$, see Corollary 1.7.8 below.

1.7.5. We claim:

Proposition 1.7.6. *Restriction along* ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$ *defines an equivalence*

$$\mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{Funct}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}, \mathrm{Spc}).$$

The inverse functor is given by first applying the left *Kan extension along*

$${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}},$$

followed by the right *Kan extension along*

$${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}.$$

Proof. By Proposition 1.4.7, it suffices to show that the following conditions on a functor

$${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Spc}$$

are equivalent:

(i) It is a left Kan extension along ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \to {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$;

(ii) Its restriction to any ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ is a left Kan extension along ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \to {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

First, it is clear that (i) implies (ii): indeed, the diagram

$$\begin{array}{ccc} \mathrm{Funct}({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}, \mathrm{Spc}) & \xrightarrow{\mathrm{LKE}} & \mathrm{Funct}({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}, \mathrm{Spc}) \\ \downarrow & & \downarrow \\ \mathrm{Funct}({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}, \mathrm{Spc}) & \xrightarrow{\mathrm{LKE}} & \mathrm{Funct}({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}, \mathrm{Spc}) \end{array}$$

is commutative.

Vice versa, let $\mathcal{Y}$ satisfy (ii). We need to show that for any $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, the map

$$\underset{S \to S'}{\mathrm{colim}}\, \mathcal{Y}(S') \to \mathcal{Y}(S) \tag{1.5}$$

is an isomorphism, where the colimit is taken over the index category

$$\left(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{S/}\right)^{\mathrm{op}}.$$

However, cofinal in the above index category is the full subcategory consisting of those $S \to S'$, for which $S' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$; indeed the embedding of this full subcategory admits a left adjoint, given by $S' \mapsto \tau^{\leq n}(S')$.

Hence, the colimit in (1.5) can be replaced by

$$\underset{S \to S'}{\mathrm{colim}}\, \mathcal{Y}(S')$$

taken over $\left(({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{S/}\right)^{\mathrm{op}}$. However, the latter colimit computes

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}({}^{\leq n}\mathcal{Y}).$$

$\square$

1.7.7. We note:

Corollary 1.7.8. *The composite functor*

$$^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \hookrightarrow {}^{\leq n}\mathrm{PreStk} \overset{\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}}{\longrightarrow} \mathrm{PreStk} \overset{\mathcal{Y}\mapsto {}^{\mathrm{conv}}\mathcal{Y}}{\longrightarrow} {}^{\mathrm{conv}}\mathrm{PreStk}$$

takes values in $\mathrm{PreStk}_{\mathrm{laft}}$.

Proof. By Proposition 1.7.6, it suffices to show that the composition of the functor in the corollary with the identification

$$^{\mathrm{conv}}\mathrm{PreStk} \simeq \mathrm{Funct}({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}, \mathrm{Spc})$$

lands in the full subcategory spanned by functors obtained as a left Kan extension from

$$^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}.$$

However, the above composition is given by left Kan extension along

$$^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}.$$

□

1.7.9. By combining Lemma 1.6.8 and Proposition 1.7.6, we obtain:

Corollary 1.7.10. *The subcategory* $\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$ *is closed under finite limits.*

1.8. **Truncatedness.**

1.8.1. For $k = 0, 1, ...$, let $\mathrm{Spc}_{\leq k} \subset \mathrm{Spc}$ denote the full subcategory of *k-truncated* spaces. I.e., it is spanned by those objects $\mathcal{S} \in \mathrm{Spc}$ such that each connected component $\mathcal{S}'$ of $\mathcal{S}$ satisfies

$$\pi_l(\mathcal{S}') = 0 \text{ for } l > k.$$

For example, for $k = 0$, we have $\mathrm{Spc}_{\leq 0} = \mathrm{Set}$.

1.8.2. The embedding

$$\mathrm{Spc}_{\leq k} \hookrightarrow \mathrm{Spc}$$

admits a left adjoint.

The corresponding localization functor

$$\mathrm{Spc} \to \mathrm{Spc}_{\leq k} \to \mathrm{Spc}$$

will be denoted $\mathsf{P}_{\leq k}$.

Remark 1.8.3. The $(\infty, 1)$-category $\mathrm{Spc}_{\leq k}$ is actually a $(k+1, 1)$-category. I.e., the mapping spaces between objects are k-truncated.

1.8.4. For $\mathcal{S} \in \mathrm{Spc}$, the assignment $k \mapsto \mathsf{P}_{\leq k}(\mathcal{S})$ is a functor

$$(\mathbb{Z}^{\geq 0})^{\mathrm{op}} \to \mathrm{Spc},$$

called the *Postnikov tower* of $\mathcal{S}$.

It is a basic fact that the natural map

$$\mathcal{S} \to \lim_{k} \mathsf{P}_{\leq k}(\mathcal{S})$$

is an isomorphism.

1.8.5. For a fixed n, and an integer $k = 0, 1, ...$ we will say that $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ is *k-truncated* if, as a functor

$$({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

it takes values in the full subcategory of $\mathrm{Spc}_{\leq k} \subset \mathrm{Spc}$ of k-truncated spaces.

1.8.6. For example, if $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ is representable, i.e., is the Yoneda image of $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, then $\mathcal{Y}$ is n-truncated.

This reflects the fact that $\mathrm{ComAlg}(\mathrm{Vect}^{\geq -n, \leq 0})$ is an $(n+1, 1)$-category, which, in turn, formally follows from the fact that $\mathrm{Vect}^{\geq -n, \leq 0}$ is an $(n+1, 1)$-category.

Remark 1.8.7. In the sequel, we will see that for any (derived) scheme, its restriction to ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ is n-truncated as an object of ${}^{\leq n}\mathrm{PreStk}$.

Similarly, for a k-Artin stack, its restriction to ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ is $(n+k)$-truncated as an object of ${}^{\leq n}\mathrm{PreStk}$.

1.8.8. *Another example.* To any object $\mathcal{K} \in \mathrm{Spc}$ we can attach the corresponding constant prestack $\underline{\mathcal{K}}$:

$$\underline{\mathcal{K}}(S) := \mathcal{K}, \quad S \in \mathrm{Sch}^{\mathrm{aff}}.$$

If $\mathcal{K}$ is k-truncated, then $\underline{\mathcal{K}}$ is k-truncated.

1.8.9. Let ${}^{\leq n}\mathrm{PreStk}_{\leq k} \subset {}^{\leq n}\mathrm{PreStk}$ denote the full subcategory of k-truncated prestacks. This embedding admits a left adjoint. The corresponding localization functor

$${}^{\leq n}\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}_{\leq k} \to {}^{\leq n}\mathrm{PreStk}$$

will be denoted $\mathsf{P}_{\leq k}$. Explicitly,

$$(\mathsf{P}_{\leq k}(\mathcal{Y}))(S) = \mathsf{P}_{\leq k}(\mathcal{Y}(S)), \quad S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}.$$

The full subcategory ${}^{\leq n}\mathrm{PreStk}_{\leq k} \subset {}^{\leq n}\mathrm{PreStk}$ is actually a $(k+1, 1)$-category.

1.8.10. When $n = 0$ and $k = 0$, the (ordinary) category ${}^{\mathrm{cl}}\mathrm{PreStk}_{\leq 0}$ is that of *presheaves of sets* on ${}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$.

When $n = 0$ and $k = 1$, we shall call objects of ${}^{\mathrm{cl}}\mathrm{PreStk}_{\leq 1}$ 'ordinary classical prestacks'. I.e., ${}^{\mathrm{cl}}\mathrm{PreStk}_{\leq 1}$ is the $(2, 1)$-category of functors from the category of classical affine schemes to that of ordinary groupoids.

2. Descent and stacks

The object of study in this section is the notion of *stack*–the result of the interaction of the general notion of prestack with a given Grothendieck topology (flat, ppf, étale or Zariski) on the category of affine schemes; see [TV2, Sect. 2.2.2].

Specifically, we will be interested in how the stack condition interacts with n-coconnectivity and the finite typeness.

2.1. **Flat morphisms.** In this subsection we will introduce the crucial notion of flatness for a morphism between (derived) affine schemes. Knowing what it means to be flat, we will give the definition of what it means to be an open embedding, étale, smooth, ppf, etc.

2.1.1. Let us recall, following [TV2], the notion of flatness for a morphism between (derived) affine schemes:

A map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ between affine schemes is said to be flat if $H^0(B)$ is flat as a module over $H^0(A)$, plus the following *equivalent* conditions hold:

- The natural map
$$H^0(B) \underset{H^0(A)}{\otimes} H^i(A) \to H^i(B)$$
is an isomorphism for every i.
- For any A-module M, the natural map
$$H^0(B) \underset{H^0(A)}{\otimes} H^i(M) \to H^i(B \underset{A}{\otimes} M)$$
is an isomorphism for every i.
- If an A-module N is concentrated in degree 0 then so is $B \underset{A}{\otimes} N$.

2.1.2. Note in particular that if $S' \to S$ is flat, then
$$S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \quad \Rightarrow \quad S' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}.$$

The following assertion is easily established by induction:

Lemma 2.1.3. *For a map $S' \to S$ between affine schemes, S' is flat over S if and only if each ${}^{\leq n}S'$ is flat over ${}^{\leq n}S$.*

2.1.4. Let $f : S' \to S$ be a morphism of affine schemes. We shall say that it is ppf[2] (resp., smooth, étale, open embedding, Zariski) if the following conditions hold:

(1) The morphism f is flat (in particular, the base-changed (derived!) affine scheme
$$\tau^{\mathrm{cl}}(S) \underset{S}{\times} S'$$
is classical and thus identifies with $\tau^{\mathrm{cl}}(S')$);

(2) The map of classical affine schemes ${}^{\mathrm{cl}}S' \to {}^{\mathrm{cl}}S$ is of finite presentation (resp., smooth, étale, open embedding, disjoint union of open embeddings).

For future reference, we quote the following basic fact that can be proved using deformation theory (see [TV2, Corollaries 2.2.2.9 and 2.2.2.10]):

Lemma 2.1.5. *For a given $S \in \mathrm{Sch}^{\mathrm{aff}}$, the operation of passage to the underlying classical subscheme defines an equivalence between the full subcategory $(\mathrm{Sch}^{\mathrm{aff}})_{/S}$ spanned by $S' \xrightarrow{f} S$ with f étale and the full subcategory of $({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/{}^{\mathrm{cl}}S}$ spanned by $\widetilde{S}' \xrightarrow{\widetilde{f}} {}^{\mathrm{cl}}S$ with $\widetilde{f}$ étale. Furthermore, f is an open embedding (resp., Zariski) if and only if $\widetilde{f}$ is.*

2.1.6. We say that a morphism is $f : S' \to S$ is a covering with respect to the flat (resp., ppf, smooth, étale, Zariski) topology, if it is flat (resp., ppf, smooth, étale, Zariski), and the induced map of classical affine schemes ${}^{\mathrm{cl}}S' \to {}^{\mathrm{cl}}S$ is surjective.

Thus, the category $\mathrm{Sch}^{\mathrm{aff}}$ acquires a hierarchy of Grothendieck topologies: flat, ppf, smooth, étale and Zariski.

[2]ppf=plat de présentation finie= flat of finite presentation

2.1.7. The property of a morphism to be flat (resp., ppf, smooth, étale, open embedding, Zariski) is obviously stable under base change.

Moreover, the property of a morphism $f : S' \to S$ to be flat (resp., ppf, smooth, étale, open embedding) it itself local with respect to any of the above topologies on S.

In addition, the property of a morphism $f : S' \to S$ to be flat (resp., ppf, smooth, étale, Zariski) is local with respect to the flat (resp., ppf, smooth, étale, Zariski) topology on S'.

Remark 2.1.8. For obvious reasons, the property of a morphism to be an open embedding is *not* Zariski-local on the source. And the property of a morphism to be Zariski is *not* étale-local on the target.

2.1.9. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be an *affine schematic* morphism in PreStk (see Sect. 1.1.6 for what this means).

We shall say that it is flat (resp., ppf, smooth, étale, open embedding, Zariski) if for every $S \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}_2}$, the corresponding map

$$S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S$$

(of affine schemes(!)) is flat (resp., ppf, smooth, étale, open embedding, Zariski).

2.2. **Digression: the Čech nerve.**

2.2.1. Let Fin denote the category of finite sets.

Let $\mathbf{C}$ be an arbitrary ∞-category with Cartesian products. Then to an object $\mathbf{c} \in \mathbf{C}$ we can attach a functor

$$\mathrm{Fin}^{\mathrm{op}} \to \mathbf{C}, \quad I \mapsto \mathbf{c}^I.$$

In terms of the Yoneda embedding, this functor is uniquely characterized by

$$\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}', \mathbf{c}^I) = \mathrm{Maps}_{\mathrm{Spc}}(I, \mathrm{Maps}_{\mathbf{C}}(\mathbf{c}', \mathbf{c})), \quad \mathbf{c}' \in \mathbf{C}'.$$

Composing with the functor $\mathbf{\Delta} \to \mathrm{Fin}$, we obtain a functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}.$$

2.2.2. Let now $\mathbf{D}$ be an ∞-category with fiber products, and $\mathbf{d} \in \mathbf{D}$ an object. Set

$$\mathbf{C} := \mathbf{D}_{/\mathbf{d}},$$

so that Cartesian products in $\mathbf{C}$ are the fiber products in $\mathbf{D}$ *over* $\mathbf{d}$.

Given an object $\mathbf{c} \in \mathbf{D}_{/\mathbf{d}}$ we thus obtain a functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{D}_{/\mathbf{d}} \to \mathbf{D}.$$

It is called the *Čech nerve* of the morphism $\mathbf{c} \to \mathbf{d}$, and denoted $\mathbf{c}^\bullet/\mathbf{d}$.

2.2.3. Thus, we have $\mathbf{c}^0/\mathbf{d} = \mathbf{c}$,

$$\mathbf{c}^1/\mathbf{d} = \mathbf{c} \underset{\mathbf{d}}{\times} \mathbf{c}.$$

In general, the object $\mathbf{c}^\bullet/\mathbf{d} \in \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{D})$ is an example of a *groupoid object* of $\mathbf{D}$; see [Lu1, Sect. 6.1.2] for what this means.

2.3. **The descent condition.** In this subsection we will impose the descent condition that singles out the class of *stacks* among all prestacks.

This discussion here is not specific to the category $\mathrm{Sch}^{\mathrm{aff}}$. It is applicable to any ∞-category (with fiber products) equipped with a Grothendieck topology. So, we can view this subsection is a summary of some results from [Lu1, Sect. 6] and [TV1].

2.3.1. Let $\mathcal{Y}$ be a prestack. We say that it satisfies flat (resp., ppf, smooth, étale, Zarski) descent if:

- $\mathcal{Y}(\emptyset) = \{*\}$;
- $\mathcal{Y}$ sends disjoint unions of affine schemes to products, i.e., the map
$$\mathcal{Y}(S_1 \sqcup S_2) \to \mathcal{Y}(S_1) \times \mathcal{Y}(S_2)$$
is an isomorphism;
- Whenever
$$f : S' \to S \in \mathrm{Sch}^{\mathrm{aff}}$$
is a flat covering, the map
$$\mathcal{Y}(S) \to \mathrm{Tot}(\mathcal{Y}(S'^{\bullet}/S))$$
is an isomorphism, where $S'^{\bullet}/S$ is the Čech nerve of the map f.

2.3.2. In what follows we will assume that our topology is chosen to be étale. However, the entire discussion equally applies to the other cases, i.e. flat, ppf, smooth or Zariski.

We shall call prestacks that satisfy the above descent condition *stacks*, and denote the corresponding full subcategory of PreStk by Stk.

As in the case of classical algebraic geometry, one shows that if an object of PreStk satisfies étale descent, then it satisfies smooth descent.

2.3.3. We say that a map $\mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk is an *étale equivalence* if it induces an isomorphism
$$\mathrm{Maps}(\mathcal{Y}_2, \mathcal{Y}) \to \mathrm{Maps}(\mathcal{Y}_1, \mathcal{Y})$$
whenever $\mathcal{Y} \in \mathrm{Stk}$.

2.3.4. The inclusion
$$\mathrm{Stk} \hookrightarrow \mathrm{PreStk}$$
admits a left adjoint making Stk a localization of PreStk.

Concretely, the functor PreStk $\to$ Stk is universal among functors that turn étale equivalences into isomorphisms, see [Lu1, Sect. 6.2.1].

We will denote by L the corresponding localization (=sheafification) functor, i.e., the composition
$$\mathrm{PreStk} \to \mathrm{Stk} \to \mathrm{PreStk}\,.$$

Tautologically, a map $\mathcal{Y}_1 \to \mathcal{Y}_2$ is an étale equivalence if and only if $L(\mathcal{Y}_1) \to L(\mathcal{Y}_2)$ is an isomorphism.

2.3.5. We have the following assertion (see [Lu1, Corollary 6.2.1.6 and Proposition 6.2.2.7][3]):

Lemma 2.3.6. *The functor L is left exact, i.e., commutes with finite limits.*

[3]For this proposition the reader should use the version of [Lu1] available on Lurie's website rather than the printed version.

2.3.7. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in PreStk.

We say that f is an étale surjection if for every $S \in \mathrm{Sch}^{\mathrm{aff}}$ and an object $y_2 \in \mathcal{Y}_2(S)$ there exists an étale cover $\phi : S' \to S$, such that $\phi^*(y_2) \in \mathcal{Y}_2(S')$ belongs to the essential image of $f(S') : \mathcal{Y}_1(S') \to \mathcal{Y}_2(S')$.

The following is [Lu1, Corollary 6.2.3.5]:

Lemma 2.3.8. *Let $\mathcal{Y}_1 \to \mathcal{Y}_2$ be an étale surjection. Then the induced map*

$$|\mathcal{Y}_1^\bullet/\mathcal{Y}_2|_{\mathrm{PreStk}} \to \mathcal{Y}_2$$

is an étale equivalence, where $\mathcal{Y}_1^\bullet/\mathcal{Y}_2$ is the Čech nerve of f, and $|-|_{\mathrm{PreStk}}$ denotes geometric realization taken in the category PreStk.

Note that the assertion of Lemma 2.3.8 can be reformulated as the statement that if $\mathcal{Y}_1 \to \mathcal{Y}_2$ is an étale surjection, then the map

$$|L(\mathcal{Y}_1^\bullet/\mathcal{Y}_2)|_{\mathrm{Stk}} \simeq |L(\mathcal{Y}_1)^\bullet/L(\mathcal{Y}_2)|_{\mathrm{Stk}} \simeq L(|\mathcal{Y}_1^\bullet/\mathcal{Y}_2|_{\mathrm{PreStk}}) \to L(\mathcal{Y}_2)$$

is an isomorphism.

2.3.9. Finally, we have:

Lemma 2.3.10. *For $\mathcal{Y} \in$ PreStk, the unit of the adjunction*

$$\mathcal{Y} \to L(\mathcal{Y})$$

is an étale surjection.

2.4. **Descent for affine schemes.** In this subsection we state (without proof) the standard, but crucial, fact that affine schemes are in fact stacks, and discuss some of its corollaries.

As in the previous subsection, the results stated in this subsection here hold also for the flat, ppf and Zariski topologies.

2.4.1. We have the following basic fact (see [TV2, Lemma 2.2.2.13]):

Proposition 2.4.2.

(a) *The image of the Yoneda embedding* $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$ *belongs to* Stk.

(b) *Let*

$$\begin{array}{ccc} \mathcal{Y} & \longleftarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\ f\ } & S' \end{array}$$

be a pullback diagram in Stk *with $S, S' \in \mathrm{Sch}^{\mathrm{aff}}$. Assume that $\mathcal{Y}'$ also belongs to* $\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{PreStk}$ *and the morphism f is an étale covering. Then $\mathcal{Y} \in \mathrm{Sch}^{\mathrm{aff}}$.*

2.4.3. As a corollary, we obtain:

Corollary 2.4.4. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be an affine schematic morphism in* PreStk*. Then the morphism $L(\mathcal{Y}_1) \to L(\mathcal{Y}_2)$ is also affine schematic.*

Proof. We need to show that for $S \in \mathrm{Sch}^{\mathrm{aff}}$ and a map $S \to L(\mathcal{Y}_2)$, the fiber product $S \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1)$ belongs to $\mathrm{Sch}^{\mathrm{aff}}$. By Proposition 2.4.2(b), it suffices to show that that the fiber product $S' \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1)$ belongs to $\mathrm{Sch}^{\mathrm{aff}}$ for some étale covering map $S' \to S$ with $S' \in \mathrm{Sch}^{\mathrm{aff}}$.

However, by Lemma 2.3.10, we can choose $S' \to S$ so that the composition $S' \to S \to L(\mathcal{Y}_2)$ factors as $S' \to \mathcal{Y}_2 \to L(\mathcal{Y}_2)$. Since the functor L commutes with fiber products (by Lemma 2.3.6), we have

$$S' \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1) \simeq L(S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1).$$

Now, by assumption, $S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \in \mathrm{Sch}^{\mathrm{aff}}$, and

$$S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to L(S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1)$$

is an isomorphism by Proposition 2.4.2(a)

□

2.4.5. The same proof also gives:

Corollary 2.4.6. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be affine flat (resp., ppf, smooth, étale, open embedding). Then so is $L(\mathcal{Y}_1) \to L(\mathcal{Y}_2)$.*

2.5. **Descent and n-coconnectivity.** In this subsection we will study how the étale descent condition interacts with the operation of restriction and left Kan extension to the (full) subcategory ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{Sch}^{\mathrm{aff}}$.

Again, the entire discussion is applicable when we replace the word 'étale' by 'flat', 'ppf' or 'Zariski'.

2.5.1. Let us denote by ${}^{\leq n}\mathrm{Stk}$ the full subcategory of ${}^{\leq n}\mathrm{PreStk}$ consisting of objects that satisfy descent for étale covers $S_1 \to S_2 \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

We obtain that ${}^{\leq n}\mathrm{Stk}$ is a localization of ${}^{\leq n}\mathrm{PreStk}$. Let ${}^{\leq n}L$ denote the corresponding localization functor

$${}^{\leq n}\mathrm{PreStk} \to {}^{\leq n}\mathrm{Stk} \to {}^{\leq n}\mathrm{PreStk}\,.$$

The analog of Lemma 2.3.6 equally applies in the present context.

2.5.2. The sheafification functor ${}^{\leq n}L$ on *truncated objects* can be described explicitly as follows (see [Lu1, Sect. 6.5.3]):

We have the following endo-functor, denoted

$$\mathcal{Y} \mapsto \mathcal{Y}^+ \tag{2.1}$$

of ${}^{\leq n}\mathrm{PreStk}$.

Namely, for $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$, the value of $\mathcal{Y}^+$ on $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ is the colimit over all étale covers $S' \to S$ of $\mathrm{Tot}(\mathcal{Y}(S'^{\bullet}/S))$.

Now, if $\mathcal{Y}$ is $(k-2)$-truncated for $k = 2, 3, ...$, then the value of $L(\mathcal{Y})$ on $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ is

$$\mathcal{Y}^{+^k}(S),$$

where $\mathcal{Y}^{+^k}$ denotes the k-th iteration of the functor (2.1).

In particular, since the colimit involved in its description is filtered, we obtain:

Lemma 2.5.3. *The functor ${}^{\leq n}L : {}^{\leq n}\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$ sends k-truncated objects to k-truncated ones.*

2.5.4. The following results from the definitions:

Lemma 2.5.5.
(a) *The restriction functor* $\mathrm{PreStk} \to {}^{\leq n}\mathrm{PreStk}$ *sends* Stk *to* ${}^{\leq n}\mathrm{Stk}$.
(b) *The functor*

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}$$

sends étale equivalences to étale equivalences.

2.5.6. Note now that the *right* Kan extension functor along ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$:

$$\mathrm{RKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}$$

tautologically sends ${}^{\leq n}\mathrm{Stk}$ to Stk. This implies that the restriction functor $\mathcal{Y} \mapsto {}^{\leq n}\mathcal{Y}$ sends étale equivalences to étale equivalences.

Thus, from Lemma 2.5.5 we obtain:

Corollary 2.5.7. *For* $\mathcal{Y} \in \mathrm{PreStk}$ *we have:*

$${}^{\leq n}L({}^{\leq n}\mathcal{Y}) \simeq {}^{\leq n}(L(\mathcal{Y})).$$

2.5.8. *Right Kan extensions from* ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$. Let $\mathcal{Y}'$ be a functor

$$({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

which we can think of as a compatible family of objects $\mathcal{Y}'_n \in {}^{\leq n}\mathrm{PreStk}$. Let

$$\mathcal{Y} := \mathrm{RKE}_{{}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}') \in \mathrm{PreStk}\,.$$

Lemma 2.5.9. *Assume that for all* n, $\mathcal{Y}'_n \in {}^{\leq n}\mathrm{Stk}$. *Then* $\mathcal{Y}$ *belongs to* Stk.

Proof. Follows from the description of the functor $\mathrm{RKE}_{{}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}$ given in the proof of Proposition 1.4.7.

□

From here we obtain:

Corollary 2.5.10. *Suppose that* $\mathcal{Y} \in \mathrm{PreStk}$ *belongs to* Stk. *Then so does* ${}^{\mathrm{conv}}\mathcal{Y}$.

2.6. **The notion of n-coconnective stack.**

2.6.1. Note that the functor

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk} \to \mathrm{PreStk}$$

does *not* send ${}^{\leq n}\mathrm{Stk}$ to Stk. Instead, the left adjoint to the restriction functor ${}^{\leq n}\mathrm{Stk} \leftarrow \mathrm{Stk}$ is given by the composition

$${}^{\leq n}\mathrm{Stk} \hookrightarrow {}^{\leq n}\mathrm{PreStk} \xrightarrow{\mathrm{LKE}} \mathrm{PreStk} \xrightarrow{L} \mathrm{Stk};$$

we denote this composite functor by ${}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}$.

2.6.2. The above left adjoint is easily seen to be fully faithful. Hence, we can identify ${}^{\leq n}\mathrm{Stk}$ with a full subcategory of Stk. We shall denote by ${}^{L}\tau^{\leq n} : \mathrm{Stk} \to \mathrm{Stk}$ the resulting colocalization functor

$$\mathcal{Y} \mapsto {}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}({}^{\leq n}\mathcal{Y}).$$

By definition, ${}^{L}\tau^{\leq n} \simeq L \circ \tau^{\leq n}$.

2.6.3. We shall call objects of Stk that belong to the essential image of ${}^L\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}$ n-coconnective stacks. I.e., $\mathcal{Y} \in \mathrm{Stk}$ is n-coconnective as a stack if and only if the adjunction map

$$ {}^L\tau^{\leq n}(\mathcal{Y}) \to \mathcal{Y} $$

is an isomorphism.

I.e., the functor ${}^L\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}$ identifies the category ${}^{\leq n}\mathrm{Stk}$ with the full subcategory of PreStk spanned by n-coconnective stacks.

We shall refer to objects of ${}^{\leq 0}\mathrm{Stk} =: {}^{\mathrm{cl}}\mathrm{Stk}$ as 'classical stacks', and also denote ${}^L\tau^{\leq 0} =: {}^L\tau^{\mathrm{cl}}$.

Remark 2.6.4. We emphasize again that, as subcategories PreStk, it is *not* true that ${}^{\leq n}\mathrm{Stk}$ is contained in ${}^{\leq n}\mathrm{PreStk}$. That is to say, that a n-coconnective stack is not necessarily n-coconnective as a prestack.

Note, however, that we do have an inclusion

$$ \mathrm{Stk} \cap {}^{\leq n}\mathrm{PreStk} \subset {}^{\leq n}\mathrm{Stk} $$

as subcategories of PreStk.

2.6.5. We shall say that a stack is eventually coconnective if it is n-coconnective for some n.

2.7. **Descent and the 'locally of finite type' condition.** In this subsection we will study how the descent condition interacts with the condition of being of finite type.

The entire discussion is applicable if we replace the étale topology by the ppf, or Zariski one.

However, the flat topology (without the finite type condition) would not do: we need finite typeness for the validity of Lemma 2.8.2.

2.7.1. Let n be a fixed integer. We can consider the étale topology on the category ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Thus, we obtain a localization of ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ that we denote ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$.

We shall denote by ${}^{\leq n}L_{\mathrm{ft}}$ the corresponding localization functor

$$ {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}\,. $$

As in Lemma 2.5.3, we have:

Lemma 2.7.2. *The functor* ${}^{\leq n}L_{\mathrm{ft}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ *sends k-truncated objects to k-truncated ones.*

2.7.3. Consider the restriction functor for ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, i.e.,

$$ {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \leftarrow {}^{\leq n}\mathrm{PreStk}\,. $$

It is clear that it sends ${}^{\leq n}\mathrm{Stk}$ to ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$. By adjunction, the functor of left Kan extension

$$ \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk} $$

sends étale equivalences to étale equivalences.

Moreover, we claim:

Lemma 2.7.4. *The functor of* right *Kan extension*

$$ \mathrm{RKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk} $$

sends ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$ *to* ${}^{\leq n}\mathrm{Stk}$.

Proof. For $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ the value of $\mathrm{RKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y})$ on $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ is given as

$$\lim_{S_0 \to S} \mathcal{Y}(S_0),$$

where the limit is taken over the category opposite to $({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/S}$.

Let $S' \to S$ be an étale cover. We need to show that the map from $\lim\limits_{S_0 \to S} \mathcal{Y}(S_0)$ to the totalization of the cosimplicial space whose m-simplices are given by

$$\lim_{S_0^m \to (S'^m/S)} \mathcal{Y}(S_0^m),$$

is an isomorphism.

However, this follows from the fact that the functor

$$({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/S} \to ({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/(S'^m/S)}, \quad S_0 \mapsto S_0^m := S_0 \underset{S}{\times} (S'^m/S),$$

is cofinal.

□

From Lemma 2.7.4 we obtain:

Corollary 2.7.5.

(a) *The restriction functor* ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \leftarrow {}^{\leq n}\mathrm{PreStk}$ *sends étale equivalences to étale equivalences.*

(b) *For* $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ *we have:*

$${}^{\leq n}L(\mathcal{Y})|_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}} \simeq {}^{\leq n}L_{\mathrm{ft}}(\mathcal{Y}|_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}).$$

2.7.6. Let us return to the functor

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \to {}^{\leq n}\mathrm{PreStk}\,.$$

It is not clear, and probably not true, that this functor sends ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$ to ${}^{\leq n}\mathrm{Stk}$. However, as we have learned from J. Lurie, there is the following partial result, proved below:

Proposition 2.7.7. *Suppose that an object* $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$ *is* k*-truncated for some* k *(see Sect. 1.8.5), and that* $\mathcal{Y} \in {}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$*. Then the object*

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y})$$

of ${}^{\leq n}\mathrm{PreStk}$ *belongs to* ${}^{\leq n}\mathrm{Stk}$.

2.7.8. In what follows we shall use the notation

$${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}} := {}^{\leq n}\mathrm{Stk} \cap {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}\,.$$

We shall refer to objects of the subcategory ${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}}$ of ${}^{\leq n}\mathrm{Stk}$ as 'n-coconnective stacks locally of finite type'.

We have the inclusion

$${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}} \subset {}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}\,.$$

Thus, Proposition 2.7.7 says that the essential image of this inclusion contains all truncated objects.

2.7.9. As a corollary of Proposition 2.7.7 and Lemma 2.7.2, we obtain:

Corollary 2.7.10. *For* $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$*, which is truncated, the natural map*

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}} \left({}^{\leq n}L_{\mathrm{ft}}(\mathcal{Y})\right) \to {}^{\leq n}L\left(\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y})\right)$$

is an isomorphism.

2.8. **Proof of Proposition 2.7.7.**

2.8.1. The proof will use the following assertion:

Let $f : S_1 \to S_2$ be an étale morphism in ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$. Consider the category of Cartesian diagrams

$$\begin{array}{ccc} S_1 & \longrightarrow & S_1' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S_2 & \longrightarrow & S_2' \end{array}$$

with $S_2', S_1' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, and f' is étale. Denote this category by f_{ft}. We have the natural forgetful functors

$$\{S_2 \to S_2', S_2' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}\} \leftarrow f_{\mathrm{ft}} \to \{S_1 \to S_1', S_1' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}\}. \tag{2.2}$$

Lemma 2.8.2. *Both functors opposite to those in* (2.2) *are cofinal.*

Proof. We first show that the functor opposite to

$$f_{\mathrm{ft}} \to \{S_1 \to S_1', S_1' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}\}$$

is cofinal.

Both categories in question are filtered: the above categories (before passing to the opposite) admit fiber products. Hence, it is enough to show that for any $S_1 \to S_1''$ with $S_1'' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, there *exists* an object of f_{ft} such that the map $S_1 \to S_1''$ factors as $S_1 \to S_1' \to S_1''$. For $n = 0$ this a standard fact in classical algebraic geometry, and for general n, it follows by induction using deformation theory (specifically, [Chapter III.1, Proposition 5.4.2(b)]).

To prove the assertion concerning

$$f_{\mathrm{ft}} \to \{S_2 \to S_2', S_2' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}\},$$

we note that the corresponding fact holds for $n = 0$, i.e., in classical algebraic geometry.

Consider the following diagram

$$\begin{array}{ccc} f_{\mathrm{ft}} & \longrightarrow & {}^{\mathrm{cl}}f_{\mathrm{ft}} \\ \downarrow & & \downarrow \\ \{S_2 \to S_2', S_2' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}\} & \longrightarrow & \{{}^{\mathrm{cl}}S_2 \to S_{2,0}', S_{2,0}' \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}\}. \end{array}$$

By Lemma 2.1.5, this is a pullback diagram. In addition, the bottom horizontal arrow is a Cartesian fibration. Hence, the cofinality of the functor opposite to the right vertical arrow implies the corresponding fact for the left vertical arrow.

□

Remark 2.8.3. An assertion parallel to Lemma 2.8.2 remains valid if we replace the word 'étale' by 'ppf', but the proof is more involved.

2.8.4. Let $\mathcal{Y}'$ be an object of ${}^{\leq n}\mathrm{NearStk}_{\mathrm{lft}}$, and let $\mathcal{Y}$ be its left Kan extension to an object of ${}^{\leq n}\mathrm{PreStk}$. Let $f : S_1 \to S_2$ be an étale cover. To prove Proposition 2.7.7, we need to check that the map

$$\mathcal{Y}(S_2) \to \mathrm{Tot}(\mathcal{Y}(S_1^\bullet/S_2)) \tag{2.3}$$

is an isomorphism.

For $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, the value of $\mathcal{Y}$ on S is calculated as

$$\underset{S\to S'}{\mathrm{colim}}\, \mathcal{Y}'(S'),$$

where the colimit is taken over the category opposite to $({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{S/}$. Recall that according to Theorem 1.5.3(b), the above category is *filtered*. This implies that if $\mathcal{Y}'$ is k-truncated, then so is $\mathcal{Y}$.

Hence, we can replace Tot in (2.3), which is a limit in Spc over the index category $\mathbf{\Delta}$, by the corresponding limit, denoted $\mathrm{Tot}^{\leq k}$, in the category $\mathrm{Spc}_{\leq k}$, over the index category $\mathbf{\Delta}^{\leq k}$ of finite ordered sets of cardinality $\leq k+1$.

2.8.5. We rewrite the left-hand side in (2.3) as

$$\underset{S_2\to S_2',\, S_2'\in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\mathrm{colim}}\, \mathcal{Y}'(S_2').$$

Applying Lemma 2.8.2 for the $\to$ functor, we rewrite the right-hand side in (2.3) as

$$\mathrm{Tot}^{\leq k}\left(\underset{(f_{\mathrm{ft}})^{\mathrm{op}}}{\mathrm{colim}}\, \mathcal{Y}(S_1'^\bullet/S_2')\right).$$

The category $(f_{\mathrm{ft}})^{\mathrm{op}}$ is filtered, as it contains push-outs. Since $\mathrm{Tot}^{\leq k}$ is a *finite* limit, we can commute the limit and the colimit in the above expression, and therefore rewrite it as

$$\underset{(f_{\mathrm{ft}})^{\mathrm{op}}}{\mathrm{colim}}\left(\mathrm{Tot}^{\leq k}(\mathcal{Y}(S_1'^\bullet/S_2'))\right).$$

By the descent condition for $\mathcal{Y}'$, the latter expression is isomorphic to $\underset{(f_{\mathrm{ft}})^{\mathrm{op}}}{\mathrm{colim}}\, \mathcal{Y}(S_2')$. Applying Lemma 2.8.2 for the $\leftarrow$ functor, we obtain that

$$\underset{(f_{\mathrm{ft}})^{\mathrm{op}}}{\mathrm{colim}}\, \mathcal{Y}(S_2') \simeq \underset{S_2\to S_2',\, S_2'\in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\mathrm{colim}}\, \mathcal{Y}'(S_2'),$$

as required.

□

2.9. **Stacks locally almost of finite type.**

2.9.1. Recall the full subcategory $\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$. In this subsection we will perceive it as the category

$$\mathrm{Funct}\left(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}, \mathrm{Spc}\right),$$

see Proposition 1.7.6.

2.9.2. Consider the étale topology on the category ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Thus, we obtain a localization of $\mathrm{PreStk}_{\mathrm{laft}}$ that we denote $\mathrm{NearStk}_{\mathrm{laft}}$.

Let us denote by L_{laft} the corresponding localization functor

$$\mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{NearStk}_{\mathrm{laft}} \to \mathrm{PreStk}_{\mathrm{laft}}.$$

2.9.3. Consider the functor

$$\mathrm{PreStk} \to \mathrm{PreStk}_{\mathrm{laft}}$$

given by restriction along

$${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}.$$

It is clear that this functor sends Stk to $\mathrm{NearStk}_{\mathrm{laft}}$. Moreover, as in Corollary 2.7.5 and Corollary 2.5.7, we obtain:

Lemma 2.9.4. *For $\mathcal{Y} \in \mathrm{PreStk}$ we have:*

$$L(\mathcal{Y})|_{{}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}} \simeq L_{\mathrm{laft}}(\mathcal{Y}|_{{}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}).$$

From Proposition 2.7.7 and Lemma 2.5.9 we obtain:

Corollary 2.9.5. *Let $\mathcal{Y}$ be an object of $\mathrm{NearStk}_{\mathrm{laft}}$, thought of as an object of* PreStk *via*

$$\mathrm{NearStk}_{\mathrm{laft}} \subset \mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$$

(see Proposition 1.7.6). Suppose that for each n, the restriction ${}^{\leq n}\mathcal{Y}$ of $\mathcal{Y}$ to ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ is k_n-truncated for some $k_n \in \mathbb{N}$. Then $\mathcal{Y} \in \mathrm{Stk}$.

2.9.6. In what follows, we will denote the intersection

$$\mathrm{Stk} \cap \mathrm{PreStk}_{\mathrm{laft}}$$

by $\mathrm{Stk}_{\mathrm{laft}}$. We shall refer to objects of the subcategory $\mathrm{Stk}_{\mathrm{laft}} \subset \mathrm{Stk}$ as 'stacks locally almost of finite type'.

We have an evident inclusion

$$\mathrm{Stk}_{\mathrm{laft}} \subset \mathrm{NearStk}_{\mathrm{laft}}.$$

Corollary 2.9.5 says that the essential image of $\mathrm{Stk}_{\mathrm{laft}}$ in $\mathrm{NearStk}_{\mathrm{laft}}$ contains all objects $\mathcal{Y}$, such that for every n, the restriction ${}^{\leq n}\mathcal{Y}$ of $\mathcal{Y}$ to ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ is truncated.

3. (Derived) schemes

In this section we introduce the basic object of study in derived algebraic geometry–the notion of (derived) scheme[4].

We investigate some basic properties of schemes: what it means to be n-coconnective and locally (almost) of finite type.

3.1. **The definition of (derived) schemes.**

Our approach to the definition of (derived) schemes (or more general algebro-geometric objects) is that they are prestacks that have some specific properties. I.e., we never need to introduce additional pieces of structure.

In the case of (derived) schemes, the relevant properties are descent and the existence of a *Zariski atlas.*

[4]In the main body of the text we drop the adjective 'derived': everything is derived unless specified otherwise.

3.1.1. Recall the notion of an affine open embedding, see Sect. 2.1.9.

Following [TV2, Sect. 2.2], we say that an object $Z \in \text{PreStk}$ is a *scheme* if:

(1) Z satisfies étale descent;

(2) The diagonal map $Z \to Z \times Z$ is affine schematic, and for every $T \in (\text{Sch}^{\text{aff}})_{/Z\times Z}$, the induced map of classical schemes ${}^{\text{cl}}(T \underset{Z\times Z}{\times} Z) \to {}^{\text{cl}}T$ is a closed embedding;

(3) There exists a collection of affine schemes S_i and maps $f_i : S_i \to Z$ (called a *Zariski atlas*), such that:
   - Each f_i (which is affine schematic by the previous point) is an open embedding;
   - For every $T \in (\text{Sch}^{\text{aff}})_{/Z}$, the images of the maps ${}^{\text{cl}}(T \underset{Z}{\times} S_i) \to {}^{\text{cl}}T$ cover ${}^{\text{cl}}T$.

We shall denote the full subcategory of Stk spanned by schemes by Sch.

Remark 3.1.2. One can show that the étale descent condition can be replaced by a weaker one: namely, it is sufficient to require that Z satisfy Zariski descent. In addition, it is not difficult to see that schemes as defined above actually satisfy flat descent.

Remark 3.1.3. Our definition gives what is usually called a *separated* scheme. The non-separated case will be covered under the rubric of Artin stacks, discussed in the next section.

3.1.4. We shall say that a scheme Z is quasi-compact if the classical scheme ${}^{\text{cl}}Z$ is. Equivalently, this means that Z admits a Zariski cover by a finite collection of affine schemes.

3.1.5. It follows from the definition that if $(S_i \xrightarrow{f_i} Z)$ is a Zariski atlas, then the map

$$\underset{i}{\sqcup} S_i \to Z$$

is an étale (and, in fact, Zariski) surjection.

Hence, from Lemma 2.3.8, we obtain:

Lemma 3.1.6. *Let Z be a scheme. For a given Zariski atlas $\underset{i}{\sqcup} S_i \to Z$, we have $Z \simeq L(|(\underset{i}{\sqcup} S_i)^\bullet/Z|_{\text{PreStk}})$.*

3.1.7. The following results from Lemma 2.1.5:

Corollary 3.1.8.

(a) *Given a Zariski morphism of affine schemes $S' \to S$, for $T \to S$, the datum of its lift to a map $T \to S'$ is equivalent to the datum of a lift of ${}^{\text{cl}}T \to {}^{\text{cl}}S$ to a map ${}^{\text{cl}}T \to {}^{\text{cl}}S'$.*

(b) *Let $Z' \to Z$ be an affine Zariski map, where $Z', Z \in \text{Sch}$. Then for $T \to Z$ with $T \in \text{Sch}^{\text{aff}}$, the datum of a lift of f to a map $f' : T \to Z'$ is equivalent to the datum of a lift of ${}^{\text{cl}}f : {}^{\text{cl}}T \to {}^{\text{cl}}Z$ to a map ${}^{\text{cl}}f' : {}^{\text{cl}}T \to {}^{\text{cl}}Z'$.*

Remark 3.1.9. Both points in Corollary 3.1.8 remain valid if we replace the word 'Zariski' by 'étale'.

3.2. **Construction of schemes.** In this subsection we will prove an assertion that provides a converse to Lemma 3.1.6.

3.2.1. First, we claim:

Proposition 3.2.2. *Let Z be an object of* Stk*, equipped with a collection of affine open embeddings $S_i \to Z$, where $S_i \in \mathrm{Sch}^{\mathrm{aff}}$. Suppose that ${}^{\mathrm{cl}}Z$ is a classical scheme*[5] *and $\underset{i}{\sqcup} {}^{\mathrm{cl}}S_i \to {}^{\mathrm{cl}}Z$ is its Zariski atlas. Then:*

(a) *Z is a scheme;*

(b) *The maps $\underset{i\in I}{\sqcup} S_i \to Z$ form a Zariski atlas of Z.*

Proof. We only have to show that the diagonal map $Z \to Z \times Z$ is affine schematic. This is equivalent to showing that for any $T, U \in (\mathrm{Sch}^{\mathrm{aff}})_{/Z}$, the fiber product $T \underset{Z}{\times} U$ is an affine scheme.

Consider the fiber products $S_i \underset{Z}{\times} T$. By assumption, these are affine schemes, and the map

$$\underset{i}{\sqcup} S_i \underset{Z}{\times} T \to T$$

is a Zariski covering. Therefore, by Proposition 2.4.2(b), it suffices to show that the fiber products

$$S_i \underset{Z}{\times} T \underset{Z}{\times} U$$

are affine schemes. However,

$$S_i \underset{Z}{\times} T \underset{Z}{\times} U \simeq (S_i \underset{Z}{\times} T) \underset{S_i}{\times} (S_i \underset{Z}{\times} U).$$

□

3.2.3. Let $S^\bullet$ be a groupoid-object of PreStk (see [Lu1, Sect. 6.1.2] for what this means).

Denote

$$Z := L(|S^\bullet|).$$

We claim:

Proposition 3.2.4. *Assume that S^0 and S^1 are of the form*

$$S^0 = \underset{i\in I}{\sqcup} S^0_i \text{ and } S^1 = \underset{j\in J}{\sqcup} S^1_j,$$

where S^0_i and S^1_i are affine schemes, and the maps $S^1 \rightrightarrows S^0$ are comprised of open embeddings $S^1_i \to S^0_j$. Assume, moreover, that ${}^{\mathrm{cl}}Z$ is a classical scheme and that $\underset{i}{\sqcup} {}^{\mathrm{cl}}S^0_i \to {}^{\mathrm{cl}}Z$ is its Zariski atlas. Then:

(a) *Z is a scheme;*

(b) *The maps $\underset{i\in I}{\sqcup} S^0_i \to Z$ form a Zariski atlas of Z.*

Proof. By Proposition 3.2.2, it is enough to show that each of the maps $S^0_i \to Z$ is an affine open embedding. By Corollary 2.4.4, it suffices to show that the each of the maps

$$S^0_i \to |S^\bullet|$$

is an affine open embedding.

Fix a map $T \to |S^\bullet|$. By definition, such a map factors as $T \to S^0 \to |S^\bullet|$. Hence, we have

$$T \underset{|S^\bullet|}{\times} S^0_i \simeq T \underset{S^0}{\times} S^0 \underset{|S^\bullet|}{\times} S^0_i.$$

[5]Following our conventions, when talking about classical schemes, we impose the hypothesis that they be separated.

Thus, it suffices to show that each of the maps $S^0 \underset{|S^\bullet|}{\times} S^0_i \to S^0$ is an affine open embedding.

We have

$$S^0 \underset{|S^\bullet|}{\times} S^0_i \simeq (S^0 \underset{|S^\bullet|}{\times} S^0) \underset{S^0}{\times} S^0_i.$$

Now,

$$S^0 \underset{|S^\bullet|}{\times} S^0 \simeq S^1,$$

and the assertion follows from the assumption on the map $S^1 \to S^0$.

□

3.2.5. Combining Proposition 3.2.4 with Lemma 2.1.5, we obtain:

Corollary 3.2.6. *Let Z be a scheme. Then the operation of passage to the underlying classical subscheme defines an equivalence between the full subcategory $\mathrm{Sch}_{/Z}$ spanned by $Z' \xrightarrow{f} Z$ with f affine Zariski and the full subcategory of ${}^{\mathrm{cl}}\mathrm{Sch}_{/{}^{\mathrm{cl}}Z}$ spanned by $\widetilde{Z}' \xrightarrow{\widetilde{f}} {}^{\mathrm{cl}}Z$ with $\widetilde{f}$ affine Zariski. Furthermore, f is an open embedding if and only if $\widetilde{f}$ is.*

Further, combining with Proposition 2.4.2(b), we obtain:

Corollary 3.2.7. *In the circumstances of Corollary 3.2.6, the scheme Z' is affine if and only if the classical scheme ${}^{\mathrm{cl}}Z'$ is affine.*

And finally:

Corollary 3.2.8. *A scheme Z is affine if and only if the classical scheme ${}^{\mathrm{cl}}Z$ is affine.*

3.3. **Schemes and n-coconnectivity.** In this subsection we study the question of how the notion of scheme interacts with the notion of n-coconnective stack.

3.3.1. Replacing the category PreStk by ${}^{\leq n}\mathrm{PreStk}$ in the definition of the notion of scheme we obtain a category that we denote by ${}^{\leq n}\mathrm{Sch}$.

For $n = 0$ we recover the category of classical (separated) schemes.

3.3.2. We claim:

Proposition 3.3.3. *Any object of ${}^{\leq n}\mathrm{Sch}$ is n-truncated as an object of ${}^{\leq n}\mathrm{PreStk}$.*

Proof. Let Z be an object of ${}^{\leq n}\mathrm{Sch}$ and let us be given a map $f_0 : {}^{\mathrm{cl}}T \to Z$, where $T \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$. We will show that the space of maps $T \to Z$ that restrict to f_0 is n-truncated.

Fix a Zariski atlas $\underset{i}{\sqcup} S_i \to Z$. Consider the induced Zariski cover ${}^{\mathrm{cl}}T \underset{Z}{\times} S_i$ of ${}^{\mathrm{cl}}T$. Since ${}^{\mathrm{cl}}T$ is quasi-compact, we can replace the initial index set by its finite subset, denoted I, so that

$$\underset{i \in I}{\sqcup} {}^{\mathrm{cl}}T \underset{Z}{\times} S_i \to {}^{\mathrm{cl}}T$$

is still a cover.

By Lemma 2.1.5, there exists a canonically defined Zariski cover $\underset{i \in I}{\sqcup} T_i = T' \to T$ such that

$$\underset{i \in I}{\sqcup} {}^{\mathrm{cl}}T \underset{Z}{\times} S_i = {}^{\mathrm{cl}}T'.$$

Now, the datum of a map $f : T \to Z$ that restricts to f_0 is equivalent to the datum of a point of

$$\mathrm{Tot}(\mathrm{Maps}(T'^\bullet/T, Z) \underset{\mathrm{Maps}({}^{\mathrm{cl}}T'^\bullet/{}^{\mathrm{cl}}T, Z)}{\times} \{f_0\}|_{{}^{\mathrm{cl}}T'^\bullet/{}^{\mathrm{cl}}T}).$$

We now claim that the above cosimplicial space is n-truncated simplex-wise.

Indeed, by Corollary 3.1.8(b), for every $m \geq 0$, the corresponding space of m-simplices is the product over the set of $(m+1)$-tuples $(i_0, ..., i_m)$ of elements of I of

$$\text{Maps}(T_{i_0} \underset{T}{\times} ... \underset{T}{\times} T_{i_m}, S_{i_0}) \underset{\text{Maps}({}^{\text{cl}}T_{i_0} \underset{{}^{\text{cl}}T}{\times} ... \underset{{}^{\text{cl}}T}{\times} {}^{\text{cl}}T_{i_m}, S_{i_0})}{\times} \{f_0\}|_{{}^{\text{cl}}T_{i_0} \underset{{}^{\text{cl}}T}{\times} ... \underset{{}^{\text{cl}}T}{\times} {}^{\text{cl}}T_{i_m}}.$$

Now, the assertion follows from the fact that mapping spaces in ${}^{\leq n}\text{Sch}^{\text{aff}}$ are n-truncated, by Sect. 1.8.6.

□

3.3.4. It is easy to see that the restriction functor for ${}^{\leq n}\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}$ sends Sch to ${}^{\leq n}\text{Sch}$ (replace the original Zariski cover S_i by ${}^{\leq n}S_i$).

We claim:

Proposition 3.3.5.
(a) *The functor*

$$^{L}\text{LKE}_{{}^{\leq n}\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}} : {}^{\leq n}\text{Stk} \hookrightarrow \text{Stk}$$

sends ${}^{\leq n}\text{Sch}$ *to* Sch.
(b) *If Z is an object of* ${}^{\leq n}\text{Sch}$ *with a Zariski atlas* $\underset{i}{\sqcup} S_i \to Z$*, then*

$$\underset{i}{\sqcup} S_i \to {}^{L}\text{LKE}_{{}^{\leq n}\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}}(Z)$$

is a Zariski atlas.

Proof. Follows from Proposition 3.2.4.

□

3.3.6. We shall call a scheme 'n-coconnective' if it is n-coconnective as an object of Stk.

We obtain that the functor $^{L}\text{LKE}_{{}^{\leq n}\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}}$ identifies the category ${}^{\leq n}\text{Sch}$ with that of n-coconnective schemes.

We emphasize that an n-coconnective scheme is *not* necessarily n-coconnective as a prestack, but it is n-coconnective as a stack.

3.3.7. We have the following characterization of n-coconnective schemes:

Proposition 3.3.8. *For $Z \in$ Sch the following conditions are equivalent:*
(i) *Z is n-coconnective.*
(ii) *For every $Z' \in$ Sch equipped with an affine open embedding $Z' \to Z$, we have $Z \in {}^{\leq n}\text{Sch}$.*
(iii) *Z admits a Zariski atlas by affine schemes belonging to ${}^{\leq n}\text{Sch}^{\text{aff}}$.*

Proof. The implication (i) ⇒ (iii) is Proposition 3.3.5(b). The implication (ii) ⇒ (iii) is tautological. We will now show that (iii) implies both (i) and (ii).

Assume first that Z admits a Zariski atlas consisting of affine schemes in ${}^{\leq n}\text{Sch}^{\text{aff}}$. Then we can write Z as

$$L(\underset{a \in A}{\text{colim}}\, S_a), \tag{3.1}$$

for some diagram of objects $S_a \in {}^{\leq n}\text{Sch}^{\text{aff}}$, see Lemma 3.1.6. Concretely, the colimit in question is the geometric realization of the Čech nerve of the given atlas.

In particular, $\underset{a \in A}{\text{colim}}\, S_a \in {}^{\leq n}\text{PreStk}$. And hence, $Z \in {}^{\leq n}\text{Stk}$.

For any affine open embedding $Z' \to Z$, the pullback of this atlas gives a Zariski atlas for Z' with a similar property. This implies that in this case Z' also belongs to ${}^{\leq n}\mathrm{Sch}$.

□

3.4. **Schemes and convergence.**

3.4.1. We claim:

Proposition 3.4.2. *A scheme, regarded as an object of* PreStk, *is convergent.*

Proof. Let Z be a scheme and let us be given a map $f_0 : {}^{\mathrm{cl}}T \to Z$, where $T \in \mathrm{Sch}^{\mathrm{aff}}$. We will show that the datum of a lift of f_0 to a map $f : T \to Z$ is equivalent to the datum of a compatible family of lifts $f_n : {}^{\leq n}T \to Z$.

Let $\underset{i}{\sqcup} S_i \to Z$ and $\underset{i \in I}{\sqcup} T_i = T' \to T$ be as in the proof of Proposition 3.3.3.

As in *loc.cit.*, the datum of a map $f : T \to Z$ that restricts to f_0 is equivalent to the datum of a point of

$$\mathrm{Tot}(\mathrm{Maps}(T'^{\bullet}/T, Z) \underset{\mathrm{Maps}({}^{\mathrm{cl}}T'^{\bullet}/{}^{\mathrm{cl}}T, Z)}{\times} \{f_0\}|_{{}^{\mathrm{cl}}T'^{\bullet}/{}^{\mathrm{cl}}T}).$$

The datum of a compatible family of maps f_n is equivalent to the datum of a point of

$$\mathrm{Tot}\left(\lim_n \mathrm{Maps}({}^{\leq n}T'^{\bullet}/{}^{\leq n}T, Z) \underset{\mathrm{Maps}({}^{\mathrm{cl}}T'^{\bullet}/{}^{\mathrm{cl}}T, Z)}{\times} \{f_0\}|_{{}^{\mathrm{cl}}T'^{\bullet}/{}^{\mathrm{cl}}T}\right).$$

Now, we claim that the restriction map

$$(3.2) \quad \mathrm{Maps}(T'^{\bullet}/T, Z) \underset{\mathrm{Maps}({}^{\mathrm{cl}}T'^{\bullet}/{}^{\mathrm{cl}}T, Z)}{\times} \{f_0\}|_{{}^{\mathrm{cl}}T'^{\bullet}/{}^{\mathrm{cl}}T} \to$$
$$\to \lim_n \mathrm{Maps}({}^{\leq n}T'^{\bullet}/{}^{\leq n}T, Z) \underset{\mathrm{Maps}({}^{\mathrm{cl}}T'^{\bullet}/{}^{\mathrm{cl}}T, Z)}{\times} \{f_0\}|_{{}^{\mathrm{cl}}T'^{\bullet}/{}^{\mathrm{cl}}T}$$

is an isomorphism simplex-wise.

Indeed, by Corollary 3.1.8(b), for every $m \geq 0$, the spaces of m-simplices in the two sides in (3.2) are products over the set of $(m+1)$-tuples $(i_0, ..., i_m)$ of elements of I of

$$\mathrm{Maps}(T_{i_0} \underset{T}{\times} ... \underset{T}{\times} T_{i_m}, S_{i_0}) \underset{\mathrm{Maps}({}^{\mathrm{cl}}T_{i_0} \underset{{}^{\mathrm{cl}}T}{\times} ... \underset{{}^{\mathrm{cl}}T}{\times} {}^{\mathrm{cl}}T_{i_m}, S_{i_0})}{\times} \{f_0\}|_{{}^{\mathrm{cl}}T_{i_0} \underset{{}^{\mathrm{cl}}T}{\times} ... \underset{{}^{\mathrm{cl}}T}{\times} {}^{\mathrm{cl}}T_{i_m}}$$

and

$$\lim_n \mathrm{Maps}({}^{\leq n}T_{i_0} \underset{{}^{\leq n}T}{\times} ... \underset{{}^{\leq n}T}{\times} {}^{\leq n}T_{i_m}, S_{i_0}) \underset{\mathrm{Maps}({}^{\mathrm{cl}}T_{i_0} \underset{{}^{\mathrm{cl}}T}{\times} ... \underset{{}^{\mathrm{cl}}T}{\times} {}^{\mathrm{cl}}T_{i_m}, S_{i_0})}{\times} \{f_0\}|_{{}^{\mathrm{cl}}T_{i_0} \underset{{}^{\mathrm{cl}}T}{\times} ... \underset{{}^{\mathrm{cl}}T}{\times} {}^{\mathrm{cl}}T_{i_m}},$$

respectively.

Now, the required isomorphism follows from the fact that each

$$\mathrm{Maps}(T_{i_0} \underset{T}{\times} ... \underset{T}{\times} T_{i_m}, S_{i_0}) \to \lim_n \mathrm{Maps}({}^{\leq n}T_{i_0} \underset{{}^{\leq n}T}{\times} ... \underset{{}^{\leq n}T}{\times} {}^{\leq n}T_{i_m}, S_{i_0})$$

is an isomorphism (the convergence of S_{i_0} as a prestack).

□

3.4.3. We have the following partial converse to Proposition 3.4.2:

Proposition 3.4.4. *Let Z be an object of* ${}^{\mathrm{conv}}\mathrm{PreStk}$*, such that for every n, the corresponding object* ${}^{\leq n}Z \in {}^{\leq n}\mathrm{PreStk}$ *belongs to* ${}^{\leq n}\mathrm{Sch}$*. Then $Z \in \mathrm{Sch}$.*

Proof. Let $\underset{i}{\sqcup} \widetilde{S}_i \to {}^{\mathrm{cl}}Z$ be a Zariski atlas of ${}^{\mathrm{cl}}Z$. By Corollary 3.1.8(b), for every i we have a compatible family of open embeddings

$$S_{i,n} \to {}^{\leq n}Z.$$

Set

$$S_i = \underset{n}{\mathrm{colim}}\, S_{i,n},$$

where the colimit is taken in $\mathrm{Sch}^{\mathrm{aff}}$. By construction, we have $S_{i,n} = {}^{\leq n}S_i$, and the convergence property of Z implies that we have a well-defined map $S_i \to Z$.

We claim now that Z is a scheme with $\underset{i}{\sqcup} S_i \to Z$ providing a Zariski atlas. Indeed, this follows from Proposition 3.2.2.

□

3.5. **Schemes locally (almost) of finite type.**

3.5.1. We shall denote by ${}^{\leq n}\mathrm{Sch}_{\mathrm{lft}}$ and $\mathrm{Sch}_{\mathrm{laft}}$ the full subcategories of Stk, given by

$$\mathrm{Sch} \cap {}^{\leq n}\mathrm{Stk}_{\mathrm{lft}} \;\text{ and }\; \mathrm{Sch} \cap \mathrm{Stk}_{\mathrm{laft}},$$

respectively.

We will denote by

$${}^{\leq n}\mathrm{Sch}_{\mathrm{ft}} \subset {}^{\leq n}\mathrm{Sch}_{\mathrm{lft}} \;\text{ and }\; \mathrm{Sch}_{\mathrm{aft}} \subset \mathrm{Sch}_{\mathrm{laft}}$$

the full subcategories corresponding to quasi-compact schemes.

3.5.2. We have:

Proposition 3.5.3. *For $Z \in {}^{\leq n}\mathrm{Sch}$ (resp., $Z \in \mathrm{Sch}$) the following conditions are equivalent:*

(i) *$Z \in {}^{\leq n}\mathrm{Sch}_{\mathrm{lft}}$ (resp., $Z \in \mathrm{Sch}_{\mathrm{laft}}$);*

(ii) *For an affine open embedding $Z' \to Z$ with $Z' \in {}^{\leq n}\mathrm{Sch}$ (resp., $Z' \in \mathrm{Sch}$), we have $Z' \in {}^{\leq n}\mathrm{Sch}_{\mathrm{lft}}$ (resp., $Z' \in \mathrm{Sch}_{\mathrm{laft}}$);*

(iii) *Z admits a Zariski atlas consisting of affine schemes from ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ (resp., $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$).*

Proof. Since schemes are convergent (see Proposition 3.4.2), it suffices to treat the case of $Z \in {}^{\leq n}\mathrm{Sch}$.

Assume first that Z admits a Zariski atlas consisting of affine schemes from ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Write

$$Z \simeq {}^{\leq n}L(\underset{a \in A}{\mathrm{colim}}\, S_a),$$

where $S_a \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Using Corollary 2.7.10, we obtain that Z lies in the image of the functor $\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}$, i.e., it belongs to ${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}}$.

Assume now that Z belongs to ${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}}$. We will show that if we have an affine open embedding $Z' \to Z$, then $Z' \in {}^{\leq n}\mathrm{Stk}_{\mathrm{lft}}$.

Let T be an object of ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$. We need to show that the map

$$\underset{a}{\mathrm{colim}}\, Z'(T_a) \to Z'(T) \tag{3.3}$$

is an isomorphism, where a runs over the (*filtered*) category $({}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{T/}$.

The map $Z'(S) \to Z(S)$ is a monomorphism for any $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$. Hence, since

$$\underset{a}{\mathrm{colim}}\, Z(T_a) \to Z(T)$$

is an isomorphism, we obtain that (3.3) is a monomorphism, by filteredness.

Hence, it remains to show that any map $T \to Z'$ can be factored as

$$T \to T_a \to Z',$$

where $T_a \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Consider the composite morphism

$$T \to Z' \to Z,$$

and let $T \to T_b \to Z$ be its factorization with $T_b \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, which exists because Z is locally of finite type.

Now set $T_a := T_b \underset{Z}{\times} Z'$.

□

3.6. **Properties of morphisms.**

3.6.1. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in PreStk. We say that f is *schematic* if for any $S \in \mathrm{Sch}^{\mathrm{aff}}$ and $S \to \mathcal{Y}_2$, the Cartesian product

$$S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$$

is representable by an object of Sch.

The class of schematic maps is tautologically stable under base change. In addition, we claim that the composition of schematic maps is schematic. This is equivalent to the next assertion:

Proposition 3.6.2. *Let Z be a scheme and let $Z' \to Z$ be a schematic map. Then Z' is also a scheme.*

Proof. It is clear that Z' satisfies étale descent.

Let $\underset{i}{\sqcup} S_i \to Z$ be a Zariski atlas of Z. By assumption, each $S_i \underset{Z}{\times} Z'$ is a scheme. Let

$$\underset{j \in J_i}{\sqcup} T_j \to S_i \underset{Z}{\times} Z'$$

be its Zariski atlas. We claim that

$$\underset{i}{\sqcup} \big(\underset{j \in J_i}{\sqcup} T_j \big) \to \underset{i}{\sqcup} S_i \underset{Z}{\times} Z' \to Z'$$

provides a Zariski atlas for Z'.

Indeed, this is true at the classical level. Hence, by Proposition 3.2.2, it suffices to show that each of the maps

$$T_j \to S_i \underset{Z}{\times} Z' \to Z'$$

is an affine open embedding. However, this is evident, since $T_j \to S_i \underset{Z}{\times} Z'$ is such by construction, and $S_i \underset{Z}{\times} Z' \to Z'$ is such being a base change of an open embedding.

□

3.6.3. The next assertion follows from Proposition 3.4.4:

Lemma 3.6.4. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map in ${}^{\mathrm{conv}}\mathrm{PreStk}$. To test the property of f of being schematic (resp., schematic flat/ppf/smooth/étale) it is enough to do so on affine schemes S belonging to ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$. If, moreover, $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{PreStk}_{\mathrm{laft}}$, then it is enough to take $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.*

3.6.5. Since the properties of a morphism in $\mathrm{Sch}^{\mathrm{aff}}$ of being flat/ppf/smooth/étale/Zariski are local in the Zariski topology of the source, they transfer to the corresponding notions for morphisms in Sch:

A morphism $Z' \to Z$ between schemes is flat/ppf/smooth/étale/Zariski if and only if for some (equivalently, any) Zariski atlas $\underset{i}{\sqcup} S'_i \to Z'$, each of the composite maps $S'_i \to Z$ (which is now a schematic affine map of prestacks) has the corresponding property.

Thus, by base change, we obtain the notion of a schematic flat/ppf/smooth/étale/Zariski morphism in PreStk.

3.6.6. The following is obtained by reduction to the affine case:

Lemma 3.6.7. *Let $Z' \overset{f}{\to} Z' \overset{g}{\to} Z''$ be morphisms between schemes. Assume that f is surjective[6] and flat (resp., ppf, smooth, étale, Zariski). If $g \circ f$ is flat (resp., ppf, smooth, étale, Zariski), then so is g.*

4. (Derived) Artin stacks

In this section we introduce the notion of k-Artin stack, $k = 0, 1, ...$ As in the case of schemes, k-Artin stacks are prestacks with some particular properties (but no additional structure).

Our definition is a variation of the definition of *k-geometric stacks* ot *geometric k-stacks* in [TV2]. Although for an individual k, our definition will be different from both these notions from [TV2], the union over all k produces the same class of objects for all three classes of objects.

We also note that from the point of view of (our version of) the hierarchy of k-Artin stacks, schemes (which are, beyond doubt, a natural object of study) are a red herring: the category of schemes properly contains the category of 0-Artin stacks and is properly contained in the category of 1-Artin stacks. As a related phenomenon, we completely bypass the other important notion: that of algebraic space.

As in the previous sections, we will only be interested in only the most formal aspects of the theory: the notions of n-coconnectivity, finite typeness and convergence.

4.1. **Setting up Artin stacks.** For $k \geq 0$, we will define a full subcategory of Stk spanned by objects that we refer to as k-Artin stacks.

In setting up Artin stacks the choice of étale topology is no longer arbitrary. It is made in order to make our system of definitions as simple as possible; see, however, Remark 4.1.4 below.

4.1.1. We start with $k = 0$. We shall say that an object $\mathcal{Y} \in \mathrm{Stk}$ is a 0-Artin stack if it is of the form $L(\underset{i}{\sqcup} S_i)$, where $S_i \in \mathrm{Sch}^{\mathrm{aff}}$. In particular,

$$\mathrm{Stk}^{0\text{-}\mathrm{Artn}} \subset \mathrm{Sch}\,.$$

[6]Surjective=surjective at the level of underlying classical schemes.

4.1.2. To define the notion of k-Artin stack for $k \geq 1$ we proceed inductively.

Along with this notion, we will define what it means for a morphism in PreStk to be k-representable, and for a k-representable morphism what it means to be flat (resp., ppf, smooth, étale, surjective). These notions have an obvious meaning in the case of $k = 0$.

We will inductively assume the following properties:

- Any $(k-1)$-Artin stack is a k-Artin stack;
- Any morphism that is $(k-1)$-representable, is k-representable;
- A $(k-1)$-representable morphism is flat (resp., ppf, smooth, étale, surjective) if and only if it is such when viewed as a k-representable morphism;
- The class of k-representable (resp., k-representable + flat/ppf/smooth/étale/surjective) morphisms is stable under compositions and base change.

It will follow inductively from the construction that the class of k-Artin stacks is closed under fiber products.

4.1.3. Suppose the above notions have been defined for $k' < k$.

We say that $\mathcal{Y} \in \text{Stk}$ is a k-Artin stack if the following conditions hold:

(1) The diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is $(k-1)$-representable.
(2) There exists $\mathcal{Z} \in \text{Stk}^{(k-1)\text{-Artn}}$ and a map $f : \mathcal{Z} \to \mathcal{Y}$ (which is a $(k-1)$-representable by the previous point), which is smooth and surjective.

We shall call the pair $f : \mathcal{Z} \to \mathcal{Y}$ a (smooth) atlas for $\mathcal{Y}$. Note that we can always choose an atlas with $\mathcal{Z} \in \text{Stk}^{0\text{-Artn}}$.

Remark 4.1.4. Here we quote two fundamental results of Toën ([To, Theorem 2.1]). One says that Artin stacks as defined above actually satisfy ppf descent. Another says that if we require ppf descent, but instead of requiring a smooth atlas, we only require a ppf atlas, we still arrive at the same class of objects.

4.1.5. We will say that $\mathcal{Y} \in \text{Stk}$ is an Artin stack if it is a k-Artin stack for some k.

We let $\text{Stk}^{k\text{-Artn}}$ (resp., Stk^{Artn}) denote the full subcategory of Stk spanned by k-Artin (resp., Artin) stacks.

Note that in our definition, schemes are 1-Artin stacks:

$$\text{Sch} \subset \text{Stk}^{1\text{-Artn}}.$$

4.1.6. We say that a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk is k-representable if for every $S \to \mathcal{Y}_2$ with $S \in \text{Sch}^{\text{aff}}$ the fiber product $S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is a k-Artin stack in the above sense.

4.1.7. Let $\mathcal{Y}$ be a k-Artin stack mapping to an affine scheme S. We shall say that this map is flat (resp., ppf, smooth, étale, surjective) if for some atlas $\mathcal{Z} \to \mathcal{Y}$, the composite map of $\mathcal{Z} \to S$ (which is $(k-1)$-representable) is flat (resp., ppf, smooth, étale, surjective). Note that Lemma 3.6.7 implies by induction that if this condition holds for one atlas, then it holds for any other atlas.

4.1.8. We shall say that a k-representable morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is flat (resp., ppf, smooth, étale, surjective) if for every $S \to \mathcal{Y}_2$ with $S \in \text{Sch}^{\text{aff}}$, the map

$$S \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to S$$

is flat (resp., ppf, smooth, étale, surjective).

4.1.9. *Quasi-compactness and quasi-separatedness.* Let $\mathcal{Y}$ be a k-Artin stack. We say that $\mathcal{Y}$ is *quasi-compact* if there exists a smooth atlas $f : S \to \mathcal{Y}$ with $S \in \mathrm{Sch}^{\mathrm{aff}}$.

For a k-representable morphism $\mathcal{Y}_1 \to \mathcal{Y}_2$ in PreStk, we say that it is quasi-compact, if its base change by an affine scheme yields a quasi-compact k-Artin stack.

For $0 \leq k' \leq k$, we define the notion of k'-quasi-separatedness of a k-Artin stack or a k-representable morphism inductively on k'.

We say that a k-Artin stack $\mathcal{Y}$ is 0-quasi-separated if the diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is quasi-compact, as a $(k-1)$-representable map. We say that a k-representable map is 0-quasi-separated if its base change by an affine scheme yields a 0-quasi-separated k-Artin stack.

For $k' > 0$, we say that a k-Artin stack $\mathcal{Y}$ is k'-quasi-separated if the diagonal map $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is $(k'-1)$-quasi-separated, as a $(k-1)$-representable map. We shall say that a k-representable map is k'-quasi-separated if its base change by an affine scheme yields a k'-quasi-separated k-Artin stack.

We shall say that a k-Artin stack is quasi-separated if it is k'-quasi-separated for all k', $0 \leq k' \leq k$. We shall say that a k-representable map is quasi-separated if its base change by an affine scheme yields a quasi-separated k-Artin stack.

4.2. **Verification of the induction hypothesis.**

4.2.1. Tautologically, the class of representable maps is stable under base change. Moreover, diagram chase shows:

Lemma 4.2.2.

(a) *Let a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in* PreStk *be k-representable. Then the diagonal morphism $\mathcal{Y}_1 \to \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is $(k-1)$-representable.*

(b) *Any map between k-Artin stacks is k-representable.*

4.2.3. We claim that the class of k-representable maps is stable under compositions. This is equivalent to the following assertion:

Proposition 4.2.4. *Let $f : \mathcal{Y}' \to \mathcal{Y}$ be a k-representable map in* PreStk *where $\mathcal{Y}$ is a k-Artin stack. Then so is $\mathcal{Y}'$.*

Proof. Consider the diagonal $\mathcal{Y}' \to \mathcal{Y}' \times \mathcal{Y}'$, and factor it as

$$\mathcal{Y}' \to \mathcal{Y}' \underset{\mathcal{Y}}{\times} \mathcal{Y}' \to \mathcal{Y}' \times \mathcal{Y}'.$$

Since f is k-representable we obtain that

$$\mathcal{Y}' \to \mathcal{Y}' \underset{\mathcal{Y}}{\times} \mathcal{Y}'$$

is $(k-1)$-representable (by Lemma 4.2.2(a)). Now, $\mathcal{Y}' \underset{\mathcal{Y}}{\times} \mathcal{Y}' \to \mathcal{Y}' \times \mathcal{Y}'$ is $(k-1)$-representable, being a base change of $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$.

We now need to construct a smooth atlas for $\mathcal{Y}'$. Let $\mathcal{Z} \to \mathcal{Y}$ be a smooth atlas for $\mathcal{Y}$ with $\mathcal{Z} \in \mathrm{Stk}^{0\text{-Artn}}$. By assumption, each $\mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Y}'$ is a k-Artin stack. Choose a smooth atlas $\mathcal{Z}' \to \mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Y}'$. We claim that the composite map

$$\mathcal{Z}' \to \mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Y}' \to \mathcal{Y}'$$

provides a smooth atlas for $\mathcal{Y}'$. Indeed, this map is smooth and surjective, being the composition of $\mathcal{Z}' \to \mathcal{Z}$ (which is smooth and surjective by assumption) and $\mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Y}' \to \mathcal{Y}'$ (which is smooth and surjective, being a base change of $\mathcal{Z} \to \mathcal{Y}$).

□

4.2.5. We claim that that the composition of representable flat/ppf/smooth/étale/surjective maps is itself a flat/ppf/smooth/étale/surjective map. This is equivalent to the following:

Proposition 4.2.6. *Let $\mathcal{Y}' \to \mathcal{Y}$ be a k-representable flat (resp., ppf. smooth, étale, surjective) map, where $\mathcal{Y}$ is a k-Artin stack, equipped with a flat (resp., ppf, smooth, étale, surjective) map to $S \in \mathrm{Sch}^{\mathrm{aff}}$. Then the composite map $\mathcal{Y}' \to S$ is flat (resp., ppf. smooth, étale, surjective).*

Proof. The required property tautologically holds for the atlas constructed in the proof of Proposition 4.2.4.

□

4.3. **Descent properties.**

4.3.1. The following results from the definitions:

Lemma 4.3.2.

(a) *If $f : \mathcal{Z} \to \mathcal{Y}$ is an atlas of a k-Artin stack, then it is an étale surjection.*

(b) *If $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a k-representable morphism, which is étale and surjective, then it is an étale surjection.*

Corollary 4.3.3. *Let $\mathcal{Y}$ be a k-Artin stack and let $f : \mathcal{Z} \to \mathcal{Y}$ be a smooth atlas. Then the natural map*

$$L(|\mathcal{Z}^\bullet/\mathcal{Y}|_{\mathrm{PreStk}}) \simeq |\mathcal{Z}^\bullet/\mathcal{Y}|_{\mathrm{Stk}} \to \mathcal{Y}$$

is an isomorphism, where the subscript Stk *(resp.,* PreStk*) indicates that the geometric realization is taken in* Stk *(resp.,* PreStk*).*

Corollary 4.3.4. *Let $\mathcal{Y}$ be a k-Artin stack. Then for any n, the restriction ${}^{\leq n}\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}$ is $(n+k)$-truncated.*

Proof. We prove the assertion by induction. The assertion for $k = 0$ is a particular case of Proposition 3.3.3. Assume now that the assertion is valid for $k' < k$.

Note that the geometric realization of a m-truncated groupoid object in Spc is $(m+1)$-truncated. Combining this with Lemma 2.5.3, we obtain that it suffices to show that the simplicial prestack $\mathcal{Z}^\bullet/\mathcal{Y}$ has the property that for every n its restriction ${}^{\leq n}(\mathcal{Z}^\bullet/\mathcal{Y})$ is $(n+k-1)$-truncated.

However, each simplex of ${}^{\leq n}(\mathcal{Z}^\bullet/\mathcal{Y})$ belongs to $\mathrm{Stk}^{(k-1)\text{-}\mathrm{Artn}}$, and the assertion follows from the induction hypothesis.

□

4.3.5. We will now prove an (amplified) converse to Corollary 4.3.3. Let $\mathcal{Y}^\bullet$ be a groupoid-object of Stk (see [Lu1, Sect. 6.1.2] for what this means).

Set

$$\mathcal{Y} := |\mathcal{Y}^\bullet|_{\mathrm{Stk}} \simeq L(|\mathcal{Y}^\bullet|_{\mathrm{PreStk}})$$

be its geometric realization. We have

$$\mathcal{Y}^1 \simeq \mathcal{Y}^0 \underset{\mathcal{Y}}{\times} \mathcal{Y}^0 \tag{4.1}$$

(indeed, this tautologically holds before sheafification, and then use the fact that the functor L preserves fiber products).

We claim:

Proposition 4.3.6.
(a) *Assume that in the above situation $\mathcal{Y}^1$ and $\mathcal{Y}^0$ are k-Artin stacks, the maps $\mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0$ are smooth and the map $\mathcal{Y}^1 \to \mathcal{Y}^0 \times \mathcal{Y}^0$ is $(k-1)$-representable. Then $\mathcal{Y}$ is a k-Artin stack.*
(b) *Let*

$$\begin{array}{ccc} \mathcal{Y} & \longleftarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\ f\ } & S' \end{array}$$

be a Cartesian square in Stk *with $S, S' \in \mathrm{Sch}^{\mathrm{aff}}$ and the morphism f being smooth and surjective. Then if $\mathcal{Y}'$ is a k-Artin stack, the so is $\mathcal{Y}$.*
(c) *Suppose that a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ in* PreStk *is k-representable (resp., k-representable and flat/ppf/smooth/étale/surjective). Then so is the morphism $L(f) : L(\mathcal{Y}_1) \to L(\mathcal{Y}_2)$.*

Remark 4.3.7. By Remark 4.1.4, statement (b) of the above lemma can be strengthened: one can relax the condition that the morphism f be ppf instead of smooth. I.e., Artin stacks satisfy ppf descent, and not just smooth descent. Statement (a) can be strengthened accordingly, by requiring that the maps $\mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0$ be ppf instead of smooth.

Remark 4.3.8. The above proposition allows to construct the familiar examples of algebraic stacks. For example, if G is a smooth group-scheme acting on a scheme X, we consider $\mathcal{Y}^1 := G \times X$ as a groupoid acting on $Z^0 := X$, and the resulting 1-Artin stack $\mathcal{Y}$ is what we usually refer to as X/G.

Proof of Proposition 4.3.6. We prove all three assertions by induction on k. The base case is $k = 1$, which we will establish together with the induction step. We note that statements (b) and (c) make sense for $k = 0$, and hold due to Proposition 2.4.2(b) and Corollary 2.4.4, respectively.

We begin by proving point (a).

Let us show that the diagonal morphism of $\mathcal{Y}$ is $(k-1)$-representable. By point (c) for $k-1$, it suffices to show that the map

$$|\mathcal{Y}^\bullet|_{\mathrm{PreStk}} \to |\mathcal{Y}^\bullet|_{\mathrm{PreStk}} \times |\mathcal{Y}^\bullet|_{\mathrm{PreStk}}$$

is $(k-1)$-representable. Fix a map $S \to |\mathcal{Y}^\bullet|_{\mathrm{PreStk}} \times |\mathcal{Y}^\bullet|_{\mathrm{PreStk}}$ with $S \in \mathrm{Sch}^{\mathrm{aff}}$. Such a map factors through a map $S \to \mathcal{Y}^0 \times \mathcal{Y}^0$. Hence,

$$S \underset{|\mathcal{Y}^\bullet|_{\mathrm{PreStk}} \times |\mathcal{Y}^\bullet|_{\mathrm{PreStk}}}{\times} |\mathcal{Y}^\bullet|_{\mathrm{PreStk}} \simeq S \underset{\mathcal{Y}^0 \times \mathcal{Y}^0}{\times} (\mathcal{Y}^0 \times \mathcal{Y}^0) \underset{|\mathcal{Y}^\bullet|_{\mathrm{PreStk}} \times |\mathcal{Y}^\bullet|_{\mathrm{PreStk}}}{\times} |\mathcal{Y}^\bullet|_{\mathrm{PreStk}} \simeq$$
$$\simeq S \underset{\mathcal{Y}^0 \times \mathcal{Y}^0}{\times} (\mathcal{Y}^0 \underset{|\mathcal{Y}^\bullet|_{\mathrm{PreStk}}}{\times} \mathcal{Y}^0) \simeq S \underset{\mathcal{Y}^0 \times \mathcal{Y}^0}{\times} \mathcal{Y}^1.$$

A similar argument shows that the map $\mathcal{Y}^0 \to \mathcal{Y}$ is smooth and surjective. Hence, if $\mathcal{Z} \to \mathcal{Y}_0$ is a smooth atlas for $\mathcal{Z}^0$, then the composition $\mathcal{Z} \to \mathcal{Y}^0 \to \mathcal{Y}^1$ is a smooth atlas for $\mathcal{Y}$.

Let us now prove point (b).

Let $\mathcal{Y}^\bullet$ be the Čech nerve of the map $\mathcal{Y}' \to \mathcal{Y}$. In particular $\mathcal{Y}^0 = \mathcal{Y}'$ is a k-Artin stack. The maps $\mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0$ are affine schematic and smooth, being base-changed from $S' \to S$. In particular, $\mathcal{Y}^1$ is also a k-Artin stack. The map $\mathcal{Y}^1 \to \mathcal{Y}^0 \times \mathcal{Y}^0$ is $(k-1)$-representable since the diagonal morphism of $\mathcal{Y}'$ is $(k-1)$-representable.

Since $\mathcal{Y}' \to \mathcal{Y}$ is an étale surjection, we have $\mathcal{Y} \simeq L(|\mathcal{Y}^\bullet|_{\mathrm{PreStk}})$, by Lemmas 4.3.2(b) and 2.3.8. Applying point (a) we obtain that $\mathcal{Y}$ is a k-Artin stack, as desired.

Finally, let us prove point (c).

Let us be given a map $S \to L(\mathcal{Y}_2)$. We need to show that the fiber product $S \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1)$ is a k-Artin stack (resp., a k-Artin stack, whose map to S is flat/ppf/smooth/étale/surjective).

Since $\mathcal{Y}_2 \to L(\mathcal{Y}_2)$ is an étale surjection, we can find an étale covering $S' \to S$ so that the composition $S' \to S \to L(\mathcal{Y}_2)$ factors as $S' \to \mathcal{Y}_2 \to L(\mathcal{Y}_2)$. Consider the Cartesian square

$$\begin{array}{ccc} S \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1) & \longleftarrow & S' \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1) \\ \downarrow & & \downarrow \\ S & \longleftarrow & S'. \end{array}$$

By point (b), it suffices to show that $S' \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1)$ is a k-Artin stack (the properties of the map $S' \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1) \to S'$ imply the corresponding properties of the map $S \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1) \to S$ by Corollary 2.4.6.)

However, since the functor L commutes with fiber products, we have

$$S' \underset{L(\mathcal{Y}_2)}{\times} L(\mathcal{Y}_1) \simeq L(S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1),$$

where

$$L(S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1) \simeq S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1,$$

since $S' \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$ is a k-Artin stack by assumption.

□

Corollary 4.3.9. *Let $\mathcal{Y}$ be an object of* Stk*, and let $f : \mathcal{Z} \to \mathcal{Y}$ be a $(k-1)$-representable, smooth and surjective morphism, where $\mathcal{Z}$ is a k-Artin stack. Then $\mathcal{Y}$ is a k-Artin stack.*

Proof. Apply Proposition 4.3.6(a) to the Čech nerve of the map $\mathcal{Z} \to \mathcal{Y}$.

□

4.4. **Artin stacks and n-coconnectivity.**

4.4.1. Replacing the category Sch by ${}^{\leq n}\mathrm{Sch}$ in the above discussion, we arrive to the definition of the category ${}^{\leq n}\mathrm{Stk}^{k\text{-Artn}}$.

It is clear that the restriction functor under ${}^{\leq n}\mathrm{Sch} \hookrightarrow \mathrm{Sch}$ sends $\mathrm{Stk}^{k\text{-Artn}}$ to ${}^{\leq n}\mathrm{Stk}^{k\text{-Artn}}$.

4.4.2. We claim:

Proposition 4.4.3.
(a) *The functor*

$${}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}} : {}^{\leq n}\mathrm{Stk} \hookrightarrow \mathrm{Stk}$$

sends ${}^{\leq n}\mathrm{Stk}^{k\text{-Artn}}$ *to* $\mathrm{Stk}^{k\text{-Artn}}$.
(b) *If $\mathcal{Z} \to \mathcal{Y}$ is a smooth atlas for an object $\mathcal{Y} \in {}^{\leq n}\mathrm{Stk}^{k\text{-Artn}}$, then*

$${}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Z}) \to {}^{L}\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y})$$

is a smooth atlas.

Proof. We will prove the proposition by induction on k, assuming its validity for $k' < k$. We note that the assertion for a given k' implies the following:

(i) If $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a k'-representable (resp., k'-representable and flat/smooth) map in ${}^{\leq n}\mathrm{PreStk}$, then the induced map in PreStk

$$^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}_1) \to {}^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}_2)$$

is also k'-representable (resp., k'-representable and flat/smooth).

(ii) If we have a Cartesian diagram in ${}^{\leq n}\mathrm{Stk}^{k'\text{-Artn}}$

$$\begin{array}{ccc} \mathcal{Y}_1' & \longrightarrow & \mathcal{Y}_1 \\ \downarrow & & \downarrow \\ \mathcal{Y}_2' & \longrightarrow & \mathcal{Y}_2 \end{array} \tag{4.2}$$

with the vertical arrows flat, then the diagram

$$\begin{array}{ccc} ^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}_1') & \longrightarrow & ^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}_1) \\ \downarrow & & \downarrow \\ ^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}_2') & \longrightarrow & ^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}_2) \end{array} \tag{4.3}$$

is Cartesian as well.

Let us now carry out the induction step.

Let $\mathcal{Y}$ be an object of ${}^{\leq n}\mathrm{Stk}^{k\text{-Artn}}$. By Corollary 4.3.3, for a given smooth atlas $\mathcal{Z} \to \mathcal{Y}$, we can write $\mathcal{Y}$ as $|\mathcal{Z}^\bullet|_{^{\leq n}\mathrm{Stk}}$, where $\mathcal{Z}^\bullet$ is the Čech nerve of $\mathcal{Z} \to \mathcal{Y}$, In particular, $\mathcal{Z}^\bullet$ is a groupoid object in ${}^{\leq n}\mathrm{Stk}^{(k-1)\text{-Artn}}$.

By (ii) above, the simplicial object of Stk given by

$$^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Z}^\bullet)$$

is a groupoid object. Moreover, by (i) above, it satisfies the assumption of Proposition 4.3.6(a). Hence,

$$\mathcal{Y}' := |{}^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Z}^\bullet)|$$

is an object of $\mathrm{Stk}^{k\text{-Artn}}$.

Furthermore, $\mathcal{Y}'$ is n-coconnective as a stack, whose restriction to ${}^{\leq n}\mathrm{Sch}$ identifies with $\mathcal{Y}$. Therefore,

$$\mathcal{Y}' \simeq {}^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}).$$

□

4.4.4. We shall say that an object of $\mathrm{Stk}^{k\text{-Artn}}$ is n-coconnective if it is n-coconnective as an object of Stk. From Proposition 4.4.3, we obtain:

Corollary 4.4.5. *The functor* $^L\mathrm{LKE}_{^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}$ *is an equivalence from* ${}^{\leq n}\mathrm{Stk}^{k\text{-Artn}}$ *to the full subcategory of* $\mathrm{Stk}^{k\text{-Artn}}$*, spanned by n-coconnective k-Artin stacks.*

Warning: We emphasize again that being n-coconnective as a stack does *not* imply being n-coconnective as a prestack.

4.4.6. We will now characterize those k-Artin stacks that are n-coconnective:

Proposition 4.4.7. *Let $\mathcal{Y}$ be a k-Artin stack. The following conditions are equivalent:*
(i) *$\mathcal{Y}$ is n-coconnective.*
(ii) *There exists an atlas $f : Z \to \mathcal{Y}$, where $Z \in {}^{\leq n}\mathrm{Stk}^{0\text{-Artn}}$.*
(iii) *If $\mathcal{Y}' \to \mathcal{Y}$ is a k-representable flat map, then $\mathcal{Y}'$ is n-coconnective as a stack.*

Proof. We argue inductively on k, assuming the validity for $k' < k$.

The implication (i) $\Rightarrow$ (ii) follows from Proposition 4.4.3(b).

Let us show that (ii) implies (i). By Corollary 4.3.3, it suffices to show that the Čech nerve of the atlas $Z \to \mathcal{Y}$ consists of $(k-1)$-Artin stacks that are n-coconnective. However, this follows from the implication (i) $\Rightarrow$ (iii) for $k-1$.

The implication (iii) $\Rightarrow$ (ii) is tautological: the assumption in (iii) implies that for any smooth atlas $Z \to \mathcal{Y}$, the scheme Z is n-coconnective.

Finally, the implication (i),(ii) $\Rightarrow$ (iii) follows by retracing the construction of the atlas in the proof of Proposition 4.2.4.

□

4.4.8. *Artin stacks and convergence.* We will now prove:

Proposition 4.4.9.
(a) *Any k-Artin stack, viewed as an object of* PreStk*, is convergent.*
(b) *Let $\mathcal{Y} \in {}^{\mathrm{conv}}\mathrm{PreStk}$ be such that for any n, we have ${}^{\leq n}\mathcal{Y} \in {}^{\leq n}\mathrm{Stk}^{k\text{-Artn}}$. Then $\mathcal{Y}$ is a k-Artin stack.*

Proof. We proceed by induction on k. For $k = 0$, point (a) follows from Proposition 3.4.2, and point (b) follows by repeating the argument of Proposition 3.4.4.

We first prove point (a), assuming the validity of both (a) and (b) for $k' < k$.

Let $f : \mathcal{Z} \to \mathcal{Y}$ be a smooth atlas $\mathcal{Y}$. By Corollary 4.3.3, we have:

$$\mathcal{Y} \simeq |\mathcal{Z}^\bullet/\mathcal{Y}|_{\mathrm{Stk}}.$$

Consider the induced map ${}^{\mathrm{conv}}f : {}^{\mathrm{conv}}\mathcal{Z} \to {}^{\mathrm{conv}}\mathcal{Y}$. We claim that ${}^{\mathrm{conv}}f$ is $(k-1)$-representable, smooth and surjective. Indeed, for $S \to {}^{\mathrm{conv}}\mathcal{Y}$ with $S \in \mathrm{Sch}^{\mathrm{aff}}$, for every n, we have

$${}^{\leq n}(S \underset{{}^{\mathrm{conv}}\mathcal{Y}}{\times} {}^{\mathrm{conv}}\mathcal{Z}) \simeq {}^{\leq n}S \underset{{}^{\leq n}\mathcal{Y}}{\times} {}^{\leq n}\mathcal{Z} \in {}^{\leq n}\mathrm{Stk}^{(k-1)\text{-Artn}}.$$

Hence, $S \underset{{}^{\mathrm{conv}}\mathcal{Y}}{\times} {}^{\mathrm{conv}}\mathcal{Z}$ is a $(k-1)$-Artin stack by the induction hypothesis. Moreover, since each ${}^{\leq n}S \underset{{}^{\leq n}\mathcal{Y}}{\times} {}^{\leq n}\mathcal{Z}$ is smooth and surjective over ${}^{\leq n}S$, by Lemma 2.1.3, we obtain that $S \underset{{}^{\mathrm{conv}}\mathcal{Y}}{\times} {}^{\mathrm{conv}}\mathcal{Z}$ is smooth and surjective over S.

In particular, by Lemma 4.3.2(b), we obtain that ${}^{\mathrm{conv}}\mathcal{Z} \to {}^{\mathrm{conv}}\mathcal{Y}$ is an étale surjection, and hence

$${}^{\mathrm{conv}}\mathcal{Y} \simeq |{}^{\mathrm{conv}}\mathcal{Z}^\bullet/{}^{\mathrm{conv}}\mathcal{Y}|_{\mathrm{Stk}}.$$

However, we claim that the map of the cosimplicial objects

$$\mathcal{Z}^\bullet/\mathcal{Y} \to {}^{\mathrm{conv}}\mathcal{Z}^\bullet/{}^{\mathrm{conv}}\mathcal{Y}$$

is an isomorphism. Indeed, for every m, we have

$${}^{\mathrm{conv}}\mathcal{Z}^m/{}^{\mathrm{conv}}\mathcal{Y} \simeq {}^{\mathrm{conv}}(\mathcal{Z}^m/\mathcal{Y}),$$

where $\mathcal{Z}^m/\mathcal{Y}$ is a $(k-1)$-Artin stack, and hence $\mathcal{Z}^m/\mathcal{Y} \to {}^{\mathrm{conv}}(\mathcal{Z}^m/\mathcal{Y})$ is an isomorphism by the induction hypothesis.

To prove point (b) we will need to appeal to deformation theory. Choose a smooth atlas $Z_0 \to {}^{\mathrm{cl}}\mathcal{Y}$ with $Z_0 \in {}^{\mathrm{cl}}\mathrm{Stk}^{0\text{-}\mathrm{Artn}}$. Then deformation theory (see [Chapter III.1, Sect. 7.4]) implies that we can we can construct a compatible system of objects $Z_n \in {}^{\leq n}\mathrm{Stk}^{0\text{-}\mathrm{Artn}}$ equipped with smooth maps $Z_n \to {}^{\leq n}\mathcal{Y}$.

Set $Z := \underset{n}{\mathrm{colim}}\, Z_n$, where the colimit is taken in ${}^{\mathrm{conv}}\mathrm{PreStk}$. By the case of $k = 0$, we have $Z \in \mathrm{Stk}^{0\text{-}\mathrm{Artn}}$, and since $\mathcal{Y}$ is convergent we have a canonically defined map $Z \to \mathcal{Y}$. Set $\mathcal{Y}^\bullet := Z^\bullet/\mathcal{Y}$. By Lemma 2.5.9, we have $\mathcal{Y} \in \mathrm{Stk}$. Hence, the map

$$|\mathcal{Y}^\bullet|_{\mathrm{Stk}} \to \mathcal{Y}$$

is an isomorphism.

Thus, by Proposition 4.3.6(a), it suffices to show that the map

$$\mathcal{Y}^1 = Z \underset{\mathcal{Y}}{\times} Z \to Z \times Z$$

is $(k-1)$-representable and its composition with either of the the projections $Z \times Z \to Z$ is smooth. By the induction hypothesis and Lemma 2.1.3, it suffices to show that the map

$$Z_n \underset{{}^{\leq n}\mathcal{Y}}{\times} Z_n = {}^{\leq n}Z \underset{{}^{\leq n}\mathcal{Y}}{\times} {}^{\leq n}Z \simeq {}^{\leq n}(Z \underset{\mathcal{Y}}{\times} Z) \to {}^{\leq n}(Z \times Z) \simeq {}^{\leq n}Z \times {}^{\leq n}Z = Z_n \times Z_n$$

has the corresponding properties. However, this follows from the fact that the map $Z_n \to {}^{\leq n}\mathcal{Y}$ is $(k-1)$-representable and smooth.

□

4.5. **Artin stacks locally almost of finite type.**

4.5.1. The goal of this subsection is to establish the following:

Proposition 4.5.2. *Let $\mathcal{Y}$ be an object of* $\mathrm{Stk}^{k\text{-}\mathrm{Artn}}$ *(resp.,* ${}^{\leq n}\mathrm{Stk}^{k\text{-}\mathrm{Artn}}$*). The following conditions are equivalent:*

(i) $\mathcal{Y} \in \mathrm{Stk}_{\mathrm{laft}}$ *(resp.,* $\mathcal{Y} \in {}^{\leq n}\mathrm{Stk}_{\mathrm{lft}}$*);*

(ii) $\mathcal{Y}$ *admits an atlas $f : Z \to \mathcal{Y}$ with* $Z \in \mathrm{Stk}^{0\text{-}\mathrm{Artn}}_{\mathrm{laft}}$ *(resp.,* $Z \in {}^{\leq n}\mathrm{Stk}^{0\text{-}\mathrm{Artn}}_{\mathrm{lft}}$*);*

(iii) *For a k-representable ppf morphism $Z \to \mathcal{Y}$ with* $Z \in \mathrm{Stk}^{0\text{-}\mathrm{Artn}}$ *(resp.,* $Z \in {}^{\leq n}\mathrm{Stk}^{0\text{-}\mathrm{Artn}}$*), we have* $Z \in \mathrm{Stk}^{0\text{-}\mathrm{Artn}}_{\mathrm{laft}}$ *(resp.,* $Z \in {}^{\leq n}\mathrm{Stk}^{0\text{-}\mathrm{Artn}}_{\mathrm{lft}}$*).*

(iv) *For a k-representable ppf morphism $\mathcal{Y}' \to \mathcal{Y}$, we have* $\mathcal{Y}' \in \mathrm{Stk}_{\mathrm{laft}}$ *(resp.,* $\mathcal{Y}' \in {}^{\leq n}\mathrm{Stk}_{\mathrm{lft}}$*)*

We will call k-Artin stacks satisfying the equivalent conditions of the above proposition 'k-Artin stacks locally almost of finite type'.

Since we know that k-Artin stacks are convergent, it is enough to treat the case of $\mathcal{Y} \in {}^{\leq n}\mathrm{Stk}^{k\text{-}\mathrm{Artn}}$.

4.5.3. The proof of the proposition proceeds by induction, so we are assuming that all four conditions are equivalent for $k' < k$.

The implications (iii) ⇒ (ii) and (iv) ⇒ (iii) are tautological. The construction of the atlas in Proposition 4.2.4 shows that (ii) and (iii) imply (iv).

4.5.4. *Implication* (*ii*) $\Rightarrow$ (*i*). By Corollary 4.3.3, we have

$$\mathcal{Y} \simeq {}^{\leq n}L(|Z^\bullet/\mathcal{Y}|_{\leq n}\mathrm{PreStk}).$$

First, the implication (i) $\Rightarrow$ (iv) for $k-1$ implies that the terms of the Čech nerve of the atlas $Z \to \mathcal{Y}$ consist of objects of ${}^{\leq n}\mathrm{Stk}_{\mathrm{lft}}$. Hence, we can rewrite the expression for $\mathcal{Y}$ as

$${}^{\leq n}L \circ \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}|_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}).$$

However, by Corollary 4.3.4, the restriction $\mathcal{Y}|_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}$ is $(n+k)$-truncated. Hence, applying Proposition 2.7.7, we obtain that

$${}^{\leq n}L \circ \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}|_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}) \simeq \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}(\mathcal{Y}|_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}})$$

(i.e., no sheafification is necessary).

Thus, $\mathcal{Y}$, viewed as an object of ${}^{\leq n}\mathrm{PreStk}$, lies in the essential image of $\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \hookrightarrow {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}}$, i.e., belongs to ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$.

4.5.5. *Implication* (*i*) $\Rightarrow$ (*iii*). (J.Lurie)

It is easy to see that we can assume that $Z = S$ is an affine scheme. Let us be given a ppf map $f : S \to \mathcal{Y}$. We wish to show that $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Since $\mathcal{Y} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$, there exists $T \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, such that f factors as

$$S \xrightarrow{h} T \xrightarrow{g} \mathcal{Y}.$$

Consider the Cartesian square:

$$\begin{array}{ccc} T \underset{\mathcal{Y}}{\times} S & \xrightarrow{g'} & S \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & \mathcal{Y}. \end{array}$$

Since the map f is ppf, so is f'. Let $Z' \to T \underset{\mathcal{Y}}{\times} S$ be an atlas with $Z' \in {}^{\leq n}\mathrm{Stk}^{0\text{-}\mathrm{Artn}}$. We obtain that Z' is ppf over T. Since T is of finite type, we obtain that $Z' \in {}^{\leq n}\mathrm{Stk}^{0\text{-}\mathrm{Artn}}_{\mathrm{lft}}$.

Since $T \underset{\mathcal{Y}}{\times} S \in {}^{\leq n}\mathrm{Stk}^{(k-1)\text{-}\mathrm{Artn}}$, by the induction hypothesis, we obtain that

$$T \underset{\mathcal{Y}}{\times} S \in {}^{\leq n}\mathrm{Stk}^{(k-1)\text{-}\mathrm{Artn}}_{\mathrm{lft}} \subset {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}.$$

Consider now the maps

$$S \xrightarrow{\mathrm{diag}} S \underset{\mathcal{Y}}{\times} S \xrightarrow{h \times \mathrm{id}} T \underset{\mathcal{Y}}{\times} S \to S,$$

where the last map if the projection on the second factor. The composition is the identity map on S. Hence, S is a retract of $T \underset{\mathcal{Y}}{\times} S$ as an object of ${}^{\leq n}\mathrm{PreStk}$. Since the subcategory ${}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \subset {}^{\leq n}\mathrm{PreStk}$ is stable under retracts, we obtain that

$$S \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}} \cap {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}.$$

Now, the assertion that $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ follows from Lemma 1.6.6.